Extending The Capacity of Ad Hoc Networks beyond Network Coding

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ABSTRACT

The protocols used in ad hoc networks today are based on the assumption that the best way to approach multiple access interference (MAI) is to avoid it. Unfortunately, as the seminal work by Gupta and Kumar has shown, this approach does not scale. We demonstrate that protocol architectures that exploit multi-packet reception (MPR) *do* increase the order of the transport capacity of random wireless ad hoc networks for multi-pair unicast applications by a factor of $\Theta(\log n)$ and $\Theta(\log(\log n))$ under the protocol and physical models, respectively, where n is the number of nodes in the network. By contrast, Liu, Goeckel, and Towsley have shown that network coding (NC) does not increase the order capacity of wireless ad hoc networks under the protocol and physical models.

Categories and Subject Descriptors

C.2.1 [**Network Architecture and Design Wireless Communication**]: [Computer-Communication Network]

General Terms

Performance, Theory

Keywords

Multipacket Reception, Ad Hoc Networks, Unicast Capacity, Multihop Wireless Networks

1. INTRODUCTION

The communication protocols used today in ad hoc networks are based on a one-to-one communication paradigm in which a given receiver is able to decode at most one transmission correctly. The main objective of this one-to-one communication approach is the avoidance of multiple access interference (MAI). Unfortunately, the seminal work by Gupta and Kumar [7] demonstrated that the per source-destination throughput in a connected random wireless ad hoc network of n nodes adhering to such a communication paradigm scales as $\Theta\left(\frac{1}{\sqrt{n\log(n)}}\right)$[1] under the protocol model [7] for multi-pair unicast applications. Intuitively, the sharp decrease in capacity experienced as the number of nodes increases can be explained in the protocol model by the fact that a single successful transmission occupies an area given by the reception radius of the receiver, and this area is a function of the minimum radius needed for the network to be connected. Hence, as nodes are added, a smaller percentage of nodes are free to transmit successfully. Clearly, without exploiting node mobility [6], the only two possible approaches to increase the order capacity of ad hoc networks consist of (a) increasing the amount of information a transmitting node relays in each transmission, or (b) enabling a receiver to decode multiple concurrent transmissions within its reception radius. Work has been carried out in both fronts.

Recently, Ahlswede et al. introduced the concept of network coding (NC) [1], which allows nodes to conduct processing and combining on received packets before forwarding them. They proved that the max-flow min-cut throughput can be achieved for single source multicast applications in a directed graph in which there are no restrictions on when a node can send and receive information. However, Liu et al. [9] recently showed that NC cannot increase the transport capacity order of wireless ad hoc networks for multi-pair unicast applications when nodes are half-duplex using either the physical or the protocol model.

On the other hand, Ghez et al. [4,5] and Mergen et al. [10] provided a framework for many-to-one communications. In this context, multiple nodes cooperate to transmit their packets simultaneously to a single node using multiuser detection (MUD), directional antennas (DA), or multiple input multiple output (MIMO) techniques [2,12,14]. The receiver node utilizes MUD and successive interference cancelation (SIC) to decode multiple packets. Toumpis and Goldsmith [13] have shown that the capacity regions for ad hoc networks are significantly increased when multiple access schemes are combined with spatial reuse (i.e., multiple simultaneous transmissions), multi-hop routing (i.e., packet relaying), and SIC.

The contribution of this paper is to demonstrate that, unlike NC, MPR increases the order of the transport capacity of a random wireless ad hoc network under the protocol and the physical models.

IWCMC'07, August 12–16, 2007, Honolulu, Hawaii, USA.

[1] Θ, Ω and O are the standard order bounds.

Section 2 shows that the per source-destination throughput of a random wireless network of three dimensions (or 3-D network) in which nodes utilize MPR is tight bounded by $\Theta(r(n))$ (upper and lower bounds) w.h.p.[2] when the protocol model is used, where $r(n)$ is the reception range of a receiver. We note that $r(n) \geq \Theta\left(\sqrt[3]{\log n/n}\right)$ to ensure connectivity in random wireless ad hoc networks. This minimum $r(n)$ results in an achievable capacity bound of $\Theta\left(\sqrt[3]{\log n/n}\right)$ when nodes are endowed with MPR, which represents a gain in the order capacity of $\Theta(\log(n))$ compared to that attained with simple multihop routing [7,8] (one-to-one communication) or NC [9].

Section 3 shows that, under the physical model, a gain of $\Theta(\log(\log n))$ can be achieved in a two-dimensional random network compared with the capacity result by Gupta and Kumar in [7].

Our results are in stark contrast to prior results in ad hoc networks that assume point-to-point communications! They state that *increasing* the communication range $r(n)$ actually *increases* the capacity of an ad hoc network. Intuitively, the reason for this is that, given that all receivers are endowed with MPR, MAI around any receiver becomes useful information and no longer decreases the capacity. Clearly, the restrictions in choosing the communication range among nodes are: (a) the need to maintain connectivity in the network, which provides a lower bound on $r(n)$; and (b) the decoding complexity of the nodes in the network, which provides a practical upper bound on $r(n)$.

2. PROTOCOL MODEL

Our analysis under the protocol model focuses on the 3-D networks in which nodes are endowed with MPR capabilities. Our model is consistent with the analytical model in [8], and considers networks represented with an undirected graph (bidirectional links) such that two nodes X_i and X_j can communicate directly only if they are connected with an edge. These graph models have traditionally been used assuming a collision channel assumption [7], which we also denote by one-to-one communication assumption. That is, two nodes can communicate directly if they are within a distance $d(n)$, and the transmission from node X_i to node X_j is successful only if there is no other transmitter within distance $(1+\Delta)d(n)$ to node X_j. This inherently implies that the disks of different concurrent receivers with radius $d(n)$ are disjoint.

Applying the same protocol model to wireless networks with MPR capability means that nodes are able to receive successfully multiple packets concurrently, as long as the transmitters are within a radius of $r(n)$ from the receiver and all other transmitting nodes have a distance larger than $(1+\Delta)r(n)$. The key difference is that MPR allows the receiver node to receive multiple packets from different nodes within its disk of radius $r(n)$ simultaneously. Note that $d(n)$ in point-to-point communication is a random variable while $r(n)$ in MPR is a predefined value. To consider such networks, we use the graph models with MPR. We assume that a node communicates in half-duplex mode which is a common assumption in wireless ad hoc networks. The MPR transmission and reception assumptions together with protocol model for MPR are shown in Fig. 1.

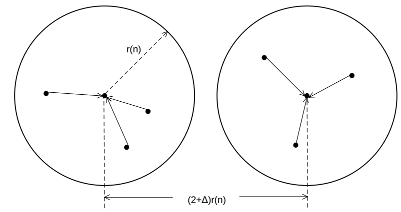

Figure 1: Protocol model for MPR scheme in wireless ad hoc networks.

Before proceeding with our discussion of capacity limits, we need to introduce a few results that we will use in our computations. First, Gupta and Kumar [8] showed that the connectivity among nodes in the 3-D model is guaranteed w.h.p. if and only if $r(n)$ is lower bounded by $\Theta\left(\sqrt[3]{\frac{\log n}{n}}\right)$. They also showed [8] that the distribution of nodes in random networks is uniform, so if there are n nodes in a unit volume, then the density of nodes equals n. Hence, if $|V|$ denotes the volume of space region V, the expected number of the nodes, $E(N_V)$, in this volume is given by $E(N_V) = n|V|$.

Let N_j be a random variable defining the number of nodes in V_j. Then, for the family of variables N_j, we have the following standard results known as the Chernoff bounds [11]:

$$P[|N_j - n|V_j|| > \delta n|V_j|] < e^{-\theta n|V_j|}, \tag{1}$$

where, θ is a function of δ. Therefore, for any $\theta > 0$, there exist constants such that deviations from the mean by more than these constants occur with probability approaching zero as $n \to \infty$. It follows that, w.h.p., we can get a very sharp concentration on the number of nodes in a volume, so we can find the achievable lower bound w.h.p., provided that the upper bound is given. In the following, we first derive the upper bound, and then use the Chernoff Bound to prove the achievable lower bound w.h.p..

Lastly, we note that the capacity results that we present also depend on the transmission bandwidth W of the network. However, given that we assume that W is independent of n, the value of W is simply a constant multiplier in capacity computations and does not change the order of capacity. Hence, we consider $W = 1$ for simplicity.

In the following, to simplify our analysis, we assume that n nodes are randomly located inside a cube of a unit area[3]. Each node selects a destination randomly, and sends $\lambda(n)$ bits/sec.

2.1 Upper Bound in 3-D

A cut Γ is a partition of a network into two connected components. The cut capacity is defined as the sum of bandwidth of all the edges crossing the cut. A min-cut is a cut whose capacity is the minimum value of all cuts. For wireless networks, we use a *sparsity cut* instead of min-cut to take into account the broadcast nature of wireless links [9].

In the 3-D case, the cut plane Γ_p is defined as the area of the cut. The cut plane that we consider has zero volume, such that no node lies on it. A sparsity cut for a random network is defined as a cut induced by the plane with the minimum

[2] In this paper, w.h.p. denotes "with probability 1 when $n \to \infty$."

[3] In order to avoid edge effects, we can use a sphere as in [8] and the results of this paper will not change.

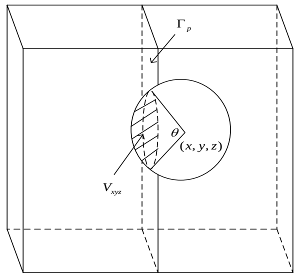

Figure 2: Cubic unit area.

area that separates the region into two subregions [9]. For the square deployment region illustrated in Fig. 2, the middle plane induces a sparsity cut Γ_p. Because nodes are uniformly deployed in a random network, such a sparsity cut captures the traffic bottleneck of a random network on average. This cut capacity constrains the information rate that the nodes from one side of the cut as a whole can deliver to the nodes at the other side. This is the maximum information (in bits per second) that can be transmitted across the cut from left to right (or from right to left). The formal definition of sparsity cut is given below.

DEFINITION 2.1. *In the 3-D model, a sparsity cut Γ_p for a random network is defined as a cut induced by the plane with the minimum area that separates the region into two volume subregions.*

The sparsity cut capacity is upper bounded by deriving the maximum number of simultaneous transmissions across the cut. In the work by Gupta and Kumar [8], spheres of radius $r(n)$ centered at each receiver are not necessarily disjoint, and the protocol model is still satisfied as long as the transmitter has the closest distance to the receiver node compared to all other transmitters in the network. In the MPR protocol model, the receiver node can receive packets simultaneously from all the nodes within a radius of $r(n)$ and all other transmitters should be outside of region of radius $r(n)$. The following lemma describes the MPR protocol model.

LEMMA 2.2. *The sphere with radius $r(n)$ centered at any receiver should be disjoint from the other spheres centered at the other receivers.*

PROOF. The proof is by contradiction and it is omitted because of space limitations. The readers can find the details in [15]. □

The protocol model assumption for the MPR scheme allows simultaneous transmissions by nodes as long as they are within a radius of $r(n)$ from the receiver and all other transmitting nodes have a distance larger than $(1 + \Delta)r(n)$. Δ is a guard zone that is a function of the physical layer characteristics.

LEMMA 2.3. *The capacity of a sparsity cut Γ for a unit region has an upper bound of $2c_1\Gamma_p r(n)n$, where $c_1 = \frac{1}{3(1+\frac{\Delta}{2})^2}$.*

PROOF. The cut capacity is upper bounded by the maximum number of simultaneous transmissions across the cut. In Fig. 2 we observe that all the nodes located in the shaded volume V_{xyz} can send their packets to the receiver node located at (x, y, z). These nodes lie in the left side of the cut Γ_p within a volume called V_{xyz} and the assumption is that all these nodes send packets to the right side of the cut Γ_p.

For a node at location (x, y, z), any node in the sphere of radius $r(n)$ can transmit information to this node simultaneously and the node can successfully decode those transmissions. To obtain an upper bound, we only need to consider edges that cross the cut. We first consider all possible nodes that can transmit to the receiver node in the V_{xyz} region. We use the fact that $E(N_V) = n|V|$ to estimate the average number of transmitters located in V_{xyz} as nV_{xyz}. The number of nodes that are able to transmit at the same time from left to right is upper bounded as a function of V_{xyz}. The volume of V_{xyz}, which is a spherical cap, is given by

$$V_{xyz} = \frac{1}{3}\pi r^3(n)\left(1 - \cos\frac{\theta}{2}\right)^2\left(2 + \cos\frac{\theta}{2}\right). \qquad (2)$$

Hence, the total number of nodes that can send packets across the cut is

$$\begin{aligned} N_{max} &= \max_{0\le\theta\le\pi}\left[\frac{\Gamma_p}{\pi(1+\frac{\Delta}{2})^2 r^2(n)\sin^2\frac{\theta}{2}}V_{xyz}n\right] \\ &= \max_{0\le\theta\le\pi}\left[c_1\Gamma_p\frac{(1-\cos\frac{\theta}{2})(2+\cos\frac{\theta}{2})}{1+\cos\frac{\theta}{2}}r(n)n\right], \qquad (3) \end{aligned}$$

where, $c_1 = \frac{1}{3(1+\frac{\Delta}{2})^2}$. This number is maximized with $\theta = \pi$. Therefore, the total number of nodes is upper bounded by $2c_1\Gamma_p r(n)n$. □

COROLLARY 2.4. *For any unit-volume 3-D random network of arbitrary shape, if the minimum cut plane Γ_p is not a function of n, then the sparsity cut capacity has an upper bound of $O(nr(n))$.*

PROOF. Regardless of the shape of the unit volume region, there exists a sparsity cut for each orientation of the cut plane. This sparsity cut capacity depends only on the minimum cut area Γ_p. If Γ_p is not a function of n, then the capacity is always upper bounded by $O(nr(n))$. □

THEOREM 2.5. *The per source-destination throughput of the MPR scheme in a 3-D random network is upper bounded by $O(r(n))$.*

PROOF. For a sparsity cut Γ in the middle of the network, on average, there are $\Theta(n)$ pairs of source-destination nodes that need to cross Γ in one direction w.h.p., i.e., $n_{\Gamma_{1,2}} = n_{\Gamma_{2,1}} = \Theta(n)$. The theorem then follows by combining this result with Corollary 2.4. □

2.2 Lower Bound in 3-D

We now prove that, when n nodes are distributed uniformly over a unit cubic volume, there are simultaneously at least $\frac{c_2\Gamma_p}{r^2(n)}$ circular regions, where $c_2 = \frac{1}{\pi(1+\Delta/2)^2}$, and each such region contains $\frac{2}{3}n\pi r^3(n)$ nodes w.h.p.. This allows us to obtain an achievable lower bound by using the Chernoff bound, such that the distribution of the number of edges across the cut plane is sharply concentrated around its mean. Therefore, in a randomly chosen network, the actual number of edges crossing the sparsity cut plane is indeed $\Theta(nr(n))$ w.h.p..

THEOREM 2.6. *Each spherical region V_j contains $\Omega(nr^3(n))$ nodes w.h.p. for all values of $j, 1 \le j \le \frac{c_2\Gamma_p}{r^2(n)}$.*

This theorem can be expressed as

$$\lim_{n\to\infty} P\left[\bigcap_{j=1}^{c_2\Gamma_p/r^2(n)} |N_j - E(N_j)| < \delta E(N_j)\right] = 1, \quad (4)$$

where, δ is a positive small value arbitrarily close to zero.

PROOF. From the Chernoff bound and Equation (1), for any given $0 < \delta < 1$, we can find $\theta > 0$ such that

$$P\left[[|N_j - E(N_j)| > \delta E(N_j)] < e^{-\theta E(N_j)}\right] = e^{-\theta n|V_j|}. \quad (5)$$

Thus, we can conclude that the probability that the value of the random variable N_j deviates by an arbitrarily small constant value from the mean tends to zero as $n \to \infty$. Hence, when all the events $\bigcap_{j=1}^{c_2\Gamma_p/r^2(n)} |N_j - E(N_j)| < \delta E(N_j)$ occur simultaneously, then all N_j's converge uniformly to their expected values. Utilizing the union bound, we obtain

$$\begin{aligned} & P\left(\bigcap_{j=1}^{c_2\Gamma_p/r^2(n)} |N_j - E(N_j)| < \delta E(N_j)\right) \\ \geq\ & \max\left\{1 - \sum_{j=1}^{c_2\Gamma_p/r^2(n)} P\left[|N_j - E(N_j)| < \delta E(N_j)\right], 0\right\} \\ >\ & \max\left\{1 - \frac{c_2\Gamma_p}{r^2(n)} e^{-\theta E(N_j)}, 0\right\}. \end{aligned} \quad (6)$$

Because $E(N_j) = \frac{2\pi}{3} n r^3(n)$, the final result is

$$\begin{aligned} & \lim_{n\to\infty} P\left[\bigcap_{j=1}^{c_2\Gamma_p/r^2(n)} |N_j - E(N_j)| < \delta E(N_j)\right] \\ & \geq \max\left\{1 - \frac{c_2\Gamma_p}{r^2(n)} e^{-\frac{2\pi\theta}{3} n r^3(n)},\ \ 0\right\}. \end{aligned} \quad (7)$$

To guarantee connectivity, $r(n) > \sqrt[3]{\frac{\log n}{n}}$ [8]. Therefore, as $n \to \infty$, we have $\frac{e^{-\frac{2\pi\theta n r^3(n)}{3}}}{r^2(n)} \to 0$. □

This theorem demonstrates that the lower bound can be achieved w.h.p.

COROLLARY 2.7. *The per source-destination throughput of the MPR scheme for a 3-D random network has an achievable lower bound of $\Omega(r(n))$ w.h.p..*

PROOF. Theorem 2.6 proves that there are $\frac{\Gamma_p}{\pi(1+\Delta/2)^2 r^2(n)}$ different circles of radius $r(n)$, each of them having $\Theta(nr^3(n))$ nodes w.h.p. Therefore, the per source-destination throughput is the multiplication of these two values divided by the total number of nodes, which proves the corollary. □

By utilizing the lower bound for $r(n)$ as $\Omega(\sqrt[3]{\log n/n})$ to guarantee connectivity of the network, the tight bound capacity can be given as

$$\Theta(r(n)) = \Theta(\sqrt[3]{\log n/n}).$$

Note that if we increase $r(n)$, the transport capacity increases as well. Constructively, therefore, the following theorem follows for the 3-D case.

THEOREM 2.8. *The per source-destination throughput $\lambda(n)$ of MPR scheme for a 3-D random network is given by $\Theta(r(n))$ as the lower and upper bounds w.h.p..*

3. PHYSICAL MODEL

In the physical model [7], a successful communication occurs if $SINR \geq \beta$, where $SINR$ is defined as

$$SINR = \frac{P_{ij}(t) g_{ij}(t)}{BN_0 + \underbrace{\sum_{k\neq i, k\in A} P_{kj} g_{kj}}_{COI} + \underbrace{\sum_{t\neq i, t\notin A} P_{tj} g_{kj}}_{DEI}}. \quad (8)$$

With MPR, each receiving node is able to decode the transmissions of all the nodes transmitting within its receiving range of distance A, and any transmission outside that range is considered interference. In Eq. (8), P_{ij} is the transmit power of the node i with closest distance to the receiver j; P_{kj} is the transmit power of a node other than i within the receiver range, which constitutes constructive interference (COI); and P_{tj} is the transmit power of node t outside the receiver range,[4] which is considered as destructive interference (DEI).

The normalized capacity between transmit node i and receive node j is defined as C_{ij} bits/sec and it is given by

$$C_{ij} = \log\left(1 + SINR\right), \quad (9)$$

3.1 Interference Analysis

We first prove that decoding occurs from the closest node to the farthest node.

LEMMA 3.1. *The transmitter receiver pair with maximum* SINR *is the nearest transmitter, after decoding and subtracting this pair from the received signal, the pair with the next highest* SINR *is the second nearest transmitter; i.e., we decode the information from the nearest node to the receiver to the last transmitter with maximum distance of A.*

PROOF. Because the channel propagation model is based on the path-loss parameter, it is clear from (8) that the node with the closest distance to the receiver has the highest $SINR$. After decoding this packet and subtracting it from the received data, it is obvious that the next packet with highest $SINR$ is from the second closest node to the receiver node and this procedure can continue. □

To compute the interference around a receiver, we first consider a differential element area $rdrd\phi$ that is distant r units from the receiver. Using a technique similar to that used in [3] and considering the fact that the nodes are uniformly distributed in the square unit area, then the interference created by this area can be computed as

$$I_{(x_0,y_0)} = \frac{2\pi\delta P_{tj} n}{(\alpha-2) r_0^{\alpha-2}}\left[1 - \frac{r_0^{\alpha-2}}{2\pi} C(x_0,y_0)\right], \quad (10)$$

where $C(x_0, y_0)$ is a function of the receiver location (x_0, y_0), δ is the sender density parameter, and r_0 is the distance between the transmitter with the highest $SINR$ and the receiver.

Assume that our communication model is a narrow band system, then interference is the dominant term compared to the additive white Gaussian noise (AWGN). Consequently, we can omit the noise term BN_0 and using (8) and (10), the

[4]Note that for the MPR model, we need to define receiver range as opposed to transmission range for point-to-point communication [7].

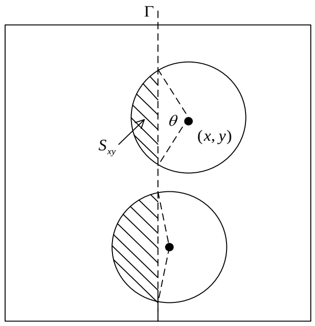

Figure 3: Feasible transmitters for node at location (x,y).

SINR is given by

$$\begin{aligned} SINR &= \frac{P_{ij}g_{ij}}{BN_0 + I} \le \frac{P_{ij}}{P_{tj}}\frac{(\alpha-2)}{2\pi\delta}\frac{1}{nr_0^2}\left[1 - \frac{r_0^{\alpha-2}}{2\pi}C(x_0,y_0)\right]^{-1} \\ &\le \frac{c_1}{nr_0^2}\left[1 - c_2 r_0^{\alpha-2}\right]^{-1}. \end{aligned} \quad (11)$$

Note that $g_{ij} = 1/r_0^\alpha$, $c_1 = \frac{P_{ij}}{P_{tj}}\frac{(\alpha-2)}{2\pi\delta}$ and c_2 is a constant value such that $c_2 \ge C(x_0,y_0)/2\pi$. Each receiver node decodes all transmitting nodes within a receiver range of A using MPR and SIC schemes.

3.2 Upper Bound on Capacity

From (9) and (11), the normalized shannon link capacity can be computed as

$$\begin{aligned} C_{ij} &= \log(1 + SINR) \\ &\le \log\left(1 + \frac{c_1}{nr_0^2}\left[1 - c_2 r_0^{\alpha-2}\right]^{-1}\right). \end{aligned} \quad (12)$$

LEMMA 3.2. *The capacity of a sparsity cut Γ for a unit square region has an upper bound of*

$$\Theta\left(\frac{l_\Gamma}{2R_0}\log\frac{(c_1 + nR_0^2)^{(c_1+nR_0^2)}}{(nR_0^2)^{(nR_0^2)}c_1{}^{c^1}}\right).$$

PROOF. The cut capacity is upper bounded by the maximum number of simultaneous transmissions across the cut. From Fig. 3, we observe that all nodes located in the shaded area S_{xy} can transmit concurrently to the receiver located at (x,y). These nodes lie in the left side of the cut Γ and the assumption is that all these nodes are sending packets to the right side of the cut Γ.

For a node at location (x,y), any node in the disk of radius R_0 can transmit information to this receiver simultaneously and the node can successfully decode those packets. In order to obtain an upper bound, we only need to consider edges that cross the cut. Let us first consider all possible nodes in the S_{xy} region that can transmit to the receiver node. Because nodes are uniformly distributed, the average number of transmitters in area S_{xy} is $n \times S_{xy}$. The number of nodes that can transmit at the same time from left to right is upper bounded as a function of S_{xy}.

Similar to the 3-D case in the previous section, it can be shown that the area of S_{xy} is maximized when $\theta = \pi$. Hence,

$$\max_{0\le\theta\le\pi}[S_{xy}] = \frac{1}{2}\pi R_0^2 \quad (13)$$

From Lemma 3.1, it is clear that the decoding sequence is from close nodes to far nodes. We can compute the total information capacity using the differential area $r_0 dr_0 d\phi$ with $nr_0 dr_0 d\phi$ transmitter nodes inside this region. The total information that can be transmitted from this region is equal to $C_{ij}nr_0dr_0d\phi$, where C_{ij} is defined in Eq. (12). Define $R_0 = f(n)$, such that $\lim_{n\to\infty} f(n) = 0$. Given that $r_0 \le R_0$, we have $\lim_{n\to\infty}[1 - c_2 r_0^{\alpha-2}]^{-1} = 1$ in (12) for $\alpha > 2$. The total throughput across the cut Γ when $n \to \infty$ can be computed as follows:

$$\begin{aligned} C_j &= \sum_{i\in\frac{1}{2}\pi R_0^2} C_{ij} = \int_0^{R_0}\int_0^{\pi} C_{ij}nr_0dr_0d\phi \\ &= \Theta\left(\log\frac{(c_1+nR_0^2)^{(c_1+nR_0^2)}}{(nR_0^2)^{(nR_0^2)}c_1{}^{c_1}}\right), \end{aligned} \quad (14)$$

where R_0 is the receiver range that separates the *COI* and *DEI* interferences.

Therefore, the total throughput capacity C across the sparsity cut is $C = \frac{l_\Gamma}{2R_0}C_j$. □

THEOREM 3.3. *The per source-destination throughput of* MPR *scheme in a 2-D random network is upper bounded by* $\Theta(\sqrt{1/n})$*, when* $R_0 = \Theta(\sqrt{1/n})$.

PROOF. There are $l_\Gamma/2R_0$ different circles of radius R_0 each of them having $\Theta(nR_0^2)$ nodes w.h.p. Therefore, the per node throughput capacity can be derived as

$$\begin{aligned} \lambda(n) &= \frac{C}{\Theta(n)} = \Theta\left(R_0\log\left(1 + \frac{c_1}{nR_0^2}\right) + c_1\frac{\log(nR_0^2/c_1)}{nR_0}\right) \\ &= \Theta\left(\log\left(1 + \frac{c_1}{nR_0^2}\right)^{R_0}\right) + \Theta\left(\frac{\log(nR_0^2/c_1)}{nR_0}\right) \end{aligned} \quad (15)$$

The first term in Eq. 15 tends to zero when $n \to \infty$. To maximize the second term of the equation, the optimum value of R_0 is $\sqrt{\frac{c_1e^2}{n}} = \Theta(\sqrt{1/n})$. When $R_0 = \Theta(\sqrt{1/n})$, the maximum of $\lambda(n)$ equals $\Theta(\sqrt{1/n})$, i.e., $\lambda_{max}(n) = \Theta(\sqrt{1/n})$. □

3.3 Lower Bound on Capacity

We now prove that, when n nodes are distributed uniformly over a square area, we have simultaneously at least $\frac{l_\Gamma}{2R_0}$ circular regions (see fig. 3), each one containing $\Theta(nR_0^2)$ nodes w.h.p..

THEOREM 3.4. *Each area* A_j *with circular shape of radius* R_0 *contains* $\Theta(nR_0^2)$ *nodes w.h.p. and uniformly for all values of* $j, 1 \le j \le \frac{l_\Gamma}{2R_0}$ *under the condition that* $R_0 \ge \Theta\left(\sqrt{\frac{\log n}{n}}\right)$. *Equivalently, this theorem can be expressed as*

$$\lim_{n\to\infty} P\left[\bigcap_{j=1}^{l_\Gamma/2R_0} |N_j - E(N_j)| < \delta E(N_j)\right] = 1, \quad (16)$$

where δ *is a positive small value arbitrarily close to zero.*

PROOF. From the definition of the Chernoff bound and Eq. (1), there exists a $\theta > 0$ for any given $0 < \delta < 1$ such that

$$P\left[[N_j - |E(N_j)| > \delta E(N_j)] < e^{-\theta E(N_j)}\right] = e^{-\theta n|A_j|} \quad (17)$$

Utilizing the union bound and using a similar technique as the one applied for the lower bound in the protocol model, we arrive at

$$
\begin{aligned}
& P\left[\bigcap_{j=1}^{l_\Gamma/2R_0} |N_j - E(N_j)| < \delta E(N_j)\right] \\
\geq \quad & \max\left(1 - \sum_{j=1}^{l_\Gamma/2R_0} P\left[|N_j - E(N_j)| > \delta E(N_j)\right], 0\right) \\
\geq \quad & \max\left(1 - \frac{l_\Gamma}{2R_0} e^{-\theta E(N_j)}, 0\right). \qquad (18)
\end{aligned}
$$

Because $E(N_j) = \frac{\pi}{2} n R_0^2$ (see (13)), then the final result is

$$
\lim_{n\to\infty} P\left[\bigcap_{j=1}^{l_\Gamma/2R_0} |N_j - E(N_j)| < \delta E(N_j)\right] \geq \max\left(1 - \frac{l_\Gamma}{2R_0} e^{-\frac{\theta\pi n R_0^2}{2}}, 0\right) \qquad (19)
$$

If $R_0 \geq \sqrt{\frac{c_5 \log n}{n}} = \Theta\left(\sqrt{\frac{\log n}{n}}\right)$ and as $n \to \infty$, then $\frac{e^{-\frac{\theta \pi n R_0^2(n)}{2}}}{R_0} \to 0$, when $\theta > 1/\pi c_5$. Here, the key constraint of R_0 is given as

$$
R_0 \geq \Theta\left(\sqrt{\frac{\log n}{n}}\right), \qquad (20)
$$

which is equivalent to the connectivity constraint in the protocol model [7]. $\square$

This theorem demonstrates that we can achieve the lower bound with the constraint in Eq. (20) w.h.p. The achievable capacity is only feasible when the receiver range of each node using MPR is at least equal to the connectivity criterion of transmission range in point-to-point communication [7]. Combining the result in (15) and (20), we have the following theorem for the lower bound of throughput capacity.

THEOREM 3.5. *The per source-destination throughput capacity of MPR scheme in a 2-D static wireless ad hoc network is lower bounded by* $\Theta\left(\frac{\log(\log n)}{\sqrt{n \log n}}\right)$.

This theorem demonstrates that a gain of $\Theta(\log(\log n))$ can be achieved compared with the result of Gupta and Kumar in [7].

4. CONCLUSION

We have shown that exploiting MPR techniques in ad hoc networks can render an increase in transport capacity for both protocol and physical models. The key significance of this result is that, with MPR, the ability of ad hoc networks to scale is no longer limited by MAI, but by the complexity of transmitters and receivers. Using recent results from [9], we have also shown that MPR is a more attractive alternative than NC.

5. ACKNOWLEDGMENTS

This work was supported in part by the US Army Research Office under grants W911NF-04-1-0224 and W911NF-05-1-0246, and by the Baskin Chair of Computer Engineering. The opinions, interpretations, conclusions and recommendations are those of the authors and are not necessarily endorsed by the Department of Defense.

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