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Journal

Physics of Fluids, 24(6)

ISSN

00319171

Author

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Publication Date

1981

DOI

10.1063/1.863497

Peer reviewed

Kink and displacement instabilities in imploding wire arrays

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(Received 28 July 1980; accepted 29 January 1981)

Cylindrical arrays of parallel wires can be imploded by the magnetic forces generated by currents through the wires to form hot, dense *Z*-pinch plasmas. Analytic growth rates of displacements and deformations of wires in imploding wire arrays are calculated. Arrays of six or more wires are reasonably stable against asymmetric displacements. The growth rate of the kink instability on a single wire of linear mass density μ , radius a , and carrying current I peaks at $\lambda_{\max} = 0.71/\mu^{1/2}ca$ at a kink wavelength of about $4a$. The most unstable kink modes of n -wire arrays are the symmetric ($l = 0$) radial modes and antisymmetric ($l = n/2$) tangential modes. The effect of a center wire is to tend to destabilize radial kink modes and stabilize tangential modes. In the experimental parameter range of the largest current generators, the number of kink growth times before collision is insensitive to values of maximum current and current pulse width. In general, the kink instability will grow nonlinearly if the initial array radius is more than a few plasma wire radii. In the limit of long wavelengths, the kink instability is shown to be equivalent to the displacement instability.

I. INTRODUCTION

A high-voltage, high-current discharge through a cylindrical array of parallel wires causes the wires to accelerate toward the axis of the array, where they coalesce into a hot, dense *Z* pinch. The symmetry of the implosion of the array affects the temperature and density of the resulting pinch. The symmetry of the implosion can be degraded by kink instabilities and by displacement instabilities. A kink instability causes the distortion in a wire to grow. A displacement instability causes the wire as a whole to move away from its time-dependent canonical position in an array. This paper analytically calculates growth rates of both kink and displacement instabilities in imploding wire arrays.

Within the last few years large pulse power generators have been used to drive implosions in cylindrical geometry of fine wire arrays, foils, and annular gas jets.¹ The 11-TW BLACKJACK 5 at Maxwell Laboratories, the 8-TW PROTO-II at Sandia Laboratories, and the 5-TW PITHON at Physics International each generate currents in the megampere range with voltages in the megavolt range. Each has been used to produce hot, dense *Z*-pinch plasmas. Electron temperatures up to about 2 keV and densities of 10^{20} to 10^{21} cm⁻³ have been produced by this means. The *Z* pinches are useful as tools for understanding the physics of hot, dense plasmas and as sources of copious x rays.

The maximum temperatures and densities that can be achieved in such *Z* pinches depend upon the symmetry of the implosion. The dependence, however, is not yet well understood. For example, the highest plasma temperatures are not necessarily produced by the most symmetrical implosions. Kinking may cause the plasma to pinch more tightly in some regions, producing higher temperatures locally than could be achieved if the plasma imploded uniformly.

The purpose of this paper is to gain a better understanding of the instabilities in imploding wire arrays so that the dependence of plasma temperature and density

on implosion symmetry can be determined experimentally and used to advantage. This paper treats only the kink and displacement instabilities of imploding wire arrays because these instabilities most directly affect the position of the wire plasmas. The position of the axis of each wire can be completely described as a linear combination of sinusoidal perturbations superimposed on a straight wire. The displacement instability causes the straight wire to move away from its canonical position in an imploding array. The kink instability causes growth of the sinusoidal perturbations. These positional instabilities are those most in evidence in x-ray pinhole photographs of imploding wire arrays.²

The analysis of these instabilities does not require a treatment of plasma magnetohydrodynamics as long as the wavelength of the kink perturbation is much longer than the diameter of the wire plasma. The motion and position of the wires is assumed to be determined solely by the Biot and Savart law. The magnetohydrodynamics of single wire pinches has been treated analytically for both the linear kink instability³ and linear sausage instability.⁴ The magnetohydrodynamics of imploding wire arrays has been treated numerically.⁵ This paper presents the first analytic treatment of kink and displacement instabilities in imploding wire arrays. A linear perturbation analysis suffices as long as the amplitude of the perturbation is much less than its wavelength.

The next section presents the zeroth order dynamics of an imploding wire array as a base state of the perturbation analysis. Section III analyzes the displacement instability of imploding wire arrays. Section IV analyzes the kink instability of imploding wire arrays. Conclusions are presented in Sec. V.

II. DYNAMICS OF AN IMPLODING WIRE ARRAY

A typical wire array before implosion consists of n parallel wires mounted symmetrically and coaxially between an anode plate and a cathode plate. Farther from the axis thick current return bars are also mounted between the anode and cathode plates symmetrically and coaxially. The force on the imploding wires produced by the currents in the return bars can usually be neglected relative to the force produced by the other imploding

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wires because it is of the order of $(r/r_b)^n$ smaller, where r is the distance of the wires from the axis and r_b is the distance of the return bars from the axis. If the anode and cathode plates are much wider than r , then image currents in the plates cause the wires to behave magnetically as though they are infinite in length.

In cylindrical coordinates (r, θ, z) , the magnetic field of an array of n infinite wires parallel to the z axis is

$$B_\theta = \frac{2I}{nc} \sum_{\beta=1}^n \frac{r - r_\beta \cos(\theta - \theta_\beta)}{r^2 + r_\beta^2 - 2rr_\beta \cos(\theta - \theta_\beta)}, \quad (1a)$$

$$B_r = \frac{2I}{nc} \sum_{\beta=1}^n \frac{r_\beta \sin(\theta - \theta_\beta)}{r^2 + r_\beta^2 - 2rr_\beta \cos(\theta - \theta_\beta)}, \quad (1b)$$

in which I/n is the current in each wire. If all the wires are located an equal distance from the axis, then the field at the α th wire is

$$B_\theta = (n-1)I/ncr, \quad (2a)$$

$$B_r = \frac{I}{ncr} \sum_{\beta=1, \beta \neq \alpha}^n \frac{\sin(\theta_\alpha - \theta_\beta)}{1 - \cos(\theta_\alpha - \theta_\beta)}. \quad (2b)$$

If, additionally, all the wires on the circle are separated by equal angles, then $B_r = 0$. Equation (2a) implies that the tangential magnetic field at each wire is the same regardless of the angular positions of the wires on a circle. This geometric fact has important implications for the displacement stability of the array discussed in the next section.

Equations (2) and the equation of motion give the radial acceleration of each wire in a symmetrical array as

$$\ddot{R} = -\alpha f^2(\tau)/R, \quad (3)$$

in which $R = r/r_0$ is the distance of the wires from the axis normalized to the initial distance, $f(\tau) = I(\tau)/I_m$ is the total current through all of the wires of the array normalized to the maximum current, $\tau = t/t_0$ is the time normalized to some characteristic time, and the dimensionless constant α is defined by

$$\alpha \equiv \frac{(n-1)I_m^2 t_0^2}{n^2 \mu c^2 r_0^2}, \quad (4)$$

where μ is the mass per unit length of each wire.

Equation (3) can be solved exactly for the case of constant current. In that case the velocity of the wires is given by

$$\dot{R} = -[2\alpha \ln(1/R)]^{1/2},$$

and their radial position by inverting

$$\tau = \left(\frac{\pi}{2\alpha}\right)^{1/2} \operatorname{erf}\left[\ln\left(\frac{1}{R}\right)\right]^{1/2}.$$

The run-in time $(\pi/2\alpha)^{1/2}$ is a lower bound on the run-in time for more realistic current waveforms. A reasonable estimate of the run-in time for $\alpha \gg 1$ (and an upper bound for all α) is found by noting that $R \lesssim 1$ most of the time during the implosion. Thus, the run-in time, τ_1 , is bounded by

$$\int_0^{\tau_1} d\tau' \int_0^{\tau'} f^2(\tau'') d\tau'' \lesssim \frac{1}{\alpha}. \quad (5)$$

Figures 1(a) and 1(b) illustrate wire motion computed from Eq. (3) for the model current waveforms shown. Figure 2 shows run-in times as a function of α for the current wave-forms of both Figs. 1(a) and 1(b). The dashed curves are the upper bounds of the run-in times given by Eq. (5). Run-in times are not sensitive to the precise shape of the current waveforms for implosions having the same value of α .

III. DISPLACEMENT INSTABILITY

This section analyzes the displacement instability of imploding wire arrays by a linear perturbation approach. The displacement of a wire from its canonical position is assumed to be much less than the distance between wires. The canonical azimuthal angle of each wire in the array is given by

$$\theta_\alpha = 2\pi\alpha/n, \quad \alpha = 1, 2, \dots, n. \quad (6)$$

The canonical distance, r , of each wire from the center is given by the solution of Eq. (3). Let the coordinates of each wire be $(r + \delta r_\alpha, \theta_\alpha + \delta \theta_\alpha)$, and let each wire carry equal current I/n . Then, using Eqs. (1), the linearized equations of motion are

$$\left(\frac{d^2}{dt^2} + \omega^2 f_n\right) \delta r_\alpha = \omega^2 \sum_{\beta=1, \beta \neq \alpha}^n \frac{\delta r_\beta}{1 - \cos(\theta_\beta - \theta_\alpha)}, \quad (7a)$$

$$\left(\frac{d^2}{dt^2} - \omega^2 f_n\right) \delta \theta_\alpha = -\omega^2 \sum_{\beta=1, \beta \neq \alpha}^n \frac{\delta \theta_\beta}{1 - \cos(\theta_\beta - \theta_\alpha)}. \quad (7b)$$

The constants in these equations are

$$\omega \equiv I/(n\mu^{1/2}cr), \quad (8)$$

$$f_n \equiv \sum_{\beta=1}^{n-1} \frac{\cos \theta_\beta}{1 - \cos \theta_\beta}.$$

Table I lists values of f_n .

The tangential modes, which are solutions of Eq. (7b), can be found from the radial modes, which are solutions

FIG. 1. Normalized radius of imploding wire arrays (solid curves) for normalized current shown (dashed curves) vs normalized time for several values of $\alpha = (n-1)I_m^2 t_0^2 / (n^2 \mu c^2 r_0^2)$. (a) Triangular-like current waveform. (b) Parabolic-like current waveform.

FIG. 2. Normalized run-in times (solid curves) and analytic upperbound fit (dashed curves) versus parameter α for same current waveforms as in Figs. 1(a) and 1(b) (shown in insets).

of Eq. (7a), by the transformation $\omega \rightarrow i\omega$. Equation (7a) represents a system of n coupled, second-order, linear differential equations. The most general solution is

$$\delta r_{\alpha} = \sum_{l=0}^{n-1} \delta r_{\alpha,l} \quad (9)$$

$$\delta r_{\alpha,l} = \exp(p_l \omega t) [a_l \cos(2\pi l \alpha / n) + b_l \sin(2\pi l \alpha / n)]$$

in which a_l and b_l are constants determined by initial conditions. The eigenvalues p_l^2 are found by substituting Eq. (9) into Eq. (7a). The sign of p_l^2 determines the stability of the radial displacement mode l . If $p_l^2 > 0$, the mode is unstable with growth rate $|p_l| \omega$; if $p_l^2 < 0$, the mode is stable; if $p_l^2 = 0$, the mode is marginally stable against the displacement instability. Table II presents values of p_l^2 for all modes of arrays having six or less wires.

The $l = 0$ mode is unstable for all arrays. This mode is just the breathing mode, however, and does not affect the symmetry of the implosion. All other modes are either stable or marginally stable. The $l = 1$ mode is always marginally stable because it represents a translation of the entire array.

The sign of p_l^2 also determines the stability of the tangential displacement modes. If a radial mode is stable, the corresponding tangential mode is unstable with growth rate $|p_l| \omega$, and vice-versa. Thus, an array is stable against rotation ($l = 0$ mode), but its other tangential displacement modes are either unstable or marginally stable. Linear instability of tangential modes does not affect the run-in times of the wires, however, nor does it change the value of B_{θ} , which drives the implosion, because B_{θ} was shown in Eq. (2a) to be independent

TABLE I. Values of f_n .

n	f_n	n	f_n	n	f_n
2	-1/2	5	0	9	16/3
3	-2/3	6	5/6	12	77/6
4	-1/2	8	7/2	24	437/6

TABLE II. Radial eigenvalues p_l^2 for mode l of n wire arrays.

$l \backslash n$	0	1	2	3	4	5
2	1	0				
3	2	0	0			
4	3	0	-1	0		
5	4	0	0	0	0	
6	5	0	-3	-4	-3	0

of the angular positions of wires at equal distances from the center. The relative harmlessness of the tangential displacement instability and the stability against radial displacements explain why imploding wire arrays work so well.

The effect of an immobile center wire carrying current I_0 in the same direction as the other wires of the array is to tend to destabilize radial modes and to stabilize tangential modes. A center wire adds a term $2I_0/cr$ to the expression for B_{θ} in Eq. (1a). The effect of this term in the linearized Eqs. (7a) and (7b) is to replace f_n by $f_n - 2nI_0/I$. The effect on the eigenvalues is to replace p_l^2 by $p_l^2 + 2nI_0/I$. The destabilizing effect on radial displacements of an immobile center wire is easily understood. A wire displaced toward the center will be subject to a stronger attractive force from the center wire, increasing the inward displacement.

IV. KINK INSTABILITY

X-ray pinhole photographs show that much braiding, twisting, and turning of wire plasmas in imploded arrays can occur.² One possible explanation of this behavior is the growth of the kink instability during run-in of the wires. The kink instability causes growth of a distortion in the axis of a current-carrying wire or plasma column. The kink instability is caused by a greater magnetic pressure acting on the concave side of a distortion than on the convex. The instability is driven both by the force on a wire of its own current and by currents of other wires in the array. This section calculates the contribution to the growth rate of each of these forces.

The analysis is a first-order perturbation approach similar to that of the preceding section. Flexible wires with constant and uniform radius are assumed to move only under the influence of the magnetic fields generated by their own currents and by currents of other wires in the array. By radius of a wire, we mean the radius of the exploded wire plasma, which seems to remain fairly constant at about 1 mm during most of the implosion of the array.

This section evaluates the growth rates of the kink instability. The model breaks down in the short-wavelength limit, in which the radius of curvature of the kink is comparable to the wire radius, because of magnetohydrodynamic effects. Magnetohydrodynamic instabilities, such as the sausage instability, are not treated by this analytic model.

First, the growth rate of the kink instability of a single wire is calculated analytically. An arbitrary, small-amplitude distortion of a wire may be Fourier-analyzed,

and each Fourier component may be treated independently. Therefore, no generality is lost by defining the coordinates of the central axis of the wire as

$$\begin{aligned}x_1 &= \delta x \cos(k_x z_1), \quad \delta x = \delta x_0 \exp(\gamma_x t), \\y_1 &= \delta y \cos(k_y z_1 + \psi), \quad \delta y = \delta y_0 \exp(\gamma_y t), \\z_1 &,\end{aligned}\quad (10)$$

where $k_x, k_y, \gamma_x, \gamma_y, \delta x_0, \delta y_0, \psi$ are constants, and $|k_x \delta x| \ll 1$ and $|k_y \delta y| \ll 1$.

The coordinates of a point on the surface of the wire are

$$\begin{aligned}x &= \delta x \cos(k_x z) + a \cos \phi, \\y &= \delta y \cos(k_y z + \psi) + a \sin \phi, \\z &,\end{aligned}\quad (11)$$

where a is the wire radius and ϕ is an azimuthal, body-centered coordinate. The displacement vector from a point on the axis to a point on the surface is

$$\mathbf{r} = (x - x_1)\hat{x} + (y - y_1)\hat{y} + (z - z_1)\hat{z}.$$

To first order

$$\hat{e} = -k_x \delta x \sin(k_x z_1)\hat{x} - k_y \delta y \sin(k_y z_1 + \psi)\hat{y} + \hat{z} \quad (13)$$

is a unit vector along the wire axis. The geometry is illustrated in Fig. 3(a).

We assume that a current $(I/n)\hat{e}$ is carried on the outside of the wire. To first order the Biot and Savart law and Ampere's law give the magnetic field at a point on the surface of the wire. The field, averaged over ϕ , is

$$\langle B_x \rangle = (Ik_y^2 / nc) \delta y \cos(k_y z + \psi) K_0(k_y a), \quad (14a)$$

$$\langle B_y \rangle = (Ik_x^2 / nc) \delta x \cos(k_x z) K_0(k_x a), \quad (14b)$$

$$\langle B_z \rangle = 0. \quad (14c)$$

Then, the net force acting on the wire per unit mass is

$$\begin{aligned}\mathbf{F} &= I^2 / n^2 \mu c^2 \{ [k_x^2 K_0(k_x a) \delta x \cos(k_x z)] \hat{x} \\&+ [k_y^2 K_0(k_y a) \delta y \cos(k_y z + \psi)] \hat{y} \},\end{aligned}\quad (15)$$

where K_n is the n th order modified Bessel function. The x and y components decouple; that is, a perturbation in the x direction does not affect the growth of a perturbation in the y direction.

FIG. 3. Geometry of wire kink perturbation. (a) Single wire. (b) 2-wire array.

The growth rate is easily found from Eq. (15) to be

$$\gamma_j = (Ik_j / n \mu^{1/2} c) K_0^{1/2}(k_j a), \quad j = x, y. \quad (16)$$

The growth rate is plotted in Fig. 4, in which the fastest growing mode is seen to be $k \approx 2\pi/4a$. The growth rate in the long wavelength limit

$$\gamma \sim (Ik / n \mu^{1/2} c) \ln^{1/2}(2/ka), \quad ka \ll 1 \quad (17)$$

agrees with the magnetohydrodynamic analysis of Kruskal *et al.*³ Although it disagrees in the short wavelength limit because our analysis permits only axial redistribution of current in perturbed wires, neither analysis is reliable in this limit, because the distribution of magnetic field and current in exploded wire plasmas is more complicated than either model assumes. Short wavelengths require nonlinear analysis.

Next, we calculate the growth rate of the perturbation of a wire driven by its interaction with the magnetic field of another wire. The result will then be generalized to n wires in an array.

Let a wire carrying current I/n at

$$\begin{aligned}x_1 &= c_x \delta x \cos k_x z_1, \quad \delta x = \delta x_0 \exp \gamma_x t, \\y_1 &= c_y \delta y \cos(k_y z_1 + \psi), \quad \delta y = \delta y_0 \exp \gamma_y t, \\z_1 &,\end{aligned}\quad (18)$$

generate a magnetic field at a second wire, also carrying current I/n , located at

$$\begin{aligned}x &= d + \delta x \cos k_x z, \\y &= \delta y \cos(k_y z + \psi), \\z &,\end{aligned}\quad (19)$$

where $d, c_x, c_y, \delta x_0, \delta y_0, k_x, k_y$, and ψ are constants, and

$$\begin{aligned}|k_x c_x \delta x| &\ll 1, \quad |k_y c_y \delta y| \ll 1, \\|k_x \delta x| &\ll 1, \quad |k_y \delta y| \ll 1.\end{aligned}\quad (20)$$

The direction of the current in the first wire is

$$\begin{aligned}\hat{e} &= -c_x k_x \delta x \sin(k_x z_1) \hat{x} \\&+ c_y k_y \delta y \sin(k_y z_1 + \psi) \hat{y} + \hat{z}.\end{aligned}\quad (21)$$

The geometry of the two-wire array is shown in Fig. 3(b). From the Biot and Savart law, we find the mag-

FIG. 4. Dimensionless growth rate of kink instability for single wire of radius a carrying current I/n versus wire radius in units of inverse wavenumber of perturbation.

netic field of the first wire at the second wire

$$B_x = \frac{2Ik_y^2 \delta y \cos(k_y z + \psi)}{nc} \times \left(c_y K_0(k_y d) + \frac{c_y K_1(k_y d)}{k_y d} - \frac{1}{k_y^2 d^2} \right), \quad (22a)$$

$$B_y = \frac{2I}{ncd} - \frac{2Ik_x^2 \delta x \cos k_x z}{nc} \left(-\frac{c_x K_1(k_x d)}{k_x d} + \frac{1}{k_x^2 d^2} \right), \quad (22b)$$

$$B_z = \frac{Ik_y^2 c_y \delta y \sin(k_y z + \psi)}{nc} K_1(k_y d). \quad (22c)$$

The perturbed magnetic force per unit mass acting on the second wire is:

$$F_x = \frac{2I^2 k_x^2}{n^2 \mu c^2} \delta x \cos k_x z \left(-\frac{c_x K_1(k_x d)}{k_x d} + \frac{1}{k_x^2 d^2} \right), \quad (23a)$$

$$F_y = \frac{2I^2 k_y^2}{n^2 \mu c^2} \delta y \cos(k_y z + \psi) \times \left(c_y K_0(k_y d) + \frac{c_y K_1(k_y d)}{k_y d} - \frac{1}{k_y^2 d^2} \right) \quad (23b)$$

$$F_z = \frac{-2I^2 c_y k_x}{n^2 \mu c^2 d} \delta x \sin k_x z. \quad (23c)$$

Neglecting the distortion of the sinusoidal waves by F_z , which produces higher-order effects in the x and y direction, we find the growth rate of its perturbations to be

$$\gamma_x = \left[\frac{2I^2 k_x^2}{n^2 \mu c^2} \left(-\frac{c_x K_1(k_x d)}{k_x d} + \frac{1}{k_x^2 d^2} \right) \right]^{1/2}, \quad (24a)$$

$$\gamma_y = \left[\frac{2I^2 k_y^2}{n^2 \mu c^2} \left(c_y K_0(k_y d) + \frac{c_y K_1(k_y d)}{k_y d} - \frac{1}{k_y^2 d^2} \right) \right]^{1/2}. \quad (24b)$$

The results of Eqs. (16) and (24) may be combined and generalized to radial and tangential displacements of the α th wire of an n -wire array as

$$\left(\frac{d^2}{dt^2} + \omega^2 [f_n - (kr)^2 K_0(ka)] \right) \delta r_\alpha = \omega^2 \sum_{\beta=1, \beta \neq \alpha}^n \frac{\delta r_\beta}{1 - \cos(\theta_\beta - \theta_\alpha)} [d_{\alpha\beta} K_1(d_{\alpha\beta}) + (kr)^2 \sin^2(\theta_\beta - \theta_\alpha) K_0(d_{\alpha\beta})], \quad (25a)$$

$$\left(\frac{d^2}{dt^2} - \omega^2 [f_n + (kr)^2 K_0(ka)] \right) \delta \theta_\alpha = -\omega^2 \sum_{\beta=1, \beta \neq \alpha}^n \frac{\delta \theta_\beta}{1 - \cos(\theta_\beta - \theta_\alpha)} \{ d_{\alpha\beta} K_1(d_{\alpha\beta}) + (kr)^2 [1 - \cos(\theta_\beta - \theta_\alpha)]^2 K_0(d_{\alpha\beta}) \}, \quad (25b)$$

in which ω and f_n have been defined in Eq. (7) and

$$d_{\alpha\beta} \equiv kr \{ 2[1 - \cos(\theta_\beta - \theta_\alpha)] \}^{1/2}.$$

Equations (25) become identical to Eqs. (7) in the long wavelength limit, $k \rightarrow 0$. In this limit, therefore, the kink instability is equivalent to the displacement instability.

The most general solution of Eq. (25a) has the same form as Eq. (9). The radial eigenvalues p_l^2 for two-wire arrays and six-wire arrays are listed in Table III.

The tangential eigenvalues may be found in a similar manner from Eq. (25b). In the limit $k \rightarrow 0$, radial and tangential eigenvalues for modes of the kink instability reduce to those of the displacement instability.

The radial and tangential growth rates of the $l = 0$ and $l = 3$ kink modes of six-wire arrays are plotted in Fig. 5 for the case $ka = 1$. The $l = 0$ radial mode and $l = 3$ tangential mode are the most unstable. In general, for arrays of any number of wires, the symmetric ($l = 0$) radial mode and antisymmetric ($l = n/2$) tangential mode are most unstable because in these modes the perturbations of each wire reinforce those of their neighboring wires. The effect of the self-induced kinking of each wire is to increase γ^2 for all values of kr and for all modes by a constant amount equal to $(\omega kr)^2 K_0(ka)$. At large array radii ($kr \gg 1$), self-induced kinking is the dominant cause of the instability. At small array radii ($kr \lesssim 1$), the effects of the other wires in the array dominate.

Because the kink growth rates depend upon array radius, the amplitude of a perturbation,

TABLE III. Radial eigenvalues p_l^2 for mode l .

(a) Two-wire array	
l	p_l^2
0	$(kr)^2 K_0(ka) - \frac{1}{2} [1 + (2kr) K_1(2kr)]$
1	$(kr)^2 K_0(ka) + \frac{1}{2} [1 - (2kr) K_1(2kr)]$
(b) Six-wire array	
l	p_l^2
0	$(kr)^2 K_0(ka) - \frac{5}{6} + 4(kr) K_1(kr) + \frac{4}{3} (3^{1/2} kr) K_1(3^{1/2} kr) + \frac{1}{2} (2kr) K_1(2kr) + (kr)^2 [3K_0(kr) + K_0(3^{1/2} kr)]$
1	$(kr)^2 K_0(ka) - \frac{5}{6} + 2(kr) K_1(kr) - \frac{2}{3} (3^{1/2} kr) K_1(3^{1/2} kr) - \frac{1}{2} (2kr) K_1(2kr) + \frac{1}{2} (kr)^2 [3K_0(kr) - K_0(3^{1/2} kr)]$
2	$(kr)^2 K_0(ka) - \frac{5}{6} - 2(kr) K_1(kr) - \frac{2}{3} (3^{1/2} kr) K_1(3^{1/2} kr) + \frac{1}{2} (2kr) K_1(2kr) - \frac{1}{2} (kr)^2 [3K_0(kr) + K_0(3^{1/2} kr)]$
3	$(kr)^2 K_0(ka) - \frac{5}{6} - 4(kr) K_1(kr) + \frac{4}{3} (3^{1/2} kr) K_1(3^{1/2} kr) - \frac{1}{2} (2kr) K_1(2kr) - (kr)^2 [3K_0(kr) - K_0(3^{1/2} kr)]$
4	$(kr)^2 K_0(ka) - \frac{5}{6} - 2(kr) K_1(kr) - \frac{2}{3} (3^{1/2} kr) K_1(3^{1/2} kr) + \frac{1}{2} (2kr) K_1(2kr) - \frac{1}{2} (kr)^2 [3K_0(kr) + K_0(3^{1/2} kr)]$
5	$(kr)^2 K_0(ka) - \frac{5}{6} + 2(kr) K_1(kr) - \frac{2}{3} (3^{1/2} kr) K_1(3^{1/2} kr) - \frac{1}{2} (2kr) K_1(2kr) + \frac{1}{2} (kr)^2 [3K_0(kr) - K_0(3^{1/2} kr)]$

FIG. 5. Dimensionless growth rate squared versus array radius normalized to inverse wavenumber for symmetric ($l=0$) and antisymmetric ($l=3$) radial and tangential kink modes of array of six wires having radii $a=1/k$. Dashed line is growth rate squared of single wire. Positive γ^2 represents unstable growth; negative γ^2 represents stable oscillations.

$$A(t) \equiv \exp\left(\int_0^t \gamma(r) dt\right)$$

depends upon the implosion history of the array. At large array radii, the growth rate of any mode is $\gamma \approx (Ik/n\mu^{1/2}c)K_0^{1/2}(ka)$. At small array radii, the growth rate of the most unstable mode, the $l=0$ radial mode, is $\gamma \approx (n-1)^{1/2}l/(n\mu^{1/2}cr)$. Most of the growth of the kink instability occurs at small array radii, even though the wires spend the least time there, because the growth rate varies inversely as radius at small array radii.

A typical amplitude of the symmetric radial kink mode for six-wire arrays is plotted as a function of time in Fig. 6 for the same current waveform as in Fig. 1(b). Because the amplitude diverges at $r=0$, an arbitrary cutoff radius of $0.01 r_0$ was chosen. Several such cal-

FIG. 6. Normalized amplitude of $l=0$ radial kink instability versus normalized time for a six-wire array of initial radius $r_0=5/k$, wire radius $a=1/k$, and $\alpha=10$. Current waveform and array radius versus time are same as in Fig. 1(b).

culations were performed for the symmetric radial kink mode using the same current waveform shown in Figs. 6 and 1(b). The final amplitudes are plotted in Fig. 7. The final amplitudes are very sensitive to the radius at which the wires collide. Nevertheless, Fig. 7 clearly shows that the final amplitude of a kink perturbation depends strongly on the initial array radius for $kr_0 \gg 1$, but in the experimental range of interest for large machines depends only weakly on the dimensionless parameter α , which was defined in Eq. (4).

Typically, the kink instability of a wire in an array is driven mainly by its own field during the early part of an implosion, as seen in Fig. 5. The fastest growing wavelength of the self-induced kink is about twice the wire plasma diameter, as shown in the preceding section. This wavelength is likely to predominate by the time the wires approach closely enough to interact. Then, the initial radius of the array should not be greater than several plasma wire radii if the instability is not to reach the nonlinear growth stage.

For example, suppose each of the wire plasmas of a six-wire array has a radius of 1 mm. Then, a kink wavelength of about 4 mm ($k \approx 16 \text{ cm}^{-1}$) is likely to establish itself on the wire during the early part of the implosion. If the initial amplitude of the perturbation is 0.1 mm, then the amplitude will grow as large as the wavelength and become nonlinear if $kr_0 \gtrsim 5$ or $r_0 \gtrsim 3 \text{ mm}$.

The effect of a straight center wire on the kink instability in an array is the same as its effect on the displacement instability: it tends to destabilize radial modes and to stabilize tangential modes. If a straight center wire carries current I_0 in the same direction as the other wires in the array, then Eqs. (24) and (25) show that the effect on the growth rates is to replace f_n by $f_n - 2nI_0/I$, just as for the displacement instability.

FIG. 7. Normalized amplitude of $l=0$ radial kink mode of six-wire array versus α for several values of initial array radius.

V. CONCLUSIONS

This paper presented analytic expressions for the growth rates of wire displacements and deformations in imploding cylindrical wire arrays. It also reviewed the dynamics of wires in imploding arrays. Analytic upper and lower bounds on the run-in time of wire arrays were found for any arbitrary current waveform. The upper bound is a fairly good approximation to the actual run-in time. Run-in times are not sensitive to the shape of the current waveform for implosions having the same value of $\alpha = (n-1)I_m^2 t_0^2 / (n^2 \mu c^2 r_0^2)$.

Arrays of six or more wires were found to be reasonably stable against asymmetric radial displacements of wires, but unstable to asymmetric tangential displacements. All arrays are stable against rotation, however. Also, all arrays are marginally stable against uniform translation of the entire array unless there is a center wire. The effect of a center wire is to destabilize radial displacement modes and stabilize tangential displacement modes. This result concerning center wires applies to the kink instability as well. The growth rate of the displacement instability for any mode of an array of any number of wires may easily be calculated from Eqs. (7). The results are summarized in Table II for arrays of up to six wires.

Growth rates were also calculated analytically for the kink instability of wires in imploding cylindrical arrays. The displacement instability was shown to be equivalent to the kink instability in the limit $k \rightarrow 0$, where k is the wavenumber of the kink perturbation on the wires. First, the growth rate of the kink instability of a single wire was calculated and found to peak at a wavelength about equal to two wire diameters. Then, the kink instability of one wire driven by the magnetic field of a neighboring wire was examined, and the results generalized to arrays of any number of wires. Growth rates of any modes of arrays of any number of wires may easily be calculated from Eqs. (25). The results for two-wire and six-wire arrays are summarized in Table III.

When the radius of the array is much greater than $1/k$, the kink growth rate, $\gamma \approx (Ik/n\mu^{1/2}c)K_0^{1/2}(ka)$, is independent of the other wires in the array, and is driven in each wire only by the field of the wire. On the other hand, if $r \ll 1/k$, then the effect of the field of a wire on its kink instability is negligible compared with the much greater effect of the other wires.

Once the wires approach closely enough to interact, the symmetric ($l=0$) radial mode and antisymmetric ($l=n/2$) tangential mode have the largest growth rates and will probably establish themselves on the wires. The wavelength of these modes will probably be the wavelength established earlier in the implosion as the fastest growing self-induced kink wave, namely, about twice the wire diameter.

The amplitude of the fastest growing kink perturbation, the $l=0$ radial mode, was calculated as a function of time for a particular current waveform in a six-wire array. The results show that just as the wires dwell at large radii for a long time before running in suddenly, the amplitude of the kink instability remains low for a long time before growing explosively near the collision time. Although the final number of growth times depends strongly on exactly when the wires collide, several general conclusions may be drawn. The final number of growth times is: (1) larger for smaller wire radii; (2) larger for larger initial array radius; and (3) fairly insensitive to the dimensionless parameter α , maximum current, and current pulse width in the range $10 \lesssim \alpha \lesssim 1000$ of experimental interest for large machines like BLACKJACK 5. A general rule seems to be that the kink perturbation will grow nonlinearly if the initial array radius is more than a few wire plasma radii.

ACKNOWLEDGMENTS

The authors thank W. Clark, J. Pearlman, J. Katzenstein, T. Peratt, and A. Wilson for stimulating discussion and reports of experimental results. This work was supported by the Defense Nuclear Agency.

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