

# Problem Proposed for the *American Mathematical Monthly*

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*Problem:* Define

$$P(x) := \sum_{k=1}^{\infty} \arctan \left( \frac{x-1}{(k+x+1)\sqrt{k+1} + (k+2)\sqrt{k+x}} \right). \quad (1)$$

- (a) Find explicit, finite-expression evaluations of  $P(n)$  for all integers  $n \geq 0$ .
- (b) Show  $\tau := \lim_{x \rightarrow -1^+} P(x)$  exists, and find an explicit evaluation for  $\tau$ .
- (c) Are there a more general closed forms for  $P$ , say at half-integers?

*Solution.* With the abbreviations

$$r := \sqrt{k+1}, \quad s := \sqrt{k+x}$$

the argument of  $\arctan$  in (1) becomes

$$\frac{s^2 - r^2}{(s^2 + 1)r + (r^2 + 1)s} = \frac{s - r}{r s + 1} = \frac{\frac{1}{r} - \frac{1}{s}}{1 + \frac{1}{r} \frac{1}{s}}.$$

Therefore, by using the addition theorem of the tangent function, the definition (1) may be written in the more convenient form

$$P(x) = \sum_{k=1}^{\infty} \left( \arctan \frac{1}{\sqrt{k+1}} - \arctan \frac{1}{\sqrt{k+x}} \right) \quad (2)$$

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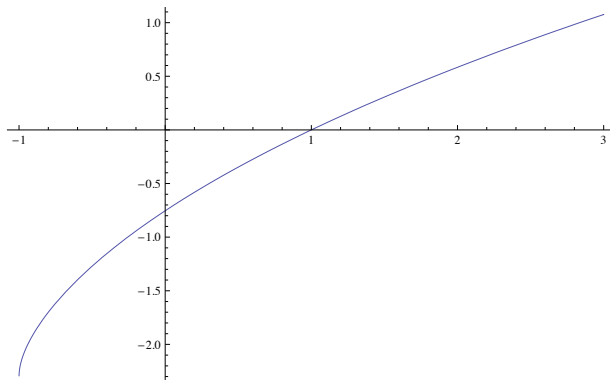


Figure 1: Plot of  $P(x)$

Now define

$$A(x) = \arctan \frac{1}{\sqrt{x+1}}$$

and note that a telescoping sum argument gives

$$P(x) + A(x) = P(x+1). \quad (3)$$

It is easy to see that the series defining  $P(x)$  is absolutely convergent by the Weierstrass M-test, and to verify that  $P(x)$  is increasing for  $x > -1$ , as shown in Figure 1. Thus,  $\tau$  exists.

(a). First observe that since  $P(1) = 0$ , the identity (3) establishes that  $P(0) = -A(0) = -\pi/4$ , which we had computationally observed. By iteratively applying (3) and applying induction, we establish that

$$\begin{aligned} P(2) &= \arctan \frac{1}{\sqrt{2}} \\ P(3) &= \arctan \frac{1}{\sqrt{2}} + \arctan \frac{1}{\sqrt{3}} \\ P(4) &= \arctan \frac{1}{\sqrt{2}} + \arctan \frac{1}{\sqrt{3}} + \arctan \frac{1}{2}, \end{aligned}$$

and indeed by induction we have, for all  $n \geq 2$ ,

$$P(n) = \sum_{k=2}^n \arctan \frac{1}{\sqrt{k}}.$$

(b). We computationally discovered that to 13-digit accuracy  $\tau = \lim_{x \rightarrow -1^+} P(x) = -3\pi/4$ . This can be rigorously established by noting that

$$\lim_{x \rightarrow -1^+} P(x) + \frac{\pi}{2} = \lim_{x \rightarrow -1^+} P(x) + \lim_{x \rightarrow -1^+} A(x) = \lim_{x \rightarrow -1^+} P(x+1) = P(0) = \frac{-\pi}{4}.$$