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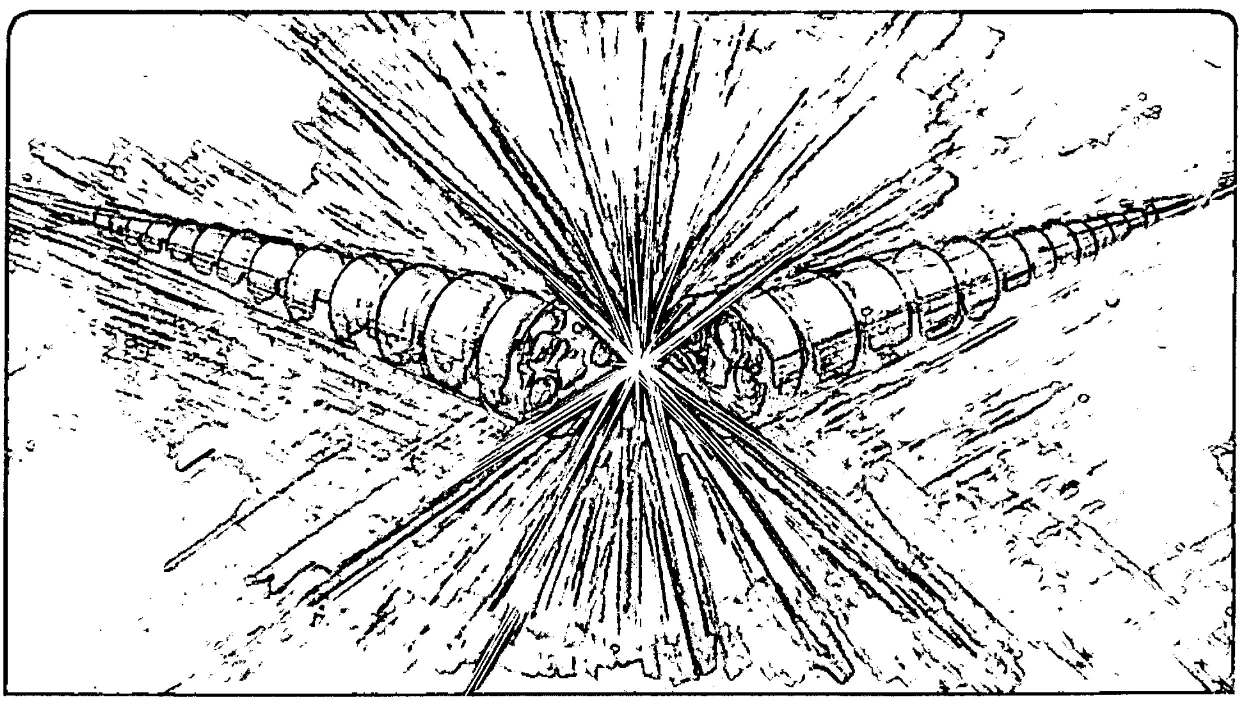

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The Covariant Lie-Transformed Plasma Action Principle*

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THE COVARIANT LIE-TRANSFORMED PLASMA ACTION PRINCIPLE

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The Lie Transform is a systematic technique [1] for obtaining the nonlinear quasi-static effects of high-frequency phenomena. We consider, in particular, a plasma of charged particles in a self-consistent electromagnetic wave. Each particle oscillates about an "oscillation center" which undergoes non-oscillatory motion [2]. We obtain the so-called "ponderomotive" Hamiltonian for the oscillation center, in relativistically covariant form, and demonstrate, via the action principle, that it determines the dielectric susceptibility and wave propagation.

We begin with the (eight) canonical variables $z = (r,p)$, which satisfy the covariant Hamiltonian equations:

$$dr^{\mu}/d\tau = \partial H/\partial p_{\mu}, \qquad dp_{\mu}/d\tau = -\partial H/\partial r^{\mu}, \tag{1}$$

with the invariant Hamiltonian function $H(z;A)$, a functional of the Maxwell potential $A(x)$. The correct particle evolution equations are obtained by the choice ($c=1$):

$$H = (p - eA(r))^2/2m, \tag{2}$$

noting that

$$A_{\mu}(r) = \int d^4x\ \delta^4(x-r)\ A_{\mu}(x). \tag{3}$$

Consider the family of all phase-space trajectories $z(\tau)$, each parameterized (smoothly) by its proper time τ. The seven-dimensional time-like surface $\tau = 0$ will be called the "initial-condition" surface. We introduce arbitrary coordinates η on this surface, and let $g(\eta)d^7\eta$ denote the number of particles (of a given species) in $d^7\eta$. With the trajectory $z(\tau;\eta)$ considered as an eight-dimensional field on the eight-dimensional phase-space (τ,η) [it's actually a mapping of phase space onto itself], we construct the action functional for the system:

$$S[z(\tau,\eta),A(x)] = \int d^7\eta\ g(\eta) \int d\tau\ [p_{\mu}(\tau,\eta)dr^{\mu}(\tau,\eta)/d\tau - H(p,r;A)]$$

$$- \int d^4x\ F_{\mu\nu}F^{\mu\nu}/16\pi, \tag{4}$$

where $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ is the Maxwell field.

Variation of S with respect to $z(\tau;\eta)$ yields the Hamiltonian equations (1), while variation with respect to A(x) yields the Maxwell equation

$$F^{\mu\nu}{}_{,\nu} = 4\pi j^{\mu}, \tag{5}$$

with

$$j^{\mu}(x) = \int d^7 n\ g(n) \int d\tau\ j^{\mu}(x;z) \tag{6}$$

and

$$j^{\mu}(x;z) = -\ \delta H(z)/\delta A_{\mu}(x). \tag{7}$$

Evaluation of (7), by (2), yields

$$j^{\mu}(x;z) = (e/m)\ \delta^4(x-r)\ (p^{\mu} - eA^{\mu}). \tag{8}$$

We now define the (invariant) Vlasov distribution

$$f(\bar{z}) = \int d^7 n\ g(n) \int d\tau\ \delta^8(\bar{z} - z(\tau,n))\ , \tag{9}$$

and derive the Ignatiev equation [3],

$$\{f(z),\ H(z)\} = 0\ , \tag{10}$$

in terms of the convariant canonical Poisson bracket:

$$\{a,b\} = (\partial a/\partial r^{\mu})(\partial b/\partial p_{\mu}) - (\partial a/\partial p_{\mu})(\partial b/\partial r^{\mu}) \tag{11}$$

To derive (10), we introduce the intermediate distribution

$$\bar{g}(\bar{z};\tau) = \int d^7 n\ g(n)\ \delta^8(\bar{z} - z(\tau,n)) \tag{12}$$

and obtain, by (1),

$$\partial g/\partial\tau = -\{g,H\}. \tag{13}$$

Integration over τ then yields (10), since g = 0 at infinite τ.

We now restrict the Maxwell potential to represent a single wave of eikonal form:

$$A_{\mu}(x) = \tilde{A}_{\mu}(x)\ \exp\ i\theta(x)/\epsilon + \text{c.c.} \tag{14}$$

where the amplitude $\tilde{A}$ and the gradient of the phase $k_{\mu}(x) = \partial_{\mu}\theta(x)$ are slowly varying fields. We substitute (14) into (2), and then invoke the Lie Transform to eliminate rapidly oscillating terms.

The new Hamiltonian K is related to the old H by the formula [1]

$$K = [\exp\ i\ \{w,\cdot\}]\ H, \tag{15}$$

where w(z) is a suitably chosen generating function. Expanding both H and K in powers of the amplitude $\tilde{A}$, we have $K^{(0)} = H^{(0)}$, and

$$K^{(1)} = H^{(1)} + \{w,\ H^{(0)}\}. \tag{16}$$

We can require $K^{(1)}$ to vanish, if we ignore the problems of resonant denominators. Solving (16) for w(z), and proceeding to second order, we obtain $K^{(2)}(z)$, which we denote by $\Psi(z;A)$, the relativistically invariant ponderomotive potential:

$$\Psi(z;A) = \int d^4x\, A^*_\mu(x)\, \Psi^\mu{}_\nu(x;z) A^\nu(x); \tag{17}$$

where

$$\Psi^\mu{}_\nu(x;z) = \delta^4(x-r)(e^2/m)(p\cdot k)^{-2}\,[k^2 p^\mu p_\nu + \delta^\mu_\nu (k\cdot p)^2 - (k\cdot p)(p^\mu k_\nu + k^\mu p_\nu)]. \tag{18}$$

This may be expressed more concisely as

$$\Psi = m\,|d\tilde{u}/d\tau|^2/(d\Theta(r)/d\tau)^2, \tag{19}$$

and reduces, in the rest-frame, to the familiar

$$\Psi = e^2|\tilde{E}|^2/m\omega^2. \tag{20}$$

Invariance of the phase-space Lagrangian under a canonical transformation:

$$\int d\tau\, [p_\mu dr^\mu/d\tau - H(p,r)] = \int d\tau\, [\bar{p}_\mu d\bar{r}^\mu/d\tau - K(\bar{p},\bar{r})], \tag{21}$$

where the overbar denotes oscillation-center variables, converts the action to

$$S = \int d^7\eta\, g(\eta) \int d\tau\, [\bar{p}_\mu d\bar{r}^\mu/d\tau - \bar{p}^2/2m - \Psi(z;A)] - \int d^4x\, F^*_{\mu\nu} F^{\mu\nu}/8\pi. \tag{22}$$

Defining the invariant oscillation-center distribution $F(z) = \int d^7\eta\, g(\eta) \int d\tau\, \delta^8(z-\bar{z}(\tau,\eta))$, we obtain, in analogy to the steps leading to (10), the corresponding Ignatiev equation:

$$\{F(z),\, p^2/2m + \Psi(z;A)\} = 0. \tag{23}$$

In order to vary S with respect to $\tilde{A}_\mu(x)$ and $\Theta(x)$, we first substitute (17) into (22), obtaining (for the terms bilinear in A)

$$S^{(2)} = \int d^4x\, \tilde{A}^*_\mu(x) D^\mu{}_\nu(x;F) \tilde{A}^\nu(x), \tag{24}$$

where the dielectric matrix $D^\mu{}_\nu$ is the sum of its vacuum part

$$D_{vac}{}^\mu{}_\nu(k(x)) = (k^\mu k_\nu - k^2 \delta^\mu_\nu)/4\pi \tag{25}$$

and the susceptibility

$$\chi^\mu{}_\nu(x;F) = -\int d^8z\, F(z)\, \Psi^\mu{}_\nu(x;z). \tag{26}$$

The relation (26) is the "K-χ theorem" [4], which is seen to be the essential ingredient of the action functional, coupling F(z) to $A_\mu(x)$.

We now express the dielectric matrix in terms of its local eigenvalues $D_\alpha(x)$ and eigenvectors $e_\alpha(x)$:

$$D^\mu{}_\nu(x) = \sum_\alpha D_\alpha(x) e^\mu_\alpha(x) e^{\alpha *}_\nu(x), \tag{27}$$

whence

$$S^{(2)} = \int d^4x\, D_\alpha(x) |A_\alpha(x)|^2, \tag{28}$$

with $A_\alpha(x) = e^\alpha{}_\nu{}^*(x)\tilde{A}^\nu(x)$, the projection of A on the α eigenvector.

We now vary $S^{(2)}$ with respect to $A_\alpha(x)$, obtaining the eikonal equation for the phase:

$$D_\alpha(x, k = \partial\theta/\partial x) = 0, \tag{29}$$

associated with polarization e_α. This yields the covariant ray equations:

$$dx^\mu/d\sigma = - \partial D_\alpha/\partial k_\mu, \qquad dk_\mu/d\sigma = \partial D_\alpha/\partial x^\mu . \tag{30}$$

Variation with respect to $\theta(x)$ yields the wave-action conservation law [5] $\partial J^\mu(x)/\partial x^\mu = 0$, where the wave-action density four-vector is

$$J^\mu(x) = - A_\alpha(x)^2 \partial D_\alpha(x,k)/\partial k_\mu \tag{31}$$

Since the eigenvalues $D_\alpha(x)$ are functionals of the oscillation-center distribution F(z), by (26), we have thus obtained a closed self-consistent set of coupled equations for F(z) and the wave amplitude and phase.

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