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A Radon-Nikodým theorem for Fréchet measures

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Abstract

We apply results in operator space theory to the setting of multidimensional measure theory. Using the extended Haagerup tensor product of Effros and Ruan, we derive a Radon-Nikodým theorem for bimeasures and then extend the result to general Fréchet measures (scalar-valued polymeasures). We also prove a measure-theoretic Grothendieck inequality, provide a characterization of the injective tensor product of two spaces of Lebesgue integrable functions, and discuss the possibility of a bounded convergence theorem for Fréchet measures.

Keywords: multidimensional measure theory, bimeasures, extended Haagerup tensor product
2000 MSC: 28A10, 46M10

1. Introduction

The origins of multidimensional measure theory (also known as multilinear measure theory) can be traced back to the work of Fréchet in 1915 [11], when he characterized the bounded bilinear functionals on $C[0,1]$. These bounded bilinear functionals later came to be identified with set functions called bimeasures [17]. Since that time, multidimensional measure theory has developed and contains many interesting and deep results (e.g., [2, 3, 7, 18]).

In higher dimensions, these set functions have been called *polymeasures* or *multimeasures*, but we prefer the name *Fréchet measures* when the set functions are scalar-valued [3], which is the case considered here. Let $(X_1, \mathcal{A}_1), \ldots, (X_n, \mathcal{A}_n)$ be measurable spaces. A *Fréchet measure*, or $\mathcal{F}_n$*-measure*, on $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ is a scalar-valued set function $\mu : \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathbb{C}$ that is a measure in each argument separately; that is, $\mu(E_1, \ldots, E_{j-1}, \cdot, E_{j+1}, \ldots, E_n)$ is a measure on $\mathcal{A}_j$ for fixed $E_k \in \mathcal{A}_k$ $(k \neq j)$. We denote by $\mathcal{F}_n = \mathcal{F}_n(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ the collection of all n-dimensional Fréchet measures on $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$.

Fréchet measures have received much attention over the years, and have recently found application in the context of stochastic processes [15], quantum

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mechanics [14], harmonic analysis [12], and functional analysis [4].

The *Fréchet variation* of $\mu \in \mathcal{F}_n(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is given by

$$\|\mu\|_{\mathcal{F}_n} = \sup \Big| \sum_{j_1,\ldots,j_n} \epsilon_{j_1} \cdots \epsilon_{j_n} \, \mu(E_{j_1}, \ldots, E_{j_n}) \Big|, \tag{1}$$

where the supremum is taken over all $\epsilon_{j_k} \in \mathbb{C}$ such that $|\epsilon_{j_k}| \leq 1$ and over finite measurable partitions $(E_{j_k})_{j_k}$ of $\mathcal{A}_k$, for $1 \leq k \leq n$. It is known that $(\mathcal{F}_n, \|\cdot\|_{\mathcal{F}_n})$ is a Banach space [3, Corollary VI.7].

Fréchet measures were introduced to characterize bounded multilinear functionals on certain spaces of continuous functions. If $K_1, \ldots, K_n$ are compact Hausdorff spaces, and $\mathcal{B}_1, \ldots, \mathcal{B}_n$ are their respective Borel fields, then there is a one-to-one correspondence between bounded n-linear functionals on $C(K_1) \times \cdots \times C(K_n)$ and elements of $\mathcal{F}_n(\mathcal{B}_1, \ldots, \mathcal{B}_n)$. A natural way of phrasing this statement is in the context of *projective tensor products*, in which case we have

$$\big(C(K_1) \widehat{\otimes} \cdots \widehat{\otimes} C(K_n)\big)^* = \mathcal{F}_n(\mathcal{B}_1, \ldots, \mathcal{B}_n). \tag{2}$$

(See [3, Theorem VI.13].) This provides an elegant generalization of the classical Riesz representation theorem.

One may wonder: In what way do other classical theorems of measure theory generalize to higher dimensions? Is there, for example, a generalization of the Radon-Nikodým theorem? Indeed, there have been some theorems in this direction [10, 13]. In this note, we prove a very natural generalization of the classical Radon-Nikodým theorem, which (following the example of (2)) casts the theorem in the context of tensor products:

Theorem 1.1. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and let $\mu \in \mathcal{F}_2(\mathcal{A}, \mathcal{B})$. Suppose ν_1 and ν_2 are positive σ-finite measure on $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, respectively. If μ is absolutely continuous with respect to $\nu_1 \times \nu_2$, then there exists a function $\psi \in L^1(X, \nu_1) \otimes_{eh} L^1(Y, \nu_2)$ such that*

$$\int \phi \, d\mu = \int_{X \times Y} \phi(x, y) \, \psi(x, y) \, (\nu_1 \times \nu_2)(dx, dy),$$

for all $\phi \in L^\infty(X, \nu_1) \widehat{\otimes} L^\infty(Y, \nu_2)$.

When we say μ is *absolutely continuous* with respect to $\nu_1 \times \nu_2$, we mean that $\mu(E, F) = 0$ whenever $\nu_1(E) = 0$ or $\nu_2(F) = 0$. The tensor product $\otimes_{eh}$ is the extended Haagerup tensor product of Effros and Ruan [9]. The function ψ is in general not an element of $L^1(\nu_1 \times \nu_2)$, but rather the limit (in a certain weak* sense) of a specific net of integrable functions. By an abuse of notation, we write this limit as a pointwise limit:

$$\psi(x, y) = \lim_K \psi_K(x, y), \quad (x, y) \in X \times Y.$$

The integral in Theorem 1.1 is also given by a limit:

$$\int \phi \, \psi \, d(\nu_1 \times \nu_2) = \lim_K \int \phi \, \psi_K \, d(\nu_1 \times \nu_2).$$

In Section 2, we introduce much of the notation and background information we will be using. In Section 3, we introduce absolute continuity for Fréchet measures and relate it to integrability. The multidimensional Radon-Nikodým theorem for bimeasures appears in Section 4. Before that, however, we recall the Haagerup and extended Haagerup tensor products. As an application of the extended Haagerup tensor product, we prove a measure-theoretic version of the Grothendieck inequality.

In Section 5, we use the Haagerup tensor product to provide a characterization of the injective tensor product of spaces of Lebesgue integrable functions:

$$L^1(X,\nu_1)\check{\otimes}L^1(Y,\nu_2) = L^1(X,\nu_1) \otimes_h L^1(Y,\nu_2).$$

Also, if $\|\cdot\|_{\check{\otimes}}$ and $\|\cdot\|_{\otimes_h}$ denote the norms on the injective and Haagerup tensor products, respectively, then $\|\cdot\|_{\check{\otimes}} \leq \|\cdot\|_{\otimes_h} \leq K_G\|\cdot\|_{\check{\otimes}}$, where K_G is the Grothendieck constant.

In Section 6, we generalize to higher dimensions. In Section 7, we close by discussing the possibility of a bounded convergence theorem for Fréchet measures. To make use of the machinery of operator theory, we will work over the scaler field $\mathbb{C}$. Many of the arguments can be adapted to $\mathbb{R}$, but with some change in constants.

2. Background

Let $(X,\mathcal{A})$ be a measurable space. We denote by $L^\infty(X)$ the collection of scalar valued bounded measurable functions on X. This forms a Banach space when equipped with the supremum norm:

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}, \quad f \in L^\infty(X).$$

Now suppose ν is a σ-finite measure on $(X,\mathcal{A})$. We let $L^\infty(X,\nu)$ denote the collection of equivalence classes of scalar-valued essentially bounded measurable functions on X. As is standard, we consider functions to be equivalent when they are equal almost everywhere with respect to ν. This forms a Banach space when equipped with the essential supremum norm:

$$\|f\|_\infty = \inf\{M \in \mathbb{R} : \nu(|f(x)| > M) = 0\}, \quad f \in L^\infty(X,\nu).$$

In the event we must distinguish the supremum norm from the essential supremum norm, we will write $\|\cdot\|_{L^\infty(X)}$ and $\|\cdot\|_{L^\infty(X,\nu)}$, respectively. We remark that every equivalence class in $L^\infty(X,\nu)$ contains a function in $L^\infty(X)$.

If $1 \leq p < \infty$, we let $L^p(X,\nu)$ be the collection of equivalence classes of scalar-valued measurable functions f on X such that $\int_X |f(x)|^p\,\nu(dx) < \infty$. Once again, we consider two functions to be equivalent in $L^p(X,\nu)$ whenever they are equal almost everywhere with respect to the σ-finite measure ν. The set $L^p(X,\nu)$ forms a Banach space when equipped with the norm

$$\|f\|_p = \Big(\int_X |f(x)|^p\,\nu(dx)\Big)^{1/p}, \quad f \in L^p(X,\nu).$$

When unambiguous, we often write $L^p(\nu)$ instead of $L^p(X,\nu)$.

For any Banach space X, the notation $\|\cdot\|_X$ will denote the norm on X. The dual space of X will be denoted X^*, and the dual action of X^* on X will be written $\langle x, x^*\rangle$, for $x \in X$ and $x^* \in X^*$. Any other notation will be defined as it appears. We mostly follow the conventions of [2].

Let $(X_1,\mathcal{A}_1),\ldots,(X_n,\mathcal{A}_n)$ be measurable spaces. Define the *projective tensor product* $\mathcal{V}_n = \mathcal{V}_n(\mathcal{A}_1,\ldots,\mathcal{A}_n) = L^\infty(X_1)\widehat{\otimes}\cdots\widehat{\otimes}L^\infty(X_n)$ to be the completion of the algebraic tensor product $L^\infty(X_1)\otimes\cdots\otimes L^\infty(X_n)$ in the projective tensor norm

$$\|\phi\|_{\mathcal{V}_n} = \inf\Big\{\sum_j \|f_j^{(1)}\|_\infty\cdots\|f_j^{(n)}\|_\infty : \phi = \sum_j f_j^{(1)}\otimes\cdots\otimes f_j^{(n)}\Big\},$$

where the infimum is taken over pointwise representations and finite sums.

It is known that if ϕ is an element of $\mathcal{V}_n$, then there exists a pointwise representation $\phi = \sum_{j=1}^\infty f_j^{(1)}\otimes\cdots\otimes f_j^{(n)}$ and $\|\phi\|_{\mathcal{V}_n}$ is obtained as the infimum of $\sum_{j=1}^\infty \|f_j^{(1)}\|_\infty\cdots\|f_j^{(n)}\|_\infty$ over such representations [5, Proposition 1.1.4].

Theorem 2.1. *There exists a well-defined integral $\int\phi\,d\mu$, for every $\phi\in\mathcal{V}_n$ and $\mu\in\mathcal{F}_n$, and $|\int\phi\,d\mu| \le \|\phi\|_{\mathcal{V}_n}\|\mu\|_{\mathcal{F}_n}$.*

A proof of this theorem can be found in [2] or [3]. (In these works, the reader will notice the presence of a 2^n term which results from defining the Fréchet variation using real scalars ϵ_{j_k} in (1).) We will give an outline of one construction of the integral; one we shall make use of later. Let $(f_1,\ldots,f_n)\in L^\infty(X_1)\times\cdots\times L^\infty(X_n)$ and $\mu\in\mathcal{F}_n(\mathcal{A}_1,\ldots,\mathcal{A}_n)$. Define a set function on $\mathcal{A}_2\times\cdots\times\mathcal{A}_n$ as follows:

$$\mu_{f_1}(A_2,\ldots,A_n) = \int_{X_1} f_1(x_1)\,\mu(dx_1,A_2,\ldots,A_n), \tag{3}$$

for all $(A_2,\ldots,A_n)\in\mathcal{A}_2\times\cdots\times\mathcal{A}_n$. The set function μ_{f_1} is a Fréchet measure on the product $\mathcal{A}_2\times\cdots\times\mathcal{A}_n$ and $\|\mu_{f_1}\|_{\mathcal{F}_{n-1}} \le \|f_1\|_\infty\|\mu\|_{\mathcal{F}_n}$ [3, Lemma VI.9]. (The integral in (3) is well-defined, since $\mu(\cdot,A_2,\ldots,A_n)$ is a countably additive measure, by assumption.)

Continuing recursively, let $m\in\mathbb{N}$ be such that $1<m<n$, and define a scalar-valued set function $\mu_{f_1\otimes\cdots\otimes f_m}$ on $\mathcal{A}_{m+1}\times\cdots\times\mathcal{A}_n$ by

$$\mu_{f_1\otimes\cdots\otimes f_m}(A_{m+1},\ldots,A_n) = \int_{X_m} f_m(x_m)\,\mu_{f_1\otimes\cdots\otimes f_{m-1}}(dx_m,A_{m+1},\ldots,A_n), \tag{4}$$

for all $(A_{m+1},\ldots,A_n)\in\mathcal{A}_{m+1}\times\cdots\times\mathcal{A}_n$. It follows that $\mu_{f_1\otimes\cdots\otimes f_m}$ is an element of $\mathcal{F}_{n-m}(\mathcal{A}_{m+1},\ldots,\mathcal{A}_n)$ and $\|\mu_{f_1\otimes\cdots\otimes f_m}\|_{\mathcal{F}_{n-m}} \le \|f_1\|_\infty\cdots\|f_m\|_\infty\|\mu\|_{\mathcal{F}_n}$.

Finally, we observe that $\mu_{f_1\otimes\cdots\otimes f_{n-1}}$ is a countably additive measure on $\mathcal{A}_n$ of finite total variation. If $(f_1,\ldots,f_n)\in L^\infty(X_1)\times\cdots\times L^\infty(X_n)$ and $\mu\in\mathcal{F}_n(\mathcal{A}_1,\ldots,\mathcal{A}_n)$, then

$$\int f_1\otimes\cdots\otimes f_n\,d\mu = \int_{X_n} f_n(x_n)\,\mu_{f_1\otimes\cdots\otimes f_{n-1}}(dx_n). \tag{5}$$

If $\phi \in \mathcal{V}_2$ and $\phi = \sum_{j=1}^{\infty} f_j^{(1)} \otimes \cdots \otimes f_j^{(n)}$ is a pointwise representation, then the integral of ϕ with respect to μ can be computed by

$$\int \phi \, d\mu = \sum_{j=1}^{\infty} \int f_j^{(1)} \otimes \cdots \otimes f_j^{(n)} \, d\mu. \tag{6}$$

The equality in (5) provides a method of iterating the integral. It is worth noting that the order of integration is irrelevant. To make this precise, we state a Fubini-type theorem, the proof of which can be found in [3, Theorem VI.10]. It suffices to state the theorem for $n = 2$:

Theorem 2.2. *If $(f_1, f_2) \in L^\infty(X_1) \times L^\infty(X_2)$ and $\mu \in \mathcal{F}_2(\mathcal{A}_1, \mathcal{A}_2)$, then*

$$\int_{X_1} f_1(x_1) \, \mu_{f_2}(dx_1) = \int_{X_2} f_2(x_2) \, \mu_{f_1}(dx_2).$$

Now define the *Grothendieck tensor product* to be the space

$$\mathcal{G}_n = \mathcal{G}_n(\mathcal{A}_1, \ldots, \mathcal{A}_n) = L^\infty(X_1) \otimes_g \cdots \otimes_g L^\infty(X_n),$$

the completion of the algebraic tensor product $L^\infty(X_1) \otimes \cdots \otimes L^\infty(X_n)$ in the norm

$$\|\phi\|_g = \inf \left\{ \Big\| \sum_j |f_j^{(1)}|^2 \Big\|_\infty^{1/2} \cdots \Big\| \sum_j |f_j^{(n)}|^2 \Big\|_\infty^{1/2} : \phi = \sum_j f_j^{(1)} \otimes \cdots \otimes f_j^{(n)} \right\}.$$

The infimum is taken over all pointwise representations.

The following theorem is taken from [2, Theorem 1.1]. It is a multilinear extension of the well-known Grothendieck inequality.

Theorem 2.3 (Blei)**.** *For all $n \geq 2$, we have the inclusion $\mathcal{G}_n \subseteq \mathcal{V}_n$ and there is a constant $c_n > 0$ that depends only on n such that $\|\phi\|_{\mathcal{V}_n} \leq c_n \|\phi\|_{\mathcal{G}_n}$ for all $\phi \in \mathcal{G}_n$. In particular, $\mathcal{G}_2 = \mathcal{V}_2$.*

The Grothendieck inequality itself is obtained when $n = 2$; $c_2 = K_G$ is the Grothendieck constant. As a consequence of Theorem 2.3, we have the following [2, Corollary 2.2]:

Theorem 2.4 (Blei)**.** *Every $\phi \in \mathcal{G}_n$ is integrable with respect to every $\mu \in \mathcal{F}_n$, and $\int \phi \, d\mu = \sum_{j=1}^{\infty} \int f_j^{(1)} \otimes \cdots \otimes f_j^{(n)} \, d\mu$, whenever $\phi = \sum_{j=1}^{\infty} f_j^{(1)} \otimes \cdots \otimes f_j^{(n)}$. Furthermore, $\left| \int \phi \, d\mu \right| \leq c_n \|\phi\|_{\mathcal{G}_n} \|\mu\|_{\mathcal{F}_n}$, where $c_n > 0$ is a constant that depends only on n.*

3. Absolute continuity of Fréchet measures

Let $(X_1, \mathcal{A}_1), \ldots, (X_n, \mathcal{A}_n)$ be measurable spaces and suppose $\nu_1, \ldots, \nu_n$ are positive σ-finite measures on $\mathcal{A}_1, \ldots, \mathcal{A}_n$ (respectively). A Fréchet measure $\mu \in \mathcal{F}_n(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ is said to be *absolutely continuous* with respect to $\nu_1 \times \cdots \times \nu_n$

if $\mu(E_1, \ldots, E_n) = 0$ whenever $\nu_k(E_k) = 0$ for any $k \in \{1, \ldots, n\}$. When μ is absolutely continuous with respect to $\nu_1 \times \cdots \times \nu_n$, we write $\mu \ll \nu_1 \times \cdots \times \nu_n$. We remark that these notions coincide with established terminology and notation when $n = 1$.

Let $\mathcal{V}_n(\nu_1, \ldots, \nu_n) = L^\infty(X_1, \nu_1) \widehat{\otimes} \cdots \widehat{\otimes} L^\infty(X_n, \nu_n)$.

Proposition 3.1. *Every element of $\mathcal{V}_n(\nu_1, \ldots, \nu_n)$ can be integrated with respect to any $\mu \in \mathcal{F}_n$, provided $\mu \ll \nu_1 \times \cdots \times \nu_n$.*

Proof. Let $\mu \in \mathcal{F}_n(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. We will show that the set function μ determines a bounded n-linear functional on $L^\infty(X_1, \nu_1) \times \cdots \times L^\infty(X_n, \nu_n)$. We will construct an iterated integral, like the one in Section 2.

Let $(f_1, \ldots, f_n) \in L^\infty(X_1, \nu_1) \times \cdots \times L^\infty(X_n, \nu_n)$. Define a set function on $\mathcal{A}_2 \times \cdots \times \mathcal{A}_n$ as follows:

$$\mu_{f_1}(A_2, \ldots, A_n) = \int_{X_1} f_1(x_1)\, \mu(dx_1, A_2, \ldots, A_n), \tag{7}$$

for all $(A_2, \ldots, A_n) \in \mathcal{A}_2 \times \cdots \times \mathcal{A}_n$. For fixed $(A_2, \ldots, A_n) \in \mathcal{A}_2 \times \cdots \times \mathcal{A}_n$, the set function $\mu(\cdot, A_2, \ldots, A_n)$ is a measure on $\mathcal{A}_1$ that is (by assumption) absolutely continuous with respect to ν_1. Consequently, the integral in (7) is well-defined.

The set function μ_{f_1} is an element of $\mathcal{F}_{n-1}(\mathcal{A}_2, \ldots, \mathcal{A}_n)$. To see this, take any $\tilde{f}_1 \in L^\infty(X_1)$ such that $f_1 = \tilde{f}_1$ a.e.(ν_1). (Such a function always exists.) For fixed $(A_2, \ldots, A_n) \in \mathcal{A}_2 \times \cdots \times \mathcal{A}_n$,

$$\int_{X_1} f_1(x_1)\, \mu(dx_1, A_2, \ldots, A_n) = \int_{X_1} \tilde{f}_1(x_1)\, \mu(dx_1, A_2, \ldots, A_n),$$

because f_1 and $\tilde{f}_1$ differ on a ν_1-null set and $\mu(\cdot, A_2, \ldots, A_n) \ll \nu_1$. We already know the integral on the right determines an element of $\mathcal{F}_{n-1}(\mathcal{A}_2, \ldots, \mathcal{A}_n)$ (refer to Section 2), and so the integral on the left must as well.

We have $\mu_{f_1} \in \mathcal{F}_{n-1}$, and so $\mu_{f_1}(\cdot, A_3, \ldots, A_n)$ is a measure on $\mathcal{A}_2$, for fixed $(A_3, \ldots, A_n) \in \mathcal{A}_3 \times \cdots \times \mathcal{A}_n$. We claim that the measure $\mu_{f_1}(\cdot, A_3, \ldots, A_n)$ is absolutely continuous with respect to ν_2. To see this, take f_1 to be a simple function. In this case, a direct computation shows that $\mu_{f_1}(A_2, A_3, \ldots, A_n) = 0$, whenever $\nu_2(A_2) = 0$. For general $f_1 \in L^\infty(X_1, \nu_1)$, the result follows from the density of simple functions.

Since $\mu_{f_1}(\cdot, A_3, \ldots, A_n) \ll \nu_2$ for every $(A_3, \ldots, A_n) \in \mathcal{A}_3 \times \cdots \times \mathcal{A}_n$, the set function

$$\mu_{f_1 \otimes f_2}(A_3, \ldots, A_n) = \int_{X_2} f_2(x_2)\, \mu_{f_1}(dx_2, A_3, \ldots, A_n)$$

is well-defined, for each $(A_3, \ldots, A_n) \in \mathcal{A}_3 \times \cdots \times \mathcal{A}_n$.

Continuing recursively, let $m \in \mathbb{N}$ be such that $2 < m < n$, and define a set function $\mu_{f_1 \otimes \cdots \otimes f_m}$ on $\mathcal{A}_{m+1} \times \cdots \times \mathcal{A}_n$ by

$$\mu_{f_1 \otimes \cdots \otimes f_m}(A_{m+1}, \ldots, A_n) = \int_{X_m} f_m(x_m)\, \mu_{f_1 \otimes \cdots \otimes f_{m-1}}(dx_m, A_{m+1}, \ldots, A_n), \tag{8}$$

for all $(A_{m+1},\dots,A_n)\in\mathcal{A}_{m+1}\times\cdots\times\mathcal{A}_n$. Using the same reasoning as above, the set function $\mu_{f_1\otimes\cdots\otimes f_m}$ is an element of $\mathcal{F}_{n-m}(\mathcal{A}_{m+1},\dots,\mathcal{A}_n)$, and the measure $\mu_{f_1\otimes\cdots\otimes f_m}(\cdot,A_{m+2},\dots,A_n)$ is absolutely continuous with respect to ν_{m+1} for each $(A_{m+2},\dots,A_n)\in\mathcal{A}_{m+2}\times\cdots\times\mathcal{A}_n$.

Now define

$$\mu_{f_1\otimes\cdots\otimes f_n}=\int_{X_n} f_n(x_n)\mu_{f_1\otimes\cdots\otimes f_{n-1}}(dx_n).$$

Certainly, the map $(f_1,\dots,f_n)\to\mu_{f_1\otimes\cdots\otimes f_n}$ is n-linear, and by construction, we have

$$|\mu_{f_1\otimes\cdots\otimes f_n}|\le\|f_1\|_\infty\cdots\|f_n\|_\infty\|\mu_n\|_{\mathcal{F}_n}.$$

Therefore, the map $(f_1,\dots,f_n)\to\mu_{f_1\otimes\cdots\otimes f_n}$ is a bounded n-linear functional on the space $L^\infty(X_1,\nu_1)\times\cdots\times L^\infty(X_n,\nu_n)$.

Bounded n-linear functionals on $L^\infty(X_1,\nu_1)\times\cdots\times L^\infty(X_n,\nu_n)$ correspond to bounded linear functionals on $L^\infty(X_1,\nu_1)\widehat{\otimes}\cdots\widehat{\otimes}L^\infty(X_n,\nu_n)$. We define the integral on elementary tensors by

$$\int f_1\otimes\cdots\otimes f_n\,d\mu=\mu_{f_1\otimes\cdots\otimes f_n},$$

and extend to elements of $\mathcal{V}_n(\nu_1,\dots,\nu_n)$ in the natural way:

$$\int\phi\,d\mu=\sum_{j=1}^\infty\int f_{1,j}\otimes\cdots\otimes f_{n,j}\,d\mu,$$

where $\phi=\sum_{j=1}^\infty f_{1,j}\otimes\cdots\otimes f_{n,j}$. $\square$

Naturally, the order of integration in the construction of the iterated integral in the proof of Proposition 3.1 is irrelevant. Certainly we can construct iterated integrals in any order, and they must all be equal by Theorem 2.2, since every function in $L^\infty(X_k,\nu_k)$ is equal almost everywhere (with respect to ν_k) to a function in $L^\infty(X_k)$ (for each $1\le k\le n$).

We will have need of the following, which is essentially a corollary to Theorem 2.2:

Proposition 3.2. *Let $(X_1,\mathcal{A}_1,\nu_1),\dots,(X_n,\mathcal{A}_n,\nu_n)$ be measure spaces such that the measures $\nu_1,\dots,\nu_n$ are positive and σ-finite. Assume $\mu\ll\nu_1\times\cdots\times\nu_n$. Then*

$$\int f_1\otimes\cdots\otimes f_n\,d\mu=\int_{X_k} f_k(x_k)\,\mu_{\bigotimes_{j\neq k}f_j}(dx_k),\quad 1\le k\le n,$$

for all $(f_1,\dots,f_n)\in L^\infty(X_1,\nu_1)\times\cdots\times L^\infty(X_n,\nu_n)$.

In the statement of Proposition 3.2, the notation $\mu_{\bigotimes_{j\neq k}f_j}$ refers to the iteratively constructed set function (as in (8)), constructed using all functions $f_1,\dots,f_n$, except for f_k.

Let $\mathcal{G}_n(\nu_1,\dots,\nu_n)=L^\infty(X_1,\nu_1)\otimes_g\cdots\otimes_g L^\infty(X_n,\nu_n)$.

Corollary 3.3. *Let* $(X_1, \mathcal{A}_1, \nu_1), \ldots, (X_n, \mathcal{A}_n, \nu_n)$ *be measure spaces such that the measures* $\nu_1, \ldots, \nu_n$ *are positive and* σ*-finite and let* $\mu \ll \nu_1 \times \cdots \times \nu_n$*. If* $\phi \in \mathcal{V}_n(\nu_1, \ldots, \nu_n)$*, then*

$$\Big| \int \phi \, d\mu \Big| \leq \|\phi\|_{\mathcal{V}_n(\nu_1,\ldots,\nu_n)} \|\mu\|_{\mathcal{F}_n}.$$

Furthermore, if ϕ *is in* $\mathcal{G}_n(\nu_1, \ldots, \nu_n)$*, then*

$$\Big| \int \phi \, d\mu \Big| \leq c_n \|\phi\|_{\mathcal{G}_n(\nu_1,\ldots,\nu_n)} \|\mu\|_{\mathcal{F}_n},$$

where $c_n > 0$ *is the constant from Theorem 2.4.*

Proof. This follows from Theorem 2.4 and Proposition 3.1. □

4. $\mathcal{F}_2$-measures

In this section, we will prove the Radon-Nikodým theorem in the special case $n = 2$. Much can be said when $n = 2$, because of the connection (in this case) between the Grothendieck tensor product and the Haagerup tensor product. We begin by recalling the Haagerup tensor product for $n = 2$ and expressing this connection.

4.1. Haagerup tensor products

Let A and B be operator spaces. (We are interested in the case when each of A and B is an L^p-space for $p = 1$ or $p = \infty$.) Define the *Haagerup tensor norm* on $A \otimes B$ by

$$\|u\|_h = \inf \Big\{ \Big\| \sum_j a_j a_j^* \Big\|^{1/2} \Big\| \sum_j b_j^* b_j \Big\|^{1/2} : u = \sum_j a_j \otimes b_j \Big\}.$$

(The sums are finite.) The completion of $A \otimes B$ in this norm is called the *Haagerup tensor product* of A and B and is denoted by $A \otimes_h B$. The Haagerup tensor norm can be defined on the algebraic tensor product of more than two spaces; in that case, the norm is defined in terms of matrix products and the interested reader is encouraged to peruse [8].

The w^*-*Haagerup tensor product* $\otimes_{w^*h}$ is the dual to the Haagerup tensor product: $A^* \otimes_{w^*h} B^* = (A \otimes_h B)^*$. The w^*-Haagerup tensor product was introduced in [1] for pairs of dual operator spaces and was generalized in [9]. The *extended Haagerup tensor product* $\otimes_{eh}$ was introduced in [9] and is defined to be $A \otimes_{eh} B = CB_m^\sigma(A^* \times B^*, \mathbb{C})$, the space of *normal completely bounded multiplicative multilinear forms.* This space can be characterized as the collection of all $\Lambda \in (A^* \otimes_h B^*)^*$ that are weak* continuous in A^* and B^* separately; that is, the maps

$$a^* \to \Lambda\big(a^* \otimes b_0^*\big) \quad \text{and} \quad b^* \to \Lambda\big(a_0^* \otimes b^*\big)$$

are continuous in the weak*-topologies on A^* and B^* (respectively) for fixed $a_0^* \in A^*$ and $b_0^* \in B^*$. (See [16] or [9].) The norm on $A \otimes_{eh} B$ is the one inherited from $(A^* \otimes_h B^*)^*$.

For dual operator spaces A^* and B^*, the extended and w^*-Haagerup tensor products coincide [9, Theorem 5.3]. In particular, this implies that

$$A \otimes_{eh} B \subseteq (A^* \otimes_h B^*)^* = A^{**} \otimes_{eh} B^{**}.$$

The extended Haagerup tensor product is injective [9, Lemma 5.4], and so $A \otimes_{eh} B$ is a closed subspace of $A^{**} \otimes_{eh} B^{**}$. Consequently, $A \otimes_{eh} B$ is a closed subspace of a dual space, and as such can be endowed with the weak*-topology of the larger space.

The next theorem comes from [9, Section 5] and [1, Theorem 3.1]:

Theorem 4.1. *Let A and B be operator spaces. If $u \in A \otimes_{eh} B$, then u has a weak*-representation $u = \sum_{i \in I} a_i \otimes b_i$, and*

$$\|u\|_{eh} = \inf \Big\{ \Big\| \sum_{i \in I} a_i a_i^* \Big\|^{1/2} \Big\| \sum_{i \in I} b_i^* b_i \Big\|^{1/2} \Big\},$$

where the infimum is taken over all possible weak*-representations of u. Furthermore, there exists a weak*-representation for which the infimum is achieved.

The index I in the above theorem may be uncountable. When we say there is a "weak*-representation" of u, we mean there is a representation of u in the weak*-topology on $A \otimes_{eh} B$ inherited from $(A^* \otimes_h B^*)^*$. Therefore, for every $(a^*, b^*) \in A^* \times B^*$,

$$\langle a^* \otimes b^*, u \rangle = \sum_{i \in I} \langle a_i, a^* \rangle \langle b_i, b^* \rangle = \lim_K \sum_{i \in K} \langle a_i, a^* \rangle \langle b_i, b^* \rangle, \tag{9}$$

where the limit is taken over the directed set of finite subsets K of I. Of course, for a given $(a^*, b^*) \in A^* \times B^*$, the sum over I in (9) must be countable, and so there must exist a countable subset J of I (which depends on a^* and b^*) such that

$$\langle a^* \otimes b^*, u \rangle = \sum_{i \in J} \langle a_i, a^* \rangle \langle b_i, b^* \rangle. \tag{10}$$

Remark 4.2. By definition, $L^\infty(X) \otimes_h L^\infty(Y) = L^\infty(X) \otimes_g L^\infty(Y)$. This is a result of the two norms having the same definition when $n = 2$, but it is not true in general (i.e., for products of three or more).

Remark 4.3. When A and B are L^1-spaces, we will use $\star$ to denote the operator space product (to distinguish it from the scalar product of two functions).

4.2. An application: a measure-theoretic Grothendieck inequality

Consider the following theorem [9, Proposition 5.6 & Theorem 5.7]:

Theorem 4.4. *Let A and B be operator spaces. Any bounded linear functional Λ on $A \otimes_h B$ can be extended to a bounded linear functional on $A \otimes_{eh} B$ having the same norm. In particular, if $\Lambda \in (A \otimes_h B)^* = A^* \otimes_{eh} B^*$ has weak*-representation $\Lambda = \sum_{i \in I} a_i^* \otimes b_i^*$, then*

$$\Lambda(u) = \sum_{i \in I} \langle a_i^* \otimes b_i^*, u \rangle = \lim_K \sum_{i \in K} \langle a_i^* \otimes b_i^*, u \rangle, \quad u \in A \otimes_{eh} B,$$

where the limit is taken over the directed set of finite subsets K of I.

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$, and suppose $\boldsymbol{f} : X \to H$ and $\boldsymbol{g} : Y \to H$ are two bounded weakly measurable functions [6, Section II.1]. Let $(e_i)_{i \in I}$ be an orthonormal basis for H and, for each $i \in I$, define

$$f_i(x) = \langle \boldsymbol{f}(x), e_i \rangle_H \quad \text{and} \quad g_i(y) = \langle \boldsymbol{g}(y), e_i \rangle_H, \quad (x, y) \in X \times Y.$$

Then $\langle \boldsymbol{f}, \boldsymbol{g} \rangle_H = \sum_{i \in I} f_j \otimes g_j$, and

$$\Big\| \sum_{i \in I} |f_i|^2 \Big\|_\infty^{1/2} \Big\| \sum_{i \in I} |g_i|^2 \Big\|_\infty^{1/2} = \sup_{x \in X} \|\boldsymbol{f}(x)\|_H \sup_{y \in Y} \|\boldsymbol{g}(y)\|_H < \infty.$$

Therefore, $\langle \boldsymbol{f}, \boldsymbol{g} \rangle_H \in L^\infty(X) \otimes_{eh} L^\infty(Y)$.

Recall that $L^\infty(X) \otimes_h L^\infty(Y) = L^\infty(X) \otimes_g L^\infty(Y)$ (Remark 4.2). By Theorem 2.4, $\mu \in \mathcal{F}_2$ determines a bounded linear functional on $L^\infty(X) \otimes_h L^\infty(Y)$, and hence (via Theorem 4.4) on $L^\infty(X) \otimes_{eh} L^\infty(Y)$. Therefore, $\langle \boldsymbol{f}, \boldsymbol{g} \rangle$ is integrable with respect to any $\mu \in \mathcal{F}_2$ and, by Theorem 4.4,

$$\int \langle \boldsymbol{f}, \boldsymbol{g} \rangle_H \, d\mu = \sum_{i \in I} \int f_i \otimes g_i \, d\mu = \lim_K \sum_{i \in K} \int f_i \otimes g_i \, d\mu,$$

where the limit is taken over the directed set of finite subsets K of I, and

$$\Big| \int \langle \boldsymbol{f}, \boldsymbol{g} \rangle_H \, d\mu \Big| \le K_G \| \langle \boldsymbol{f}, \boldsymbol{g} \rangle_H \|_{eh} \|\mu\|_{\mathcal{F}_2}. \tag{11}$$

This provides a measure-theoretic version of the Grothendieck inequality.

4.3. The Radon-Nikodým theorem for $\mathcal{F}_2$-measures

Presently, we will provide the first formulation of the Radon-Nikodým theorem for $\mathcal{F}_2$-measures (Proposition 4.5). First, we introduce the following notation:

$$\mathcal{E}_2^1 \ = \ \mathcal{E}_2^1(\nu_1, \nu_2) \ = \ L^1(X, \nu_1) \otimes_{eh} L^1(Y, \nu_2). \tag{12}$$

Recall that $\mathcal{E}_2^1$ is a subspace of $\mathcal{G}_2(\nu_1, \nu_2)^* = \mathcal{V}_2(\nu_1, \nu_2)^*$.

Proposition 4.5. *Let $(X, \mathcal{A}, \nu_1)$ and $(Y, \mathcal{B}, \nu_2)$ be measure spaces, where ν_1 and ν_2 are positive σ-finite measures. Then $\mu \in \mathcal{F}_2(\mathcal{A}, \mathcal{B})$ is absolutely continuous with respect to $\nu_1 \times \nu_2$ if and only if μ corresponds to an element $u \in \mathcal{E}_2^1(\nu_1, \nu_2)$ and*

$$\int f \otimes g \, d\mu = \langle f \otimes g, u \rangle, \quad (f, g) \in L^\infty(X, \nu_1) \times L^\infty(Y, \nu_2). \tag{13}$$

Proof. First, let $\mu \in \mathcal{F}_2(\mathcal{A}, \mathcal{B})$ be such that $\mu \ll \nu_1 \times \nu_2$. We wish to show that u, defined by (13), is an element of $L^1(X, \nu_1) \otimes_{eh} L^1(Y, \nu_2)$. By Corollary 3.3, the bimeasure μ determines a bounded linear functional on $\mathcal{G}_2$. It remains to show that u satisfies the required weak*-continuity property. To that end, let

$$\hat{\mu}(f, g) = \int f \otimes g \, d\mu, \quad (f, g) \in L^\infty(X, \nu_1) \times L^\infty(Y, \nu_2).$$

It suffices to show that $\hat{\mu}$ is weak*-continuous in each argument separately.

Let $g^{(0)} \in L^\infty(Y, \nu_2)$ be given. Define a bounded linear functional Λ_1 on $L^\infty(X, \nu_1)$ by

$$\Lambda_1(f) = \int f \otimes g^{(0)} \, d\mu, \quad f \in L^\infty(X, \nu_1). \tag{14}$$

By Proposition 3.2,

$$\Lambda_1(f) = \int_X f(x) \, \mu_{g^{(0)}}(dx).$$

The measure $\mu_{g^{(0)}}$ is absolutely continuous with respect to ν_1 (see the proof of Proposition 3.1). Therefore, by the (classical) Radon-Nikodým theorem, there exists a function $h \in L^1(X, \nu_1)$ such that

$$\int_X f_k(x) \, \mu_{g^{(0)}}(dx) = \int_X f(x) \, h(x) \, \nu_1(dx).$$

Consequently,

$$\Lambda_1(f) = \langle h, f \rangle, \quad f \in L^\infty(X, \nu_1),$$

where $\langle \cdot, \cdot \rangle$ represents the dual action of $L^\infty(X, \nu_1)$ on $L^1(X, \nu_1)$. This type of linear functional defines the weak*-topology on $L^\infty(X, \nu_1)$, and so is weak*-continuous. A similar argument shows that $g \to \hat{\mu}(f^{(0)}, g)$ is weak* continuous for fixed $f^{(0)}$. Therefore, μ determines an element of $\mathcal{E}_2^1$, as required.

Now let $u \in \mathcal{E}_2^1$. We show u determines an $\mathcal{F}_2$-measure μ that is absolutely continuous with respect to $\nu_1 \times \nu_2$. Recall that $\mathcal{E}_2^1 = L^1(X, \nu_1) \otimes_{eh} L^1(Y, \nu_2)$ is (by definition) a subspace of $\mathcal{G}_2^* = \big(L^\infty(X, \nu_1) \otimes_h L^\infty(Y, \nu_2)\big)^*$. Denote this dual action by $\langle \cdot, \cdot \rangle$.

Define a set function μ on $\mathcal{A} \times \mathcal{B}$ by $\mu(A, B) = \langle \mathbf{1}_A \otimes \mathbf{1}_B, u \rangle$. We show that μ is countably additive in each argument separately. It suffices to check the first argument.

Let $(A_j)_{j=1}^\infty$ be a sequence of pairwise disjoint measurable sets in $\mathcal{A}$. Let $A = \bigcup_{j=1}^\infty A_j$. For every $h \in L^1(X, \nu_1)$,

$$\lim_{N \to \infty} \int_X \sum_{j=1}^N \mathbf{1}_{A_j}(x) \, h(x) \, \nu_1(dx) = \int_X \mathbf{1}_A(x) \, h(x) \, \nu_1(dx),$$

by the Lebesgue dominated convergence theorem. Therefore, $\sum_{j=1}^N \mathbf{1}_{A_j}$ converges to $\mathbf{1}_A$ in the weak*-topology on $L^\infty(X, \nu_1)$. By assumption, the map

$f \to \langle f \otimes g, u\rangle$ is continuous in the weak*-topology on $L^\infty(X,\nu_1)$, for fixed $g \in L^\infty(Y,\nu_2)$. Consequently, for each $B \in \mathcal{B}$,

$$\lim_{N\to\infty} \sum_{j=1}^{N} \langle \mathbf{1}_{A_j} \otimes \mathbf{1}_B, u\rangle = \langle \mathbf{1}_A \otimes \mathbf{1}_B, u\rangle.$$

Therefore, $\sum_{j=1}^{\infty} \mu(A_j, B) = \mu(A, B)$, as required.

Next, we show that μ has finite Fréchet variation. Let $(A_j)_j$ and $(B_k)_k$ be finite measurable partitions of $\mathcal{A}$ and $\mathcal{B}$, respectively. Without loss of generality, we may assume $j, k \in \{1, \dots, N\}$ for some $N \in \mathbb{N}$. Let ϵ_j and δ_k be complex numbers with modulus 1 for each $1 \le j, k \le N$. Finally, let $f = \sum_{j=1}^{N} \epsilon_j \mathbf{1}_{A_j}$ and $g = \sum_{k=1}^{N} \delta_k \mathbf{1}_{B_k}$. Then

$$\sum_{j,k=1}^{N} \epsilon_j \, \delta_k \, \mu(A_j, B_k) = \sum_{j,k=1}^{N} \epsilon_j \, \delta_k \, \langle \mathbf{1}_{A_j} \otimes \mathbf{1}_{B_k}, u\rangle = \langle f \otimes g, u\rangle.$$

By duality, $|\langle f \otimes g, u\rangle| \le \|f\|_\infty \|g\|_\infty \, \|u\|_{\mathcal{E}_2^1} = \|u\|_{\mathcal{E}_2^1}$, and so

$$\Big| \sum_{j,k=1}^{N} \epsilon_j \, \delta_k \, \mu(A_j, B_k) \Big| \le \|u\|_{\mathcal{E}_2^1}. \tag{15}$$

The choice of partitions $(A_j)_j$ and $(B_k)_k$, as well as complex numbers ϵ_j and δ_k with modulus 1, was arbitrary, and hence we conclude that $\|\mu\|_{\mathcal{F}_2} \le \|u\|_{\mathcal{E}_2^1}$. Therefore, μ has finite Fréchet variation, and so is a Fréchet measure.

It remains to show $\mu \ll \nu_1 \times \nu_2$. (Note that (13) follows from the density of simple functions.) By Theorem 4.1, u has a weak*-representation $u = \sum_{i\in I} h_i \otimes k_i$, where $(h_i, k_i) \in L^1(X, \nu_1) \times L^1(Y, \nu_2)$ for each $i \in I$. Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$ be measurable sets. Then $\mathbf{1}_A$ and $\mathbf{1}_B$ are bounded measurable functions, and so there exists a countable index J such that

$$\mu(A,B) = \langle \mathbf{1}_A \otimes \mathbf{1}_B, u\rangle = \sum_{i\in J} \langle h_i, \mathbf{1}_A\rangle \langle k_i, \mathbf{1}_B\rangle = \sum_{i\in J} \int_{A\times B} h_i(x)\, k_i(y)\, (\nu_1 \times \nu_2)(dx, dy).$$

Suppose that $\nu_1(A) = 0$ or $\nu_2(B) = 0$. For each $i \in J$, the functions h_i and k_i are integrable, and hence

$$\int_{A\times B} \big|h_i(x)\, k_i(y)\big| (\nu_1 \times \nu_2)(dx, dy) = 0.$$

Thus,

$$|\mu(A,B)| \le \sum_{i\in J} \int_{A\times B} \big|h_i(x)\, k_i(y)\big| (\nu_1 \times \nu_2)(dx, dy) = 0. \tag{16}$$

Therefore, μ is absolutely continuous with respect to $\nu_1 \times \nu_2$. □

If $(X, \mathcal{A}, \nu_1)$ and $(Y, \mathcal{B}, \nu_2)$ are positive σ-finite measure spaces, then denote by $\mathcal{F}_2(\nu_1, \nu_2)$ the collection of all bimeasures on $\mathcal{A} \times \mathcal{B}$ that are absolutely continuous with respect to $\nu_1 \times \nu_2$.

Corollary 4.6. *Let $(X,\mathcal{A},\nu_1)$ and $(Y,\mathcal{B},\nu_2)$ be measure spaces with ν_1 and ν_2 positive σ-finite measures. Then $\mathcal{F}_2(\nu_1,\nu_2) = L^1(X,\nu_1)\otimes_{eh} L^1(Y,\nu_2)$. Furthermore, if $\mu \in \mathcal{F}_2(\nu_1,\nu_2)$ corresponds to $u \in \mathcal{E}_2^1$ (as in Proposition 4.5), then*

$$\|\mu\|_{\mathcal{F}_2} \leq \|u\|_{\mathcal{E}_2^1} \leq K_G\,\|\mu\|_{\mathcal{F}_2}.$$

Proof. It follows from Proposition 4.5 that the two spaces coincide. It remains only to show the relationship between the norms. For $A \in \mathcal{A}$ and $B \in \mathcal{B}$,

$$\mu(A,B) = \int \mathbf{1}_A \otimes \mathbf{1}_B \, d\mu = \langle \mathbf{1}_A \otimes \mathbf{1}_B, u\rangle,$$

and consequently $\|\mu\|_{\mathcal{F}_2} \leq \|u\|_{\mathcal{E}_2^1}$, using the argument that μ is an $\mathcal{F}_2$-measure from Proposition 4.5. (See (15).)

Next, recall that the norm on the extended Haagerup tensor product $\mathcal{E}_2^1 = L^1(\nu_1)\otimes_{eh} L^1(\nu_2)$ is inherited from $\mathcal{G}_2^* = (L^\infty(\nu_1)\otimes_h L^\infty(\nu_2))^*$. Consequently, $\|u\|_{\mathcal{E}_2^1} = \sup|\langle\phi,u\rangle|$, where the supremum is taken over all $\phi \in \mathcal{G}_2$ such that $\|\phi\|_{\mathcal{G}_2} \leq 1$. Since the algebraic tensor product $L^\infty(\nu_1)\otimes L^\infty(\nu_2)$ is dense in $\mathcal{G}_2$, it suffices to consider ϕ as the finite sum of elementary tensors. In this case, it is easy to see that $\langle\phi,u\rangle = \int\phi\,d\mu$. By Corollary 3.3,

$$\left|\int \phi\, d\mu\right| \leq c_2\|\phi\|_{\mathcal{G}_2}\,\|\mu\|_{\mathcal{F}_2}.$$

Therefore, $\|u\|_{\mathcal{E}_2^1} \leq c_2\,\|\mu\|_{\mathcal{F}_2}$, as required, recalling that $c_2 = K_G$. $\square$

Suppose $u \in \mathcal{E}_2^1(\nu_1,\nu_2)$. By Theorem 4.1, there exists a weak*-representation of the element u, say $u = \sum_{i\in I} h_i \otimes k_i$, where $h_i \in L^1(X,\nu_1)$ and $k_i \in L^1(Y,\nu_2)$ for each $i \in I$, and

$$\Big\|\sum_{i\in I} h_i \star h_i^*\Big\|_\infty^{1/2}\Big\|\sum_{i\in I} k_i^* \star k_i\Big\|_\infty^{1/2} < \infty.$$

This representation need not be unique, and I, which we call the *associated index set*, may be uncountable. We define a function ψ on $X\times Y$ by

$$\psi(x,y) = \sum_{i\in I} h_i(x)\,k_i(y), \quad (x,y)\in X\times Y. \tag{17}$$

The sum in (17) is generally uncountable, and so there is no reason to assume that $\psi(x,y)$ exists in a pointwise sense. We call ψ a function and write $\psi(x,y)$ merely for convenience. We express ψ in terms of nets. Consider the directed set $\{K : K \subseteq I\}$ of finite subsets K of I. Then

$$\psi(x,y) = \lim_K \sum_{i\in K} h_i(x)\,k_i(y), \quad (x,y)\in X\times Y,$$

where the limit is actually a weak* limit taken over all finite subsets K of I. For each finite subset K of I, define

$$\psi_K(x,y) = \sum_{i\in K} h_i(x)\,k_i(y), \quad (x,y)\in X\times Y. \tag{18}$$

Then $\psi = \lim_K \psi_K$, where K is taken from the finite subsets of I. Note that ψ_K does determine an integrable function on $X \times Y$, and so $\psi_K(x, y)$ does have a meaning.

Let ψ be given by a weak*-representation of u, as in (17). Let

$$\|\psi\|_{\rho(u)} = \Big\| \sum_{i \in I} h_i \star h_i^* \Big\|_\infty^{1/2} \Big\| \sum_{i \in I} k_i^* \star k_i \Big\|_\infty^{1/2}.$$

Denote by $\rho(u)$ the collection of all functions ψ that are weak*-representation of u, as defined in (17), for which $\|\psi\|_{\rho(u)} < \infty$. Then $\|u\|_{\mathcal{E}_2^1} = \inf\{\|\psi\|_{\rho(u)} : \psi \in \rho(u)\}$ and there exists a ψ which achieves this infimum (see Theorem 4.1). In particular, $\rho(u)$ is not empty.

Remark 4.7. If ψ_K is defined as in (18), then $\psi_K \in L^1(X \times Y, \nu_1 \times \nu_2)$ whenever K is finite. Thus, ψ is the limit of a net of integrable functions, even though ψ itself will in general not be integrable.

Remark 4.8. Let $u \in \mathcal{E}_2^1$ and let $\psi \in \rho(u)$ be such that $\|\psi\|_{\rho(u)} = \|u\|_{\mathcal{E}_2^1}$. We think of ψ as being a representative of u in $\mathcal{E}_2^1$, and so we (inaccurately) say $\psi \in \mathcal{E}_2^1$ and write $\|\psi\|_{\mathcal{E}_2^1}$ for $\|u\|_{\mathcal{E}_2^1}$.

We now state and prove the Radon-Nikodým theorem for bimeasures:

Theorem 4.9. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be arbitrary measurable spaces and let μ be in $\mathcal{F}_2(\mathcal{A}, \mathcal{B})$. Suppose ν_1 and ν_2 are positive σ-finite measures on $\mathcal{A}$ and $\mathcal{B}$ (respectively). If $\mu \ll \nu_1 \times \nu_2$, then there exists a $\psi \in \mathcal{E}_2^1$ such that*

$$\int \phi \, d\mu = \int_{X \times Y} \phi(x, y) \, \psi(x, y) \, (\nu_1 \times \nu_2)(dx, dy),$$

for all $\phi \in \mathcal{V}_2(\nu_1, \nu_2)$, and $|\int \phi \, d\mu| \le \|\phi\|_{\mathcal{V}_2} \|\psi\|_{\mathcal{E}_2^1}$.

Proof. Because $\psi(x, y)$ is shorthand for a limit, the integral appearing in the statement of the theorem is also a limit. What we mean to say is

$$\int \phi \, d\mu = \lim_K \int_{X \times Y} \phi(x, y) \, \psi_K(x, y) \, (\nu_1 \times \nu_2)(dx, dy).$$

This limit exists, by assumption.

By Proposition 4.5, we view μ as an element of $L^1(X, \nu_1) \otimes_{eh} L^1(Y, \nu_2)$. By Theorem 4.1, μ can be given a weak*-representation $\psi = \sum_{i \in I} h_i \otimes k_i$, where I is an index set and $h_i \in L^1(X, \nu_1)$ and $k_i \in L^1(Y, \nu_2)$ for each $i \in I$, and such that $\|\psi\|_{\rho(\mu)} = \|\mu\|_{\mathcal{E}_2^1}$. Then, for $(f, g) \in L^\infty(X, \nu_1) \times L^\infty(Y, \nu_2)$,

$$\begin{aligned}
\int f \otimes g \, d\mu = \langle f \otimes g, \psi \rangle &= \sum_{i \in I} \langle h_i, f \rangle \langle k_i, g \rangle \\
&= \lim_K \sum_{i \in K} \int_{X \times Y} f(x) g(y) \, h_i(x) k_i(y) \, (\nu_1 \times \nu_2)(dx, dy) \\
&= \lim_K \int_{X \times Y} f(x) g(y) \Big(\sum_{i \in K} h_i(x) k_i(y) \Big) \, (\nu_1 \times \nu_2)(dx, dy) \\
&= \lim_K \int_{X \times Y} f(x) g(y) \, \psi_K(x, y) \, (\nu_1 \times \nu_2)(dx, dy),
\end{aligned} \tag{19}$$

where the limit is taken over finite subsets K of I. In (19), we implicitly made use of Fubini's theorem, which can be done because $\psi_K \in L^1(\nu_1 \times \nu_2)$ for each finite index set K.

Now let $\phi \in L^\infty(X, \nu_1) \otimes L^\infty(Y, \nu_2)$. For any $\epsilon > 0$, the function ϕ can be given a pointwise representation $\phi(x, y) = \sum_{j=1}^N f_j(x) \, g_j(y)$, where $(x, y) \in X \times Y$, $N \in \mathbb{N}$, $f_j \in L^\infty(X, \nu_1)$ and $g_j \in L^\infty(Y, \nu_2)$ for each $j \in \{1, \ldots, N\}$, and such that $\sum_{j=1}^N \big\|f_j\big\|_\infty \big\|g_j\big\|_\infty \leq \|\phi\|_{\mathcal{V}_2} + \epsilon$. Because the sum is finite, we have

$$\begin{aligned}
\int \phi \, d\mu = \sum_{j=1}^N \int f_j \otimes g_j \, d\mu &= \sum_{j=1}^N \langle f_j \otimes g_j, \psi \rangle \\
&= \lim_K \int_{X \times Y} \Big(\sum_{j=1}^N f_j(x) g_j(y) \Big) \psi_K(x, y) \, (\nu_1 \times \nu_2)(dx, dy) \\
&= \lim_K \int_{X \times Y} \phi(x, y) \, \psi_K(x, y) \, (\nu_1 \times \nu_2)(dx, dy).
\end{aligned}$$

We thus conclude that

$$\int \phi \, d\mu = \int_{X \times Y} \phi(x, y) \, \psi(x, y) \, (\nu_1 \times \nu_2)(dx, dy).$$

By choice of representations,

$$\Big| \int \phi \, d\mu \Big| = \Big| \sum_{j=1}^N \langle f_j \otimes g_j, \psi \rangle \Big| \leq \sum_{j=1}^\infty \big\|f_j\big\|_\infty \big\|g_j\big\|_\infty \, \|\psi\|_{\mathcal{E}_2^1} \leq \big(\|\phi\|_{\mathcal{V}_2} + \epsilon \big) \, \|\psi\|_{\mathcal{E}_2^1}.$$

Taking the infimum over pointwise representations of ϕ, we have $\big| \int \phi \, d\mu \big| \leq \|\phi\|_{\mathcal{V}_2} \, \|\psi\|_{\mathcal{E}_2^1}$. Because $L^\infty(\nu_1) \otimes L^\infty(\nu_2)$ is dense in $\mathcal{V}_2(\nu_1, \nu_2)$, the proof is complete. □

Definition 4.10. If $\psi \in \mathcal{E}_2^1$ corresponds to $\mu \in \mathcal{F}_2(\nu_1, \nu_2)$ as in Theorem 4.9, we call ψ a *derivative* of μ with respect to $\nu_1 \times \nu_2$.

Corollary 4.11. *Let $(X, \mathcal{A}, \nu_1)$ and $(Y, \mathcal{B}, \nu_2)$ be positive σ-finite measure spaces. If ψ is a function in $\mathcal{E}_2^1(\nu_1, \nu_2)$ and*

$$\mu(A, B) = \int_{A\times B} \psi(x, y)\, \nu_1(dx)\, \nu_2(dy), \quad (A, B) \in \mathcal{A} \times \mathcal{B},$$

then $\mu \in \mathcal{F}_2(\nu_1, \nu_2)$ and ψ is a derivative of μ with respect to $\nu_1 \times \nu_2$.

Proof. This is a consequence of Proposition 4.5 and Theorem 4.9. □

In Theorem 4.9, the integral $\int \phi\, d\mu$ was written in terms of integration over the product space $(X \times Y, \nu_1 \times \nu_2)$. The integral can also be iterated in the following sense:

Proposition 4.12. *Let $(X, \mathcal{A}, \nu_1)$ and $(Y, \mathcal{B}, \nu_2)$ be positive σ-finite measure spaces. If $\psi \in L^1(\nu_1) \otimes_{eh} L^1(\nu_2)$, then*

$$\int_{X\times Y} f(x)\, g(y)\, \psi(x, y)\, (\nu_1 \times \nu_2)(dx, dy) = \int_X f(x) \left(\int_Y g(y)\, \psi(x, y)\, \nu_2(dy) \right) \nu_1(dx),$$

for all $(f, g) \in L^\infty(\nu_1) \times L^\infty(\nu_2)$.

Proof. For each $x \in X$, let $\psi_g(x) = \int_Y g(y)\, \psi(x, y)\, \nu_2(dy)$. Let ψ have weak* representation $\psi = \sum_{i\in I} h_i \otimes k_i$. Then

$$\psi_g(x) = \lim_K \int_Y g(y) \sum_{i\in K} h_i(x) k_i(y)\, \nu_2(dy) = \lim_K \sum_{i\in K} h_i(x) \int_Y g(y)\, k_i(y)\, \nu_2(dy),$$

where the limit is taken over finite subsets K of I. Thus,

$$\psi_g = \lim_K \sum_{i\in K} h_i \langle k_i, g\rangle = \sum_{i\in I} h_i \langle k_i, g\rangle. \tag{20}$$

The object in (20) is known as a *right slice*. (See, for example, [8, 16].) It is known (e.g., [16, Theorem 2.4]) that the right slice of an element in the extended Haagerup tensor product $A \otimes_{eh} B$ lies in the operator space A. It follows that $\psi_g \in L^1(\nu_1)$. In fact, if we choose $\epsilon : X \to \mathbb{C}$ with $|\epsilon(x)| = 1$ a.e.(ν_1) such that

$$\int_X \left| \int_Y g(y)\, \psi(x, y)\, \nu_2(dy) \right| \nu_1(dx) = \int_X \epsilon(x) \left(\int_Y g(y)\, \psi(x, y)\, \nu_2(dy) \right) \nu_1(dx),$$

then

$$\|\psi_g\|_{L^1(\nu_1)} = \langle \epsilon \otimes g, \psi\rangle \le \|\epsilon\|_\infty \|g\|_\infty \|\psi\|_{\mathcal{E}_2^1} = \|g\|_\infty \|\psi\|_{\mathcal{E}_2^1}.$$

It remains to show the equality, but this follows from the dominated convergence theorem, because, for given f and g, the sum defining ψ can be taken to be countable. □

Corollary 4.13. *If $\mu \in \mathcal{F}_2(\nu_1, \nu_2)$ has derivative ψ and $g \in L^\infty(\nu_2)$, then $\mu_g(dx) = \psi_g\, \nu_1(dx)$.*

Remark 4.14. From Corollary 4.6, we see that the $\mathcal{F}_2$-measures that are absolutely continuous with respect to $\nu_1 \times \nu_2$ coincide with the elements of $\mathcal{G}_2(\nu_1, \nu_2)^*$ that satisfy a certain weak*-continuity property. By Theorem 2.3, we may identify $\mathcal{G}_2(\nu_1, \nu_2)$ with $\mathcal{V}_2(\nu_1, \nu_2)$, and consequently, we see that $\mathcal{F}_2(\nu_1, \nu_2)$ corresponds to the bounded bilinear functionals on $L^\infty(\nu_1) \times L^\infty(\nu_2)$ that are weak*-continuous in each argument separately. It is the weak*-continuity property that coincides with countable additivity. It is easy to see, then, that the full space $\mathcal{G}_2(\nu_1, \nu_2)^*$ corresponds to the space of *finitely* additive Fréchet measures on $\mathcal{A} \times \mathcal{B}$ that are absolutely continuous with respect to $\nu_1 \times \nu_2$.

5. A characterization of $L^1(\nu_1) \otimes_h L^1(\nu_2)$

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and let ν_1 and ν_2 be positive σ-finite measures on $\mathcal{A}$ and $\mathcal{B}$, respectively. Denote by $\mathcal{P}_2 = L^1(\nu_1) \check{\otimes} L^1(\nu_2)$ the completion of $L^1(\nu_1) \otimes L^1(\nu_2)$ in the *injective tensor norm*:

$$\|\psi\|_{\mathcal{P}_2} = \sup\left\{|\psi(f, g)| : \|f\|_\infty \leq 1, \|g\|_\infty \leq 1\right\}.$$

(We use the letter $\mathcal{P}$ because of the relationship to Pettis integrable functions; see, for example, [6, Chapter VIII].)

Suppose $\psi \in L^1(\nu_1) \otimes L^1(\nu_2)$, say $\psi = \sum_{j=1}^m h_i \otimes k_i$. Then, for (f, g) in $L^\infty(\nu_1) \times L^\infty(\nu_2)$,

$$\psi(f, g) = \sum_{j=1}^m \langle h_i, f\rangle\langle k_i, g\rangle = \int_{X \times Y} f(x) g(y)\, \psi(x, y)\, (\nu_1 \times \nu_2)(dx, dy).$$

Since the sum is finite, the element ψ determines an L^1-function on $X \times Y$, and so the set function

$$\mu(E, F) = \int_{X \times Y} \mathbf{1}_E(x)\, \mathbf{1}_F(y)\, \psi(x, y)\, (\nu_1 \times \nu_2)(dx, dy)$$

determines a measure on $\mathcal{A} \times \mathcal{B}$.

Let us compute the Fréchet variation of μ: Let $(E_m)_{m=1}^M$ be a finite measurable partition of $\mathcal{A}$, and let $(F_n)_{n=1}^N$ be a finite measurable partition of $\mathcal{B}$. Next, let ϵ_m and δ_n be complex numbers having modulus 1 for each $1 \leq m \leq M$ and $1 \leq n \leq N$. Furthermore, let $f = \sum_{m=1}^M \epsilon_m\, \mathbf{1}_{E_m}$ and $g = \sum_{n=1}^N \delta_n\, \mathbf{1}_{F_n}$. Then

$$\sum_{m,n} \epsilon_m\, \delta_n\, \mu(E, F) = \sum_{m,n} \int_{X \times Y} \epsilon_m \mathbf{1}_{E_m} \otimes \delta_n \mathbf{1}_{F_n}\, \psi\, d\nu \;=\; \psi(f, g).$$

Consequently,

$$\Big|\sum_{m,n} \epsilon_m\, \delta_n\, \mu(E_m, F_n)\Big| \leq \|\psi\|_{\mathcal{P}_2}.$$

The choice of partitions and scalars was arbitrary, and so we conclude that $\|\mu\|_{\mathcal{F}_2} \leq \|\psi\|_{\mathcal{P}_2}$.

We have that $\|\psi\|_{\mathcal{P}_2} \leq \|\psi\|_{L^1(\nu_1)\otimes_h L^1(\nu_2)} = \|\psi\|_{L^1(\nu_1)\otimes_{eh} L^1(\nu_2)}$. (The last equality following from the fact that the Haagerup tensor product maps into the extended Haagerup tensor product by a completely isometric injection; see [9].) By Corollary 4.6, $\|\psi\|_{L^1(\nu_1)\otimes_{eh} L^1(\nu_2)} \leq K_G\|\mu\|_{\mathcal{F}_2}$. Putting all of these inequalities together, we have

$$\|\psi\|_{\mathcal{P}_2} \leq \|\psi\|_{L^1(\nu_1)\otimes_h L^1(\nu_2)} \leq K_G\|\mu\|_{\mathcal{F}_2} \leq K_G\|\psi\|_{\mathcal{P}_2}.$$

Thus, the norms $\|\cdot\|_{L^1(\nu_1)\otimes_h L^1(\nu_2)}$ and $\|\cdot\|_{\mathcal{P}_2}$ are equivalent. Since both $L^1(\nu_1)\otimes_h L^1(\nu_2)$ and $\mathcal{P}_2$ are completions of $L^1(\nu_1)\otimes L^1(\nu_2)$, and the norms are equivalent, we conclude the two spaces are the same. To summarize:

Proposition 5.1. *If $(X,\mathcal{A},\nu_1)$ and $(Y,\mathcal{B},\nu_2)$ are positive σ-finite measure spaces, then $L^1(\nu_1)\check{\otimes}L^1(\nu_2) = L^1(\nu_1)\otimes_h L^1(\nu_2)$ and $\|\cdot\|_{\mathcal{P}_2} \leq \|\cdot\|_{L^1(\nu_1)\otimes_h L^1(\nu_2)} \leq K_G\|\cdot\|_{\mathcal{P}_2}$, where K_G is the Grothendieck constant.*

Remark 5.2. We let $\mathcal{F}_2(\mathbb{N},\mathbb{N})$ denote the Fréchet measures on $\mathbb{N}\times\mathbb{N}$. It is known (for example, [3, Equation IV.6.9]) that $\mathcal{F}_2(\mathbb{N},\mathbb{N}) = \ell^1\check{\otimes}\ell^1$. This is a consequence of the density of the algebraic tensor norm $\ell^1\otimes\ell^1$ in the space $\mathcal{F}_2(\mathbb{N},\mathbb{N})$ [3, Theorem IV.6]. Proposition 5.1 can be viewed as a non-discrete analogue of this result.

6. The general case

Our goal is to generalize the arguments of Section 4.3 to $\mathcal{F}_n$-measures for $n \geq 3$. In dimensions greater than two, however, the Grothendieck and Haagerup tensor products of L^∞-spaces do not coincide, and so we can no longer work within the framework of the extended Haagerup tensor product. We will overcome this obstacle by defining the extended Grothendieck tensor product for L^1-spaces.

Let $(X_1,\mathcal{A}_1,\nu_1),\ldots,(X_n,\mathcal{A}_n,\nu_n)$ be positive σ-finite measure spaces. We define the *extended Grothendieck tensor product* $L^1(X_1,\nu_1)\otimes_{eg}\cdots\otimes_{eg}L^1(X_n,\nu_n)$ to be the collection of all bounded linear maps $\Lambda \in (L^\infty(X_1,\nu_1)\otimes_g\cdots\otimes_g L^\infty(X_n,\nu_n))^* = \mathcal{G}_n(\nu_1,\ldots,\nu_n)^*$ such that, for each j, the map

$$f_j \to \Lambda\big(f_1\otimes\cdots\otimes f_j\otimes\cdots\otimes f_n\big)$$

is continuous in the weak*-topology on $L^\infty(X_j,\nu_j)$ for fixed $f_k \in L^\infty(X_k,\nu_k)$, $k\neq j$. We take the norm $\|\cdot\|_{eg}$ on $L^1(X_1,\nu_1)\otimes_{eg}\cdots\otimes_{eg}L^1(X_n,\nu_n)$ to be the one inherited from $\mathcal{G}_n(\nu_1,\cdots,\nu_n)^*$.

Let

$$\mathcal{E}_n^1 \;=\; \mathcal{E}_n^1(\nu_1,\ldots,\nu_n) \;=\; L^1(X_1,\nu_1)\otimes_{eg}\cdots\otimes_{eg}L^1(X_n,\nu_n). \tag{21}$$

In Section 4.3, we introduced the symbol $\mathcal{E}_2^1$ to denote the space $L^1(X,\nu_1)\otimes_{eh} L^1(Y,\nu_2)$. (See (12).) This does not result in any ambiguity, however, because the extended Haagerup and extended Grothendieck tensor products coincide in this case.

If $u \in \mathcal{E}_n^1(\nu_1, \ldots, \nu_n)$, we denote by $\rho(u)$ the collection of all functions

$$\psi(x_1, \ldots, x_n) = \sum_{i \in I} g_i^{(1)}(x_1) \cdots g_i^{(n)}(x_n), \quad (x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n, \tag{22}$$

such that $u = \sum_{i \in I} g_i^{(1)} \otimes \cdots \otimes g_i^{(n)}$ is a weak*-representation of u in $\mathcal{E}_n^1$. As in the case $n = 2$, the sum in (22) is generally uncountable and ψ may not actually be a function. For convenience, we think of ψ as being a function that represents u in $\mathcal{E}_n^1$ and we say $\psi \in \mathcal{E}_n^1$ and write $\|\psi\|_{\mathcal{E}_n^1}$ for $\|u\|_{\mathcal{E}_n^1}$.

Let $\mathcal{F}_n(\nu_1, \ldots, \nu_n)$ denote the collection of $\mu \in \mathcal{F}_n(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ that are absolutely continuous with respect to $\nu_1 \times \cdots \times \nu_n$.

Theorem 6.1. *Let $(X_1, \mathcal{A}_1, \nu_1), \ldots, (X_n, \mathcal{A}_n, \nu_n)$ be positive σ-finite measure spaces. Then $\mu \in \mathcal{F}_n(\nu_1, \ldots, \nu_n)$ if and only if there exists a $u \in \mathcal{E}_n^1(\nu_1, \ldots, \nu_n)$ such that*

$$\int f_1 \otimes \cdots \otimes f_n \, d\mu = \langle f_1 \otimes \cdots \otimes f_n, u \rangle, \quad f_k \in L^\infty(X_k, \nu_k), \quad 1 \leq k \leq n.$$

If $\mu \in \mathcal{F}_n$ corresponds to $u \in \mathcal{E}_n^1$, then $\|\mu\|_{\mathcal{F}_n} \leq \|u\|_{\mathcal{E}_n^1} \leq c_n \|\mu\|_{\mathcal{F}_n}$, where c_n is the constant in Theorem 2.3. Furthermore, if $\psi \in \rho(u)$, then

$$\int \phi \, d\mu = \int_{X_1 \times \cdots \times X_n} \phi(x_1, \ldots, x_n) \, \psi(x_1, \ldots, x_n) \, (\nu_1 \times \cdots \times \nu_n)(dx_1, \ldots, dx_n),$$

for all $\phi \in \mathcal{V}_n(\nu_1, \ldots, \nu_n)$, and $|\int \phi \, d\mu| \leq \|\phi\|_{\mathcal{V}_n} \|\psi\|_{\mathcal{E}_n^1}$.

Proof. Let $\mu \in \mathcal{F}_n(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ be such that $\mu \ll \nu_1 \times \cdots \times \nu_n$. We wish to show that the corresponding u is an element of $L^1(X_1, \nu_1) \otimes_{eg} \cdots \otimes_{eg} L^1(X_n, \nu_n)$. Let

$$\hat{\mu}(f_1, \ldots, f_n) = \int f_1 \otimes \cdots \otimes f_n \, d\mu,$$

for all $(f_1, \ldots, f_n) \in L^\infty(X_1, \nu_1) \times \cdots \times L^\infty(X_n, \nu_n)$. It suffices to show that $\hat{\mu}$ is weak*-continuous in each argument separately. Fix some $k \in \mathbb{N}$ such that $1 \leq k \leq n$. For each $j \neq k$, let $f_j^{(0)} \in L^\infty(X_j, \nu_j)$ be given. Define a bounded linear functional Λ_k on $L^\infty(X_k, \nu_k)$ by

$$\Lambda_k(f_k) = \int f_1^{(0)} \otimes \cdots \otimes f_k \otimes \cdots \otimes f_n^{(0)} \, d\mu, \quad f_k \in L^\infty(X_k, \nu_k). \tag{23}$$

By Proposition 3.2,

$$\Lambda_k(f_k) = \int_{X_k} f_k(x_k) \, \mu_{\bigotimes_{j \neq k} f_j^{(0)}}(dx_k).$$

The measure $\mu_{\bigotimes_{j \neq k} f_j^{(0)}}$ is absolutely continuous with respect to ν_k (Proposition 3.1), and so (by the Radon-Nikodým theorem) there exists a function $g_k \in L^1(X_k, \nu_k)$ such that

$$\int_{X_k} f_k(x_k) \, \mu_{\bigotimes_{j \neq k} f_j^{(0)}}(dx_k) = \int_{X_k} f_k(x_k) \, g_k(x_k) \, \nu_k(dx_k).$$

Consequently,

$$\Lambda_k(f_k) = \langle g_k, f_k \rangle, \quad f_k \in L^\infty(X_k, \nu_k),$$

where $\langle \cdot, \cdot \rangle$ represents the dual action of $L^\infty(X_k, \nu_k)$ on $L^1(X_k, \nu_k)$. Since this is weak*-continuous, and the choice of k was arbitrary, it follows that u is an element of $\mathcal{E}_n^1$, as required. The remaining claims are proved using similar arguments to those used in the proofs of Proposition 4.5, Corollary 4.6, and Theorem 4.9. □

As in the case $n = 2$, if $\psi \in \mathcal{E}_n^1$ corresponds to $\mu \in \mathcal{F}_n$ as in Theorem 6.1, we call ψ a *derivative* of μ with respect to $\nu_1 \times \cdots \times \nu_n$.

7. A bounded convergence theorem?

The purpose of this note was to generalize the Radon-Nikodým theorem to Fréchet measures. This complements the generalization of the Riesz representation theorem mentioned in the introduction. (See (2).) It is natural to ask if the bounded convergence theorem can be similarly generalized.

Question 7.1. *Let $(X_1, \mathcal{A}_1, \nu_1), \ldots, (X_n, \mathcal{A}_n, \nu_n)$ be positive σ-finite measure spaces, and suppose that $\mu \in \mathcal{F}_n(\nu_1, \ldots, \nu_n)$. Let $(\phi_k)_{k=1}^\infty$ be a sequence of functions that converge a.e.$(\nu_1 \times \cdots \times \nu_n)$ and are uniformly bounded in $\mathcal{V}_n(\nu_1, \ldots, \nu_n)$; i.e., $\sup_k \|\phi_k\|_{\mathcal{V}_n} < \infty$. Is it true that $\phi = \lim_{k\to\infty} \phi_k$ a.e.$(\nu_1 \times \cdots \times \nu_n)$ is integrable with respect to μ, and does it follow that $\int \phi \, d\mu = \lim_{k\to\infty} \int \phi_k \, d\mu$?*

Functions of the type ϕ in Question 7.1 are members of the *tilde algebra* of $\mathcal{V}_n(\nu_1, \ldots, \nu_n)$. Recall that $\mathcal{V}_n = \mathcal{V}_n(\nu_1, \ldots, \nu_n) = L^\infty(\nu_1) \widehat{\otimes} \cdots \widehat{\otimes} L^\infty(\nu_n)$. The tilde algebra of $\mathcal{V}_n$, which is denote by $\widetilde{\mathcal{V}}_n$, is the collection of functions on $X = X_1 \times \cdots \times X_n$ that are almost everywhere (with respect to $\nu = \nu_1 \times \cdots \times \nu_n$) limits of functions $\phi_k \in \mathcal{V}_n(\nu_1, \ldots, \nu_n)$, where the sequence $(\phi_k)_{k=1}^\infty$ is uniformly bounded in the $\mathcal{V}_n$ norm. That is,

$$\widetilde{\mathcal{V}}_n = \Big\{ \phi \in L^\infty(X, \nu) : \phi = \lim_{k\to\infty} \phi_k \text{ a.e.}(\nu), \ \sup_k \|\phi_k\|_{\mathcal{V}_n} < \infty \Big\}. \tag{24}$$

The space $\widetilde{\mathcal{V}}_n$ becomes a Banach algebra when equipped with the norm

$$\|\phi\|_{\widetilde{\mathcal{V}}_n} = \inf \Big\{ \sup_k \|\phi_k\|_{\mathcal{V}_n} : \phi = \lim_{k\to\infty} \phi_k \text{ a.e.}(\nu), \ \sup_k \|\phi_k\|_{\mathcal{V}_n} < \infty \Big\}, \tag{25}$$

Because Question 7.1 is a question about the integrability of functions in the tilde algebra of $\mathcal{V}_n$, it is sometimes called the *tilde problem*.

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