

UNIVERSITY OF CALIFORNIA,
IRVINE

Sums and products of Cantor sets and
separable two dimensional quasicrystal
models

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Yuki Takahashi

Dissertation Committee:
Professor Anton Gorodetski, Chair
Professor Svetlana Jitomirskaya
Professor Abel Klein

2017

Table of Contents

Acknowledgements	iv
Curriculum Vitae	v
Abstract of the Dissertation	xi
Introduction	1
1 Products of two Cantor sets	3
1.1 Introduction	3
1.1.1 Sums and products of Cantor sets	3
1.1.2 Main results	5
1.1.3 Structure of this paper	8
1.2 Preliminaries	8
1.3 Estimates of thickness	11
1.4 Proof of the main results	18
1.5 Other cases	28
1.6 Connection with questions on intersections of two Cantor sets	30
1.7 Open problems	31
2 Quantum and spectral properties of the Labyrinth model	32
2.1 Introduction	32
2.1.1 Quasicrystal and the Labyrinth model.	32
2.1.2 Structure of the paper	36
2.2 Preliminaries	36
2.2.1 Linearly recurrent sequences	36
2.2.2 Metallic mean sequence	37
2.2.3 Necessary results	39
2.2.4 The off-diagonal model	40
2.3 The Labyrinth Model	47
2.3.1 The Labyrinth model	47
2.3.2 The spectrum of the Labyrinth model	48
2.3.3 Density of states measure of the Labyrinth model	50

3	Mixed spectral regimes for square Fibonacci Hamiltonian	53
3.1	Introduction	53
3.1.1	Models and main results	54
3.2	Proof of Theorems 3.1.1, 3.1.2, and 3.1.3	58
3.2.1	Background	59
3.2.2	Proof of Theorem 3.1.1	63
3.2.3	Proof of Theorem 3.1.2	69
3.2.4	Proof of Theorem 3.1.3	77
	BIBLIOGRAPHY	82

Acknowledgements

I would like to sincerely thank my supervisor, Professor Anton Gorodetski, for his guidance and support throughout this study. I would also like to thank Professor Svetlana Jitomirskaya and Professor Abel Klein for serving as members of my advancement and thesis committee. Their comments and questions are beneficial in the completion of my work. I also want to thank Willam Yessen for allowing me to include Figure 3.1 and Figure 3.4.

This research was partially supported by the NSF DMS-1301515. I would like to thank the National Science Foundation for the support.

Curriculum Vitae

Yuki Takahashi

Basic Information

citizenship: Japan

address: 440V Rowland Hall, Irvine, CA, 92697

email: takahasy@uci.edu

webpage: <http://www.math.uci.edu/~takahasy>

Education

- Doctor of Philosophy in Mathematics, University of California, Irvine, May 2017
- Master of Science in Mathematics, the University of Tokyo, March 2012
- Bachelor of Science in Mathematics, the University of Tokyo, March 2009

Research Interests

Dynamical Systems, Spectral Theory, Mathematical Physics, Fractal Geometry

Honors and Awards

- Dissertation Fellowship (UC Irvine, department fellowship, Spring 2017)
- Department Fellowship (UC Irvine, department fellowship, Winter 2017)
- Connelly Award (UC Irvine, department award, Spring 2014)
- Euler Outstanding Promise as a Graduate Student Award (UC Irvine, department award, Spring 2013)

Publications and Preprints

1. Y. Takahashi, *Sums of two homogeneous Cantor sets*, in preparation
2. Y. Takahashi, *Quantum and spectral properties of the Labyrinth model*, *J. Math. Phys.* **57** (2016)
3. Y. Takahashi, *Products of two Cantor sets*, to appear in *Nonlinearity*
4. J. Fillman, Y. Takahashi, W. Yessen, *Mixed spectrum regimes of the square tridiagonal Fibonacci Hamiltonian*, to appear in *J. Fractal Geom.*
5. M. Kanki, Y. Takahashi, T. Tokihiro, *Graphs emerging from the solutions to the periodic discrete Toda equation over finite fields*, to appear in *Nonlinear Theory and Its Applications*

Talks and Conference Participation

Conference Talks

- *On a class of Cantor sets that appear in dynamics but are not dynamically dyfined* (Workshop on Dynamical Systems and Related Topics, University of Maryland, March 2017)
- *Products of two Cantor sets and application to the Labyrinth model* (poster session, Midwest Dynamical Systems Seminar, Indiana University Purdue University Indianapolis, November 2016)
- *Sums and products of Cantor sets and two-dimensional quasicrystal models* (Semi-annual Workshop in Dynamical Systems and Related Topics, Pennsylvania State University, October 2016)

- *Products of two Cantor sets and application to the Labyrinth model* (poster session, Doctoral School, Applications of Ergodic Theory in Number Theory, CIRM, Marseille Luminy, France, October 2016)
- *Products of two Cantor sets and application to the Labyrinth model* (poster session, Frontiers in Mathematical Physics, CRM, Montreal, Canada, August 2016)
- *Calculation of the spectrum for the action of the Lamplighter group* (learning talk, Connections Between Complex Dynamics, Statistical Physics, and Limiting Spectra of Self-similar Group Actions, Indiana University Purdue University Indianapolis, August 2016)
- *Products of two Cantor sets and application to the Labyrinth model* (poster session, School on Algebraic, Geometric and Probabilistic Aspect of Dynamical Systems and Control Theory, ICTP, Trieste, Italy, July 2016)
- *Stable intersections of homogeneous Cantor sets* (Summer School on Fractal Geometry & Complex Dimensions, California Polytechnic State University, San Luis Obispo, June 2016)
- *Products of two Cantor sets and application to the Labyrinth model* (Great Lakes Mathematical Physics Meeting, Michigan State University, June 2016)
- *The Labyrinth model and products of two Cantor sets* (Between Dynamics and Spectral Theory, Stony Brook University, June 2016)
- *Products of two Cantor sets and application to the Labyrinth model* (poster session, Fractal Geometry, Hyperbolic Dynamics and Thermodynamical Formalism Semester Workshop, Brown University, March 2016)
- *Products of two Cantor sets and application to the Labyrinth model* (poster session, The 2015 Midwest Dynamics Meeting, The Ohio State University, October 2015)
- *Labyrinth model and products of two Cantor sets* (AMS Sectional Meeting, Special Session on Spectral Theory of Ergodic Schrödinger Operators and Related Models, California State University, Fullerton, October 2015)
- *Labyrinth model and products of two Cantor sets* (Spectral Properties of Quasicrystals via Analysis, Dynamics, and Geometric Measure Theory, Oaxaca, Mexico, September 2015)
- *Products of two Cantor sets and application to the Labyrinth model* (Rocky Mountain Dynamical Systems Conference, Brigham Young University, June 2015)

- *Products of Cantor sets and Spectral Properties of the Labyrinth model* (AMS Western Spring Sectional Meeting, University of New Mexico, Special Session on Hyperbolic Dynamics, Dynamically Defined Fractals, and Applications II, April 2014)

Seminar Talks

- *On the dimension of Furstenberg measure for $SL_2(\mathbb{R})$ random matrix products I, II* (UC Irvine, Ergodic Schrödinger Operator seminar, May 2017)
- *On self-similar sets with overlaps and inverse theorems for entropy I, II, III* (UC Irvine, Dynamical System seminar, March - April 2017)
- *Furstenberg correspondence principle I, II* (UC Irvine, Ergodic Schrödinger Operator seminar, March 2017)
- *Recent developments on projections of self-similar fractals I, II* (UC Irvine, Dynamical System seminar, January - March 2017)
- *Proof of the Furstenberg's theorem I, II* (UC Irvine, Ergodic Schrödinger Operator seminar, September - October)
- *Sums of two homogeneous Cantor sets I, II* (UC Irvine, Dynamical System seminar, April 2016)
- *Repetitive Delone sets and quasicrystals I, II* (UC Irvine, Ergodic Schrödinger Operator seminar, March - April 2016)
- *An Ergodic Theorem for Delone Dynamical Systems and Existence of the Integrated Density of States* (UC Irvine, Ergodic Schrödinger Operator seminar, March 2016)
- *Stable Intersections of Regular Cantor sets with Large Hausdorff Dimensions IV, V, VI, VII, VIII, IX, X* (UC Irvine, Dynamical System seminar, January - March 2016)
- *Stable Intersections of Regular Cantor sets with Large Hausdorff Dimensions I, II, III* (UC Irvine, Dynamical System seminar, November 2015)
- *Opening Gaps in the Spectrum of Strictly Ergodic Schrödinger Operators I, II* (UC Irvine, Ergodic Schrödinger Operator seminar, October 2015)
- *Products of Two Cantor sets and Application to the Labyrinth model* (UC Irvine, Future Faculty Program, November 2014)
- *Products of Two Cantor sets I, II, III* (UC Irvine, Dynamical System seminar, November 2014)

- *Equilibrium Measures and Capacities in Spectral Theory I, II, III* (UC Irvine, Ergodic Schrödinger Operator seminar, November 2014)
- *Products of Cantor sets and Spectral Properties of the Labyrinth model* (UC Irvine, Advancement to Candidacy exam, May 2014)
- *The Spectrum of Quasi-Periodic Schrödinger Operator I, II, III* (UC Irvine, Ergodic Schrödinger Operator seminar, February 2014)

Colloquium Talks

- *Spectra of self-similar groups* (UC Irvine, Math Grad Students Colloquium, October 2016)
- *Sums of two Cantor sets and Palis conjecture* (UC Irvine, Math Grad Students Colloquium, May 2016)
- *Products of two Cantor sets and application to the Labyrinth model* (UC Irvine, AGS Colloquium, April 2016)
- *The Mathematical Connection of Juggling* (UC Irvine, sponsored by UCI Illuminations and Juggle Buddies, November 2015)
- *Products of two Cantor sets* (UC Irvine, Grad Slum competition, semi-final, April 2015)
- *Quasicrystals, Labyrinth model, and products of two Cantor sets* (UC Irvine, Math Grad Students Colloquium, April 2015)
- *Products of two Cantor sets and Application to the Labyrinth model* (UC Irvine, AGS Symposium, March 2015)

Undergraduate Math Club Talks

- *Products of two Cantor sets and Application to the Labyrinth model* (UC Irvine, January 2016)
- *Products of Cantor sets and Application to the Labyrinth model* (UC Irvine, April 2014)

Teaching Experience

Instructor

- Calculus, Math 2A (UC Irvine, Fall 2014)
- Calculus, Math 2B (UC Irvine, Summer 2014)

Teaching Assistant

- Introduction to Linear Algebra, 3A (UC Irvine, Summer 2016)
- Introduction to Graduate Analysis, 205C (UC Irvine, Spring 2016)
- Introduction to Graduate Analysis, 205B (UC Irvine, Winter 2016)
- Introduction to Graduate Analysis, 205A (UC Irvine, Fall 2015)
- Calculus, 2B (UC Irvine, Summer 2015)
- Introduction to Graduate Analysis, accelerated version of 205ABC for incoming Ph.D. students (UC Irvine, Summer 2015)
- Elementary Analysis, 140B (UC Irvine, Spring 2015)
- Introduction to Topology, 141 (UC Irvine, Spring 2015)
- Dynamical Systems, 117 (UC Irvine, Winter 2015)
- Probability and Stochastic Process, 130A (UC Irvine, Winter 2015)
- Calculus, Math 2B (UC Irvine, Summer 2014)
- Introduction to Graduate Analysis, 205C (UC Irvine, Spring 2014)
- Vector Calculus, Math 2E (UC Irvine, Winter 2014)
- Elementary Analysis, 140A (UC Irvine, Fall 2013)
- Vector Calculus, Math 2E (UC Irvine, Summer 2013)
- Multivariable Calculus, Math 2D (UC Irvine, Summer 2013)
- Calculus, Math 2B (UC Irvine, Summer 2013)
- Calculus, Math 2B (UC Irvine, Spring 2013)
- Calculus, Math 2B (UC Irvine, Winter 2013)

Grader

- Algebra, 230A graduate course (UC Irvine, Fall 2017)
- Introduction to Abstract Mathematics, Math13 (UC Irvine, Fall 2012)

Abstract of the Dissertation

Sums and products of Cantor sets and separable two dimensional quasicrystal models

By

Yuki Takahashi

Doctor of Philosophy in Mathematics
University of California, Irvine, 2017
Professor Anton Gorodetski, Chair

The spectra of tridiagonal square Fibonacci Hamiltonians, which are two-dimensional quasicrystal models, are given by sums of two Cantor sets, and the spectrum of the Labyrinth model, which is another two-dimensional quasicrystal model, is given by products of two Cantor sets. We consider spectral properties of those models and also obtain the optimal estimates in terms of thickness that guarantee that products of two Cantor sets is an interval.

Introduction

Arithmetic sums of two Cantor sets appear naturally in dynamical systems, number theory, and also spectral theory. Indeed, the spectrum of the tridiagonal square Fibonacci Hamiltonians, which is a separable two-dimensional quasicrystal model, is given by sums of two Cantor sets. On the other hand, the spectrum of the Labyrinth model, which is another separable two-dimensional quasicrystal model, is given by products of two Cantor sets.

For the first theme, we consider products of two Cantor sets and give the optimal estimates in terms of thickness that guarantee that products of two Cantor sets is an interval. We also discuss the connection with the question on the structure of intersections of two Cantor sets which was considered by many authors previously. This work is in [69].

For the second part, we consider the Labyrinth model, and show that the spectrum is an interval for small values of the coupling constant. We also consider the density of states measure of the Labyrinth model, and show that it is absolutely continuous with respect to Lebesgue measure for almost all values of coupling constants in the small coupling regime. This work is in [70].

For the third part, we consider the square tridiagonal Fibonacci Hamiltonian, and

prove existence of an open set of parameters which yield mixed interval-Cantor spectra (i.e. spectra containing an interval as well as a Cantor set), as well as mixed density of states measure (i.e. one whose absolutely continuous and singular continuous components are both nonzero). Using the methods developed in this paper, we also show existence of parameter regimes for the square continuum Fibonacci Schrödinger operator yielding mixed interval-Cantor spectra. These examples provide the first instance of an interesting phenomenon that has not hitherto been observed in aperiodic models. Moreover, while we focus only on the Fibonacci model, our techniques are equally applicable to models based on any two-letter primitive invertible substitution. This is a joint work with J. Fillman and W. Yessen [20].

Chapter 1

Products of two Cantor sets

1.1 Introduction

1.1.1 Sums and products of Cantor sets

Sums of Cantor sets have been considered in many papers and in many different settings (e.g., [1], [7], [11], [13], [18], [22], [25], [26], [27], [45], [46], [47], [50], [52], [61]). It arises naturally in dynamical systems in the study of homoclinic bifurcations [50]. It also arises in number theory in connection with continued fractions as initiated by Hall [22]. In [22], the author proved that any real number can be written as a sum of two real numbers whose continued fractional coefficients are at most 4. It is also connected to spectral theory (e.g., [7], [11], [13], [49]). The spectra of certain types of two dimensional quasicrystal models can be written as sums of two dynamically defined Cantor sets [11]. The study of sums of Cantor sets also has natural connection to the study of intersections of Cantor sets (e.g., [27], [28], [32], [33], [44], [48], [71]).

In [48], Newhouse proved the following so-called Gap Lemma (for the definition of thickness $\tau(\cdot)$, see section 1.2):

Lemma 1.1.1 (Gap Lemma) *Let K, L be Cantor sets with $\tau(K) \cdot \tau(L) > 1$. Then, if neither K nor L lies in a complementary domain of the other, $K \cap L$ contains at least one element.*

In fact, if Newhouse’s proof is slightly altered the condition $\tau(K) \cdot \tau(L) > 1$ may be replaced with $\tau(K) \cdot \tau(L) \geq 1$. The following is a direct consequence of Gap Lemma:

Theorem 1.1.1 *Suppose K and L are Cantor sets with $\tau(K) \cdot \tau(L) \geq 1$. Assume also that the size of the largest gap of K is not greater than the diameter of L , and the size of the largest gap of L is not greater than the diameter of K . Then $K + L$ is a closed interval.*

Using Theorem 1.1.1 as a tool, we consider products of two Cantor sets. This problem arises naturally in the study of the spectrum of the Labyrinth model [70]. For any two Cantor sets $K, L > 0$, we have

$$K \cdot L = \exp(\log K + \log L).$$

Using this equality, if K and L do not contain 0, some results of products of Cantor sets can be immediately obtained by that of sums of Cantor sets. For example, in [2], the authors obtained an estimate for products of two or more Cantor sets to be an interval.

The main difficulty arises when K or L contain 0. To the best of our knowledge, this case has never been discussed before. If K contains 0, then $\log K$ is “stretched to negative infinity”, making products of Cantor sets different from sums of Cantor sets. Indeed, for example, in section 1.2 we will show that under the condition of $\tau(K) \cdot \tau(L) > 1$, products of two Cantor sets $K \cdot L$ may contain countably many disjoint closed intervals. This is a phenomena which never appears in sums of two Cantor sets under the condition of $\tau(K) \cdot \tau(L) \geq 1$.

Initially this work was motivated by the question on spectral properties of the Labyrinth model [70]. As mentioned above, the spectrum of the Labyrinth model is a product of two Cantor sets, and in fact, these two Cantor sets both contain the origin. Using our results, we can show that the spectrum of the Labyrinth model is an interval for the small coupling constant regime. See [70].

1.1.2 Main results

For any Cantor set K , we denote $K \cap (0, \infty)$ by K_+ , and $-(K \cap (-\infty, 0))$ by K_- .

Let us give the following definition:

Definition 1.1.1 *Let K be a Cantor set. We call K a*

- (1) 0-Cantor set if $K_+, K_- \neq \phi$, $\inf K_+ = 0$, and $\inf K_- = 0$;
- (2) 0^+ -Cantor set if $\min K = 0$;
- (3) 0^\times -Cantor set if $K_+, K_- \neq \phi$, and $0 \notin K$;
- (4) 0_-^+ -Cantor set if $K_+, K_- \neq \phi$, $\inf K_+ = 0$, and $\inf K_- > 0$.

Our main results are the following:

Theorem 1.1.2 *Let K, L be 0^+ -Cantor sets. Then, $K \cdot L$ is an interval if*

$$(1.1.1) \quad \tau(L) \geq \frac{2\tau(K) + 1}{\tau(K)^2}, \text{ or } \tau(K) \geq \frac{2\tau(L) + 1}{\tau(L)^2}.$$

In particular, if

$$\tau(K) = \tau(L) \geq \frac{1 + \sqrt{5}}{2},$$

then $K \cdot L$ is an interval. Furthermore, let $M, N > 0$ be real numbers with

$$(1.1.2) \quad N < \frac{2M + 1}{M^2}, \text{ and } M < \frac{2N + 1}{N^2}.$$

Then

- (1) *there exist 0^+ -Cantor sets K, L , such that $\tau(K) = M$, $\tau(L) = N$, and $K \cdot L$ is a disjoint union of $\{0\}$ and countably many closed intervals;*
- (2) *for any $k \geq 2$, there exist 0^+ -Cantor sets K, L , such that $\tau(K) = M$, $\tau(L) = N$, and $K \cdot L$ is a disjoint union of k closed intervals.*

Similarly, we have

Theorem 1.1.3 *Let K be a 0^+ -Cantor set and let L be a 0-Cantor set. Then, $K \cdot L$ is an interval if*

$$(1.1.3) \quad \tau(L) \geq \frac{2\tau(K) + 1}{\tau(K)^2}.$$

Furthermore, let $M, N > 0$ be real numbers with

$$(1.1.4) \quad N < \frac{2M + 1}{M^2}.$$

Then

- (1) *there exists a 0^+ -Cantor set K and a 0-Cantor set L such that $\tau(K) = M$, $\tau(L) = N$, and $K \cdot L$ is a disjoint union of countably many closed intervals;*
- (2) *for any $k \geq 2$, there exists a 0^+ -Cantor set K and a 0-Cantor set L such that $\tau(K) = M$, $\tau(L) = N$, and $K \cdot L$ is a disjoint union of k closed intervals.*

We also have the following:

Theorem 1.1.4 *Let K, L be 0-Cantor sets. Then, if*

$$(1.1.5) \quad 2(\tau(K) + 1)(\tau(L) + 1) \leq (\tau(K)\tau(L) - 1)^2,$$

$K \cdot L$ is an interval. In particular, if

$$\tau(K) = \tau(L) \geq 1 + \sqrt{2},$$

then $K \cdot L$ is an interval. Furthermore, let $M, N > 0$ be real numbers with

$$(1.1.6) \quad 2(M + 1)(N + 1) > (MN - 1)^2.$$

Then, there exist 0-Cantor sets K, L , such that $\tau(K) = M$, $\tau(L) = N$, and $K \cdot L$ is a disjoint union of two intervals.

We believe that Theorem 1.1.4 does not hold if we replace “disjoint union of two intervals” with “disjoint union of countably many closed intervals”, or “disjoint union of k (≥ 3) closed intervals”. It would be interesting to prove that this is indeed true.

To ensure that the product may contain countably many disjoint closed intervals, we have the following estimate:

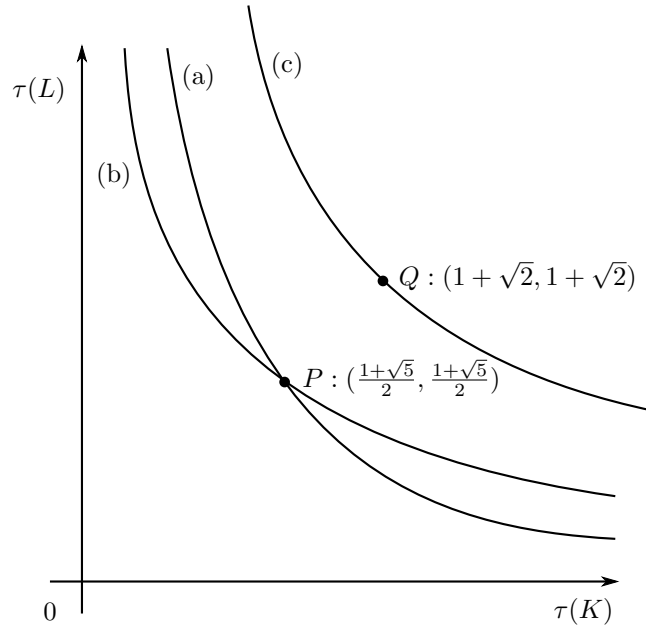


Figure 1.1: (a), (b) are the graphs of (1.1.1), and (c) is the graph of (1.1.5).

Theorem 1.1.5 *Let $M, N > 0$ be real numbers with $M \geq N$. Then, if*

$$(1.1.7) \quad M < \frac{N^2 + 3N + 1}{N^2}, \text{ or } N < \frac{(2M + 1)^2}{M^3},$$

there exist 0-Cantor sets K, L , such that $\tau(K) = M$, $\tau(L) = N$, and $K \cdot L$ is a disjoint union of $\{0\}$ and countably many closed intervals.

We are not sure whether the estimate in Theorem 1.1.5 is optimal. We leave this as an open problem.

As stated above, the study of intersections of Cantor sets is naturally connected to the study of sums of Cantor sets. In fact, it is also connected to the study of products of Cantor sets. Indeed, the estimate in Theorem 1.1.5 is exactly the same as the estimate that appears in [28] and [32], which is the optimal estimate that two interleaved Cantor sets may have a one point intersection ([28] and [32] obtained the same results independently). Our method provides different proofs to some of the results presented there. On the other hand, we do not believe that our results follow from the techniques in [28], [32]. We discuss this connection in more detail in section 1.6.

1.1.3 Structure of this paper

In section 1.2, we give necessary definitions and prove that the product of two Cantor sets $K \cdot L$ may contain countably many disjoint closed intervals in the case of $\tau(K) \cdot \tau(L) > 1$. In section 1.3 we prove the key lemma, which is the optimal estimate of the thickness of $\log K$ for certain type of Cantor set K . Using the results established in section 1.3, we prove Theorem 1.1.2, 1.1.3, and 1.1.4 in section 1.4. In section 1.5, we present some analogous results which are not stated in section 1.4. In section 1.6, we discuss the connection between the question on products of two Cantor sets and the question on intersections of two Cantor sets. In section 1.7, we state some open problems.

1.2 Preliminaries

Definition 1.2.1 *For any Cantor set $K \subset \mathbb{R}$, we denote the right and left endpoint of K by K^R and K^L , respectively. We denote $K^R - K^L$ by $|K|$. If two Cantor sets K_1, K_2 satisfy $K_1^R < K_2^L$, we write $K_1 < K_2$. For any gaps U, U_1 and U_2 , we define $U^R, U^L, |U|$ and $U_1 < U_2$ analogously. For any set $A \subset (0, \infty)$, we denote $\log A$ by \tilde{A} .*

Definition 1.2.2 *We call K an extended Cantor set if K is a closed, perfect, and nowhere dense set which is bounded from above and unbounded from below.*

The following is immediate:

Lemma 1.2.1 *If a Cantor set K satisfies $K > 0$, \tilde{K} is again a Cantor set. If K is a 0^+ -Cantor set, then \tilde{K}_+ is an extended Cantor set.*

Definition 1.2.3 *Let K be a Cantor set, or an extended Cantor set. Define the thickness of K by*

$$\inf_{U_1 < U_2} \max \left\{ \frac{U_2^L - U_1^R}{|U_1|}, \frac{U_2^L - U_1^R}{|U_2|} \right\},$$

where the infimum is taken for all pairs of gaps of K , with at least one of them being a finite gap. We denote this value by $\tau(K)$.

Remark 1.2.1 *Definition 1.2.3 is not the most standard, but if K is a Cantor set it is easy to see that it coincides with the usual definition of thickness. Compare the definition in chapter 4 of [50].*

If we drop the assumption of sizes of K and L in Theorem 1.1.1, we obtain Theorem 1.2.1. For the reader's convenience, we include the proof (Theorem 1.1.1 can be shown in a similar way).

Theorem 1.2.1 *Let K and L be Cantor sets with $\tau(K) \cdot \tau(L) \geq 1$. Then, $K + L$ is the disjoint union of finitely many closed intervals.*

Proof: It is enough to show that $K + L$ has at most finitely many open gaps in $[K^L + L^L, K^R + L^R]$. Suppose $x \in [K^L + L^L, K^R + L^R]$ and $x \notin K + L$. It is easy to see that

$$x \notin K + L \iff K \cap (x - L) = \phi.$$

Note that $x - L$ is again a Cantor set. Since we have

$$\tau(K) \cdot \tau(x - L) = \tau(K) \cdot \tau(L) \geq 1,$$

the Gap Lemma implies that there are only three possibilities:

- (1) the intervals $[K^L, K^R]$ and $[x - L^R, x - L^L]$ are disjoint;
- (2) the set K is contained in a finite gap of the set $(x - L)$;
- (3) the set $(x - L)$ is contained in a finite gap of the set K .

Case (1) contradicts the assumption that $x \in [K^L + L^L, K^R + L^R]$. It is easy to see that the set of points which satisfy (2) or (3) is a union of finitely many open intervals.

The following is immediate from the definition of thickness:

Lemma 1.2.2 *Let K be a Cantor set, and let U be a gap of the maximal size of K . Let $K_1 = K \cap [K^L, U^L]$ and $K_2 = K \cap [U^R, K^R]$. Then $\tau(K_1) \geq \tau(K)$ and $\tau(K_2) \geq \tau(K)$.*

Lemma 1.2.3 *Let K be a Cantor set, and let $C_1, C_2 > 0$ be real numbers. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a strictly increasing continuous function. Assume further that for any $p_1, p_2, p_3 \in K$ with $p_1 < p_2 < p_3$, we have*

$$C_1 \frac{p_3 - p_2}{p_2 - p_1} < \frac{f(p_3) - f(p_2)}{f(p_2) - f(p_1)} < C_2 \frac{p_3 - p_2}{p_2 - p_1}.$$

Write $L = f(K)$. Then, we have $C_1\tau(K) \leq \tau(L) \leq C_2\tau(K)$.

Proof: Let $\epsilon > 0$. By the definition of thickness there exist two gaps of L , say W_1, W_2 , which satisfy $W_1 < W_2$ and

$$\tau(L) + \epsilon > \max \left\{ \frac{W_2^L - W_1^R}{|W_1|}, \frac{W_2^L - W_1^R}{|W_2|} \right\}.$$

Let U_1, U_2 be the gaps of K with $f(U_1) = W_1$ and $f(U_2) = W_2$. Then,

$$\max \left\{ \frac{W_2^L - W_1^R}{|W_1|}, \frac{W_2^L - W_1^R}{|W_2|} \right\} > C_1 \max \left\{ \frac{U_2^L - U_1^R}{|U_1|}, \frac{U_2^L - U_1^R}{|U_2|} \right\} \geq C_1\tau(K).$$

Since $\epsilon > 0$ was arbitrary, $\tau(L) \geq C_1\tau(K)$. The other inequality can be shown analogously.

Corollary 1.2.1 *Let $\epsilon, c > 0$ be real numbers and assume that $0 < c < 1$. Then there exists $\delta > 0$ such that for any Cantor set K with $|K| < \delta$ and $K^L > \epsilon$, $\tau(\tilde{K}) > c \cdot \tau(K)$.*

Lemma 1.2.4 *Let K and L be Cantor sets with $\tau(K) \cdot \tau(L) > 1$. Assume that $0 \notin K, L$. Then $K \cdot L$ is the disjoint union of finitely many closed intervals.*

Proof: By Lemma 1.2.2, there exist Cantor sets K_1, K_2, \dots, K_m such that

- (1) $K = K_1 \sqcup \dots \sqcup K_m$;
- (2) $\tau(K_i) \geq \tau(K)$ ($i = 1, 2, \dots, m$);
- (3) either $K_i \subset (0, \infty)$ or $K_i \subset (-\infty, 0)$;
- (4) $|K_i|$ is sufficiently small.

Write $L = L_1 \sqcup \dots \sqcup L_n$ analogously. By Theorem 1.1.1 and Corollary 1.2.1, $K_i \cdot L_j$ ($i = 1, 2, \dots, m, j = 1, 2, \dots, n$) is a union of finitely many closed intervals. Therefore, $K \cdot L$ is again a union of finitely many closed intervals. The result follows from this.

Corollary 1.2.2 *Let K and L be Cantor sets with $\tau(K) \cdot \tau(L) > 1$. Suppose that K or L contain 0. Then, one of the following occurs:*

- (1) $(K \cdot L) \cap [0, \infty)$ is the disjoint union of finitely many closed intervals;
- (2) $(K \cdot L) \cap [0, \infty)$ is the disjoint union of $\{0\}$ and countably many closed intervals which accumulate to 0.

1.3 Estimates of thickness

Lemma 1.3.1 *Let us define a function $f_k : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by*

$$f_k(x) = \frac{\log\left(1 + \frac{kx}{1+x}\right)}{\log(1+x)},$$

where k is a positive real number. Then f_k is a strictly decreasing function. In particular, since $\lim_{x \rightarrow 0^+} f_k(x) = k$, we have $f_k(x) < k$ for all $x \in \mathbb{R}^+$.

Proof: We have

$$f'_k(x) = \frac{(1+k)\log(1+x) - \left(1 + \frac{kx}{1+x}\right)\log(1+x+kx)}{(1+x+kx)\{\log(1+x)\}^2}.$$

Let us denote the numerator by $g_k(x)$. Then, since $g_k(0) = 0$ and

$$g'_k(x) = -\frac{k\log(1+x+kx)}{(1+x)^2} < 0,$$

we get $g_k(x) < 0$. This implies $f'_k(x) < 0$.

Definition 1.3.1 *Let K be a Cantor set with $K_+ \neq \emptyset$, and let C be a positive number. If a gap U of K_+ satisfies*

$$(1.3.1) \quad \frac{U^L}{|U|} \geq C,$$

we call U a C -nice gap of K_+ , and a C -bad gap of K_+ otherwise. If the inequality (1.3.1) is in fact equality, we call U a C -gap of K_+ . Furthermore, we call \widetilde{U} a log- C -nice gap of \widetilde{K}_+ if U satisfies (1.3.1). Define a log- C -bad gap of \widetilde{K}_+ , and a log- C -gap of \widetilde{K}_+ analogously. If U_1, U_2 are C -bad gaps of K_+ which satisfy

(1) $U_1 < U_2$;

(2) every gap contained in (U_1^R, U_2^L) is a C -nice gap of K_+ ;

we call the Cantor set $V = [U_1^R, U_2^L] \cap K_+$ a C -nice Cantor set of K_+ , and \tilde{V} a log- C -nice Cantor set of \tilde{K}_+ . If

(1) K is a 0^+ -Cantor set;

(2) there is no C -bad gap of K_+ in $(0, K^R)$;

we call K a C -nice 0^+ -Cantor set, and \tilde{K}_+ a log- C -nice extended Cantor set.

Lemma 1.3.2 *Let K be a Cantor set with $K_+ \neq \phi$, and let C be a positive number.*

Assume that U_1 and U_2 are gaps of K_+ such that

(1) $U_1 < U_2$, $|U_1| \leq |U_2|$;

(2) U_1 is a C -nice gap of K_+ .

Then, we have

$$\frac{\tilde{U}_2^L - \tilde{U}_1^R}{|\tilde{U}_1|} \geq \frac{\log\left(1 + \frac{\tau(K)}{1+C}\right)}{\log\left(1 + \frac{1}{C}\right)}.$$

Proof: Let us write

$$x = \frac{|U_1|}{U_1^L}.$$

Note that $x \leq \frac{1}{C}$. By Lemma 1.3.1, we have

$$\begin{aligned} \frac{\tilde{U}_2^L - \tilde{U}_1^R}{|\tilde{U}_1|} &\geq \frac{\log(U_1^L + |U_1| + \tau(K)|U_1|) - \log(U_1^L + |U_1|)}{\log(U_1^L + |U_1|) - \log U_1^L} \\ &= \frac{\log\left(1 + \frac{\tau(K)x}{1+x}\right)}{\log(1+x)} \\ &\geq \frac{\log\left(1 + \frac{\tau(K)\frac{1}{C}}{1+\frac{1}{C}}\right)}{\log\left(1 + \frac{1}{C}\right)} = \frac{\log\left(1 + \frac{\tau(K)}{1+C}\right)}{\log\left(1 + \frac{1}{C}\right)}. \end{aligned}$$

Lemma 1.3.3 *Let $K \geq 0$ be a Cantor set, and let C be a positive number. Suppose that every gap of K is a C -nice gap. Then*

$$\tau(\widetilde{K}_+) \geq \frac{\log\left(1 + \frac{\tau(K)}{1+C}\right)}{\log\left(1 + \frac{1}{C}\right)}.$$

Remark 1.3.1 *If $0 \in K$, then \widetilde{K}_+ is an extended Cantor set.*

Proof: Let U_1 and U_2 be gaps of K with $U_1 < U_2$.

Case 1) Suppose that $|U_1| \leq |U_2|$. By Lemma 1.3.2, we have

$$\begin{aligned} \max\left\{\frac{\widetilde{U}_2^L - \widetilde{U}_1^R}{|\widetilde{U}_1|}, \frac{\widetilde{U}_2^L - \widetilde{U}_1^R}{|\widetilde{U}_2|}\right\} &\geq \frac{\widetilde{U}_2^L - \widetilde{U}_1^R}{|\widetilde{U}_1|} \\ &\geq \frac{\log\left(1 + \frac{\tau(K)}{1+C}\right)}{\log\left(1 + \frac{1}{C}\right)}. \end{aligned}$$

Case 2) Suppose that $|U_1| > |U_2|$. By the mean value theorem, we have $|\widetilde{U}_1| > |\widetilde{U}_2|$.

Therefore,

$$\begin{aligned} \max\left\{\frac{\widetilde{U}_2^L - \widetilde{U}_1^R}{|\widetilde{U}_1|}, \frac{\widetilde{U}_2^L - \widetilde{U}_1^R}{|\widetilde{U}_2|}\right\} &= \frac{\widetilde{U}_2^L - \widetilde{U}_1^R}{|\widetilde{U}_2|} \\ &> \tau(K) \\ &> \frac{\log\left(1 + \frac{\tau(K)}{1+C}\right)}{\log\left(1 + \frac{1}{C}\right)}. \end{aligned}$$

It follows that

$$\begin{aligned} \tau(\widetilde{K}_+) &= \inf_{U_1 < U_2} \max\left\{\frac{\widetilde{U}_2^L - \widetilde{U}_1^R}{|\widetilde{U}_1|}, \frac{\widetilde{U}_2^L - \widetilde{U}_1^R}{|\widetilde{U}_2|}\right\} \\ &\geq \frac{\log\left(1 + \frac{\tau(K)}{1+C}\right)}{\log\left(1 + \frac{1}{C}\right)}. \end{aligned}$$

Lemma 1.3.4 *Let K be a Cantor set with $K_+ \neq \emptyset$, and let C be a positive number.*

Assume that V is a log- C -nice Cantor set of \widetilde{K}_+ . Then,

- (1) $|V| > \log\left(1 + \frac{\tau(K)}{1+C}\right)$;
- (2) $\tau(V) \geq \frac{\log\left(1 + \frac{\tau(K)}{1+C}\right)}{\log\left(1 + \frac{1}{C}\right)}$;
- (3) *for any gap U of V , $|U| \leq \log\left(1 + \frac{1}{C}\right)$.*

Proof: Let U_1, U_2 be the C -bad gaps of K_+ such that

$$U_1 < U_2, \widetilde{U}_1^R = V^L, \text{ and } \widetilde{U}_2^L = V^R.$$

If $|U_1| \leq |U_2|$, we get

$$\begin{aligned} |V| &= \log \left(1 + \frac{U_2^L - U_1^R}{U_1^R} \right) > \log \left(1 + \frac{U_2^L - U_1^R}{(1+C)|U_1|} \right) \\ &\geq \log \left(1 + \frac{\tau(K)}{1+C} \right). \end{aligned}$$

The other case can be shown similarly. This proves (1). Lemma 1.3.3 implies (2), and (3) is straightforward.

Recall that a Cantor set K is called a 0^\times -Cantor set if $K_+, K_- \neq \emptyset$, and $0 \notin K$.

Definition 1.3.2 *Let C be a positive number and let K be a 0^\times -Cantor set. Assume that U_1, U_2 are the gaps of K which satisfy*

- (1) $0 \in U_1$;
- (2) $U_1 < U_2$;
- (3) every gap in (U_1^R, U_2^L) is a C -nice gap of K_+ .

Then, we call the Cantor set $V = [U_1^R, U_2^L] \cap K$ a C -nice 0^\times -Cantor set of K_+ , and \widetilde{V} a log- C -nice 0^\times -Cantor set of \widetilde{K}_+ .

The following can be shown analogously to Lemma 1.3.4.

Lemma 1.3.5 *Let K be a 0^\times -Cantor set, and let C be a positive number. Assume that V is the log- C -nice 0^\times -Cantor set of \widetilde{K}_+ . Then,*

- (1) $|V| > \log(1 + \tau(K))$;
- (2) $\tau(V) \geq \frac{\log \left(1 + \frac{\tau(K)}{1+C} \right)}{\log \left(1 + \frac{1}{C} \right)}$;
- (3) for any gap U of V , $|U| \leq \log \left(1 + \frac{1}{C} \right)$.

Lemma 1.3.6 *Let K be an extended Cantor set. Then for any $\epsilon > 0$, there exist a decreasing sequence $\{k_n\}$ which satisfies*

(1) $k_n \rightarrow -\infty$;

(2) $K \cap [k_n, K^R]$ is a Cantor set with $\tau(K \cap [k_n, K^R]) > \tau(K) - \epsilon$.

Proof: Without loss of generality, we can assume that $K^R = 0$. Let

$$A = \lim_{N \rightarrow -\infty} \sup \{|U| \mid U \text{ is a gap of } K \text{ contained in } (-\infty, N]\}.$$

Case 1) Suppose that $A = \infty$. We can find a sequence of gaps $\{U_n\}$ ($n = 1, 2, \dots$) such that

(a) $|U_n| > |U|$ for every gap U contained in $(U_n^R, 0)$;

(b) $|U_{n+1}| > |U_n|$;

(c) $U_{n+1} < U_n$.

Let $k_n = U_n^R$ ($n = 1, 2, \dots$). By the definition of thickness, it is easy to see that $\tau(K \cap [k_n, 0]) \geq \tau(K)$. Therefore, $\{k_n\}$ satisfies the desired properties.

Case 2) Suppose that $0 < A < \infty$. Let us take $\epsilon_0 > 0$ such that

$$\tau(K) \cdot \frac{A - \epsilon_0}{A + \epsilon_0} > \tau(K) - \epsilon.$$

Let us choose $N < 0$ which satisfies

$$\sup\{|U| \mid U \text{ is a gap contained in } (-\infty, N]\} < A + \epsilon_0.$$

Then, we can take a sequence of gaps $\{U_n\}$ ($n = 1, 2, \dots$) such that

(a) every U_n is contained in $(-\infty, N]$;

(b) $A - \epsilon_0 < |U_n| < A + \epsilon_0$;

(c) $U_{n+1} < U_n$.

Let $k_n = U_n^R$ ($n = 1, 2, \dots$). Let us show that this $\{k_n\}$ satisfies the desired properties. Retaking N and $\{U_n\}$ if necessary, we can further assume that for any gap S contained

in $[N, 0]$, $\frac{S^L - U_1^R}{|S|}$ is sufficiently large. Therefore, it is enough to show that for any U_n we have

$$\frac{U^L - U_n^R}{|U|} > \tau(K) - \epsilon,$$

where U is a gap which is contained in $(-\infty, N]$ and satisfies $U_n < U$. Let U and U_n be such gaps. Then,

$$\frac{U^L - U_n^R}{|U|} = \frac{U^L - U_n^R}{|U_n|} \cdot \frac{|U_n|}{|U|} \geq \tau(K) \cdot \frac{A - \epsilon_0}{A + \epsilon_0} > \tau(K) - \epsilon,$$

which was to be proved.

Case 3) Suppose that $A = 0$. Let us assume that we cannot find such a sequence $\{k_n\}$, to derive a contradiction. Let $N < 0$ be a number such that

$$\sup\{|U| \mid U \text{ is a gap contained in } (-\infty, N]\} < 1.$$

By assumption, retaking N if necessary, we can assume that $\tau(K \cap [U^R, 0]) \leq \tau(K) - \epsilon$ for any gap U contained in $(-\infty, N]$. Let us choose a gap U_0 contained in $(-\infty, N]$ in such a way that $N - B$ is sufficiently large, where

$$B = U_0^R + (1 + \tau(K) - \epsilon) \frac{\tau(K)}{\epsilon}.$$

Let us show by induction that we can choose a sequence of gaps U_n ($n = 1, 2, \dots$) contained in $(-\infty, B]$ which satisfies

$$(1.3.2) \quad \begin{aligned} (a) \quad & U_n < U_{n+1}; \\ (b) \quad & |U_n| \leq \frac{\tau(K) - \epsilon}{\tau(K)} |U_{n+1}|; \\ (c) \quad & U_{n+1}^L - U_n^R \leq (\tau(K) - \epsilon) |U_{n+1}|; \end{aligned}$$

for all $n = 0, 1, 2, \dots$. If this is shown, this apparently leads to a contradiction since it implies $|U_n|$ goes to infinity.

Suppose that U_0, U_1, \dots, U_n are already chosen. Since

$$\tau(K \cap [U_n^R, 0]) \leq \tau(K) - \epsilon$$

and $N - B$ is sufficiently large, there exists a gap U_{n+1} contained in $(-\infty, N]$ which satisfies

$$\begin{aligned} (a) \quad & U_n < U_{n+1}; \\ (b) \quad & \frac{U_{n+1}^L - U_n^R}{|U_{n+1}|} \leq \tau(K) - \epsilon; \\ (c) \quad & \frac{U_{n+1}^L - U_n^R}{|U_n|} \geq \tau(K). \end{aligned}$$

Therefore,

$$|U_n| \leq \frac{\tau(K) - \epsilon}{\tau(K)} |U_{n+1}|, \quad \text{and} \quad U_{n+1}^L - U_n^R \leq (\tau(K) - \epsilon) |U_{n+1}|.$$

Since we have

$$|U_{i+1}| + (U_{i+1}^L - U_i^R) \leq (1 + \tau(K) - \epsilon) |U_{i+1}| \quad (i = n, n-1, \dots, 0),$$

we get

$$\begin{aligned} U_{n+1}^R &= U_0^R + \{|U_{n+1}| + (U_{n+1}^L - U_n^R) + |U_n| + (U_n^L - U_{n-1}^R) + \dots + |U_1| + (U_1^L - U_0^R)\} \\ &< U_0^R + \{(1 + \tau(K) - \epsilon) |U_{n+1}| + (1 + \tau(K) - \epsilon) |U_n| + \dots\} \\ &< U_0^R + (1 + \tau(K) - \epsilon) \left\{ 1 + \frac{\tau(K) - \epsilon}{\tau(K)} + \left(\frac{\tau(K) - \epsilon}{\tau(K)} \right)^2 + \dots \right\} \\ &= B \end{aligned}$$

(from the second to the third inequality, we used the fact that $|U_{n+1}| < 1$). Therefore, U_{n+1} is indeed contained in $(-\infty, B]$.

By induction, we can choose a sequence of gaps $\{U_n\}$ which satisfies the condition (1.3.2), a clear contradiction.

Lemma 1.3.7 *Let $C > 0$ be a real number, and let K be a C -nice 0^+ -Cantor set. Then, there exists a decreasing sequence $\{k_n\}$ which satisfies*

- (1) $k_n \rightarrow -\infty$;
- (2) $\widetilde{K}_+ \cap [k_n, \widetilde{K}_+^R]$ is a Cantor set with

$$\tau\left(\widetilde{K}_+ \cap [k_n, \widetilde{K}_+^R]\right) \geq \frac{\log\left(1 + \frac{\tau(K)}{1+C}\right)}{\log\left(1 + \frac{1}{C}\right)}.$$

Proof: Let

$$A = \lim_{N \rightarrow -\infty} \sup \left\{ |U| \mid U \text{ is a gap of } \widetilde{K}_+ \text{ contained in } (-\infty, N] \right\}.$$

Note that $A \leq \log(1 + \frac{1}{C})$.

Case 1) Suppose that $A = \log(1 + \frac{1}{C})$. Then, there exists a sequence of gaps $\{U_n\}$ ($n = 0, 1, 2, \dots$) of \widetilde{K}_+ such that

- (a) $|U_n| \uparrow \log\left(1 + \frac{1}{C}\right)$;
- (b) $U_0 > U_1 > U_2 > \dots$;
- (c) $|U_n| \geq |U|$ for every gap U in $[U_n^R, \widetilde{K}_+^R]$.

By Lemma 1.3.3, $k_n = U_n^R$ ($n = 0, 1, 2, \dots$) satisfies the desired properties.

Case 2) Suppose that every gap in K is a C' -nice gap for some $C' > C$. Then, since we have

$$\tau(\widetilde{K}_+) \geq \frac{\log\left(1 + \frac{\tau(K)}{1+C'}\right)}{\log\left(1 + \frac{1}{C'}\right)} > \frac{\log\left(1 + \frac{\tau(K)}{1+C}\right)}{\log\left(1 + \frac{1}{C}\right)}$$

by Lemma 1.3.3 and Lemma 1.3.1, the claim follows from Lemma 1.3.6.

Case 3) Suppose that $A < \log(1 + \frac{1}{C})$, and K has a C -gap. Let U be the log- C -gap of \widetilde{K}_+ such that each gap in $(-\infty, U^L]$ is not a log- C -gap of \widetilde{K}_+ . Then, there exists $C'' > C$ such that every gap of \widetilde{K}_+ in $(-\infty, U^L]$ is a log- C'' -nice gap of \widetilde{K}_+ . Therefore, we get

$$\tau\left((-\infty, U^L] \cap \widetilde{K}_+\right) \geq \frac{\log\left(1 + \frac{\tau(K)}{1+C''}\right)}{\log\left(1 + \frac{1}{C''}\right)} > \frac{\log\left(1 + \frac{\tau(K)}{1+C}\right)}{\log\left(1 + \frac{1}{C}\right)},$$

so again the claim follows from Lemma 1.3.6.

1.4 Proof of the main results

In this section, we will prove our main results.

Proof: [Proof of Theorem 1.1.2] Without loss of generality, we can assume that

$$\tau(L) \geq \frac{2\tau(K) + 1}{\tau(K)^2}.$$

Let $C = \tau(K)\tau(L) - 1$. Note that $C > 0$. It is easy to see that

$$1 + \frac{\tau(L)}{1+C} = 1 + \frac{1}{\tau(K)}, \quad \text{and} \quad 1 + \frac{1}{C} \leq 1 + \frac{\tau(K)}{1+\tau(K)}.$$

Case 1) Suppose that every gap in L is a C -nice gap. Note that every gap in K is a $\tau(K)$ -nice gap. Since

$$\frac{\log\left(1 + \frac{\tau(K)}{1+\tau(K)}\right)}{\log\left(1 + \frac{1}{\tau(K)}\right)} \cdot \frac{\log\left(1 + \frac{\tau(L)}{1+C}\right)}{\log\left(1 + \frac{1}{C}\right)} \geq 1,$$

By Theorem 1.1.1 and Lemma 1.3.7, $\widetilde{K}_+ + \widetilde{L}_+$ is a half line.

Case 2) Suppose that there is a C -bad gap in L . Let V be the C -nice Cantor set of L such that $V^R = L^R$. By Lemma 1.3.4,

$$\begin{aligned} |\widetilde{V}| &> \log\left(1 + \frac{\tau(L)}{1+C}\right) \\ &= \log\left(1 + \frac{1}{\tau(K)}\right), \end{aligned}$$

so for any gap U of \widetilde{K}_+ , we have $|U| \leq |\widetilde{V}|$. Therefore, Theorem 1.1.1 and Lemma 1.3.7 imply that $\widetilde{K}_+ + \widetilde{L}_+$ is a half line.

Next, let $M, N > 0$ be real numbers with condition (1.1.2). Without loss of generality, we can assume that $M \geq N$. With this additional assumption, (1.1.2) is equivalent to

$$N < \frac{2M+1}{M^2}.$$

Let $C = \frac{M(1+N)}{1+M}$. Then, it is easy to see that

$$1 + \frac{1}{M} > 1 + \frac{N}{1+C}, \quad \frac{1+2M}{M} = \frac{1+C+N}{C}, \quad \text{and} \quad C \geq N.$$

Let K be the middle- $\frac{1}{1+2M}$ Cantor set whose convex hull is $[0, 1]$. Write

$$K_0 = K \cap \left[\frac{1+M}{1+2M}, 1 \right].$$

Then

$$K = \{0\} \sqcup \bigsqcup_{n=0}^{\infty} \left(\frac{M}{1+2M} \right)^n K_0.$$

Let L_0 be a Cantor set with sufficiently large thickness whose convex hull is $[\frac{1+C}{1+C+N}, 1]$.

Let us define a 0^+ -Cantor set L as follows:

$$L = \{0\} \sqcup \bigsqcup_{n=0}^{\infty} \left(\frac{C}{1+C+N} \right)^n L_0.$$

Since $\tau(L_0)$ is sufficiently large, $\tau(L) = N$. Note that

$$\begin{aligned} \widetilde{K}_+ + \widetilde{L}_+ &= \bigcup_{m,n \geq 0} \left\{ -m \log \left(\frac{1+2M}{M} \right) - n \log \left(\frac{1+C+N}{C} \right) + (\widetilde{K}_0 + \widetilde{L}_0) \right\} \\ &= \bigcup_{n \geq 0} \left\{ -n \log \left(\frac{1+2M}{M} \right) + (\widetilde{K}_0 + \widetilde{L}_0) \right\}. \end{aligned}$$

Note that by Theorem 1.1.1 and Lemma 1.3.3, $\widetilde{K}_0 + \widetilde{L}_0$ is an interval. Therefore, since

$$\begin{aligned} (\widetilde{K}_0^R + \widetilde{L}_0^R) - (\widetilde{K}_0^L + \widetilde{L}_0^L) &= (\widetilde{K}_0^R - \widetilde{K}_0^L) + (\widetilde{L}_0^R - \widetilde{L}_0^L) \\ &= \log \left(1 + \frac{M}{1+M} \right) + \log \left(1 + \frac{N}{1+C} \right) \\ &< \log \left(1 + \frac{M}{1+M} \right) + \log \left(1 + \frac{1}{M} \right) \\ &= \log \left(\frac{1+2M}{M} \right), \end{aligned}$$

$\widetilde{K}_+ + \widetilde{L}_+$ is a disjoint union of countably many closed intervals.

Next, let us also show that $K \cdot L$ can be a disjoint union of k intervals, for any $k \geq 2$. The construction is almost the same as the construction above. Consider exactly the same K and L_0 , and modify L to be

$$L = L_1 \sqcup \bigsqcup_{n=0}^{k-2} \left(\frac{C}{1+C+N} \right)^n L_0,$$

where L_1 is a Cantor set which satisfies

- (1) the convex hull of L_1 is $\left[0, \left(\frac{C}{1+C+N} \right)^{k-1} \right]$;
- (2) $\tau(L_1)$ is sufficiently large.

Then, $\tau(K) = M$, $\tau(L) = N$, and it is easy to see that $K \cdot L$ is a disjoint union of k intervals.

Remark 1.4.1 *Since every gap in K and L is a $\tau(K)$ -nice gap and a $\tau(L)$ -nice gap, respectively, $K \cdot L$ is an interval if*

$$\frac{\log\left(1 + \frac{\tau(K)}{1+\tau(K)}\right)}{\log\left(1 + \frac{1}{\tau(K)}\right)} \cdot \frac{\log\left(1 + \frac{\tau(L)}{1+\tau(L)}\right)}{\log\left(1 + \frac{1}{\tau(L)}\right)} \geq 1,$$

by Lemma 1.3.7. But interestingly this is not the optimal estimate.

Next, let us show Theorem 1.1.3. The proof is essentially the same as Theorem 1.1.2.

Proof: [proof of Theorem 1.1.3] The first half is a verbatim repetition of Theorem 1.1.2. To show that the estimate is optimal, we consider L' instead of L , which has the same positive part as L and has the negative part of sufficiently large size and thickness.

To prove Theorem 1.1.4, we need a series of lemmas. Recall that for any Cantor set K and real number $C > 0$, we defined log- C -nice Cantor set of K_+ , and log- C -nice extended Cantor set of K_+ in Definition 1.3.1. We define log- C -nice Cantor set of K_- , and log- C -nice extended Cantor set of K_- analogously.

Lemma 1.4.1 *Let K be a Cantor set, and let $C < \tau(K)$ be a positive number. Suppose that U is a log- C -bad gap of \widetilde{K}_+ . Then, there exists a set $X \subset \widetilde{K}_-$ such that*

- (1) X is a log- C -nice Cantor set of \widetilde{K}_- , or a log- C -nice extended Cantor set of \widetilde{K}_- ;
- (2) $X^R - U^R > \log\left(\frac{\tau(K) - C}{1 + C}\right)$;
- (3) $U^L - X^L > \log\left(\frac{\tau(K) - C}{1 + C}\right)$

(if X is a log- C -nice extended Cantor set, we set $X^L = -\infty$).

Remark 1.4.2 *In general, this set X is not unique.*

Proof: Let W be the gap of K_+ satisfying $\widetilde{W} = U$. Let U_1 be the C -bad gap of K_- which satisfies

- (1) $|W| \leq |U_1|$;

(2) the size of every C -bad gap of K_- in $(0, U_1^L)$ is less than $|W|$.

Then, by the definition of thickness, we have

$$U_1^L > \tau(K)|W| - C|W|.$$

Therefore,

$$\begin{aligned} \widetilde{U}_1^L - U^R &= \log U_1^L - \log W^R \\ &> \log(\tau(K) - C)|W| - \log(1 + C)|W| \\ &= \log\left(\frac{\tau(K) - C}{1 + C}\right). \end{aligned}$$

Suppose next that U_2 is the C -bad gap of K_- such that

- (1) $|U_2| < |W|$;
- (2) every gap in $(U_2^R, U_1^L) \cap K_-$ is a C -nice gap. (If such U_2 does not exist, then $X = \log((0, U_1^L] \cap K_-)$ is a desired set.)

Then, by the definition of thickness, we have

$$W^L > \tau(K)|U_2| - C|U_2|.$$

Therefore,

$$\begin{aligned} U^L - \widetilde{U}_2^R &= \log W^L - \log U_2^R \\ &> \log(\tau(K) - C)|U_2| - \log(1 + C)|U_2| \\ &= \log\left(\frac{\tau(K) - C}{1 + C}\right). \end{aligned}$$

Then $X = \log(K_- \cap [U_2^R, U_1^L])$ satisfies the desired properties.

We call the set X given in Lemma 1.4.1 a \log - C -cover of U .

Lemma 1.4.2 *Let x, y be positive real numbers with $2(x + 1)(y + 1) \leq (xy - 1)^2$.*

Write

$$C_{x,y} = \frac{x + 1}{xy - 1}, \text{ and } C_{y,x} = \frac{y + 1}{xy - 1}.$$

Then, $0 < C_{x,y} < x$ and $0 < C_{y,x} < y$. Furthermore, we have

$$\frac{x - C_{x,y}}{1 + C_{x,y}} \cdot \frac{y - C_{y,x}}{1 + C_{y,x}} \geq 1.$$

Proof: It is easy to see that

$$2(x+1)(y+1) \leq (xy-1)^2 \iff y \geq \frac{2x+1+(x+1)\sqrt{2x+1}}{x^2}.$$

This implies that $xy > 1$. Therefore, since

$$\frac{x+1}{xy-1} < x \iff \frac{2x+1}{x^2} < y,$$

the first claim follows. Also, since we have

$$\begin{aligned} \frac{x-C_{x,y}}{1+C_{x,y}} \cdot \frac{y-C_{y,x}}{1+C_{y,x}} \geq 1 &\iff xy-1 \geq C_{y,x}(x+1) + C_{x,y}(y+1) \\ &\iff xy-1 \geq \frac{(x+1)(y+1)}{xy-1} + \frac{(x+1)(y+1)}{xy-1} \\ &\iff (xy-1)^2 \geq 2(x+1)(y+1), \end{aligned}$$

the second claim follows.

For any Cantor sets K and L with $\tau(K) \cdot \tau(L) > 1$, define $C_{K,L}$ and $C_{L,K}$ by

$$C_{K,L} = \frac{\tau(K)+1}{\tau(K)\tau(L)-1}, \text{ and } C_{L,K} = \frac{\tau(L)+1}{\tau(K)\tau(L)-1}.$$

Lemma 1.4.3 *Let K and L be Cantor sets with $\tau(K) \cdot \tau(L) > 1$. Assume that V is a log- $C_{K,L}$ -nice Cantor set of \widetilde{K}_+ , or a log- $C_{K,L}$ -nice 0^\times -Cantor set of \widetilde{K}_+ , or a log- $C_{K,L}$ -nice extended Cantor set of \widetilde{K}_+ . Similarly, suppose that T is a log- $C_{L,K}$ -nice Cantor set of \widetilde{L}_+ , or a log- $C_{L,K}$ -nice 0^\times -Cantor set of \widetilde{L}_+ , or a log- $C_{L,K}$ -nice extended Cantor set of \widetilde{L}_+ . Then $V+T$ is an interval, or a half line.*

Proof: Let us consider the case where V is a log- $C_{K,L}$ -nice Cantor set of \widetilde{K}_+ , and T is a log- $C_{L,K}$ -nice Cantor set of \widetilde{L}_+ . Since

$$1 + \frac{\tau(K)}{1+C_{K,L}} = 1 + \frac{1}{C_{L,K}}, \text{ and } 1 + \frac{\tau(L)}{1+C_{L,K}} = 1 + \frac{1}{C_{K,L}},$$

by Theorem 1.1.1 and Lemma 1.3.4, $V+T$ is an interval. Other cases can be shown analogously.

Lemma 1.4.4 *Let K, L be Cantor sets with condition (1.1.5). Let U_1, U_2 be a log- $C_{K,L}$ -bad gap of \widetilde{K}_+ , and a log- $C_{L,K}$ -bad gap of \widetilde{L}_+ , respectively. Let X, Y be a log- $C_{K,L}$ -cover of U_1 , and a log- $C_{L,K}$ -cover of U_2 , respectively. Then, we have $X+Y \supset U_1+U_2$.*

Proof: By Lemma 1.4.2, $\tau(K) > C_{K,L}$, $\tau(L) > C_{L,K}$, and

$$\frac{\tau(K) - C_{K,L}}{1 + C_{K,L}} \cdot \frac{\tau(L) - C_{L,K}}{1 + C_{L,K}} \geq 1.$$

Therefore, by Lemma 1.4.1 we have

$$\begin{aligned} (X^R + Y^R) - (U_1^R + U_2^R) &= (X^R - U_1^R) + (Y^R - U_2^R) \\ &\geq \log \frac{\tau(K) - C_{K,L}}{1 + C_{K,L}} + \log \frac{\tau(L) - C_{L,K}}{1 + C_{L,K}} \\ &\geq 0. \end{aligned}$$

Similarly, $(U_1^L + U_2^L) - (X^L + Y^L) \geq 0$. The claim follows from this and Lemma 1.4.3.

Definition 1.4.1 *Let K be a Cantor set with $K_+ \neq \emptyset$, and let C be a positive number. Let us take the sequence of log- C -bad gaps $\{U_n\}$ ($n = 0, 1, \dots, k$) of \widetilde{K}_+ , where k is either finite or infinite, in the following way:*

- (1) $U_0 = (\widetilde{K}_+^R, \infty)$;
- (2) $U_0 > U_1 > U_2 > U_3 > \dots$;
- (3) $V_n = [U_{n+1}^R, U_n^L] \cap \widetilde{K}_+$ ($n = 0, 1, 2, \dots$) are log- C -nice Cantor sets of \widetilde{K}_+ . (If k is finite, set $V_k = (-\infty, U_k^L]$.)

Then, we call $\{U_n\}$ and $\{V_n\}$ the log- C -split gaps of \widetilde{K}_+ and log- C -split Cantor sets of \widetilde{K}_+ , respectively. If k is finite, we say this split is finite.

Remark 1.4.3 *Note that if the split is finite in the above definition, V_k is either a log- C -nice extended Cantor set, or a log- C -nice 0^\times -Cantor set.*

Using these lemmas, we can complete the proof of Theorem 1.1.4.

Proof: [proof of Theorem 1.1.4] Let $\{U_n\}$ ($n = 0, 1, \dots, k$) and $\{V_n\}$ ($n = 0, 1, \dots, k$) be the log- $C_{K,L}$ -split gaps of \widetilde{K}_+ , and the log- $C_{K,L}$ -split Cantor sets of \widetilde{K}_+ , respectively. Similarly, let $\{S_n\}$ ($n = 0, 1, \dots, l$) and $\{T_n\}$ ($n = 0, 1, \dots, l$) be

the log- $C_{L,K}$ -split gaps of \widetilde{L}_+ , and the log- $C_{L,K}$ -split Cantor sets of \widetilde{L}_+ , respectively. Then, by Lemma 1.4.3,

$$V_i + T_j \quad (i = 0, 1, \dots, k, \quad j = 0, 1, \dots, l)$$

are intervals, or half lines. Let X_n ($n = 1, 2, \dots, k$) and Y_n ($n = 1, 2, \dots, l$) be log- $C_{K,L}$ -cover of U_n , and log- $C_{L,K}$ -cover of S_n , respectively. Then, by Lemma 1.4.3 and Lemma 1.3.7,

$$X_i + Y_j \quad (i = 1, 2, \dots, k, \quad j = 1, 2, \dots, l)$$

are intervals, or half lines. Note that, by Lemma 1.4.4, we have

$$(1.4.1) \quad X_i + Y_j \supset U_i + S_j \quad (i = 1, 2, \dots, k, \quad j = 1, 2, \dots, l).$$

Consider the case that k and l are both infinite. Other cases can be shown similarly.

We get

$$\begin{aligned} (\widetilde{K}_+ + \widetilde{L}_+) \cup (\widetilde{K}_- + \widetilde{L}_-) &\supset \bigcup_{i=0}^{\infty} (V_i + T_i) \cup \bigcup_{i=1}^{\infty} (X_i + Y_i) \\ &\supset (-\infty, \widetilde{K}_+^R + \widetilde{L}_+^R]. \end{aligned}$$

Similarly, $(\widetilde{K}_+ + \widetilde{L}_+) \cup (\widetilde{K}_- + \widetilde{L}_-) \supset (-\infty, \widetilde{K}_-^R + \widetilde{L}_-^R]$. Therefore,

$$(\widetilde{K}_+ + \widetilde{L}_+) \cup (\widetilde{K}_- + \widetilde{L}_-) = \left(-\infty, \max \left\{ \widetilde{K}_+^R + \widetilde{L}_+^R, \widetilde{K}_-^R + \widetilde{L}_-^R \right\} \right],$$

which implies the first claim of the Theorem.

Next, let us show that this estimate is optimal. Let M, N be positive real numbers with condition (1.1.6). Let

$$C_1 = \frac{MN(M+1)}{3MN+2M+2N+1}, \quad \text{and} \quad C_2 = \frac{MN(N+1)}{3MN+2M+2N+1}.$$

Note that (1.1.6) implies $C_1 < M$ and $C_2 < N$. It is easy to see that

$$(1.4.2) \quad \frac{1+C_1+M}{M-C_1} \cdot \frac{1+C_2+N}{N-C_2} = 1 + \frac{1+M}{C_1} = 1 + \frac{1+N}{C_2}.$$

Also, (1.1.6) implies

$$(1.4.3) \quad 1 + \frac{1}{C_1} > 1 + \frac{N}{1+C_2}, \quad 1 + \frac{1}{C_2} > 1 + \frac{M}{1+C_1}.$$

Let K be a 0-Cantor set such that

- (1) $K = K_1 \sqcup K_2$ and $0 \in K_1$;
- (2) K_1, K_2 are Cantor sets with sufficiently large thickness;
- (3) the convex hull of K_1 and K_2 are $[C_1 - M, C_1]$ and $[1 + C_1, 1 + C_1 + M]$, respectively.

Let us define a 0-Cantor set $L = L_1 \sqcup L_2$ analogously, with C_2 instead of C_1 and N instead of M . Note that $\tau(K) = M$ and $\tau(L) = N$. Let $U = (K_1^R, K_2^L)$ and $S = (L_1^R, L_2^L)$. By (1.4.2), we have

$$|\widetilde{K}_2| + |\widetilde{U}| = |\widetilde{L}_2| + |\widetilde{S}|, \text{ and } \widetilde{K}_-^R + \widetilde{L}_-^R = \widetilde{U}^L + \widetilde{S}^L.$$

Also, (1.4.3) implies

$$|\widetilde{K}_2| < |\widetilde{S}|, \text{ and } |\widetilde{L}_2| < |\widetilde{U}|.$$

Therefore,

$$(\widetilde{K}_+ + \widetilde{L}_+) \cup (\widetilde{K}_- + \widetilde{L}_-) = (-\infty, \widetilde{U}^L + \widetilde{S}^L] \sqcup [\widetilde{K}_2 + \widetilde{L}_2].$$

Next, let us show Theorem 1.1.5. We need the following definition and lemma:

Definition 1.4.2 *Let K be a 0-Cantor set, and let C, M be positive numbers. If*

- (1) K_0 is a Cantor set whose convex hull is $\left[\frac{1+C}{1+C+M}, 1 \right]$;
- (2) $\tau(K_0) > \max \left\{ 1, M, C + \sqrt{C(1+C+M)} \right\}$;
- (3) $K_+ = \bigsqcup_{n=0}^{\infty} \left(\frac{C}{1+C+M} \right)^n K_0$;
- (4) $K_- = \sqrt{\frac{C}{1+C+M}} K_+$;

we call K a (C, M) -Cantor set. Note that

$$\widetilde{K}_+ = \bigsqcup_{n=0}^{\infty} \left(\widetilde{K}_0 - nd \right), \text{ and } \widetilde{K}_- = \bigsqcup_{n=0}^{\infty} \left(\widetilde{K}_0 - \left(n + \frac{1}{2} \right) d \right),$$

where

$$d = \log \left(1 + \frac{1+M}{C} \right).$$

See Figure 1.2.

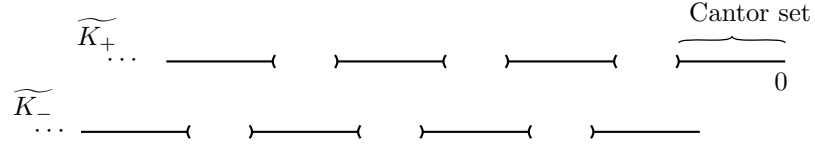


Figure 1.2: K is a (C, M) -Cantor set (line segments represent Cantor sets)

Lemma 1.4.5 *Let $C, M > 0$ be real numbers and let K be a (C, M) -Cantor set. We have*

- (1) *if $C \geq \frac{M^2}{3M+1}$, then $\tau(K) = M$;*
- (2) *if $C \leq \frac{M^2}{3M+1}$, then $\tau(K) = C + \sqrt{C(1+C+M)}$.*

Proof: Write $U = \left(\frac{C}{1+C+M}, \frac{1+C}{1+C+M}\right)$. By the definition of (C, M) -Cantor set, we have

$$\begin{aligned} \tau(K) &= \min \left\{ \frac{K^R - U^R}{|U|}, \frac{U^L - K^L}{|U|} \right\} \\ &= \left\{ M, C + \sqrt{C(1+C+M)} \right\}. \end{aligned}$$

By a simple computation, we get

$$C + \sqrt{C(1+C+M)} \geq M \iff C \geq \frac{M^2}{3M+1}.$$

The result follows from this.

Proof: [proof of theorem 1.1.5] First, let us assume that

$$(1.4.4) \quad M \geq N, \text{ and } N < \frac{(2M+1)^2}{M^3}.$$

Let

$$C_1 = \frac{M^2}{1+3M}, \text{ and } C_2 = \frac{M^2}{(1+M)(1+3M)}(1+N).$$

Then, we have

$$1 + \frac{1+M}{C_1} = 1 + \frac{1+N}{C_2}.$$

Also, it is easy to see that (1.4.4) implies

$$C_2 \geq \frac{N^2}{3N+1}, \text{ and } 1 + \frac{1}{C_1} > 1 + \frac{N}{1+C_2}.$$

Let K, L be a (C_1, M) -Cantor set and a (C_2, N) -Cantor set, respectively. By Lemma 1.4.5 we have $\tau(K) = M$ and $\tau(L) = N$. It is easy to see that $K \cdot L$ is a disjoint union of $\{0\}$ and countably many closed intervals.

Next, let us assume that

$$M \geq N, \text{ and } M < \frac{N^2 + 3N + 1}{N^2}.$$

Write

$$C'_1 = \frac{MN}{3N + 1}, \text{ and } C'_2 = \frac{N^2}{3N + 1}.$$

Let K' be a $(C'_1, M + \frac{M}{N} - 1)$ -Cantor set and L' be a (C'_2, N) -Cantor set. Then, by arguing analogously, $\tau(K') = M$, $\tau(L') = N$, and $K' \cdot L'$ is a disjoint union of $\{0\}$ and countably many closed intervals.

Remark 1.4.4 *It is immediate from the construction above that if the condition (1.1.7) is satisfied, there exist 0-Cantor sets K and L , such that*

- (1) $\tau(K) = M$, $\tau(L) = N$;
- (2) neither K nor L lies in a complementary domain of the other;
- (3) $K \cap L$ consists of exactly one element.

In fact, [28] and [32] independently showed that the condition (1.1.7) is the optimal estimate that guarantees the existence of such K and L . See section 1.6.

1.5 Other cases

In this section, we consider the cases that we have not yet discussed. The proofs are analogous, so we only state the results. Recall that a Cantor set K is a 0_-^+ -Cantor set if $K_+, K_- \neq \emptyset$, $\inf K_+ = 0$, and $\inf K_- > 0$.

Theorem 1.5.1 *Suppose that either of the following holds:*

- (1) K is a 0^+ -Cantor set, and L is a 0_-^+ -Cantor set;

(2) K is a 0^+ -Cantor set, and L is a 0^\times -Cantor set.

Then, if the condition (1.1.3) is satisfied, $K \cdot L$ is an interval. Furthermore, let $M, N > 0$ be real numbers that satisfy the condition (1.1.4). Then, for any $k \geq 2$, there exist Cantor sets K, L , such that K and L satisfy one of the conditions above, $\tau(K) = M$, $\tau(L) = N$, and $K \cdot L$ is a disjoint union of k closed intervals.

Theorem 1.5.2 *Suppose that one of the following holds:*

- (1) K is a 0 -Cantor set, and L is a 0_+^+ Cantor set;
- (2) K is a 0 -Cantor set, and L is a 0^\times -Cantor set;
- (3) K is a 0_-^+ -Cantor set, and L is a 0^\times -Cantor set;
- (4) K and L are both 0_-^+ -Cantor sets.

Then, if the condition (1.1.5) is satisfied, $K \cdot L$ is an interval. Furthermore, let $M, N > 0$ be real numbers with condition (1.1.6). Then, there exist Cantor sets K, L , such that K and L satisfy one of the conditions above, $\tau(K) = M$, $\tau(L) = N$, and $K \cdot L$ is a disjoint union of two intervals.

If K and L are both 0^\times -Cantor sets, the best we can hope for is $K \cdot L$ to become a disjoint union of two intervals.

Theorem 1.5.3 *Let K, L be 0^\times -Cantor sets. Then, if the condition (1.1.5) is satisfied, $K \cdot L$ is a disjoint union of two intervals. Furthermore, let $M, N > 0$ be real numbers with condition (1.1.6). Then there exist 0^\times -Cantor sets K and L such that $\tau(K) = M$, $\tau(L) = N$, and $K \cdot L$ is a disjoint union of three intervals.*

So far, we have not considered the case that $\min K > 0$ or $\min L > 0$. In fact, this turns out to be very simple. We have the following:

Theorem 1.5.4 *For any real numbers $M, N > 0$, there exist a Cantor set K and a 0^+ -Cantor set L such that $\min K > 0$, $\tau(K) = M$, $\tau(L) = N$, and $K \cdot L$ is a disjoint union of $\{0\}$ and countably many closed intervals.*

Proof: [Outline of the proof] Let us take sufficiently small $\epsilon > 0$. Let K be a Cantor set such that

- (1) $K \subset [1, 1 + \epsilon]$ and $\tau(K) = M$;
- (2) $K = K_1 \sqcup K_2$, where K_1, K_2 are Cantor sets with sufficiently large thickness.

Let L be the middle $\frac{1}{1+2N}$ -Cantor set whose convex hull is $[0, 1]$. It is easy to see that $K \cdot L$ is the disjoint union of $\{0\}$ and countably many closed intervals.

1.6 Connection with questions on intersections of two Cantor sets

In this section, we discuss the connection between questions on products of two Cantor sets and questions on intersections of two Cantor sets. For any Cantor sets K and L , if neither K nor L lies in a complementary domain of the other we say K and L are *interleaved*. Williams showed the following in [71] (this result was later extended by [28] and [32], independently):

Theorem 1.6.1 (Theorem 1 of [71]) *Let K, L be interleaved Cantor sets. Then if $\tau(K), \tau(L) \geq 1 + \sqrt{2}$, $K \cap L$ contains infinitely many elements. Furthermore, for any $M, N < 1 + \sqrt{2}$ there exist interleaved Cantor sets K and L such that $\tau(K) = M$, $\tau(L) = N$, and $K \cap L$ consists of exactly one element.*

Remark 1.6.1 *In fact, Williams showed much more. For example, he also showed that if $\tau(K), \tau(L) > 1 + \sqrt{2}$, $K \cap L$ contains a Cantor set. We are not sure whether our method can be applied to prove this statement.*

To illustrate the connection, we present a completely different proof of Theorem 1.6.1 using our method. See also Remark 1.4.4.

Proof: [outline of the proof of Theorem 1.6.1] For the sake of simplicity, we only consider the case $\tau(K) = \tau(L) = 1 + \sqrt{2}$. Write $C = \frac{1}{\sqrt{2}}$.

By the Gap Lemma, $K \cap L \neq \emptyset$. Translating K and L if necessary, we can assume that $0 \in K \cap L$. Let us only consider the case that K and L are both 0-Cantor sets.

Let $\{U_n\}$ ($n = 0, 1, \dots$) and $\{V_n\}$ ($n = 0, 1, \dots$) be the log- C -split gaps of \widetilde{K}_+ and the log- C -split Cantor sets of \widetilde{K}_+ , respectively. Let X_n ($n = 1, 2, \dots$) be log- C -covers of U_n ($n = 1, 2, \dots$). Similarly, let $\{S_n\}$ and $\{T_n\}$ be the log- C -split gaps of \widetilde{L}_+ and the log- C -split Cantor sets of \widetilde{L}_+ , respectively. Let Y_n be a log- C -cover of S_n . For the sake of simplicity, we assume that both splits are infinite. By Lemma 1.3.4 and the Gap Lemma, we have

- (1) for all $n \in \mathbb{N}$, $|X_n| \geq |U_n|$ and $|Y_n| \geq |S_n|$;
- (2) for all $n, m \in \mathbb{N}$, $V_n \cap T_m \neq \phi$ if $\text{con}(V_n) \cap \text{con}(T_m) \neq \phi$;
- (3) for all $n, m \in \mathbb{N}$, $X_n \cap Y_m \neq \phi$ if $\text{con}(X_n) \cap \text{con}(Y_m) \neq \phi$;

where $\text{con}(A)$ is the convex hull of a set A . The claim follows from this.

1.7 Open problems

In this section, we state a few questions and open problems that are suggested by the results of this paper.

1. In [2], the author generalized Theorem 1.1.1 for sums of three or more Cantor sets. It is natural to try to extend their results to products of three or more Cantor sets.
2. It is also natural to consider the product of middle α -Cantor set and middle β -Cantor set instead of general Cantor sets.
3. Similarly, we can consider products of two dynamically defined Cantor sets instead of general Cantor sets. In this case, we believe that the estimate in Theorem 1.1.4 will be different; but this question is beyond the scope of our paper.

Chapter 2

Quantum and spectral properties of the Labyrinth model

2.1 Introduction

2.1.1 Quasicrystal and the Labyrinth model.

The Fibonacci Hamiltonian is a central model in the study of electronic properties of one-dimensional quasicrystals. It is given by the following bounded self-adjoint operator in $l^2(\mathbb{Z})$:

$$(2.1.1) \quad (H_{\lambda,\beta}\psi)(n) = \psi(n+1) + \psi(n-1) + \lambda\chi_{[1-\alpha,1)}(n\alpha + \beta \pmod{1})\psi(n),$$

where $\alpha = \frac{\sqrt{5}-1}{2}$ is the *frequency*, $\beta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the *phase*, and $\lambda > 0$ is the *coupling constant*. By the minimality of the circle rotation and strong operator convergence, the spectrum is easily seen to be independent of β . With this specific choice of α , when $\beta = 0$ the potential of (2.1.1) coincides with the *Fibonacci substitution sequence* (for the precise definition, see section 2.2). Papers on this model include [?], [?], [11], [12], [14]. In [11], the authors showed that for sufficiently small coupling constant, the spectrum is a dynamically defined Cantor set, and the *density of states measure* is exact dimensional. Later, this result was extended for all values of the coupling constant [14].

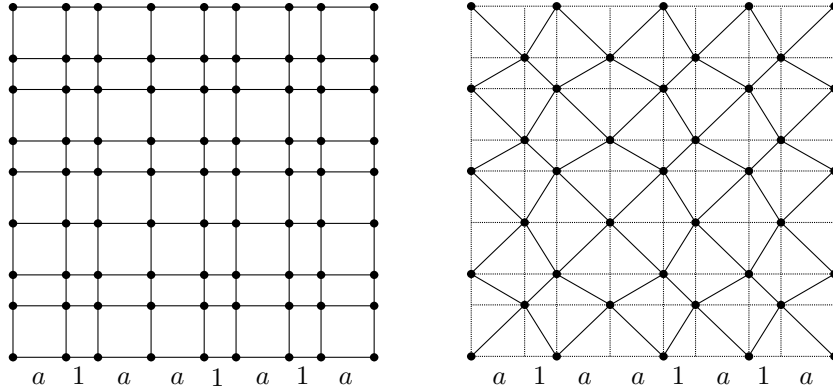


Figure 2.1: The square tiling (left) and the Labyrinth model (right).

In physics papers, it is more traditional to consider off-diagonal model, but in fact they are known to be very similar. The operator of the off-diagonal model has the following form:

$$(2.1.2) \quad (H_\omega \psi)(n) = \omega(n+1)\psi(n+1) + \omega(n)\psi(n-1),$$

where the sequence ω is in the *hull* of the Fibonacci substitution sequence. For the precise definition of hull, see (2.2.1). This sequence takes two positive real values, say 1 and a . Let

$$(2.1.3) \quad \lambda = \frac{|a^2 - 1|}{a},$$

and call this the *coupling constant*. The spectral properties of H_ω do not depend on the particular choice of ω , and depend only on the coupling constant λ . Recent mathematics papers discussing this operator include [11], [43], [72]. In [43] the authors considered tridiagonal substitution Hamiltonians, which include both (2.1.1) and (2.1.2) as special cases.

It is natural to consider higher dimensional models, but that is known to be extremely difficult. To get an idea of spectral properties of higher dimensional quasicrystals, simpler models have been considered. In two dimensional case, we have, for example, the square Fibonacci Hamiltonian [13], the square tiling, and the Labyrinth model. The Labyrinth model is the main subject of this paper. These models are

separable, so the existing results of one-dimensional models can be applied in the study of their spectral properties.

The square Fibonacci Hamiltonian is constructed by two copies of the Fibonacci Hamiltonian. Namely, this operator acts on $l^2(\mathbb{Z}^2)$, and is given by

$$[H_{\lambda_1, \lambda_2, \beta_1, \beta_2} \psi](m, n) = \psi(m+1, n) + \psi(m-1, n) + \psi(m, n+1) + \psi(m, n-1) \\ + (\lambda_1 \chi_{[1-\alpha, 1)}(m\alpha + \beta_1 \pmod{1}) + \lambda_2 \chi_{[1-\alpha, 1)}(n\alpha + \beta_2 \pmod{1})) \psi(m, n),$$

where $\alpha = \frac{\sqrt{5}-1}{2}$, $\beta_1, \beta_2 \in \mathbb{T}$, and $\lambda_1, \lambda_2 > 0$. It is known that the spectrum of this operator is given by the sum of the spectra of the one-dimensional models, and the density of states measure of this operator is the convolution of the density of states measures of the one-dimensional models. See, for example, the appendix in [13]. Recently, it was shown that for small coupling constants the spectrum of the square Fibonacci Hamiltonian is an interval [11]. Furthermore, it was shown that for almost all pairs of the coupling constants, the density of states measure is absolutely continuous with respect to Lebesgue measure in weakly coupled regime [13].

The square tiling is constructed by two copies of off-diagonal models. The operator acts on $l^2(\mathbb{Z}^2)$, and is given by

$$[H_{\omega_1, \omega_2} \psi](m, n) = \omega_1(m+1)\psi(m+1, n) + \omega_1(m)\psi(m-1, n) \\ + \omega_2(n+1)\psi(m, n+1) + \omega_2(n)\psi(m, n-1),$$

where the sequences ω_1 and ω_2 are in the hull of the Fibonacci substitution sequence. All vertices are connected horizontally and vertically. See Figure 2.1. It has been mainly studied numerically by physicists (e.g., [19], [21], [49]). By repeating the argument from [13], one can show that the analogous results of the square Fibonacci Hamiltonian hold for the square tiling. Recently, [20] considered the square tridiagonal Fibonacci Hamiltonians, which include the square Fibonacci Hamiltonian and the square tiling as special cases.

The operator of the Labyrinth model is given by:

$$\begin{aligned}
 (2.1.4) \quad \left[\hat{H}_{\omega_1, \omega_2} \psi \right] (m, n) &= \omega_1(m+1)\omega_2(n+1)\psi(m+1, n+1) \\
 &+ \omega_1(m+1)\omega_2(n)\psi(m+1, n-1) \\
 &+ \omega_1(m)\omega_2(n+1)\psi(m-1, n+1) \\
 &+ \omega_1(m)\omega_2(n)\psi(m-1, n-1),
 \end{aligned}$$

where the sequences ω_1 and ω_2 are in the hull of the Fibonacci substitution sequence. It is constructed by two copies of off-diagonal models. All vertices are connected diagonally, and the strength of the bond is equal to the product of the sides of the rectangle. See Figure 2.1. Compare with Figure 2 from [59]. Without loss of generality, we can assume that ω_1 and ω_2 take values in $\{1, a_1\}$ and $\{1, a_2\}$, respectively. We denote the corresponding coupling constants by λ_1 and λ_2 . It can be shown that the spectral properties do not depend on the specific choice of ω_1 and ω_2 , and only depend on the coupling constants λ_1 and λ_2 . Unlike the square Fibonacci Hamiltonian or the square tiling, the spectrum is the product (not the sum) of the spectra of the two one-dimensional models, and the density of states measure is not the convolution of the density of states measures of the one-dimensional models. This model was suggested in the late 1980s in [59], and so far this has been studied mostly by physicists, and their work is mainly relied on numerics [6], [58], [59], [60], [64], [65], [66], [67], [68], [74]. Sire considered this model in [59] and the numerical experiments suggested that the density of states measure is absolutely continuous for small coupling constants and singular continuous for large coupling constants. By a heuristic argument, the author also estimated the critical value of which the transition from zero measure spectrum to positive measure spectrum occurs, and showed that it agrees with numerical experiment. In some papers, other substitution sequences, e.g., silver mean sequence or bronze mean sequence, are used to define the Labyrinth model. We consider more general cases in this paper, and give rigorous proofs to some of the physicists' conjectures. In physicists' work, the coupling constants of two substitution sequences ω_1 and ω_2 are set as equal, but we consider the case that they may be different. We denote the spectrum of (2.1.4) by $\hat{\Sigma}_{\lambda_1, \lambda_2}$, and the density of states

measures of (2.1.2) and (2.1.4) by ν_λ and $\hat{\nu}_{\lambda_1, \lambda_2}$, respectively. The following theorems are the main results of this paper.

Theorem 2.1.1 *The spectrum $\hat{\Sigma}_{\lambda_1, \lambda_2}$ is a Cantor set of zero Lebesgue measure for sufficiently large coupling constants and is an interval for sufficiently small coupling constants.*

Theorem 2.1.2 *For any $E \in \mathbb{R}$,*

$$(2.1.5) \quad \hat{\nu}_{\lambda_1, \lambda_2}((-\infty, E]) = \iint_{\mathbb{R}^2} \chi_{(-\infty, E]}(xy) d\nu_{\lambda_1}(x) d\nu_{\lambda_2}(y).$$

The density of states measure $\hat{\nu}_{\lambda_1, \lambda_2}$ is singular continuous for sufficiently large coupling constants. Furthermore, there exists $\lambda^ > 0$ such that for almost every pair $(\lambda_1, \lambda_2) \in [0, \lambda^*) \times [0, \lambda^*)$, the density of states measure $\hat{\nu}_{\lambda_1, \lambda_2}$ is absolutely continuous with respect to Lebesgue measure.*

2.1.2 Structure of the paper

In section 2, we introduce metallic mean sequences and prove some lemmas. We then define off-diagonal model and discuss necessary results. In section 3 we define the Labyrinth model, and using the results in section 2, we prove Theorem 1.1 and 1.2.

2.2 Preliminaries

2.2.1 Linearly recurrent sequences

We recall some basic facts about subshifts over two symbols.

An *alphabet* is a finite set of symbols called *letters*. A *word* on \mathcal{A} is a finite nonempty sequence of letters. Write \mathcal{A}^+ for the set of words. For $u = u_1 u_2 \cdots u_n \in \mathcal{A}^+$, $|u| = n$ is the *length* of u . Define the *shift* T on $\mathcal{A}^{\mathbb{Z}}$ by

$$(Tx)_n = x_{n+1}.$$

Assume that $\mathcal{A}^{\mathbb{Z}}$ is equipped with the product topology. A *subshift* (X, T) on an alphabet \mathcal{A} is a closed T -invariant subset X of $\mathcal{A}^{\mathbb{Z}}$, endowed with the restriction of

T to X , which we denote again by T . Given $u = u_1u_2\cdots u_n \in \mathcal{A}^+$ and an interval $J = \{i, \dots, j\} \subset \{1, 2, \dots, n\}$, we write u_J to denote the word $u_iu_{i+1}\cdots u_j$. A *factor* of u is a word v such that $v = u_J$ for some interval $J \subset \{1, 2, \dots, n\}$. We extend this definition in obvious way to $u \in \mathcal{A}^{\mathbb{Z}}$. The *language* $\mathcal{L}(X)$ of a subshift (X, T) is the set of all words that are factors of at least one element of X .

Definition 2.2.1 *Let (X, T) be a subshift. We say that $x \in X$ is linearly recurrent if there exists a constant $K > 0$ such that for every factor u, v of x , $K|u| < |v|$ implies that u is a factor of v .*

We say that a subshift is *linearly recurrent* if it is minimal and contains a linearly recurrent sequence. Note that if a subshift is linearly recurrent, then by minimality, all sequences belonging to X are linearly recurrent.

2.2.2 Metallic mean sequence

Let $\mathcal{A} = \{a, b\}$ be an alphabet, and consider the following substitution:

$$\mathcal{P}_s : \begin{cases} a \longrightarrow a^s b \\ b \longrightarrow a, \end{cases}$$

where s is a positive integer. Consider the iteration of \mathcal{P}_s on a . For example, if $s = 1$,

$$a \longrightarrow ab \longrightarrow aba \longrightarrow abaab \longrightarrow abaababa \longrightarrow \dots$$

Let us write the n th iteration as $\mathcal{C}_s(n)$. It is easy to see that

$$\mathcal{C}_s(n+1) = (\mathcal{C}_s(n))^s \mathcal{C}_s(n-1).$$

Therefore, for any $s \in \mathbb{N}$ we can define a sequence $\{u_s(k)\}_{k=1}^{\infty}$ by $u_s = \lim_{n \rightarrow \infty} \mathcal{C}_s(n)$. They are called *metallic mean sequences*. In particular, when $s = 1$, it is called the *Fibonacci substitution sequence* or *golden mean sequence*. When $s = 2, 3$, they are called the *silver mean sequence* and *bronze mean sequence*, respectively.

We define the *hull* $\Omega_{a,b}^{(s)}$ of u_s by

$$(2.2.1) \quad \Omega_{a,b}^{(s)} = \{\omega \in \{a, b\}^{\mathbb{Z}} \mid \text{every factor of } \omega \text{ is a factor of } u_s\}.$$

It is well known that $\Omega_{a,b}^{(s)}$ is compact and T -invariant and $(\Omega_{a,b}^{(s)}, T)$ is linearly recurrent. See for example, [53] and references therein.

Remark 2.2.1 *Let us define a rotation sequence $v_{a,b,s,\beta}$ by*

$$v_{a,b,s,\beta}(n) = \begin{cases} a & \text{if } n\alpha + \beta \pmod{1} \in [1 - \alpha, 1) \\ b & \text{o.w.,} \end{cases}$$

where α is given by

$$\alpha = \frac{1}{1 + s + \frac{1}{s + \frac{1}{s + \frac{1}{\ddots}}}} = \frac{s + 2 - \sqrt{s^2 + 4}}{2s}.$$

It is easy to see that the potential of the Fibonacci Hamiltonian (2.1.1) is $v_{\lambda,0,1,\beta}$. It is well known that $v_{a,b,s,0} = u_s$, so there is no need to distinguish the rotation sequence and substitution sequence. However, it seems that it is more common to use the rotation sequence in the definition of the on-diagonal model and use the substitution sequence in the definition of the off-diagonal model. It is also known that

$$\Omega_{a,b}^{(s)} = \bigcup_{\beta \in \mathbb{T}} v_{a,b,s,\beta}.$$

See, for example [42].

Remark 2.2.2 *There seems to be a minor confusion about substitution sequences and rotation sequences in some papers. Let*

$$\alpha^* = \frac{1}{s + \frac{1}{s + \frac{1}{s + \frac{1}{\ddots}}}}.$$

Using α^* , define $v_{a,b,s,\beta}^*$ analogously. In some papers it is stated that $v_{a,b,s,0}^* \in \Omega_{a,b}^{(s)}$, but this is obviously not true. What is true is that $v_{a,b,1,0}^* = v_{b,a,1,0}$, so when $s = 1$ there is no actual harm.

We simply write $\Omega_{a,b}^{(s)}$ as $\Omega^{(s)}$ below when there is no chance of confusion.

2.2.3 Necessary results

We will need the following definition and subsequent lemmas later.

Definition 2.2.2 *Let (X, T) be a linearly recurrent subshift, and let $x, y \in \mathcal{L}(X)$. If there exist disjoint intervals J_1 and J_2 such that*

- 1) $J_i \subset \{1, 2, \dots, |x|\}$ for $i = 1, 2$,
- 2) $J_1 = J_2 + k$ for some odd number k , and
- 3) $x_{J_1} = x_{J_2} = y$,

we say that y is odd-twin in x . Define even-twin analogously. For example, in the case of the subshift $(\Omega^{(1)}, T)$, ab is odd-twin in $abaab$, and even-twin in $abab$.

Lemma 2.2.1 *For any $k \geq 1$, there exists $x \in \mathcal{L}(\Omega^{(s)})$ such that $|x| \leq 3|\mathcal{C}_s(k)|$ and $\mathcal{C}_s(k)$ is odd-twin in x .*

Proof: In the proof below, we simply write $\mathcal{C}_s(n)$ as $\mathcal{C}(n)$. Recall that $\mathcal{C}(n)$ satisfies the concatenation rule

$$\mathcal{C}(n+1) = \mathcal{C}(n)^s \mathcal{C}(n-1).$$

Therefore, it is easy to see that $\mathcal{C}(k)\mathcal{C}(k)$ and $\mathcal{C}(k)\mathcal{C}(k-1)\mathcal{C}(k)$ are both factors of $\mathcal{C}(k+3)$. If $|\mathcal{C}(k)|$ is odd, $x = \mathcal{C}(k)\mathcal{C}(k)$ satisfies the desired properties. Suppose $|\mathcal{C}(k)|$ is even. Note that the sequence

$$\{|\mathcal{C}(n)| \pmod{2}\}$$

repeats $1, 1, 1, 1, \dots$ if s is even, and $1, 0, 1, 1, 0, 1, \dots$ if s is odd. Since $|\mathcal{C}(k)|$ is even, s has to be odd. Therefore $|\mathcal{C}(k-1)|$ is odd, so $x = \mathcal{C}(k)\mathcal{C}(k-1)\mathcal{C}(k)$ satisfies the desired properties.

Lemma 2.2.2 *For every $s \in \mathbb{N}$, there exists a constant $K_s > 0$ such that for any $x, y \in \mathcal{L}(\Omega^{(s)})$, $K_s|y| < |x|$ implies y is odd-twin in x . Analogous results hold for even-twins.*

Proof: Let us show the statement for odd-twin. The latter statement is immediate. Let $K > 0$ be a number such that for any $x, y \in \mathcal{L}(\Omega^{(s)})$, y is a factor of x whenever $K|y| < |x|$. Let $y \in \mathcal{L}(\Omega^{(s)})$. Take $k > 0$ such that

$$|\mathcal{C}_s(k-1)| \leq K|y| < |\mathcal{C}_s(k)|.$$

Then y is a factor of $\mathcal{C}_s(k)$, and since $|\mathcal{C}_s(k)| < (s+1)|\mathcal{C}_s(k-1)|$, we have $|\mathcal{C}_s(k)| < (s+1)K|y|$. Therefore, by Lemma 2.2.1, there exists $v \in \mathcal{L}(\Omega^{(s)})$ such that $|v| < 3(s+1)K|y|$ and $\mathcal{C}_s(k)$ is odd-twin in v . Since y is a factor of $\mathcal{C}_s(k)$, y is odd-twin in v . Therefore,

$$K_s := K \cdot 3(s+1)K = 3(s+1)K^2$$

satisfies the desired properties.

2.2.4 The off-diagonal model

Let $a, b > 0$ be real numbers, and let s be a positive integer. Let $\omega \in \Omega_{a,b}^{(s)}$. We define a Jacobi matrix H_ω acting on $l^2(\mathbb{Z})$ by

$$(H_\omega \psi)(n) = \omega(n+1)\psi(n+1) + \omega(n)\psi(n-1),$$

and set

$$\lambda = \left| \frac{a^2 - b^2}{ab} \right|.$$

We call this λ the *coupling constant*. We only consider the case that $a > b$. The argument is completely analogous in the case $a < b$. By appropriate scaling, we can always assume $b = 1$. We assume this scaling all throughout this section. Note that this coincides with the definition (2.1.3). See also Remark 2.2.3 below. We call this family of self-adjoint operators $\{H_\omega\}$ the *off-diagonal model*. By a well known argument (minimality of the subshift and strong operator convergence), one can see that the spectrum of H_ω is independent of the specific choice of ω and depends only on λ and s .

Definition 2.2.3 We define the trace map T_s by

$$T_s = U^s \circ P,$$

where

$$U \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2xz - y \\ x \\ z \end{pmatrix} \quad \text{and} \quad P \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ z \\ y \end{pmatrix}.$$

Let ℓ_λ be the line given by

$$\ell_\lambda = \left\{ \left(\frac{E^2 - a^2 - 1}{2a}, \frac{E}{2a}, \frac{E}{2} \right) : E \in \mathbb{R} \right\},$$

and call this the *line of initial condition*. We define the map $J_\lambda(\cdot)$ by

$$(2.2.2) \quad J_\lambda : E \mapsto \ell_\lambda(E).$$

The function

$$G(x, y, z) = x^2 + y^2 + z^2 - 2xyz - 1$$

is invariant under the action of T_s and hence preserves the family of surfaces

$$S_V = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 - 2xyz - 1 = \frac{V^2}{4} \right\}.$$

It is easy to see that $\ell_\lambda \subset S_\lambda$.

The following can be proven by repeating the argument of [43].

Theorem 2.2.1 We have

$$\sigma(H_\omega) = \{E \in \mathbb{R} : \text{the forward semi-orbit of } J_\lambda(E) \text{ is bounded}\}.$$

Notice that it is clear from this theorem that the spectrum of H_ω depends only on λ and s . We denote it by Σ_λ below.

Remark 2.2.3 Let us define ℓ'_λ by

$$\ell'_\lambda = \left\{ \left(\frac{E^2 - \lambda E - 2}{2}, \frac{E - \lambda}{2}, \frac{E}{2} \right) : E \in \mathbb{R} \right\},$$

and the map $J'_\lambda(\cdot)$ by

$$J'_\lambda : E \mapsto \ell'_\lambda(E).$$

It is easy to see that $\ell'_0 = \ell_0$ and $\ell'_\lambda \subset S_\lambda$. Then, the exact same statement in Theorem 2.2.1 holds with $H_{\lambda,\beta}$ and J' instead of H_ω and J , respectively. See, for example, [11]. This is why we defined the coupling constant of off-diagonal model by (2.1.3).

By Theorem 2.2.1, we immediately get the following:

Corollary 2.2.1 *The spectrum Σ_λ contains 0.*

Proof: Note that

$$(2.2.3) \quad J_\lambda(0) = \left(-\frac{a^2 + 1}{2a}, 0, 0 \right).$$

It is easy to see that this point is periodic under the action of T_s .

In what follows we are going to use some notations and results from the theory of hyperbolic dynamical systems, see [30] for some background on this subject.

Let us denote by Λ_λ the set of points whose orbits are bounded under T_s . The following theorem was first proven in [62] for the Fibonacci Hamiltonian.

Theorem 2.2.2 ([4], see also **Theorem 4.1** from [42]) *The set Λ_λ is a compact locally maximal T_s -invariant transitive hyperbolic subset of S_λ , and the periodic points of T_s form a dense subset of Λ_λ .*

We also have the following:

Theorem 2.2.3 (Corollary 2.5 of [16]) *The forward semi-orbit of a point $p \in S_\lambda$ is bounded if and only if p lies in the stable lamination of Λ_λ .*

The following theorem was proven in [14] for the Fibonacci Hamiltonian case, and recently it was extended to tridiagonal Fibonacci Hamiltonians in [20]. It follows by repeating the argument of [14].

Theorem 2.2.4 *For all $\lambda > 0$, the intersections of the curve of initial condition ℓ_λ with the stable lamination is transverse.*

Corollary 2.2.2 *The spectrum Σ_λ is a dynamically defined Cantor set.*

Now we define the density of states measure. The definition is analogous for higher dimensional models.

Definition 2.2.4 *Denote by $H_\omega^{(N)}$ the restriction of H_ω to the interval $[0, N-1]$ with Dirichlet boundary conditions. The density of states measure ν_λ of H_ω is given by*

$$\nu_\lambda((-\infty, E]) = \lim_{N \rightarrow \infty} \frac{1}{N} \# \{ \text{eigenvalues of } H_\omega^{(N)} \text{ that are in } (-\infty, E] \},$$

where $E \in \mathbb{R}$.

The limit does not depend on the specific choice of ω , and depends only on λ and s . In fact, the convergence is uniform in ω . This was shown in a more general setting [35].

It is well known that $\Sigma_0 = [-2, 2]$, and

$$(2.2.4) \quad \nu_0((-\infty, E]) = \begin{cases} 0 & E \leq -2 \\ \frac{1}{\pi} \arccos(-\frac{E}{2}) & -2 < E < 2 \\ 1 & E \geq 2. \end{cases}$$

Let us write

$$\mathbb{S} = S_0 \cap \{(x, y, z) \in \mathbb{R}^3 \mid |x| \leq 1, |y| \leq 1, |z| \leq 1\}.$$

The trace map T_s restricted to \mathbb{S} is a factor of the hyperbolic automorphism \mathcal{A} of $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ given by

$$\mathcal{A} : \begin{pmatrix} \theta \\ \varphi \end{pmatrix} \mapsto \begin{pmatrix} s & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ \varphi \end{pmatrix}.$$

The semi-conjugacy is given by the map

$$F : (\theta, \varphi) \mapsto (\cos 2\pi(\theta + \varphi), \cos 2\pi\theta, \cos 2\pi\varphi).$$

A Markov partition of $\mathcal{A} : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ when $s = 1$ is shown in Figure 2.2. Compare with Figure 5 from [5]. For other values of $s \in \mathbb{N}$, the only difference is the slope of

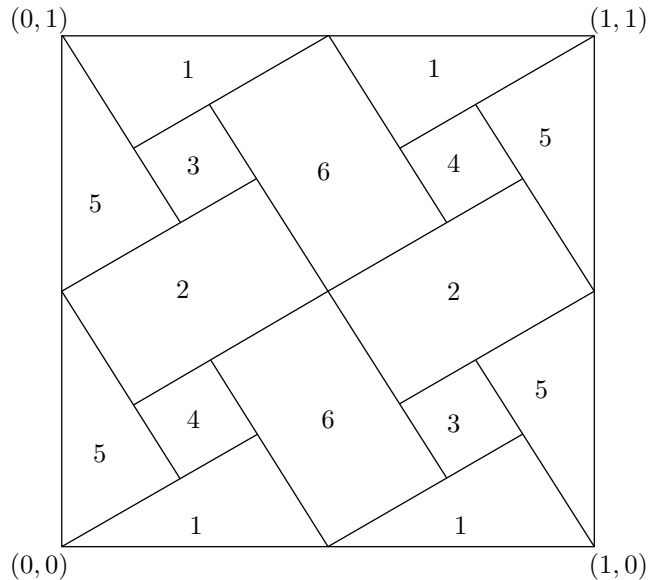


Figure 2.2: The Markov partition for the map \mathcal{A} .

the stable and unstable manifolds. Its image under the map $F : \mathbb{T}^2 \rightarrow \mathbb{S}$ is a Markov partition for the pseudo-Anosov map $T_s : \mathbb{S} \rightarrow \mathbb{S}$. Write

$$I = \{(t, t) \mid 0 \leq t \leq 1/2\}.$$

Note that $F(I) \subset \ell_0$. The following lemma is immediate:

Lemma 2.2.3 *The push-forward of the normalized Lebesgue measure on I under the semi-conjugacy F , which is a probability measure on $\ell_0 \cap \mathbb{S}$, corresponds to the free density of states measure (2.2.4) under the identification (2.2.2).*

Consider the union of elements of the Markov partition of \mathbb{T}^2 , 1, 2, 4, 5, and 6, as in Figure 2.3. Compare with Figure 3 from [12]. Let us denote the image of this union of elements under F by \mathcal{R}_0 , and the continuation of \mathcal{R}_0 in $\lambda > 0$ by \mathcal{R}_λ . The following statement can be proven by repeating the proof of Claim 3.2 of [12].

Proposition 2.2.1 *Consider the measure of maximal entropy of $T_s|_{\Lambda_\lambda}$ and restrict it to \mathcal{R}_λ . Normalize this measure and project it to ℓ_λ along the stable manifolds. Then, the resulting probability measure on ℓ_λ corresponds to the density of states measure ν_λ under the identification (2.2.2).*

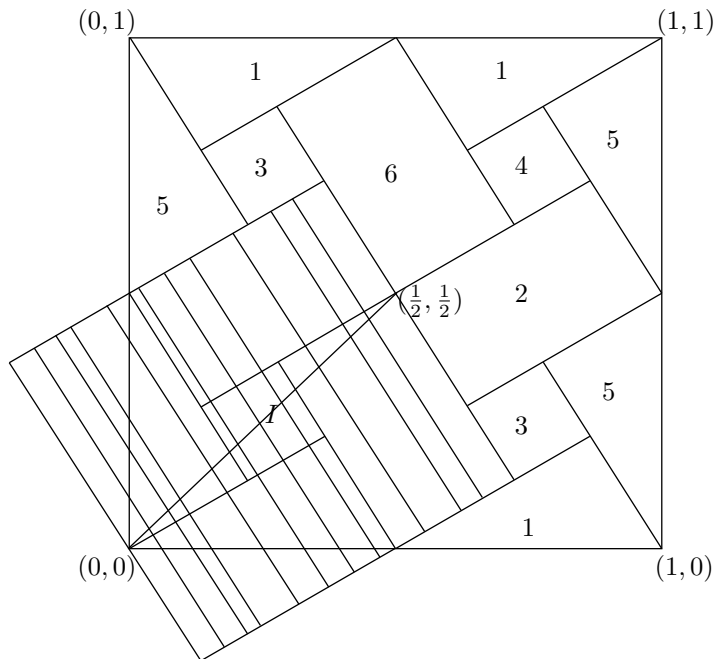


Figure 2.3: A Markov partition for the map \mathcal{A} , line segment I , and the stable manifolds.

This immediately implies the following:

Theorem 2.2.5 *For every $\lambda > 0$, the density of states measure ν_λ is exact-dimensional.*

That is, for ν_λ -almost every $E \in \mathbb{R}$, we have

$$\lim_{\epsilon \downarrow 0} \frac{\log \nu_\lambda(E - \epsilon, E + \epsilon)}{\log \epsilon} = d_\lambda,$$

where d_λ satisfies

$$\lim_{\lambda \downarrow 0} d_\lambda = 1.$$

Proof: The first claim is an immediate consequence of Proposition 2.2.1. The second claim follows verbatim from the repetition of Theorem 1.1 of [12].

We also have the following:

Proposition 2.2.2 ([51]) *The stable and unstable Lyapunov exponents are analytic functions of $\lambda > 0$.*

For any Cantor set K , we denote the *thickness* of K by $\tau(K)$. For the definition of thickness, see for example, chapter 4 of [50]. By Theorem 2 of [?] and by repeating the proof of Theorem 1.1 of [11], we obtain the following:

Theorem 2.2.6 *We have*

$$\lim_{\lambda \rightarrow \infty} \mathbf{dim}_H \Sigma_\lambda = 0, \text{ and } \lim_{\lambda \downarrow 0} \tau(\Sigma_\lambda) = \infty.$$

Proposition 2.2.3 *The density of states measure ν_λ is symmetric with respect to the origin. In particular, the spectrum Σ_λ is symmetric with respect to the origin.*

Proof: Denote by $H_\omega^{(N)}$ the restriction of H_ω to the interval $[0, N-1]$ with Dirichlet boundary conditions. Let ψ be an eigenvector of $H_\omega^{(N)}$ and E be the corresponding eigenvalue. Let us define $\phi \in l^2([0, N-1])$ by

$$\phi(n) = (-1)^n \psi(n) \quad (n = 0, 1, \dots, N-1).$$

Then, since

$$\begin{aligned} (H_\omega^{(N)}\phi)(n) &= \omega(n+1)\phi(n+1) + \omega(n)\phi(n-1) \\ &= (-1)^{n+1}\omega(n+1)\psi(n+1) + (-1)^{n-1}\omega(n)\psi(n-1) \\ &= (-1)^{n+1}E\psi(n) \\ &= -E\phi(n), \end{aligned}$$

$-E$ is also an eigenvalue of $H_\omega^{(N)}$. Therefore, the set of eigenvalues of $H_\omega^{(N)}$ is symmetric with respect to the origin. Therefore, for any interval $A \subset (0, \infty)$,

$$\begin{aligned} \nu_\lambda(A) &= \lim_{N \rightarrow \infty} \frac{1}{N} \# \{ \text{eigenvalues of } H_\omega^{(N)} \text{ that are in } A \} \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \# \{ \text{eigenvalues of } H_\omega^{(N)} \text{ that are in } (-A) \} \\ &= \nu_\lambda(-A). \end{aligned}$$

This concludes the first claim. Since the spectrum is the topological support of the density of states measure, the second claim also follows.

2.3 The Labyrinth Model

Let $a_i, b_i > 0$ ($i = 1, 2$) be real numbers, and let s be a positive integer. Let $\omega_i \in \Omega_{a_i, b_i}^{(s)}$ ($i = 1, 2$) and λ_i be the corresponding coupling constants.

2.3.1 The Labyrinth model

We define the Labyrinth model. Write

$$A^e = \{(m, n) \in \mathbb{Z}^2 \mid m + n \text{ is even}\}, \text{ and } A^o = \{(m, n) \in \mathbb{Z}^2 \mid m + n \text{ is odd}\}.$$

Using ω_1, ω_2 , we realign the lattices of A^e and A^o . See Figure 2.4. We denote this again by A^e and A^o (we use this identification freely). We define the operator $\hat{H}_{\omega_1, \omega_2}$, which acts on $l^2(A^e \cup A^o)$, by

$$\begin{aligned} \left[\hat{H}_{\omega_1, \omega_2} \psi \right] (m, n) &= \omega_1(m+1)\omega_2(n+1)\psi(m+1, n+1) \\ &\quad + \omega_1(m+1)\omega_2(n)\psi(m+1, n-1) \\ &\quad + \omega_1(m)\omega_2(n+1)\psi(m-1, n+1) \\ &\quad + \omega_1(m)\omega_2(n)\psi(m-1, n-1). \end{aligned}$$

Every lattice is connected diagonally and the strength of the bond is equal to the product of the sides of the rectangle. With appropriate scaling, we can always assume that $b_i = 1$ ($i = 1, 2$). We assume this scaling throughout this section. In a similar way, we define the operators $\hat{H}_{\omega_1, \omega_2}^e$ and $\hat{H}_{\omega_1, \omega_2}^o$, which act on $l^2(A^e)$ and $l^2(A^o)$ respectively. From here, we drop the subscripts ω_1, ω_2 if no confusion can arise. Note that

$$\hat{H} = \hat{H}^e \oplus \hat{H}^o.$$

It is natural to expect that the spectral properties of \hat{H}^e and \hat{H}^o are the same, and in fact, the spectra and the density of states measures coincide for the three operators. For the proof, we need the notion of *Delone dynamical systems*, and *linear repetitivity* of Delone dynamical systems. See, for example, [34].

Proposition 2.3.1 *Let us denote the density of states measures of \hat{H}^e , \hat{H}^o and \hat{H} by $\hat{\nu}^e$, $\hat{\nu}^o$ and $\hat{\nu}$, respectively. Then, $\hat{\nu}^e$, $\hat{\nu}^o$ and $\hat{\nu}$ define the same measure. In particular, the spectra of \hat{H}^e , \hat{H}^o and \hat{H} all coincide.*

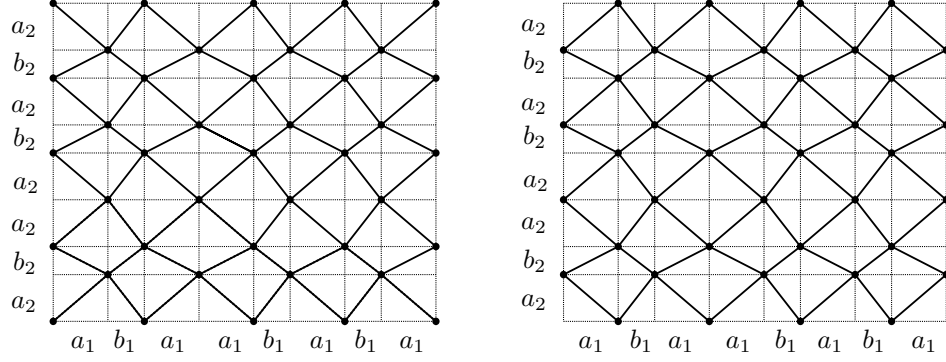


Figure 2.4: A^e (left) and A^o (right).

Proof: By Lemma 2.2.2, A^e and A^o are linearly repetitive. Therefore, by Theorem 6.1 of [34], Theorem 3 of [35] and Lemma 2.2.2, we have $\hat{\nu}^e = \hat{\nu}^o$. Since $\hat{H} = \hat{H}^e \oplus \hat{H}^o$, we get $\hat{\nu}^e = \hat{\nu}^o = \hat{\nu}$.

By this proposition, we will restrict our attention to \hat{H} below.

2.3.2 The spectrum of the Labyrinth model

We start by proving that the spectrum of $\hat{H}_{\omega_1, \omega_2}$ is given by the product of the spectra of off-diagonal models.

Proposition 2.3.2 *We have*

$$\sigma(\hat{H}_{\omega_1, \omega_2}) = \Sigma_{\lambda_1} \cdot \Sigma_{\lambda_2}.$$

In particular, the spectrum $\sigma(\hat{H}_{\omega_1, \omega_2})$ does not depend on particular choice of ω_1 and ω_2 and only depends on the coupling constants λ_1, λ_2 .

In the proof below, we simply write $H_{\omega_1}, H_{\omega_2}$ and $\hat{H}_{\omega_1, \omega_2}$ as H_1, H_2 and \hat{H} , respectively.

Proof: Let U be the unique unitary map from $l^2(\mathbb{Z}) \otimes l^2(\mathbb{Z})$ to $l^2(\mathbb{Z}^2)$ so that for $\psi_1, \psi_2 \in l^2(\mathbb{Z})$, the elementary tensor $\psi_1 \otimes \psi_2$ is mapped to the element ψ of $l^2(\mathbb{Z}^2)$

given by $\psi(m, n) = \psi_1(m)\psi_2(n)$. We have

$$\begin{aligned}
\left[\hat{H}U(\psi_1 \otimes \psi_2) \right] (m, n) &= \omega_1(m+1)\omega_2(n+1)\psi_1(m+1)\psi_2(n+1) \\
&\quad + \omega_1(m+1)\omega_2(n)\psi_1(m+1)\psi_2(n-1) \\
&\quad + \omega_1(m)\omega_2(n+1)\psi_1(m-1)\psi_2(n+1) \\
&\quad + \omega_1(m)\omega_2(n)\psi_1(m-1)\psi_2(n-1) \\
&= [H_1\psi_1] (m) [H_2\psi_2] (n) \\
&= [U(H_1 \otimes H_2)(\psi_1 \otimes \psi_2)] (m, n),
\end{aligned}$$

for all $(m, n) \in \mathbb{Z}^2$. Therefore,

$$(U^* \hat{H}U)(\psi_1 \otimes \psi_2) = (H_1 \otimes H_2)(\psi_1 \otimes \psi_2).$$

Since the linear combinations of elementary tensors are dense in $l^2(\mathbb{Z}) \otimes l^2(\mathbb{Z})$, we get $U^* \hat{H}U = H_1 \otimes H_2$. Therefore, the result follows from Theorem VIII 33 of [54].

Let us denote the spectrum of $\hat{H}_{\omega_1, \omega_2}$ by $\hat{\Sigma}_{\lambda_1, \lambda_2}$.

Proof: [Proof of Theorem 2.1.1] By Theorem 2.2.6, we have

$$\tau(\Sigma_{\lambda_1}), \tau(\Sigma_{\lambda_2}) > 1 + \sqrt{2}$$

for sufficiently small coupling constants. Therefore, by Theorem 1.4 of [69], $\Sigma_{\lambda_1} \cdot \Sigma_{\lambda_2}$ is an interval for sufficiently small λ_1, λ_2 . Combining this with Proposition 2.3.2, $\hat{\Sigma}_{\lambda_1, \lambda_2}$ is an interval for sufficiently small coupling constants.

By Theorem 2.2.6, we get

$$\mathbf{dim}_H \Sigma_{\lambda_1} + \mathbf{dim}_H \Sigma_{\lambda_2} < 1$$

for sufficiently large λ_1, λ_2 . Notice that, by the symmetry of Σ_{λ_1} and Σ_{λ_2} , we have

$$\hat{\Sigma}_{\lambda_1, \lambda_2}^+ = \exp(\log \Sigma_{\lambda_1}^+ + \log \Sigma_{\lambda_2}^+),$$

where A^+ denotes $A \cap (0, \infty)$. Therefore, since

$$\mathbf{dim}_H \log \Sigma_{\lambda_i}^+ = \mathbf{dim}_H \Sigma_{\lambda_i} \quad (i = 1, 2),$$

by Proposition 1 of chapter 4 in [50], $\log \Sigma_{\lambda_1}^+ + \log \Sigma_{\lambda_2}^+$ has zero Lebesgue measure. Hence, for sufficiently large λ_1, λ_2 , $\hat{\Sigma}_{\lambda_1, \lambda_2}$ is a Cantor set of zero Lebesgue measure.

2.3.3 Density of states measure of the Labyrinth model

In this section, we will prove Theorem 2.1.2. If there is no chance of confusion, we simply write the density of states measures of \hat{H} , H_1 and H_2 as $\hat{\nu}$, ν_1 and ν_2 , respectively.

Proof: [proof of (2.1.5)] The proof is essentially the repetition of the proof of Proposition A.3 of [13]. For the reader's convenience, we will repeat the argument.

Denote by $H_j^{(N)}$ ($j = 1, 2$) the restriction of H_j to the interval $[0, N - 1]$ with Dirichlet boundary conditions. Denote the corresponding eigenvalues and eigenvectors by $E_{j,k}^{(N)}$, $\phi_{j,k}^{(N)}$, where $j = 1, 2$ and $1 \leq k \leq N$. Recall that we have

$$(2.3.1) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \# \left\{ 1 \leq k \leq N \mid E_{j,k}^{(N)} \in (-\infty, E] \right\} = \nu_j((-\infty, E])$$

for $E \in \mathbb{R}$.

Similarly, we denote by $\hat{H}^{(N)}$ the restriction of \hat{H} to $[0, N - 1]^2$ with Dirichlet boundary conditions. Denote the corresponding eigenvalues and eigenvectors by $E_k^{(N)}$, $\phi_k^{(N)}$ ($1 \leq k \leq N^2$). Then, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \# \left\{ 1 \leq k \leq N^2 \mid E_k^{(N)} \in (-\infty, E] \right\} = \hat{\nu}((-\infty, E]).$$

The eigenvectors $\phi_{j,k}^{(N)}$ of $H_j^{(N)}$ form an orthonormal basis of $l^2([0, N - 1])$. Thus, the associated elementary tensors

$$(2.3.2) \quad \phi_{1,k_1}^{(N)} \otimes \phi_{2,k_2}^{(N)} \quad (1 \leq k_1, k_2 \leq N)$$

form an orthonormal basis of $l^2([0, N - 1]) \otimes l^2([0, N - 1])$, which is canonically isomorphic to $l^2([0, N - 1]^2)$. Moreover, the vector in (2.3.2) is an eigenvector of $\hat{H}^{(N)}$, corresponding to the eigenvalue $E_{1,k_1}^{(N)} \cdot E_{2,k_2}^{(N)}$. By counting dimensions, these eigenvalues exhaust the entire set $\left\{ E_k^{(N)} \mid 1 \leq k \leq N^2 \right\}$. Therefore, for $E \in \mathbb{R}$,

$$\# \left\{ 1 \leq k \leq N^2 \mid E_k^{(N)} \in (-\infty, E] \right\} = \# \left\{ 1 \leq k_1, k_2 \leq N \mid E_{1,k_1}^{(N)} \cdot E_{2,k_2}^{(N)} \in (-\infty, E] \right\}.$$

Let $\nu_j^{(N)}$ ($j = 1, 2$) be the probability measures on \mathbb{R} with $\nu_j^{(N)}(E_{j,k}^{(N)}) = 1/N$ ($k = 1, 2, \dots, N$). Similarly, Let $\hat{\nu}^{(N)}$ be the probability measure on \mathbb{R} with $\hat{\nu}^{(N)}(E_k^{(N)}) =$

$1/N^2$ ($k = 1, 2, \dots, N^2$). Then, by the above argument, we get

$$\hat{\nu}^{(N)}((-\infty, E]) = \iint_{\mathbb{R}^2} \chi_{(-\infty, E]}(xy) d\nu_1^{(N)}(x) d\nu_2^{(N)}(y).$$

By (2.3.1), $\nu_i^{(N)}$ converges weakly to ν_i (see, for example, chapter 13 of [31]). Therefore, $\nu_1^{(N)} \times \nu_2^{(N)}$ converges weakly to $\nu_1 \times \nu_2$. By Theorem 13.16 of [31], we have

$$\lim_{N \rightarrow \infty} \iint_{\mathbb{R}^2} \chi_{(-\infty, E]}(xy) d\nu_1^{(N)}(x) d\nu_2^{(N)}(y) = \iint_{\mathbb{R}^2} \chi_{(-\infty, E]}(xy) d\nu_1(x) d\nu_2(y).$$

The result follows from this.

Let us define Borel measures $\bar{\nu}_i$ ($i = 1, 2$) on \mathbb{R} by

$$\bar{\nu}_i(A) = \nu_i(e^A),$$

where $A \subset \mathbb{R}$ is a Borel set. Then, the following holds.

Lemma 2.3.1 *The density of states measure of the Labyrinth model $\hat{\nu}$ is given by*

$$\hat{\nu}(A) = 2 \{ (\bar{\nu}_1 * \bar{\nu}_2)(\log A^+) + (\bar{\nu}_1 * \bar{\nu}_2)(\log A^-) \},$$

where A is a Borel set, and $A^+ = A \cap (0, \infty)$ and $A^- = (-A) \cap (0, \infty)$.

Proof: Let $A \subset (0, \infty)$ be a Borel set. Using Fubini's Theorem and change of coordinates, we get

$$\begin{aligned} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \chi_A(xy) d\nu_1(x) d\nu_2(y) &= \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^+} \chi_A(xy) d\nu_1(x) \right) d\nu_2(y) \\ &= \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}} \chi_A(e^x y) d\bar{\nu}_1(x) \right) d\nu_2(y) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_A(e^x e^y) d\bar{\nu}_1(x) d\bar{\nu}_2(y) \\ &= \iint_{\mathbb{R}^2} \chi_{\log A}(x + y) d\bar{\nu}_1(x) d\bar{\nu}_2(y) \\ &= (\bar{\nu}_1 * \bar{\nu}_2)(\log A). \end{aligned}$$

Combining this with Proposition 2.2.3, the result follows. Therefore, the absolute continuity of $\hat{\nu}$ is equivalent to the absolute continuity of $\bar{\nu}_1 * \bar{\nu}_2$.

Theorem 3.2 from [13] implies the following:

Theorem 2.3.1 *Let $\mathcal{J} \subset \mathbb{R}$ be an interval. Assume that for $\lambda \in \mathcal{J}$, ν_λ is the density of states measure of H_λ . Let $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ be a diffeomorphism, and define a Borel measure μ_λ by*

$$\mu_\lambda(A) = \nu_\lambda(\gamma(A)),$$

where $A \subset \mathbb{R}$ is a Borel set. Then, for any compactly supported exact-dimensional measure η on \mathbb{R} with

$$\mathbf{dim}_H \eta + \mathbf{dim}_H \mu_\lambda > 1$$

*for all $\lambda \in \mathcal{J}$, the convolution $\eta * \mu_\lambda$ is absolutely continuous with respect to Lebesgue measure for almost every $\lambda \in \mathcal{J}$.*

Proof: [proof of theorem 2.1.2] By Theorem 2.1.1, $\hat{\Sigma}_{\lambda_1, \lambda_2}$ is a Cantor set of zero Lebesgue measure for sufficiently large coupling constants. Therefore, since the density of states measure $\hat{\nu}_{\lambda_1, \lambda_2}$ is supported on $\hat{\Sigma}_{\lambda_1, \lambda_2}$, it has to be singular continuous for sufficiently large λ_1, λ_2 .

By Theorem 3.2.10, there exists $\lambda^* > 0$ such that $\mathbf{dim}_H \nu_\lambda > \frac{1}{2}$ for all $\lambda \in [0, \lambda^*)$. Recall that $0 \in \Sigma_\lambda$. Recall also that Σ_λ is the set of intersections between the stable laminations and the line ℓ_λ . Therefore, since the stable laminations and ℓ_λ both depend smoothly on λ , we can write

$$\Sigma_{\lambda_i} \cap (0, \infty) = \bigsqcup_{n=1}^{\infty} K_n^{(i)}(\lambda_i) \quad (i = 1, 2),$$

where $K_n^{(i)}(\lambda_i)$ are Cantor sets which depend naturally on λ_i . Let us define Borel measures $\bar{\nu}_{\lambda_i}^{(n)}$ ($i = 1, 2, n \in \mathbb{N}$) by

$$\bar{\nu}_{\lambda_i}^{(n)}(A) = \nu_{\lambda_i}|_{K_n^{(i)}(\lambda_i)}(e^A),$$

where $A \subset \mathbb{R}$ is a Borel set. Then, by Theorem 2.3.1, for each $(m, n) \in \mathbb{N} \times \mathbb{N}$, $\bar{\nu}_{\lambda_1}^{(m)} * \bar{\nu}_{\lambda_2}^{(n)}$ is absolutely continuous for almost all (λ_1, λ_2) . This implies that $\bar{\nu}_{\lambda_1} * \bar{\nu}_{\lambda_2}$ is absolutely continuous for almost all (λ_1, λ_2) .

Chapter 3

Mixed spectral regimes for square Fibonacci Hamiltonian

3.1 Introduction

This is a joint work with J. Fillman and W. Yessen. The purpose of this work is to investigate an interesting phenomenon of mixed interval-Cantor spectra occurring in some natural higher-dimensional versions of the one-dimensional substitution models. More precisely, as our models we chose the square of the extensively studied Fibonacci Hamiltonian and its generalizations (compare [13, 14, 43]). The reason for this choice is three-fold: (1) the one-dimensional versions have been extensively studied as prototypical examples of one-dimensional quasicrystals and the square models present a natural next step towards honest models of higher-dimensional quasicrystals; (2) the square models have also been considered in a physical setting [29, 38, 39, 40], and while numerical studies have appeared, analytical results are scarce (to the best of our knowledge, the only comprehensive work in this direction to date is [13]); (3) while the more commonly accepted models of two-dimensional quasicrystals (such as those based on the Penrose tiling) are currently out of reach, the square models are amenable to some of the modern tools. Our techniques are based on spectral theory, smooth dynamical systems, and geometric measure theory. Our techniques are equally applicable to models based on any two-letter primitive invertible substitution

(compare [?]).

3.1.1 Models and main results

Define the parameter space \mathcal{R} by

$$(3.1.1) \quad \mathcal{R} = \{(\mathbf{p}, \mathbf{q}) \in \mathbb{R}^2 : \mathbf{p} \neq 0\}.$$

Given $\theta \in \mathbb{T} \stackrel{\text{def}}{=} \mathbb{R}/\mathbb{Z}$, we define a Sturmian sequence $\{\omega_n\}$ via

$$\omega_n \stackrel{\text{def}}{=} \chi_{[1-\alpha, 1)}(n\alpha + \theta \pmod{1}), \quad n \in \mathbb{Z},$$

where $\alpha = \frac{\sqrt{5}-1}{2}$, the inverse of the golden mean. Given $\lambda = (\mathbf{p}, \mathbf{q}) \in \mathcal{R}$, we define the coefficients $\{p_n\}$ and $\{q_n\}$ by

$$(3.1.2) \quad p_n \stackrel{\text{def}}{=} (\mathbf{p} - 1)\omega_n + 1 = \begin{cases} 1 & \text{if } \omega_n = 0 \\ \mathbf{p} & \text{if } \omega_n = 1 \end{cases} \quad \text{and} \quad q_n \stackrel{\text{def}}{=} \mathbf{q}\omega_n,$$

and then the operator H_λ is defined on $\ell^2(\mathbb{Z})$ via

$$(3.1.3) \quad (H_\lambda \phi)_n = p_{n+1}\phi_{n+1} + p_n\phi_{n-1} + q_n\phi_n.$$

Then the so-called *square Hamiltonian*, which acts on $\ell^2(\mathbb{Z}^2)$, is given by

$$(3.1.4) \quad (H_{(\lambda_1, \lambda_2)}^2 \phi)_{n,m} = (H_{\lambda_1} \phi(\cdot, m))_n + (H_{\lambda_2} \phi(n, \cdot))_m,$$

where $\phi(\cdot, m)$ and $\phi(n, \cdot)$ denote elements of $\ell^2(\mathbb{Z})$ defined by $n \mapsto \phi(n, m)$ and $m \mapsto \phi(n, m)$, respectively. Equivalently, $\ell^2(\mathbb{Z}^2)$ is canonically isomorphic to the tensor product $\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$; under the canonical identification, $H_{(\lambda_1, \lambda_2)} \cong H_{\lambda_1} \otimes I_2 + I_1 \otimes H_{\lambda_2}$, where I_j denotes the identity operator acting on the j th factor of the tensor product.

It is known that the spectrum of the operator H_λ is independent of the choice of $\theta \in \mathbb{T}$, hence its suppression in the notation. The spectrum, the spectral measures, and the density of states measure of this operator have been studied in the context of electronic transport properties of quasicrystals, as well as their magnetic properties (see [72, 73], and the survey [43]); this operator is also particularly attractive as a generalization of the extensively studied Fibonacci Schrödinger operator (see [7, 14]

and references therein). It is known that the spectrum of H_λ is a Cantor set of zero Lebesgue measure whenever it is different from the free Schrödinger operator, i.e., whenever $\lambda \in \mathcal{R} \setminus \{(1, 0)\}$ [72, Theorem 2.1]. Moreover, in this case, the spectral measures and the density of states measure are purely singular continuous. It is worth noting that in contrast to the Schrödinger Fibonacci Hamiltonian (i.e. the case in which $\mathfrak{p} = 1$), the spectrum of H_λ in general exhibits rich multifractal structure (compare [14, Theorem 1.1] and [72, Theorem 2.3]).

Remark 3.1.1 *Notice certain geometric restrictions in [72, Theorem 2.3]. Similar restrictions in the Schrödinger case were recently lifted in [14]. In this paper, we lift those restrictions in full generality (Theorem 3.2.1 below), so that [72, Theorem 2.3] holds without any restrictions on the parameters.*

Like in the one-dimensional case, the interest in the square model is driven by the desire to understand fine quantum-dynamical properties of two-dimensional quasicrystals. Indeed, the square model was proposed as a model that is simpler than the commonly accepted model for a two-dimensional quasicrystal, and unlike the general model, the square model is susceptible of a rigorous study since its spectral data can be expressed in terms of the spectral data of the one-dimensional model. Moreover, like the general model, the square model is physically motivated (e.g. [29, 38, 39, 40]). However, even this simplified case presents substantial challenges, and some of the resulting problems present purely mathematical interest (e.g. sums of Cantor sets and convolutions of measures defined on them; see the introduction in [13] for a deeper discussion and references).

Let us denote the spectrum of H_λ by Σ_λ , and that of $H_{(\lambda_1, \lambda_2)}^2$ by $\Sigma_{(\lambda_1, \lambda_2)}^2$. By general spectral-theoretic arguments (see [11, Appendix A], for example),

$$\Sigma_{(\lambda_1, \lambda_2)}^2 = \Sigma_{\lambda_1} + \Sigma_{\lambda_2} \stackrel{\text{def}}{=} \{a + b : a \in \Sigma_{\lambda_1}, b \in \Sigma_{\lambda_2}\},$$

and the relevant measures supported on $\Sigma_{(\lambda_1, \lambda_2)}^2$ (i.e. the spectral and the density of states measures) are convolutions of the respective measures of the one-dimensional Hamiltonian.

Regarding the topology of $\Sigma_{(\lambda_1, \lambda_2)}^2$, some partial results are available. For example, if $\lambda_1 = (1, \mathbf{q}_1)$ and $\lambda_2 = (1, \mathbf{q}_2)$ with \mathbf{q}_i , $i = 1, 2$, sufficiently close to zero, $\Sigma_{(\lambda_1, \lambda_2)}^2$ is an interval, while for all \mathbf{q}_i , $i = 1, 2$, sufficiently large, $\Sigma_{(\lambda_1, \lambda_2)}^2$ is a Cantor set of Hausdorff dimension strictly smaller than one, and hence also of zero Lebesgue measure [8, 11]. These properties are established via the dynamical properties of the associated trace map (we recall the basics in Section 3.2.1; see [7] for details). Similar results hold for parameters of the form $(\mathbf{p}, 0)$ ([11, Appendix A] shows that all the spectral properties of the operators $H_{(1, \mathbf{q})}$ that can be obtained from the trace map dynamics can also be obtained via the same methods, without modification, for the operator of the form $H_{(\mathbf{p}, 0)}$). The topological picture in the intermediate regimes is currently unclear.

Let us adopt the following terminology for convenience: a set $A \subset \mathbb{R}$ is said to be *interval-Cantor mixed* provided that A has nonempty interior, and for some $a, b \in A$, $a < b$, $[a, b] \cap A$ is a (nonempty) Cantor set. In this paper, we prove

Theorem 3.1.1 *There exists a (nonempty) open set $\mathcal{U} \subset \mathcal{R}$ such that for every $\lambda_1, \lambda_2 \in \mathcal{U}$, $\Sigma_{(\lambda_1, \lambda_2)}^2$ is interval-Cantor mixed.*

Remark 3.1.2 *We suspect that for all $\lambda_1, \lambda_2 \in \mathcal{R} \setminus \{(1, 0)\}$ sufficiently close to $(1, 0)$, $\Sigma_{(\lambda_1, \lambda_2)}^2$ is an interval; at present, however, our techniques only show that $\Sigma_{(\lambda_1, \lambda_2)}^2$ is a union of compact intervals. More details are given in Remark 3.2.2.*

In [13] the authors proved that for almost all pairs $(\mathbf{q}_1, \mathbf{q}_2)$ sufficiently close to $(0, 0)$, the convolution of the density of states measures of the Hamiltonians $H_{(1, \mathbf{q}_1)}$ and $H_{(1, \mathbf{q}_2)}$, supported on $\Sigma_{(1, \mathbf{q}_1)} + \Sigma_{(1, \mathbf{q}_2)}$, is absolutely continuous. In this paper, using the aforementioned multifractality of Σ_λ and appealing to the techniques of [13], we prove the following measure-theoretic analog of Theorem 3.1.1.

Denote the density of states measure for the Hamiltonian $H_{(\lambda_1, \lambda_2)}^2$ by

$$(3.1.5) \quad dk_{(\lambda_1, \lambda_2)}^2 = dk_{\lambda_1} * dk_{\lambda_2},$$

where dk_λ is the density of states measure for H_λ (see, for example, [63, Chapter 5] for definitions), and $*$ is the convolution of measures (see [13, Appendix A] for the

relation (3.1.5)). For $\lambda_1, \lambda_2 \in \mathcal{R}$, let us denote by $(dk_{(\lambda_1, \lambda_2)}^2)_{ac}$ and $(dk_{(\lambda_1, \lambda_2)}^2)_{sc}$ the absolutely continuous and the singular continuous components (with respect to Lebesgue measure) of $dk_{(\lambda_1, \lambda_2)}^2$, respectively.

Theorem 3.1.2 *There exists a (nonempty) open set $\mathcal{U} \subset \mathcal{R}$ such that for every $\lambda_1 \in \mathcal{U}$, there exists a full-measure subset $\mathcal{V} = \mathcal{V}(\lambda_1) \subseteq \mathcal{U}$ with the property that $(dk_{(\lambda_1, \lambda_2)}^2)_{ac} \neq 0$ and $(dk_{(\lambda_1, \lambda_2)}^2)_{sc} \neq 0$ whenever $\lambda_2 \in \mathcal{V}$.*

Remark 3.1.3 *A nonempty open $\mathcal{U} \subset \mathcal{R}$ can be chosen so as to satisfy Theorems 3.1.1 and 3.1.2 simultaneously; see Remark 3.2.4 below.*

We suspect that one can choose \mathcal{U} in such a way that the conclusion of Theorem 3.1.2 holds for all pairs $\lambda_1, \lambda_2 \in \mathcal{U}$, not just a full-measure set, but our proof does not yield this stronger conclusion.

As another application of the methods of this paper, we can also construct separable two-dimensional continuum quasicrystal models which exhibit mixed topological structures in the spectrum. To construct an explicit example of a continuum quasicrystal model, take $\lambda > 0$ (note the change in parameter space) and $\theta \in \mathbb{T}$. Define ω_n as before, take

$$V_\lambda(x) = \sum_{n \in \mathbb{Z}} \lambda \omega_n \chi_{[n, n+1)}(x),$$

and define the self-adjoint operator H_λ on $L^2(\mathbb{R})$ by

$$H_\lambda \phi = -\phi'' + V_\lambda \phi.$$

Models of this type were considered in [9]. By the results therein, we know that $\sigma(H_\lambda)$ does not depend on θ and is a (noncompact) Cantor set of zero Lebesgue measure for all $\lambda > 0$. We denote this common spectrum by Σ_λ . As before, we define the square operator on $L^2(\mathbb{R}^2) \cong L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ by $H_{(\lambda_1, \lambda_2)}^2 = H_{\lambda_1} \otimes I_2 + I_1 \otimes H_{\lambda_2}$; the spectrum of the square Hamiltonian is given by $\Sigma_{(\lambda_1, \lambda_2)}^2 = \Sigma_{\lambda_1} + \Sigma_{\lambda_2}$, as above.

Theorem 3.1.3 *For all $\lambda_1, \lambda_2 > 0$, the interior of $\Sigma_{(\lambda_1, \lambda_2)}^2$ is nonempty. Moreover, for all λ_1, λ_2 sufficiently large, there exists an interval I_0 such that $I_0 \cap \Sigma_{(\lambda_1, \lambda_2)}^2$ is a nonempty Cantor set.*

Remark 3.1.4 *A few remarks are in order here.*

1. *We suspect that more is true, namely, for all $\lambda_1, \lambda_2 > 0$, there exists $E_1 = E_1(\lambda_1), E_2 = E_2(\lambda_2) > 0$ such that $\Sigma_{(\lambda_1, \lambda_2)}^2$ contains the ray $[E_1 + E_2, \infty)$, but our methods do not prove this. If this is the case, the small-coupling behavior of E_i is of interest – for example, does one have $E_i(\lambda_i) \rightarrow 0$ as $\lambda_i \rightarrow 0$? Some details on this are contained in Remark 3.2.7.*
2. *It follows directly from the results of [9] that for all $\lambda \geq 0$, the Hausdorff dimension of Σ_λ is equal to one. It is natural to ask where in the spectrum the Hausdorff dimension accumulates. For example, in the discrete Jacobi setting, the Hausdorff dimension may accumulate only at one of the extrema of the spectrum [72]. In the continuum case, it turns out, the Hausdorff dimension accumulates at infinity but, for certain values of λ , it may also accumulate at finitely many points in the spectrum. A more extensive discussion is given in Remark 3.2.5.*
3. *In the continuum quasicrystal setting, there is no need to restrict to locally constant potential pieces. We do this so that we can control the location of the ground state and appeal to explicit expressions for the Fricke–Vogt invariant, but it is interesting to ask whether general results of this type hold for any choice of L_{loc}^2 continuum potential modelled on the Fibonacci subshift.*

3.2 Proof of Theorems 3.1.1, 3.1.2, and 3.1.3

Our proof relies on the dynamics of the associated renormalization map, the Fibonacci trace map. Detailed discussions on the dynamics of this map are contained in [4, 5, 10, 55, 56, 57] and references therein, and are complemented by [73, Section 4.2]. We use freely standard notions from the theory of (partially) hyperbolic dynamical systems; [23, 24] can serve as comprehensive references, while sufficient background is also given in [10] and the appendices of [73]. Let us briefly describe our approach.

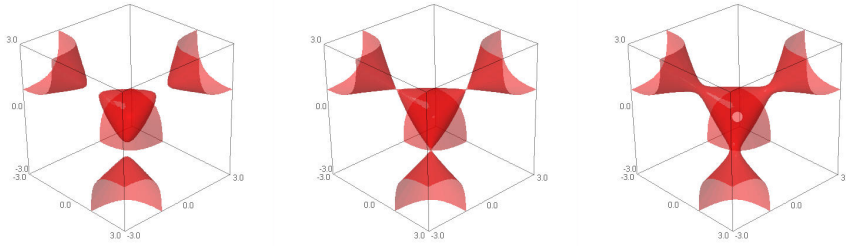


Figure 3.1: Plots of the Fricke-Vogt invariant surfaces for a few values of V .

3.2.1 Background

Given $\lambda = (\mathbf{p}, \mathbf{q}) \in \mathcal{R}$, define

$$(3.2.1) \quad \ell_\lambda(E) \stackrel{\text{def}}{=} \left(\frac{E - \mathbf{q}}{2}, \frac{E}{2\mathbf{p}}, \frac{1 + \mathbf{p}^2}{2\mathbf{p}} \right), \quad E \in \mathbb{R}.$$

In what follows, we use the notation ℓ_λ to also denote the image of \mathbb{R} under ℓ_λ , that is, the line $\ell_\lambda(\mathbb{R})$.

Given $x, y, z \in \mathbb{R}$, define the so-called Fricke-Vogt invariant by

$$(3.2.2) \quad I(x, y, z) \stackrel{\text{def}}{=} x^2 + y^2 + z^2 - 2xyz - 1,$$

and, for $V \geq 0$, consider its level surfaces

$$(3.2.3) \quad S_V \stackrel{\text{def}}{=} \{(x, y, z) \in \mathbb{R}^3 : I(x, y, z) = V\}$$

(see Figure 3.1 for some plots). Finally, define the *Fibonacci trace map* $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by

$$f(x, y, z) = (2xy - z, x, y).$$

It is easily verified that $f(S_V) = S_V$ for every V . Furthermore, for every $V > 0$, $f|_{S_V}$ is an Axiom A diffeomorphism [4, 5, 10]. It is known that $E \in \mathbb{R}$ is in the spectrum Σ_λ if and only if the forward orbit of $\ell_\lambda(E)$, $\{f^n(\ell_\lambda(E))\}_{n \in \mathbb{N}}$, is bounded (see the proof of Theorem 2.1 in [72], which is a generalization of [62]). While there exist $E \in \mathbb{R}$ with $\ell_\lambda(E) \notin S_V$ for any $V \geq 0$, it turns out that such E do not belong to Σ_λ (see Section 3.2.1 in [72]). Denoting by Ω_V the nonwandering set of $f|_{S_V}$, we have

that $\bigcup_{V>0} \Omega_V$ is partially hyperbolic (see [73, Proposition 4.10] for the details). It is then proved that $\ell_\lambda(\Sigma_\lambda)$ is precisely the intersection set of ℓ_λ with the center-stable lamination [72].

This implies that Σ_λ is Cantor set with quite rich multifractal structure; see [72, Theorem 2.3]. It is this multifractality that we exploit to prove Theorems 3.1.1, 3.1.2, and 3.1.3.

Many of the results of the aforementioned works suffer from certain geometric restrictions. Namely, those results were established only for $\lambda \in \mathcal{R}$, where the line ℓ_λ intersects the center-stable lamination transversally. On the other hand, transversality was established for some rather restricted values of the parameters $\lambda \in \mathcal{R}$ (see [43, 72]). Recently, transversality has been extended to all parameters of the form $(1, \mathbf{q})$ in [14], and the case for $(\mathbf{p}, 0)$ follows from [14] without modification to the proofs (combine with the appendix in [11]). In this paper we extend transversality to *all values* $(\mathbf{p}, \mathbf{q}) \in \mathcal{R}$ as the following

Theorem 3.2.1 *For all $\lambda \in \mathcal{R}$ with $\lambda \neq (1, 0)$, the line ℓ_λ intersects the center-stable lamination of the Fibonacci trace map transversally.*

As a consequence, the said geometric restrictions can be lifted for all previous results (many of which are surveyed in [43], and some for other models appear in [16, 17, 73]; see also [14]).

Proof: [Proof of Theorem 3.2.1] It is easy to verify that for all $\mathbf{q} \in \mathbb{R}$, $\ell_{(1, \mathbf{q})}$ lies entirely in $S_{\frac{\mathbf{q}^2}{4}}$, and it is known from [14] that in this case $\ell_{(1, \mathbf{q})}$ intersects the stable lamination on $S_{\frac{\mathbf{q}^2}{4}}$ transversally. From (3.2.7), we see that for all $\mathbf{p} \neq 0$, the line $\ell_{(\mathbf{p}, 0)}$ lies on the surface $S_{V_{(\mathbf{p}, 0)}}$, where $V_{(\mathbf{p}, 0)} = \frac{(\mathbf{p}-1)^2}{4\mathbf{p}^2}$. The arguments from [14, Section 2] apply without modification to show that for all $\mathbf{p} \neq 0$, $\ell_{(\mathbf{p}, 0)}$ intersects the stable lamination on $S_{V_{(\mathbf{p}, 0)}}$ transversally. On the other hand, it is known from [73, Proposition 4.10] that the center-stable manifolds intersect the surfaces $\{S_V\}_{V>0}$ transversally (the intersection of the center-stable manifolds with S_V gives the stable lamination on S_V). Since the points in the intersection of ℓ_λ with the center-stable manifolds

correspond to the points in the spectrum, and since the spectrum is compact, it follows by continuity that for each fixed $\mathbf{p}_0 \notin \{0, 1\}$, there exists $\delta > 0$ such that for all $\mathbf{q} \in (-\delta, \delta)$, $\ell_{(\mathbf{p}_0, \mathbf{q})}$ intersects the center-stable manifolds transversally; similarly, for each fixed $\mathbf{q}_0 \in \mathbb{R}$, there exists $0 < \delta < 1$ such that for all $\mathbf{p} \in (1 - \delta, 1 + \delta)$, $\ell_{(\mathbf{p}, \mathbf{q}_0)}$ intersects the center-stable manifolds transversally (compare with [72, Theorem 2.5]).

Let us argue by contradiction that tangencies cannot occur. Fix $\mathbf{p}_0 \notin \{0, 1\}$. Let us assume that $\mathbf{q}_0 > 0$ (respectively, $\mathbf{q}_0 < 0$) is such that for all $\mathbf{q} \in [0, \mathbf{q}_0)$ (respectively, $\mathbf{q} \in (\mathbf{q}_0, 0]$), $\ell_{(\mathbf{p}_0, \mathbf{q})}$ intersects the center-stable manifolds transversally, while $\ell_{(\mathbf{p}_0, \mathbf{q}_0)}$ is tangent to some center-stable manifold. In what follows, let us assume without loss of generality, that $\mathbf{q}_0 > 0$.

Denote by p a point of tangency between $\ell_{(\mathbf{p}_0, \mathbf{q}_0)}$ and a center-stable manifold. We claim that $p \notin S_0$. Indeed, it is easy to see from (3.2.7) that for all $\lambda \in \mathcal{R}$ with $\lambda \neq (1, \mathbf{q})$ and $\lambda \neq (\mathbf{p}, 0)$, ℓ_λ intersects S_V transversally for every V (differentiate (3.2.7) with respect to E and notice that the result is zero if and only if $\mathbf{q} = 0$ or $\mathbf{p} = 1$, independently of E). On the other hand, if the intersection of ℓ_λ and the center-stable lamination contains a point $p_0 \in S_0$, then p_0 must lie on the strong stable manifold of one of the singularities of S_0 (see [16, Lemma 2.2], and use equations (3.2.1) and (3.2.2) to see that $p_0 \notin [-1, 1]^3 \subset \mathbb{R}^3$); on the other hand, those center-stable manifolds that intersect S_0 along the aforementioned strong-stable manifolds are tangent to S_0 (and this tangency is quadratic; see [43, Lemma 4.8] for the details); combined with the fact that ℓ_λ is transversal to S_0 , we see that ℓ_λ is transversal to the center-stable manifold at p_0 .

Now, we finish the proof by ruling out the possibility $p \notin S_0$.

It is known that for all $\lambda \in \mathcal{R}$ with $\lambda \neq (1, 0)$, tangential intersections of ℓ_λ with the center-stable manifolds away from S_0 are isolated, if they exist (see the proof of [72, Theorem 2.3(i)]). Let U be a small open neighborhood of p such that p is the only tangency in U , and $U \cap S_0 = \emptyset$.

It can be shown that the center-stable manifolds are analytic away from S_0 (see [3, Section 2] for a discussion). Let $W(p)$ denote the (part of the) center-stable manifold in U containing p . Perform an analytic change of coordinates to map p to

the origin in \mathbb{R}^3 , and $W(p) \cap U$ to a part of the xy plane. Let us call this change of coordinates Φ . Then $g : E \mapsto \pi_z \circ \Phi \circ \ell_{(p_0, \mathfrak{q})}(E)$, $\mathfrak{q} \in [0, \mathfrak{q}_0]$ where π_z is the projection onto the z coordinate, is analytic and depends analytically on \mathfrak{q} . Moreover, if E_p is such that $g(E_p) = \Phi(p)$, then $g'(E_p) = 0$ and there exists $\epsilon > 0$ such that for all $E \in (E_p - \epsilon, E_p + \epsilon)$, with $E \neq E_p$, $g'(E) \neq 0$. The arguments from [14, Section 2] apply without modification to guarantee that either the tangency at p is quadratic (i.e. E_p is a root of g of order 2), or that E_p is a root of g of order $k > 2$ and for all $\mathfrak{q} \in [0, \mathfrak{q}_0)$, g has precisely k roots E_1, \dots, E_k with $g'(E_j) \neq 0$ for all $j = 1, \dots, k$. Moreover, these roots approach the origin as \mathfrak{q} approaches \mathfrak{q}_0 . Combined with the fact that the center-stable manifolds vary continuously in the C^2 topology (see [73, Section B.1.2]), the arguments from [14, Section 2] again apply without modification to guarantee a tangency of $\ell_{(p_0, \mathfrak{q})}$ with a center-stable manifold in U for some $\mathfrak{q} \in (0, \mathfrak{q}_0)$, which contradicts the assumption that no tangencies occur for all $\mathfrak{q} \in [0, \mathfrak{q}_0)$. Thus the tangency at p must be quadratic. On the other hand, the arguments in [14, Section 2] guarantee that the tangency at p cannot be quadratic, as this would lead to either an isolated point in the spectrum (which is precluded by the fact that the spectrum is known to be a Cantor set; see [72]), or to closure of a gap in the spectrum, which is precluded by Claim 3.2.1 below (which states that no two center-stable manifolds that mark the endpoints of the same gap can intersect away from S_0). Thus p cannot be a point of tangency. This completes the proof; let us only remark that in the application of [14, Section 2], one needs to know that all the intersections between the complexified ℓ_λ and the complexified center-stable manifolds are real. This follows, just like in [14], from an application of the complexified version of Sütő's arguments (see [72]) which guarantees that if $E \in \mathbb{C}$ is such that $\ell_\lambda(E)$ has a bounded forward orbit (and it does provided that $\ell_\lambda(E)$ is a point on a (complexified) center-stable manifold), then E is an element of the spectrum; on the other hand, since H_λ is self-adjoint, its spectrum is real.

Claim 3.2.1 *For every gap G of the Cantor set given by the intersection of $\ell_{(p_0, 0)}$ with the center-stable manifolds inside U , there exist center-stable manifolds W_1 and W_2 such that the endpoints of G are given by the intersection of $\ell_{(p_0, 0)}$ with W_1 and*

W_2 , and $W_1 \cap W_2 = \emptyset$ away from the surface S_0 .

Proof: This follows from [14, Theorem 1.3] (the theorem is stated for the Fibonacci Hamiltonian of the form $H_{(1,q)}$, but the proof applies to the case $(\mathbf{p}_0, 0)$ without modification).

The notions of thickness of a Cantor set, as well as the notions of (local) Hausdorff and (upper) box-counting dimensions will play a crucial role in the rest of the paper; these notions are discussed in detail in [50, Chapter 4], for example. The following notation will be used throughout the remainder of the paper.

- The Hausdorff dimension of a set A : $\mathbf{dim}_H(A)$.
- The box-counting dimension of a set A : $\mathbf{dim}_B(A)$; the upper box-counting dimension of A will be denoted by $\overline{\mathbf{dim}}_B(A)$.
- The thickness of a set A : $\tau(A)$.
- For $\star \in \{\mathbf{dim}_H, \tau\}$, the local \star of a set A at $a \in A$ is denoted by $\star^{\text{loc}}(A, a)$.

We are now ready to prove Theorems 3.1.1, 3.1.2, and 3.1.3.

3.2.2 Proof of Theorem 3.1.1

Given $\lambda = (\mathbf{p}, \mathbf{q}) \in \mathcal{R}$, denote by Π_λ the plane $\left\{ (x, y, z) \in \mathbb{R}^3 : z = \frac{1+\mathbf{p}^2}{2\mathbf{p}} \right\}$; note that Π_λ only depends on \mathbf{p} . Clearly, ℓ_λ is contained in Π_λ for all $\lambda \in \mathcal{R}$. Set $y = y(E) = \frac{E}{2\mathbf{p}}$ and $x = x(E) = \frac{E-\mathbf{q}}{2}$. Then we have

$$(3.2.4) \quad y = y(x) \stackrel{\text{def}}{=} \frac{1}{\mathbf{p}}x + \frac{\mathbf{q}}{2\mathbf{p}},$$

and

$$\ell_{(\mathbf{p},\mathbf{q})}(E) = \left(x, y(x), \frac{1+\mathbf{p}^2}{2\mathbf{p}} \right).$$

Thus, for any fixed $\lambda \in \mathcal{R}$, ℓ_λ can be viewed as a line in Π_λ with parameterization $(x, y(x))$, $x \in \mathbb{R}$.

Lemma 3.2.1 *For all $\Delta > 0$ there exists \mathbf{p} with $|\mathbf{p}| > \Delta$ and \mathbf{q} depending on \mathbf{p} , such that there exists $E \in \Sigma_{(\mathbf{p},\mathbf{q})}$ with $\ell_{(\mathbf{p},\mathbf{q})}(E) \in S_0$.*

Proof: Let us denote the point $(1, 1, 1) \in S_0$ by P_1 ; P_1 is a partially hyperbolic fixed point with one-dimensional stable, unstable, and center directions (Df acts as an isometry on the center subspace). Obviously, since the surfaces S_V are invariant, the strong-stable manifold of P_1 lies on S_0 . Notice that P_1 is a cut point of its strong-stable manifold; after removing P_1 from the strong-stable manifold, we end up with one of the two branches, which we denote by W , which satisfies the following. For all $p \in W$, $|f^{-n}(p)| \xrightarrow{n \rightarrow \infty} \infty$ (see [4, Section 5]; in the terminology of [4], P_1 is an *s-one-sided* or *stably one-sided point*). Consequently, every $p \in W$ escapes to infinity in every coordinate under f^{-1} (see Section 3 in [55]). Since $W = f^{-1}(W)$, it follows that there exist points of W with arbitrarily large z coordinate in absolute value. It follows by continuity (indeed, W is smooth), that for any arbitrarily large $\Delta > 0$ there exists \mathbf{p} with $|\mathbf{p}| > \Delta$ such that $\Pi_{(\mathbf{p},\mathbf{q})}$ intersects W ; pick such a \mathbf{p} . Now from (3.2.4), we see that for any such choice of \mathbf{p} we can find \mathbf{q} such that $\ell_{(\mathbf{p},\mathbf{q})}$ intersects W . Indeed, Π_λ is determined only by \mathbf{p} , and the slope of the line ℓ_λ when viewed as a line in Π_λ in terms of (3.2.4) is also determined only by \mathbf{p} . It follows that for any point p in Π_λ , we can choose \mathbf{q} such that the line in (3.2.4) passes through p .

Lemma 3.2.2 *For all $\Delta > 1$ and $\tau_0 > 0$ there exists $(\mathbf{p}, \mathbf{q}) \in \mathcal{R}$ with $|\mathbf{p}| > \Delta$, and $\epsilon > 0$, such that for all $\lambda \in \mathcal{R}$ with $\|(\mathbf{p}, \mathbf{q}) - \lambda\| < \epsilon$, there exists $E \in \Sigma_\lambda$ with $\tau^{\text{loc}}(\Sigma_\lambda, E) > \tau_0$.*

Proof: It is known that the center-stable manifold that contains P_1 is tangent to S_0 ; its intersection with S_0 is precisely the strong-stable manifold of P_1 (see [43, Lemma 4.8]). Let us denote this center-stable manifold by $W^{\text{cs}}(P_1)$.

Now pick \mathbf{p}_0 with $|\mathbf{p}_0| > \Delta$ and a suitable \mathbf{q}_0 (guaranteed by Lemma 3.2.1), such that the corresponding $\ell_{(\mathbf{p}_0, \mathbf{q}_0)}$ intersects $W^{\text{cs}}(P_1)$ at some point along W . Notice that $\mathbf{q}_0 \neq 0$, since $\ell_{(\mathbf{p}_0, 0)}$ lies entirely in the surface S_V with $V = \frac{(\mathbf{p}_0^2 - 1)^2}{4\mathbf{p}_0^2}$ (see equation (3.2.7) below). Observe that for all $\delta > 0$ or $\delta < 0$ with $|\delta|$ sufficiently small, $\ell_{(\mathbf{p}_0, \mathbf{q}_0 + \delta)}$ still intersects $W^{\text{cs}}(P_1)$ in some point $\ell_{(\mathbf{p}_0, \mathbf{q}_0 + \delta)}(E_\delta)$, $E_\delta \in \Sigma_{(\mathbf{p}_0, \mathbf{q}_0 + \delta)}$, but this intersection

no longer occurs on S_0 , but on some S_{V_δ} with $V_\delta > 0$ and $V_\delta \rightarrow 0$ as $\delta \rightarrow 0$. Indeed, since W is analytic, if for infinitely many small $\delta > 0$, $\ell_{(\mathfrak{p}_0, q_0 + \delta)}(E_\delta) \in S_0$, then we must have W lying entirely in $\Pi_{(\mathfrak{p}_0, q_0 + \delta)}$ (which, recall, depends only on \mathfrak{p}_0), which is impossible since $\Pi_{(\mathfrak{p}_0, q_0 + \delta)}$ does not contain P_1 , while W contains points arbitrarily close to P_1 .

In what follows, let us assume, without loss of generality, that $\delta > 0$. Moreover, let us remark, for future reference, that

$$(3.2.5) \quad V_\delta = \min \{ I(\ell_{(\mathfrak{p}_0, q_0 + \delta)}(E)) : E \in \Sigma_{(\mathfrak{p}_0, q_0 + \delta)} \}.$$

Indeed, this follows by monotonicity (see (3.2.7) below).

By continuity, the intersection of $\ell_{(\mathfrak{p}_0, q_0 + \delta)}$ with $W^{cs}(P_1)$ at $\ell_{(\mathfrak{p}_0, q_0 + \delta)}(E_\delta)$ is transversal uniformly in δ for all sufficiently small $\delta > 0$. Thus, in a sufficiently small neighborhood of $\ell_{(\mathfrak{p}_0, q_0 + \delta)}(E_\delta)$, $\ell_{(\mathfrak{p}_0, q_0 + \delta)}$ intersects the center-stable manifolds uniformly transversally (since the center-stable manifolds form a continuous family in the C^1 topology away from S_0 ; see [73, Proposition 4.10] for the details). Let us take such a neighborhood of $\ell_{(\mathfrak{p}_0, q_0 + \delta)}(E_\delta)$ and denote it by \mathcal{V}_δ . Pick a plane Λ containing $\ell_{(\mathfrak{p}_0, q_0 + \delta)}$ and transversal to the surfaces S_V , $V \geq 0$, as well as to the center-stable manifolds, inside the neighborhood \mathcal{V}_δ . Observe that the intersection of $\Lambda \cap \mathcal{V}_\delta$ with the surfaces $\{S_V\}_{V \geq 0}$ produces a smooth foliation of $\Lambda \cap \mathcal{V}_\delta$. We shall denote this foliation by \mathcal{S} . Let us further assume that this foliation has been rectified (notice that after the rectification, $\ell_{(\mathfrak{p}_0, q_0 + \delta)}$ is in general no longer a line, but we abuse the notation and use the same symbol $\ell_{(\mathfrak{p}_0, q_0 + \delta)}$). The intersection of the center-stable manifolds with $\Lambda \cap \mathcal{V}_\delta$ produces a lamination with smooth leaves. We shall call this lamination \mathcal{L} .

Let us parameterize the leaves of \mathcal{S} by V (same as for S_V), and call the leaves L_V . We know that for every V , the intersection of L_V with the leaves of the lamination \mathcal{L} is a Cantor set.

Pick an arbitrary but small $\epsilon > 0$, such that if $J_{\delta, \epsilon}$ is a compact interval of length ϵ along $\ell_{(\mathfrak{p}_0, q_0 + \delta)}$ having $\ell_{(\mathfrak{p}_0, q_0 + \delta)}(E_\delta)$ as one of its endpoints, $J_{\delta, \epsilon} \subset \mathcal{V}_\delta$ and $J_{\delta, \epsilon}$ intersects the leaves of \mathcal{L} . Denote $\mathcal{C}_{\delta, \epsilon} \stackrel{\text{def}}{=} J_{\delta, \epsilon} \cap \mathcal{L}$, which, by adjusting ϵ , can be ensured to be a Cantor set (indeed, for any given ϵ , $\mathcal{C}_{\delta, \epsilon}$ is either a Cantor set or a Cantor set together

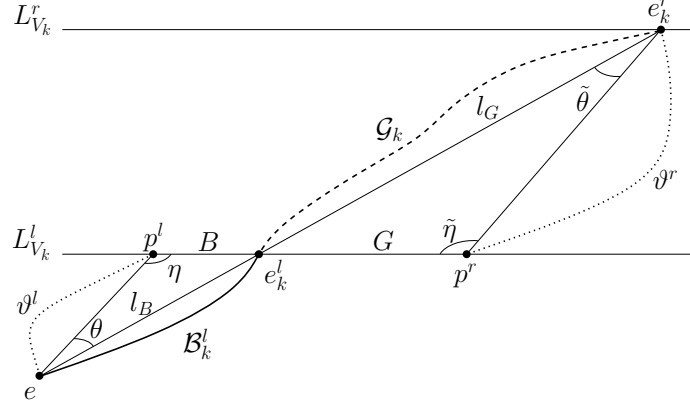


Figure 3.2:

with an isolated point on the boundary of $J_{\delta,\epsilon}$). Assume also without loss of generality that $J_{\delta,\epsilon}$ is the convex hull of $C_{\delta,\epsilon}$. For what follows, refer to Figure 3.2 for visual guidance.

Let $\{\mathcal{G}_i\}$ be the set of ordered gaps of the Cantor set $C_{\delta,\epsilon}$. Let $\{\mathcal{B}_i^l, \mathcal{B}_i^r\}$ be the set of the corresponding bands (that is, \mathcal{B}_k^l and \mathcal{B}_k^r are the subintervals of $J_{\delta,\epsilon} \setminus \bigcup_{i=1}^k \mathcal{G}_i$ immediately to the left and immediately to the right of \mathcal{G}_k , respectively). For any k , denote the two endpoints of \mathcal{G}_k by e_k^l and e_k^r , with the assumption that $e_k^l \in \mathcal{B}_k^l$ and $e_k^r \in \mathcal{B}_k^r$. Let us write also $L_{V_k}^\bullet$, $\bullet \in \{l, r\}$, for the leaf of \mathcal{S} containing e_k^\bullet . Identify by p^r the point $\vartheta^r \cap L_{V_k}^l$, where ϑ^r is the leaf of \mathcal{L} whose intersection with $\ell_{(p_0, q_0 + \delta)}$ gives e_k^r . If $\vartheta^l \in \mathcal{L}$ is the leaf whose intersection with $\ell_{(p_0, q_0 + \delta)}$ gives the other endpoint of \mathcal{B}_k^l , say e , denote by p^l the point $\vartheta^l \cap L_{V_k}^l$, and denote by l_B the line segment connecting e and e_k^l . Denote by B the line segment connecting p^l and e_k^l . Denote the line segment connecting e_k^l and e_k^r by l_G . There exist constants $C_1, C_2 > 0$ independent of δ and ϵ such that

$$(3.2.6) \quad C_1 |l_G| \geq |\mathcal{G}_k|_\ell \quad \text{and} \quad C_2 |l_B| \leq |\mathcal{B}_k^l|_\ell$$

(here $|\cdot|$ denotes the length of the line segment, and $|\cdot|_\ell$ denotes the length along the curve $\ell_{(p_0, q_0 + \delta)}$); see [72, Lemma 3.3] for a simple justification of this.

Now (refer to Figure 3.2) observe that since $\vartheta^{l,r}$ are transversal to the leaves of \mathcal{S} as well as $\ell_{(p_0, q_0 + \delta)}$ (albeit the angle of intersection may depend on δ), assuming that ϵ was initially chosen sufficiently small (depending on δ only) we can guarantee that

the angles θ , η , $\tilde{\theta}$ and $\tilde{\eta}$ are such that

$$\frac{\sin(\eta) \sin(\tilde{\theta})}{\sin(\tilde{\eta}) \sin(\theta)} > \frac{1}{2}.$$

Combining this with (3.2.6), we obtain

$$\frac{|\mathcal{B}_k^l|_\ell}{|\mathcal{G}_k|_\ell} \geq \frac{C_2 |l_B|}{C_1 |l_G|} = \frac{|B| C_2 \sin(\eta) \sin(\tilde{\theta})}{|G| C_1 \sin(\tilde{\eta}) \sin(\theta)} > \frac{C_2}{2C_1} \frac{|B|}{|G|}.$$

Similar bounds are obtained for the quotient $|\mathcal{B}_k^r|_\ell / |\mathcal{G}_k|_\ell$. On the other hand, since C_2 and C_1 are universal constants, we can guarantee that

$$\frac{C_2}{2C_1} \frac{|B|}{|G|} > \tau_0$$

by choosing δ suitably small, and choosing ϵ (depending on δ) sufficiently small so that all the bounds above hold; indeed, this follows since the thickness of the Cantor sets obtained by intersecting the leaves of \mathcal{S} with those of \mathcal{L} tends to infinity at S_0 (see [11]). Since these estimates are independent of the gap index k , we conclude that, for suitably small δ and ϵ , the thickness of $\mathcal{C}_{\delta,\epsilon}$ is larger than τ_0 . This also holds for all parameters (\mathbf{p}, \mathbf{q}) sufficiently close to $(\mathbf{p}_0, \mathbf{q}_0 + \delta)$.

Lemma 3.2.3 *For every $\Delta, \delta > 0$, there exists a nonempty open set $\mathcal{U} \subset \mathcal{R}$, such that the following hold for all $\lambda \in \mathcal{U}$.*

- (1) *There exists $E_0 \in \Sigma_\lambda$ such that $\tau^{\text{loc}}(\Sigma_\lambda, E_0) > \Delta$.*
- (2) *At one of the extrema of Σ_λ , E_{end} , we have $\mathbf{dim}_{\text{H}}^{\text{loc}}(\Sigma_\lambda, E_{\text{end}}) < \delta$.*

Proof: Recall that the spectral radius of a self-adjoint operator is equal to its norm, and that $\|H_{(\mathbf{p}, \mathbf{q})}\|$ is of order $\max\{|\mathbf{p}|, |\mathbf{q}|\}$. Consequently, for all $C > 0$, $\Delta > 1$, and $\tau_0 > 0$, we can always find $\lambda = (\mathbf{p}, \mathbf{q}) \in \mathcal{R}$, and $E_0 \in \Sigma_\lambda$ so that the conclusion of Lemma 3.2.2 holds and such that $I(\ell_\lambda(E_1)) > C$ for some $E_1 \in \Sigma_\lambda$. Indeed, this follows since

$$(3.2.7) \quad V_\lambda(E) \stackrel{\text{def}}{=} I(\ell_\lambda(E)) = \frac{\mathbf{q}(\mathbf{p}^2 - 1)E + \mathbf{q}^2 + (\mathbf{p}^2 - 1)^2}{4\mathbf{p}^2}$$

for all E , which can easily be computed from (3.2.3). Evidently, V_λ is a monotone function of E , so we can take E_1 to be one of the extrema of Σ_λ .

On the other hand, for any $E \in \Sigma_\lambda$,

$$\mathbf{dim}_H^{\text{loc}}(\Sigma_\lambda, E) = \frac{1}{2} \mathbf{dim}_H(\Omega_{V_\lambda(E)}),$$

where Ω_V is the nonwandering set of $f|_{S_V}$ (see the proof of in [72, Theorem 2.1(iii)]). Moreover, from [8], we have

$$\lim_{V \rightarrow \infty} \mathbf{dim}_H(\Omega_V) = 0,$$

so we can get (1) and (2) for λ by taking $C > 0$ sufficiently large above. That small perturbations of λ do not destroy these bounds follows from Lemma 3.2.2 and the fact that $\mathbf{dim}_H(\Omega_V)$ is continuous in V .

Remark 3.2.1 *It is not difficult to see from the proof of Lemma 3.2.2 that E_0 can be taken arbitrarily close to the other extremum of the spectrum. In fact, \mathcal{U} can be adjusted so that E_0 can be taken to be an extremum of Σ_λ (that is, we need to make sure that $\ell_\lambda(E_0)$ does not lie on S_0 in the proof of Lemma 3.2.2).*

Proof: [Proof of Theorem 3.1.1] Take $\Delta > 1$ and $\delta \in (0, 1/2)$. With this choice of Δ and δ , let \mathcal{U} be as in Lemma 3.2.3. For $\lambda_i \in \mathcal{U}$ with $i = 1, 2$, let $E_0(\lambda_i)$ and $E_{\text{end}}(\lambda_i)$ be as in the Lemma.

Let $E_{\text{end}}^2 = E_{\text{end}}(\lambda_1) + E_{\text{end}}(\lambda_2)$. Notice that E_{end}^2 is an extremum of $\Sigma_{(\lambda_1, \lambda_2)}^2$; let us assume, without loss of generality, that it is the supremum (so $E_{\text{end}}(\lambda_1)$ and $E_{\text{end}}(\lambda_2)$ are also the suprema, provided that \mathcal{U} is chosen sufficiently small in the beginning). Take $\epsilon > 0$ such that

$$\mathbf{dim}_H([E_{\text{end}}(\lambda_i) - \epsilon, E_{\text{end}}(\lambda_i)] \cap \Sigma_{\lambda_i}) < \delta.$$

We claim that there exists $\tilde{\epsilon} > 0$ such that

$$\mathbf{dim}_H([E_{\text{end}}^2 - \tilde{\epsilon}, E_{\text{end}}^2] \cap \Sigma_{(\lambda_1, \lambda_2)}^2) < 1.$$

Indeed, for arbitrary sets $A, B \subset \mathbb{R}$, we have

$$(3.2.8) \quad \mathbf{dim}_H(A + B) \leq \min \{ \overline{\mathbf{dim}}_B(A) + \mathbf{dim}_H(B), 1 \}$$

(see [41, Theorem 8.10(2)]). On the other hand, we have

$$(3.2.9) \quad \begin{aligned} & \mathbf{dim}_H([E_{\text{end}}(\lambda_i) - \epsilon, E_{\text{end}}(\lambda_i)] \cap \Sigma_{\lambda_i}) \\ &= \overline{\mathbf{dim}}_B([E_{\text{end}}(\lambda_i) - \epsilon, E_{\text{end}}(\lambda_i)] \cap \Sigma_{\lambda_i}), \end{aligned}$$

which follows from the transversality of intersection of ℓ_λ with the center-stable manifolds (see [72, Proposition 3.2]). Thus $[E_{\text{end}}^2 - \tilde{\epsilon}, E_{\text{end}}^2] \cap \Sigma_{(\lambda_1, \lambda_2)}^2$ has Hausdorff dimension strictly smaller than one, hence is of zero Lebesgue measure. Since it is also compact and does not contain any isolated points (except possibly the boundary $E_{\text{end}}^2 - \tilde{\epsilon}$, which can be remedied by decreasing $\tilde{\epsilon}$), it follows that it is a Cantor set.

Finally, given that $\Delta > 1$, it follows from the Gap Lemma of S. Newhouse (see, e.g., [2, Theorem 2.2] for a derivation) that $\Sigma_{(\lambda_1, \lambda_2)}^2$ contains an interval around $E_0(\lambda_1) + E_0(\lambda_2)$.

Remark 3.2.2 *As mentioned in Remark 3.1.2, we suspect that for all $\lambda_1, \lambda_2 \in \mathcal{R}$ sufficiently close to $(1, 0)$, $\Sigma_{(\lambda_1, \lambda_2)}^2$ is an interval. A proof of this would involve control of the global thickness of Σ_λ for λ close to $(1, 0)$. Such control can be obtained via finer estimates than those used in the proof above, but the arguments would be far more technical. We have decided to delegate this task to later investigations.*

3.2.3 Proof of Theorem 3.1.2

In what follows, we need a slight generalization of [13, Proposition 2.3], namely Proposition 3.2.1 below. For the sake of convenience and completeness, let us first recall the general setup of [13].

Let \mathcal{B} be a finite set of cardinality $|\mathcal{B}| \geq 2$, and consider $\mathcal{B}^{\mathbb{Z}_+}$, the standard symbolic space, equipped with the product topology. Here $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. Let \mathcal{E} be a Borel subset of $\mathcal{B}^{\mathbb{Z}_+}$, and let μ be a probability measure on \mathcal{E} .

Let $J \subset \mathbb{R}$ be a compact nondegenerate interval. We assume that we are given a family of continuous maps

$$\Pi_j : \mathcal{E} \rightarrow \mathbb{R}, \quad j \in J,$$

and

$$\nu_j = \Pi_j(\mu) \stackrel{\text{def}}{=} \mu \circ \Pi_j^{-1}.$$

For a word $u \in \mathcal{B}^n$, $n \geq 0$, we denote by $|u| = n$ its length and by $[u]$ the cylinder set of elements of \mathcal{E} that have u as a prefix. More precisely, $[u] = \{\omega \in \mathcal{E} : \omega_0 \cdots \omega_{n-1} = u\}$. For $\omega, \tau \in \mathcal{E}$, we write $\omega \wedge \tau$ for the maximal common prefix of ω and τ , which is empty if $\omega_0 \neq \tau_0$; we set the length of the empty word to be zero. Furthermore, for $\omega, \tau \in \mathcal{E}$, let

$$\phi_{\omega, \tau}(j) \stackrel{\text{def}}{=} \Pi_j(\omega) - \Pi_j(\tau).$$

We write \mathcal{L}^1 for one-dimensional Lebesgue measure on \mathbb{R} .

Proposition 3.2.1 *Let η be a compactly supported Borel measure on the real line. Suppose that the following holds. For every $\epsilon > 0$ there exists $\mathcal{E}_0 \subset \mathcal{E}$ with $\mu(\mathcal{E}_0) > 1 - \epsilon$ and constants $C_1, C_2, C_3, \alpha, \beta, \gamma > 0$ and $k_0 \in \mathbb{Z}_+$ such that*

$$(3.2.10) \quad \mathbf{dim}_H^{\text{loc}}(\eta, x) \stackrel{\text{def}}{=} \lim_{r \downarrow 0} \frac{\log \eta(B_r(x))}{\log r} \geq d_\eta \quad \text{for } \eta\text{-a.e. } x,$$

where $B_r(x) = [x - r, x + r]$ and d_η satisfies

$$(3.2.11) \quad d_\eta + \frac{\gamma}{\beta} > 1 \quad \text{and} \quad d_\eta > \frac{\beta - \gamma}{\alpha};$$

$$(3.2.12) \quad \max_{j \in J} |\phi_{\omega, \tau}(j)| \leq C_1 |\mathcal{B}|^{-\alpha|\omega \wedge \tau|} \quad \text{for all } \omega, \tau \in \mathcal{E}_0, \omega \neq \tau;$$

$$(3.2.13)$$

$$\sup_{v \in \mathbb{R}} \mathcal{L}^1(\{j \in J : |v + \phi_{\omega, \tau}(j)| \leq r\}) \leq C_2 |\mathcal{B}|^{|\omega \wedge \tau|/\beta} r \quad \text{for all } \omega, \tau \in \mathcal{E}_0, \omega \neq \tau$$

such that $|\omega \wedge \tau| \geq k_0$, and

$$(3.2.14) \quad \max_{u \in \mathcal{B}^n, [u] \cap \mathcal{E}_0 \neq \emptyset} \mu([u]) \leq C_3 |\mathcal{B}|^{-\gamma n} \quad \text{for all } n \geq 1.$$

Then, $\eta * \nu_j \ll \mathcal{L}^1$ for Lebesgue-a.e. $j \in J$.

The only difference between Proposition 3.2.1 and [13, Proposition 2.3] is condition (3.2.10). In [13], the measure η is exact dimensional. Nevertheless, Proposition 3.2.1 can be proved by repeating verbatim the proof of [13, Proposition 2.3].

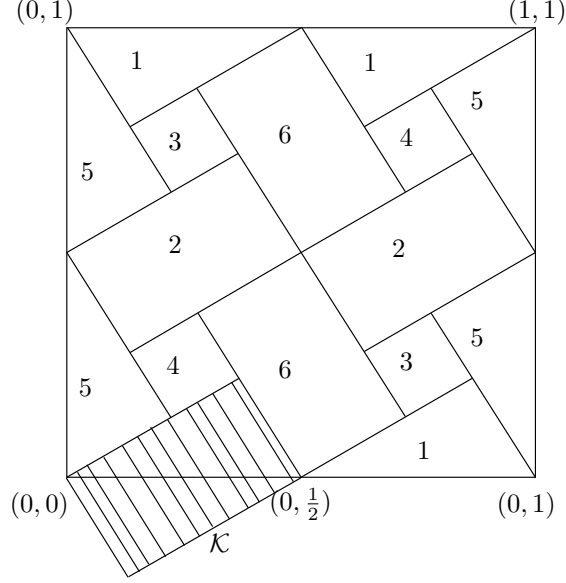


Figure 3.3: The Markov Partition for the map \mathcal{A} , the rectangle \mathcal{K} , and the stable manifolds which lie inside \mathcal{K} .

Let

$$(3.2.15) \quad \mathbb{S} \stackrel{\text{def}}{=} S_0 \cap \{(x, y, z) \in \mathbb{R}^3 : |x| \leq 1, |y| \leq 1, |z| \leq 1\}.$$

The trace map f restricted to \mathbb{S} is a factor of the hyperbolic automorphism of $\mathbb{T}^2 \stackrel{\text{def}}{=} \mathbb{R}^2 / \mathbb{Z}^2$ given by

$$(3.2.16) \quad \mathcal{A}(\theta, \varphi) = (\theta + \varphi, \theta).$$

The semi-conjugacy is given by the map

$$(3.2.17) \quad F : (\theta, \varphi) \mapsto (\cos 2\pi(\theta + \varphi), \cos 2\pi\theta, \cos 2\pi\varphi).$$

Let us fix a Markov partition of the map \mathcal{A} (for example, see Figure 3.3). Pick one element of this Markov partition, denote it by \mathcal{K} , and let \mathcal{K}_0 denote the projection of \mathcal{K} to \mathbb{S} via the map F . The Markov partition for $f : \mathbb{S} \rightarrow \mathbb{S}$ can be continued to a Markov partition for the map $f|_{\Omega_V} : \Omega_V \rightarrow \Omega_V$. One can check that the elements of this induced Markov partition on Ω_V are disjoint for every $V > 0$. Let us denote the continuation of \mathcal{K}_0 along the parameter V by \mathcal{K}_V .

Suppose that $\sigma : \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}}$ is the two-sided topological Markov chain conjugate to $f|_{\Omega_V}$ via the conjugacy $H_V : \tilde{\mathcal{E}} \rightarrow \Omega_V$. Let $\tilde{\mu}$ be the measure of maximal entropy for $\sigma : \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}}$, and let $\tilde{\mu}_V$ denote the induced measure on Ω_V , that is, $\tilde{\mu}_V = H_V(\tilde{\mu})$.

Consider the stable manifolds which lie inside \mathcal{K}_0 . These stable manifolds have natural continuations for all $V > 0$ (see [73, Proposition 4.10] for details), and so they form a lamination of two-dimensional smooth (away from S_0) manifolds. We denote the resulting lamination of these *center-stable manifolds* by $\tilde{\Omega}$ (these are the same center-stable manifolds as in Section 3.2.1 and the proof of Theorem 3.1.1 above). Let us restrict the measure $\tilde{\mu}_V$ to \mathcal{K}_V and normalize it. This naturally induces a measure on $\tilde{\Omega}$; we abuse the notation and denote this measure by $\tilde{\mu}$.

Proposition 3.2.2 *For every $\lambda \in \mathcal{R}$, the projection of $\tilde{\mu}$ to ℓ_λ is proportional to the density of states measure dk_λ under the identification (3.2.1). In particular, if \mathcal{K} is the element of the Markov partition in Figure 3.3 which contains the interval $[0, 1/2] \times \{0\}$, then the projection of $\tilde{\mu}$ to ℓ_λ corresponds to the density of states measure under the identification (3.2.1).*

For a proof of Proposition 3.2.2, see Claim 3.16 and the discussion preceding it in [73], which is an extension to the Jacobi case of the original result given in [12] for the Schrödinger operators.

Let $\ell_{(p,q)}$ be as in (3.2.1), let J be a compact interval in \mathbb{R} , and assume that \mathcal{U} is a subset of \mathcal{R} , where \mathcal{R} is as in (3.1.1). Assume further that $\alpha : J \rightarrow \mathcal{U}$ is analytic, so that $\{\ell_{\alpha(t)}\}_{t \in J}$ is an analytic family of lines.

Let \mathcal{E}^+ be the one-sided topological Markov chain which is obtained by deleting the negative side of $\tilde{\mathcal{E}}$. Let \mathcal{E} be the subset of \mathcal{E}^+ which corresponds to $\tilde{\Omega}$. Note that $\tilde{\mu}$ naturally induces a probability measure on \mathcal{E} , which we denote by μ .

For any $\omega \in \mathcal{E}$, we denote the corresponding leaf of the center-stable manifold by $\tilde{\omega}$. Let $\pi_{\alpha(t)} : \tilde{\Omega} \rightarrow \ell_{\alpha(t)}$ be the map mapping each center-stable manifold to the point of its intersection with $\ell_{\alpha(t)}$. Let us define $\Pi_{\alpha(t)} : \mathcal{E} \rightarrow \mathbb{R}$ by

$$\Pi_{\alpha(t)}(\omega) = \ell_{\alpha(t)}^{-1} \circ \pi_{\alpha(t)}(\tilde{\omega}).$$

Set $\nu_{\alpha(t)} = \Pi_{\alpha(t)}(\mu)$. Note that the measure $\nu_{\alpha(t)}$ is a probability measure supported on the spectrum $\Sigma_{\alpha(t)}$.

For any $\omega \in \mathcal{E}$ and $t \in J$, if we have $\pi_{\alpha(t)}(\tilde{\omega}) \in S_V$ for some $V > 0$, then we sometimes express this dependency explicitly and write this V as $V(\omega, t)$ below.

Let us denote by $\text{Lyap}^u(\mu_V)$ the unstable Lyapunov exponent of $f|_{S_V}$ with respect to the measure μ_V . The following statement can be proved by applying the proofs of Propositions 3.8 and 3.9 of [13] without modification.

Proposition 3.2.3 *Assume that for all $t \in J$ and $\omega \in \mathcal{E}$, $|\frac{d}{dt}\text{Lyap}^u(\mu_{V(\omega,t)})| \geq \delta > 0$. Then for every $\epsilon > 0$, there exist $N^* \in \mathbb{Z}_+$ and a set $\mathcal{E}_0 \subset \mathcal{E}$ such that $\mu(\mathcal{E}_0) > 1 - \frac{\epsilon}{2}$, and such that for $t \in J$, $\omega \in \mathcal{E}_0$, and $N \geq N^*$, we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \left\| Df^n(\pi_{\alpha(t)}(\tilde{\omega}))|_{\ell_{\alpha(t)}} \right\| = \text{Lyap}^u(\mu_{V(\omega,t)}),$$

and

$$\left| \frac{d}{dt} \left(\frac{1}{N} \log \left\| Df^N(\pi_{\alpha(t)}(\tilde{\omega}))|_{\ell_{\alpha(t)}} \right\| \right) \right| > \frac{\delta}{4}.$$

We can now prove

Proposition 3.2.4 *Let η be a compactly supported Borel measure on \mathbb{R} such that $\mathbf{dim}_H^{\text{loc}}(\eta, x) \geq d_\eta$ for η -a.e. x . Assume that $d_\eta + \mathbf{dim}_H \nu_{\alpha(t)} > 1$ for all $t \in J$. Assume also that there exists $\delta > 0$ such that $|\frac{dV(\omega,t)}{dt}| > \delta$ for all $\omega \in \mathcal{E}$ and $t \in J$. Then $\eta * \nu_{\alpha(t)}$ is absolutely continuous with respect to Lebesgue measure for a.e. $t \in J$.*

Proof: The proof is essentially the same as the proof of [13, Theorem 3.7]. We need to check that the conditions in Proposition 3.2.3 are satisfied. According to the hypothesis of Proposition 3.2.4, for all $\omega \in \mathcal{E}$, $V(\omega, t)$ is strictly monotone with uniformly (in t) nonzero derivative. On the other hand, $\text{Lyap}^u(\mu_{V(\omega,t)})$ is analytic in V (this follows from general principles, since $f_V = f|_{S_V}$ is analytic in V). It follows that for any given ω , away from a set of nonaccumulating points t , the derivative $\frac{d}{dt}\text{Lyap}^u(\mu_{V(\omega,t)})$ is nonzero. By continuity, using compactness of \mathcal{E} , this can be ensured for all ω by restricting t to a sufficiently small nongenerate interval.

Let $\mathcal{E}_0 \subset \mathcal{E}$ and $N^* \in \mathbb{Z}_+$ be as in Proposition 3.2.3. Let us take $\omega, \omega' \in \mathcal{E}_0$ in such a way that $p(t) = \pi_{\alpha(t)}(\tilde{\omega})$ and $q(t) = \pi_{\alpha(t)}(\tilde{\omega}')$ are sufficiently close. Let

$n \in \mathbb{Z}_+$ be such that the distance between $f^n(p)$ and $f^n(q)$ is of order one. Let us introduce coordinates on each curve $\ell_{\alpha(t)}, f(\ell_{\alpha(t)}), \dots, f^n(\ell_{\alpha(t)})$, using the original parametrization and taking $p \in \ell_{\alpha(t)}, f(p) \in f(\ell_{\alpha(t)}), \dots, f^n(p) \in f^n(\ell_{\alpha(t)})$ to be the origin. We consider the map $f^{-1} : f^i(\ell_{\alpha(t)}) \rightarrow f^{i-1}(\ell_{\alpha(t)})$ in these coordinates, and write the resulting map by $k_{\alpha(t)}^{(i)} : \mathbb{R} \rightarrow \mathbb{R}$. Let $l^{(i)} = \frac{\partial k_{\alpha(t)}^{(i)}}{\partial x}(0)$.

Then, by the Lemma 3.11 of [13] and by the Proposition 3.2.3, we have

$$\frac{d}{dt} \mathbf{dist}(p(t), q(t)) \leq \left(\prod_{s=1}^n l^{(s)} \right) (C - \delta' n)$$

for some $C, \delta' > 0$. The rest of the argument is a verbatim repetition of the proof of [13, Theorem 3.7] with our Proposition 3.2.1 in place of [13, Proposition 2.3] and Proposition 3.2.3 in place of [13, Propositions 3.8 and 3.9].

We now prove the following measure-theoretic analog of Lemma 3.2.3.

Lemma 3.2.4 *For every $\Delta, \delta \in (0, 1)$ there exists a nonempty open set $\mathcal{U} \subset \mathcal{R}$ such that we have the following for every $\lambda \in \mathcal{U}$.*

- (1) *There exists $E_0 \in \Sigma_\lambda$ such that $\mathbf{dim}_H^{\text{loc}}(dk_\lambda, E) > \Delta$ for all $E \in \Sigma_\lambda$ in a sufficiently small neighborhood of E_0 (where $\mathbf{dim}_H^{\text{loc}}(dk_\lambda, E_0)$ is as defined in (3.2.10) with dk_λ and E_0 in place of η and x).*
- (2) *At one of the extrema of Σ_λ , E_{end} , we have $\mathbf{dim}_H^{\text{loc}}(\Sigma_\lambda, E_{\text{end}}) < \delta$.*

Proof:

First, the existence of $\mathbf{dim}_H^{\text{loc}}(dk_\lambda, E)$ for dk_λ -a.e. $E \in \Sigma_\lambda$, that is, the existence of the limit in (3.2.10), is proved in [72, Proposition 3.15]. We can now apply the proof of [72, Theorem 2.6] without modification to obtain the following (in [72] the proof is given for the entire set Σ_λ with λ close to $(1, 0)$; however, the same proof gives the following local result).

Claim 3.2.2 *For all $\Delta \in (0, 1)$ there exists $\epsilon > 0$ such that for all $\lambda \in \mathcal{R}$, if $E_0 \in \Sigma_\lambda$ is such that $0 < I(\ell_\lambda(E_0)) = V_\lambda(E_0) < \epsilon$ (with V_λ as in (3.2.7)), then $\mathbf{dim}_H^{\text{loc}}(dk_\lambda, E) > \Delta$ for dk_λ -a.e. E in a sufficiently small neighborhood of E_0 (any open neighborhood of E_0 is necessarily of nonzero dk_λ measure).*

At the same time, just as in the proof of Lemma 3.2.3, we know that there exists $C > 0$ such that if $\lambda \in \mathcal{R}$ such that for one of the extrema of Σ_λ , E_{end} , $V_\lambda(E_{\text{end}}) > C$, then $\mathbf{dim}_{\mathbb{H}}^{\text{loc}}(\Sigma_\lambda, E_{\text{end}}) < \delta$.

Now, let $\epsilon > 0$ as in Claim 3.2.2 and $C > 0$ as in the preceding paragraph. Just as in the proof of Lemma 3.2.3, let $\mathcal{U} \subset \mathcal{R}$ be an open set such that for all $\lambda \in \mathcal{U}$, there exists $E_0 \in \Sigma_\lambda$ and one of the extrema E_{end} of Σ_λ satisfying $V_\lambda(E_0) \in (0, \epsilon)$ and $V_\lambda(E_{\text{end}}) > C$.

Remark 3.2.3 *Just as in Lemma 3.2.3, \mathcal{U} can be adjusted so that E_0 can be taken to be the other extremum of the spectrum (see Remark 3.2.1).*

In what follows, we fix a nonempty open set $\mathcal{U} \subset \mathcal{R}$ as in Lemma 3.2.4 with $\Delta > \frac{1}{2}$, and so that $(1, 0) \notin \mathcal{U}$. Pick and fix $\lambda \in \mathcal{U}$. We see that for every $\tilde{\lambda} \in \mathcal{U}$, the singular continuous component of $dk_\lambda * dk_{\tilde{\lambda}}$ is nonzero. Indeed, since by Lemma 3.2.4 (see also Lemma 3.2.3), $\Sigma_{(\lambda, \tilde{\lambda})}^2$ contains a Cantor set \mathcal{C} of zero Lebesgue measure which is the sum of Cantor subsets of Σ_λ and $\Sigma_{\tilde{\lambda}}$ of nonzero dk_λ and $dk_{\tilde{\lambda}}$ measure, respectively, we have $(dk_\lambda * dk_{\tilde{\lambda}})(\mathcal{C}) > 0$.

Now fix any $\tilde{\lambda} \in \mathcal{U}$. Without loss of generality, let us assume that \mathcal{U} is an open ball in \mathcal{R} . Let $\alpha : [0, 1] \rightarrow \mathcal{R}$ be an analytic curve with the following properties.

- (1) $\alpha(0) = (1, 0)$ and for all $t \in (0, 1]$, $\alpha(t) \neq (1, 0)$.
- (2) $\alpha(1) = \lambda$.
- (3) For some $t_0 \in (0, 1)$, $\alpha(t_0) = \tilde{\lambda}$.
- (4) The arc $\alpha([t_0, 1])$ is contained entirely in \mathcal{U} .

Now take $E \in \Sigma_{\alpha(0)}$ and its continuation $E(t)$ along $t \in (0, 1]$. Appealing again to the analyticity of the center-stable manifolds (proof of Theorem 3.2.1) and to transversality (Theorem 3.2.1), we have that $E(t)$ is analytic on $(0, 1]$; therefore, $V_{\alpha(t)}(E(t))$ is also analytic. This implies

Lemma 3.2.5 *Let $E(t)$ be as above, and assume in addition that for all t , $E(t)$ is not an extremum of $\Sigma_{\alpha(t)}$. Then $V'_{\alpha(t)}(E(t)) = \frac{dV_{\alpha(t)}(E(t))}{dt}$ admits at most finitely many zeros in the interval $[t_0, 1]$.*

Proof: Assume for the sake of contradiction that V' admits infinitely many zeros in $[t_0, 1]$. By analyticity, this means that $V_{\alpha(t)}(E(t))$ is constant on $[t_0, 1]$. Using analyticity again, it then must be constant on $(0, 1]$. Since $E(t)$ is not an extremum of $\Sigma_{\alpha(t)}$, we have $V_{\alpha(t)}(E(t)) > 0$ by monotonicity of $V_{\alpha(t)}$ in E , which we know from (3.2.7). Consequently, $|E(t)| \rightarrow \infty$ as $t \rightarrow 0$, since $\ell_{\alpha(t)} \rightarrow \ell_{\alpha(0)}$ in the C^1 topology as $t \rightarrow 0$, and $\ell_{\alpha(0)}$ lies in S_0 . On the other hand, by compactness of $[0, 1]$ and continuity (in operator norm topology) of the map $t \mapsto H_{\alpha(t)}$, $\Sigma_{\alpha(t)}$ is uniformly bounded for $t \in [0, 1]$.

Proof: [Proof of Theorem 3.1.2] Now with $E_0 \in \Sigma_\lambda$ as in Lemma 3.2.4, take $E \in \Sigma_\lambda$ in a neighborhood of E_0 so that E is not an extremum of Σ_λ and (1) of Lemma 3.2.4 is satisfied. By Lemma 3.2.5, there exist finitely many points $\{p_1, \dots, p_n\} \subset [t_0, 1]$ such that $[t_0, 1] \setminus \{p_1, \dots, p_n\}$ can be written as a union of compact nondegenerate intervals $\{J_n\}_{n \in \mathbb{N}}$ such that there exists a sequence $\{\delta_n > 0\}_{n \in \mathbb{N}}$ such that for all $t \in J_n$, $V'_{\alpha(t)}(E(t)) > \delta_n$. By continuity it follows that for all n there exist $E_n^+, E_n^- \in \Sigma_\lambda$ with $E \in (E_n^+, E_n^-)$, such that for all $t \in J_n$ and $\tilde{E} \in \Sigma_\lambda \cap [E_n^-, E_n^+]$, $\tilde{E}(t) \in [E_n^-(t), E_n^+(t)]$ and $V'_{\alpha(t)}(\tilde{E}(t)) > \delta_n$ (here $\tilde{E}(t)$ and $E_n^\pm(t)$ are defined similarly to $E(t)$ as the continuations of \tilde{E} and E_n^\pm , respectively).

Now let us replace \mathcal{E} in Proposition 3.2.3 with $\mathcal{E}_n \subset \mathcal{E}$ such that $\Pi_{\alpha(1)}(\mathcal{E}_n) \subset \Sigma_\lambda \cap [E_n^-, E_n^+]$ so that $\Pi_{\alpha(t)}(\mathcal{E}_n) \subset \Sigma_{\alpha(t)} \cap [E_n^-(t), E_n^+(t)]$ and $dk_{\alpha(t)}(\Pi_{\alpha(t)}(\mathcal{E}_n)) > 0$, J with J_n , δ with δ_n , η with dk_λ , and $\nu_{\alpha(t)}$ with $dk_{\alpha(t)}$ restricted to $[E_n^+(t), E_n^-(t)]$ and normalized. We can do this by taking a sufficiently fine refinement of the original Markov partition so that $\nu_{\alpha(t)}$ is a normalized restriction of $dk_{\alpha(t)}$ to the intersection of $\Sigma_{\alpha(t)}$ with a compact interval of nonzero $dk_{\alpha(t)}$ measure. Then from Proposition 3.2.4 we have $dk_\lambda * dk_{\alpha(t)} \ll \mathcal{L}^1$ for Lebesgue almost every $t \in J_n$. Since this holds for every analytic curve α which satisfies items (1)–(4), Theorem 3.1.2 follows.

Remark 3.2.4 *It is now easy to see, combining Lemmas 3.2.4 and 3.2.3 that there exists a nonempty open set $\mathcal{U} \subset \mathcal{R}$ that satisfies the conclusions of both Theorem 3.1.1 and Theorem 3.1.2.*

3.2.4 Proof of Theorem 3.1.3

Damanik, Fillman, and Gorodetski have computed the curve of initial conditions for this version of the continuum Fibonacci Hamiltonian in [9].

(3.2.18)

$$\begin{aligned} x(E) &= \cos \sqrt{E}, & y(E) &= \cos \sqrt{E - \lambda}, \\ z(E) &= \cos \sqrt{E} \cos \sqrt{E - \lambda} - \frac{1}{2} \left(\sqrt{\frac{E}{E - \lambda}} + \sqrt{\frac{E - \lambda}{E}} \right) \sin \sqrt{E} \sin \sqrt{E - \lambda}. \end{aligned}$$

In particular, $\ell_\lambda(E) = (x(E), y(E), z(E))$ is an analytic curve in \mathbb{R}^3 and \mathbb{C}^3 both, in E and in λ , for each $\lambda > 0$. Using the expressions for x , y , and z , we can compute the Fricke-Vogt invariant as a function of E . We get

$$(3.2.19) \quad I(E, \lambda) \stackrel{\text{def}}{=} I(x(E), y(E), z(E)) = \frac{\lambda^2}{4E(E - \lambda)} \sin^2 \sqrt{E} \sin^2 \sqrt{E - \lambda}.$$

Notice that the expressions in (3.2.18) and (3.2.19) are analytic in E and in λ , with removable singularities at $E = 0$ and $E = \lambda$. Note also that for all λ , $I(E, \lambda) \rightarrow 0$ as $E \rightarrow \infty$, so the analytic curve in (3.2.18) approaches the Cayley cubic in the high-energy regime. As before, the proof of Theorem 3.1.3 relies on two pieces:

1. Near the bottom, the spectrum is thin in the sense of Hausdorff dimension for large enough coupling.
2. For any fixed coupling, the spectrum has large local thickness at sufficiently high energies.

We can use [9] to control the Hausdorff dimension of the spectrum near the bottom. First, we need some control on the ground state energy.

Lemma 3.2.6 Denote $E_0(\lambda) = \inf(\Sigma_\lambda)$. There exists a constant $C \in (0, 3)$ such that $0 \leq E_0(\lambda) \leq C$ for all $\lambda \geq 0$.

Proof: Since $V_\lambda(x) \geq 0$ for all $x \in \mathbb{R}$ and all $\lambda \geq 0$, the inequality $E_0(\lambda) \geq 0$ follows immediately. Notice that there is an interval $I \subset \mathbb{R}$ of length two such that V_λ vanishes on I for all $\lambda \geq 0$ (obviously, there are infinitely many such intervals). Let φ be a smooth function with compact support contained in I such that $\|\varphi\|_2 = 1$. Then

$$\langle \varphi, H_\lambda \varphi \rangle = \int \bar{\varphi}(-\varphi'' + V_\lambda \varphi) = \int |\varphi'|^2 < \infty.$$

The second equality follows from $V|_{\text{supp}(\varphi)} \equiv 0$ and integration by parts. Since H_λ is self-adjoint, we have

$$\inf \Sigma_\lambda = \inf_{\|\psi\|=1} \langle \psi, H\psi \rangle \leq \langle \varphi, H_\lambda \varphi \rangle,$$

so we may take $C = \|\varphi'\|_2^2$. By choosing φ to be a smooth function which is suitably close to a (the square root of) a tent function on an interval of length two, we see that we can make $\|\varphi'\|_2^2 < 3$.

Lemma 3.2.7 In the large coupling regime, we have

$$\lim_{\lambda \rightarrow \infty} \mathbf{dim}_H^{\text{loc}}(\Sigma_\lambda, E_0(\lambda)) = 0.$$

Proof: This follows from [9, Theorem 6.5], (3.2.19), and Lemma 3.2.6. Notice that we need the C from Lemma 3.2.6 to be bounded away from π^2 to effectively bound $I(E_0(\lambda), \lambda)$ from below as $\lambda \rightarrow \infty$.

Remark 3.2.5 As mentioned in the introduction, it follows directly from [9] that for all $\lambda \geq 0$, $\mathbf{dim}_H(\Sigma_\lambda) = 1$. It also follows from [9] that \mathbf{dim}_H accumulates at infinity (that is, for all $E > 0$, $\mathbf{dim}_H([E, \infty) \cap \Sigma_\lambda) = 1$). On the other hand, there may exist other points where the dimension accumulates. Indeed, $E \in \Sigma_\lambda$ is such a point if and only if $\ell_\lambda(E) \in S_0$ (see [16, Section 2]). Now, if λ is of the form $4\pi^2(a^2 - b^2)$ for $a, b \in \mathbb{N}$ with $a > b$, then with $E = 4a^2\pi^2$, it is evident from (3.2.18) that $\ell_\lambda(E) = (1, 1, 1)$, which is a point on S_0 that is fixed under the action by f . Thus we have $\mathbf{dim}_H^{\text{loc}}(\Sigma_\lambda, E) = 1$.

Lemma 3.2.8 *For all $\lambda_1, \lambda_2 > 0$ sufficiently large, there exists an interval J such that $J \cap \Sigma_{(\lambda_1, \lambda_2)}^2$ is a (nonempty) Cantor set.*

Proof:

Take λ_1, λ_2 so that $\mathbf{dim}_H^{\text{loc}}(\Sigma_{\lambda_i}, E_0(\lambda_i)) < \frac{1}{4}$. Observe that $E_0 \stackrel{\text{def}}{=} E_0(\lambda_1) + E_0(\lambda_2)$ marks the bottom of $\Sigma_{(\lambda_1, \lambda_2)}^2$. It follows that for all $\epsilon > 0$ sufficiently small, there exist $\delta_i > 0$ and sets

$$J_i^{(k)}(\delta_i) \subset \Sigma_{\lambda_i} \cap (E_0(\lambda_i), E_0(\lambda_i) + \delta_i), \quad i = 1, 2, k \in \mathbb{N},$$

such that

$$(3.2.20) \quad \Sigma_{(\lambda_1, \lambda_2)}^2 \cap (E_0, E_0 + \epsilon) \subseteq \bigcup_{k \in \mathbb{N}} [J_1^{(k)}(\delta_1) + J_2^{(k)}(\delta_2)].$$

It may happen that ℓ_{λ_i} intersects a center-stable manifold tangentially at the point $\ell_{\lambda_i}(E_0(\lambda_i))$; however, since tangencies cannot accumulate (see [72, Section 3.2.1]), there exist $\epsilon_1, \epsilon_2 > 0$ such that for all $E_i \in \Sigma_{\lambda_i} \cap (E_0(\lambda_i), E_0(\lambda_i) + \epsilon_i)$, the intersection of ℓ_{λ_i} with a center-stable manifold at the point $\ell_{\lambda_i}(E_i)$ is transversal. It follows from [72, Proposition 3.2] that for all i and k such that $J_i^{(k)} \subset \Sigma_{\lambda_i} \cap (E_0(\lambda_i), E_0(\lambda_i) + \epsilon_i)$, $\mathbf{dim}_H(J_i^{(k)}) = \mathbf{dim}_B(J_i^{(k)})$.

Now choose $\epsilon_i > 0$ in the previous paragraph so small, that in addition to the transversality condition being satisfied, we have

$$\mathbf{dim}_H(\Sigma_{\lambda_i} \cap (E_0(\lambda_i), E_0(\lambda_i) + \epsilon_i)) < \frac{1}{4}.$$

Take $\epsilon > 0$ above sufficiently small, ensuring that we can take $\delta_i \in (0, \epsilon_i)$. Then just as in (3.2.9) we have

$$\mathbf{dim}_H(J_1^{(k)}(\delta_1) + J_2^{(k)}(\delta_2)) \leq \mathbf{dim}_H(J_1^{(k)}(\delta_1)) + \mathbf{dim}_H(J_2^{(k)}(\delta_2)) < \frac{1}{2} \text{ for all } k.$$

But then from (3.2.20) we have $\mathbf{dim}_H(\Sigma_{(\lambda_1, \lambda_2)}^2 \cap (E_0, E_0 + \epsilon)) \leq \frac{1}{2} < 1$, ensuring that $\Sigma_{(\lambda_1, \lambda_2)}^2$ is a Cantor set near E_0 .

Remark 3.2.6 *We do not suspect that tangencies between ℓ_λ and the center-stable manifolds occur; however, a proof of absence of tangencies would introduce unnecessary technical difficulties. For this reason, we have decided to omit it.*

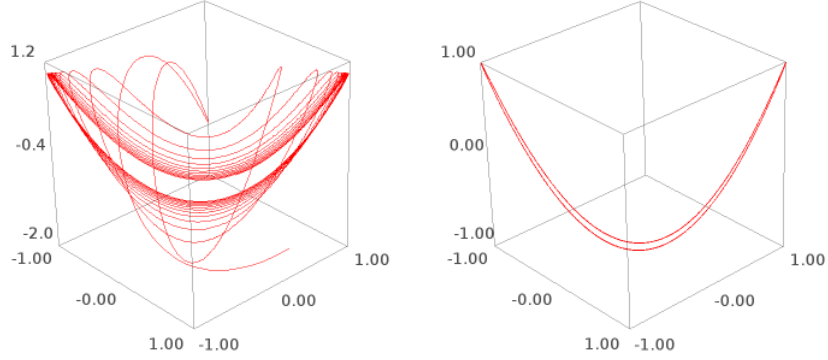


Figure 3.4:

To prove that the interior of Σ_λ^2 is nonempty for all $\lambda > 0$, we first prove the following lemma (in what follows, \mathbb{S} , \mathcal{A} , and F are as in (3.2.15), (3.2.16), and (3.2.17)).

Lemma 3.2.9 *For all $\lambda > 0$ and $\tau_0 > 0$ there exists $M > 0$ and infinitely many $E \in (M, \infty) \cap \Sigma_\lambda$, such that $\tau^{\text{loc}}(\Sigma_\lambda, E) > \tau_0$.*

Proof: Observe that ℓ_0 passes through the point $(0, 0, -1)$, which is periodic. Its continuation to the surfaces S_V , $V \neq 0$, is of the form $(0, 0, -\sqrt{V+1})$. The center-stable manifold containing the point $(0, 0, -1)$ is transversal to \mathbb{S} ; denote this manifold by W .

It is easy to see that ℓ_0 lies in \mathbb{S} , and that $F^{-1}(\ell_0)$ is horizontal in \mathbb{T}^2 . On the other hand, since the eigendirections of \mathcal{A} do not align with $(1, 0)$, $F^{-1}(\ell_0)$ is transversal to the stable foliation of \mathcal{A} on \mathbb{T}^2 . Now by [10, Lemma 3.1] we conclude that ℓ_0 is transversal to the stable foliation of f on \mathbb{S} . It follows that ℓ_0 is transversal to W at the point $(0, 0, -1)$.

Let $\mathbf{dist}(A, B)$ denote the distance between the sets A and B . It is easy to see from the expression for ℓ_λ that away from a neighborhood of the points $(1, 1, 1)$ and $(-1, -1, 1)$, $\ell_\lambda((E_{\text{end}}, \infty))$ approaches $\ell_0(\mathbb{R})$ in the C^1 topology as $E_{\text{end}} \rightarrow \infty$ (see Figure 3.4). It follows that there exists $\delta \in (0, \frac{\pi}{2})$ such that for all $\lambda > 0$ there exists $\epsilon_0 > 0$, such that for every $\epsilon \in (0, \epsilon_0)$, every segment L of $\ell_\lambda(\mathbb{R})$ that satisfies $\mathbf{dist}(L, \{(0, 0, -1)\}) < \epsilon$ intersects the center-stable manifolds uniformly transversally at an angle of size at least δ . Now the method of (the proof of) Theorem 3.1.1 applies

and yields $\tau^{\text{loc}}(\Sigma_\lambda, E) > \tau_0$ for infinitely many sufficiently large E .

Now with τ_0 in Lemma 3.2.9 chosen in $(1, \infty)$, it follows, just as in the proof of Theorem 3.1.1, that for all $\lambda_i > 0$, $i = 1, 2$, $\Sigma_{(\lambda_1, \lambda_2)}^2$ contains an interval. Combining this with Lemma 3.2.8, we obtain Theorem 3.1.3.

Remark 3.2.7 *As mentioned in Remark 3.1.4 (2), we suspect that for all $\lambda_i > 0$, $i = 1, 2$, there exists $E_i \in \Sigma_{\lambda_i}$ such that $[E_1 + E_2, \infty) \subset \Sigma_{(\lambda_1, \lambda_2)}^2$, which is obviously stronger than $\Sigma_{(\lambda_1, \lambda_2)}^2$ just having nonempty interior. A proof of this would involve control of the global thickness of $\Sigma_\lambda \cap [E, \infty)$ as $E \rightarrow \infty$. The global thickness could be controlled via finer estimates than those used for the control of the local thickness in the proofs above, but the arguments would be far more technical. For this reason we have decided to delegate this task to later investigations.*

BIBLIOGRAPHY

Bibliography

- [1] R. Anisca, C. Chlebovec, On the structure of arithmetic sums of Cantor sets with constant ratios of dissection, *Nonlinearity* **22** (2009), 2127–2140.
- [2] S. Astels, Cantor sets and numbers with restricted partial quotients, *Trans. Amer. Math. Soc.* **352** (2000), 133–170.
- [3] G. T. Buzzard, K. Verma, Hyperbolic automorphisms and holomorphic motions in \mathbb{C}^2 , *Michigan Math. J.* **49** (2001), 541–565.
- [4] S. Cantat, Bers and Hénon, Painlevé and Schrödinger, *Duke Math. J.* **149** (2009), 411–460.
- [5] M. Casdagli, Symbolic dynamics for the renormalization map of a quasiperiodic Schrödinger equation, *Commun. Math. Phys.* **107** (1986), 295–318.
- [6] V.Z. Cerovski, M. Schreiber, U. Grimm, Spectral and diffusive properties of silver-mean quasicrystals in one, two, and three dimensions, *Physical Review B* **72** (2005).
- [7] D. Damanik, M. Embree, A. Gorodetski, *Spectral properties of Schrödinger operators arising in the study of quasicrystals*, Mathematics of aperiodic order, Birkhaeuser, 2015.
- [8] D. Damanik, M. Embree, A. Gorodetski, S. Tcheremchantsev, The fractal dimension of the spectrum of the Fibonacci Hamiltonian, *Commun. Math. Phys.* **280** (2008), 499–516.

- [9] D. Damanik, J. Fillman, A. Gorodetski, Continuum Schrödinger operators associated with aperiodic subshifts, *Ann. Henri Poincaré* **15** (2014), 1123–1144.
- [10] D. Damanik, A. Gorodetski, Hyperbolicity of the trace map for the weakly coupled Fibonacci Hamiltonian, *Nonlinearity* **22** (2009), 123–143.
- [11] D. Damanik, A. Gorodetski, Spectral and quantum dynamical properties of the weakly coupled Fibonacci Hamiltonian, *Commun. Math. Phys.* **305** (2011), 221–277.
- [12] D. Damanik, A. Gorodetski, The density of states measure of the weakly coupled Fibonacci Hamiltonian, *Geom. Funct. Anal.* **22** (2012), 976–989.
- [13] D. Damanik, A. Gorodetski, B. Solomyak, Absolutely continuous convolutions of singular measures and an application to the Square Fibonacci Hamiltonian, *Duke Math. J.* **164** (2015), 1603–1640.
- [14] D. Damanik, A. Gorodetski, W. Yessen, The Fibonacci Hamiltonian, *Invent. Math.* **206** (2016), 629–692.
- [15] D. Damanik, D. Lenz, Linear repetitivity. I. Uniform subadditive ergodic theorems and applications, *Discrete Comput. Geom.* **26** (2001), 411–428.
- [16] D. Damanik, P. Munger, W. Yessen, Orthogonal polynomials on the unit circle with Fibonacci Verblunsky coefficients, I. The essential support of the measure, *J. Approx. Theory* **173** (2013), 56–88.
- [17] D. Damanik, P. Munger, W. Yessen, Orthogonal polynomials on the unit circle with Fibonacci Verblunsky coefficients, II. Applications., *J. Stat. Phys.* **153** (2013), 339–362.
- [18] K. Eroglu, On the arithmetic sums of Cantor sets, *Nonlinearity* **20** (2007), 1145–1161.
- [19] S. Even-Dar Mandel, R. Lifshitz, Electronic energy spectra of square and cubic Fibonacci quasicrystals, *Philosophical Magazine* **88** (2008), 2261–2273.

- [20] J. Fillman, Y. Takahashi, W. Yessen, Mixed spectral regimes for square Fibonacci Hamiltonians, *J. Fract. Geom.* **3** (2016), 377-405.
- [21] R. Ilan, E. Liberty, S. Even-Dar Mandel, R. Lifshitz, Electrons and phonons on the square Fibonacci tiling, *Ferroelectrics* **305** (2004), 15–19.
- [22] M. Hall, On the sum and product of continued fractions, *Ann. of Math.* **48** (1947), 966–993.
- [23] B. Hasselblatt, Handbook of Dynamical Systems: Hyperbolic Dynamical Systems, vol. 1A, Elsevier B. V., Amsterdam, The Netherlands, 2002.
- [24] B. Hasselblatt, Y. Pesin, Partially hyperbolic dynamical systems, Handbook of dynamical systems, *1B* (2006), 1–55, Elsevier B. V., Amsterdam (Reviewer: C. A. Morales).
- [25] M. Hochman, On self-similar sets with overlaps and inverse theorems for entropy, *Ann. of Math.* **180** (2014), 773–822.
- [26] M. Hochman, P. Shmerkin, Local entropy averages and projections of fractal measures, *Ann. of Math.* **175** (2012), 1001–1059.
- [27] B. Honary, C. Moreira, M. Pourbarat, Stable intersections of affine Cantor sets, *Bull. Braz. Math. Soc.* **36** (2005), 363–378.
- [28] B. Hunt, I. Kan, J. Yorke, When Cantor sets intersect thickly, *Trans. Amer. Math. Soc.* **339** (1993), 869–888.
- [29] R. Ilan, E. Liberty, S. Even-Dar Mandel, R. Lifshitz, Electrons and phonons on the square Fibonacci tilings, *Ferroelectrics* **305** (2004), 15–19.
- [30] A. Katok, B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, Cambridge, 1996.
- [31] A. Klenke, *Probability Theory: A Comprehensive Course*, Universitext, 2014.

- [32] R. Kraft, Intersections of thick Cantor sets, *Mem. Amer. Math. Soc.* **97** (1992), 468.
- [33] R. Kraft, Random intersections of thick Cantor sets, *Trans. Amer. Math. Soc.* **352** (2000), 1315–1328.
- [34] C. Lagarias, B. Pleasants, Repetitive Delone sets and quasicrystals, *Ergod. Th. and Dynam. Sys.* **23** (2003), 831–867.
- [35] D. Lenz, P. Stollmann, An ergodic theorem for Delone dynamical systems and existence of the integrated density of states, *J. Anal. Math.* **97** (2005), 1–24.
- [36] D. Lenz, P. Stollmann, Delone dynamical systems and associated random operators, *Operator Algebras and Mathematical Physics, Theta Bucharest* (2003), 267–285.
- [37] R. Lifshitz, The square Fibonacci tiling, *Journal of Alloys and Compounds* **342** (2002), 186–190.
- [38] S. Even-Dar Mandel, R. Lifshitz, Bloch-like electronic wave functions in two-dimensional quasicrystals, preprint.
- [39] S. Even-Dar Mandel, R. Lifshitz, Electronic energy spectra and wave functions on the square Fibonacci tiling, *Philosophical Magazine* **86** (2006), 759–764.
- [40] S. Even-Dar Mandel, R. Lifshitz, Electronic energy spectra of square and cubic Fibonacci quasicrystals, *Philosophical Magazine* **88** (2008), 2261–2273.
- [41] P. Mattila, *Geometry of sets and measures in Euclidean spaces, Cambridge studies in mathematics*, Cambridge University Press **44** (1995).
- [42] M. Mei, Spectra of discrete Schrödinger operators with primitive invertible substitution potentials, *J. Math. Phys.* **55** (2014).
- [43] M. Mei, W. Yessen, Tridiagonal substitution Hamiltonians, *Math. Model. Nat. Phenom.* **9** (2014), 204–238.

- [44] C. Moreira, There are no C^1 -stable intersections of regular Cantor sets, *Acta Math.* **206** (2011), 311–323.
- [45] C. Moreira, E. Morales, Sums of Cantor sets whose sum of dimensions is close to 1, *Nonlinearity* **16** (2003), 1641–1647.
- [46] C. Moreira, E. Muñoz, J. Letelier, On the topology of arithmetic sums of regular Cantor sets, *Nonlinearity* **13** (2000), 2077–2087.
- [47] C. Moreira, J.-C. Yoccoz, Stable intersections of regular Cantor sets with large Hausdorff dimensions, *Ann. of Math.* **154** (2001), 45–96.
- [48] S. Newhouse, The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms, *Inst. Hautes Études Sci. Publ. Math.* **50** (1979), 101–151.
- [49] R. Lifshitz, The square Fibonacci tiling, *Journal of Alloys and Compounds* **342** (2002), 186–190.
- [50] J. Palis, F. Takens, *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations*, Cambridge University Press, Cambridge, 1993.
- [51] M. Pollicott, Analyticity of dimensions for hyperbolic surface diffeomorphisms, *Proc. Amer. Math. Soc.* **143** (2015), 3465–3474.
- [52] Y. Peres, B. Solomyak, Self-similar measures and intersections of Cantor sets, *Trans. Amer. Math. Soc.* **350** (1998), 4065–4087.
- [53] M. Queffélec, *Substitution Dynamical Systems - Spectral Analysis*, Lecture Notes in Mathematics, 2010.
- [54] M. Reed, B. Simon, *Methods of Modern Mathematical Physics I*, Cambridge University Press, Cambridge, 1990.
- [55] J. A. G. Roberts, Escaping orbits in trace maps, *Physica A: Stat. Mech. App.* **228** (1996), 295–325.

- [56] J. A. G. Roberts, M. Baake, The Dynamics of Trace maps, Hamiltonian Mechanics: Integrability and Chaotic Behavior, *ed. J. Seimenis, NATO ASI Series B: Physics (Plenum Press, New York)* (1994), 275–285.
- [57] L. A. G. Roberts, M. Baake, Trace maps as 3D reversible dynamical systems with an invariant, *J. Stat. Phys.* **74** (1994), 829–888.
- [58] S. Rolof, S. Thiem, M. Schreiber, Electronic wave functions of quasiperiodic systems in momentum space, *The European Physical Journal B* **86** (2013), 372.
- [59] C. Sire, Electronic spectrum of a 2D quasi-crystal related to the octagonal quasiperiodic tiling, *Europhys. Lett.* **10** (1989), 483–488.
- [60] C. Sire, R. Mosseri, J.-F. Sadoc, Geometric study of a 2D tiling related to the octagonal quasiperiodic tiling, *J. Physique* **55** (1989), 3463–3476.
- [61] B. Solomyak, On the measure of arithmetic sums of Cantor sets, *Indag. Math.* **8** (1997), 133–141.
- [62] A. Sütö, The spectrum of a quasiperiodic Schrödinger operator, *Commun. Math. Phys.* **111** (1987), 409–415.
- [63] G. Teschl, *Jacobi operators and completely integrable nonlinear lattices*, AMS mathematical surveys and monographs **vol. 72**, American Mathematical Society, Providence, RI.
- [64] S. Thiem, M. Schreiber, Generalized inverse participation numbers in metallic-mean quasiperiodic systems, *Physics of Condensed Matter* **84** (2011).
- [65] S. Thiem, M. Schreiber, Quantum diffusion in separable d-dimensional quasiperiodic tilings, preprint (arXiv: 1212.6337).
- [66] S. Thiem, M. Schreiber, Renormalization group approach for the wave packet dynamics in golden-mean and silver-mean Labyrinth tilings, *Physical Review B* **85** (2012), 224205.

- [67] S. Thiem, M. Schreiber, Similarity of eigenstates in generalized Labyrinth tilings, *Journal of Physics Conference Series* **226** (2010), 012029.
- [68] S. Thiem, M. Schreiber, Wave functions, quantum diffusion and scaling exponents in golden-mean quasiperiodic tilings, *Journal of Physics Condensed Matter* **25** (2013), 075503.
- [69] Y. Takahashi, Products of two Cantor sets, *Nonlinearity* **30** (2017), 2114–2137.
- [70] Y. Takahashi, Quantum and spectral properties of the Labyrinth model, *J. Math. Phys.* **57** (2016).
- [71] R. Williams, How big is the intersection of two thick Cantor sets?, *Contemp. Math.* **117** (1991).
- [72] W. Yessen, Spectral analysis of tridiagonal Fibonacci Hamiltonians, *J. Spectr. Theory* **3** (2013), 101–128.
- [73] W. Yessen, On the spectrum of 1D quantum Ising quasicrystal, *Annal. H. Poincaré* **15** (2014), 419–467.
- [74] H.Q. Yuan, U. Grimm, P. Repetowicz, M. Schreiber, Energy spectra, wave functions and quantum diffusion for quasi periodic systems, *Physical Review B* **62** (2000), 15569.