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Large expanders in high genus unicellular maps

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Abstract. We study large uniform random maps with one face whose genus grows linearly with the number of edges. They can be seen as a model of discrete hyperbolic geometry. In the past, several of these hyperbolic geometric features have been discovered, such as their local limit or their logarithmic diameter. In this work, we show that with high probability such a map contains a very large induced subgraph that is an expander.

Keywords. Combinatorial maps, high genus, expander graphs

Mathematics Subject Classifications. 05C10, 05C48, 60C05, 60D05

1. Introduction

Combinatorial maps Combinatorial maps are discrete geometric structures constructed by gluing polygons along their sides to form (compact, connected, oriented) surfaces. They appear in various contexts, from computer science to mathematical physics, and have been given a lot of attention in the past few decades. The first model that was extensively studied is *planar maps* (or maps of the sphere), starting with their enumeration [Tut62, Tut63] by generating function methods. Later on, explicit constructions put planar maps in bijection with models of decorated trees [Sch98, BDFG04, BF12, AP15], and geometric properties of large random uniform planar maps have been studied [AS03, CS04, LG13, Mie13]. All these works were later extended to maps on fixed surfaces of any genus (see for instance [WL72, BC86] for enumeration, [CMS09, Lep19] for bijections, and [Bet16] for random maps).

High genus maps Much more recently, another regime of maps has been studied: high genus maps, that is (sequences of) maps whose genus grows linearly in the size of the map. The main goal is to study the geometric properties of a random uniform such map as the size tends to infinity. By the Euler formula, the high genus implies that these maps have negative average discrete curvature (or, equivalently, that the average degree is higher than in planar maps). They

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must therefore have hyperbolic features, some of whose have been identified in previous works [ACCR13, Ray15, BL21, BL22, Lou22].

Mostly, two types of models of high genus maps have been dealt with. First, *unicellular maps*, i.e. maps with one face, who are easier to tackle thanks to an explicit bijection [CFF13], and then more general models of maps like triangulations or quadrangulations. It is believed that both models have a similar behaviour.

The local behaviour of high genus maps around their root is now well understood ([ACCR13] in the unicellular case, [BL21, BL22] in the general case), and some global properties have been tackled: the planarity radius [Lou22] (see also [Ray15] for unicellular maps) and the diameter ([Ray15] for unicellular maps, still open for other models).

Large expanders: a result and a conjecture In this paper we deal with yet another property: the presence of large expanders inside our map, in the case of unicellular maps. Contrary to the previous properties, this involves the whole geometric structure of the map. Expander graphs are very well connected graphs, in which every set of vertices has a large number of edges going out of it, which is a typical hyperbolic behaviour[1]. Unfortunately, it is impossible that the whole map itself is an expander, since it can be shown that finite but very large trees (which have very bad influence on the expansion of the graph) can be found somewhere in the map. However, we will show that most of the map is an expander, in the following sense.

Let $\frac{g_n}{n} \to \theta \in (0, 1/2)$, and let $\mathbf{U}_{n,g_n}$ be a uniform unicellular map with n edges and genus g_n.

Theorem 1.1. *For all $\varepsilon > 0$, there exists a constant $\kappa > 0$ depending only of ε and θ such that the following is true. With high probability[2], there exists an induced subgraph G_n of $\mathbf{U}_{n,g_n}$ that has at least $(1-\varepsilon)n$ edges and is a κ-expander.*

It is natural to conjecture that a similar result holds for more general models of maps (i.e., without a fixed number of faces). For instance, let $\mathbf{T}_{n,g_n}$ be a uniform triangulation of genus g_n with $3n$ edges. The following conjecture (and the present work) comes from a question of Itai Benjamini (private communication) about large expanders in high genus triangulations.

Conjecture 1.2. For all $\varepsilon > 0$, there exists a $\kappa > 0$ depending only on ε and θ such that the following is true. With high probability, there exists an induced subgraph G_n of $\mathbf{T}_{n,g_n}$ that has at least $(1-\varepsilon)3n$ edges and is a κ-expander.

This conjecture deals with the entire structure of the map, therefore we believe it is a very ambitious open problem about the geometry of high genus maps. Some other conjectures might be easier to tackle, see [Lou22].

Structure of the paper The proof of the main result involves the refinement/extension of several known results (Lemma 4.1, Proposition 4.3 and Lemma 5.2). We chose to push all the technical proofs to the appendix, in order to make it clear how we combine these results to obtain the proof of our main theorem. However, each time we include a paragraph explaining the main idea of the proof.

[1] indeed, for each set, a large proportion of its mass is contained on its boundary.

[2] throughout the paper, we will write *with high probability* or *whp* in lieu of *with probability tending to* 1 *as* $n \to \infty$.

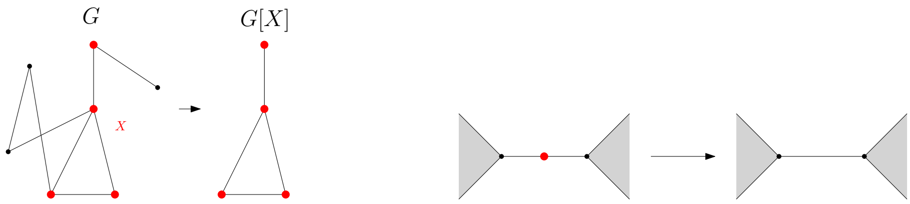

Figure 2.1: Left: A graph G, with a vertex subset X, and the induced subgraph $G[X]$. Right: Smoothing a vertex of degree 2.

The main objects are defined in the next section, then we give an outline of the proof. The proof consists roughly of two halves: showing that the kernel is an expander (Section 4) and showing that a well defined "almost kernel" is still an expander while having a large proportion of the edges (Section 5). Finally, in the appendix, we give the proofs of the technical lemmas.

2. Definitions

We begin with some definitions about graphs. Note that here we allow graphs to have loops and multiple edges, such objects are also commonly called multigraphs. The *degree* of a vertex v is the number of half-edges it is incident to. In other words, each loop incident to v adds 2 to its degree, and multiple edges contribute to 1 each to the degree of v. We will write $e(G)$ for the number of edges in a graph G. An *induced subgraph* of a graph G is a graph obtained from G by deleting some of its vertices and all the edges incident to these vertices. Given a subset X of vertices of G, we write $G[X]$ for the induced subgraph of G obtained by deleting all vertices that do not belong to X (see Figure 2.1 left). A *topological minor* of a graph G is a graph obtained from G by deleting some of its vertices, some of its edges, and by "smoothing" some of its vertices of degree 2 as depicted in Figure 2.1 right.

We will now define the notion of *expansion*. More precisely, we will work with edge-expansion. Often an alternative version, vertex-expansion, is used. Given a graph G and a subset X of its vertices, we define $\mathrm{vol}(X) = \sum_{v \in X} \deg(v)$ and $\partial_G(X)$ as the number of edges of G with exactly one endpoint in X. Then we set

$$h_G(X) = \frac{\partial_G(X)}{\min(\mathrm{vol}(X), \mathrm{vol}(\overline{X}))},$$

where $\overline{X}$ is the set of vertices of G that do not belong to X. The *Cheeger constant* h_G of a graph G is

$$h_G = \min_{\substack{X \subset V(G) \\ X, \overline{X} \neq \varnothing}} h_G(X).$$

It is easily verified that the min can be taken only on subsets X such that $G[X]$ is connected (see for instance [MR04], Section 3.1). A graph G is said to be a *κ-expander* if $h_G(X) \geqslant \kappa$.

A *map* is the data of a collection of polygons whose sides are glued two by two to form a compact oriented surface. The interior of the polygons define the *faces* of the map. After the gluing, the sides of the polygons become the *edges* of the map, and the vertices of the polygons

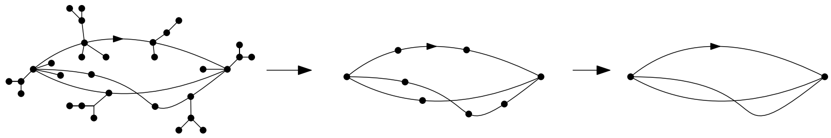

Figure 2.2: The kernel decomposition of a unicellular map.

become the *vertices* of the map. Alternatively, a map is the data of a graph endowed with a *rotation system*, i.e. a clockwise ordering of half-edges around each vertex. A *unicellular map* of size n is the data of a $2n$-gon whose sides were glued two by two to form a compact, connected, orientable surface. The genus g of the surface is also called the *genus* of the map. We will consider *rooted maps*, i.e. maps with a distinguished oriented edge called the *root*. Let $\mathcal{U}_{n,g}$ be the set of rooted unicellular maps of size n and genus g. A map of $\mathcal{U}_{n,g}$ has exactly $n+1-2g$ vertices by Euler's formula. We will denote by $\mathbf{U}_{n,g}$ a random uniform element of $\mathcal{U}_{n,g}$.

A (plane) *tree* is a unicellular map of genus 0. A *bipointed tree* is a plane tree with a ordered pair of distinct marked vertices (but without a distinguished oriented edge). The *size* of a bipointed tree is its number of edges. We set bt_n to be the number of bipointed trees of size n. The *kernel* of a unicellular map m, denoted $\ker(m)$, is the map obtained from m by iteratively deleting all its leaves, then smoothing all its vertices of degree 2 (see Figure 2.2). We do not make precise here how the root of $\ker(m)$ is obtained from the root of m, we will only explain it in Section C (the only place where the root matters, for enumeration purposes, everywhere else in the paper we will only need the graph structure of the kernel).

Reciprocally, m can be obtained from $\ker(m)$ in a unique way by replacing each edge of $\ker(m)$ by a bipointed tree. More precisely, given a bipointed tree t and its two distinguished vertices v_1 and v_2, there is a unique simple path p going from v_1 to v_2. Let e_1 (resp. e_2) be the edge of p that is incident to v_1 (resp. v_2), and let c_1 (resp. c_2) be the corner that comes right before e_1 (resp. e_2) in the counterclockwise order around v_1 (resp. v_2). Now, we can remove an edge e from $\ker(m)$ to obtain a map with a pair of marked corners c and c', and we can glue c_1 on c and c_2 on c' (see Figure 2.3). The list of bipointed trees used to construct m from $\ker(m)$ will be called the *branches* of m. We also define $\ker^{<M}(m)$ to be the map obtained from m by replacing all its branches of size larger or equal to M by a single edge. Notice that both $\ker(m)$ and $\ker^{<M}(m)$ are topological minors of m.

For any sequence of integers $\mathbf{d} = d_1 \leqslant d_2 \leqslant \dots \leqslant d_k$, we set $|\mathbf{d}| = \sum_{i=1}^{k} d_i$. Let $\mathcal{U}(\mathbf{d})$ be the set of rooted unicellular maps with vertex degrees given by $\mathbf{d}$[3]. If m is a unicellular map, let $\mathbf{d}(m)$ be the sequence of its vertex degrees in increasing order. Now, we introduce the *configuration model*. The random map $\mathrm{CM}(\mathbf{d})$ is defined in the following way: let $v_1, v_2, \dots, v_k$ be vertices such that v_i has d_i distinguishable dangling half-edges arranged in clockwise order around it. Now $\mathrm{CM}(\mathbf{d})$ is the random map obtained by taking a random uniform pairing of all the dangling half-edges, and then picking a uniform oriented edge as the root (see Figure 2.4).

[3]notice that if $|\mathbf{d}|$ is odd, or if $|\mathbf{d}|/2+k$ is even, then $\mathcal{U}(\mathbf{d})$ is empty.

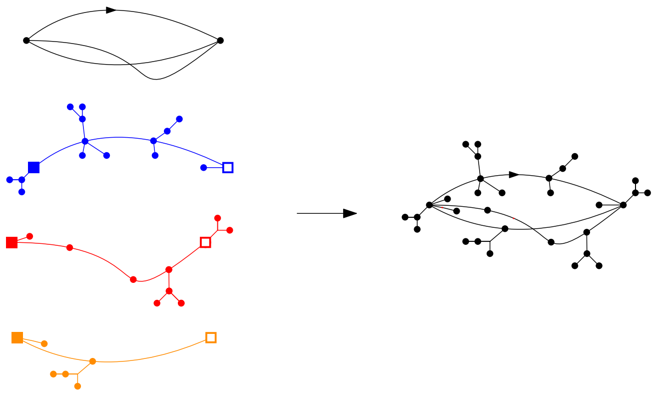

Figure 2.3: Reconstructing a map from its kernel and branches. The squares correspond to the marked vertices of the bipointed trees.

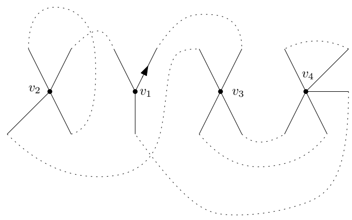

Figure 2.4: The map configuration model for $\mathbf{d} = (3, 4, 4, 5)$.

3. Strategy of proof

In this section, we give the outline of the proof of Theorem 1.1. First, the problem reduces to finding a good topological minor, because of the following theorem:

Theorem 3.1 ([LS21], Theorem 1)**.** *For all $\kappa, \alpha > 0$, and for all $0 < \alpha' < \alpha$, there exists a constant $\kappa' > 0$ such that the following holds for every (multi)graph G.*

If there exists a graph H satisfying the following conditions:

- $e(H) \geqslant \alpha e(G)$,
- *H is a topological minor of G,*
- *H is a κ-expander,*

then there exists a graph H^* satisfying the following conditions:

- $e(H^*) \geqslant \alpha' e(G)$,
- *H^* is an induced subgraph of G,*
- *H^* is a κ'-expander.*

Furthermore, we will prove the following:

Proposition 3.2. *For all $\varepsilon > 0$, there exist M and $\kappa > 0$ that depend only on θ and ε such that the following is true whp:*

- $\ker^{<M}(\mathbf{U}_{n,g_n})$ *has more than $(1-\varepsilon)n$ edges,*
- $\ker^{<M}(\mathbf{U}_{n,g_n})$ *is a κ-expander.*

It is clear that Theorem 1.1 is an immediate corollary of Theorem 3.1 and Proposition 3.2, since $\ker^{<M}(\mathbf{U}_{n,g_n})$ is a topological minor of $\mathbf{U}_{n,g_n}$.

A key tool of the proof of Proposition 3.2 is the following result.

Proposition 3.3. *There exists a constant $0 < \delta < \frac{1}{15}$ such that, for all $g_n \xrightarrow[n\to\infty]{} \infty$, whp, $\ker(\mathbf{U}_{n,g_n})$ is a δ-expander.*

Remark 3.4. The kernel $\ker(\mathbf{U}_{n,g_n})$ is always an expander (as long as $g_n \xrightarrow[n\to\infty]{} \infty$), however, $\ker(\mathbf{U}_{n,g_n})$ has size $\Theta(g_n)$, hence in the case $g_n = o(n)$, this provides us only with a very small expander inside $\mathbf{U}_{n,g_n}$ and Theorem 1.1 is not true anymore.

In the next section, we will prove Proposition 3.3, and Section 5 is devoted to the proof of Proposition 3.2.

4. The kernel is an expander

Here, we will prove Proposition 3.3 by comparing $\ker(\mathbf{U}_{n,g_n})$ to a well chosen map configuration model. We begin with two results about this model. We will consider sequences $\mathbf{d}^{(s)} = d_1 \leqslant d_2 \leqslant \ldots \leqslant d_k$ that do not contain any 1's or 2's (i.e. $d_1 \geqslant 3$), with $|\mathbf{d}^{(s)}| = 2s$, such that $k+s$ is odd.

Lemma 4.1. *The map $\mathrm{CM}(\mathbf{d}^{(s)})$ is unicellular with probability larger than*

$$\frac{1+o(1)}{3s}$$

as $s \to \infty$, where the $o(1)$ is independent of $\mathbf{d}^{(s)}$.

Idea of proof In [BCP19], the authors study the model dual to $\mathrm{CM}(\mathbf{d}^{(s)})$: they have a collection of polygons whose sides are given by $\mathbf{d}^{(s)}$, and they glue the sides two by two in a uniform fashion. In particular, they study the number of vertices (for us, the number of faces) of the resulting map. Their strategy is to consider the gluings one after the other, as a process. Throughout the process, the current map is possibly disconnected and has *boundaries* which consist of the sides of the polygons that were not glued yet. At each step, two sides are picked and are glued together to form an edge of the map, until every side has been glued to another, and the resulting map has no boundary.

To count the number of vertices, the main idea is the following: at some time τ_s, the current map has only one boundary (with size $2(s-\tau_s)$ since τ_s pairs of sides have been glued together). Let us say that this current map has X_{τ_s} vertices outside its boundary. This single boundary is a $2(s-\tau_s)$-gon, and the pairing of the remaining sides is exactly equivalent to constructing a uniform unicellular map on $(s-\tau_s)$ edges, let us call $V_{2(s-\tau_s)}$ the number of vertices in such a random map. The total number of vertices in the final map is then

$$X_{\tau_s} + V_{2(s-\tau_s)}.$$

We will lower bound the probability that $X_{\tau_s} = 0$ and $V_{2(s-\tau_s)} = 1$.

Remark 4.2. In [CP16], the authors consider the structure of the faces of $\mathrm{CM}(\mathbf{d}^{(s)})$, encoded by a permutation $\sigma_{\mathbf{d}}^{(s)}$ of $\mathfrak{S}_{2s}$ (with an odd number of cycles), and they show that, as $s \to \infty$,

$$d_{TV}(\sigma_{\mathbf{d}}^{(s)}, u_s) \to 0, \tag{4.1}$$

where d_{TV} is the total variation distance and u_s is a random uniform permutation of $\mathfrak{S}_{2s}$ with an odd number of cycles.

Heuristically, this means that $\mathrm{CM}(\mathbf{d}^{(s)})$ should be unicellular with probability $\frac{1+o(1)}{s}$, unfortunately the rate of convergence in (4.1) is of order C/s, with $C > 1$ which does not help us to prove Lemma 4.1. It does however provide an upper bound (which is not needed in this work): $\mathrm{CM}(\mathbf{d}^{(s)})$ is unicellular with probability $O(1/s)$.

This next proposition states that $\mathrm{CM}(\mathbf{d}^{(s)})$ is an expander with very high probability.

Proposition 4.3. *The map $\mathrm{CM}(\mathbf{d}^{(s)})$ is a δ-expander (with the same δ as in Proposition 3.3) with probability*

$$1 - \mathbb{P}(\mathrm{CM}(\mathbf{d}^{(s)}) \textit{ is disconnected}) - o\left(\frac{1}{s}\right)$$

as $s \to \infty$, where the $o\left(\frac{1}{s}\right)$ is independent of $\mathbf{d}^{(s)}$.

Idea of proof The proof is rather technical, but it is heavily inspired by [HLW06, KW14].

We consider a set of vertices $V = \{v_1, v_2, \dots v_k\}$ such that for all i, v_i has d_i dangling half-edges attached. Given a subset $X \subset V$, it breaks the condition of δ-expansion if and only if "too few" half-edges inside X get paired with half-edges outside X. For each subset, X we give an upper bound of the probability that it is a "bad subset" that depends only on its volume $\mathrm{vol}(X)$ (the sum of the degrees of the vertices of X).

The idea is then to show that for δ small enough, if $\mathrm{vol}(X) \leqslant 14$, then X is a bad subset if and only if it is disconnected from the rest of the map, and that for all the X with $\mathrm{vol}(X) \geqslant 15$, the probabilities are small enough so that their sum is $o\left(\frac{1}{s}\right)$, and thus we can conclude by a union bound.

We are now ready to prove Proposition 3.3.

Proof of Proposition 3.3. Let s_n be the random variable that is equal to the number of edges in $\ker(\mathbf{U_{n,g_n}})$, and $\mathbf{d}^{(s_n)} = \mathbf{d}(\ker(\mathbf{U_{n,g_n}}))$. The map $\ker(\mathbf{U}_{n,g_n})$ has genus $g_n \to \infty$ and by Euler's formula, $s_n \geqslant 2g_n$, hence $s_n \to \infty$.

Now, conditionally on $\mathbf{d}^{(s_n)}$, $\ker(\mathbf{U}_{n,g_n})$ is uniform in $\mathcal{U}(\mathbf{d}^{(s_n)})$. Indeed, by definition of the kernel, if c is a unicellular map of genus g_n, the probability that $c = \ker(\mathbf{U}_{n,g_n})$ is proportional to the number of lists of s bipointed trees whose total size is n, where s is the size of c. Hence, this probability only depends on $\mathbf{d}(c)$.

Also, $\mathrm{CM}(\mathbf{d}^{(s_n)})$ conditioned on having one face is uniform in $\mathcal{U}(\mathbf{d}^{(s_n)})$. Indeed, $\mathrm{CM}(\mathbf{d}^{(s_n)})$ is uniform in the set of maps with degrees given by $\mathbf{d}^{(s_n)}$ (for each such map, there is exactly one pairing of the half-edges that constructs this map).

Hence, the probability of $\ker(\mathbf{U}_{n,g_n})$ not being a δ-expander is

$$\begin{aligned}
&\mathbb{P}\left(\mathrm{CM}(\mathbf{d}^{(s_n)}) \text{ is not a } \delta\text{-expander} | \mathrm{CM}(\mathbf{d}^{(s_n)}) \text{ is unicellular}\right) \\
&= \frac{\mathbb{P}\left(\mathrm{CM}(\mathbf{d}^{(s_n)}) \text{ is not a } \delta\text{-expander and is unicellular}\right)}{\mathbb{P}(\mathrm{CM}(\mathbf{d}^{(s_n)}) \text{ is unicellular})}. \qquad (4.2)
\end{aligned}$$

But we also have

$$\begin{aligned}
&\mathbb{P}\left(\mathrm{CM}(\mathbf{d}^{(s_n)}) \text{ is not a } \delta\text{-expander and is unicellular}\right) \\
&\leqslant \mathbb{P}\left(\mathrm{CM}(\mathbf{d}^{(s_n)}) \text{ is not a } \delta\text{-expander and is connected}\right) \\
&= 1 - \mathbb{P}\left(\mathrm{CM}(\mathbf{d}^{(s_n)}) \text{ is a } \delta\text{-expander or is not connected}\right) \\
&= 1 - \mathbb{P}\left(\mathrm{CM}(\mathbf{d}^{(s_n)}) \text{ is a } \delta\text{-expander}\right) - \mathbb{P}\left(\mathrm{CM}(\mathbf{d}^{(s_n)}) \text{ is not connected}\right) \\
&= o(1/s),
\end{aligned}$$

where in the second line we used the fact that all unicellular maps are connected, in the fourth line we used the fact that all expanders are connected and in the last line we used Proposition 4.3.

Using this, along with (4.2) and Lemma 4.1 shows

$$\mathbb{P}\left(\mathrm{CM}(\mathbf{d}^{(s_n)}) \text{ is not a } \delta\text{-expander} | \mathrm{CM}(\mathbf{d}^{(s_n)}) \text{ is unicellular}\right) = o(1).$$

This $o(1)$ does not depend on $\mathbf{d}^{(s_n)}$, hence the proof is finished. □

5. Almost-kernel decomposition

In this section, we prove Proposition 3.2. Our strategy is the following: now that we know that $\ker(\mathbf{U}_{n,g_n})$ is an expander, we will add back to it the "small" branches of $\mathbf{U}_{n,g_n}$ to get very close to the size of $\mathbf{U}_{n,g_n}$ without penalizing the expansion too much. We start with technical lemmas. The first one states that replacing edges by small bipointed trees does not change the Cheeger constant too much.

Lemma 5.1. *Let H be a graph and G be constructed by replacing each edge of H by a bipointed tree of size M or less. Then*

$$h_G \geqslant \frac{h_H}{2M+1}.$$

Proof. This proof is very similar to the proof of Lemma 5 of [LS21].

In G, color in red the vertices that come from H, and the rest in black. Let Y be a subset of $V(G)$ such that $G[Y]$ is connected (recall that we only need to consider connected subsets).

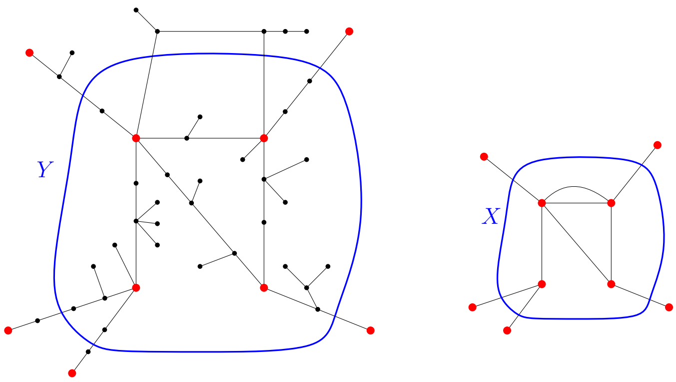

Figure 5.1: Comparing the edge expansions of Y in G and X in H. Here, $M = 7$.

Let X be the set of red vertices in Y, and $\overline{X} = V(H) \setminus X$, i.e. it is the set of red vertices in $\overline{Y}$. We want to lower bound $h_G(Y)$ in terms of $h_H(X)$. See Figure 5.1 for an illustration.

If $X = \varnothing$, then $G[Y]$ is a tree on at most $M - 1$ vertices and so $\mathrm{vol}_G(Y) \leqslant 2(M-1)$ and $e_G(Y, \overline{Y}) \leqslant 2$. Hence

$$h_G(Y) \geqslant \frac{e_G(Y,\overline{Y})}{\mathrm{vol}_G(Y)} \geqslant \frac{1}{M-1}.$$

Similarly if $\overline{X} = \varnothing$ then we use the fact that $h_G(Y) = h_G(\overline{Y})$, and hence $h_G(Y) \geqslant 1/(M-1)$.

Now, consider the case $X \neq \varnothing$ and $\overline{X} \neq \varnothing$. The number of edges of H which are incident to a vertex of X is $e_H(X) + e_H(X, \overline{X}) \leqslant \mathrm{vol}_H(X)$. Each edge of H is replaced by a tree with at most M edges, thus of volume at most $2(M-1)$. Therefore the total degree of the black vertices in Y can be bounded above by $2(M-1)\,\mathrm{vol}_H(X)$. Hence

$$\mathrm{vol}_G(Y) \leqslant \mathrm{vol}_H(X) + 2(M-1)\,\mathrm{vol}_H(X) = (2M-1)\,\mathrm{vol}_H(X) \tag{5.1}$$

and similarly

$$\mathrm{vol}_G(\overline{Y}) \leqslant (2M-1)\,\mathrm{vol}_H(\overline{X}). \tag{5.2}$$

Now, each edge counted in $e_H(X, \overline{X})$ corresponds to a bipointed tree in G between Y and $\overline{Y}$, therefore

$$e_G(Y, \overline{Y}) \geqslant e_H(X, \overline{X}). \tag{5.3}$$

Hence, by (5.1), (5.2) and (5.3):

$$h_G(Y) \geqslant \frac{1}{2M-1} h_H(X).$$

This concludes the proof. □

The next lemma states that the large branches of $\mathbf{U}_{n,g_n}$ only make up for a very small proportion of its size.

Lemma 5.2. *For all $\varepsilon > 0$, there exists a constant M such that, whp, the total size of the branches of $\mathbf{U}_{n,g_n}$ that are larger than M is less than εn.*

Idea of proof Conditionally on $\ker(\mathbf{U}_{n,g_n})$ having s_n edges, the branches are just a random uniform list of s_n bipointed trees with total size n (with the subtlety that the first branch in the list, the one that will contain the root, carries an extra marked edge).

Using Poissonization and the saddle point method (see [FS09], Chapter VIII), we can replace the list of sizes of the branches by i.i.d variables. We then show that the probability that a given branch is larger than k decreases exponentially in k, which then yields the result.

Remark 5.3. The local limit result of [ACCR13] allows to prove a weaker version of Lemma 5.2, where the result holds in expectation instead of whp. Roughly speaking, a direct corollary of [ACCR13] is that for a uniform edge of $\mathbf{U}_{n,g_n}$, the probability that it belongs to a large branch is small, but we do not have information about how pairs of edges are correlated.

We are now ready to prove Proposition 3.2.

Proof of Proposition 3.2. By Lemma 5.2, we know that there exists an M depending only on ε and θ such that $\ker^{<M}(\mathbf{U}_{n,g_n})$ contains at least $(1-\varepsilon)n$ edges whp. By Proposition 3.3, $\ker(\mathbf{U}_{n,g_n})$ is a δ-expander whp. Now, $\ker^{<M}(\mathbf{U}_{n,g_n})$ can be constructed out of $\ker(\mathbf{U}_{n,g_n})$ by replacing each edge by a bipointed tree of size M or less, hence if we set $\kappa = \frac{\delta}{2M-1}$, then $\ker^{<M}(\mathbf{U}_{n,g_n})$ is a κ-expander whp, by Lemma 5.1. □

A. Proof of Lemma 4.1

Recall that we have a sequence $\mathbf{d} = d_1 \leqslant d_2 \leqslant \ldots \leqslant d_k$ that does not contain any 1's or 2's, with $|\mathbf{d}| = 2s$, such that $k+s$ is odd. We want to show that $\mathrm{CM}(\mathbf{d}^{(s)})$ has only one face with probability $\Omega\left(\frac{1}{s}\right)$. Our proof consists in estimating some quantities carefully in an argument of [BCP19]. We however do not know of a more direct proof.

In [BCP19], the authors consider a model that is dual to ours, i.e. they glue polygons together, with the condition that there are few one-gons and digons (our case fits into their assumptions, since we have none). More precisely, they have as a parameter a list $\mathcal{P}_s$ of sizes of polygons that sum to $2s$ (this corresponds to our $\mathbf{d}^{(s)}$). The list $\mathcal{P}_s$ contains $\#\mathcal{P}_s$ elements (this corresponds to our k). An important parameter in their proofs is a random number $0 \leqslant \tau_s \leqslant s$.

In section 4 of [BCP19], they control the number of vertices of their map, which is the number of faces in $\mathrm{CM}(\mathbf{d}^{(s)})$. More precisely, their equation 13 writes the number of vertices as

$$X_{\tau_s} + V_{2(s-\tau_s)}. \tag{A.1}$$

Let us define the notions used in (A.1).

It is shown in Section 4.3 that

$$X_{\tau_s} \overset{d_{\mathrm{TV}}}{=} (1+o(1))\,\mathrm{Poisson}\left(\log\left(\frac{n}{s-\#\mathcal{P}_s}\right)\right),$$

(note that this $o(1)$ can be made uniform in $\mathcal{P}_s$ by a classical diagonal argument). In particular, since we have no one-gons or digons, we have $\#\mathcal{P}_s \leqslant \frac{2}{3}s$. This implies that

$$\mathbb{P}(X_{\tau_s} = 0) \geqslant 1/3 + o(1). \tag{A.2}$$

Now let us turn to $V_{2(s-\tau_s)}$. For any p, V_{2p} is the number of vertices in a uniform unicellular map on p edges. We can calculate $\mathbb{P}(V_{2p}=1)$ for even p :

$$\mathbb{P}(V_{2p}=1)=\frac{\#\text{ of unicellular maps on }p\text{ edges with one vertex}}{\#\text{ of unicellular maps on }p\text{ edges}}. \tag{A.3}$$

The denominator in the formula above is easy to enumerate, it is $(2p-1)!!$ (number of ways to pair the edges in a $2p$-gon). To enumerate the numerator, we will use [GS98], more precisely equation 14 for $x=1$, and then Corollary 4.2 for $g=p/2$:

Proposition A.1 ([GS98])**.** *There are*

$$\frac{(2p)!}{2^p p!(p+1)}$$

unicellular maps with p edges and one vertex.

Therefore, we have exactly

$$P(V_{2p}=1)=\frac{1}{p+1}$$

if p is even. At the end of Section 4 in [BCP19] (proof of Theorem 3), it is shown that $X_{\tau_s}+s-\tau_s+1$ has the same parity as $s+\#\mathcal{P}_s$, which corresponds to $n+k$ in our case (and we require it to be odd), therefore $s-\tau_s$ is even, if we condition on $X_{\tau_s}=0$. Hence, using the fact that $V_{2(s-\tau_s)}$ and X_{τ_s} are independent conditionally on τ_s,

$$\mathbb{P}(V_{2(s-\tau_s)}=1|X_{\tau_s}=0)\geqslant\frac{1}{s-\tau_s+1}\geqslant\frac{1}{s+1}. \tag{A.4}$$

We are ready to conclude the proof of Lemma 4.1. Conditionally on τ_s, by (A.2) and (A.4), the probability that $\mathrm{CM}(\mathbf{d}^{(s)})$ has exactly one face is

$$\mathbb{P}(X_{\tau_s}=0)\mathbb{P}(V_{2(s-\tau_s)}=1|X_{\tau_s}=0)\geqslant\frac{1+o(1)}{3s}.$$

This quantity is independent of τ_s and uniform in $\mathbf{d}^{(s)}$, hence it finishes the proof of Lemma 4.1.

B. Proof of Proposition 4.3

We recall that we work with a sequence of vertex degrees $\mathbf{d}^{(s)}$ such that $|\mathbf{d}^{(s)}|=2s$. We want to prove that $\mathrm{CM}(\mathbf{d}^{(s)})$ is a δ-expander with very high probability, for some **universal** $\delta>0$. For the sake of simplicity, we will not make δ explicit, but we will show that it exists.

In what follows, we will consider list of vertices $(v_1,v_2,\dots,v_k)$ equipped with distinguishable dangling half-edges, where v_i has degree d_i. For all $I\subset[k]$, we set $\mathrm{vol}(I)=\sum_{i\in I}d_i$, and $N_V(\mathbf{d}^{(s)})$ is the number of sets $I\in[k]$ such that $\mathrm{vol}(I)=V$. We will also write $v_I=\{v_i|i\in I\}$.

We begin by estimating $N_V(\mathbf{d}^{(s)})$. All fractions are to be understood as their floor values, which we do not write to make the notation less cumbersome.

Lemma B.1. *Let* $0 < V \leqslant s$*, then*

$$N_V(\mathbf{d}^{(s)}) \leqslant V/3\binom{2s/3}{V/3}.$$

Proof. Since for all i we have $d_i \geqslant 3$, we have $\mathrm{vol}(I) \geqslant 3|I|$, and thus if I is such that $\mathrm{vol}(I) = V$, then $|I| \leqslant V/3$. Hence

$$N_V(\mathbf{d}^{(s)}) \leqslant \sum_{i=1}^{V/3} \binom{k}{i}.$$

But, since $d_i \geqslant 3$ for all i, we have $k \leqslant 2s/3$, thus

$$N_V(\mathbf{d}^{(s)}) \leqslant \sum_{i=1}^{V/3} \binom{2s/3}{i}.$$

Finally, since $V/3 \leqslant \frac{1}{2}(2n/3)$, the sequence $\binom{2s/3}{i}$ is increasing in i in the range $[1, V/3]$, therefore

$$N_V(\mathbf{d}^{(s)}) \leqslant V/3\binom{2s/3}{V/3}.$$ □

Now, we will define a set of bad events. Let $\mathcal{E}_V$ be the event that, in $\mathrm{CM}(\mathbf{d}^{(s)})$, there exists a set I with $\mathrm{vol}(I) = V$ and such that among all the dangling half-edges of v_I, strictly less than δV get paired with dangling half-edges of vertices outside v_I. Notice that $\mathrm{CM}(\mathbf{d}^{(s)})$ is not a δ-expander iff at least one of the $\mathcal{E}_V$ happens. We will separate the analysis in two regimes, depending on the size of V. We introduce a universal, small enough $\eta > 0$. We do not make it explicit, for the sake of simplicity, but we will show that it exists later on.

Small subsets We will tackle the case $V \leqslant \eta s$. This proof follows the lines of the argument in [HLW06][Theorem 4.16] in the case of regular graphs.

First we treat the case of $V \leqslant 14$. Since $\delta < \frac{1}{15}$, then if $\mathcal{E}_V$ happens for $V \leqslant 14$, it implies that $\mathrm{CM}(\mathbf{d}^{(s)})$ is disconnected (note that in particular $\mathcal{E}_V$ cannot happen for odd V). Hence

$$\mathbb{P}\left(\bigcup_{V \leqslant 14} \mathcal{E}_V\right) \leqslant \mathbb{P}(\mathrm{CM}(\mathbf{d}^{(s)}) \text{ is disconnected}). \tag{B.1}$$

From now on, $V \geqslant 15$. Given $I \subset [k]$ of volume V and H a subset of the half-edges of v_I we define the following event

$$Y_{I,H} := \text{all half-edges of } H \text{ are matched along themselves.}$$

If H has cardinality h, then

$$\mathbb{P}\left(Y_{I,H}\right) = \frac{(h-1)!!(2s-h-1)!!}{(2s-1)!!}.$$

By a union bound and Lemma B.1, we have

$$\begin{aligned}\mathbb{P}(\mathcal{E}_V) &\leqslant N_V(\mathbf{d}^{(s)}) \sum_{(1-\delta)V<h\leqslant V} \binom{V}{h} \frac{(h-1)!!(2n-h-1)!!}{(2s-1)!!} \\ &\leqslant \sum_{(1-\delta)V<h\leqslant V} V/3 \binom{2s/3}{V/3}\binom{V}{h} \frac{(h-1)!!(2s-h-1)!!}{(2s-1)!!}.\end{aligned}$$

Note that at this point we have a bound that only depends on s and not on $\mathbf{d}^{(s)}$. By the classical inequality $\binom{a}{b} \leqslant \left(\frac{ae}{b}\right)^b$, we obtain

$$\begin{aligned}\mathbb{P}(\mathcal{E}_V) &\leqslant \sum_{(1-\delta)V<h\leqslant V} V/3 \left(\frac{2se/3}{V/3}\right)^{V/3} \left(\frac{Ve}{h}\right)^h \frac{h-1}{2s-1}\frac{h-3}{2s-3}\cdots\frac{1}{2n-h+1} \\ &\leqslant \sum_{(1-\delta)V<h\leqslant V} V/3 \left(\frac{2se}{V}\right)^{V/3} \left(\frac{Ve}{h}\right)^h \left(\frac{h}{s}\right)^{h/2} \\ &\leqslant \sum_{(1-\delta)V<h\leqslant V} V/3e^{V/3+h}2^{V/3} \left(\frac{s}{V}\right)^{V/3-h/2} \left(\frac{V}{h}\right)^{h/2} \\ &\leqslant \sum_{(1-\delta)V<h\leqslant V} V/3(2e)^{4V/3} \left(\frac{s}{V}\right)^{V/3-h/2},\end{aligned}$$

where in the last inequality, we used the crude bound $V/2 \leqslant h \leqslant V$. Now, since $\delta < \frac{1}{15}$, we have $V/3 - h/2 < -\frac{V}{7}$ when $(1-\delta)V < h \leqslant V$, hence, since there are less than V terms in the sum above,

$$\mathbb{P}(\mathcal{E}_V) \leqslant V^2/3(2e)^{4V/3}\left(\frac{V}{s}\right)^{V/7}.$$

For $\eta > 0$ small enough (and s large enough), the RHS in the inequality above is decreasing in V in the range $V \in [15, \eta s]$, and hence

$$\sum_{V=15}^{\eta s} \mathbb{P}(\mathcal{E}_V) \leqslant \sum_{V=15}^{\eta s} \mathbb{P}(\mathcal{E}_{15}) = O(s) \times O\left(\frac{1}{s^{15/7}}\right) = o(1/s) \tag{B.2}$$

uniformly in $\mathbf{d}^{(s)}$.

Large subsets We will now care about larger bad subsets, this time we need to control probabilities more carefully. The following proof is adapted from [KW14][Section 7]. Given $0 \leqslant y < u \leqslant 1$, let $X_{u,y}$ be the number of subsets $I \subset [k]$ of volume us that have exactly ys half-edges that are paired with half-edges outside I. We have (using Lemma B.1)

$$\begin{aligned}\mathbb{E}(X_{u,y}) &= N_{us}(\mathbf{d})\binom{us}{ys}\binom{2s-us}{ys}\frac{(ys)!(2s-us-ys-1)!!(us-ys-1)!!}{(2s-1)!!} \\ &\leqslant us/3\binom{2s/3}{us/3}\binom{us}{ys}\binom{2s-us}{ys}\frac{(ys)!(2s-us-ys-1)!!(us-ys-1)!!}{(2s-1)!!}.\end{aligned}$$

Therefore, by Stirling's formula, we have

$$\log \mathbb{E}(X_{u,y}) \leqslant s(f(u,y) + o(1)), \tag{B.3}$$

where

$$f(u,y) = \log\left[\left(\frac{2^2}{u^u(2-u)^{2-u}}\right)^{1/3} \frac{u^u}{y^y(u-y)^{u-y}} \frac{(2-u)^{2-u}}{y^y(2-u-y)^{2-u-y}}\right] + \log\left[\frac{y^y}{2}\left((2-u-y)^{2-u-y}(u-y)^{u-y}\right)^{1/2}\right].$$

Notice that

$$f(u,0) = \frac{1}{6}\log(u^u(2-u)^{2-u}) - \frac{1}{3}\log 2.$$

It is easily verified that this function is decreasing and tends to zero as $u \to 0$. Therefore, taking our $\eta > 0$ from earlier, let $-c = f(\eta, 0)/2$. We have, for all $u \geqslant \eta$, $f(u,0) \leqslant -2c$. By continuity of $f(u,y)$, there exists $\delta > 0$ small enough such that for all $u \geqslant \eta$ and $y < \eta\delta$

$$f(u,y) < -c.$$

Hence, by (B.3), the first moment method and a union bound, we have that (again, uniformly in $\mathbf{d}^{(s)}$):

$$\sum_{V \geqslant \eta n} \mathbb{P}(\mathcal{E}_V) = o\left(\frac{1}{s}\right) \tag{B.4}$$

(the probabilities are actually exponentially small, but we don't need it here).

Concluding the proof Combining (B.1), (B.2) and (B.4) yields the proof of Proposition 4.3.

C. Proof of Lemma 5.2

This section is devoted to the proof of Lemma 5.2. The general idea of the proof is to approximate the sizes of the branches in $\mathbf{U}_{n,g_n}$ by random i.i.d. variables. This method is often called *Poissonization* by abuse of language, and it relies on the saddle point method. We will directly apply the results of [FS09][Chapter VIII.8].

Counting bipointed trees We first need to compute the generating function of bipointed trees counted by edges. Let

$$D(z) = \sum_{k \geqslant 1} bt_s z^k,$$

(recall that bt_k is the number of bipointed trees with s edges, and let

$$T(z) = \frac{1-\sqrt{1-4z}}{2z} - 1$$

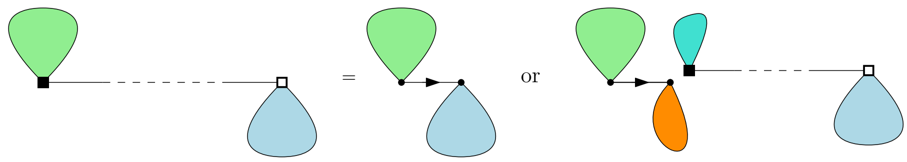

Figure C.1: Decomposing a bipointed tree along the path between its roots. The first (resp. second root) is a box (resp. square).

be the generating function of planar rooted trees with at least one edge, counted by edges. Then, we can prove the following formula

$$D = T + TD$$

by considering the path between the two roots of a bipointed tree (see Figure C.1). This directly implies

$$D(z) = \frac{-2z + 1 - \sqrt{1-4z}}{4z - 1 + \sqrt{1-4z}}. \tag{C.1}$$

Also, $C = \frac{z\partial}{\partial z} D$ is the series of bipointed trees with a marked edge, and

$$C(z) = \frac{z(2 - 2\sqrt{1-4z} - 4z)}{(4z - 1 + \sqrt{1-4z})^2 \sqrt{1-4z}}. \tag{C.2}$$

Studying the Poissonized law: defining the law For any $0 < \beta < 1/4$, we define the two random variables X_β and Y_β with laws

$$\mathbb{P}(X_\beta = k) = \frac{([z^k]C(z))\beta^k}{C(\beta)}$$

and

$$\mathbb{P}(Y_\beta = k) = \frac{([z^k]D(z))\beta^k}{D(\beta)}.$$

Now, fix a constant $\theta \leqslant c \leqslant 1$ and $s_n \leqslant n$ such that $s_n \sim cn$. We fix β such that

$$\mathbb{E}(X_\beta) + (s_n - 1)\mathbb{E}(Y_\beta) = n. \tag{C.3}$$

Let us first show that this β exists. The equation above rewrites

$$(1 + o(1))c\frac{C(\beta)}{D(\beta)} = 1.$$

Now, for $0 < c \leqslant 1$, the equation $c\frac{C(\beta)}{D(\beta)} = 1$ has a root in $[0, 1/4)$, that is

$$-\frac{c\left(\frac{c}{4} + \frac{\sqrt{c^2+8c}}{4}\right)}{8} - \frac{c}{8} + \frac{1}{4}.$$

Hence, by continuity, for n large enough, (C.3) has a solution $\beta \in [0, 1/4)$. Notice that β is decreasing in s_n, hence there exists $\beta^* < 1/4$ such that for all $s_n \geqslant \theta n$, we have $\beta \leqslant \beta^*$.

Studying the Poissonized law: depoissonization probability Now, the probability generating functions (in a variable u) for X_β and Y_β are respectively $\frac{C(\beta u)}{C(\beta)}$ and $\frac{D(\beta u)}{D(\beta)}$. Let $Y_{\beta,1}$, $Y_{\beta,2}$,…,Y_{β,s_n-1} be i.i.d random variables distributed like Y_β, and let

$$S = X_\beta + Y_{\beta,1} + Y_{\beta,2} + \ldots + Y_{\beta,s_n-1}.$$

We have $\mathbb{E}(S) = n$, therefore, by [FS09][Corollary VIII.3], we have

$$\mathbb{P}(S = n) = \Theta\left(\frac{1}{\sqrt{n}}\right) \tag{C.4}$$

uniformly in $s_n \in [\theta n, n]$.

Studying the Poissonized law: large deviations Next we will estimate the large deviations of the Y_i's. Fix an $A > 1$ such that $A\beta^* < 1/4$ (recall that $1/4$ is the radius of convergence of D). We have

$$\mathbb{E}(A^{Y_\beta}) = \frac{D(A\beta)}{D(\beta)} \leqslant \frac{D(A\beta)}{A\beta}\frac{\beta}{D(\beta)}.$$

But $D(z)/z$ is a series with positive coefficients, and hence increasing. When $z \to 0$, $D(z)/z \to 1$ (the number of bipointed trees with one edge). Therefore

$$\mathbb{E}(A^{Y_\beta}) \leqslant \frac{D(A\beta^*)}{A\beta^*} =: W.$$

Hence, by the Markov inequality,

$$P(Y_\beta \geqslant k) \leqslant \frac{W}{A^k}. \tag{C.5}$$

Now, let $Y_\beta^{(M)} = \mathbb{1}_{Y_i > M} Y_i$ and, for $1 \leqslant i \leqslant s_n - 1$, $Y_{\beta,i}^{(M)}$ be independent instance of $Y_\beta^{(M)}$. Let also

$$L^{(M)} = \mathbb{1}_{X_\beta > M} X_\beta + \sum_{i=1}^{s_n-1} Y_{\beta,i}^{(M)}.$$

Let us also fix $1 < B < A$, and set $r = B/A$, we have

$$\mathbb{E}\left(B^{Y_i^{(M)}}\right) = \sum_{k \geqslant M} P(Y_\beta \geqslant k) B^k \leqslant W \sum_{k \geqslant M} r^k = \frac{W r^M}{1-r},$$

where the inequality follows from (C.5). From now on, and until the end of the proof, let us fix M large enough such that $\frac{W r^M}{1-r}$ is small enough to guarantee $\mathbb{E}(B^{Y_\beta^{(M)}})^c \leqslant B^\varepsilon/2$ for all $c \in [\theta, 1]$ in the inequality above. Note that this M is independent of s_n as long as $s_n \in [\theta n, n]$.

We can also show $\mathbb{E}(B^{\mathbb{1}_{X_\beta > M} X_\beta}) = O(1)$, hence by the Markov inequality we obtain

$$\mathbb{P}(L^{(M)} > \varepsilon n) \leqslant O(1) \frac{\mathbb{E}\left(B^{Y_\beta^{(M)}}\right)^{s_n-1}}{B^{\varepsilon n}} \leqslant 2^{-(1+o(1))n} \tag{C.6}$$

uniformly in s_n.

Combining (C.6) with (C.4), we obtain the following for $s_n \in [\theta n, n]$ and n large enough

$$\mathbb{P}(L^{(M)} > \varepsilon n | S = n) \leqslant \sqrt{2}^{-n}. \tag{C.7}$$

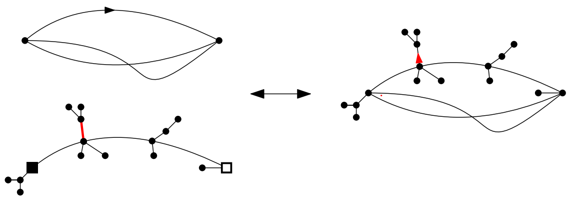

Figure C.2: The kernel decomposition for the branch that contains the root. An edge is marked, and the root is obtained by orienting this marked edge towards the second root in the clockwise exploration order around the branch.

Sizes of the branches To go from a map to its kernel, one removes each branch and replaces it by an edge. Since a rooted map has no automorphisms, these branches can be put in a list of bipointed tree whose total size is the number of edges in the map. One actually needs to be a little more careful for the branch containing the root: mark the (unoriented) edge of the root in the bipointed tree, and replace it by a root edge with a coherent orientation (see Figure C.2). This operation is bijective, therefore the list of sizes of the branches of $\mathbf{U}_{n,g_n}$, conditionally on $\ker(\mathbf{U}_{n,g_n})$ having s_n edges is exactly the list of variables $(X_\beta, Y_1, Y_2, \ldots, Y_{s_n-1})$, conditionally on $S = n$.

We are ready to prove Lemma 5.2.

Proof of Lemma 5.2. Since $\ker(\mathbf{U}_{n,g_n})$ is of genus g_n and has one face, by the Euler formula, it has s_n edges, with $s_n \geqslant 2g_n > \theta n$ (for n large enough). The number of edges of $\ker^{<M}(\mathbf{U}_{n,g_n})$ is exactly $n - L^{(M)}$ (conditionally on $S = n$). We can apply (C.7) conditionally on the number of edges of $\ker(\mathbf{U}_{n,g_n})$, and it uniformly gives

$$\mathbb{P}(\ker^{<M}(\mathbf{U}_{n,g_n}) \text{ has less than } (1-\varepsilon)n \text{ edges}) \leqslant \sqrt{2}^{-n}$$

which finishes the proof. □

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