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Teacher Change in the Context of a Proof-centered Professional Development

A dissertation submitted in partial satisfaction of the requirements for the
degree Doctor of Philosophy

in

Mathematics and Science Education

by

Osvaldo D. Soto

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2010

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Chair

University of California, San Diego

San Diego State University

2010

DEDICATION

This work is dedicated to all who have helped me reach this point, especially my mother. Without your shining example of overcoming adversity, I would not be the person I am today.

I'd also like to thank Dr. Guershon Harel and Dr. Alfred Manaster for all of their mentoring and for bringing me to realize this field as a possible area of study seven years ago.

To all of my friends, fellow graduate students, committee members, and CRMSE staff I'd like to extend my deepest appreciate. Chris and Joanne, you have made a difference in so many students' lives. Add one more to the list. Thank you.

TABLE OF CONTENTS

Signature Page	iii
Dedication	iv
Table of Contents	v
List of Figures	vi
List of Tables	ix
Vita	x
Abstract of the Dissertation	xi
Chapter 1: Rationale	1
Chapter 2: Literature Review	41
Chapter 3: Methodology	72
Chapter 4: Analysis – Proof Schemes	99
Chapter 5: Analysis II	285
Chapter 6: Conclusions	447
Appendix	463
References	478

LIST OF FIGURES

Figure 2.1: Outline of Literature Review.....	42
Figure 2.2: Outline of Richardson and Placier's (2001) Review of Teacher Change Literature.....	45
Figure 4.1: TV Rating Problem Data With Answers.....	110
Figure 4.2: Maggie's diagram for the Cat and Mouse Problem	132
Figure 4.3: Maggie's second diagram for the Cat and Mouse Problem (more detail)....	134
Figure 4.4: Cat and Mouse Problem without mid-segment	137
Figure 4.5: Cat and Mouse Problem with mid-segment	138
Figure 4.6 – Maggie's equations for the Cat and Mouse Problem	141
Figure 4.7: A Stair-like Structure.....	152
Figure 4.8: Markings on stair-like structure	156
Figure 4.9: Maggie's construction of the horizontal segments in a 4-level stair-like structure.....	159
Figure 4.10: Maggie's construction of the vertical segments in a 4-level stair-like structure.....	160
Figure 4.11: A stair-like structure with odd numbered steps shaded.....	171
Figure 4.12: Maggie's first attempt to prove William's conjecture.....	173
Figure 4.13: Maggie's strategy for computing the sum of the series for $n = 19$ and $n = 20$	175
Figure 4.14: Maggie's strategy for computing the sum of the series of odd positive integers in the case when n and k are both even.	177
Figure 4.15: Maggie adding a number composed of all 9's and the number 1.	205

Figure 4.16: Maggie's table showing the value of the first angle, given the number of sides of a polygon.	211
Figure 4.17: Maggie's inscriptions on her copy of TR's graph.	256
Figure 5.1: Variables of interest in sub-question 3.	286
Figure 5.2: Variables in of interest in sub-question 1	286
Figure 5.3: Three students' WoU $(7-3)^2 + \frac{12}{4}$	301
Figure 5.4: Maggie's WoU $(7-3)^2 + \frac{12}{4}$	302
Figure 5.5: Maggie's WoU $\frac{t^3}{t^6}$	308
Figure 5.6: Some students' WoU mental multiplication	315
Figure 5.7: Sunny's initial WoU 6(52)	315
Figure 5.8: Sunny's corrected WoU 6(52).	315
Figure 5.9: Map of a fictitious city	322
Figure 5.10: Number of toothpicks	357
Figure 5.11: Student 1's first illustration of $\frac{1}{4} + \frac{3}{5}$	366
Figure 5.12: Student 1's illustration of $\frac{4}{20} + \frac{12}{20}$	366
Figure 5.13: Student 2's WoU $\frac{1}{4} + \frac{3}{5}$	368
Figure 5.14: Maggie's WoU incorporating student 2's WoU $\frac{1}{4} + \frac{3}{5}$	370
Figure 5.15: Maggie's representation of one-fourth and one-third.	374

Figure 5.16: Maggie's representation of $\frac{3}{12}$ and $\frac{4}{12}$	374
Figure 5.17: Student 1's additive WoU the pattern for water and juice	381
Figure 5.18: Student 1's multiplicative WoU the pattern of water and juice	382
Figure 5.19: Student 2's multiplicative WoU the pattern of water and juice	383
Figure 5.20: An application of student 2's multiplicative WoU the pattern of water and juice.....	385
Figure 5.21: The table and graph used in task 1, lesson 12	388
Figure 5.22: Maggie's pairing of factors for 56.....	406
Figure 5.23: Paired factors of 72 illustrating Maggie's method	407
Figure 5.24: A student's WoU Task 4	411
Figure 5.25: A student's WoU $\frac{2}{6}$	419
Figure 5.26: Student 3's WoU the Rest Stops Problem.....	429
Figure 5.27: Student 1's WoU $\frac{1}{2} \div 2$	431
Figure 5.28: Student 3's WoU the Division Patterns task	432
Figure 5.29: Number of teaching practices observed by lesson	438

LIST OF TABLES

Table 4.1: Maggie’s table for stair-like structures in problem #3.....	157
Table 4.2: Maggie’s table computing the total number of tiles in the odd columns of a stair-like structure	172
Table 4.3: Maggie’s WoU Problem 12	238
Table 4.4: Maggie’s new WoU Problem 12	240
Table 4.5: Maggie’s final WoU Problem 12.....	240
Table 4.6: Maggie’s WoU Problem 16	245
Table 4.7: Proof Schemes Timeline.....	1
Table 4.8: Proof Schemes Timeline.....	266
Table 5.1: The Pythagorean Theorem missing values	421
Table 5.2: Observed codes by lesson.....	438
Table 5.3: Percent of lessons with five or more teaching practices by year	440
Table 6.1: Maggie’s Teaching Practices.....	453
Table 6.2: Number of Teaching Practices per lesson	453
Table 6.3: Percent of lessons with five or more teaching practices by year	454

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ABSTRACT OF THE DISSERTATION

Teacher Change in the Context of a Proof-Centered Professional Development

by

Osvaldo D. Soto

Doctor of Philosophy in Mathematics and Science Education

University of California, San Diego, 2010

San Diego State University, 2010

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Though it is widely acknowledged that teachers' knowledge of mathematics is a cornerstone on which their instructional practices are based, little research exists documenting the impact of changes in teachers' mathematical content knowledge on their teaching practices. As proving is a central activity in the study of mathematics, a teacher's proof schemes (in the sense of Harel and Sowder, 1998) enable and constrain instructional approaches. For professional developers hoping to better understand the impact of teachers' proof schemes on their instructional practices, examinations of specific cases, with special attention to the nature and mechanisms of change in proof schemes and teaching practice, provide insight into the organization of effective professional development (PD) for teachers in the domain of proving.

The case study reported here examines the development of proof schemes and teaching practices of one in-service secondary mathematics teacher who participated in

an off-site PD for two years. Two data sources were examined: video of the participant doing mathematics at the PD and footage of her own teaching. The analysis of proof schemes focuses on proof production during the PD. Development of the teacher's practice was also investigated during the two academic years following each summer. The study includes theoretical connections (using Harel's DNR Theoretical Framework) between developments in the teacher's proof schemes and teaching practices.

Specifically, this study asked: (1) What changes were observed in one participant's proving and proof schemes as she participated in the PD? (2) What connections can be found between her experiences at the PD (including changes in her proof schemes) and the evolution of her teaching practices in a whole class setting? It was found that she became increasingly able to identify pivotal statements in her own proofs that had previously been left unattended. The participant showed evidence of a transition from empirical to deductive proof schemes. The greatest developments in teaching practices were observed in the practices of handling students' solutions by encouraging student-to-student talk, asking students to prove conjectures, soliciting alternative solutions in the presence of correct solutions, and attending to mathematical detail in correct solutions.

CHAPTER 1:

RATIONALE

I. A Recognized Need for Teacher Change

“Investigations seeking to identify alternative pathways toward new forms of practice would substantially fill out our understanding of the terrain of teachers’ development...” (Goldsmith & Schifter, 1997, p. 40).

Recent reports and policy documents, like the *A Nation at Risk* (1983), The *No Child Left Behind Act* (2001), The “*Rising Above the Gathering Storm*” report (2007), the *America Competes Act* (2006), and the *Math Now Act* (2007), embody governmental calls for school reform and higher student achievement in mathematics over the past two decades.

Long before the publication of “*A Nation at Risk*”, it had been acknowledged that the quality of a nation’s teaching corps is an important factor in student achievement. The *Missouri Effectiveness Project* (Good and Grouws, 1979) concluded that, “...teachers and/or teaching methods can exert a significant difference on student progress in mathematics” (p. 355). Seizing on these and similar findings linking student achievement in mathematics to the kind of instruction received, researchers have recognized a need for a more adequately prepared teaching corps as a means of bringing about higher student achievement in mathematics.

While politicians have cited low test scores on international assessments (e.g. TIMSS, 2007; PISA, 2007) and an increasingly unmet need for a mathematically savvy

American workforce as evidence of a need to reform American mathematical education, mathematics educators and mathematicians have voiced other reasons for concern.

Harel (1994) pointed out several “symptoms” of a crisis in the mathematical education of American students, but provided a different source of evidence for the crisis and defined the crisis itself in a different, but related way. Using the data of several groups concerned with student achievement at higher levels of education¹, he noted that during the early 1990’s interest among entering college freshmen in studying mathematics was low, few students were studying advanced mathematics, and there was a decline in the number of students persisting to the PhD level in mathematics. A source of this crisis is “... the quality of the school mathematics teacher” (Harel, 1994; p. 114).

An important quality defining mathematics teachers is their understanding of the subject matter. Mathematical understanding depends on several factors (e.g. beliefs about mathematics, knowledge of rules and procedures, and conceptual understanding). It is difficult for students to relinquish their conceptions and beliefs once formed (Harel, 1994). A teacher’s early experiences with mathematics education are formative, both in terms of content and teaching. There is much reason to believe that these conceptions and beliefs help shape a teacher’s understandings of mathematics and how it should be taught (Ball, 1988; Goldsmith and Schifter, 1997; Putnam and Borko, 2000).

Harel (2001), Knuth (2002), and Phillip et al (2007) have provided evidence demonstrating that it is difficult to help pre-service and in-service teachers relinquish their conceptions or beliefs about either the subject matter of mathematics or the teaching

¹ This data included reports from: A joint report from the Mathematical Association of America and the Association of American Colleges (1990) and the National Research Council (1991).

of mathematics. Harel (2001) provided an example of an intervention guided by a research-based theory of learning and teaching that helped a group of undergraduate students² develop their *proof schemes*³, but explained that he was not able to catalyze the level of change one might hope for. Knuth's (2002) study of 17 secondary school teachers' conceptions of the role of proof in secondary school mathematics found that even teachers who participated in professional development efforts and had exposure to documents like NCTM (2000) viewed proof as an appropriate activity for only a small minority of students - mainly in the context of a high school Geometry course.

These findings demonstrate that the problem of low student achievement has a self-reinforcing nature. That is, conceptions and beliefs about mathematics are formed by students during their primary and secondary school experiences. If colleges and universities are unsuccessful in their attempts to catalyze change in their students' conceptions and beliefs about mathematics and teaching, after leaving colleges and universities these students will become teachers whose conceptions and beliefs influence a new generation of students in way that is similar to the way they were influenced.

Politicians and researchers agree that elementary and secondary teachers are an influential group with the potential to make a difference in students' understandings of mathematics. However, there is a need to understand how changes in teachers' knowledge can influence instruction. As Goldsmith & Schifter (1997) have pointed out, the "terrain of teacher development" is not adequately mapped out.

² Since some of these students were mathematics majors, it is possible that some many have become mathematics teachers.

³ The term "proof schemes" has a specific meaning which will be defined later in this chapter.

There exists a recognized⁴ need for reform of mathematics instruction. However, little is known about how teachers go about developing conceptions and beliefs consistent with recent national studies in mathematics education that could lead to desirable changes in instruction. According to Schifter (1998, p. 57), such changes in instruction would require a “qualitatively different and significantly richer understanding of mathematics than most teachers currently possess”. Schifter further noted that it is unclear how teachers’ mathematical understandings develop or how they affect instruction. Although she assumed that changes in mathematical understandings would allow for an effect on instruction. Harel (2001) has demonstrated how one might go about changing one aspect of undergraduates’ mathematical understandings, but did not show how this would affect instruction. Knuth and others (e.g. Shulman, 1986; Fennema et al, 1996; Schifter, 1998; Ball & Bass, 2003), have pointed out other kinds of knowledge that would require attention as well.

II. A Need to Study Teacher’s Content Knowledge

There exists a lack of consensus regarding the role that mathematical content knowledge plays in teachers’ practice (cf. Begle, 1979; Monk et al, 1994; Ball, 1991). Monk (1994) showed mixed results concerning the link between teachers’ content knowledge and student achievement on standardized testing - the traditional indicator of teacher effectiveness. Meanwhile, Ball and Bass (2003) have pointed out that 8% of cases investigated by Begle (1979) indicated higher teacher mathematical knowledge⁵ was negatively associated with student performance.

⁴ Recognized by politicians and researchers alike.

⁵ As measured by number of courses taken past the calculus series.

On the surface, empirical evidence does not seem to support the assertion that mathematics teachers' teaching practices rest on their knowledge of mathematics. This point has been acknowledged by Ball (1991).

“Although few would disagree with the assertion that, in order to teach mathematics effectively, teachers must understand mathematics themselves, past efforts to show the relationship of teachers' mathematical knowledge to their teaching of mathematics have been largely unsuccessful. How can this be? My purpose here is to unravel this *intuitively indisputable yet empirically unvalidated*⁶ requirement of teaching by revisiting what it means to "understand mathematics" and the role played by such understanding in teaching” (Ball, 1991, p. 1).

Nevertheless, mathematics educators (cf. Shulman, 1986; Ball, 1991; Harel, 1994; Fennema et al, 1996; Schifter, 1998; Ma, 1999; Sowder, 2007), have been reluctant to dismiss the intuitive argument that teachers need to know their subject matter well to be effective. Thompson's (1984) findings support the claim that, “...what teachers' know about math affects what they do” (Ball, 1998; p. 5). Noting that the empirical evidence previously cited does not seem to support this claim (or only supports it weakly, at best), mathematics educators have explained that it is important to reframe the debate by rethinking the nature of the variables used to determine content knowledge (Shulman, 1986; Ball, D. L., Lubienski, S. T., & Mewborn, D. S., 2001; Hill, Rowan, & Ball, 2005).

How can it be explained that when teachers take more mathematics courses, their students either do no better or actually worse (except in 10% of cases⁷) than students of teachers with less mathematics training? One hypothesis is that such findings may be indicators of an ‘expert's blind spot’. If so, one might argue that it is possible to promote

⁶ Emphasis added.

⁷ According to Begle, 1979

knowledge of pedagogy divorced from knowledge of mathematics as a means of improving mathematics instruction. Indeed, this argument has been made.

Shulman (1986) recognized that research on teaching during the 1970's and early 1980's tended to focus on knowledge of pedagogy rather than content.

“The emphasis [of most research on teaching] is on how teachers manage their classrooms, organize activities, allocate time and turns, structure assignments, ascribe praise and blame, formulate the levels of their questions, plan lessons, and judge general student understanding” (p. 8).

He explained that content knowledge had been taken for granted in these studies because it was assumed as a prerequisite for entrance into teacher preparation programs (p. 3, 6).

As such, knowledge of one's subject matter was treated as a constant rather than a variable in many studies while knowledge of pedagogy was treated as the variable.

Shulman noted that researchers had not asked how a teacher translates her knowledge of content into the lessons that they teach. In order to investigate “...the knowledge that grows in the minds of teachers, with special emphasis on content...,” he proposed the need to categorize types of content knowledge of mathematics teachers as either *subject matter* content knowledge, *pedagogical* content knowledge, or *curricular* knowledge (p. 7).

Shulman (1986, p. 9) and Ball (1998) followed Schwab (1961/1978) in defining *subject matter* content knowledge as consisting of substantive knowledge of mathematics and what Schwab (1961/ 1978) called syntactic knowledge of mathematics. Ball (1998) referred to the latter kind of subject matter content knowledge as “knowledge about mathematics”. Shulman (1986) explained that for Schwab (1978), substantive knowledge of mathematics referred to knowledge of the variety of ways in which the basic concepts

and principles of the discipline are organized to incorporate its facts (the substantive structures) together with knowledge of the set of ways in which truth or falsehood, validity or invalidity, are established (the syntactic structure). Applying Schwab's term to mathematics, Ball (1998, p. 6) explained that,

“Substantive knowledge includes ... understandings of particular topics (e.g., fractions and trigonometry), procedures (e.g., long division and factoring quadratic equations), and concepts (e.g., quadrilaterals and infinity), and the relationships among these topics, procedures, and concepts.”

Ball (1998, p. 6, 7) applied Schwab's notion of knowledge of the syntactic structures of a discipline to mathematics; knowledge *about* mathematics⁸. This form of subject matter content knowledge includes

“... understandings about the nature of knowledge in the discipline--where it comes from, how it changes, and how truth is established, what it means to "know" and "do" mathematics, the relative centrality of different ideas, what is arbitrary or conventional versus what is necessary or logical, and a sense of the philosophical debates within the discipline.”

These distinctions between kinds of subject matter content knowledge are important because they help begin to explain phenomenon why subject matter content knowledge is worth investigating. Investigating more refined forms of content knowledge might help researchers find a link between teachers' content knowledge to student achievement.

Hill, Rowan, and Ball (2005) asked what it means to know mathematics and how teachers need to know mathematics differently than others, including mathematicians. For the authors, answers to these questions lie in what differentiates the work of mathematics teachers from the work of others. Hill, Rowan, and Ball pointed out specific

⁸ Ball (1998) italicized as follows: knowledge *of* mathematics and knowledge *about* mathematics.

aspects of teachers' work that requires them to make use of mathematical knowledge. Teachers need to explain terms and concepts, interpret students' statements and solutions, judge and correct textbook treatments of particular topics, use representations accurately in the classroom, and provide examples of mathematical concepts, algorithms, or proofs. In order to do so, teachers must draw on certain aspects of their mathematical knowledge that others do not need to access with the same regularity.

The kinds of mathematical knowledge that teachers call upon when performing one of the actions listed above is what Hill, Rowan, and Ball refer to as mathematical knowledge for teaching. Though limited in some ways, Hill et al's (2005) findings indicate that this kind of knowledge "...positively predicted [elementary school] student gains in mathematics achievement...".

Pedagogical content knowledge refers to "...the ways of representing and formulating the subject that makes it comprehensible to others" (p. 9). Shulman explained that for him this kind of knowledge includes both practical and theoretical elements.

"...Within the category of pedagogical content knowledge I include, for the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations... Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult..." (p. 9).

Recognizing the need to connect teachers' knowledge and practice, researchers have attempted to: explain how it is developed in pre-service teachers (Ball, 1988), distinguish it from pedagogical content beliefs (Carpenter, Fennema, Peterson, & Carey, 1988),

create subcategories that can be used to measure growth (Grossman, 1988), and diminish ambiguities in the definition (Marks, 1990⁹).

Marks (1990) explained that pedagogical content knowledge represents the relationship between subject matter knowledge and pedagogical knowledge in a unidirectional sense. For example, Shulman wrote about “transforming” subject matter knowledge into pedagogical content knowledge. Dewey wrote about “psychologizing” the subject matter and Ball described this adaptation process as “representing” the subject matter for pedagogical purposes. Marks contrasted pedagogical content knowledge with what he called content-specific pedagogical knowledge which represents the relationship a transformation of general pedagogical knowledge to a particular subject matter context. His interviews of eight fifth-grade teachers demonstrated the difficulty in determining when knowledge is pedagogical content knowledge or content-specific pedagogical knowledge; highlighting the importance of compatibility between research methods and research questions when investigating sources of pedagogical content knowledge.

Recognizing the difference between pedagogical content knowledge and content-specific pedagogical knowledge helps make Hill, Rowan, & Ball’s (2005) notion of mathematical knowledge for teaching more meaningful because it helps explicate the point that knowledge used in teaching may emerge from several sources.

Mathematics teachers have many kinds of knowledge which are all necessary and useful. However, many of the essential tasks of mathematics instruction depend on a deep knowledge of mathematics. Harel (2008), Ball (1988), Marks (1990), Goldsmith &

⁹ For a more detailed discussion of ambiguities in the definition of pedagogical content knowledge see Marks, 1990.

Schifter (1997), Putnam & Borko (2000), Hill, Rowan, & Ball (2005) and many others have demonstrated a history of concern for understanding how teachers use their knowledge of mathematics in teaching rather than how teachers adapt general pedagogical knowledge in teaching mathematics and a need to continue investigating this topic. Furthermore, investigations into how teachers transform mathematical knowledge into other forms of knowledge entail an initial understanding of teachers' mathematical knowledge.

Curricular knowledge refers to knowledge of the "...full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs, and the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum and program materials in particular circumstances" (p. 10). Of these forms of content knowledge pointed out by Shulman, pedagogical content knowledge has received the most attention by researchers.

By focusing his investigation on the growth of teachers' content knowledge into categories of content knowledge, Shulman's work calls into question the very notion that researchers can account for teachers' content knowledge using the proxies for content knowledge that Begle and Monk had used. Admittedly, Shulman focused his investigation on the teacher, rather than the learner; stating that, "The cognitive psychology of learning has focused almost exclusively on such questions in recent years, but strictly from the perspective learners" (p. 8)¹⁰. However, Shulman clearly stated that

¹⁰ Such questions refers to: "Where do teacher explanations come from? How do teachers decide what to teach, how to represent it, how to question students about it and how to deal with problems of misunderstanding?" (Shulman, 1986, p. 8)

teachers must be aware of students' "misconceptions" and be able to make use of other findings about learners of mathematics from research. A focus on the growth of content knowledge in the mind of the teacher raises process oriented questions about the nature of teachers' content knowledge.

Ball (1991) has asked whether or not further mathematical preparation via number of pure mathematics courses taken prepares teachers with the pedagogical tools they need to teach mathematics in an attempt to account for one possible reason why the intuitively based claim that teachers' knowledge of mathematical content matters has not been confirmed. Schifter's (1998) work supports Ball's perspective and demonstrates the deep connection that exists between what Shulman called pedagogical content knowledge and subject matter content knowledge.

"A successful practice grounded in the principles that guide the current mathematics education reform effort requires a qualitatively different and significantly richer understanding of mathematics than most teachers currently possess." (Schifter, 1998, p. 57)

Schifter (1998), Ball (1991), Ball, D. L., Lubienski, S. T., & Mewborn, D. S. (2001), and Sowder (2007) explain that effectiveness of mathematics teachers has been measured in narrow terms; in terms of *procedural knowledge*, as opposed to *principled knowledge*. Sowder (2007) cited Spillane (2000) explaining the difference between procedural and principled knowledge as follows:

"Whereas *procedural knowledge* centers on computational procedures and involves memorizing and following predetermined steps to compute answers, *principled knowledge* focuses on the mathematical ideas and concepts that undergird mathematical procedures".

A different vision of what constitutes student knowledge of mathematics that is principled might lead to different results as to whether or not more content knowledge of

mathematics leads to higher student achievement. For example, Fennema et al (1996) measured student knowledge in terms of problem-solving ability and conceptual knowledge, as well as computational ability.

Ball and Bass (2003) have also argued that taking more mathematics courses¹¹ with traditional goals and means of instruction may bring the mathematics knowledge of teachers into a state of increased compression. In such a state teachers may have difficulties “unpacking” their understandings in ways that would be helpful for them to communicate effectively with students. Teachers spending more time taking mathematics courses taught in “conventional approaches to teaching mathematics” may have these approaches impressed upon them by their experiences (Ball and Bass, 2003). Working under the assumption that teachers tend to teach as they have been taught, Putnam & Borko (2000) explained that teachers who have learned mathematics under a conventional approach need new learning experiences to gain visions of mathematics instruction that are qualitatively different.

Knowledge of mathematics content need not be thought of solely in terms of “specific [mathematical] skills, concepts, and symbol manipulations” (Harel, 1994).

“If it is agreed that a solid background in mathematics is an indispensable component in the teachers’ knowledge base ...teachers’ knowledge of mathematics should be promoted and evaluated in terms of mathematics values¹², not specific skills, concepts, and symbol manipulations.” (Harel, 1994)

Reform-oriented approaches to teaching mathematics have been characterized as “student-centered” (e.g. - Schifter (1998), Fennema et al (1996), and Steffe and

¹¹ In order to speak about teachers taking “more mathematics courses” it is necessary to define a set of essential courses for teachers. I am aware of the issue, but will not make that decision for the reader.

¹² The term “mathematical values” has since been expanded by Harel into the term “ways of thinking”. The latter term is defined in the section introducing the reader to DNR terms below.

Thompson (2000)). In such approaches it is imperative that a teacher be able to build models of students' mathematics and make valid connections between their own understanding of mathematics and their students'. Student-centered instructional practices require the teacher to "unpack" her mathematical understandings in order to see the validity of what students say or to appreciate the relevance of student utterances and inscriptions.

Seeing the potential application of a problem solving approach in multiple contexts, understanding that mathematics is a human endeavor (Lakoff & Nunez, 2000), or knowing what constitutes a mathematically valid source of justification (Harel and Sowder, 1998) are all examples of "mathematical values" which need to be promoted as part of a teachers' content knowledge. If mathematical values are considered a part of teachers' knowledge of mathematics, then their development in teachers' and their influences on a teacher's instructional practice merits investigation. These statements are in agreement with the findings of Hill, Rowan, and Ball (2005) and the statements of Shulman (1986). Of particular importance in this dissertation is an aspect of knowledge about mathematics; how truth is established. This is related to what Harel and Sowder (1998) have called an individual's *proof schemes*; a topic which will be explored below.

Such changes do not occur in a vacuum. Putnam & Borko (2000), Sowder (2007), Schifter (1998) and others have pointed out a need to study the professional development efforts whose intent is to bring about changes in teachers' knowledge of mathematics and its potential influence on instruction.

In light of the mixed empirical results (e.g., Begle, 1979; Monk, 1994; Hill, Rowan, and Ball, 2005) and intuitive arguments raised supporting and contradicting

assumptions about what teachers need to know in order to be effective, it seems necessary to investigate more qualitative aspects of in-service teachers' knowledge of mathematics as well as how these teachers use their developing knowledge during instruction. Doing so within the context of an ongoing professional development effort might provide insight into how content knowledge grows in the mind of the teacher.

Within this section I have referred to a need for delineating what teachers need to know in order to be effective mathematics instructors; emphasizing the importance of content knowledge. Knowledge of content is certainly one important aspect of what teachers need to know. However, it has been pointed out by several researchers that using proxies for content knowledge, such as the number of courses a teacher has taken, does not provide the type of data needed to determine how to help raise student achievement. If it is acknowledged that what teachers know influences their instruction, which in turn influences student achievement, then it is to our advantage to understand how different forms of knowledge grow in the teacher – not only knowledge of pedagogy, but also knowledge of content. Since much of the research on teaching has already focused on issues of classroom management, it will be important to focus on the growth of content knowledge in teachers without ignoring what they do in the classroom.

The field of mathematics education is in the process of redefining its variables of study in an effort to relate teachers' subject matter knowledge and student achievement (Hill, Rowan, & Ball, 2005). While teachers need to know about many things (e.g. learning theory, pedagogy, classroom management, subject matter, curriculum, etc.) in order to teach effectively, it has been acknowledged by many researchers and professional developers that there exists a need for more studies about what teachers

know about the content they teach and how a teacher's knowledge of the subject grows with instructional (or professional development) experience.

III. A Need for Research on Teachers' Knowledge and Beliefs about Proof

Within the field of mathematics, proving is a central activity. Rav (1999) states that proofs are the “heart of mathematics, the generators, bearers, and guarantors of mathematical knowledge” (p.31). Proof can be seen as a form of problem solving and draws on many aspects of mathematics. Furthermore, proving draws in many other *mental acts*¹³.

Among mathematics educators there is agreement that proof and its means of instruction deserve the attention of researchers. Harel and Sowder (2007, p.806) cited Ball & Bass (2003), Haimo (1995), and Schoenfeld (1994) as support for the claim that, “No one questions the importance of proof in mathematics, and in school mathematics.” Jones (1997) stated that the role of proof in mathematics is what sets it apart from other disciplines.

Traditionally, proof has not been given adequate attention. Regardless of the support on the part of mathematicians and mathematics educators for the centrality of proof in mathematics, Knuth (2002) has pointed out that “... surprisingly, the role of proof in school mathematics within the United States has been peripheral at best” (p. 61). In contrast to this, Knuth points out that in the NCTM standards (2000) the role of proof has been elevated.

In their investigations on students' understanding of proof, mathematics education researchers' studies have focused on various populations. Research by Harel & Sowder

¹³ More will be said about mental acts. For now, it will be treated as an undefined term.

(1998), Harel (2002), Sowder & Harel (2003), and Housman & Porter (2003) has focused on undergraduate math majors' proof schemes while Martin, G., & Harel, G. (1989) and Jones (1997) have centered on pre-service teachers' notions of proof. Knuth (2002) has focused on in-service secondary school teachers' conceptions and beliefs about proof.

In his study of 17 in-service secondary mathematics teachers, Knuth (2002) asked teachers what constitutes proof in school mathematics and researched their conceptions about the nature and role of proof in school mathematics. His results suggested that “teachers viewed proof as appropriate for the mathematics education of a minority of students...teachers tended to view proof ... as a topic of study rather than as a tool for communicating and studying mathematics” (p. 1).

Research into teachers' conceptions of proof is necessary. It is documented that students' performance in proof at the secondary and undergraduate level is weak (Weber, 2003). Harel and Sowder (2007) have pointed out that while some students' sources of difficulties with proof are known (e.g., “lack of logical maturity and understanding of the need for proof”), other important sources of difficulty are still unknown (p. 807).

Another possible source of student difficulties with proof may relate to their teachers' views of proof. Knuth (2002) has shown that teachers “tend to view proof in a limited way” (p. 61). At the secondary level, students' understanding of proof stems from their experiences as learners in which they only study proof in the context of geometry. According to Weber, expecting pre-service teachers to make up for lost time as an undergraduate has not generally been an effective strategy.

Current and historical findings about student and teacher conceptions of proof, as well as a more nuanced approach to proof, will be presented in chapter 2. For now, it

suffices to explain that proof and teachers' ways of thinking about proof are central areas of concern in this dissertation. These areas are under-explored. In order to conduct an investigation on these topics, it will be useful to find a theoretical framework with language and constructs that are helpful for discussing student and teacher thinking about proof. It would be helpful if this theoretical framework also specified the *teaching actions*¹⁴ that show promise for helping students make positive changes to their existing conceptions and beliefs about proof.

IV. The DNR Theoretical Framework and Its Constructs.

In this section I will introduce the DNR theoretical framework (henceforth referred to as simply **DNR**) with some of its existing constructs, language, empirically based theory, and synthetic approach. Harel and Sowder's PUPA (proof understanding, production, and appreciation) project "produced two complementary products: (a) a conceptual framework for students' proof schemes [See Harel & Sowder, 1998] and (b) a system of pedagogical principles called DNR..." (Harel, 2002). I will reserve a more detailed discussion of (a) for more appropriate places (see chapter 2 and 3); attempting to introduce the essential language and theory here.

The Triad of Mental Acts, Ways of Understanding, and Ways of Thinking

In DNR, *mental acts* are described as "basic elements of human cognition". Examples are "... representing, defining, interpreting, computing, conjecturing, inferring, proving, structuring, symbolizing, transforming, generalizing, applying, modeling, connecting, predicting, reifying, classifying, formulating, searching, anticipating, and

¹⁴ Italics are use here, and in other places, to draw attention to the fact that this term is still undefined. However, it will be defined shortly.

problem solving” (Harel, in press, p. 3). Within this theory it is assumed that a person’s “statements and actions are products of her or his mental acts” (Harel, 2008, p. 4, 5). Mental acts are taken to be basic elements of human cognition used to “...describe, analyze, and communicate about humans’ intellectual activities”.

A person’s *ways of understanding* (henceforth WoU) are defined as the products of her or his mental acts. According to Harel (2008, p. 5), “Repeated observations of one’s ways of understanding associated with a given mental act may reveal certain characteristics – persistent features – of the act. These characteristics are referred to as *ways of thinking*.” A way of thinking is a model inferred of an observer from repeated observations of the various products produced by an individual performing the same mental act. A way of thinking will be denoted WoT. The terms, way of understanding and way of thinking can only be understood in connection with a particular mental act.

Mental acts are also important in other researchers’ theoretical frameworks; though they are referred to by other names. In writing about algebra as an activity, Kieran (2007) refers to a list of activities that are almost identical to the elements of Harel’s list of mental acts as “meta-level” activities. For Kieran, these “meta-level” activities form an important part of her framework for understanding Algebra.

Perhaps the nearest relative to Harel’s triad of mental acts, WsoU, and WsoT is Cuoco, Goldenberg, and Mark’s (1996) notion of generalized, mathematical, geometric and algebraic habits of mind. Cuoco et al explained that general habits of mind are used in many domains outside mathematics. Harel has written the same regarding mental acts. “Humans perform mental acts, and they perform them in every domain of life, not just in science and mathematics” (Harel, in press, p. 4). However, in mathematics education

researchers are ultimately concerned the characteristics of habits of mind which distinguish a mathematician's use of them from the ways others use them.

Examples of general habits of mind include searching for patterns, experimentation, description of situations, invention of language and notation, visualizing, conjecturing, guessing, and generalizing; a list that is very similar to Harel's list of mental acts and Kieran's meta-level activities.

Cuoco et al (1996) use the term mathematical habits of mind to refer to "ways mathematicians approach things" (Cuoco et al, 1996). Examples of mathematical habits of mind are descriptions of how mathematicians apply many general habits of mind. Cuoco et al included thinking about general statements using special cases as a mathematical habit of mind. This characteristic of the mental act of generalizing is a problem-solving approach specific to a way mathematicians go about generalizing.

Other examples of mathematical habits of mind characterize mathematicians' special abilities, penitents, and preferred ways of carrying out general habits of mind. Mathematicians are good at building abstract theories and models, using functions to study change, look for similarities in seemingly different objects of interest, use multiple points of view, mix deduction and experimentation to prove results, and push the use of the language of mathematics. Cuoco et al included further descriptions of how Geometers and Algebraists use mathematical habits of mind in their specific fields. Several general problem solving approaches have been described (e.g. "Geometers use proportional reasoning"). However, Cuoco et al have also included characteristics of habits of mind that help describe what mathematics is, how it is created, and its intellectual or practical

benefits¹⁵ for mathematicians, such as the belief that it is advantageous to have multiple interpretations of a mathematical concept. Indeed, these authors point out that geometers often use multiple techniques and languages (e.g., the languages of turtle geometry, vectors, analytic geometry, and algebraic number fields) to solve problems as an instantiation of the multiple points of view they take in solving problems.

Harel (2008) has categorized W_soT into problem-solving approaches, proof schemes, and beliefs about mathematics. The triad of mental acts, W_{so}U and W_{so}T is an abstraction of the relationship between the mental act of proving, proofs as products of the mental act of proving, and proof schemes as ways to characteristics of a person's mental act of proving. I now turn to this important topic.

Proof, Proving and Proof Schemes

Five terms have been defined by Harel and Sowder (1998) in an effort to provide a taxonomy of W_{so}T about proof. These definitions rest on the issues of certainty and conviction from the prover's perspective.

“A **conjecture** is an observation made by a person who has doubts about its truth. A person's observation ceases to be a conjecture and becomes a fact in her or his view once the person becomes certain of its truth... **Proving** is the process employed by an individual to remove or create doubts about the truth of an observation... **Ascertaining** is the process an individual employs to remove her or his own doubts about the truth of an observation. **Persuading** is the process an individual employs to remove others' doubts about the truth of an observation... A person's **proof scheme** consists of what constitutes ascertaining and persuading for that person” (p. 244).

¹⁵ Harel (in press...see DNR System as a Conceptual ...p.6) has called these beliefs about mathematics. In Ball's language this is knowledge *about* mathematics; a form of syntactic knowledge of the mathematics (Schwab, 1961/1978).

When the focus of attention is the mental act of proving, the corresponding *way of thinking* is a proof scheme. There are other perspectives on proof that will be explored in a later chapter.

Teaching practices, actions, and behaviors.

In DNR, “a *teaching action* is a curricular or instructional measure or decision a teacher carries out for the purpose of achieving a cognitive objective, establishing a new didactical contract (Brousseau, 1997), or implementing an existing one” (Harel, in press). Characteristics of teaching actions are called *teaching behaviors*. Taken together, teaching actions and their associated teaching behaviors constitute *teaching practices*.

The Duality Principle

The duality principle, one of the three foundational principles of DNR, states that one’s ways of understanding ($W_s o U$) and ways of thinking ($W_s o T$) constrain each other. According to Harel (1998, p. 497), students’ $W_s o T$ act as filters for what teachers attempt to teach them. Teachers’ access to students’ $W_s o T$ is not direct. In order for a teacher to help students refine their $W_s o T$ toward more desirable $W_s o T$, she must design activities and participate in interactions with students that challenge the students’ to change their current $W_s o U$.

From a DNR perspective “... the ultimate goal of instruction must be unambiguous; to help students develop ways of understanding and ways of thinking that are compatible with those that are currently accepted by the mathematical community at large” (Harel, 2008, p. 9). If the focus is on the mental act of proving, this statement can be expressed in terms of proof schemes. That is, one goal of instruction should be to help students make changes to their proof schemes so that arguments will eventually seem

incomplete to students without a deductive component. This does not mean that other forms of proof schemes can not be set as intermediate goals by an instructor.

V. Professional Development Efforts and the Teachers' Knowledge Base

Calls for reform in mathematics instruction have spawned a variety of efforts to help practicing teachers make changes that might help produce students with mathematical knowledge of higher quality (Richardson and Placier, 2001; NRC Report, 2001; Sowder, 2007). During the late 80's and early 90's, many teachers participated in professional development (PD) designed to help them make changes in their teaching that are in line with findings of recent national studies (Schifter, 1995). These PD efforts continue to involve many pre-service and in-service teachers and come in a variety of forms (Richardson & Placier, 2001; Sowder, 2007).

Sowder (2007) has described the reform-oriented vision of mathematics education shared by many professional developers as being "based on principles rather than procedures". Sowder explains that researchers do not offer a set of 'one-size fits all' solutions to the problems that math teachers, professional developers, and curriculum developers are facing because of the individualistic nature of learning and teaching. Rather, reformers view teachers as needing flexible knowledge that can be adapted to different settings (Putnam & Borko, 2000).

One defining characteristic of the reform-oriented PD efforts reviewed by Sowder (2007) and Richardson & Placier (2001) is that they emphasize the importance of a student-centered approach to instruction. Three specific PD efforts have been mentioned repeatedly. Cognitively Guided Instruction (e.g., Fennema et al, 1996), the work of Fosnot, Schifter, and Simon (e.g., Simon & Schifter, 1991), and the work of Cobb,

Yackel, Wood, et al.. (e.g., Cobb et al., 1991) have all reported on attempts to bolster teachers' knowledge about their students' thinking as a potential source for changes in their teaching practice.

The PD efforts previously mentioned assume that teachers need to know about student thinking in order to be effective. The question of what teachers need¹⁶ in order to teach is complex and has been approached from several perspectives. It is not the goal of this section to give an exhaustive list or to describe in depth the different ways researchers have decided to define what teachers need in order to teach. Rather, in this section I introduce two important areas of interest of this dissertation; the teachers' knowledge base and the professional development of in-service teachers. Researchers' definitions of the former inform their approaches to the latter (Sowder, 2007). Both areas of research are related to the notion of teacher change, which is another concern of this dissertation.

Citing Borasi and Fonzi (2002), Sowder (2007) explained that teachers have a variety of needs that must be taken into account in the planning of professional development efforts:

“... they included the needs to develop a vision and commitment to school mathematics reform, to become familiar with exemplary curriculum materials, to understand equity issues and their implications for the classroom, to cope with the emotional aspects of engaging in reform, and to develop an attitude of inquiry towards one's practice.”

Borasi and Fonzi have done well to point out that teachers' needs exceed building knowledge of different types. However, since knowledge can come in many forms the

¹⁶ Temporarily, I have chosen to use the phrase “what teachers need” to include knowledge, beliefs, and disposition.

development of knowledge is rich enough to merit its own investigations. This is not to say that teachers' needs are limited to the development or acquisition of knowledge. However, attempts to bolster or change the teachers' knowledge base are, for many, the focal point of professional development.

The knowledge teachers need to develop practices compatible with the findings of recent national studies has several categories which are intertwined. As previously mentioned, teachers need to know about their subject matter. However, this is only one component of the teachers' knowledge base.

Clearly there are many roles and duties a secondary school teacher has. A secondary school teacher should: be able to plan lessons, manage student-student interactions, participate in student-teacher interactions, be aware of students' alternative conceptions and adjust lesson plans accordingly, access a wide variety of examples and problems of different types for students, write and score exams, create scoring rubrics, record and track student grades, communicate effectively with parents, and justify instructional choices to administrators. This is by no means an exhaustive list of teachers' duties and roles. The point is that it would be extremely difficult to account for all aspects of a teacher's knowledge simultaneously using any one framework. Furthermore, a framework of such a broad scope would lack the explanatory power that a theoretical framework must have.

Shulman (1986) cautioned against narrowing the focus of research to only pedagogical knowledge in order to account for student achievement because many important questions remained unasked as a result. Shulman's research questions focused on the growth of content knowledge in the mind of the teacher in terms of the subject

matter, knowledge of how to deliver the subject matter (referred to as *pedagogical content knowledge*), and knowledge of the curriculum.

Goldsmith & Schifter (1997) have suggested that teachers need to know what the big ideas of the curriculum for a given grade level are, what the ideas children bring with them regarding these “big ideas” are, and what kinds of experiences can be used to help stimulate students mathematically. Goldsmith & Schifter’s suggestions are closely connected to a teacher’s understanding of the content taught. In fact, these suggestions assume (or require) a deep understanding of content on the part of teachers coupled with an understanding of learning theories that values student thinking.

Shulman (1986, p. 9) asked, “... How... are content knowledge and general pedagogical knowledge related?” While Shulman included knowledge of a wide variety of curricula as a form of content knowledge (curricular knowledge), others may see it as a form of pedagogical knowledge (e.g. – Harel, 1994) or may view it as something teachers have little control over in certain school districts¹⁷. Choices of which examples to use, problems to assign, or other ways to effectively teach students were included as a form of pedagogical content knowledge by Shulman.

“Within the category of pedagogical content knowledge I include, for the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations... Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult...” (p. 9).

¹⁷ Within school districts teachers may be mandated to use a set, district approved curriculum and pulled out for training using only one set of curriculum (e.g. – the San Diego Unified School District, one of the largest in the United States regularly implements such practices).

Certainly a student-centered approach entails teachers knowing about their students' thinking and being able to design instruction to suit students' needs; including the selection of appropriate examples and problems for students.

In order to understand what makes the learning of specific topics difficult, teachers must know more than just what misconceptions¹⁸ their students have. For example, Zaslavsky et al (2002) have demonstrated how taking an analytical versus a visual perspective on slope on slope (of lines) which may lead students to make different conclusions. In the hands of an expert teacher, the ability to create uncertainty where there is none can be a powerful tool.

Since affecting a teacher's knowledge base is a goal of many PD efforts, it is important to define the term carefully. What types of teacher knowledge are being targeted? Ball and Cohen (1999) explained that teachers need to understand how students learn. Teachers also need to realize the implications of that knowledge for their practice (Fennema et al, 1996, Schifter, 1998).

Remaining consistent with Shulman (1986), Harel (1994) defined the *teacher's knowledge base* as consisting of a teacher's knowledge of mathematics, knowledge of student learning and knowledge of pedagogy.

Recall that Harel (2008) defined *knowledge of mathematics* as the set of all ways of understanding ($W_s o U$) and ways of thinking ($W_s o T$) associated with the mental acts that mathematicians engage in while doing mathematics. Harel (2008) describes $W_s o U$ as,

¹⁸ The term "misconceptions" coincides with both Shulman's (1986) language and an entire genre of literature.

“...a collection, or structure, of structures consisting of particular axioms, definitions, theorems, proofs, problems, and solutions [which the mathematical community at large has accepted throughout history as correct and useful in solving mathematical and scientific problems]¹⁹”.

Connections can be made between $W_s o U$ in DNR and substantive knowledge of mathematics as described by Schwab (1961/1978), Shulman (1986), and Ball (1998) demonstrating the consistency that exists between these two ways of defining mathematics. Both Harel and Schwab describe mathematics as a structure of structures. Schwab has written about substantive knowledge in terms of structures that organize a discipline’s facts. Substantive knowledge includes understandings of particular topics, procedures, and concepts as well as the relationships among them (Ball, 1998). In mathematics these topics, procedures, and concepts are instantiated in sets of axioms, definitions, theorems, proofs, problems and solutions; as described by Harel (2008) and similarly by Ball (1998).

Recall that $W_s o T$ are characteristics of particular $W_s o U$ associated with a given mental act. Connections can also be made between $W_s o T$ and syntactic knowledge of mathematics (or knowledge *about* mathematics). In this dissertation, I will focus on the mental act of proving. Proofs which have been accepted by the mathematical community can be characterized as deductive in nature. As such, it is fair to say that for a proof to be considered valid to a mathematician it must have this characteristic. That is, it must make use of logic. According to Schwab (1961/1978), syntactic knowledge consists of “...a set of rules for determining what is legitimate to say in a disciplinary domain...”. When Ball (1998) described knowledge *about* mathematics as understandings about the nature of

¹⁹ Information in square brackets were taken from a footnote in Harel (in press) and used here to complete a thought.

knowledge in the discipline, she specifically included, "...how it is established", as a part of knowledge *about* mathematics.

Harel (1994) defined the *teacher's knowledge base* as consisting of a teacher's knowledge of mathematics, knowledge of student learning and knowledge of pedagogy. In Harel (2008) *knowledge of mathematics* is defined as a teacher's W_{s0U} and W_{s0T} about mathematics. *Knowledge of student learning* "refers to the teacher's understanding of fundamental psychological principles of learning..." *Knowledge of pedagogy* refers to the teacher's understanding of the implications of the aforementioned principles of learning for teaching.

Unlike Shulman, in Harel's definition of the teacher's knowledge base, knowledge of curriculum can be thought of as a subset of pedagogical knowledge. In a student-centered approach to instruction, understanding the implications of "fundamental psychological principles of learning" leads to a need to rearrange existing curriculum, select an appropriate curriculum from among several that a teacher is aware of, or write new curriculum to address the needs of students. Knowledge of *how to*²⁰ rearrange, select, or write curriculum may be considered a subset of the knowledge that created the need to do so (Harel's knowledge of pedagogy) or it may be seen as a separate kind of knowledge (Shulman's curricular knowledge). Similarly, knowledge of fundamental psychological principles of learning may be viewed as guiding pedagogical content knowledge (Shulman) or it may be viewed as an entirely separate form of knowledge (Harel). In using the DNR theoretical framework, I will use Harel's labels for these kinds of knowledge when I use the term teacher's knowledge base (TKB).

²⁰ Italics added for emphasis.

A teacher's theory about learning coupled with the quality of his content knowledge informs instructional decision making (Fennema, 1996; Schifter, 1998). As Schifter (1998) and others have pointed out, mathematical content knowledge can constrain or enhance a teacher's ability to understand student thinking. Therefore, from a DNR perspective mathematical content knowledge (thought of in terms of the teacher's $W_{sO}U$ and $W_{sO}T$) influences a teacher's knowledge of student learning and pedagogy.

The three professional development efforts pointed out by Fennema et al (1996) were all examples of how professional development efforts can make the advancement of the teachers' knowledge base (TKB) a stated goal, but go about achieving their objectives by focusing more on a particular component of the TKB. In the case of CGI researchers, the focus of PD efforts has been (and continues to be) on teachers' knowledge of student thinking. Simon & Schifter have focused on teachers' knowledge of mathematics as a source of influence on their ability to understand their students' thinking. Schifter (1998) reported that in-service teacher participants in her "Thinking to the Big Ideas" institute solved at an adult level mathematics problems that were based on elementary school mathematics. Wood, Cobb, & Yackel (1991) focused on pedagogical content knowledge when working with a teacher of a second grade classroom.

According to Goldsmith & Schifter (1997), teachers need to move from procedural conceptions of mathematics toward an understanding that emphasizes sense-making – from a fixed set of rules and procedures, to a human endeavor. In the NCTM standards the nature of mathematics instruction is based on principles rather than procedures. Doing mathematics needs to involve constructing a reasoned argument that illuminates observed understanding of quantitative and spatial relationships en route to a

solution of a posed problem. In terms of content knowledge, PD may aim to help teachers make changes to their understandings of the subject matter along these lines.

Additionally, many teachers have a transmission model of teaching wherein mathematical meaning can be transferred, intact, from the teacher to the student (Goldsmith & Schifter, 1997). Researchers have provided evidence that this is not how many people learn to understand mathematics and use mathematics (Carraher, Carraher, and Schliemann, 1985; Lave, de la Rocha, 1984). Therefore, helping teachers develop alternative learning theories has become an important part of the current goals of many PD efforts. As such, this goal can be restated as an attempt to bolster the TKB.

So far I have described the foci of reform-oriented PD efforts whose goals are to affect teachers' knowledge in terms of the components of the TKB. I have also tried to clarify the different shades of meaning ascribed to the components of the TKB by different researchers; highlighting that the similar phenomena may be given different labels by different authors. Within the DNR theoretical perspective, the TKB has three components which have been described. The phenomena described in Harel's definition of the TKB are similar to those described by Shulman and others, with a difference in labeling of the phenomena.

Any PD effort must choose what to focus on from among the needs of the teachers involved. A theoretical framework should be used to guide the focus of the PD effort. Since theories of learning and teaching vary among researchers, each will have variations in emphasis even though they are based on similar findings from national studies. Choosing to focus on one component of the TKB does not mean dismissing the others. Indeed, it is very important to understand that these components are inseparable.

One such PD effort was conducted using the DNR theoretical framework: The Algebraic Thinking Institute (ATI). The stated objective of this institute was to bolster the TKB of participating teachers as defined in Harel (1994). This PD effort followed a line of teaching experiments conducted by Harel et al (Harel and Sowder, 1998). It made use of the findings from the PUPA (Proof Understanding, Production, and Appreciation) Project; a taxonomy of proof schemes and the DNR theoretical framework.

Within the focus of the institute was the stated objective to help teachers make changes in their proof schemes that were desirable relative to the theoretical framework. As such, data from ATI has the potential to be useful for investigating how teachers make changes to their existing WsoU and WsoT about proof²¹. Stylianides and Silver (2009) supports the need for this kind of investigation.

“...the existing research knowledge base provides insufficient guidance about the ways in which proficiency with reasoning and proving might be developed over time and how to organize effective professional development for teachers in the domain of reasoning and proving” (pp. 23,24).

VI. Teacher Change in Terms of Teachers’ Content Knowledge, Beliefs and Practices.

There is evidence to believe that teachers’ teaching behaviors influence student thinking (Good and Grouws, 1979) and that teachers’ teaching practices are influenced by both their knowledge and beliefs about mathematics, teaching, and learning (Philipp et al, 2007). From a constructivist perspective on learning, students use their past experiences to make meaning of the things their math teachers do and say, as well as the activities they are asked to engage in (Steffe and Thompson, 2000). For this reason, it has

²¹ For more details on ATI, see chapter 3 (Methodology).

been explained that teachers need to know about student thinking and that such knowledge has great potential to influence teachers' practices (Fennema et al, 1996). For example, the CGI project described by Fennema et al (1996), focused on changes in student achievement in the presence of changes in instruction informed by knowledge about student thinking using a scale measuring how much teachers tended to use their students' thinking in their instruction.

Other researchers have focused their attention on finding connections between some combination of teachers' subject matter knowledge, teachers' beliefs, and the nature of the those teachers' teaching actions (Goldsmith & Schifter, 1997; Schifter, 1998; Harel & Lim, 2004; Philip et al, 2007). Researchers have noted a need to move beyond a focus on the relationship between teaching actions and their effects on student learning; emphasizing a need to study the relationship between teachers' beliefs and knowledge about mathematics, teaching and learning (Richardson, V., & Placier, P., 2001; Sowder, 2007; Philipp et al, 2007).

Research on teachers' knowledge growth and changes in beliefs need not consider student achievement explicitly or simultaneously. It is assumed that changes in instruction which are informed by and in line with recommendations of national studies on mathematics education are desirable and have a good chance of leading to improved student achievement. This is not to say that investigations into the student achievement of teachers who transform their practice in the aforementioned direction are unnecessary. My argument is not that investigations into teacher change should or should not be separated from student achievement. Rather, my point is that in scientific experimentation it is acceptable, and in many instances desirable, to forgo the attainment

of a larger goal in order to achieve intermediary goals which can in turn be used to inform an overall strategy for achieving the larger goal. Such a strategy for investigation seems most effective when one wishes to gain traction on a difficult problem or when little is known about a given phenomenon

For example, Philipp et al (2007) investigated the change in beliefs and content knowledge of 159 pre-service teachers who were randomly assigned to one of four related instructional treatments. Philipp's research focus demonstrates the relevance and importance of investigation into the relationship between teachers' subject matter knowledge and their beliefs about the subject matter, teaching, and learning without making an immediate connection to student achievement. It might be possible to follow these pre-service teachers in the future and investigate the knowledge of their students. Without this preliminary work, Philipp et al would have no chance to complete such an undertaking.

Several researchers have attempted to delineate connections between teachers' beliefs and their teaching practices by offering models of change (Guskey, 1989; Goldsmith & Schifter, 1997). In the case of Guskey (1989), teachers engaged in a professional development effort showed change in teaching actions *before*²² there were indications of change in their beliefs. Guskey noted that when teachers saw positive changes in their students' achievements on tests, the practices they had made changes in tended to become more stable. The result was that teachers changed beliefs about teaching and learning after they implemented new practices and observed changes in students' achievement on assessments.

²² Italics added for emphasis.

In contrast to the work of Guskey, Goldsmith & Schifter (1997) proposed a model of teacher change in which beliefs and practices changed together using findings from cognitive psychology. Theories of cognitive development describe stages of understanding that differ from one another and culminate in some form of understanding that allows for complex thought and action.

With regard to change in mathematics teachers' beliefs and actions, many researchers have attempted to create frameworks for change (e.g. Fennema et al., 1996; Cooney, 1993; Schifter and Simon, 1992; Schram, Wilcox, Lappan, & Lanier, 1989; Thompson, 1991; Philipp et al, 2007). Each of the frameworks referred to have a common starting point; a description of an initial stage of understanding mathematics teaching that is grounded in a transmission model of learning and a procedural understanding of the nature of mathematics. Each of the frameworks also discuss subsequent changes in which teachers move away from the transmission model of instruction, start attending more closely to what their students already know, and begin using what they learn about their students' existing knowledge to guide their instruction. Each of these frameworks culminates in a vision of instruction that is consistent with the vision of reformers (see NCTM, 1989, 1991, 2000; NRC, 2001); the focus of these frameworks being how teachers help make mathematics learning possible for their students.

Goldsmith and Schifter (1997) expressed their view that beliefs and behaviors are inseparable. They point out that in order for teachers to change, it is important for them to have powerful images of what reformed mathematics instruction looks like. Because teachers' experiences are grounded in the kind of instruction they are trying to change,

they may need to have new experiences with mathematics instruction in order to make meaningful changes in their own practice. This point was echoed by Putnam & Borko (2000) who explained that teachers' might need new experiences as learners of mathematics in order for them to make changes in their instruction and beliefs.

It is not only important that teachers have powerful images of desirable teaching, but also that they have images of a possible trajectories of change. Speaking about what researchers and professional developers could do to help meet teachers' needs, Goldsmith and Schifter (1997) noted,

“...descriptions of the development of mathematics teaching will have to capture the process by which teachers (re)invent their teaching practice to create classroom cultures that promote learning for understanding” (p. 25).

It is important to consider the professional development effort that forms a part of the context in which teachers change their practice and develop new understandings about mathematics, learning and teaching, student thinking, and curriculum when reporting the development of participating teachers.

The case that has been made follows this reasoning. First, several authors have pointed out that studies foregrounding teachers as the focus of analysis are needed in order to better understand how teachers go about making changes to their practice. Second, foregrounding teachers does not mean removing students. These studies can pave the way to further understand the complex relationship between teachers' practices and student achievement. What we know is that there exists a need to help teachers change their practices. What we don't know is how teachers go about doing so. The case has also been made that teaching practices, knowledge and beliefs are intricately connected and multifaceted.

It is important to note that there are still many aspects of teacher change that need further exploration because of the variety of interactions that exist among the myriad of variables influencing teachers' teaching practices. I have chosen to focus on the relationship between changes in a teacher's subject matter knowledge and teaching practice during the course of a longitudinal PD effort by focusing on changes in her proof production; a central activity in mathematics and mathematics education.

Philipp et al (2007) explained that, "...teachers need more than content knowledge to be effective... teachers' beliefs about mathematics, teaching, and learning affect the ways they think about and teach mathematics..." (p. 439). Applying Philipp's statement to the domain of proof, teachers need to know more than how to produce a correct proof. Since proof schemes are a guiding force behind proof production (i.e. a WoT with respect to the mental act of proving), one can hypothesize the existence of a connection between a teacher's proof schemes (a WoT with respect to the mental act of proving) and the way she teaches mathematics; especially when she has chosen to involve herself in a longitudinal PD effort whose focus is bolstering the TKB. Depending on the design of the data set, one may be able to produce evidence for a causal link. However, before addressing the mechanisms of change it is important to establish the existence of a link between changes in a teacher's proof schemes and teaching practices. Here I speak of correlation rather than causality – a next step in the line of research on the connection between teachers' content knowledge and teaching practices rather than a first step or a final step.

Others have paved the way for this research and there is room for many others to contribute in the future. In the next section I state, formally, the research questions of this dissertation.

VIII. Research Questions

In order to state the research questions clearly, I must first familiarize the reader with a few important aspects of the data.

Algebraic Thinking Institute (ATI) is a DNR-based PD effort which generated two separate, but connected, streams of data I will use in my investigation. The first stream of data consists of video recordings of a group of in-service middle school and high school teachers who were given mathematical training for one month each summer during two consecutive summers. I will refer to the period beginning with the first summer and terminating with the end of the second summer's instruction at ATI as period A. The second stream of data generated by DNR researchers consists of video from the classrooms of several participants' instruction of their own students approximately once every three weeks during each school year after the respective summer sessions. One of these participants was of special interest for reasons I will explain in chapter 3. I will call this participant Maggie. I will refer to the time period starting with the first observation in the participants' classrooms and terminating with the last observation in the participants' classrooms as period B.

The overarching goal of this dissertation is to demonstrate the explanatory power of DNR in analyzing teacher change in terms of proofs, proof schemes and teaching practices. In order to address this goal, I ask the following questions from a DNR perspective:

1. What changes were observed in Maggie's *proving and proof schemes* during her participation in ATI?
2. What is the evolution of particular teaching practices in Maggie's classroom? Given a set of *teaching practices* selected from those observed at ATI, which of these teaching practices are reflected (or not) in Maggie's teaching and to what extent do Maggie's teaching practices reflect the nature and quality of those teaching practices selected from ATI?
3. What connections can be made between the changes observed in Maggie's proof schemes during period A and those changes observed her teaching practices during period B?

Because ATI was the first major PD effort in which the DNR theoretical framework was implemented, researchers hoping to use this theoretical lens need to know what possible connections might exist between the changes in participants' proof schemes and changes in their teaching practices.

What is the significance of these questions?

Researchers who wish to influence the content knowledge of teachers might be interested to know more about an intervention in which a teacher's knowledge of proof changed and how her proof schemes changed. If these researchers share the belief that the nature of a teacher's subject matter knowledge forms a basis for other forms of "knowledge *for* teaching" (in the sense of Cochran-Smith and Lytle, 1999), this research could be useful for their efforts as well. Results indicating desirable changes in proof schemes would show that there is a theoretical perspective that can be used to guide the development of teachers' proof schemes.

Schifter (1998), Shulman (1986), Putnam & Borko (2000), Hill, Rowan, & Ball (2005) and others have expressed the need for such case studies of teacher change that help illuminate the processes by which teachers learn mathematics in a qualitatively different way and change their practice accordingly. Such detailed information about teachers' proof schemes can contribute to work of researchers focusing on teachers' knowledge of mathematics which can in turn be used to investigate how this knowledge is used in teaching.

This research also has the potential to demonstrate a case in which a participant's teaching practices have developed along a path influenced by the goals of reform-minded professional developers. With the understanding that there may have been other influential variables involved in these changes, evidence for change in the participant's teaching practices during period B constitutes a good starting point to look for correlations between changes in proof schemes and changes in teaching practices which can be investigated further in future studies.

For researchers using the DNR perspective, this research has the potential to provide empirical evidence demonstrating the power of the DNR-theoretical perspective to account for the development of participants' mathematical knowledge and teaching practices.

What is new about this question is an emphasis on the role of proof schemes in the investigation of change in teaching practices. Connections that are found between participants' ways of thinking and teaching practices can be used to refine the DNR theoretical perspective about how teachers learn to teach. In other PD efforts, it is

acknowledged that little is known about the role that teachers' ways of thinking plays in their teaching practice (Sowder, 2007). As Sowder has explained,

“Research on teachers' knowledge, beliefs, preparation, and professional development is a fairly new phenomenon” (p. 149).

This research shows potential to make several contributions to a relatively new field.

CHAPTER 2:

LITERATURE REVIEW

Introduction

In chapter 1, I explained that this dissertation is concerned with investigating the changes that occurred in one teacher's practice who attended a DNR-based PD effort aimed at bolstering participants' knowledge base. Such an investigation attempts to contribute to the general literature on cases of teacher change within the field of mathematics education, the knowledge base of professional developers, and to the further development of the DNR theoretical framework.

Having read several extant literature reviews, and keeping in mind the questions of this dissertation as stated in chapter 1, this literature review is organized by asking the following broad questions:

- What changes do researchers feel the need to bring about in teachers?
- How have researchers gone about catalyzing change in teachers?

Below is an outline of the ideas and relationships that have guided this literature review with more explanation to follow.

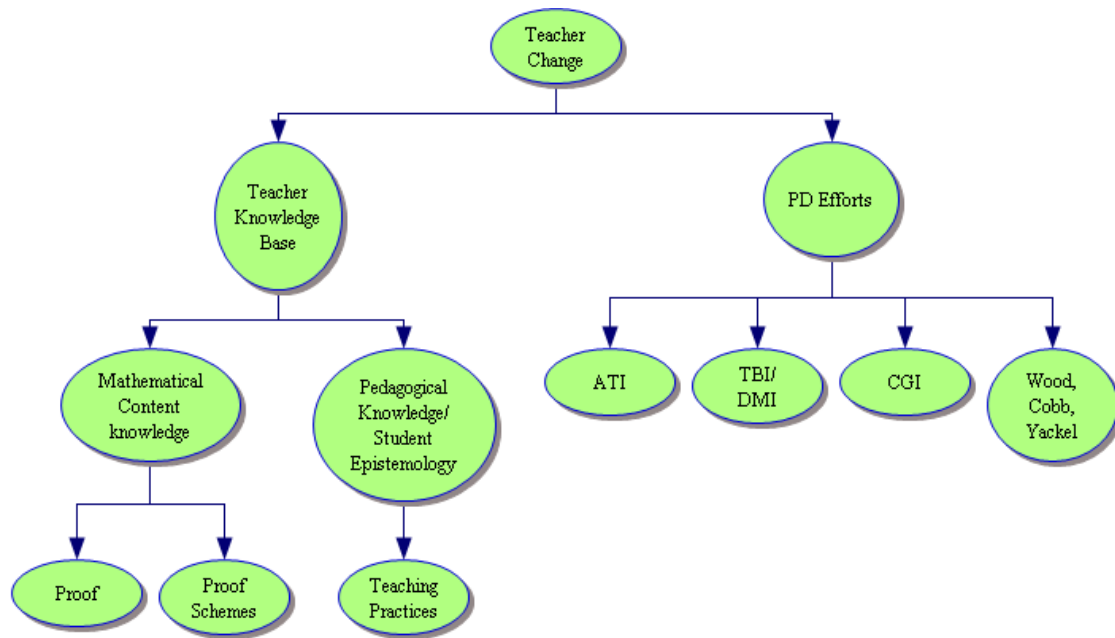


Figure 2.1: Outline of Literature Review

Conceptions of teacher learning

In an attempt to elucidate the goals of prominent professional development, Sowder (2007) wrote, “Professional growth is... marked by change in teachers’ knowledge, beliefs, and instructional strategies” (p. 11). Researchers envision teaching as a profession that should be marked by “...a climate of inquiry and sense making, [requiring] a change in the role of the teacher as the sole authority, and a focus on reasoning and problem solving.” Goldsmith & Schifter (1997) expressed the view that the aforementioned vision of instruction stands in contrast to traditional instruction in the United States.

For many researchers, efforts to help teachers make changes to instructional practices begin with some form of changes in knowledge and beliefs (e.g. - Goldsmith & Schifter, 1997; Philipp et al, 2007; Carpenter et al, 1988). However, knowledge and

beliefs are not so easily teased apart. Researchers' approaches to distinguishing knowledge from beliefs vary because the distinction between the two is fine. For example, Shulman (1986), Ball (1998) and Harel and Lim (2004) consider teachers' understandings of the nature of mathematics a part of their content knowledge. Some researchers view beliefs about mathematics to be a part of content knowledge (Schwab, 1961/1978; Schulman, 1986; Ball, 1998; Harel, 1994). On the other hand, Phillip et al (2007) saw beliefs about mathematics as separate from, but connected to content knowledge.

“Few doubt that teachers' mathematical content knowledge plays a critical role in their instruction (Fenemma & Franke, 1992; Hill, Sleep, Lewis, & Ball, 2007), but most realize, also, that teachers need more than content knowledge to be effective. In particular, teachers' beliefs about mathematics, teaching, and learning affect the ways they think about and teach mathematics (Phillip, 2007; A.G. Thompson, 1992)” (p. 493).

As an example consider the statement, “Mathematics is a web of interrelated concepts and procedures (school mathematics should be too),” (Phillip et al, 2007, p.455). In their study of pre-service elementary school teachers Phillip et al treated the statement as a belief and distinguished it from content knowledge or pedagogical knowledge while Harel (1994, 1998) and Ball (1998) has referred to beliefs about mathematics as forms of content knowledge.

The point is not to split hairs about what is a belief or what is knowledge. Rather, it is to acknowledge a need to attend to differences in meaning between the terminologies that different researchers use in referring to related phenomena. In many cases, the terminology used must be unpacked in order to understand: (1) the nuances of difference

between the foci of each effort and (2) what assumptions about knowledge, knowing, learning, and teaching are used to guide the development of a given PD effort.

Existing Frameworks for Categorizing the Literature on Teacher Change

Richardson and Placier (2001) provided an overview of educational research, while Cochran-Smith and Lyttle (1999), Goldsmith & Schifter (1997), the NRC report (2001), Sowder (2007), and Wilson and Berne (1999) have provided overviews of the professional development and teacher change literature that are more specific to math education. This chapter continues by reviewing each and exploring particular PD efforts noted in these reviews.

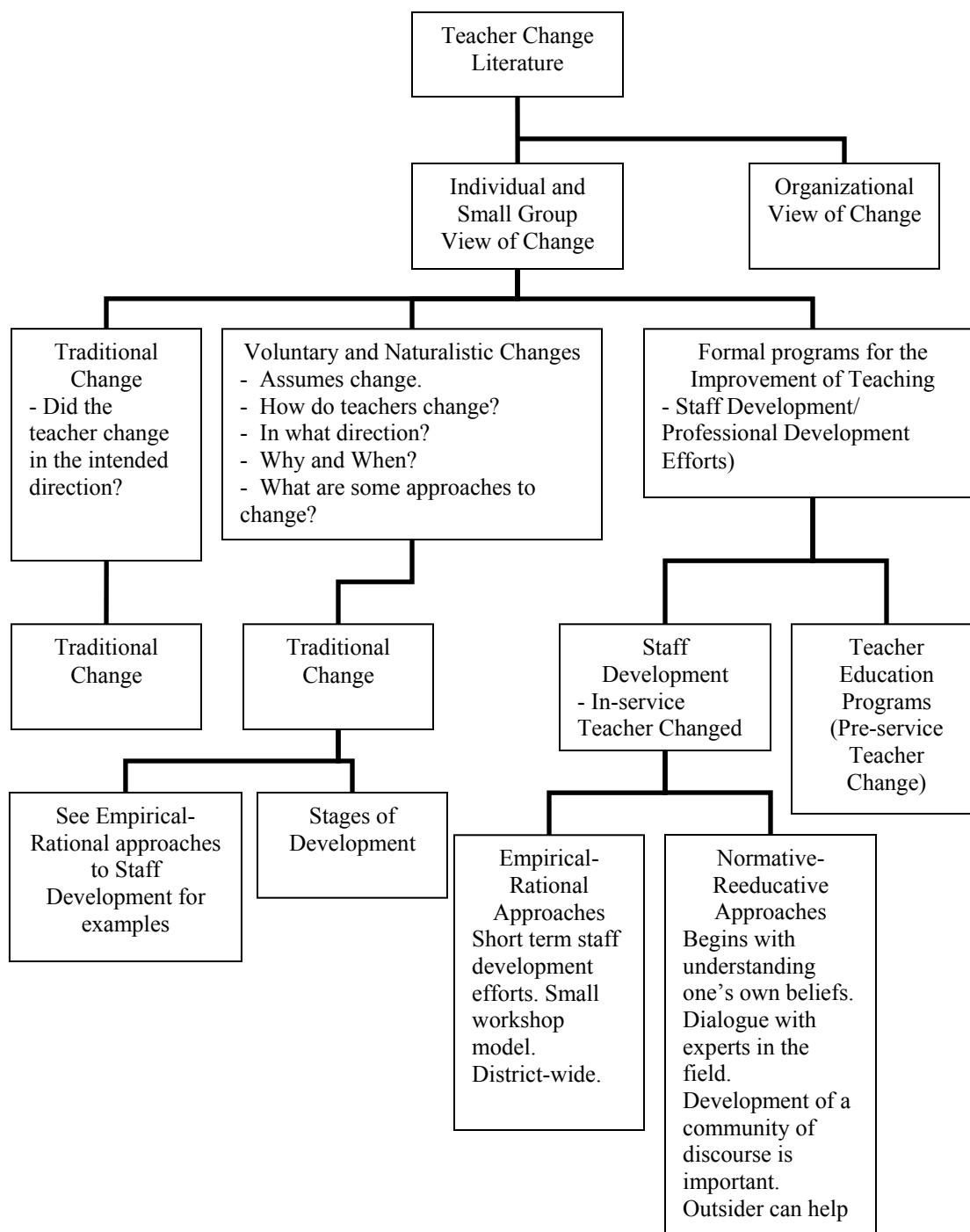


Figure 2.2: Outline of Richardson and Placier's (2001) Review of Teacher Change Literature

Richardson and Placier (2001)

Handbook of research on teaching (4th edition) chapter 43 entitled "Teacher Change" (p. 905 to 947) provides an overview of recent - meaning within 10 to 15 years prior to 2001 - research in the field of teacher change. While this chapter is not focused on the field of math education, it does provide a categorization for types of research that have been conducted in the field of teacher change and descriptions of specific studies within each category.

The literature can be divided into two categories. Literature concerning individual and small group cognitive, affective, and behavioral change processes, or literature that takes an organizational view of change; linking structural, cultural, and political aspects of the school organization to changes in teachers and teaching. An individual and small group view of teacher change has a place in psychology (behavioral, cognitive, and social), while an organizational view is better suited for the fields of sociology, anthropology, political science, and organization theory. The authors claim that these two types of literature have not been well connected and conclude with points of contact and departure for the individual and organizational approaches to teacher change.

Placier and Richardson (p. 905) found Chin and Benne's (1969) description of types of planned strategies for change helpful in organizing the "vast and scattered" literature within the field of teacher change; especially in organizing staff development literature. Two primary strategies for catalyzing teacher change were identified by Chin and Benne.

In the Empirical-rational strategy, teachers are viewed as the consumers of research and practice. In this approach, models of change for behavior, ways of thinking, or curriculum comes from an outside source (i.e. researcher, policymaker, other educator). The model is explained or demonstrated to the teacher and she is expected to implement appropriate changes because she is a rational human being and these proposed changes are explained to be based on “research, theory or both” (p.906).

In the Normative-reeducative strategy, change is assumed to come from reflection on beliefs and practices. Because within this approach individuals decide what changes need to be made for themselves, the authors describe this approach as more “naturalistic” than the empirical-rational approach. Such an approach requires discussion with others. As such, the normative-reeducative approach assumes that teachers act according to the socio-cultural norms they are committed to.

Rather than emphasizing conformity with a system, this approach emphasizes autonomy and power of individual choice; ultimately strengthening the educational system as a whole by empowering teachers by allowing them to determine the direction of change.

Individual and Small Group Change

In the current era of reform, initial attempts at reform focused on change in curriculum (during the 70's), but few teachers used the new materials because they felt that the new materials were unsuitable for actual classroom use; such an approach to reform instantiates the empirical-rational strategy. In the next wave of reform more focus was placed on teachers. However, during this stage of reform successful teacher change was determined by how well and how often teachers were able to implement new

curriculum or methods created by others. An empirical-rational strategy was still dominant.

Voluntary and Naturalistic Change

According to the authors, only recently has “voluntary change” been considered in the literature. Before that, research was concerned primarily with change that is “determined by someone other than the teacher who is going through the change process” (p. 907). There are many possible catalysts for voluntary change such as change in subject/ grade taught by the teacher’s choice, deciding that a given instructional practice is ineffective, etc. Whether changes are made because they are mandatory or because the teacher chooses them herself, the degree to which change occurs depends on the individual teacher.

In these studies (cf. Munby & Russell, 1992; Raymond, McCue and Yamagishi, 1992; Zahorik, 1990) it is assumed that teachers are autonomous and make their own choices. It is also assumed that some change is made. These studies tend to ask questions like: How do teachers change? In what direction? Why and When? What are some different approaches to change?

Because the focus is on teacher initiated change, a teacher’s biography, personal and professional experiences outside, and inside the classroom are all important. For example, Bullough and Baughman (1997) found that the teacher they studied “began teaching by drawing on her experience as a mother” (p.908). “...Research on this form of change suggests that biography, experience, perhaps personality, and context play a role in the change choices that individuals make” (p. 909).

Stages of Development

Placier and Richardson (p. 909 – 913) consider literature on stages of development a subcategory of the naturalistic change literature. A variety of stage theories about how teachers progress through their careers have been proposed. These theories range from theories proposing invariance of movement through stages along with a hierarchical status of stages proposed (developmental stage theories; cf. Fuller, 1969; Fuller & Brown, 1975), to theories that are much more flexible regarding how individuals move through stages and whether or not stages should be considered hierarchical (contextually dependent stage theory; cf. Goldsmith & Schifter, 1994).

While developmental stage theories assume the generalization of their stages beyond the teachers studied, contextually dependent stage theories do not. That is, these studies “...examine stage theory within a particular context such as a specific program of change...” (p. 911). These studies do not assume generalizability of the stage theory. In the case of Schifter (1995), four stages of mathematics teaching conceptions are described for teachers who are engaged in professional development efforts with a constructivist orientation (see Schifter, 1995). Wood, Cobb and Yackel (1991) found that the teacher they were trying to help “create a more constructivist environment” went through three major reconceptualizations of her role as teacher of mathematics (p. 912). Both Schifter and Wood et al’s stage theories are demonstrative of how teachers going through their respective intervention efforts might progress toward becoming a particular kind of teacher.

Within the work on stage theories of learning to teach, little is known about what prompts a teacher to move from one state to another. Schifter (1995) and Goldsmith & Schifter (1994) also cite the same need for further investigation into catalysts for change

in their existing models of change along the “strands” they have investigated. Placier & Richardson explained that the recent move within stage theoretical work from developmental stage theories to more flexible approaches suggests that

“... a number of factors affect either the movement from one state to another or the acquisition of another phase. These factors include biography, experience, context, personality (or stance), and beliefs” (p. 912).

Staff Development

Placier and Richardson offer a landscape view of staff development literature including areas of mathematics, science, and reading. However, they explain that content-areas handbooks offer at least one chapter on staff development. “Reports of research on staff development practices fit quite neatly into these two categories [empirical-rational and normative-reeducative], in regard to both the type of staff development being examined and the research approach used in the inquiry” (p. 917). Implicit in this statement is the notion that the categorization used is intended to work for this broad view of staff development approaches and might need refinement within a given content area.

Empirical-rational approaches to change

Empirical-rational approaches to staff development generally emphasize teaching as a transmission of skills and thinking processes. As a result, it is assumed that teaching behaviors and techniques can be learned by teachers in relatively short-term workshops with limited follow-up activities and can be implemented in a class setting at the appropriate time by teacher-participants when they return to their classrooms.

These more traditional models of staff development are being replaced by a more contemporary view which focus more on “ways of thinking and teacher action rather than on behaviors” (p. 918). Several studies are cited demonstrating the results of such “directed development” efforts (cf. Mayer, 1988).

Normative-reeducative approaches to change

According to Richardson and Placier (2001), the normative-reeducative approach to change has been implemented in helping teachers develop a “constructivist orientation toward teaching a particular subject matter...” in mathematics and science (p.918).

Placier and Richardson considered CGI a major project which had received much attention. Not only did CGI assess PCK of teachers, but it also assessed student achievement in several ways. CGI researchers found that “CGI teachers taught problem solving significantly more and taught number facts significantly less than control teachers. They were more constructivist”(p. 918). Differences were also found between experimental and control students with respect to problem solving and knowledge of number facts, with students in CGI classrooms faring better than their counterparts. It was noted that CGI teachers listened to student thinking and built on it. In addition to some mixed methods studies, the CGI group has written several case studies about these teachers.

Other constructivist oriented professional development efforts cited include Civil (1995), D. Ball (1990) and Cohen (1990), Barnett and Sather (1992), Schifter and Simon (1992) and Wood, Cobb, and Yackel (1991). While all are examples of constructivist oriented professional development efforts, D. Ball (1990) and Cohen (1990), were cited as specific studies in which teacher’s perceptions of their own changes did not agree with

researchers' perceptions. This finding seems very valuable and has also been observed in my own work (research on DNR teachers).

“A major question in the teacher change literature revolves around whether changes in beliefs precede or follow changes in practices,” (p. 919). While Guskey (1986) and others asserted that changes in practice precede changes in beliefs, Richardson, Anders, Tidwell, and Lloyd's (1991) case study found that changes in beliefs could precede changes in practice. Whether the professional development approach is of the Empirical-rational type or the Normative-reeducative type seems to influence the order of change, to some extent. Whether or not one order is more favorable (possibly because it leads to lasting change??) is not discussed. “Community was found to be important in change.”

Conclusions

The normative-reeducative approach begins with the individual examining her own “tacit beliefs and understandings” (p. 921). The change process is enhanced when the individual has a discourse community to interact with in which there are others who can be confided in, understand practice, and understand the context in which the teacher practices. Through dialogue with this community, teachers begin examine ineffective practices and possibly begin to make changes in their teaching. Changes in beliefs and practices might then co-emerge. When teachers begin discussing what doesn't work, it seems valuable to be in contact with someone who can help them develop alternative ideas and bring in language to help discuss important phenomena.

Nelson (1997) and NRC (2001)

The NRC (2001a) report, *Adding it Up*, cites Shulman (1987) in categorizing the kinds of knowledge they deem crucial for teaching school mathematics: knowledge of mathematics, knowledge of students, and knowledge of instructional practices. I will refer to these three components of teachers' knowledge as the *teacher's knowledge base* following Harel (1994). According to the NRC's report (2001a), many PD efforts can be viewed as focusing on one of the three components of the teacher's knowledge base and addressing others. However, Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, (1996, p. 404) noted that a focus on one component of the teacher's knowledge base does not necessarily exclude attention to the other components.

According to Barbara Scott Nelson (*Mathematics Teachers in Transition*, 1997, p. 4,5), research into the professional development (PD) of math teachers in the 1980's followed four lines. Shifter and her colleagues (Shifter & Fosnot, 1993; Shifter & Simon, 1992) focused on the need for teachers to reorganize their understanding of "the nature of learning and mathematics" (p.4). Of course, a reorganization of a teacher's understanding of learning has pedagogical implications, as well.

During the 1990's a growing body of work whose focus was on the nature of teacher change grounded in established theoretical perspectives emerged (Nelson, 1997). Cochran-Smith & Lytle (1999) have explained that, "...within various change efforts, there are radically different views of what "knowing more" and "teaching better" mean" (p. 249). Citing Fenstermacher (1994), Wilson and Berne (1999) wrote,

"Stories of teacher learning presume that teachers learn something. The "what" of teacher learning needs to be identified, conceptualized, and assessed. This will require making informed decisions about one's assumptions regarding the nature of teaching knowledge..." (p. 203).

A discussion about teacher change must begin by stating clear assumptions about knowledge, knowing, learning, and teaching. It stands to reason that differences in assumptions contribute to differences in approaches to PD.

Cochran-Smith and Lyttle (1999), Sowder (2007) and Wilson and Berne (1999)

Researchers' reviews of literature in the field of teacher change have acknowledged that this literature is "vast and scattered" (Placier and Richardson, 2001; Sowder, 2007). Cochran-Smith and Lyttle (1999) proposed that "three significantly different conceptions of teacher learning drive many of the prominent and widespread initiatives intended to promote teacher learning" (p. 251). Sowder (2007) found it helpful to sort the myriad of existing reported efforts using Cochran-Smith and Lyttle's three conceptions of teacher learning; knowledge-*for*-practice, knowledge-*of*-practice, and knowledge-*in*-practice. These three conceptions of teacher learning are associated with the kind of knowledge teachers are assumed to need in order to "teach better".

Knowledge-*for*-practice refers to the knowledge teachers need to know to teach mathematics. This knowledge is generated by university-based researchers and includes, but is not limited to, the teachers' knowledge base as described in chapter 1. Knowledge-*in*-practice refers to practical knowledge. Here teachers involved in professional learning communities discuss and reflect on teaching practice, as is the case in forms of Japanese lesson study. Finally, knowledge-*of*-practice refers to knowledge learned from practice by using one's own classroom as a site of inquiry, as is the case in action research or when a teacher participates in a lesson study as the teacher of the lesson and a discussant of the lesson.

A focus on efforts Guided by the Knowledge-for-Practice Conception of Teacher Change

Sowder's (2007) approach is only one way to structure the amorphous field of teacher change literature. Wilson and Berne (1999) narrowed their focus to teacher acquisition of professional knowledge. In terms of Cochran-Smith and Lyttle's (1999) framework, Wilson and Berne (1999) focused on efforts guided by the *knowledge-for-practice* conception of teacher learning. The knowledge base Wilson and Berne envisioned teachers' acquiring elements of was referred to as professional knowledge. Here is a description of some forms of knowledge included in professional teaching knowledge.

“...professional teaching knowledge might include, at the very least, knowledge of subject matter, of individual students, of cultural differences across groups of students, of learning, and of pedagogy (Ball & Cohen, in press; Shulman, 1986, 1987)” (p. 177).

After reviewing a wide variety of research reports Wilson and Berne (1999) found that the knowledge teachers were acquiring in well documented and thoughtfully reported PD efforts fell into three “knowledge” categories: “(a) opportunities to talk about (and “do”) subject matter, (b) opportunities to talk about student learning, and (c) opportunities to talk about teaching” (p. 177). Note that these three knowledge categories are consistent with Harel's definition of the teachers' knowledge base; knowledge of mathematics, knowledge of student epistemology, and knowledge of pedagogy. Similarly, Borasi and Fonzi (2002) provided a list of teacher needs which can be used to establish the goals of a PD effort. These needs include, but are not limited to, types of *knowledge-for-practice* or components of the teachers' knowledge base: understanding of mathematics for the level taught, an understanding of how students learn mathematics, and deep pedagogical content knowledge.

While Sowder (2007) did not limit her review of PD efforts to those guided by the knowledge-*for*-practice conception of teacher learning, she did find a robust group of PD efforts guided by this conception of teacher learning. Within this category, she sorted efforts into four approaches with differing foci: a focus on student thinking, a focus on curriculum, a focus on classroom activities and artifacts, and a focus on knowledge needed for teaching well taught to participants via formal coursework.

Sowder and Wilson and Berne both explained that a focus on one area of the teachers' knowledge base does not exclude professional developers from addressing other areas. It is simply a matter of emphasis, not a matter of exclusion or inclusion. All efforts reviewed acknowledged the need for teachers to learn about content, student thinking, and implications for teaching.

This section has addressed conceptions of teacher learning, with a focus on the knowledge-*for*-practice conception of teacher learning. Within this conception of teacher learning it is assumed that teachers access their knowledge to inform instructional decision making. What has not yet been addressed extensively in this literature review is what researchers hope teachers will do with the knowledge they acquire or construct. Nor have I reviewed researchers' theories about how teachers transform this knowledge into practice, although I have alluded to these theories as foci of PD efforts or goals that guide PD efforts. Just as it is difficult to separate knowledge from beliefs, separating knowledge from practice can also prove difficult.

Some Theoretical Perspectives Guiding Teacher Change Efforts in Mathematics Education

Barbara Scott Nelson (1997) explains that within the field of mathematics education several theoretical perspectives have been guided by some branch of psychology, by sociology, or a blend of psychology and sociology. Goldsmith & Schifter (1997, p. 20) described three such approaches that have been taken in researching how teachers change their practice during the late 1980's through the mid 1990's: (a) a developmental approach taking a more individualistic approach to how teachers go about changing their teaching practice (Goldsmith & Schifter, 1997), (b) a "knowledge based approach" to teacher learning in which researchers help teachers learn about research findings of student thinking and help them incorporate this knowledge into their teaching (Carpenter et al, 1996), and (c) a sociocultural approach focusing on how "teachers and students" negotiate mathematical meaning constructed by all participants of the community (Putnam & Borko, 2000; Cobb et al, 1991). Richardson & Placier (2001), Sowder (2007) and Fennema et al (1996) have also cited the three aforementioned perspectives in their reviews of literature on teacher change. The theory and practice of these three approaches to teacher change form the focus of this literature review.

These three theoretical perspectives and their associated PD efforts do not constitute the entire set of ways to understand teacher change. Indeed, Sowder (2007) and Richardson & Placier (2001) describe many other efforts. However, these three perspectives on teacher change are compatible with findings of national studies in mathematics education and are well documented.

Goldsmith & Schifter (1997) – Teacher Change: Influenced by Developmental Psychology

According to Goldsmith & Schifter (1997), current reform efforts in mathematics education in line with the recommendations of the national council of teachers of mathematics (NCTM) call for students to learn “mathematics for understanding” which entails “curricular changes” as well as changes in the instructional goals and methods of teachers. These changes would require new understandings of “teaching, learning, and the nature of mathematics” on the part of most mathematics teachers whose teaching has “...tended to reinforce the perception that mathematics is mysterious and conceptually inaccessible.”

Exactly how teachers undergo the kinds of changes being called for is still unknown. Goldsmith & Schifter suggested looking to developmental psychology for guidance in the formation of models for teacher change. Goldsmith & Schifter identified three core values that most views of development had in common (c.f., Case, 1985; deRibaupierre, 1989; Erikson, 1950; Fischer, 1980; Piaget, 1970; Werner, 1948) and added a fourth: (a) qualitative reorganizations of understanding; (b) orderly progressions of changes; (c) mechanisms and socio-cultural contexts that support transitions; and (d) the influence of motivational and dispositional factors.

Qualitative reorganizations of understanding

Theories of cognitive development describe stages of understanding that differ from one another and culminate in some form of understanding that allows for complex thought and action (Goldsmith & Schifter, 1997). With regard to change in mathematics teachers' beliefs and practice, many researchers have attempted to create frameworks for change (eg, Fennema et al., 1996; Cooney, 1993; Schifter and Simon, 1992; Schram, Wilcox, Lappan, & Lanier, 1989, Thompson, 1991). Each of the frameworks referred to

have a common starting point; a description of an initial stage of understanding mathematics teaching that is grounded in a “transmission model of learning”²³ and a procedural understanding of the nature of mathematics (Goldsmith & Schifter, 1997, p. 22, 23). Each of the frameworks also discuss subsequent changes in which teachers move away from a transmission model of instruction, begin to rely on what students know and begin to use students’ existing knowledge to guide their instruction. Each of these frameworks culminates in a vision of instruction that is consistent with the vision of reformers (see NCTM, 1989, 1991). The focus of these frameworks is how teachers help make mathematics learning possible for their students.

Different frameworks being developed focus on either beliefs about teaching, learning and mathematics, or on teaching behaviors. Goldsmith and Schifter expressed their view that beliefs and behaviors are inseparable. They point out that in order for teachers to change, it is important for them to have powerful images of what reformed mathematics instruction looks like. Because teachers’ experiences are grounded in the kind of instruction they are trying to change, they may need to have new experiences with mathematics instruction in order to make meaningful changes in their own practice (Putnam & Borko, 2000; Goldsmith & Schifter, 1997).

Beliefs and behavior are mutually defining aspects of teaching. While change in a teacher’s practice can begin with the incorporation of new pedagogical tools (e.g. using small groups or manipulatives) it is also necessary for teachers to examine their beliefs about “teaching and learning mathematics” (p.26). Listening to students’ thinking and

²³ Teachers who have a transmission model of learning believe that knowledge can be transferred in tact from a teacher to students without much interaction with students’ prior knowledge. Consequently, students can receive the message sent by a teacher in tact without much reorganization of their prior knowledge.

examining their own understandings of mathematics and teaching are also keys to change. While Goldsmith and Schifter explain that it takes time for change to come about, they do not indicate either the amount or intensity of the time needed for change to take place²⁴.

The authors explained that changes in teaching practice are accompanied by changes in beliefs and knowledge. Models for changes in two of these “strands” were explored: epistemology and personal understanding of mathematics. Goldsmith and Schifter looked at both strands as they were enacted in the classroom.

Enacted Conceptions of Epistemology

A starting point for change in most teachers is the observation that in a traditional instruction style²⁵, students are not learning what teachers are intending to teach them. Listening to students explain their thinking seems to be important in making this realization (Fennema et al, 1996; Schifter, 1998). According to Goldsmith & Schifter, this realization takes an extended period of time. As a result, teachers change their role to one of “guide or director of student learning”, rather than transmitter of information, but seem to live in two worlds. This change is gradual. They find it hard to break away from old practices compatible with their old role of teacher as transmitter of information and procedures.

Resolution to the newly introduced state of disequilibrium in teachers’ practice and beliefs is signaled by the ability to design and conduct lessons that are compatible

²⁴ Change can be thought of in several ways by asking several questions. Is it lasting? Do teachers take on leadership roles?

²⁵ A traditional style of instruction can be characterized by the teacher playing a primary role of authority in the classroom. There is a large amount of lecture and note-taking. The teacher is the arbiter of correctness (Harel, Rabin, and Fuller in preparation).

with their new beliefs about how students learn mathematics and their newly envisioned role of the teacher as a guide or director of student learning.

Another conceptual shift occurs when teachers realize that they can use student thinking as both a “guide and a goal of instruction”. The teacher assumes less responsibility for delivering clear, concise instruction and become more concerned with focusing on students’ constructions of mathematical knowledge. Because Goldsmith & Schifter take a constructivist perspective on the learning of mathematics, full implementation of a constructivist perspective into teaching marks the final stage in their developmental model.

In summary, Goldsmith & Schifter described a three stage process of change with respect to enacted conceptions of epistemology. In the first stage, mathematical knowledge emanates from the teacher. In the second stage, individual students share their understanding with the teacher and the teacher shares his understanding of the students’ understanding with other students. There is no emphasis on peer-to-peer mathematical communication. In the last stage, the teacher is a part of the community and can share insights as a member of the community with more expertise, but students debate ideas with each other and talk to each other directly. Note that the teacher still has certain responsibilities (e.g. - assessing students and planning lessons). In order for teachers to change their practice it is crucial that they “understand the mathematical terrain that they are making available to their students” (p. 32).

Having explained that teachers’ understandings of epistemology can be reorganized, Goldsmith & Schifter continued by looking at how teachers’ understandings of mathematics can be reorganized.

Enacted Conceptions of Mathematics

1. What are the big ideas of the curriculum for a given grade level?
2. What are the ideas children bring with them regarding these “big ideas”?
3. What kinds of experiences can be used to help stimulate students

mathematically?

4. Many teachers need to revisit the mathematics they know and how they know it. Also, what constitutes the domain of mathematics seems necessary.

“Traditionally, the goal of most mathematics instruction has been to teach students to become fluent in applying a set of discrete procedures in order to solve problems. With this orientation, mathematics is considered to be a set of memorized rules and procedures for correctly solving particular quantitative problems.” (Goldsmith & Schifter, 1997, p. 32).

Goldsmith and Schifter explained that for many teachers, mathematics is thought of mainly in terms of what Schwab (1961/1978) called substantive structures. However, the NCTM and many mathematics education researchers (e.g., Harel, 1994; Schifter, 1995; Steffe & Thompson, 2000; Hill, Rowan, & Ball, 2005) have emphasized a need for teachers to move from an understanding of the nature of mathematics as a fixed set of rules and procedures, to an understanding of mathematics as a “... a body of knowledge that has been created, refined, challenged, modified and appended by hundreds of generations...” (Goldsmith & Schifter, 1997, p. 33).

Several consequences arise from reorganizing the nature of mathematics to reflect its history. First, it may become more difficult for teachers to determine the big ideas of a curriculum when they are not relying on the word of an authority to tell them what the big ideas are. This may require a teacher with a traditional orientation to reorganize his own personal understandings of the subject matter in order to decide for herself what

these big ideas are. Second, when the focus of mathematics instruction is shifted away from developing students to be efficient and accurate problem solvers and placed on actively making sense of problematic situations for themselves, the teacher's role changes. The teacher may have previously taken a role of fact checker, but may now feel free to become a guide or moderator.

Among many consequences that arise as a result of this changed view of the nature of mathematics, the previous two have been highlighted because they point to a tension that exists for teachers who change their views of the nature of mathematics in this way. Teachers are challenged to find a balance in dealing with the mathematical community's shared knowledge and students' developing understandings. To do so, the teacher must be able to see possibilities for connecting the two. Teachers who have come to understand what is at the core of the subject matter they are teaching stand a better chance of making personal connections between students' developing understandings and the mathematical community's shared understandings. As such, these teachers have the opportunity to help students develop their understanding of mathematics in a desirable way.

When teachers don't have a strong enough understanding of the mathematics that underlies the mathematics they are teaching, there can be a tendency toward 'mathematical show and tell', in which each student gets a chance to express his ideas. In this situation, social need to allow every student to have a voice trumps intellectual need to explore fundamental mathematical ideas. Furthermore, these teachers might not know how to assess the solutions generated because of ill-defined goals. Rather than direct

students toward more general ideas, discussions can get stuck at the level of simply answering the problems posed.

Designing lessons compatible with reform oriented views of mathematics, responding to students, and taking advantage of teachable moments all require a deep understanding of the mathematics being taught. As teachers' understandings of the nature of mathematics and their own personal knowledge of mathematics (in terms of depth and breadth) changes, their means of assessing students change as well, but little is known about how.

Orderly Progressions of Stages

“Investigations seeking to identify alternative pathways toward new forms of practice would substantially fill out our understanding of the terrain of teachers' development... such a mapping can focus attention on understanding the conditions... for understanding the mechanisms that help teachers progress from stage to stage” (p. 40).

In developmental psychology, transitions through stages are seen to be “sequential and invariant”. Goldsmith & Schifter (p.37, 38) believe that this need not apply to the development of reform oriented mathematics instruction. The authors suggested that given a prior understanding of mathematics and epistemology, mapping the issues teachers face as they attempt to enact their new beliefs, would contribute greatly to current understandings of teachers' development. These alternative paths are dependent on the nature of a teacher's prior understanding of mathematics and epistemology, the kind of support she receives, and other motivational and dispositional factors.

Transition Mechanisms

Mechanisms of change from one stage of development to another are a key component of a developmental view of learning. These “mechanisms” provide explanation for what stimulates growth in individuals and comes in two forms; psychological or socio-cultural.

From a Piagetian perspective, the mechanisms used by individuals to move between stages of development are accommodation and assimilation. Teachers go through equilibrium and disequilibrium phases as they encounter problematic situations that can be provided by a number of sources to any aspect of a teacher’s knowledge base (i.e. knowledge of mathematics, knowledge of pedagogy, and knowledge of student epistemology). Thus teachers’ development, according to this theory, “proceeds through confronting and resolving problematic aspects of their practice” (p. 42). Such constructions of knowledge are viewed in individualistic terms, though the sources of disequilibrium may be social in nature.

Two areas of investigation within the psychological perspective are: (1) understanding the factors that seem to initiate change in teachers in order to help other teachers begin changing their practice, and (2) understanding the relationship between beliefs and classroom practices (see p. 44).

From a socio-cultural perspective, knowledge is socially constructed – meaning, individuals appropriate tools and signs from their surrounding culture for use in organizing thought. Particular ways of reasoning and “socially valued relationships” are used by individuals (possibly in idiosyncratic ways) as means of intellectual growth. Investigating changes in the use of language by teachers who are undergoing change in their practice, the different “kinds of educational cultures in which teachers work”, and

the influence of student interactions on professional development seem to be three productive lines of investigation within this perspective.

General Directions for Future Research

According to Goldsmith & Schifter, the goal of research in teacher change, from a developmental psychology perspective, is the development of a map for transition together with the mechanisms of change that drive teachers to make progress toward a vision of reformed mathematics education. Such a plan would be helpful in planning teacher training and professional development efforts. Of course, studying a large cohort of teachers over a long period of time would give the best information concerning the variety of paths that teachers take in transition.

Future work could focus on:

1. Deepening current understandings of the relationship between teachers' beliefs/knowledge of epistemology and their practice,
2. Deepening current understandings of the relationship between teachers' beliefs/knowledge of epistemology and their practice,
3. Identifying other "major teaching strands" (1 and 2 being two such strands) and describing reorganizations in them.
4. Identification of factors that make it difficult for teachers to enact consistent change in their practice (e.g. the kinds of pedagogical and content knowledge necessary to teach within the new paradigm).

Research Questions Raised by Goldsmith & Schifter's work:

1. How do teachers develop new forms of practice?

2. Assuming a developmental model can be made for teacher change, what does this model look like?
3. What do researchers believe teachers need to know to help students learn?
4. For a given perspective on learning, psychological or socio-cultural, what are some mechanisms of change and what productive questions can be answered within each perspective?

Prominent Professional Development Efforts

Within the literature on professional development, three PD efforts form the basis of the review because they are widely cited and are informed by national studies on mathematics education; the work of Fennema et al (Cognitively Guided Instruction), the work of Schifter et al (Thinking to the Big Ideas), and the work of Wood, Cobb, and Yackel. Additionally, I have included a description of the IMA project.

CGI

Sowder's (2007) review of Cognitively Guided Instruction (CGI) literature focused on the evolution of CGI reporting from an early study of a control and treatment group, to two case studies of teachers implementing CGI principles in their instruction, to a longitudinal study of 21 CGI teachers.

In the first group of studies (Carpenter et al., 1988; Carpenter et al., 1989), it was found that teachers began to listen to students while they were solving problems and student achievement rose. In the case studies (Carpenter & Fennema, 1992; Fennema, Carpenter, Franke, & Carey, 1992), researchers studied how teachers used knowledge of students' thinking about subtraction and addition problems and teachers' implementation of that knowledge in their teaching along with changes in their beliefs. "Overall, the gains

in students' concepts and ability to solve problems were substantial..." (Sowder, p. 46). A longitudinal study (Fennema et al., 1996) was also done in which it was determined that students in CGI classrooms did better conceptually and with respect to problem solving if they had this instruction for more than one year.

Placier and Richardson considered CGI a major project which had received much attention. Not only did CGI assess the pedagogical content knowledge of teachers, but it also assessed student achievement in several ways. CGI researchers found that "CGI teachers taught problem solving significantly more and taught number facts significantly less than control teachers. They were more constructivist"(p. 918). Differences were also found between experimental and control students with respect to problem solving and knowledge of number facts, with students in CGI classrooms faring better than their counterparts (Fennema et al, 1996). It was noted that CGI teachers listened to student thinking and built on it. In addition to some mixed methods studies, the CGI group has written several case studies about these teachers.

From this account of CGI research, it is possible to glean answers the first question posed. CGI researchers aimed to bolster teachers' knowledge of students' thinking, to help teachers learn to how to listen to their students, to help teachers learn to use what students say and do to inform their instruction, to teach teachers that students do not approach problem solving activities in the same ways that adult do, to teach teachers about models CGI researchers had created for students' problem solving approaches to addition/subtraction problems, to emphasize the value of students learning concepts over procedures.

TBI/DMI

The TBI (Teaching to the Big Ideas) project attempted to teach teachers how to listen and make sense of their students' mathematical ideas (Schifter, 1998, pp. 78-79). Furthermore, it was hoped that teachers would become curious about new ways to see mathematics since they have been educated in a system that lacks an understanding of the "conceptual complexity" involved in elementary mathematics (p. 84). Researchers believed that student thinking is a powerful site for learning mathematics. Participants solved mathematics problems at an adult level that were based on elementary school mathematics. They also spent a great deal of time examining student thinking and discussing how to use it in their practice by journaling, watching videotapes and reading transcripts (Schifter, 1998).

TBI spawned DMI (Developing Mathematics Ideas); a set of K-6 curriculum materials based on children's thinking.

Sowder (2007, p.48), taking from Cohen (2004, p.34), wrote:

Three aspects of DMI that, together, "help describe DMI's particular place within the new genre of professional development" (S. Cohen, 2004, p. 34) are "(1) solidity and complexity of the mathematics under study, (2) the concurrent examination of teachers' and students' mathematics; and (3) the parallel between the seminar's pedagogy and elementary classroom pedagogy as envisioned by both national Standards and the DMI designers" (pp. 33-34)."

Cohen (2004, p.245) found that teachers who had attended DMI seminars reported positive affective changes and a desire to delve more deeply into the mathematics on the part their students. Cohen also noted that researchers' observations concurred with teachers' reports.

IMA

Integrated Mathematics Assessment or IMA (Gearhart & Saxe, 2004; Saxe, Gearhart, & Nasir, 2001) is another example of a project designed to help elementary school teachers learn to understand student thinking and use that knowledge to guide their instruction. Researchers involved in the IMA project assumed that in mathematical problem solving children access prior knowledge in their development of new understandings and that effectiveness of instruction is dependent on the previous assumption.

According to Sowder (2007, p. 49), the IMA treatment involved researchers working with teachers by replacing a traditional curriculum unit on fractions with an alternative unit. In order to teach teachers the mathematics and what to expect in terms of student thinking, researchers had teachers assume the roles of learner of mathematics, researcher of student thinking about mathematics, and then instructor for students, in that order. Sowder explained that in the third phase, "...several ways to assess children's understanding were practiced" (p.49).

Comparisons were made between the students of teachers who had the IMA treatment, students of teachers in a PD effort not focused on student thinking, and students of teachers who had not participated in a PD effort. Researchers found that students with IMA training had the same procedural fluency, but higher conceptual understanding compared with the other two groups of students. It was also found that curriculum replacement alone did not lead to student improvement. Rather, teachers' interventions (in terms of building meaningful activities for their students) and teachers' assessment of students was critical for student progress.

Concluding Remarks

This chapter has pointed out that the nature of the literature on teacher change professional development is large and difficult to organize. Though researchers have created frameworks for organizing the literature, it remains vast with many connections remaining to be explored (Richardson & Placier, 2001). Several PD efforts have been reported in this chapter. However, it is still acknowledged that much more must be done to understand what trajectories teachers follow on their paths toward incorporating new practices into their existing teaching practice and how their knowledge develops as a result of particular kinds of professional development. Nathan and Knuth (2003) have explained that while there exist accounts of expert teachers using reform-oriented practices (e.g., Lubienski, Ball, McClain, & Cobb), it is not well known how the practices of ordinary teachers are transformed in ordinary settings given professional developers' efforts to help these teachers make changes to their existing knowledge base and practice. Chapter 3 explains the methods that will be used to investigate research questions addressing these gaps in existing literature.

CHAPTER 3:

METHODOLOGY

In this chapter I describe the methods used to construct a model of one PD participant's proof schemes, to investigate her teaching practices, and to connect the aforementioned model of her proof schemes with trends in her teaching practices over the two year observation period. These descriptions include information about the settings in which the data was gathered, desirable characteristics of the research subject in the case study, the nature of the professional development effort in which the teacher participated, and the processes used to analyze the data. Because the study makes use of an existing theoretical framework and taxonomy of proof schemes which guided the PD, it is important to describe this framework and the proof schemes taxonomy. The chapter closes by highlighting the reasons for conducting a case study.

Goals of the analysis

The goals of this analysis are three-fold: (a) to describe the transition(s) in proof schemes of one participant in the context of a DNR-based PD, (b) to investigate the nature of development in teaching practices of that participant, and (c) to find connections between elements of (a) and (b). The first goal will be addressed in research question 1, while the second and third goals will be addressed in research question 2 through its sub-questions.

Research questions

Research Question 1: What changes were observed in Maggie's proving and proof schemes as she participated in ATI?

Research question 2: What connections can be found between Maggie's experiences at the PD and the evolution of her teaching practices in a whole class setting?

- A. What is the evolution of the $W_s o U / W_s o T$ Maggie promoted in whole class discussions during her two years of instruction?
 - i. What $W_s o U$, and their corresponding $W_s o T$, emerged?
 - ii. When they were presented, how did Maggie attend to students' $W_s o U$, and their corresponding $W_s o T$?
 - iii. Which $W_s o U$, and their corresponding $W_s o T$, did Maggie promote?
- [Answers to question 1 provide a characterization of a set of her TPs. Harel, Manaster, Fuller, and Soto (in prep) have already discussed the teaching practices Maggie experienced in PD.].
- B. To what extent do Maggie's teaching practices reflect the PD teaching practices?
- C. What relationships can be drawn between the changes observed in Maggie's proof schemes during the PD period and the observed evolution in her teaching practices during the period of time she was teaching?

Algebraic Thinking Institute (ATI): A DNR-based professional development effort

The Algebraic Thinking Institute (ATI) was a two year PD in which 26 middle school and high school teachers participated the first year and 14 participated the second year. The institute was designed to bolster the teachers' knowledge base with an emphasis on proof. As such, ATI (henceforth PD) focused more on teachers' mathematical knowledge than on other aspects of the teachers' knowledge base. However, this does not mean that pedagogical issues, curricular issues, or cases of student thinking were not discussed. In fact, there were many such discussions, but the

focus remained first and foremost the teachers' knowledge of mathematics with emphasis on encouraging a transition in teachers' proof schemes toward more desirable proof schemes.

The PD summer institute met for 4 weeks, 5 days per week, 6 hours per day, over 2 summers. Additionally, there were 3 one day, 5 hour follow-up sessions during the academic years following each summer institute. The instructor, Guershon Harel, was a mathematician and mathematics educator. To highlight Harel's dual roles as teacher and researcher of the PD, he will be referred to as TR when he is acting as instructor. The institute was taught on the campus of a major university in the Southwestern United States. Furthermore, the instructor of the institute has written about the DNR theoretical framework (Harel & Sowder, 1998; Harel, 2000; Harel, 2007). Three facilitators were present at the PD who were participants at previous PD summer institutes. These facilitators were in-service mathematics teachers. Additionally, two other mathematicians familiar with the DNR theoretical framework were also involved in the planning of the institute. These mathematicians were actively involved in mathematics education research and were familiar with the DNR theoretical perspective.

PD participants came from a variety of southern California school districts and varied in experience. All participants were in-service teachers who held valid teaching credentials with varying backgrounds in mathematics and instruction. Participation in the PD was voluntary. Participants were given course credit, if they desired, and a stipend for their participation. No additional work was required to receive course credit.

The PD summer institute and follow up sessions were problem-based. Participants worked on series of novel problems in small groups, sharing solutions and solution

strategies with their peers in small group and whole class discussions. Many problems were designed so that the participants were likely to notice and generalize patterns.

Typically, after each problem was introduced to the class the instructor of the institute met with small groups of participants as they solved problems and discussed solutions among themselves before presenting their solutions to the class. This afforded the instructor insight into the participants' strategies and the imagery participants used to solve the problems which may have been lost by the time participants wrote up final solutions or gave presentations. Multiple solutions were discussed, compared, and contrasted by having participants share and discuss their solutions in a whole class setting.

TR facilitated these discussions about the presented solutions and mathematical issues related to those solutions. Often solution presentations for the same problem or series of problems took multiple class sessions to discuss before a new problem was posed. There were cases when the instructor posed new problems to the participants as a result of discussions about previous solutions to given problems; sometimes as an extension of a given problem and sometimes as a new problem entirely.

TR did not give teachers prescriptions for teaching. Though the teachers were allowed to use any PD problems they wished in their own classroom instruction, it was explicitly stated that each participant is responsible to make curricular decisions based on knowledge of his or her own students' needs. Furthermore, it was emphasized that the problems assigned at the PD and during the follow-up sessions were designed under certain assumptions about teachers' knowledge of mathematics based on previous research findings and supported by more current findings (Marin & Harel, 1989; Harel &

Sowder, 1998; Harel, 2001; Knuth, 2002a). As such, each participant would have to make her own decision about the appropriateness of particular PD problems for her own students.

Data Collection

The PD was taught during the summers of 2003 and 2004 after several pilot institutes were offered. Data collected from the PD and its follow-ups includes: pre-tests, post-tests, field notes, hard copies of participants' homework, an individual interview, and videotaped lessons of selected small groups and whole class discussion.

Eleven participants were selected to be observed teaching their own students at three week intervals during the academic years following each summer of the PD. Each lesson taught by teachers was videotaped and field notes were taken. Conversations were held with teachers at roughly three week intervals and recorded in audio form only.

Choosing elements of the preexisting data set

In this section I will present a rationale for choosing to use only the videotaped lessons from the PD summer institute, the PD follow-up sessions, and the classroom teaching observations from the preexisting data set in order to address the goals of the analysis. A description of the nature of each form of data selected follows, along with the reasoning behind choosing to eliminate or keep that particular form of data in light of the goals of the analysis.

Only by examining every hour of video, every page of notes, every handout collected, every moment of audiotape, and so on, could I be entirely certain that no contradictions or extra contributions were present therein. However, with such a large data set choices must be made. For this reason, the experiences of a research team proved

instrumental in guiding my choices. Subsets with the most potential contributions should be investigated first and all conclusions should be contextualized in the following manner. These are conclusions made from a particular subset of the data chosen for particular reasons.

Pre-Test / Post-Tests and Homework

Before the first PD summer institute and after the second summer institute, participants were administered a mathematics exam; pre and post- tests. During the institute participants were assigned homework. Examples of pre / post-test questions and homework will be provided in an appendix. The actual pre and post-tests will not be examined here, nor will the particular homework questions. However, what will be explained is the nature of written work compared to videotaped work. Each has advantages and disadvantages. I argue that the advantages of making conclusions based on class presentations, small group interactions, and instructor-participant interactions found on video offers a lower level of inference with respect to determining proof schemes than written answers to a predetermined set of questions given to an entire group for several reasons.

Where pre and post-test questions are administered in an effort to gather information about an individual's W_{soU} and W_{soT} without the influence of others, a skilled interviewer or interactions with other group members can provide better information by asking specific questions informed by a participant's answers to prior questions. The interviewer (or other participant) can alter the problem situation in ways that help give better insight into a participant's proof schemes than pre and post-tests by asking the participant specifically how she ascertained the truth of an observation or by

providing a variety of levels of expertise for the individual to interact with and attempt to persuade about the truth of an observation.

After examining the written work of many participants (see Harel, Manister, and Koicho, manuscript in preparation), it was determined that this work often lacked clarity. Multiple interpretations were often viable for written statements or diagrams. Even with a team of individuals, conclusions about $W_s o T$ were difficult to reach because of the multiple potential meanings of the inscriptions of participants. For example, it was noted that the difference between the symbolic referential proof scheme and the symbolic non-referential proof scheme in written work is often undetectable because there was no real way of telling how the participant interpreted the meaning of the algebraic symbols used in her solution without interviewing her. That is, many things participants communicated with obvious gestures and intonation were missing. The participants also made assumptions about the audience they were writing their solutions to. Therefore, many details necessary to determine a proof scheme were left out of solutions to pre and post-tests.

There are also potential drawbacks to data in which individuals interact in a group or with an instructor when trying to determine his or her proof schemes. For example, when an individual is involved in a group it may be difficult to separate the knowledge of the participant under observation from that of the others involved in constructing the argument. However, it is important to note that at the PD participants generally took time to work individually when starting the problems before interacting with the group. This is a fundamental feature of the PD class culture that allowed participants to state conjectures, explain how they rendered these conjectures facts (ascertainment), and

attempt to persuade others of the truth of these observations (persuasion) on many occasions.

Field notes

The value of field notes is constrained by the lens the note taker was looking through at the time those notes were taken. Unless the individual note taker was focused on proof schemes, field notes taken at the PD (summer or follow-up sessions) were unlikely to contain such information. Furthermore, these notes are likely to pertain to the class as a whole or to presentations made by individuals. Such information is caught, to a great extent by the videographer. Private conversations with an individual would only have been noted if the individual was chosen in advance for the note taker to pay special attention to. Given the grounded nature of this study, it is clear that this was not the case.

Conversational Data

The conversational data collected in conjunction with classroom observations was intended to be a teacher led session in which a teacher could raise any questions he or she wanted. Members of the research team informed me that the conversations recorded with the teachers at their sites were fragmented and difficult to follow. Unlike the PD classroom sessions, this data was not focused on problem solving of mathematical tasks unless the teacher brought these questions into the discussion. The broader point to be made is that while the participants' questions and comments might be interesting to listen to in general, it is unlikely that the participants had the three research questions of this dissertation in mind when deciding what to talk about. This does not mean that participants did not choose to discuss issues pertaining to their proof schemes, their teaching practices, or the connections between them. However, the conversations were

not structured to specifically discuss these issues. This lowers the likelihood that the teachers chose to discuss issues suitable to be used as data to support conclusions to these research questions. These points provide a rationale for not using the conversations as data to make conclusions regarding the research questions I intend to pursue.

Desirable Characteristics of the Research Subject:

Considering the goals of the analysis to connect a teacher's mathematical knowledge to her practice and investigate the role of a proof-centered PD in the participant's development, it was desirable to select a participant with an associated data set who was articulate in her explanations of solutions to problems, seemed eager to implement the principles of the PD in his or her teaching practice, had the opportunity to implement the principles of the PD, whose classroom data had minimal distractions from teaching mathematics (i.e. dealing with uncooperative students' discipline issues), and whose subject matter taught had a potential to overlap with the subject matter taught at the PD.

When studying changes in proof schemes of individuals, the ability to articulate ones' solutions and reasoning in front of a camera during small group and whole class discussions is an essential trait in a research subject. During the research team's analysis of TR's teaching practices, members of the research team noted that one participant in particular, Maggie, provided clear explanations of her thinking, attended to mental imageries guiding solutions, and demonstrated potential signs of change in her proof schemes. However, more research was needed to determine if this was truly the case. This observation inspired the first research question. What changes were observed in Maggie's proving and proof schemes as she participated in the PD?

For the purposes of the entire study, the aforementioned trait is necessary, but not sufficient. Additionally, Maggie was not the only PD participant who could articulate her solutions well. Among participants who were articulate and had promising classroom data, Maggie was one of few choices. The choice of Maggie, and not others, as a research subject entailed interviewing the principal and co-principal investigators, Harel and Manaster, about other potential research subjects. Due to the size of the entire data set associated with the project, their expertise served as a crucial starting point in ruling out other participants.

Statements made at the institute and comments from members of the research team who had either collected classroom teaching data or viewed parts of that data set pointed out that Maggie's classroom teaching data had the potential to be a record of a teacher 'tinkering' with her practice. That is, preliminary evidence in several forms²⁶ indicated that Maggie had both motive and opportunity to make changes in her teaching practice and that these changes would be visible in her teaching practices as shown in the classroom observation video. This gave me further reason to believe that she might be a better choice than others, as these forms of self-reported data at the PD were sparser for other participants.

After watching all video of Maggie's teaching. It was clear that she had control over her class. If she asked for a solution, students would give it to her. For example, if Maggie asked for multiple solutions and students offered them, she had the potential to help students connect their thinking to other students' thinking. According to Stake

²⁶ These forms include: impressions of research team members, Maggie's statements, and my own preliminary analysis.

(1995) it is important to “maximize what we can learn.” If Maggie had spent much of her time disciplining students, valuable time would have been lost and the data would not have had much potential to provide answers to questions about connections between the development of her teaching and changes in her proof schemes.

An example of a statement which indicated Maggie’s unique characteristics was made during a follow-up session to the first summer of the PD approximately one month after Maggie began teaching seventh and eighth grade math for the first time:

Maggie: I love math and I wanted to teach it, but this is the first year I'm formally teaching math ... so not having taught it before, I thought oh, I'm so for this. I can do it. ... And so I went in trying to do it... totally opened to it and um, I found that ...some of [my students] were doing multiplication on their fingers. So I'm trying to do word problems that have to do with algebra and then I have to back-pedal and ... make it fit to their needs. So, I'm just in this dilemma. Do I go back to word problems that have multiplication as ... the concept that's being targeted when I'm supposed to be teaching algebra or I mean that's not really an ATI concern. It's just how do I deal with the level of my kids? ...but I have one class that's like my guinea pigs...

The class Maggie was referring to as her “guinea pigs” were a group of eighth grade students who were preparing to take an algebra course. This allowed Maggie some autonomy to design curriculum and make instructional decisions she felt comfortable with, alleviating some of the institutional constraints she might have faced when trying to implement curricular reforms in her instruction and allowing her to tinker with her practice. This inspired the second research question. How had her teaching practices developed during the two years she was observed?

Of the 11 possible participants, some of the reasons for not choosing other participants included: the participant was already under investigation by members of the research team, the participant was less articulate than Maggie at the PD, the member was

in a teaching situation where he or she did not have as much academic freedom to make curricular decisions as Maggie did, the participant did not have a sufficient level of behavioral control over her classroom, and the participant's classroom teaching data set was not as complete as Maggie's. Still, even under these criteria, there were still other possible choices. However, the final decision came down, to some extent, to one of convenience.

In summary, the characteristics that made Maggie a good choice to study were: her articulate nature, her positive attitude toward the subject matter, the fact that she was a relatively new teacher in mathematics (first year), her willingness to share the nature of her pedagogical struggles, her indications that she was able to try out a variety of instructional approaches, her attention to students' needs as learners, and her indication of having academic freedom. Maggie's statements at the PD served as preliminary evidence that she was a good research subject to explore how a teacher's proof schemes influence her teaching practice and how the development of her teaching practices were influenced by her experiences at the PD. Though self-reported data suffers for issues of reliability, these statements were confirmed initially by watching PD footage, interviewing the principal and co-principal investigators, and then independently by watching Maggie's teaching on my own.

Returning to Stake's (1995) comment that, "The first criterion [in a case study] should be to maximize what we can learn" (p.4), after consulting the research team members, it was determined that of the 11 participants observed, Maggie was the best possible choice available. The research on two participants was already underway, and another participant had unruly classes. Of the remaining participants, Maggie was the

most articulate and seemed the most amenable to implementing what she had learned in her teaching practice. Thus, I decided that Maggie represented the strongest potential of the remaining participants to inform the research questions.

Harel and Sowder's (1998) Proof Schemes Taxonomy:

Harel & Sowder's (1998) notion of proof schemes rests on five key terms:

Conjecture - an observation made by a person who has doubts about its truth. A person's observation ceases to be a conjecture and becomes a fact in her or his view once the person becomes certain of its truth.

Proving - Proving is the process employed by an individual to remove or create doubts about the truth of an observation.

Ascertaining - Ascertaining is the process an individual employs to remove her or his own doubts about the truth of an observation.

Persuading - Persuading is the process an individual employs to remove others' doubts about the truth of an observation.

Proof Scheme - A person's proof scheme consists of what constitutes ascertaining and persuading for that person.

Three major categories with subcategories of proof schemes have been laid out and will be described below. For more detail see Harel & Sowder (1998).

External Conviction Proof Schemes: Schemes by which doubts are removed by:

- (a) Ritual - the ritual of the argument presentation
- (b) Authoritarian - the word of an authority
- (c) Symbolic - the symbolic form of the argument

Empirical Proof Scheme: Conjectures are validated, impugned, or subverted by appeals to physical facts or sensory experiences.

- (a) Inductive – ascertainment and persuasion about the truth of a conjecture by quantitatively evaluating a conjecture in one or more specific cases.
- (b) Perceptual – Perceptual observations are made by means of rudimentary mental images – images that consist of perceptions and a coordination of

perceptions, but lack the ability to transform or to anticipate the results of a transformation.

Analytical (or Deductive) Proof Scheme: Conjectures are validated by means of logical deductions .

- (a) Transformational- Observations involve goal oriented operations on objects and anticipations of the operations' results. When changes occur, the observer intends to anticipate it and intends to apply operations to compensate for the change. Images (in the sense of Thompson, 1994) are transformed – perhaps verbally or in writing. Transformational proof schemes can take on several forms described in Harel and Sowder (1998, p.).
- (b) Axiomatic- When a person understands that at least in principle a mathematical justification must have started originally from undefined terms and axioms (facts, or statements accepted without proof). Axiomatic proof schemes can take on several forms described in Harel and Sowder (1998).

Coding Maggie's Proof Schemes: Addressing Research Question 1

Video footage of Maggie engaged in problem solving at the PD were used as the primary data source to make conclusions about her proving and proof schemes. Harel & Sowder's (1998) taxonomy of proof schemes were used to guide observations. In particular, the focus of analysis was Maggie's solutions or comments on solutions in small group and whole class settings. Because the PD data set contains nearly 260 hours of video, transcript searches were used initially to help locate episodes in which instances of Maggie's proving acts can be found.

The approach taken follows the given structure for 12 episodes (Episodes 1-4 come from year 1 while Episodes 5-12 come from year 2 of the PD): (1) Question, (2) Background and Significant Events, (3) Synopsis of Maggie's Solution, (4) Claims about proof schemes, (5) Analysis, (6) Synopsis.

The mathematical task (question) Maggie solved is presented at the beginning of each episode. Due to the size of the data corpus it was important to condense events to their most relevant parts. For this reason, any important background and significant events which might help lower the level of inference are reviewed briefly in the section entitled, "Background and Significant Events," following the problem statement. Next, a synopsis of Maggie's solution is presented providing overview of her WoU the problem. Subsequently, claims are presented to help readers anticipate the assertions of the analysis section. The analysis section provides a detailed account (usually including transcript with annotation of gestures²⁷ if necessary) of pertinent events and connections between events that support assertions about Maggie's proof schemes. Each episode concludes with a discussion of the results relevant to that episode. Appendix A lists all of the problems presented at the institute.

Chapter 4 concludes with a section entitled "Results" that weaves together a final response to the research question from evidence presented throughout the chapter. A table presents the reader an overview of observations regarding changes observed in Maggie's proving and proof schemes. These observations provide data for the

²⁷ One danger in noting gestures of the participant is that this analysis does not include an analytical framework for interpreting gestures. For this reason, gestures are only intended to provide the reader, who does not have access to video, with a more colorful account of events that may help lower the level of inference.

conclusions of Chapter 5 in which Maggie's teaching practices, the teaching practices she experienced at the PD as a participant, and changes in her proof schemes are connected.

The methods used in this dissertation for determining Maggie's proof schemes were previously used by a team of researchers (Harel, Manaster, Fuller, and Soto). With these researchers I learned the criteria necessary to apply definitions of proof schemes reliably through an apprenticeship model. Coding participant solutions using Harel and Sowder's proof schemes taxonomy proceeded along the following lines. After a solution was observed, a synopsis of the participants' solutions was written. During the apprenticeship, research team members presented, attacked, and defended evidence for or against compatibility between participants' solutions and the definitions of proof schemes provided in the taxonomy. The research team established the norm that in the presence of multiple interpretations of participants' utterances, interpretations would be made by selecting the interpretation with the lowest level of inference and would be subject to reinterpretation in light of future findings.

When proving complex results, participants often stated various conjectures, employing different procedures for removing doubt in each case. For a participant, solution can include several observations. The validity of different observations can be determined by the participant in different ways. Therefore, a proof has the potential of constituting evidence for multiple proof schemes.

Not all statements in a proof have equal weight. Within an argument of any kind, though logic requires the validity of each statement in the deductive chain, some statements are more important than others to the overall argument. For this reason, the synopses served as valuable starting points from which to consider a participant's

argument in its totality. The Cat and Mouse Problem (episode 3, chapter 4) exemplified this point. In this case, Maggie's solution rested on a visual observation whose validity Maggie had no doubt about. Therefore, for Maggie, the observation was a fact. Through interactions with TR and her classmates, Maggie became destabilized about the status of her observation. The rest of her solution rested on this result and was deductive in nature. Therefore, in the analysis of the Cat and Mouse Problem, both the Empirical and Deductive proof schemes were evidenced at different points in her solution.

Consistent with the observation that each participant solution can have multiple conjectures, arguments were segmented by considering these key observations. These observations required individual proofs within the overall proof. Condensing and segmenting solutions allowed for proof schemes to be applied to parts of proofs.

From experience coding participants' solutions it was anticipated that there would be times when a proof scheme could not be assigned. There were a few such instances (e.g., parts of the Hill Problem did not have sufficient evidence from which to pick one interpretation over another). These situations were set aside during the process of analysis, more of the video was watched in order to find out if later (or earlier) statements made by Maggie or other participants could provide insight into her statements. However, there were still instances (the Hill problem in particular) in which it was not possible to apply the proof schemes codes because data was inconclusive.

A local change in proof schemes is possible while solving a particular problem. Sometimes Maggie was convinced of the truth of an assertion and stayed with that source of evidence. Other times, she was perturbed by TR, other participants, a facilitator, or the

problem itself. When a change occurred, it was noted in the descriptive analysis of the episode.

Consistent with Piagetian Constructivism, DNR defines learning as a process arising out of disequilibrium states brought on by perturbations to individuals' existing schemes. When participants are in a state of equilibrium and TR hopes to necessitate a WoU, he first had to destabilize the participants. Instances of disequilibrium were scrutinized closely in the descriptive analyses of chapter 4 since they served as potential indicators of change.

Finding Connections Between the Development of Maggie's Teaching Practices and Her Experiences at the PD

Focus question: What connections can be found between Maggie's experiences at the PD and the evolution of her teaching practices in a whole class setting?

1. What is the evolution of the W_soU/W_soT Maggie promoted in whole class discussions during her two years of instruction?
 - a. What W_soU, and their corresponding W_soT, emerged?
 - b. When they were presented, how did Maggie attend to students' W_soU, and their corresponding W_soT?
 - c. Which W_soU, and their corresponding W_soT, did Maggie promote?

[Answers to question 1 provide a characterization of a set of Maggie's teaching practices. Harel, Manaster, Fuller, and Soto (manuscript in preparation) have discussed the teaching practices Maggie experienced at the PD.].
2. To what extent do Maggie's teaching practices reflect the PD teaching practices?

3. What relationships can be drawn between the changes observed in Maggie's proof schemes during the PD period and the observed evolution in her teaching practices during the period of time she was teaching?

The three sub-questions of the focus question represent an approach to identifying sought-after connections between Maggie's proof schemes and her teaching practice. Analysis began by viewing all of footage of Maggie's classroom teaching in a compressed period of time. Initial attempts to frame Maggie's teaching practices entailed comparing her teaching practices to TR's. This comparison led to a focus on how Maggie handled students' W_soU in whole class discussions. Because it is not possible to characterize teaching practices related to handling students' W_soU without describing students' W_soU, it was decided that this needed to be done in order to answer the research question. Thus, sub-question 1a was created.

In addition to the work of Harel, Manaster, Fuller, and Soto into TR's teaching practices, Reid and Zack (2009), and Harel and Soto (manuscript in preparation), also served as starting points to focus the analysis of development in Maggie's teaching practices. Each of these papers helped identify teaching practices of interest, whose development could be connected to changes in Maggie's proof schemes and other experiences at the PD.

Reid and Zack (2009) found common aspects of instruction that seem important to the success of developing children's proving by comparing the teaching of Lampert (1990) and Boero et al (1996) with their own. The researchers found that all three assigned problems that allowed for multiple solution strategies, assigned interrelated series of problems, asked students to communicate their thinking (including mental

imagery) in small group and whole class discussions, encouraged students to state their conjectures and prove them, and established “community standards” of communication which included learning to challenge each other’s statements.

A set of teaching practices can be derived from the aspects of instruction mentioned by Reid and Zack. For example, assigning problems is a teaching action every mathematics teacher engages in. Because this teaching action is so prevalent, it is important to say more about the kinds of problems that are assigned. A more specific teaching action might be assigning problems that allow for multiple solutions.

A related teaching action is handling students’ solutions. Teacher’s can handle their students’ solutions in many different ways. Goldsmith and Schifter (1997) pointed out that when multiple solutions are present in the classroom teachers can engage in “mathematical show and tell”, where no ultimate mathematical point is made. A different teaching behavior might be for a teacher to handle students’ solutions by connecting them to make a bigger mathematical point.

Reid and Zack’s work points out that the teaching practice of assigning problems with multiple solutions is closely related to the teaching behavior of soliciting multiple solutions. Helping students focus on similarities and differences between solutions is a strategy that TR used in an attempt to help catalyze change in participants’ proof schemes.

Associated with these teaching actions are a set of behaviors or characteristics of that teaching action which may be more or less desirable when attempting to develop children’s proving. For example, when multiple solutions are presented, what does the teacher do with them? Does she lead a discussion or simply give students credit. During

this discussion, how many are presented, in what order, and is a bigger mathematical point made, or are the solutions left disjoint? Is a preference expressed for certain kinds of solutions? Are similarities and differences brought out? Other teaching actions of note are: asking students to present solutions, leading a discussion about solutions, and listening to student thinking.

Harel and Soto (manuscript in preparation) discussed several categories of teaching practices that are relevant to the instruction of all teachers who underwent professional development guided by DNR. Incorporating and soliciting of students' ideas, introducing procedures, handling student errors, handling solutions to problems, and assigning problems are all teaching actions teachers of mathematics routinely engage in. Teachers who have undergone DNR-based PD have experienced these teaching actions through the PD. Therefore, the development of teaching practices associated with these teaching actions is of interest because it is possible that the PD may have had some influence over them.

With these starting points in mind, and attempting to connect Maggie's proof schemes and PD experiences to observed developments in her teaching practice, analysis proceeded in four rounds (see figure 3.1) until coding was possible.

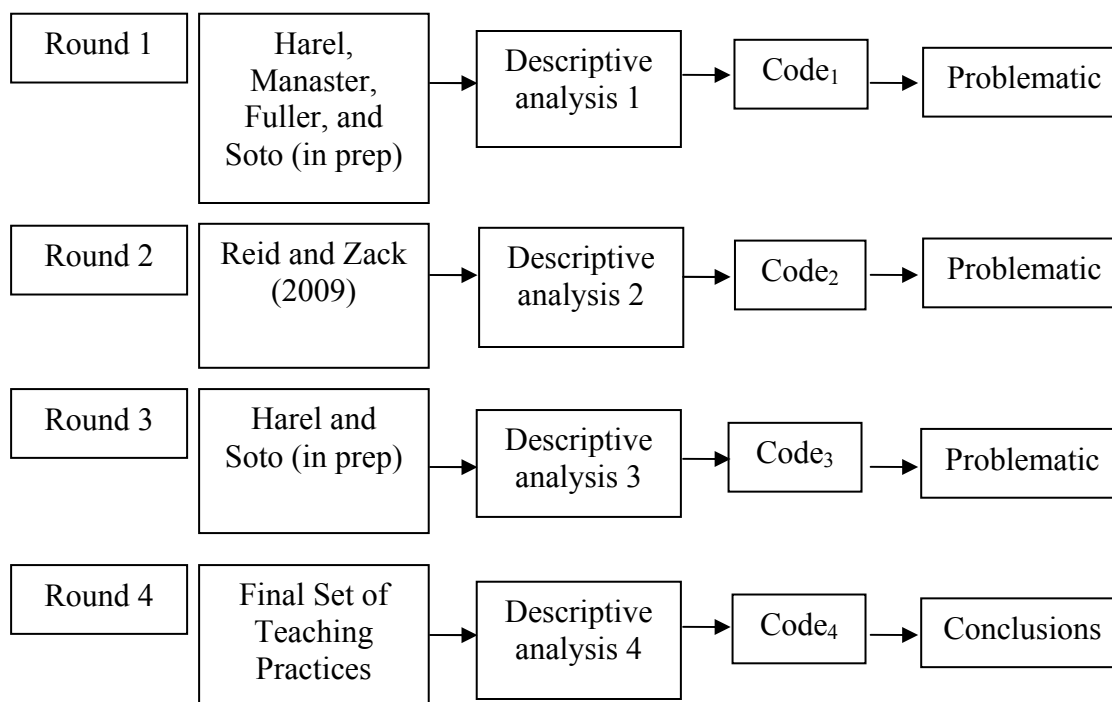


Figure 3.1 – Rounds of coding

Each round of analysis presented its own challenges that rendered attempts to code the data either unstable or impossible. Rather than describe what made each attempt fail, this section proceeds by describing the method of analysis used in round 4.

The analysis of each lesson begins with an explanation of any relevant background events necessary to understand the lesson being analyzed. Next, answers to each of the three sub-questions pertaining to Maggie’s teaching practices are provided with evidence including transcript. Following the analysis of the teaching practices in each episode there is a section entitled “Observed Teaching Practices, Connections to the PD, and Connections to Proof Schemes”. One goal of this section is to make connections between Maggie’s observed teaching practices and those teaching practices observed at PD. Another goal of the section was to find connections within the lesson between

Maggie's teaching practices and her proof schemes. These conclusions were made on a lesson-by-lesson basis and discussed at the end of the chapter.

Through the four rounds of analysis, the following teaching practices were observed as related to issues discussed at the PD, consistent with observations of other teachers involved in DNR-based PD, and resonant with literature about teaching students about proof (Reid and Zack, 2009; Stylianides and Silver, 2009). Observations about Maggie's handling of students' $W_s o U$ in class discussions are discussed within three major categories: attention to mathematical details, extending the locus of authority, attention to students' mental images. When viewed by subcategories, these three categories were considered as follows²⁸:

1. Attention to overlooked mathematical details
2. Extending the locus of authority
 - a. Encouraging Student to Student Talk
 - b. Encouraged students to state conjectures
 - c. Encouraged students to prove their conjectures
 - d. Allowing an error to persist
3. Attending to students' mental images
 - a. Assigning pattern problems
 - b. Ask students to communicate their thinking about their solutions
 - c. Gathering Distinct $W_s o U$

²⁸ Precise meanings of the subcategories are offered in Chapter 3 and at the end of this chapter in the "Results" section.

- d. Pointing out differences between $W_s o U$ and mental images (either within the individual or across individuals)
- e. Asking for alternative $W_s o U$ in the presence of correct solutions.

When the set of codes was developed, two sample lessons were chosen and two different graduate students used the existing coding scheme to code the lessons. In each case, only minor differences were found between their conclusions and mine.

When an instance of a teaching practice was observed within a lesson, it was noted in terms of presence or absence within the lesson. The number of times a teaching practice occurs does not necessarily give that teaching practice more weight. Teaching practices were noted in the descriptive analyses of each lesson. Since all teaching practices were observed in TR's teaching, the presence of more teaching practices within each lesson can be taken as an initial measure of compatibility with TR's teaching practices observed at the PD. However, descriptive analyses were also used to characterize development in a different ways. For example, in the case that a teaching practice was observed with the same frequency in both years, it was still of interest to know how the instances differed from each other and under what circumstances the teaching practices were observed. Thus, descriptions of Maggie's teaching practices played a crucial role in the final discussion about the development of Maggie's teaching practices.

The determination of which proof scheme is being promoted in a classroom can be difficult because a teacher's words can be taken at face value, but her actions can be contradictory in the eyes of her students. Because Maggie's students rarely asked questions or made comments at times when she summarized a solution or promoted a

particular WoU, without interviews of students it was not possible to tell how students interpreted her teaching practices. Therefore, the analysis of promoted W_soU and W_soT was viewed from the perspective of the observer. Given a classroom discussion, Maggie's promoted argument was the unit of analysis.

Reasons for conducting a case study

Goldsmith and Schifter (1997) pointed to a need for models of teacher change. Such models depend on the existence of researchers' accounts of changes in teachers' practices that can be connected in some way to the professional development efforts which those participants attended. Asking about the connection between teacher cognition and teaching practices finds its roots in the work of Shulman (1986) and others as a reaction to research in which investigators focused more on teachers' teaching actions than on their associated qualities (Sowder, 2007).

Mathematics education researchers (e.g. - Carpenter et al, Schifter, Cobb) have used case study methodology to investigate the connection between teachers' knowledge, beliefs and practices. The questions proposed follow in this tradition of using qualitative methods to investigate questions of development or change and make use of researcher's inferences from observable behaviors outside of a laboratory setting as a source of data. Questions #1 and #3 call for what Yin (2003) refers to as an exploratory case study while question #2 calls for a more descriptive study²⁹.

Shulman (1986) described cases as providing "knowledge of specific, well-documented, and richly described events" (p. 11). One advantage of studying cases can be found in their specificity. As Rav (1999) expressed so well, proofs are the "...heart of

²⁹ Yin (2003) explains that the line between types of case studies can be blurry.

mathematics, the generators, bearers, and guarantors of mathematical knowledge”.

Similarly, a case study can generate specific and contextualized knowledge that can generate hypotheses for future studies. Furthermore, understanding a case in its entirety can be a way to understand other cases as well.

Case studies have been used in the training of lawyers, doctors, and teachers. As such they form a valuable part of scientific knowledge. From cases, professionals in many fields learn to reason and to recognize important phenomena. As such, knowledge for these individual trained in this way is not a set of isolated facts. Rather it is a web of interconnected cases. Indeed, from a situated perspective on learning, case studies are invaluable because knowledge and its context are not separable (Brown, Collins, and Digid, 1989; Lave and Wagner, 1991).

In particular, in speaking of a person’s proof schemes it is important to note that within the context of solving a problem the same person can demonstrate a variety of proof schemes or even show evidence of transition from one proof scheme to another. Case studies are well suited for these descriptive kinds of questions (Yin, 2003; Stake, 1995; Berg, 2004). Furthermore, when a study takes place in a complex environment like a classroom whether the unit of analysis is the teacher as a participant or the teacher as an instructor, case studies can be useful in conveying the important nuances and subtleties present in these environments. For example, when analyzing TA’s asking for duration alone or number of instances trims away valuable information from which conclusions can be drawn.

Case studies do not have to be seen in isolation. Stake (1995) refers to collections of case studies as collective case studies. For ATI teachers other case studies are also

being conducted by DNR researchers (manuscripts in preparation). Taken together these cases of teachers learning mathematics and teaching mathematics can provide a rich set of evidence used to inform the DNR theoretical frameworks on teacher learning as well as professional development efforts.

In previous chapters I explained that little is known about how teachers' notions of proof change in the context of professional development efforts or how their teaching practices change as their understanding of proof changes. Aspects of the proposed case study can inform future professional development efforts that intend to bolster teachers' understanding of proof or possibly eventually help determine the effectiveness of ATI as a professional development effort.

In Chapter 4, I report on the changes observed in Maggie's proof schemes. These changes took place in the presence of TR's teaching practices intended to encourage a transition from empirical to deductive proof schemes. Though it is not the goal of chapter 4, a byproduct of the descriptive analysis is a set of examples of TR's teaching practices. The selected set of teaching practices used in chapter 5 to discuss the development of Maggie's teaching was compatible with TR's.

CHAPTER 4:

ANALYSIS – Proof Schemes

Introduction

Research Question 1: What changes were observed in Maggie's proving and proof schemes as she participated in ATI?

While the entire study focuses on the interactions between this triad of variables, here the focus is momentarily on changes in Maggie's proof schemes. The results of the study indicate a change from result pattern generalization (RPG) to process pattern generalization (PPG), the consistent observation of the referential symbolic proof scheme throughout, and the development of the ability to identify important unproven observations on which her arguments rested. The chapter illustrates the teaching practices that circumstances in which Maggie became uncertain about what constitutes a proof, including her interactions with peers and TR's that brought about her state of disequilibrium. Chapter 4 closes with evidence of a resolution to her state disequilibrium in the form of a series of deductive solutions and the rejection of her own RPG solution to the Arithmetic Polygon Problem.

In previous chapters, it has been explained that researchers have pointed out a need for continued study of each in order to better understand the roles these variables play in teacher development (e.g. Monk et al, 1994; Ball & Bass, 2003; Putnam & Borko, 2000; Stylianides & Silver, 2009). By documenting the development of one teacher's mathematical content knowledge in the context of a proof-centered professional development effort – as opposed to using other proxies for mathematical content knowledge such as number of mathematics courses taken past Calculus (e.g., Begel,

1979) this study takes an ontogenetic approach intended to produce a rich picture of how Maggie's teaching practice, learning experiences, and proof schemes are connected by examining her learning.

Recall that in DNR,

“A **conjecture** is an observation made by a person who has doubts about its truth. A person's observation ceases to be a conjecture and becomes a fact in her or his view once the person becomes certain of its truth... **Proving** is the process employed by an individual to remove or create doubts about the truth of an observation... **Ascertaining** is the process an individual employs to remove her or his own doubts about the truth of an observation. **Persuading** is the process an individual employs to remove others' doubts about the truth of an observation... A person's **proof scheme** consists of what constitutes ascertaining and persuading for that person” (p. 244).

These terms are important to keep in mind as the analysis proceeds since they inform the goal of this analysis. Conjectures serve as starting points for analysis with potential to inform the observer about an individual's proof schemes.

As mentioned in Chapter 3 (Methods) the approach taken herein follows the given structure for 12 episodes (Episodes 1-4 come from year 1 while Episodes 5-12 come from year 2 of the PD): (1) Question, (2) Background and Significant Events, (3) Synopsis of Maggie's Solution, (4) Claims, (5) Analysis, (6) Synopsis.

A brief recapitulation of the structure of analysis follows³⁰. First, the mathematical task (question) Maggie solved is presented. Due to the size of the data corpus it was important to condense events to their most relevant parts. For this reason, any important background and significant events which might help lower the level of inference are reviewed briefly in the section entitled, “Background and Significant Events.” Next, a synopsis of Maggie's solution is presented for a brief overview of her

³⁰ For a more detailed version of the structure of analysis, see Chapter 3 (Methods).

approach. Then claims are presented to help readers anticipate the assertions of the forthcoming analysis section. Though the claims section is presented after the problem and events have been contextualized, the reader may choose to read this section first. The analysis section provides a detailed account (usually including transcript with annotation of gestures) of pertinent events and connections between events that support assertions about Maggie's proof schemes. Each episode concludes with a discussion of the results relevant to a particular question. Appendix A lists all of the problems presented at the institute.

Chapter 4 concludes with a section entitled "Results" that weaves together a final response to the research question from evidence presented throughout the chapter. A table presents the reader an overview of observations regarding changes in Maggie's proving and proof schemes. These observations provide data for the conclusions of Chapter 5 in which Maggie's teaching practices, TR's teaching practices, and her proof schemes are connected.

Episode 1 – The TV Rating Problem

Question

TV Rating Problem #1

During the evening prime time, between 7pm and 9pm 70% of all TVs in Greenville are turned on, whereas during the morning prime time, between 6am and 8am only 60% of all TVs in Greenville are turned on. Greenville has only four local TV stations: Star Trek Enterprise STE, Fair and Balance News FBN, Public Broadcasting Service PBS, and Quality Family Broadcasting QFB. John, a local news reporter, obtained from an advertisement firm the "rating" and "share" of each station for each of

the prime times. The "rating" is the percentage of all TVs that are tuned to a particular show. The share is the percentage of all TVs that are turned on that are tuned to this show.

Unfortunately for John he lost some of the data he obtained from the advertisement firm. Here is the data he has left (see chart). The advertisement firm closed down before John completed his article. Can John derive the missing information from what he has left? Help John fill in as many of the other entries in the table as possible.

	STE		FBN		PBS		QFB	
	AM	PM	AM	PM	AM	PM	AM	PM
Rating	25%			14%		16.75%		
Share		30%			27.5%		11.2%	365/14%

Figure 4.1 – TV Rating Problem Data

Synopsis of Maggie's Solution

In this episode, Maggie worked in a group with three other participants. Early in the episode Maggie struggled to understand the meanings of the terms rating and share. She shaded in portions of two circles to keep track of the rating and share for STE (see Fig. 4.2 and 4.3 below). Maggie explained that for her, "The rating is 25% of the total number of TV's that exist [see fig. 1 below]... whether they're on or off." That is, Maggie saw rating as a ratio relating the number of TV's turned on and the total number of TV's in existence, regardless of whether they were on or off at a particular time.

To explain her WoU the share, Maggie drew a second circle (see Fig. 4.3) and shaded 60% of it, distinguishing the televisions that were on from those that were off. Later she reiterated that the entire circle stood for the total number of TV's in

Greeneville. Using Fig. 4.3, she pointed to a smaller portion of the 60% of the televisions turned on and said, “Of the TV’s that are actually on, the share is the percentage of this [pointing to the shaded region] that are on this station [see fig. 3].” Maggie wrote, “both”, to designate a group of TV’s that were both on and tuned in to STE in the morning.

Maggie conjectured that if the rating was 25%, then $25/100 = x/60$, where x is the percent of 60% that were turned on and tuned in to STE in the morning. She looked at The teacher-researcher (TR) for confirmation, but he referred her to her group mates saying, “Go ahead. You are looking at me for confirmation? What do you think? I mean convince other friends here.” As Maggie finished explaining that the share could be represented by the overlap in the circle (see fig. 3 below) between the TV’s turned on in Greeneville and the unknown percentage of TV’s tuned in to STE, Matt added that the overlap would be 15% of the TV’s.

A brief conversation between Maggie and Matt followed in which Maggie’s conjecture became a fact for her. Consequently, she began using the formula $(\text{rating})/100 = (\text{share})/60$ to fill in the rest of the AM data in the table and $(\text{rating})/100 = (\text{share})/70$ for the PM data.

When TR left the group Matt explained that the group had different ways to do the same thing and Maggie strongly agreed saying, “[Be]cause you come out with 15 anyways.” As Matt shared how to solve for different values in the chart from the given information using the W_{soU} , $\text{rating} = [(\text{share})/(\% \text{ of TV's on})]*100\%$ and $\text{share} = [\text{rating}*(\% \text{ of TV's on})]/100 \%$, Maggie concurred with his findings.

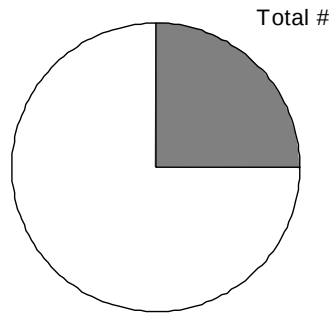


Figure 4.2: Maggie's representation showing the rating for STE in the morning.

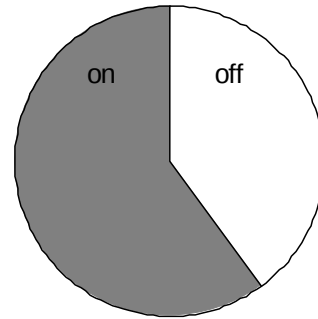


Figure 4.3: Maggie's representation showing the percent of televisions turned on in the morning.

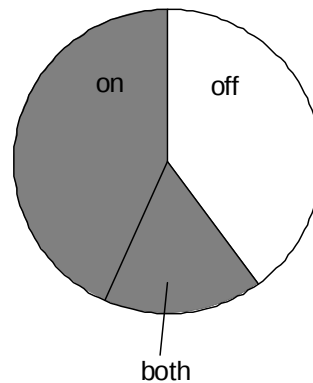


Figure 4.4: Maggie's representation showing her WoU the share for STE in the morning.

Claims

When solving this part of the TV rating problem, Maggie evidenced W_{soU} compatible with the external conviction proof scheme of the authoritative kind in two instances, both when dealing with TR and during interactions with Matt. There is no question that Maggie spent a substantial amount of time and effort to understand the definitions of rating and share and to understand Matt's WoU the problem. Despite

demonstrating an accurate WoU rating and share, Maggie generated the formula, $(\text{rate})/100 = (\text{share})/(\% \text{ of TV's on})$, and became convinced about its validity. Although her conviction lay primarily in a group member's approval, she initially searched for the teacher's confirmation. Both sources of conviction can be characterized as instantiations of the authoritative form of the external conviction proof scheme.

Analysis:

In this episode, Maggie conjectured that finding the share of televisions tuned in to STE in the morning can be found using the equation $25/100 = x/60$, where x is the share for STE in the morning. In the forthcoming analysis, several factors must be taken into consideration. The first is social in nature. Since Maggie was working with three other participants in a group, what was Maggie's WoU the terms rating and share? Proving and conjecturing are individual constructs. How can we be certain that Maggie was not merely following the thinking of others to their logical conclusions?

Another factor that must be considered is more individual in nature and central to Harel and Sowder's definition of proof, personal doubt. In order for an instance to be considered an act of proving an individual must make an observation and have doubt about its certainty. Clearly, such an assertion was present on the part of Maggie. However, what is the evidence that Maggie doubted the aforementioned assertion? That is, what evidence supports the claim that Maggie doubted the veracity of the equation $25/100 = x/60$?

Finally, it is necessary to investigate the manner in which Maggie ascertained the truth of her assertion. Previously I have claimed that in this episode Maggie evidenced an

authoritative proof scheme. Below, two particular instances will be examined to support this claim.

Maggie's WoU the terms rating and share

This episode takes place over the course of approximately 10 minutes. During this time, the group's conversation centered on Matt's WoU for the first 7 minutes. At

00:07:20 Maggie expressed her own WoU and contrasted it to what Matt explained.

Maggie: Yeah, **I** understand [his definition]. That's not what **I** initially thought.

TR: How do you understand it?

Maggie: **I** understood that the rating was, okay cause of the pie chart [drawing figure 1]; the rating is twenty-five percent of the total [motioning along the entire circle] number of TVs...

TR: Okay

Maggie: that exist, whether they're on or off.

TR: Okay,

TR and Maggie: That's a rating, okay.

Maggie: And then the share being [drawing figure 2], okay, of sixty percent ... of the TVs that are actually on, okay.

TR: So, you look only at those-

Maggie: the share is the percentage of this [pointing to the shaded portion of the circle in figure 2]-

TR: that are turned on?

Maggie: that are, on this station.

Early on, Maggie used the pronoun, "I", 3 times in the span of 3 seconds. The use of the word "I" is an indicator of personal ownership over the expressed WoU. It is also significant that this was the first time during the conversation when Maggie expressed her personal point of view. For the first 7 minutes, Maggie agreed, nodded, and occasionally interjected. However, up until this point it was not clear if she shared Matt's WoU or not.

Maggie continued by pointing out a clear contrast between her WoU and Matt's. She explained that she understood Matt's WoU, but had a different WoU. As Maggie explicated her WoU, she pointed out an awareness of the fact that a given television

could be considered on or off (see Fig. 4.3) and that a channel's rating compares the number of televisions tuned to a station to the number of TV's in existence. However, the televisions turned on could also be tuned in to STE or not (see Fig. 4.2) in a given time period. While Fig. 4.2 attends to the portion of televisions tuned in to STE in the morning out of all televisions in existence, Fig. 4.3 attends to the portion of televisions turned on in the morning out of all the televisions in existence regardless of what station a television is tuned in to. Both are meaningful attributes of televisions in the given context. Eventually, Maggie shaded a region corresponding to a portion of the televisions that are both turned on and tuned in to STE in the morning. She referred to this region as the share.

After TR clarified the question, "You want to know the portion that are tuned to a particular channel," Maggie took on personal ownership of the question saying, "That's how I understand it... share being what percentage of this 60%. Right?" At this point Maggie conjectured the following:

Maggie: Well, if twenty-five out of a hundred is twenty-five percent, that's the rating [glancing at a group mate and at TR]. It should be proportionate to [shrugs shoulders], no [in a questioning tone]... some percentage of-. If we say out of a hundred then of sixty [looking up from her paper Maggie looks directly at TR].

TR: Go ahead. You're looking at me for confirmation? [Maggie smiles, chuckling slightly] What do you think?

Maggie: Um

TR: I mean, convince other friends here.

Maggie: (00:09:01) Let me see. Well, you know, I'm, I'm not a hundred percent about the definition, though. That's my problem.

TR: Oh, okay. So, let's go back to the definition.

Though Maggie attributed her doubt to disequilibrium with respect to her WoU rating and share, when the group returned to the definitions she expressed confidence in her WoU the terms.

Evidence that Maggie doubted her assertion

In the previous conversation between Maggie and TR, several aspects of the conversation are not captured by the text alone. Glances, gestures, and vocal intonation all indicate that Maggie was searching for confirmation of her conjecture. These verbal and nonverbal forms of communication also indicate a sense of doubt in the conjecture. Just as Maggie explained at 00:09:01, her glances at a group mate and TR indicate that she was uncertain about the meaning of the terms and searching for confirmation. Her shoulder shrug and upward intonation when saying “no” indicate that even as she asserted the proportion $25/100 = x/60$ she was in doubt. TR noticed that she was looking for confirmation, not only of the meaning of rating and share, but also of the validity of her conjecture. Maggie’s reaction serves as evidence that she was indeed seeking TR’s approval to remove her doubt in the assertion. Although TR would not confirm or deny that her conjecture was valid, he did confirm her WoU the terms rating and share when the group returned to the definitions immediately after the conversation presented in the transcript above.

Evidence for a potential authoritative proof scheme

The previous interaction between Maggie and TR constitutes evidence that Maggie was looking to TR for assurance that her conjecture was true. However, TR did not provide the desired confirmation. As Maggie repeated her WoU rating and share, Matt interjected commenting that the value of x in Maggie’s conjectured equation was 15% as he had previously claimed. Eventually Matt made the point that given either her WoU or his, the result was still the same.

Maggie: (00:09:45) So, this is on, this is off, right? And the share would be where it overlaps [pointing to Figure 4.4]. The number that are on, that are on this station. Matt: Which is fifteen.

Maggie: [Maggie looks at Matt's paper] Which is fifteen. [Matt nods his head affirmatively] Okay, so this ratio is correct, then [upward intonation]? [Maggie looked back at Matt]

Matt: [Matt nods his head affirmatively] Mmhmm.

Maggie: So, if you can put everything in that ratio, if you have one number; then you can find the other number [looks back at Matt] in every situation, right?

It is important to note that Maggie did not confirm the result until she looked at Matt's paper. Then she looked back at Matt three times. Twice Matt gave an affirming nod.

Later Maggie extended her conjecture to other missing data in the table. Even in the face of strong opposition from another group member, Maggie held firm to the belief that her conjecture was correct and could be used to derive the rest of the missing information in the table. However, it was not until after her interaction with Matt that Maggie began speaking of her conjecture with confidence. This is evidence that for Maggie, her interaction with Matt was what finally convinced her to change the status of her observation from conjecture to fact.

Significance/ Discussion of Analysis

Referring to Tall and Vinner's (1981) notion of concept image and concept definition, Alcock and Weber (2004) pointed out that,

"students' images of mathematical concepts are often inconsistent with their formal definitions... even when students can accurately state the definition of a concept, an inaccurate concept image may persist and can significantly hinder their formal reasoning" (p. 232).

Even though Maggie's concept definition (as explained to TR) was correct, she did not apply it correctly in the context of the TV rating problem. From her words, it is clear that

she believed the rating was proportional to the (share)/(% of televisions on in a given time period).

In their investigation of undergraduate mathematics majors proof production Alcock and Weber (2004) noted that undergraduate mathematics students' prototypical representations of a concept, a subset of students' concept images, influence their proof production. In fact, Alcock and Weber explain that it is both useful and desirable to make use of one's concept image when producing a proof rather than try to produce formal proofs by merely unfolding a definition. Though Alcock and Weber spoke of the production of what Harel and Sowder (1998) refer to as "mathematical proof" or formal proof, it is possible that Maggie's frequent encounters with certain types of typical school percentage problems led her to use a part/whole WoU percents to write a proportion with one missing value of the form, (some percent)/100 = (part/whole). Such a prototypical set of examples may have led Maggie to apply an existing scheme for handling percentage problems to this situation without fully incorporating the definitions she struggled to understand.

As a final note, the following table documents the solutions participants eventually produced in a whole-class setting.

[The figure looks like this: **Indicates solutions**]

	STE		FBN		PBS		QFB	
	AM	PM	AM	PM	AM	PM	AM	PM
Rating	25%	21%	11.78%	14%	16.5%	16.75%	6.72%	18.25%
Share	41 2/3%	30%	19 19/30%	20%	27.5%	335/14%	11.2%	365/14%

Figure 4.1: TV Rating Problem Data with Answers

Episode 2 - Train Station Problem

Question

Train Station Problem

Jill, Jack's wife, commutes by train from her work to her hometown. She always arrives from work to the train station at five PM. Jack always leaves home by car at a time so that he too arrives at the train station at five o'clock, and without delay he and Jill drive home together. One day Jill took an earlier train without notifying Jack and arrived at the train station at four o'clock. Upon her arrival at the train station she began walking towards home, and when she met Jack, who was on his way to pick her up, they immediately turned and drove home. They arrived at their home twenty minutes earlier than the time that they usually arrive. Jill wonders if she can determine how long she walked.

Background and Significant Events

The events of this episode transpired in the context of a small group. The group worked together to create a joint solution for approximately 1 hour and 50 minutes without arriving at a final conclusion. A partial solution was presented to the class by Maggie and another group member. Due to the limited nature of Maggie's participation during the public presentation, the focus of this analysis is on the small group data. While this analysis focuses on Maggie's proof schemes, in this case it is essential to point out that the episode took place in the context of especially intricate group dynamics. While the focus of this dissertation is cognitive in nature, it is important to acknowledge that group members' status issues matter and this episode provides a meaningful case of why.

Since allowing status issues to take center stage would detract from the focus of the dissertation, I have decided to strike a balance between the two aforementioned extremes, focusing on status issues and ignoring them. To this end, the following paragraphs attempt to describe, in a rudimentary way, the issues I have chosen to focus on in reaching temporary conclusions with regard to group members' status. Further descriptions regarding status issues are beyond the scope of this dissertation, but have been identified as a meaningful complementary lens to take on in future investigation. Recall that participants of the PD worked in small groups to solve problems on a daily basis. Groups had either three or four members. The composition of these small groups differed from day to day. On this particular day Maggie worked with Laura, Marc, and Penny. Laura established her position as the mathematical authority and group leader early on with a few key observations about the problem. On several occasions she clearly expressed that she would defer to Marc for leadership of the group, but never extended the same offer to Maggie or Penny.

As the group worked together to provide a joint solution, many of their interactions centered around: interpretation of the problem statement, choice of mental images to focus on as guides for their solution efforts, choice of problem-solving approaches, evaluation of the status of particular pieces of evidence, and evaluation of overall arguments.

For the purposes of this analysis, the relevant claim regarding each group member's status in the group is that Laura's decisions held a disproportionate weight in each of the aforementioned categories, followed by Marc, then Maggie, and finally Penny. This does not mean that Laura alone set the agenda for the group. However, she

often managed to reject or redirect Maggie, Penny, and Marc's contributions within the aforementioned categories. In some cases, her reasons were mathematically valid and in other cases not. She acted as spokesperson for the group during interactions with TR and controlled the paper that Marc, Maggie, and Penny were focused on where expressions were written and equations were solved as a group. Marc and Maggie had their own papers, but group discussions usually focused on Laura's paper. On several occasions, when group discussions centered around the work on Maggie's paper, Laura began writing on Maggie's paper. However, Maggie did not write on Laura's paper. Other qualities of Laura's interactions with group members played into the establishment of her status (e.g. – tone, volume, response time, and gestures).

While Maggie may have evidenced a particular proof scheme in the presence of her group mates, it is possible that she reconsidered her position at a later point in time. That is, these results should be taken in context as tentative based on observed behavior in the presence of the particular group dynamics. The conclusions reached here should be taken as an attempt to see Maggie's sources of conviction within this overall context. Furthermore, conclusions should be viewed within the overall context of the two years worth of data and the fact that this data was collected in the context of Maggie's work with various personalities. One question for further studies to consider is, "Do Maggie's proof schemes correlate in some way to her status in a group?"

Synopsis of Maggie's Group's Approach

To properly contextualize the forthcoming analysis it is important to note that Maggie, Laura, Marc, and Penny (with limited interaction by Penny) worked together to create a solution to the problem, discussing many ideas without spending much time in

silent reflection before proposing them to the group. In many instances of the PD group collaboration, members work separately for extended periods of time before sharing mental images, interpretations, problem-solving approaches, conjectures, and proofs with each other. Consequently, attributing ideas to individual group members in this data is not always possible. Still, on several occasions the author and defender of conjectures are identifiable.

The group did not produce a final solution to the problem. Nevertheless, several conjectures can be identified. Some of these conjectures are Maggie's and others may belong to other group members. Because of the interwoven nature of the group's work, I will describe here the most relevant parts of the group's work as they pertain to Maggie's demonstrated WoU the problem and proof schemes.

Maggie and her group mates eventually decided to view the problem from Jill's perspective. The group chose to focus on a one-way trip from the train station to home composed of two legs, walking and driving. However, they decided that Jack's trip in one direction was 20 minutes shorter from the pick-up point to home. Maggie and her group mates attempted to solve the problem using a familiar approach, defining several variables to represent the rate, time, and distance and writing and solving a system of equations involving them. Laura, Marc, and Maggie each contributed to and demonstrated an understanding of the meaning of each of the following variables, which they used to write a system of equations:

- t is the number of minutes it took to drive from the Train Station (TS) to home,
- r_1 is the driving rate for both husband and wife,
- r_2 is the walking rate for the wife,

- 10 and d_1 were used interchangeably for the distance from TS to home (group members expected this distance to be inconsequential in the end),
- d_2 is the distance the wife walked,
- $10-d_2$ is the distance the husband drove.

Guided by the imagery of Jill's trip as a one-way trip in two legs traveled at two different rates Maggie, Laura, and Marc decided to pursue a problem-solving strategy that required defining the average rate over the whole trip in terms of the variables. With the help of TR, they eventually concluded that this average rate could be represented by:

$$\text{Average Rate} = \frac{10}{\frac{10-d_2}{r_1} + \frac{d_2}{r_2}}$$

or

$$\text{Average Rate} = \frac{10r_1r_2}{r_2(10-d_2) + r_1d_2}.$$

The group discussed and agreed that on the day Jill arrived at the TS early, her trip was 40 minutes longer than usual while Jack's trip was 20 minutes shorter from the pick-up point to home. Consequently, for Jill the entire distance of the trip from TS to home could be described by multiplying the average rate by the elapsed time of the trip, expressed by writing $d_1 = \left(\frac{10r_1r_2}{r_2(10-d_2) + r_1d_2} \right)(t+40)$. According to the group, Jack's entire distance from pick-up point to home could be expressed by writing $10-d_2 = r_1(t-20)$. However, on a normal day the distance from TS to home could be expressed by writing $10 = r_1t$.

Solving the system of equations led the group – momentarily – to the conclusion that Jill walked for 60 minutes, though this conclusion was later abandoned.

Maggie did not go along with everything Laura and Marc said at all times. For example, when Laura explained that it was necessary to find the average rate of the trip for Jill in order to arrive at a conclusion, Maggie offered an alternative problem-solving approach. She explained that Jill's time spent walking could be seen as the difference between the duration of her entire trip ($t+40$) and the duration of her car ride ($t-20$). Long before the group arrived at the conclusion by solving a system of equations, Maggie concluded that Jill walked for 60 minutes in this way. However, her idea was dismissed on the grounds that her result stood for the amount of time Maggie had arrived early at the train station.

Later, when the group arrived at the same conclusion by solving the system of equations Maggie proved that Jill could not have walked for 60 minutes because the husband would have been at the train station 60 minutes after Jill began walking from the train station. Therefore, Jack must have passed Jill some time before 5 o'clock and the group could be certain that Jill walked for fewer than 60 minutes.

As the episode neared its conclusion, Marc revisited the conclusion that Jack's trip was 20 minutes shorter in one direction. Realizing that Jack saved 20 minutes in two directions rather than in one direction, Marc convinced Laura and Maggie that Jack's trip was only 10 minutes shorter in one direction.

Maggie concluded that Jill's walking distance took 10 minutes by car. Consequently, she wrote the following equation, $d_2 = 10r_1$ and adjusted the group's previous equation, $10 - d_2 = r_1(t - 20)$, changing it to $10 - d_2 = r_1(t - 10)$.

The episode ended without a final conclusion. Maggie and Laura presented their work to this point to the class. They reported that they had not yet found a way to incorporate their new assumption about Jack's trip into their system of equations in a way that would allow them to solve the problem.

Claims

In the analysis that follows, I will focus on Maggie's conjecture that Jill walked for sixty minutes and her eventual rejection of this conjecture. The group chose an approach that entailed declaration of variables, writing equations, and solving a system of equations. Maggie questioned the group's problem-solving approach, which provided insight into her proof schemes. In the forthcoming analysis, I will demonstrate how Maggie's decision to reject her initial conjecture was guided by a transformational proof scheme and how the manner in which she questioned the validity of algebraic expressions provided evidence of a referential symbolic proof scheme.

Finally, this section closes with further evidence supporting these conclusions by connecting two important instances within the episode by means of a theoretical discussion regarding the internalized proof scheme.

Analysis

1. How did Maggie become convinced that Jill did not walk for sixty minutes?

Maggie raised the conjecture that Jill walked for sixty minutes by subtracting the amount of time Jill spent driving home from the duration of her entire trip. At the time, the group had just finished a discussion with TR regarding how to find an average rate for Jill's entire one-way trip symbolically. TR had suggested that it was in their best interest to focus on the quantity, d_2/r_2 , Jill's time spent walking, rather than engage in the

complex symbolic manipulation they were about to begin. Noticing that finding d_2/r_2 should be the group's overall goal and that d_2/r_2 is in fact an amount of time, Maggie proposed a time-only approach. Focusing on symbolic manipulation without regard to the meanings of the variables involved, Laura and Marc explained that they felt Maggie's computations had produced a quantity that was not a time. The group returned to their symbolic approach for the moment, but eventually came to understand Maggie's point much later.

Scene A:

Maggie: ... **can I just say something really fast and just tell me if this is totally erroneous then we'll move on to doing the hard stuff.** But, the time that it took him to get home was T minus twenty, the time it took her to get home from four o'clock was T plus forty.

Marc: Uh-hum.

Maggie: Well if you took her total time that it took her to get home minus the time she was in the car with him, wouldn't that leave you with the time that she was walking?

...

Maggie: ... **T plus forty minus T minus twenty ... equals sixty.**

Laura: That's from when she started... she waited for sixty minutes.

Marc: That's, that's what...

Maggie: Yeah [softly].

Marc: ... that represents, yeah.

In scene A Maggie clearly stated that her solution was tentative saying, "...**Just tell me if this is totally erroneous** then we'll move on to doing the hard stuff." Also, at 1:36:25 (see transcript below) Maggie specifically said to Laura, "...she wasn't walking for a whole hour...". This indicates that the conclusion that Jill walked for an hour was a conjecture for Maggie rather than a fact. When Maggie concluded that Jill walked for an hour, Laura said that the sixty minutes stood for the time Jill waited. However, the problem does not indicate that the Jill waited anywhere for any amount of time. The

question to be explored here is whether or not Maggie was convinced by Laura and Marc.

To understand what Laura meant, whether or not Maggie really accepted it, and why, it is important to return to a previous scene in the data where Laura helped institutionalize the image of the trip being taken in one direction by Jill rather than choosing to coordinate the chronological aspect of Jack's two-way trip with Jill's one-way trip. During this scene Laura explained that Jill's trip on the day she arrived early was 40 minutes longer than usual.

Scene B:

Laura: ... she didn't leave at five ok, she left at four o'clock. So for her it was going to, **she has a whole extra hour added on to her trip now.**

Maggie: Yeah.

Laura: Ok, but, so **basically ... she has a whole extra hour of waiting.** Pretend that the situation was going to be normal, she would have waited for a whole hour and then got home at T. She would have had sixty minutes plus her ... , that time T that she normally did.

Maggie: Right.

Laura: Ok.

Maggie: **But she wasn't walking for a whole hour** because...

...

01:36:41 Laura: ... ignore the fact, if she weren't walking, she would have waited here for sixty minutes to wait for him.

Maggie: Right.

Laura: Sixty minutes, and then it would have taken her the normal time to get home right.

Maggie: Right.

Laura: Which is T.

Maggie: Plus T.

Laura: Plus T ok. However, this did not take her the normal time to get home, it took her twenty minutes less, so take away twenty, which equals out to T one plus forty.

Marc: Right.

Maggie: Right, right, right, ok, I got it.

For Laura and Maggie there were two ways to think about the extra hour for Jill's trip, either time spent waiting at the train station or time spent walking from the train station.

In her explanation Laura used the image of the wife waiting at the train station until 5 o'clock to help her reach a conclusion about elapsed time. She added that on the given day Jill arrived home 20 minutes sooner than usual. Attributing the 20-minute time saving to the car trip, Laura concluded that Jill's trip was 40 minutes longer than usual. In this scene Maggie interpreted Laura's statement about waiting figuratively rather than literally. Indeed, when Laura said, "... she has a whole extra hour of waiting. Pretend that the situation was going to be normal, she would have waited for a whole hour and then got home at T," Maggie replied, "But she wasn't walking for a whole hour, because..." Before she was cut off by Laura, Maggie was prepared to provide an argument for why Jill could not have been walking for an hour.

In light of the previous conversation regarding how Laura and Maggie understood waiting, from Maggie's perspective there are two possible ways to interpret Laura's grounds for dismissing Maggie's conclusion that Jill walked for 60 minutes in scene A. If Laura meant that the 60 minutes were spent walking to the car she would not have dismissed Maggie's conclusion. The only other previously discussed interpretation of waiting was that Jill stayed at the train station for 60 minutes.

Given that Maggie had already explained that she believed Jill couldn't have walked for a whole hour, and she was prepared to defend this position, it is curious that Maggie would attempt to convince the group later that Jill had walked for an hour. What these scenes show with certainty is that Maggie was in disequilibrium about the amount of time Jill walked and that she was not convinced that her solution was correct even as she communicated it to her group mates. Her agreement with Laura and Marc seemed to be more of an agreement to return to the symbolic manipulation approach than an

expression of conviction that her solution was incorrect. For an analysis of how Maggie was finally convinced that Jill did not walk for sixty minutes, I turn to the events of scene C below.

Immediately prior to scene C (below) the group represented the average rate of Jill's entire trip symbolically, represented the situation using a system of equations, and found that the solution of the system of equations indicated that she had walked for 60 minutes. The group laughed loudly when they found that the solution to the system agreed with Maggie's previous solution. Upon further reflection, different members of the group took different positions about the reasonableness of the solution. Laura proposed that she could have walked for 60 minutes while Maggie initially said that she didn't know. Maggie promptly changed her position and explained why it could not be so.

Scene C:

TR: [Explain] why it cannot be sixty minutes?

Maggie: Ok, because at four o'clock she started walking home, ok, if this is the train station here, she started walking home. Well if he times it so that he always arrives at the train station at five o'clock, so sixty minutes would have passed before, if sixty minutes had passed he would have been at the train station, but she had already left at four o'clock, she would be somewhere along this route, and he in sixty minutes would be at the train station, but we know that they met up sometime before that cause she had that head start. So her head start, so it couldn't be sixty minutes, I know that, even though I thought it was earlier. So it has to be something less than sixty minutes.

Note that Maggie created a contradiction by assuming that Jill did walk for 60 minutes. Then Maggie followed the logical consequences by referring to relevant aspects of the problem statement. She demonstrated her understanding of the problem statement in diagrams, representing the positions of Jack and Jill at different times. Her goal was to

show that Jill didn't walk for 60 minutes in the end by noticing that the wife could not have been in her original position 60 minutes after she had arrived there.

Maggie explained that at some point in the past she believed Jill walked for 60 minutes. However, by attending to Jack's round trip, she was able to persuade herself and her group members saying that if Jill had walked for 60 minutes there would have been a contradiction. She was able to do so by visualizing the physical situation and applying deduction to her understanding of the situation, ultimately concluding that Jack must pass Jill in less than one hour.

Harel and Sowder (1998) explain that,

“Transformational observations involve operations on objects and anticipations of the operations' results. The operations are goal oriented. They may be carried out for the purpose of leaving certain relationships unchanged, but when a change occurs, the observer intends to anticipate it and, accordingly, intends to apply operations to compensate for the change” (p. 258).

It has been demonstrated, by analyzing scenes A and B, that Maggie's observation that Jill walked for 60 minutes was indeed a conjecture for her. She entered a series of equilibrium/disequilibrium phases regarding the status of her conjecture as fact.

However, what ultimately convinced her – and the manner in which she chose to persuade her group mates – was by pointing out a contraction to support her conclusion.

In the end, Maggie was able to couple her mental imagery of the problem statement with deductive reasoning. Thus this episode demonstrates evidence of a deductive proof scheme of the transformational sort on Maggie's part.

2. Examining and characterizing Maggie's sources of conviction for the validity of algebraic expressions and symbolic manipulation.

In the analysis that follows I will characterize Maggie's symbolic manipulations in the context of the Train Station Problem, focusing on Maggie's sources of conviction with respect to the validity of certain algebraic expressions group members offered as potential representations for various aspects of the problem statement. As described above in the claims section, I claim that the referential symbolic proof scheme best characterizes Maggie's symbolic manipulation. Harel and Sowder (2007) describe the referential symbolic proof scheme, a form of the deductive proof scheme, as follows:

“...in the symbolic referential proof scheme, to prove or refute an assertion or to solve a problem, students learn to represent the statement algebraically and perform symbol manipulation on the resulting expressions. The intention in these symbolic representations and manipulations is to derive relevant information that deepens one's understanding of the statement, and that can potentially lead her or him to a proof or refutation of the assertion or to a solution of the problem. ... a significant feature of the referential symbolic proof scheme... is that one possesses the ability to pause at will to probe into the meaning (quantitative or geometric, for example) of the symbols.”

This analysis begins with Maggie and Laura's self-report of their partial solution as found in TR20030717pm. In this report, Maggie characterizes her group's problem-solving approach. After reviewing Maggie's self-reporting, the analysis returns to the group's work that morning.

Maggie's Problem-solving Approach

By Maggie's account during her whole class presentation with Laura, the group's approach was to define variables to represent the rate and distance of Jack and Jill for different portions of their respective trips and relate these variables in some way to create a system of equations they could solve for the time Jill walked. While the group had originally declared several variables to represent the walking and driving times of Jack

and Jill, they settled on defining time in terms of distance and rate. As their ultimate goal they decided to search for d_2/r_2 , Jill's walking distance over her walking rate.

Maggie: ... the distance that Jill had traveled walking was d_2 . So since we said $d_1=10$ [the distance from the train station to home], the distance that Jack had gone or will take to go home, is $10-d_2$. **So our approach, really, is to come up with all these ways that rate and time are related.**

Maggie: ... we said Jack's speed in the car is r_1 **So we just defined a lot of different variables.** And then r_2 then is Jill's walking speed. And we didn't want to start doing, like, t_1 and t_2 and all these times so we tried to do time in terms of those variables; so we said that Jack's..

TR: Well, tell the story you did at the beginning with t_1 and all that stuff.

(00:03:54) Maggie: Right, right. So, time really is d/r so Jack's total time on a regular day without this Jill arriving earlier is $10/r_1$, right? The distance over his rate. That's a normal day when he picks her up at 5:00. That's how much time it takes him to get home from the train station. So we said, on this day, this exception, the time that it took him to meet Jill was really $(10-d_2)/r_1$ divided by that same rate.

Group Work

After reading the problem statement and discussing their interpretations the group decided that they would let 10 be the distance from home to train station. They declared that Jill walked d_2 distance and consequently drove $10-d_2$ distance. The group also decided to let r_1 be the rate that Jack drove and r_2 be the rate that Jill walked. Laura decided to write the equations $d_2 = r_2(t+40)$ and $10-d_2 = r_1(t-20)$ and asked if it was correct. Though Marc agreed with what was written, Maggie paused to attribute meaning to the symbols in the equation, exerting a form of control over them by connecting the meanings of the variables to problem situation.

Laura: ... is that right [Laura wrote $d_2 = r_2(t+40)$ and $10-d_2 = r_1(t-20)$]?

Marc: Right.

Laura: Do we have anything there?

Maggie: So you're saying, her distance [pointing at d_2] is her walking rate [Maggie pauses]...

Laura: Plus the time it would have taken her to go the whole way. And his distance is his driving rate plus the time it would have taken him to go the whole

way, it should be right? Shouldn't it be [replacing $t-20$ with t]? Yeah, and then these distances together...

Laura tried manipulating the two equations, but Maggie interjected asking,

Maggie: Are you guys now starting to go like this [gesturing with fingers from train station and home toward each other on paper], like looking at his [trip] and hers [trip] like that? To see where they meet?

Maggie verbalized her feeling that the group would not be able to solve their system of equations saying, "I'm lost...It just looks like there's too many variables. How would you even solve for one of them?" Laura and Marc assured her that the plan was to substitute and cancel out the variables they were not searching for.

Overwhelmed by a myriad of symbols and expressions, Maggie and Marc asked Laura to begin her explanation from the beginning in order to validate her thinking and make progress. Maggie and Marc asked questions and added their own insights as Laura tried to refine her explanation and conclusions thus far. Eventually, Laura returned to the two previously stated equations: $10 = r_2(t+40)$ and $10 - d_2 = r_1(t-20)$. Once again Maggie attempted to connect Laura's algebraic representation of the problem to her own mental imagery of the problem situation.

Maggie: Ok, **your two equations** are...

Laura: This and this [pointing to $10 = r_2(t+40)$ and $10 - d_2 = r_1(t-20)$].

Maggie: If she would have walked home [pointing to $10 = r_2(t+40)$], **ok, if she would have walked home the whole distance** right, and **then this is the distance that he drove her home** [pointing to $10 - d_2 = r_1(t-20)$]...

...

Maggie: **If you take the total distance [pointing to $10 = r_2(t+40)$], minus [pointing to $10 - d_2 = r_1(t-20)$]... his distance, we'd be left with the distance that she actually walked.**

[Laura changes the subject]

Eventually, the Maggie came to understand that if r_2 was the walking rate for Jill, it could not be used in the equation, $10 = r_2(t+40)$. Instead, they would have to explore the notion of average speed.

In light of Maggie's previous statement, "I'm lost," and her previous request for clarification, Maggie's comments can be taken as indicative of an attempt to assign mental imagery of the problem situation to Laura's expressions and equations. Returning to Harel and Sowder's (2007) definition of the referential symbolic proof scheme, it is important to note that for Maggie the symbols did not have "lives of their own". Rather, Maggie exerted control over them, treating Laura's algebraic expressions as representative of quantities relevant to an understanding of the problem statement.

Maggie and Laura proposed different ways to represent the average speed for Jill's entire trip as a function of r_1 , and r_2 , $(r_1 + r_2)/2$ and $(r_1 + r_2)$ respectively. Though neither Maggie nor Laura's proposed representations were correct, the group was able to create a counterexample to both using a correct way of understanding average speed over an entire trip. Through a discussion with TR the group arrived at, $\frac{10r_1r_2}{r_210 - r_2d_2 + r_1d_2}$, as an algebraic representation of the average speed and used it to solve their system of equations, concluding that Jill traveled for 60 minutes. Though Laura proposed that Jill could have walked for 60 minutes, Maggie's explanation (see previous analysis) indicated that she was not convinced by the symbolic manipulation. It was previously explained that Maggie relied on a transformational proof scheme to ascertain that Jill could not have walked for 60 minutes.

Synopsis

What is most noteworthy about the aforementioned events is that while Maggie did relinquish control over the meaning of the expressions as she solved the system of equations (involving Jill's average speed), she did not ultimately accept a conclusion that

could have been used to corroborate her previous finding that Jill walked for 60 minutes.

Referring to characteristics of an individual's symbolic manipulations and representations

Harel and Sowder (2007) wrote,

“In such an activity, one does not necessarily form referential representations for each of the intermediate expressions and relations that occur in the symbolic manipulation process, but has the ability to attempt to do so in any stage in the process. It is only at crucial stages – viewed as such by the person who is carrying out the process – that one forms, or attempts, to form such representations.”

Certainly it is the case that arriving at a conclusion is a “crucial stage” of symbolic manipulation. In sum these events offer a unique window into Maggie's sources of conviction in the context of this problem. While Maggie did not solve the problem two characteristics of her proving did become increasingly evident. First, Maggie demonstrated the use of mental imagery rooted in repeated attempts to understand the problem statement – whether these images were originally hers or not – as a primary source of conviction. Second, though Maggie could have used a result generated by symbolic representation and manipulation to increase confidence in a previously generated result, she did not place full faith in the result as a person demonstrating the non-referential symbolic proof scheme might. Instead, Maggie relied on a personally meaningful interpretation of the problem involving the coordination of a set of actors together with their movements in space and time, even in the presence of a solution produced through symbolic manipulation.

It is also notable that Maggie was able to abandon what appears to be a very powerful scheme for her and many other teachers – the rate, time, distance scheme – when she concluded briefly that Jill walked for 60 minutes. A closely related observation is that Maggie was also able to argue entirely without the use of variables when she

explained why Jill could not have walked for 60 minutes. While these observations may seem haphazard, they are not. Indeed, as simple as the observations seem they help to illustrate the connection between the transformational and referential symbolic proof schemes.

Harel and Sowder (1998, p. 262) refer to a *proof heuristics* as a method of proof abstracted by students from repeated application of an approach they have often found to be successful for rendering a conjecture into a fact. A transformational proof scheme is deemed an internalized [transformational] proof scheme when it has been encapsulated into a proof heuristic (p. 262). According to Harel and Sowder, “An indication that a transformational proof scheme has been encapsulated into a proof heuristic rather than a ready made formula is that the individual applies it selectively.” Harel and Sowder continue by explaining that, “A particularly important instance of the internalized [transformational] proof scheme lies in the transformational use of symbols” (p. 264). Therefore, the referential symbolic proof scheme is a form of the internalized [transformational] proof scheme which characterizes an individual’s use of symbolic representation and manipulation as a proof heuristic.

While the rate, time, distance formula is known by most teachers and is often used to solve problems when individuals are in motion, according to Harel and Sowder (1998) one way to observe that it is a proof heuristic is its selective application. Maggie’s recognition that the problem could be solved without the use of the rate, time, distance formula constitutes a form of evidence that for her the rate, time, distance schema is an internalized proof scheme. The second observation – that Maggie abandoned the use of

symbols altogether to make her argument – gives further credence to the conjecture that Maggie's need for symbolic representation is referential symbolic as described above.

Episode 3 - Cat and Mouse Problem

Question

Cat and Mouse Problem:

A cat and a mouse were standing on two adjacent vertices of a rhombus, ABCD-- the cat on A and the mouse on B. The mouse's safe house is located at O, the intersection point of the rhombus' diagonals. A fence surrounds the rhombus along its sides with only three holes through which the mouse can get to his safe house: one hole at B, one hole at E, which is the midpoint of the route BC, and one hole at C. The moment the cat and the mouse notice each other, the cat starts running toward B along route AB, and the mouse towards C along route BC. Had the mouse chosen route BO, he would have reached his safe house when the cat is halfway between A and B. Out of excitement, the mouse misses the hole at E, and continues instead towards C. Had he taken route EO he would have reached his safe house at the same time the cat reaches B. The mouse turns at C and continues running along route CO. Since the cat cannot see the mouse, she enters the hole at B, and runs along the shortest way towards route CO. The moment she reaches route CO, the mouse is six yards away from his safe house. Upon reaching route CO, the cat realizes that with her current speed, she is not going to reach the mouse, and so she increases her speed by 1 minute per yard. The mouse, however, reaches his safe house

15 seconds before the cat does. How far did the cat and mouse run? What was the speed of each?

Background and Significant Events

After beginning the problem on camera in a small group during the afternoon session, Maggie returned the following morning, worked with another group off camera for approximately 45 minutes, and then presented her solution to the class on camera. Because video data was available for both Maggie's small group work and her class presentation, both data sets were reviewed. No contradictions were found. While proof schemes are defined in terms one's ascertainment and persuasion, there were no signs of significant differences between the two. Therefore, Maggie's class presentation was selected as the primary source of analysis because it provided richer, more coherent data. During her presentation, questions were raised by class members and TR that were not raised in her small group, contributing to the depth of evidence used to characterize her sources of conviction.

While presentations may represent a presenter's final solution to a problem, as in other analyses (e.g. analysis of the Train Station Problem), they need not. In fact, class presentations at the PD often represented a participant's understanding of the problem at a given point in time rather than final conclusions about the problem. With these comments in mind it is important to position the following analysis in the context of statements Maggie made regarding the tentative nature of her findings. Maggie acknowledged a sense of discomfort with her solution. According to Maggie, her discomfort was due – at least in part – to her lack of content knowledge of Geometry.

Indeed, there are several instances in which Maggie demonstrated a lack of familiarity with terms, definitions, theorems and other institutionalized ways of understanding rhombuses and isosceles triangles. Nevertheless, even with limited recollection of institutionalized geometric ways of understanding Maggie was able to generate conjectures and attempt to support assertions about the figure in question.

In light of the aforementioned comments regarding Maggie's geometric $W_s o U$ and $W_s o T$, it is also important to note that during a discussion with TR and in comments made to the entire class she explained that had he not encouraged her to state conjectures and move on, she did not believe she would have arrived at a conclusion. Consequently, she explained that even as she presented her solution, she was aware that it still needed more work.

Finally, there are a few miscellaneous pieces of background information to note as well. In contrast to the events of the train station problem, no single member of the group played a dominant role in the group dynamics. Also, this problem was given to students during the fourth week of the first summer.

Synopsis of Maggie's Solution to the Cat and Mouse Problem

Maggie began by drawing a rhombus with its diagonals (as shown below). She continued by claiming that the diagonals of a rhombus are perpendicular bisectors of each other. She made it clear that she would not be comfortable trying to prove the aforementioned claim. However, there were no indications of doubt on Maggie's part regarding the statement. Indeed, the fact had been proven in her small group³¹.

³¹ When working in a small group Maggie addressed this conjecture, noticing that triangle BCD is isosceles within the rhombus. Then she claimed that segment OC is perpendicular to BD because O is the midpoint

Assuming that the diagonals of a rhombus are perpendicular bisectors of each other, Maggie labeled the diagram, defined the following variables, and wrote an algebraic expression relating her variables as shown below.

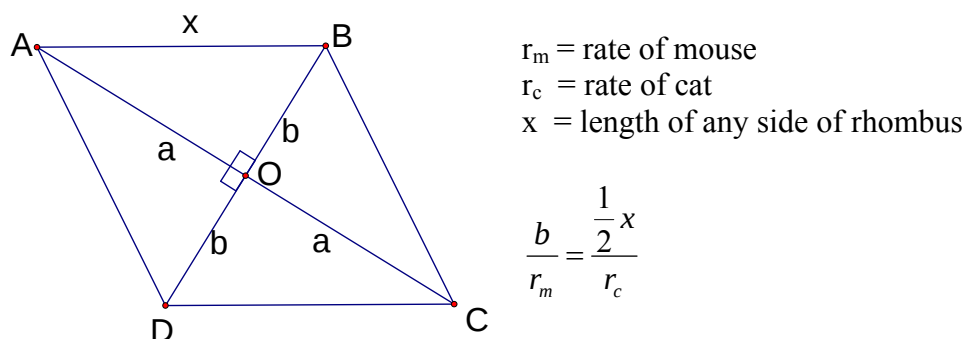


Figure 4.2: Maggie's diagram for the Cat and Mouse Problem

She explained that $\frac{b}{r_m}$ represented the time it took the mouse to run from B to O at a rate

of r_m . Maggie also explained that $\frac{\frac{1}{2}x}{r_c}$ represented the time it took the cat to run $\frac{1}{2}x$ at

his rate, r_c . Finally, she recognized that according to the problem statement these two quantities must be equal because they both represented the same amount of time.

During her presentation Maggie coordinated the positions of the cat and mouse at different points in time – tracing their respective paths out with her hands. She then created a system of equations she could use to find x , and used the value of x to answer the questions raised in the problem statement.

of BD. However, the premise – point O is the midpoint of segment BD had not been proven. She presented a similar proof to TR. In contrast, TR proved the fact for the group in a fully deductive manner.

During her presentation, Maggie derived information about the structure by drawing the various paths the cat and mouse could have taken to get to the safe house according to the problem statement (see figure below). The class made various requests for justification of multiple claims made by Maggie regarding properties of the rhombus. In some cases Maggie attempted to prove her claims. In other cases TR wrote these statements on the board as conjectures to be proven at a later time, depending on whether or not Maggie was able to prove them.

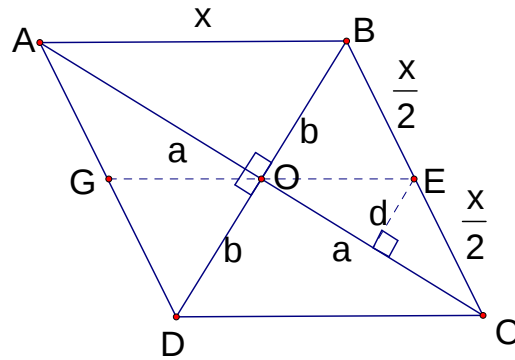


Figure 4.3: Maggie's second diagram for the Cat and Mouse Problem (more detail)

Whether her claims were provable or not, Maggie assumed their validity and continued representing them algebraically. She numbered the equations in her system of equations from 1 to 5 as follows:

$$1. \quad \frac{b}{r_m} = \frac{\frac{1}{2}x}{r_c}$$

$$2. \quad \frac{x}{r_c} = \frac{x}{r_m}$$

$$3. \quad a^2 + \left(\frac{x}{2}\right)^2 = x^2$$

$$4. \quad d^2 + \left(\frac{\sqrt{3}}{4}x\right)^2 = \left(\frac{x}{2}\right)^2$$

$$5. \quad \frac{7}{4}x = x + \frac{\sqrt{3}}{2}x - 6$$

Figure 4.8 – Maggie's equations for the Cat and Mouse Problem

Maggie explained that equation 1 was valid directly from the problem statement. Next, she explained that that had the mouse began at point B, ran to point E, and ended at point

O, it would have ran a distance of x since EO was $\frac{x}{2}$. Maggie had difficulty defending the claim that EO was $\frac{x}{2}$. Nevertheless, she continued. From equations 2 she ascertained that the cat and mouse ran the same rate until the cat increased his speed. This allowed her to support the claim that $b = \frac{x}{2}$. Concluding that $b = \frac{x}{2}$, Maggie wrote equation 3. She solved for a in terms of x in equation 3, wrote equation 4, and solved for d in terms of x . Finally, Maggie returned to the problem statement, tracking the path of the cat and mouse at different points in time, and wrote equation 5 as an expression relating the total distances run by the cat and mouse.

Of all her claims, one particular unproven claim seemed extremely important to her argument, $EO = \frac{x}{2}$. Maggie's proof of this claim will be examined in the analysis section.

Claims

Maggie's solution made use of claims about the structure of the rhombus to support the validity of algebraic expressions used. One claim, $EO = \frac{x}{2}$, stood out from others for several reasons.

First, it played a pivotal role in her solution as it was the basis of equation 2, which helped establish that the cat and mouse ran at the same rate for most of their trip. In small group data and during Maggie's presentation, participants were aware that an examination of the relative rates of the cat and mouse must be a crucial component of any solution to this problem. Second, Maggie's attempted proof of this claim foregrounds the

role of certainty within the context of the proof schemes construct, offering insight into potential mechanisms for change in her proof schemes (i.e. – the establishment of socio-mathematical norms (in the sense of Cobb and Yackel, 1996) and the role of the instructor in the renegotiation of the didactical contract Bruousseau (1997)). Third, Maggie’s attempt to prove the claim revealed information about her expectations for what constitutes an acceptable support for claims, and consequently, provides evidence regarding a potential change in her proof schemes.

The analysis below highlights an important characteristic of Maggie’s proving, her awareness of the status of claims (e.g. – definition, fact, conjecture, assumption, provable fact, unprovable fact, previously proven theorem, etc.). This analysis documents what appears to be a local change in $W_s o U$ indicating an increased awareness of the status of claims with respect to the aforementioned characteristic of the proving act.

This data also provides evidence of $W_s o U$ consistent with two different proofs schemes. An empirical proof scheme of the perceptual sort was evidenced when Maggie attempted to prove that $EO = \frac{x}{2}$. $W_s o U$ consistent with the referential-symbolic proof scheme were evidenced in the handling of the aforementioned system of equations 1 through 5 (see *Synopsis of Maggie’s Solution* section).

Analysis

Evidence of a Perceptual (Visual) Empirical Proof Scheme

Maggie began her presentation by drawing the rhombus described in the problem statement and stating that diagonals of a rhombus are perpendicular bisectors of each other. Because Maggie indicated that she could not prove this statement, TR wrote it as a

theorem to be proven on the side board. Maggie explained that the aforementioned property of rhombuses would now be treated as an assumption before continuing. Next, she declared three variables and wrote the meaning of each.

Maggie's diagram:

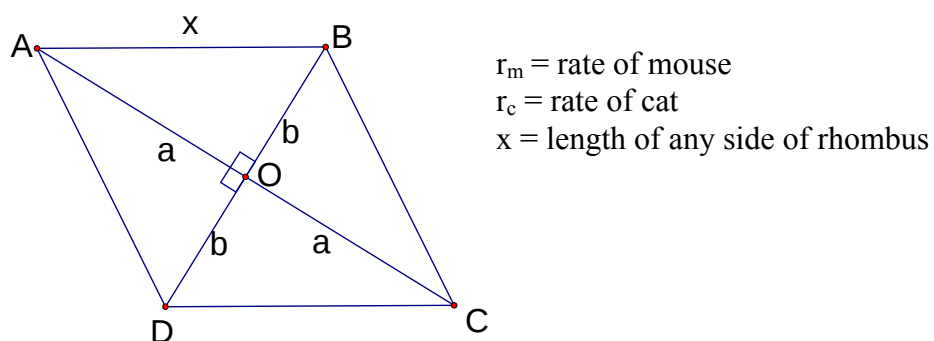


Figure 4.4: Cat and Mouse Problem without mid-segment

Maggie wrote $\frac{b}{r_m} = \frac{x/2}{r_c}$ explaining that it was the distance over the rate. The goal was to

get a system of equations.

Maggie drew the following diagram next:

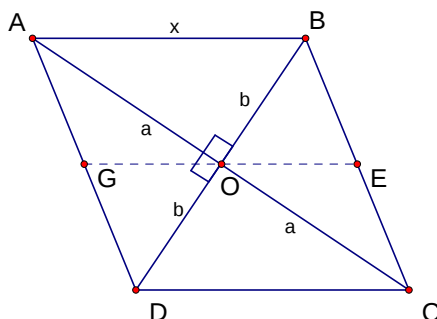


Figure 4.5: Cat and Mouse Problem with mid-segment

Scene A: How did Maggie determine that segment GE must contain point O when points G and E are midpoints of opposite sides of a rhombus?

Maggie claimed that segment GE must be congruent to segments DC and AB.

When questioned by TR and classmates, Maggie attempted to persuade TR and classmates that it was so. TR began by asking Maggie to clarify how she had constructed segment GE. Maggie explained her construction in a way that differed from the way she had originally constructed segment GE. Her argument follows:

TR: But how do you create GE? That's what I don't understand. What did you do to create GE? What are you doing exactly?

Maggie: I'm saying that this [Maggie motioned along segment GE starting at G moving toward E], this line here is...

TR: But how do you construct it? ... where are you beginning on AD? At what point?

Maggie: Well the midpoint [motioning along segment AD].

...

TR: All right. So G is the midpoint of AD? ... So and then the next thing you do?

...

Maggie: It says that E is a midpoint also... **So if I go from the midpoint on this line [pointing at G]... To the midpoint on that line that's parallel to it [pointing at E]...**

...

Mary: How did she know... how do you know it crosses at...(ina.)...

...

TR: You have control on G and E, right? ... **How do you know that it goes through the O?**

...

Maggie: I'm assuming that it does because these...the diagonals...

...

Maggie: **The diagonals are bisecting each other so it has to be at also [holding fingers horizontally in a manner indicating the length of segment GE]...**

unknown p3: At the center.

Maggie: **Yeah. In the center of that rhombus. It has to be at a midpoint.**

TR: Yeah, we just want to know why.

Maggie: So that's not why? [Maggie laughs]

Before any questions arose Maggie drew a line from point E to point O, representing the path the mouse took from point E to the safe house. She continued drawing the line so that it met segment AD at G, but did not label the point of intersection at first. She said that, "The distance between a pair of parallel lines is always the same," concluding that the distance from E to O must be $\frac{1}{2}x$. TR interrupted to ask the class if they had any questions.

TR asked for clarification about how segment GE had been constructed. Maggie's response to TR differed from how she had created segment GE on the board. The second time, she began at point G – the midpoint of segment AD – instead of point E. This is an indication that Maggie considered the following two propositions equivalent: the segment containing the midpoints of opposite sides of a rhombus contains its "center", and the line containing one midpoint of a rhombus and the "center" of the rhombus must contain the midpoints of both sides. This distinction was not made at the time in the class.

One class member noticed that Maggie's construction passed through point O and asked how she knew that segment connecting the midpoints of opposite sides of a rhombus must pass through "center" of the rhombus. The discussion then moved to explaining why GE must pass through point O.

Maggie explained that the diagonals pass through point O and bisect each other. She considered this point the center of the rhombus and said that it has to be at a midpoint. When asked why, Maggie said, “So, that’s not why?” Maggie made it clear that for her the point of intersection of the diagonals was the “center” and that the “center” was the midpoint of segment GE. Furthermore, her original construction of segment GE depended on O. Considering her original construction and her question, “So, that’s not why,” indicates that Maggie believed she had produced a persuasive argument and that this argument was based in the diagram itself.

Scene B: How did Maggie determine that segment EO is one-half the length of segment AB?

Shortly after Scene A concluded, a member of the class returned to Maggie’s claim that segment EO is one-half the length of segment AB. TR rephrased the question, making an additional point.

TR: So that's, that's question [that] has not been answered. How do you know that O is exactly in the middle?

Maggie: Haha [Maggie throws her head back and laughs].

TR: ...That theorem will tell us that, that GE will go through O...

Maggie: Right.

TR: But it can go through it in many different points.

Maggie: Right. I, well, I saw it that if this equals...if I'm saying that that equals x [motioning along segment GE], well then, if this [motioning along AB] is the same length, the midpoint of this [pointing at the midpoint of segment AB and motioning toward point O], O is at the midpoint. **Haha. Because it is.**

TR: Yeah there's something to do here, huh? All right. So let me put a little bit spin. Do you mind? Go ahead.

Maggie began laughing as she understood the question being posed. This was an acknowledgement that even accepting the conjecture from Scene A – that the segment connecting the two midpoints of opposite sides of a rhombus must pass through the point

of intersection of its diagonals – was not sufficient to justify the claim that point O must be the midpoint of segment GE. After attempting once again to defend her position Maggie laughed again, realizing that she could not. Rather, she was relying on the diagram to support her statement and could not offer any further evidence that it was so.

In both Scene A and Scene B, Maggie's claims and her argument did not differ substantially because her evidence was the diagram itself. For this reason these instances of Maggie's attempts at proving are characterized herein as visually perceptive, an instantiation of the empirical proof scheme.

Evidence of Referential Symbolic Proof Scheme

During her presentation, Maggie wrote and solved a system of equations as follows:

$$\begin{aligned}
 1. \quad & \frac{b}{r_m} = \frac{\frac{1}{2}x}{r_c} \\
 2. \quad & \frac{x}{r_c} = \frac{x}{r_m} \\
 3. \quad & a^2 + \left(\frac{x}{2}\right)^2 = x^2 \\
 4. \quad & d^2 + \left(\frac{\sqrt{3}}{4}x\right)^2 = \left(\frac{x}{2}\right)^2 \\
 5. \quad & \frac{7}{4}x = x + \frac{\sqrt{3}}{2}x - 6
 \end{aligned}$$

Figure 4.6 – Maggie's equations for the Cat and Mouse Problem

Scene C: Maggie coordinated

Maggie: ...I had to keep going back to the problem and finding out what my goal was is because ok, this is what...(inaudible)...explained to me. Um, when the cat

was here [motioning along segment AB from point A to the midpoint of segment AB with left hand], the mouse had gotten to this point [motioning along segment BC from point B to point E with right hand]. And the cat got to b [motioning from midpoint of AB to point B with left hand] and the mouse had gotten to c [motioning from point E to point C with right hand], um, and then they're running at the same rate. So by the time the cat get's to E [moving left hand to point E], the mouse has run along this line segment [motioning from C toward O], a, here and is, has to be one half of x distance whatever that is on this line [holding left hand at point E while motioning from B to E with right hand]. And then the cat at this point turns into E and instead of taking this path to the safe house. It takes the shortest path it could possibly take. I'm sure this is going to be another theorem. And I assume that that would be...this is an isosceles triangle [tracing out triangle EOC with both hands]. Right? These 2 sides, uh, sides [EO and OC] are equal. So I assume that would be just this line segment that creates the 90 degree angle.

...

Maggie: ...let's go through and find the cat's distance cause that one's kinda easier. The cat went x or actually, no I haven't done this yet, right? I need to find this distance right here [writing "d" along the perpendicular segment from point E to segment OC] from E to... To the line. Right? And I have another triangle right here and um, haha [turning to class and laughing]... And this line right here [motioning along the altitude of triangle EOC] divides this line in half [motioning along segment OC]. Ok? I'm assuming that also.

TR: Because you're saying it's an isosceles.

Maggie: Right.

TR: So what you're saying is that the altitude in an isosceles triangle is also a median? That what you're saying? Right? So this is [coming to the board]...

Maggie: I don't understand the altitude.

TR: Oh. Uh, this is, you said this is isosceles [tracing out triangle EOC].

Maggie: Yes.

TR: That's what your focus is. This is the altitude.

Maggie: Oh ok. Yes [nodding affirmatively].

TR: All right, but you want very much to say that this [motioning from O to the midpoint of OC] is equal to this [motioning from the midpoint of OC to C].

Maggie: Yes.

TR: In other words you are arguing that [in] an isosceles triangle, the altitude is also the median.

Maggie: Yes [nodding affirmatively].

TR: So do we have another theorem? [Maggie laughing mildly]

Maggie later explained that the purpose of coordinating the positions of the cat and mouse (as shown in scene C above) was to support equation 5. Her attention to positions of the cat and mouse at different points in time was for the purpose of

accomplishing the goal of supporting a relationship of equality between the distances traveled by the cat and the mouse. Furthermore, she explained that she anticipated solving equation 5 for the purpose of finding the length of each side of the rhombus. Her symbolic manipulation was driven by a goal, supported by her mental imagery of the problem's protagonists moving through space and time, and grounded in her understanding of the geometric structure on which the protagonists were moving. Thus, in scene C Maggie's persuasion is characterized as referential symbolic.

A change in awareness of the status of observations:

As mentioned in the section above, entitled Synopsis of Maggie's solution, Maggie was not entirely certain that her solution was valid in the sense that there were many unproven conjectures that her presentation was reliant upon. During a private conversation with TR and in front of the entire class, she professed that her solution was based on perception at many points. This reliance on perception formed the basis of her doubt.

Maggie's private conversation with TR took place after her presentation.

TR: Yeah. You did wonderful. Everything is nicely written.

Maggie: But **everything is based on things that I feel are right, but I didn't know.**

TR: Yeah, we are going to prove all of them.

Maggie: So, **but I think if you didn't go over there and say just keep doing the problem, I would have stayed...**

TR: **You would have stopped.**

Maggie: **At each one and said, well, I don't know if this is true for sure. I can't prove it so...**

TR: Right. Right. It is important sometimes to pause...

At a later point in time, Maggie reported to the class that she was uncomfortable about her solution...

TR: As Maggie indicated, maybe you can say this again what you told me...what

happened with this problem with respect to how you got stuck. Go ahead if you can...

Maggie: ... I forgot a lot of geometry. I'm like, I can't go any further because I can't assume that that's true and he came over and he said, well you know what, just assume it's true and put a big question mark there and then continue with your problem and if I didn't do that, I probably would have been somewhere up there still. So ...but **I feel like it's all based on my assumption so how can I say this is the answer?**

TR: Right. This, this all tentative. All that you did here, with all due respect, is still tentative.

Maggie was aware that she was unable to prove some of her conjectures, but seemingly unaware that one could make assumptions, establish their status as unproven, and return to them at a later time. Still, it is an important development in her proving that she learned to identify conjectures and continue. However, even with her growing awareness that she could state an observation, label it as a conjecture to be proven, and move on, her presentation shows that there were still times when she was overly reliant on visual sources of evidence. Scene A (above) shows one such instance.

In scene C Maggie clarified the status of her observations as assumptions in three instances. In the first case she explained, "I assume that that would be...this is an isosceles triangle [tracing out triangle EOC with both hands]." In the second case she mentioned that the shortest distance from point E to segment OC would be found along the perpendicular segment saying, "I assume that would be just this line segment that creates the 90 degree angle." In the third case she realized that she was assuming a property of isosceles triangles, the altitude of from the vertex angle to the base of an isosceles triangle is also a median.

The three aforementioned claims demonstrated an increasing awareness of the need to assign a status to a claim, clarifying whether a claim is being assumed without

proof (versus being a provable claim). Claiming an observation was an assumption also indicated at least a moderate level of certainty in the observation. Whether this need was social or intellectual is of paramount importance from a DNR perspective.

Scene C can be viewed from two different perspectives. In the language of Bruousseau (1997), TR was attempting to renegotiate an element of the didactical contract by clarifying that Maggie was citing a property of isosceles triangles and noting that this property needed to be proven, saying, “So do we have another theorem? [Maggie laughing mildly]”. In other words, the instructor was trying to show Maggie that when she cites a property of geometric figures she should be prepared to defend her position using previously known facts or clarify that the property is assumed, but needs to be proven later. This is a common thread throughout scenes A, B, and C. Though this analysis does not focus on TR, it is important to note his goals for Maggie as they may contribute to a change in her proof schemes if she sees the intellectual need for the mathematics he is attempting to teach her.

From a social perspective (Cobb and Yackel, 1996), Maggie’s repeated laughter can be taken as a sign that she was aware of a violation of a socio-mathematical norm. While it can be very difficult to prove the existence of norms, their presence is most often indicated by the reactions of community members when individuals realize they are in violation of one. In this case, the socio-mathematical norm in question relates to standards of proof for the PD classroom community.

As in scene A, class members had already established a pattern of asking Maggie why a given claim about the geometric figure is valid. That is, they could ask her for proof of any statement she made in her argument. Maggie began to anticipate their

questions. Her laughter at the end of scene C indicated her growing awareness that claiming an observation was an assumption meant that she would not be expected to prove it on the spot. However, it was an expectation that she would do so.

What is significant about these observations is that they highlight the relationship between three variables: Maggie's participation in classroom socio-mathematical norms related to proving, the instructor's attempt to renegotiate the didactical contract, and the potential for refinement of Maggie's proof schemes. The necessity principle in DNR-based instruction states that in order for an individual to learn mathematics she must see the intellectual need for it. The duality principle stipulates that $W_s o U$ and $W_s o T$ are mutually influential. Returning for a moment to the triad of proving, proofs, and proof schemes, one aspect of deductive proof schemes is demonstrating understanding that the status of observations must always be clear. According to the necessity principle and assuming for a moment that Maggie's proofs are not generally characterized as deductive, in order for Maggie to make a transition to primarily deductive proof schemes she must show signs of understanding the reason for this standard of proof. The duality principle stipulates that demonstrating a particular proving behavior influences her proof schemes, just as proof schemes influence her proving behavior. However, in scenes A, B, and C, we see that Maggie's growing awareness of, and participation in, a set of classroom socio-mathematical norms as mediated, at least in part, by the instructor's attempts to renegotiate the didactical contract contributed to a change in $W_s o U$, an indicator of change in $W_s o T$.

The necessity principle does not exclude the possibility that individuals learn to participate productively in proving activities before they see the reasons for expected

patterns of behavior. Indeed, as social learning theorists claim, Maggie's productive participation in the activity of proving is observable evidence of learning. From a DNR perspective, evidence that the necessity principle is being implemented is observable via the success of perturbations to introduce a series of equilibrium and disequilibrium phases within the individual. To substantiate claims that the necessity principle has been implemented for Maggie, evidence must be found supporting the claim that Maggie understands why members of the classroom community expect her to clarify the status of observations. While her laughter and changing patterns of participation in proving activities are indicators of perturbation and disequilibrium, it remains to be seen whether or not she understands and embraces the reason for the aforementioned classroom expectation beyond simply seeing it as a means for proceeding with her argument when she does not know how to prove what she considers a factual, but unprovable statement.

Synopsis

The Cat and Mouse Problem represented a situation in which Maggie's mathematical knowledge was demonstrated in the context of several key variables. The problem situation presented her with mathematical content (i.e. – geometry) she was uncomfortable with, as demonstrated in her self-reports and her proving acts. A defining feature of this episode was a learned problem-solving strategy relating to proving. Namely, Maggie learned that when solving a problem or even attempting to prove a result it can be productive to make plausible assumptions, set them aside for a time before proving them later, and continuing with what seems right for her at the time. Though she explained that this strategy left her with a sense of insecurity about her overall solution,

she realized that without it she would not have arrived at a conclusion, no matter how tentative.

No doubt, the geometric nature of the problem combined , important problem-solving strategy freed Maggie to convey her thinking about the problem, thinking that ranged from stating conjectures, making assumptions, and attempting to prove small results along the way. As Maggie presented she began to participate in a classroom socio-mathematical norm wherein she clarified what she felt she could prove and what she would take as an assumption. In one particular instance (see scene A above), when Maggie felt she could prove a particular result her proving acts could be characterized as reflective of an empirical proof scheme. In another instance (see scene C), Maggie's symbolic manipulation was consistent with a deductive proof scheme (referential symbolic).

From a DNR perspective, changes in proof schemes are claims about learning. According to the necessity principle, learning is conditioned by the intellectual need of the individual. From a more social perspective on learning, it was clear that Maggie's patterns of participation were changing as she interacted with the instructor and her classmates. Certain behaviors (e.g., the repeated laughter and increasingly classifying observations as assumptions when they were not deductively provable for Maggie) demonstrated changes in proving which were indicative of changes in her proof schemes according to the duality principle. What remains to be demonstrated, with respect to supporting the claim that Maggie's proving became more deductive, is whether or not Maggie understood the reasons her classmates and TR did not consider Maggie's sources of evidence convincing (referring specifically to the end of scene A as a case in point).

Finally, this episode as a whole demonstrates that Maggie has been perturbed about what constitutes an acceptable proof for the PD classroom community. She seems to have entered a state of disequilibrium and demonstrated some changes. It remains to be seen whether or not she has internalized the needs for communication and clarification in proving.

Episode 4 - The Quarterback Problem

Question

The Quarterback Problem (20031018pm)

A football team finds itself on its own forty yard line, in possession of the ball, with five minutes left in the game. The score is three to zero in favor of the opposing team. The quarterback, Alfred, knows the team should make three yards on each running play, but will use thirty seconds per play. He can make twenty yards on a successful pass play, which uses fifteen seconds. However, he usually completes only one pass out of three. Not only was Alfred an excellent quarterback but also a very good math student. Alfred chose a combination of plays that assured his team's victory. What was it?

Background and Significant Events:

Marcia came to the board to explain that a system of inequalities (and equalities) can be “optimized” by graphing and checking solutions along the boundaries of the “feasibility region”, using a strategy of shading and observing overlapping regions. She claimed that the point (7,6) was an optimal solution in some sense, but could not explain in what sense. TR asked the class to state what they understood about her solution and what questions they might have about it.

Marcia’s solution entailed writing and solving the following system:

$$p = 20 - 2r$$

$$p \geq 9 - \frac{9r}{20}$$

Maggie's was called upon to interpret the sense in which (7,6) could be considered optimal or to refute Marcia's loosely made claim. Later, Maggie was asked to share her group's results and to persuade her classmates about the validity of her observations.

Therefore, this episode constitutes an instance in which the products of two mental acts, interpreting and proving, were demonstrated simultaneously. The focus of this analysis will be on Maggie's persuasive efforts rather than her interpretation of Marcia's claim.

Claims:

Maggie's presentation constitutes evidence of a referential symbolic proof scheme because of her coherent interpretation of the variables in terms of the task she was given, trying to interpret the optimal nature of the solution (7,6).

Synopsis of Maggie's Way of Understanding Marilyn's Solution:

Maggie began by explaining that all points with integral coordinates between (0,20) and (7,6), satisfying $p = 20 - 2r$, meet the constraints of the problem. She began writing and checking combinations of running and passing plays by making a list, using the equation, $p = 20 - 2r$. When Maggie checked run/pass combinations she did so by confirming that each point satisfied both time and yardage constraints. Focusing on the point (4,12) as an example, Maggie explained that the reason (4,12), and other points along $p = 20 - 2r$ met the time constraints was that the equation Marcia had written, $5 = \frac{1}{2}r + \frac{1}{4}p$, incorporated the temporal aspect of the problem statement. However, Maggie noticed that when the team ran four running plays and twelve passing plays, they had

accumulated more yards than the game would allow for³²³³. She explained that the team would have had to turn the ball over after moving the ball more than 65 yards – assuming the end zone is 5 yards long.

Maggie interpreted the meaning of the equations in terms of the problem statement and considered the goal of understanding the sense in which (7,6) was optimal. In order to do so, Maggie considered the time and the number of yards earned in terms of the number of running and passing plays. She concluded that (7,6) was optimal because it produced the fewest yards over 60. According to Maggie's argument, this was valuable because a team could not continue making more plays once they had crossed the end zone – a constraint brought in by the reality of the problem situation.

Episode 5 - Stair-Like Structure Problem

Introduction

The stair-like structure problem, stated in its entirety below, consists of six inter-related questions that occupied three days worth of time at the summer institute. Due to the amount of data generated documenting Maggie's conjecturing, proving, and commenting on other participants' solutions, analysis of the stair-like structure problem will be presented in two parts. Part one focuses on problem #3, while part two of this analysis will focus on problems #5 and 6. A synopsis connecting part one and part two follows part two.

Stair-Like Structure Problem (Part One)

³² At 3 yards per running play and 20 yards every 3 pass plays, the team would have moved the ball 92 yards.

³³ Maggie's group computed 252 yards the combination of 4 running and 12 passing plays. It is most likely that they gave the team 20 yards per passing play by mistake, rather than 20 yards for every 3 passing plays.

Question

A figure such as the one below is called a *stair-like structure*.

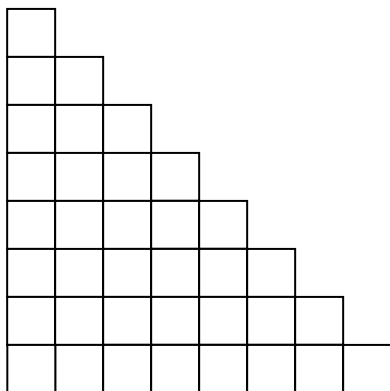


Figure 4.7: A Stair-like Structure

1. You have 1176 identical square pieces. Can you use all the pieces to construct a stair-like structure?
2. You have constructed a stair-like structure with a base of 98 pieces (squares).
How many squares does your structure have?
3. You want to build a stair-like structure out of toothpicks. Is it possible to use a total of 2628 identical toothpicks to create a stair-like structure?
4. The figure above has eight steps. You want to use between 2345 and 8789 toothpicks to build a stair-like structure. What is the minimum number of steps your structure can have? What is the maximum number of steps your structure can have?
5. Make up a new problem on “stair-like structure.”

6. In a stair-like structure the number of squares staring from the top is 1,2,3,... This sequence is an example of an arithmetic progression or of an arithmetic progression.

Other examples of arithmetic progressions are

a. 1, 6, 11, 16, ...

b. 2, -5, -12, -19, ...

c. 8, 6.5, 5, 3.5, ...

In the first arithmetic progression, find the number in the 88th place. In the second, find the number in the 100th place and in the third, find the number in the 73rd place.

Is it possible that the first two terms of an arithmetic progression are integers and that all subsequent terms are not?

Background and Significant Events

The events of this episode (part 1) transpired across three class sessions, in three different settings, and covered roughly 5 hours. For problem #1, there were five solutions presented to the entire class (for a description of publically presented solutions see TR, Manaster, Fuller, & Soto in press). Proofs demonstrated ranged from empirical (RPG guided) to deductive (PPG guided). However, all presenters concluded that the number of squares in a stair-like structure can be expressed as $\frac{n(n+1)}{2}$, where n is the number of rows in the structure.

Maggie was not present for several of the presentations for problem #1 which took place in the morning of the first day in the second year. However, she was present for the final presentation of problem #1. TR said that he would sit with her at lunch to help familiarize

her with these presentations. Problem #2 was given as a homework problem and the class went immediately to problem #3.

In Scene A Maggie worked in a small group to generate a solution to problem #3. This solution was created without the help of other group mates but was shared with others. Maggie listened to the solutions of at least two other group members and explained that she would rethink her solution on the basis of what she heard.

Scene B took place in a public setting (whole class discussion) after the first part of William's presentation. Maggie commented on the difference between a conjecture and a proof, a comment that connected closely with her own experience solving the problem. She also reflected on her own sources of uncertainty regarding proof in general.

After Maggie created her own solution and shared it with a facilitator and a fellow group member, she had a chance to listen to the approaches taken by her fellow group members. That night she was assigned problem #2. The following morning, Maggie and her classmates took 30 minutes to gather their thoughts and discuss in small groups before presentations began. Maggie listened to three presentations before presenting hers after the lunch break. It is not surprising that there were some changes in her solution. By this point, she also became more self-aware regarding which part of her solution was the source of her uncertainty.

In Scene C, Maggie presented her solution to the problem in a whole class setting. This solution differed from her solution created in the small group setting. Maggie's private solution to problem #3 incorporated the proven fact – from problem #1 – that the number of squares in an n -level stair-like structure is given by the expression $\frac{n(n+1)}{2}$.

Her public solution made use of a similar, but more general fact the sum of a finite arithmetic series can be found using the function $S_n = \frac{n(a_1 + a_n)}{2}$, where n is the number of terms in the sequence, a_1 is the first term in the sequence, a_n is the last term in the sequence, and S_n is the sum of the arithmetic series. However, this more general formula was introduced by a fellow participant. It had had not been proven.

This dissertation foregrounds Maggie's proof schemes in the context of a particular instructional intervention. However, in this episode the interplay between the instructor's teaching practices and Maggie's level of certainty is of particular importance as she attributes her disequilibrium to these teaching practices. For this reason, a brief description of the TR's teaching practices during the stair-like structure problems is included in the next paragraph.

Harel, Manaster, Fuller, and Soto (in press) investigated TR's teaching practices as seen during class presentations. The authors noted the significance of teaching practices that emphasized a need for causality in addition to the need for certainty as teaching practices with potential for encouraging a transition in students' proof schemes from the Result Pattern Generalization (RPG) to Processes Pattern Generalization (PPG) proof schemes. TR attempted to attend to the tension between emphasis on the importance of the use of inductive problem-solving approaches for deriving conjectures, such as tables, and the use of deductive reasoning to prove conjectures whenever students derive conjectures by empirical means. In so doing, he pointed out the role that empirical problem-solving approaches can play in the proving act, namely that conjecturing can be an important forerunner to proving.

The aforementioned point was made during presentations of problem #1 and reiterated during the presentations of problem #3. It was also emphasized that there are instances in which conjecturing and proving can be indistinguishable. However, as he commented on participants' solutions TR clarified that these two mathematical activities are not necessarily³⁴ the same and their products do not have the same status³⁵ in mathematics.

Synopsis of Maggie's Solutions

Scene A: Maggie's small group solution

By the time the camera focused on her, Maggie had already drawn a 4 level stair-like structure, marked certain segments off camera (as indicated in figure 1), and written the expression, $\frac{n(n+1)}{2} + n$.

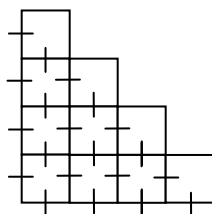


Figure 4.8: Markings on stair-like structure

She declared the variable x to represent the number of levels in a particular stair-like structure and y to represent the total number of toothpicks in an x -level structure. Then

she changed her expression, $\frac{n(n+1)}{2} + n$, to $2\left(\frac{n(n+1)}{2}\right) + 2n$ and created the following

two-column table:

³⁴ TR also pointed out that in some cases individuals can bypass conjecturing, moving directly to proving.

³⁵ The term, 'status', refers to the standards for determining when a participant's work constitutes a complete proof at the PD.

Table 4.1: Maggie's table for stair-like structures in problem #3

x	y
1	4
2	10
3	18
4	28

In two instances Maggie repeated the same solution. When a facilitator came to ask Maggie about her work, she explained that she had plugged values of x (or n) into her expression and counted segments by hand to confirm that the results matched for the first four stair-like structures. Her first solution consisted of counting 2 toothpicks for every square. Maggie explained that according to the solutions presented to problem #1, $\frac{n(n+1)}{2}$ is the number of squares in an n -level stair-like structure. She also explained that she added x (or n) more horizontal segments and x (or n) more vertical segments to account for the remaining toothpicks needed to complete the structure.

Though her diagram is consistent with focusing on the lower left corner of each square and adding $2x$ (or $2n$) more horizontal segments and vertical segments along the jagged edge of the figure, in her explanations Maggie focused on the upper-right corner of each square, leaving the $2x$ (or $2n$) segments along the extreme left and bottom of the stair-like structure to be counted last. Both solution strategies are similar. This explains the intended meaning of both expressions, $\frac{n(n+1)}{2} + n$ and $2 \cdot \left(\frac{n(n+1)}{2} \right) + 2n$, as well as

why she changed her original expression from $\frac{n(n+1)}{2} + n$ to $2 \cdot \left(\frac{n(n+1)}{2} \right) + 2n$. Next,

$$2 \cdot \left(\frac{n(n+1)}{2} \right) + 2n = n(n+1) + 2n$$

she simplified her expression:

$$= n^2 + n + 2n .$$

$$= n^2 + 3n$$

Finally, she wrote and solve the equation, $2628 = n^2 + 3n$, using the quadratic equation.

She explained to a facilitator that she checked her formula using the table and it “seemed to match”. Finally, she noted that because her solution for n was not a whole number, there could not be such a stair-like structure.

In a conversation with a group mate, Bill, Maggie expressed uncertainty about whether or not she had a proof. Bill explained to Maggie that she had created a proof and why. Maggie explained that finding the sum of a finite arithmetic sequence was the source of her uneasiness, indicating that Maggie was still uncomfortable with the result from problem #1.

Scene B: Maggie’s comment on William’s solution

The first whole-class presentation to problem #3 for the stair-like structure sequence was given by a participant named William. William’s solution³⁶ was done in two different ways. In this scene Maggie commented on both parts of his solution. In the first part, a table was used and a pattern was observed and generalized in a manner consistent with the RPG. The formula generated, $n^2 + 3n$, was then used to answer problem #3. TR mentioned that the manner in which the pattern was generalized did not

³⁶ A more detailed description and analysis of William’s solution is provided in Harel, Manaster, Fuller, and Soto (in press).

guarantee a correct solution and asked Maggie if she could explain why. Maggie's response is analyzed below.

After William returned to give the second half of his presentation in which he derived the same result in a deductive manner, Maggie commented publically on the difference between William's two problem-solving approaches. Though this is not an instance of Maggie's proving, it does give insight into what she considers convincing. Because Scene B also contains statements pertaining to Maggie's beliefs about teaching, Scene B will also be revisited briefly in the chapter concerning Maggie's teaching practices.

Scene C: Maggie's Public Solution

Maggie presented a short solution to the class that differed from her privately created solution in that her method of counting had changed. During her public solution, she explained that she looked at horizontal lines and vertical lines. She began by drawing the horizontal segments only for the first 4 stair-like structures and counting the number of horizontal segments on each level (see figure 2).

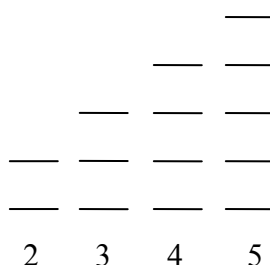


Figure 4.9: Maggie's construction of the horizontal segments in a 4-level stair-like structure.

Next, she drew and counted the vertical segments, noting the symmetry that allowed her to double the total of either the vertical or horizontal segments in order to find the total for the entire structure (see figure 3 below).

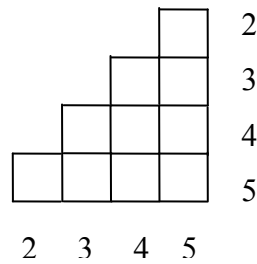


Figure 4.10: Maggie's construction of the vertical segments in a 4-level stair-like structure.

Maggie explained that the number of toothpicks can be found by computing the sum:

$2[2 + 3 + \dots + (n+1)]$ using the formula for computing the sum of a finite arithmetic

series, $S_n = \frac{n(a_1 + a_n)}{2}$, where n is the number of terms in the sequence, a_1 is the first

term in the sequence, a_n is the last term in the sequence, S_n is the sum of the arithmetic

series, and $2S_n$ is the total number of toothpicks needed. She substituted and solved as

follows:

$$2S_n = 2\left(\frac{n(a_1 + a_n)}{2}\right)$$

$$2S_n = (n)(2 + (n+1))$$

$$2S_n = (n)(n+3)$$

$$2S_n = n^2 + 3n$$

TR asked Maggie what was missing in the proof. Maggie pointed out that the formula,

$S_n = \frac{n(a_1 + a_n)}{2}$ was unproven and represented a “major link” in her solution.

Claims

In the process of solving the stair-like structure problem, sharing her results with group mates, watching the solutions of TR and other students, and presenting her own solution Maggie became increasingly aware of her discomfort with the notion of what constitutes a proof. In this episode Maggie also explained the sources of her disequilibrium. By the end of the episode, Maggie had resolved her conflict temporarily. However, her presentation – while primarily compatible with the PPG proof scheme – demonstrates simultaneously her reliance on an unproven fact and a growing ability to pinpoint a linchpin in her solution. While she is able to do so first with the help of a group mate and later with the help of TR, she did not demonstrate the ability to isolate the pivotal unproven part of her solution spontaneously.

In a conversation with a group mate Maggie explored the difference between relying on multiple examples for proof versus using examples as confirmation. Furthermore, this episode also demonstrates instances when Maggie explicitly rejected proofs relying on RPG and behaved in a manner that can be characterized as PPG primarily.

Analysis

The DNR theoretical perspective stipulates that in order for students to learn mathematics, they must have an intellectual need for what a teacher intends to teach. Based on multiple observations of TR's teaching practice Harel, Manaster, Fuller, and Soto (in press) noted that one of TR's cognitive objectives for students was to help them develop and attend to the need for causality in proving. Therefore, a central concern of this analysis is whether or not TR was successful in implementing the necessity principle for Maggie in this case. Since the DNR perspective views learning as a series of

equilibrium and disequilibrium phases that can be instigated by situations established by the teacher, as well as other social interactions, it is important to view the forthcoming analysis with TR's goal in mind, watching for signs of disequilibrium (or equilibrium) in Maggie with respect to the need for causality.

Scene A: Maggie's small group work

When asked, Maggie explained her solution to a group mate, Bernie, as follows.

Bernie: What did you do?

Maggie: ... you say there's two toothpicks for each one.

Bernie: You count two for every one.

Maggie: Then you get that. And then in the end all get is n because it's the n th level. But you have two of those. Right? That's for these inside ones.

Bernie: So, then you just have to add...

Maggie: And then here [pointing to the vertical segments on the extreme left of the structure] there are how many? n . And here [pointing to the horizontal segments on the bottom of the structure] there are how many?

Bernie: But that tells you... this tells you have many squares. This does [pointing to the formula, $\frac{n(n+1)}{2}$].

Maggie: Yeah. Yeah.

Bernie: Okay. And then you double it because for every square,

Maggie: You have two.

Bernie: you have two that are on the... like on the North-East.

Maggie: Right. Right. Plus two times that number of squares.

Bernie: So then you're adding... you're adding...

Maggie: And then add... and then add the left-overs. There's n number and then another n number.

Bernie: Uh-huh so you add...

Maggie: So you add.

Maggie's solution represented a systematic way to count that relied only on her counting technique and the solution to problem #1 of the stair-like structure problems. She explained that she felt uneasy about declaring her solution a proof. Though she had used a valid system for counting and verified that her method worked in the first four cases, Maggie explained that she "[didn't] know what proves and what doesn't prove." She

attributed her uncertainty to past experiences³⁷ in which she believed she had proven something, but it turned out that she had not.

Bernie: And it works? The answer...

Maggie: Yeah it worked for the first 4.

Bernie: Yeah. No. I think it should work for more than just the first 4.

Maggie: It should. Well, now I'm afraid to say well that's it. That it's the formula.

'Cause it's like well, how are you going to prove it? Do you know what I mean?

Bernie: You proved it. I think.

Maggie: **I don't really know what proves and what doesn't prove. Sometimes I think I've proven** (inaud.)...

Bernie: It works for all those, but...

Maggie: Would it work for a bigger one?

Bernie: Yeah, because...

Maggie: It would.

Bernie: If you don't have to prove this. If you've already proven that that's the number of squares.

Maggie: Oh. Okay.

Bernie: Which you did. Or it's been proven anyway. For every square, you're going to have two toothpicks.

Maggie: Right.

Bernie: Plus, you know that which make up the upper-right in every case.

Maggie: Right.

Bernie: You're gonna have one [holding arm horizontally] for along the base and one for along the height [holding arm vertically].

Maggie: Right (inaud)

Bernie: Yeah. I don't think you need to prove it. I think that's kind of proven because you're working in... in...

Maggie: In terms of that...

Bernie: In terms of any number n.

Maggie: Right.

Bernie: You're just... you didn't do anything with the... with the...

Maggie: For this particular structure... (inaud.)

Bernie: Nah. You didn't do anything with the little t-chart numbers other than verify the... just to give you four test points. I guess.

Maggie: But, and then I was looking at this one and I'm trying to see...

Bernie: Oh, what what I did. So now...

Maggie: what the difference this one...

Bernie: Yeah.

Maggie: this one is. You know. And then we can find out.

³⁷ The Cat and Mouse problem was one such experience.

Maggie's discussion with Bernie, her comments in scene B below, and her solution, show that she is not ultimately convinced by RPG-based solutions. Still, she is in disequilibrium about what constitutes an acceptable proof at the PD. In this case, Bernie pointed out the crucial link in her deductive chain. He noticed, and Maggie agreed, that her counting system would work for a stair-like structure of any size.

Maggie's proof rested on the fact that $\frac{n(n+1)}{2}$ represents the number of squares in an n-leveled stair-like structure. This fact was proven when Maggie was out of the room. She may not have had sufficient time to internalize that proof. While Maggie's solution was essentially deductive, at this point in time, she had difficulty identifying the part of the solution she was uncomfortable with.

Scene B: Maggie's comment on William's solution

When William presented his the first part of his solution, he used result pattern generalization (RPG) to justify his formula. TR explained that William did not know that his formula was correct and asked Maggie why not. Maggie responded by explaining that a generalization based solely on four observations serves only as a conjecture.

TR: So, just to summarize, um, what's happening here William obtained a possible formula, R_k equal $k^2 + 3k$. He doesn't know that this is a correct formula. Why he doesn't know? Maggie. Why don't we know that $R_k = 3k + k^2$ is the formula?

Maggie: Because it's a conjecture?

TR: Why? Why is it just a conjecture?

Maggie: Because it's based on his table and there are about 4 specific cases.

TR: Right. It is based on finite number of cases. It doesn't matter it's 4 or 4 hundred or 4 billion, right? ... it's a finite number? Right? ... Just to get a conjecture, but before he establishes the conjecture he went on and said okay. I know this is going to be true... But the important thing, what he is going to do now is to really have an understanding of why this is true. What makes this formula to be true. That's what we are going to do.

TR emphasized the difference between conjecturing that a result holds given a finite number of examples in which the result holds versus knowing what it is about the structure that causes the result to hold. To conclude his remarks later he said, “I am shifting the attention from conviction to being enlightened.”

After William returned to give the second half of his presentation in which he derived the same result in a manner consisted with the PPG proof scheme, Maggie commented publically on her thoughts regarding the differences between the two approaches.

Maggie: I thought of a couple of things while you were talking **about the difference between these two, um, derivations... I thought, well, if we aren't even motivated, I should speak for myself, to find, you know the proof for the first one where there is a conjecture, how are we going to get our students motivated to do that. Um, and I guess it starts with us first...**

TR: Right.

Maggie: ... secondly, **this has been my problem the whole time. I thought I knew algebra. I thought I could do it well, ... I can find patterns very easily, I can come up with formulas, but I don't truly know the difference between when I've derived something... and when I just have a conjecture. Sometimes I think I'm finished and you come along and you say, well what about this and I didn't think about it. I don't really know the difference between when I'm actually done and when I still have some work to do.**

...

just yesterday in our groups, I was still... I did a solution similar to what you did ... and I had to ask. You know, am I done yet? ... [Bernie] said, well yeah, because we proved already, ... how to get the sum of a sequence. So, because you use that... You know, you can say that you're done.

TR: If we proved it, we proved it. Once we proved it, it was established.

Maggie: **Right. But I wasn't really sure. I was like, so do I have more to do, you know or not?**

TR: But that uncertainty and that struggle is very natural. To me, if a person is in that struggle and feels like feels uncomfortable, feels um, perturbation, then that's a very good sign. Regarding the first point that you made; again, I want you to compare for yourself, your understanding of this relationship before we showed this approach and now. Just **compare it for yourself. Which one [do] you feel it's more superior in some sense than the other and why? I mean this is really a kind of a personal thing that you have to compare and say, well before I got to that stage and now I have something else. Did I gain anything?**

Maggie explained clearly that her concept of what it means to do mathematics was changing and she attributed the source of her perturbation to her experience at the PD.

She explained that for her the shift from conviction to enlightenment entails aspects of motivation to make a change in one's proving activity as well as knowing when "you're done". She continued by reporting that she had to rely on Bernie to determine whether or not she had produced a proof.

"...just yesterday in our groups, I was still... I did a solution similar to what you [referring to TR] did ... and I had to ask...am I done yet? ... [Bernie] said, well yeah, because we proved already, ... how to get the sum of a sequence. So, because you use that... You know, you can say that you're done.

TR: If we proved it, we proved it. Once we proved it, it was established.

Maggie: Right. But I wasn't really sure. I was like, so do I have more to do, you know or not?

Recall that Maggie's proof in scene A appeared to be deductive. However, here she reported that her ultimate conviction at the time rested on Bernie's opinion. Still, this does not necessarily point to an authoritative proof scheme on Maggie's part. It is also possible that Maggie was uncertain about whether or not she should go back and validate the formula $\frac{n(n+1)}{2}$. That is, she may simply have been asking how far back she must go to produce a convincing argument for TR. The main point to be taken from this scene is that Maggie has entered a state of disequilibrium about what constitutes proof. Furthermore, she has downplayed the convincing power of the empirical proof scheme (of the inductive sort).

Scene C:

Maggie presented her solution to her classmates as follows:

Maggie: ... I just looked at horizontal lines and vertical lines and I noticed that

the pattern was... There are 2 horizontal lines. When I go to the second level, there will be 3 and the third level will have 4 and then the forth level will have 5 horizontal lines... if you look at the vertical lines, 2 uh vertical lines... 2 on the first level, 3, 4, and 5. So really it's mirrored either way... it's a sequence. 2,3,4,5 all the way up ... if I go to the fifth level, there will be 6 lines. So basically, that last level will have $n + 1$ number of lines and then the sequence starts at 2. Cause you need 2 for the first level. So I just saw that and I said, well, if I were to write, um, the sum of the sequence, it would be... I didn't just know it like when Derrick [a previous presenter] did it. He said, oh, it was S of n equals n over 2 and he said a_1 plus a_n or a_n plus a_1 . So now that I'm a little bit more comfortable with this [pointing to the formula $S_n = \frac{n(a_1 + a_n)}{2}$], I can put this [pointing at the structure shown in figure 3 above] into this form here [pointing at $S_n = \frac{n(a_1 + a_n)}{2}$] and I can say that if my first term is 2 and my last is $n + 1$ [pointing at bottom of figure 3], then, um... S of n looks like... I'll still have n terms, n over 2, um and then it will be $(2 + n + 1)$... so S of n would be n over 2 times $(n + 3)$... Oh, I need to multiply this by 2 because I'm doing it with horizontal lines as well as vertical lines. So actually, this whole thing [changing $S_n = \frac{n(a_1 + a_n)}{2}$ to $S_n = \left[\frac{n(a_1 + a_n)}{2} \right]^2$] should be multiplied by 2. I kind of forgot that part. So this cancels out and you're left with n squared plus $3n$.

In both scene A and scene C, Maggie presented a solution that counts the number of toothpicks systematically. Though the counting techniques differed in the two scenes, Maggie's use of a system was at the heart of the deductive nature of both solutions. However, the difference between these two scenes is Maggie's source of uncertainty. In scene A, Maggie was unclear about what made her reluctant to declare her work a finished product on the basis of past experiences alone. Furthermore, she had used a table to verify her formula in four cases. It was Bernie who pointed at to her that her work was done for a generic stair-like structure and did not rely on the four instances she had verified.

In scene C, Maggie concludes her presentation without mentioning that anything is missing from her explanation. However, TR asked her what was missing from her solution. She hesitated for a moment before responding.

Maggie: Well, um, I guess this [pointing to $S_n = \frac{n(a_1 + a_n)}{2}$], if this hasn't been proven, the fact that you can use this here [smiling and pointing to $S_n = \frac{n(a_1 + a_n)}{2}$],

TR: Good.

ML: then that's you know,

TR: You have not proved that.

ML: a major link in the whole thing.

Maggie identified the weakest link in her deductive chain. That is, scene C differs from scene A in that Maggie was now aware of what she could not prove. According to Maggie's self-report and her conversation with Bernie, she was previously unaware if there was more to do or not. In scene C, though she did not offer the explanation spontaneously, Maggie was pleased (as evidenced by her smile) with her ability to explain what was still left to do and TR affirmed that the general result she was using had not been proven.

Synopsis

In this episode Maggie entered a state of disequilibrium with respect to what constitutes a proof or not. She explained that she was not sure when she was done with a proof and she attributed this uncertainty to repeated experiences of believing she had produced proofs when she had not³⁸. This form of disequilibrium was captured in the analysis of her interactions with class mates during her presentation of the Cat and Mouse problem (see year one data).

³⁸ See the TV Rating Problem and Cat and Mouse Problem for instances Maggie may have been referring to.

During the stair-like structure problem #3 presentations Maggie also commented on William's RPG proof, explaining that his formula was a generalization from four examples and that this type of generalization did not support the claim that the formula would hold in all instances. Maggie also generated four examples when she created her solution. However, as Bernie pointed out to her, her solution only used those four examples as a means of verifying a conclusion that was not based on specific instances entirely. Rather, Maggie's solution relied on a counting technique that was grounded in a process used to construct the stair-like structure. As such, this solution was primarily guided by the PPG proof scheme. Additionally, from Maggie's comment regarding William's proof in scene B, it appears that Bernie's comments in scene A were within Maggie's zone of proximal development because she was able to make use of it in another context, indicating that Maggie had internalized Bernie's comment.

Maggie was not convinced that she had a solution until Bernie told her so. Bernie pointed out a pivotal point in her solution but explained that it had already been proven. However, when she presented her solution to the class, Maggie presented a slightly different solution that made use of more general result. When TR pointed out that something was missing in her proof, Maggie was able to pinpoint the weak link in her argument. With the exception of proving that $S_n = \frac{n(a_1 + a_n)}{2}$, Maggie's public solution is consistent with PPG.

Stair-Like Structure Problem (Part 2)

As was the case in part 1, the events of this episode (part 2) also transpired across three class sessions, in three different settings, and covered roughly 5 hours. Part 2 of the

stair-like structure problem focuses on Maggie's creation of her own stair-like structure problem, her solution to the decontextualized question abstracted from that question, and her solution to problem #6a, b, and c. Problem #6 differs from problems #1 – 5 in that it offers no geometric structure, emphasizing a more abstract conceptualization of arithmetic sequences as functions with a constant difference whose domain is the natural numbers.

It is important to note that immediately prior to her presentation, Maggie discussed her solution with TR on camera in a small group setting. While TR helped Maggie with a few computational issues and discussed the question of whether or not she would get the same results in different cases, Maggie's public presentation was consistent with her private conversations with TR. The solution lasted for approximately 72 minutes and was presented entirely without the use of notes. During her public presentation, she reported all relevant issues that had arisen in private.

Synopsis of Maggie's Solution

Below is an abbreviated version of the contents for each scene in this episode. Following the scene summary is a more intricate synopsis of the contents of each scene. Scene D: Maggie worked with William in private to generate a solution to her problem. Scene E: Maggie and William search for a way to explain why William's conjecture is generalizable.

Scene F: Maggie presents her solution in a whole-class setting.

Scene G: Maggie applies her WoU arithmetic sequences to solve problem #6 a, b, and c.

Scene D: Maggie worked with William in private to generate a solution to her problem.

In question #5, participants were asked to make up a new problem on “stair-like structure.” In a small group setting Maggie asked, “If the builder is going to include a design on all odd numbered steps, how many square pieces with a design will be needed for any similar stair-like structure?” (see figure 4)

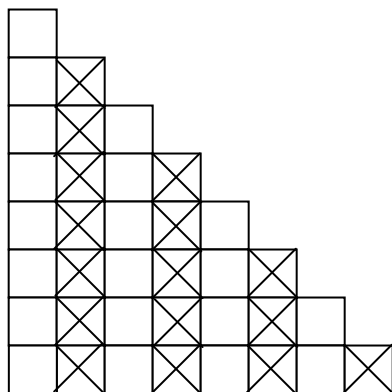


Figure 4.11: A stair-like structure with odd numbered steps shaded.

William, a group mate, interacted with Maggie in an attempt to solve her problem. Recall that William had previously presented a solution to problem #3 and Maggie had commented that a tabular solution generalized from four examples is incomplete.

Maggie’s problem-solving approach included the use of a table and an attempt to relate n to the number of tiles with designs. Noticing that the number of designed tiles necessary is always a perfect square, Maggie wrote a function that incorporated this fact (see figure 5).

Table 4.2: Maggie's table computing the total number of tiles in the odd columns of a stair-like structure

n	f(n)	# of tiles with designs
1	$(n-0)^2$	1
2	$(n-1)^2$	1
3	$(n-1)^2$	4
4	$(n-2)^2$	4
5	$(n-2)^2$	9
6	$(n-3)^2$	9
7	$(n-3)^2$	16
8	$(n-4)^2$	16
9		25
10		25

Using Maggie's table and expressions, William generalized from the examples as

$$\text{follows: } f(n) = \begin{cases} \left(n - \frac{n}{2}\right)^2 & \text{if } n \text{ is even} \\ \left(n - \frac{n-1}{2}\right)^2 & \text{if } n \text{ is odd} \end{cases} \quad \text{explaining that the function was created using}$$

only the examples in the table. To convince Maggie that this function modeled the intended phenomenon, William verified that $f(3) = 4$, $f(5) = 9$, and $f(7) = 16$. Maggie explained that she was convinced this function was correct, saying, "I think this is it. I'm happy with this."

Scene E: Maggie and William search for a way to explain why William's conjecture is generalizable.

Maggie and William moved beyond the table as a means of justification, each attempting their own problem-solving approach. Using the $n = 8$ case, William and Maggie manipulated the figure as shown below (see figure 5), duplicating and rotating it to form a rectangle. Together, both participants searched for a way to relate the size of the structure ($n = 8$) to the number of squares with designs using the diagram, but they could not think of a way to use the structure as a means of supporting their conjecture.

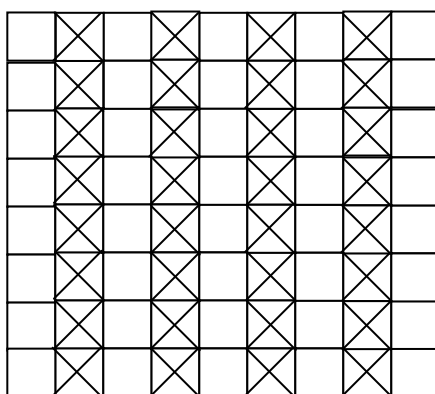


Figure 4.12: Maggie's first attempt to prove William's conjecture.

In their next attempt, Maggie and William listed the number of design squares in each column and attempted to pair the odd numbers in an effort to compute the number of design squares necessary at stage n (e.g. – In the $n = 8$ case, Maggie and William noticed that there were 16 design squares needed because $1+7 = 8$ and $3 + 5 = 8$). Maggie explained that if there was parity in the number of addends, the total sum could be found by multiplying the number of pairs of odd numbers in the set by the value of the sum of

the first and the last³⁹. If there was not parity, she noticed that the term without a pair would have a value equal to half the value of each pair. Maggie invented her own language to describe her ability (or inability) to pair the odd integers from 1 to a certain odd integer. She called some odd integers, “even odds,” and others, “odd odds”, referring to the parity of the cardinality of the set of addends, followed by the parity of n .

Maggie eventually grew frustrated with this approach. She explained to TR that she knew she needed more than the tabular approach, but didn’t know how to create a complete proof like those she had seen previously in different situations. TR capitalized on Maggie’s approach of finding the sum of odd integers in an effort to help her gain traction on the problem, encouraging her to continue rather than abandon her strategy. He also suggested that she define the last term as $2k-1$.

Scene F: Maggie presents her solution in a whole-class setting.

TR asked Maggie to present a solution to the question, “Given a positive integer n , find the sum of the odd integers from 1 to n .” Maggie’s presentation took approximately one hour to deliver, including class questioning. The presentation was delivered in two parts.

In the first part of Maggie’s presentation, she investigated four examples – 17, 18, 19, 20 – from which she was able to abstract formulas for the four cases she was intending to exemplify. Maggie credited a classmate with having shown her a way to pair the terms in the series by selecting numbers on the ends and working her way inward so that the sums would be constant (see figure 6). Maggie noticed that sum of the series could be

³⁹ In this episode Maggie did not question why the sum of any pair chosen in this manner is constant.

expressed as $20 \cdot 5$ or $20\left(\frac{20}{4}\right)$. From this observation, Maggie claimed that the sum of the series of odd integers less than any even positive integer n with an even number of terms is given by: $\frac{n^2}{4}$. By comparing to the $n = 20$ case, Maggie determined that if $n = 19$ she could use $\frac{(n+1)^2}{4}$ instead. She explained that this was significant because it was a case where n is odd but has an even number of terms to be added.

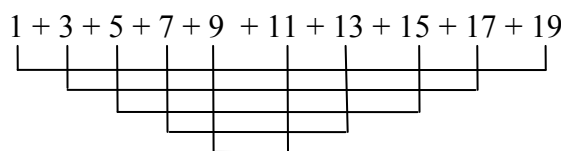


Figure 4.13: Maggie's strategy for computing the sum of the series for $n = 19$ and $n = 20$.

By focusing on the $n = 17$ and $n = 18$ cases, the cases when she could not pair up all the terms of the series, Maggie held the middle term aside and noticed that the value of the sum of each pair was still n when the value of n is even and $n+1$ when the value of n is odd. In the case where the value of n was even, she said that there would be $\frac{(n-2)}{4}$ pairs⁴⁰. Adding the middle term, which she concluded would be $\frac{n}{2}$, Maggie concluded that the sum of the series when the value of n is even and has an odd number of terms is given by: $n \cdot \frac{(n-2)}{4} + \frac{n}{2}$ or $\frac{n^2}{4}$. When the value of n is odd and there are an odd number

⁴⁰ Maggie explained that she did not remember how she found that there would be $\frac{(n-2)}{4}$ pairs.

of terms in the series, Maggie noticed that the value of the sum of each pair is $n+1$ and that there would be $\frac{n-1}{4}$ pairs. She computed the middle term as $\frac{n+1}{2}$ and the overall sum as $(n+1)\left(\frac{n-1}{4}\right) + \frac{n+1}{2}$ or $\frac{(n-1)^2}{4}$.

In the second part of her presentation, Maggie justified the conjecture in a deductive manner, considering the four possibilities for n in general rather than choosing particular instances⁴¹. Recall that Maggie considered whether or not n is even. Then she asked if there was parity (or not) for the set of odd integers less than or equal to n . These questions generated her four cases. Maggie was careful to confirm that regardless of the parity of the cardinality of the set of odd integers in question, the formulas generated matched if n is even. She did the same when n was odd.

Maggie began by letting k be the number of terms in the sequence of odd integers and n be the value of the positive number chosen. Then she related the number of terms to the value of the integer n in each of the four cases. Explaining that the sequence contains $k-1$ terms, begins at 1, ends with $n-1$ or n , and has a constant common difference of 2, Maggie wrote $2(k-1)+1 = n-1$ when n is even and $2(k-1)+1 = n$ when n is odd. In each case, she solved for k . When n is even, she found that $k = \frac{n}{2}$. When n is odd, she found that $k = \frac{n+1}{2}$.

Next, Maggie made assumptions on the parity of k , given the parity of n . In the case where the value of n is even and k is odd, Maggie concluded that the value of each pair of

⁴¹ Maggie discussed her solution with TR prior to presenting to the class. Because her class presentation was consistent with the private work, only the class presentation will be discussed here.

terms is n and the number of pairs is $\frac{k}{2}$. Therefore, the sum of series is

$$n \cdot \frac{k}{2} = n \cdot \left(\frac{\frac{n}{2}}{2}\right) = \frac{n^2}{4}.$$

Using a similar scheme Maggie decided that if k is odd she would hold the last term of the sequence aside while she added the rest by pairing them (see figure 7 below).

$$\underbrace{1 + 3 + 5 + \dots + (n-3)} + (n-1)$$

Figure 4.14: Maggie's strategy for computing the sum of the series of odd positive integers in the case when n and k are both even.

She explained that in accordance with the aforementioned strategy the sum of each pair must be $1 + (n-3)$ and the number of pairs is $\frac{k-1}{2}$. Therefore, the sum of the series is

$$[1 + (n-3)]\left(\frac{k-1}{2}\right) + (n-1) = (n-2)\left(\frac{\frac{n}{2}-1}{2}\right) + (n-1) = \frac{n^2}{4}.$$

Finally, Maggie considered the two cases when the value of n is odd to prove that her formulas were correct. She repeated the same reasoning as before with the understanding that the sequence now ends in n rather than $n-1$, arriving at the conclusion that the sum is $\frac{(n+1)^2}{4}$ when the value of n is odd, regardless of the parity of k .

Scene G: Maggie applies her WoU arithmetic sequences to solve problem #6 a, b, and c.

Maggie was given three arithmetic progressions as follows:

a. 1, 6, 11, 16, ...

b. 2, -5, -12, -19, ...

c. 8, 6.5, 5, 3.5, ...

In the first arithmetic progression, find the number in the 88th place. In the second, find the number in the 100th place and in the third, find the number in the 73rd place.

Maggie immediately viewed the value of the k^{th} term as a function of its position writing k = the number of the term and n = the k^{th} integer. When she spoke about the meaning of n , Maggie referred to it as the “value” of the k^{th} integer. She did not use particular cases. Rather, Maggie considered the value of the common difference and noticed that the k^{th} term could be found by adding $(k-1)$ common differences to the first term. For problem 6a, she wrote $n = 5(k-1)+1$ and continued by substituting $k = 88$ to find the corresponding value of n . For 6b and 6c, Maggie repeated the same reasoning.

Claims

The events documented in scene D constitute evidence of a RPG proof scheme and an Authoritative proof scheme because Maggie was persuaded by a finite set of results but felt the need to do more with her solution to please TR. However, scene E shows that when William pointed out a need for causality, Maggie engaged immediately and meaningfully in problem-solving attempts aimed at providing a reason why the function should be correct beyond the fact that it matched a finite number of results.

During scene F, Maggie explained that she had (at least in part) continued because she felt that TR would be uncomfortable with her solution, consistent with the analysis of scene D. In scene F, Maggie showed another instantiation of the RPG proof scheme in the first part of her presentation. However, she continued by demonstrating a deductive

proof and explaining why she felt her previous solution was incomplete. Therefore, it is claimed that scene F is consistent with forms of the deductive proof scheme. Finally, in scene G Maggie autonomously and spontaneously used a WoU arithmetic sequences developed in her solution to problem #5 (scenes D, E, and F). Thus there is evidence that the PPG proof scheme became internalized, at least in the context of arithmetic sequences and series.

Through interactions with fellow participants and TR, Maggie continued in a state of disequilibrium throughout this episode. Nevertheless, Maggie's proving acts in this episode demonstrate a local change in her proof schemes from empirical toward deductive proof schemes. While the episode demonstrates autonomous behavior that can be characterized as deductive, this behavior was not spontaneous early on. Only in her solution to problems 6a, b, and c could she be seen operating in a deductive manner spontaneously.

Analysis

After creating her table and sharing it with William, Maggie searched for a way to express a function modeling the data. It is important to note that even though Maggie's work with William included the use of a table, it does not necessarily constitute evidence of an RPG proof scheme. For this reason, it is important to focus on what convinced Maggie that the function was correct once it was created rather than focus exclusively on how the function was created. William created the function,

$$f(n) = \begin{cases} \left(n - \frac{n}{2}\right)^2 & \text{if } n \text{ is even} \\ \left(n - \frac{n-1}{2}\right)^2 & \text{if } n \text{ is odd} \end{cases}, \text{ and persuaded Maggie that it was valid. Below is his}$$

explanation:

William: Okay. If it's an odd number, it's equal to $\left(n - \frac{n-1}{2}\right)^2$. If it's 1, 1-1 is 0,

over 2. 1-0 is 1 square it.

Maggie: Uh-huh.

William: 3. 3-1 is 2 over 2. 3-1 is 2, square it ... 4. Let's go to a higher number, 7. 7 minus, 7-1 is 6, over 2... 3. 7-3 is 4, squared ... So there's the n [referring to $(n-1)^2$ on the third line of the table]. And then how did you get the 1? ... you took 3-1 is 2 and divided by 2. You took 5-1 is 4 divide by 2. 7-1 divided by 2 is 3.

William's explanation of how he knew his function was valid consisted of verifying that it worked in a few cases, but did not initially attend to a need for causality. However, after he confirmed empirically that the function modeled the data in their table, William addressed the question, "Why will it always be right?"

Maggie: You take any structure and say I can plug it in here and it will always be right.

William: Why will it always be right?

...

Maggie: Right [looking at William and shaking head in agreement, but speaking to another group mate]. Right. ... this is the 8 step structure, right? What about the 500 step structure? How can you prove that this works for? Why does this work for every situation? ... How would you do that? See because and we used a lot of that... that summation...

William: Yeah.

Maggie: And that really isn't included in this. Is it?

William: Uh. Not yet.

...

Maggie: I think this is it... **I'm happy with this**, [pointing to William's formulas]... but you know...

Maggie embraced William's question, though she was already convinced that the formula would continue to work. However, she did not know how to explain why the function would always be true.

Maggie's statement, "I think this is it... I'm happy with this, [pointing to William's formulas]... but you know..." reflects a sense of personal conviction that what William produced was acceptable, but that she felt she was expected to provide more. The following conversation with TR provides further evidence that for Maggie, William's observation had reached the status of fact.

Maggie: They came up with a good solution.

William: A conjecture.

Maggie: A good conjecture.

Maggie's agreement that the function was a conjecture, rather than a solution, weakens her previous statement about her certainty in William's observation. Later she repeated her agreement that the work done by William was only a starting point rather than an ending point. In doing so, Maggie expressed two feelings simultaneously. First, she expressed a feeling of satisfaction with William's solution. Second, Maggie expressed a sense of frustration with her inability to meet a perceived standard of evidence required at the PD.

Maggie: I mean, this is a major first like beginning point [referring to the table and function abstracted from it]... But how do you go from this to... especially when we feel like oh, okay, this seems to work pretty well? How does this turn into one of those proofs [expressing a sense of frustration]?

In summary, in some instances Maggie demonstrated evidence that she had been persuaded by an empirical argument of the inductive sort and in other instances she acknowledged a need for causality. However, in no instances during this scene did

Maggie express a personal desire to address the need for causality spontaneously. Once she acknowledged that William's work was not complete, she explained that she could not envision a way to justify his conjecture. Maggie made reference to an idealized proof. Presumably, this kind of proof was her target. Still, there are signs that the need to produce this kind of proof was for the purpose of conforming with a standard foreign to her.

During the presentation of her public solution (see scene G below) TR asked Maggie why she continued after generalizing from a few examples.

TR: ... Were you happy at this stage?

Maggie: ... I wasn't happy.

TR: Why?

Maggie: ... I knew that it wasn't enough...

TR: Why not?

Maggie: Because it's me **using a specific case that I came up with** to make sense of it **and then I just generalized from there, which I know you can't do**, but I needed something to start from. But I didn't take it any further. **I didn't really know what to do with it.**

TR: So you see the status... of what she did so far...? Up to this point... we have generated a conjecture and we stopped there and we said. Done... **At this point she was feeling uncomfortable. Not because she knows I will feel uncomfortable. I hope not. Is it?**

Maggie: Partly. [Maggie giggling]

Maggie explained that she felt generalizations from specific examples were unacceptable in the PD classroom. She continued, at least in part, for authoritative purposes rather than for the purpose of satisfying an internal need for causality. While Maggie continued and ultimately succeeded at creating a deductive proof, the influences of William and TR played a crucial role in providing perturbations for Maggie, leading her into a state of disequilibrium with regard to how to produce a proof that addressed the need for causality.

It was William who suggested that she needed to know why. It was also William who mentioned that his work was a conjecture rather than a proof. Maggie's behavior reflects an appreciation for William's statements. TR helped her pursue and formalize her image that the number of design squares was the sum of a set of odd positive integers. This gave Maggie an experience in turning a WoU related to a RPG WoT into a WoU related to a PPG WoT without abandoning the former. According to the duality principle, this is the manner in which $W_s o T$ can change.

This scene demonstrates that at this point, Maggie had an image of a prototypical proof which was not RPG-based. While the need to provide a deductive proof may not have been entirely intrinsic for Maggie at this point, the ability to do so was within her grasp. There is much evidence to support the claim that the ability to produce a deductive proof was within her zone of proximal development. Indeed, her ability to produce the solution in scene F, in public with no notes, and her response to questions about her solution indicate that she internalized the solution. In terms of the need for causality, Maggie's explanation of why she continued provided valuable insight. She said, "...it's me using a specific case that I came up with to make sense of it and then I just generalized from there, which I know you can't do...". Maggie did not return to her statements, "What about the 500 step structure," or "Why does this work for every situation," as motivating factors. She explained that she felt what she had done was unacceptable rather than explain that she had a personal need to know why.

As explained in the previous section, *Synopsis of Maggie's Solution (scene F)*, Maggie's presentation was presented in two parts. In the first part, Maggie explained how she came to view the problem in four cases deductively, explaining that her motivation

was taken from a need to pair the odd integers in a sequence in order to find the sum of the sequence. For this reason, Maggie invented the four terms: even-even, odd-even, even-odd, and odd-odd. It is notable that she had an experience while learning at the PD in which she felt free to invent mathematical terminology, giving her an opportunity to experience mathematics as man-made endeavor. Maggie's terms embodied both her solution strategy and focus of her attention.

In the second part of her solution, Maggie explained each of the four possibilities for n and represented each case symbolically in an effort to show that her exemplary cases would hold in general. The section entitled, *Synopsis of Maggie's Solution (scene F)*, describes in detail how Maggie went about doing so. Maggie explained clearly her mental imagery and repeated similar reasoning in each of the cases, explaining why she could or could not pair up terms in each case and making appropriate adjustments to compensate for cases when k was odd. These adjustments are hallmarks of transformational proof schemes.

In order for Maggie's strategy to work in the general cases, several crucial tools were used. By naming the number of terms in the series, k , Maggie was able to represent an unknown as if it was temporarily known. Another crucial component of her solution was her ability to relate the number of terms, k , to the value of the positive integer, n , that determined the end of the sequence of odd integers less than or equal to n . Maggie's ability to relate n and k successfully in each of the cases reflected attention to meaning of the symbols. It is important to observe carefully how Maggie justified some of her expressions. For example, in determining that $2(k-1) + 1 = n-1$ for the cases when n is even, Maggie justified the left-hand side of the equation by explaining that if there were k

terms, there would be one $k - 1$ differences between the first and last term of the sequence. Since the difference was constant she explained that $2(k - 1)$ represented how much was added to the first term in order to arrive at the last term. Finally, she explained that $2(k-1) + 1$ represents the value of the last term because the sequence began at 1. Maggie's solution attended to the meaning of the symbols, the reasons why the symbols were related in the ways they were, and her overall goals.

For the aforementioned reasons, Maggie's solution can be characterized as referential symbolic because she attended to the meaning of the symbols as she wrote her expressions and manipulated the equations, anticipating and compensating for changes to forms of her expressions. Additionally, her solution attended to, in a deductive manner, a process which explained why the patterns observed were generalizable in all cases. As such, Maggie's solution is also characteristic of the PPG proof scheme, a form of the transformational proof scheme describing how generalizations are justified.

In scene G, Maggie used her WoU arithmetic sequences developed in problem #5 to address questions a, b, and c for problem #6. While other members of her group provided answers that were specific to the particular cases of each part of problem #6, Maggie worked in general terms immediately. She declared the same variables with the same meanings as she had done in problem #5 and created a function for n in terms of k immediately, explaining the meaning of each term and justifying her expressions as she went. It is notable that TR was the one to apply empirical reasoning to her function as a means of forming a sense of personal conviction. He emphasized that checking a few cases is also important.

Scene G demonstrates the stability of Maggie's WoU arithmetic sequences in general terms as a type of function relating the position of the terms in sequence to the value of those terms.

Maggie: Let me see what I'm calling it. The k is the number of the term. Like... [Maggie underlines 88th... on her paper]... k . This is the k th term.

TR: Right. Cause that's what we want to get. Right.

Maggie: So n is the actual, is the integer in that...the value of the number there.

TR: Okay. Okay.

Maggie: It's the k th integer.

TR: Okay.

Maggie: So the relationship between these is that,

TR: Uh-huh.

Maggie: um, if I'm always adding 5...

TR: Uh-huh.

Maggie: Then on the 88th, I'm going to have $(k-1)$ times 5.

TR: Okay.

Maggie: But I started at 1.

TR: Uh-huh.

Maggie: So, I have to add that 1 in there. So this will equal n .

This spontaneous and autonomous use of her WoU is indicative of a more global change in proof schemes. Maggie's approach to solving these problems demonstrated a thorough explanation in a deductive manner compatible with the PPG proof scheme.

Synopsis

This episode demonstrates how Maggie's natural tendencies were to rely on results as a means of justifying generalizations about patterns. Interactions with other participants and TR within her ZPD led Maggie to the autonomous production of a deductive solution. Still, the need for causality was not demonstrated spontaneously until she operated in a similar context on problem #6. It is clear that Maggie is participating in what she perceived to be expectations about what constitutes acceptable justification at

the PD and that her participation is helping to shape her proof schemes. What is still not clear is whether or not she sees a personal need for causality.

This episode shows Maggie living in two worlds simultaneously. While she was personally convinced about the generality of William's conjecture. She did not know how to provide a deductive solution without help, but understood that she should do so. Maggie could visualize some kind of prototypical proof as a target, but she reported that she was initially unable to proceed without TR's help. However, investigation into her work with William in scene E showed fledgling attempts at deductive solutions without TR's help. The collision of these two worlds – the world of personal conviction and the expectations of the PD classroom – provide an analogy from which to understand the source of Maggie's disequilibrium at this point in time.

Merging part 1 and part 2 of the stair-like structure problem, Maggie has demonstrated an overall trend of moving back and forth between empirical and deductive proof schemes. While she is personally convinced by what “works pretty well,” she is also aware of what she “can't do” in the PD classroom. All the while, Maggie has demonstrated that she is not simply a student of mathematics. Each PD participant is also a teacher of students. As a teacher, Maggie has mentioned that she sees the connection to her students as a leader in mathematical ways of thinking (see stair-like structure problem part 1):

Maggie: I thought of a couple of things while you were talking about the difference between these two, um, derivations... I thought, well, if we aren't even motivated, I should speak for myself, to find, you know the proof for the first one where there is a conjecture, how are we going to get our students motivated to do that. Um, and I guess it starts with us first... secondly, this has been my problem the whole time. I thought I knew algebra. I thought I could do it well, ... I can find patterns very easily, I can come up with formulas, but I don't truly know the

difference between when I've derived something... and when I just have a conjecture. Sometimes I think I'm finished and you come along and you say, well what about this and I didn't think about it. I don't really know the difference between when I'm actually done and when I still have some work to do... just yesterday in our groups, I was still... I did a solution similar to what you did ... and I had to ask. You know, am I done yet? ... [Bernie] said, well yeah, because we proved already, ... how to get the sum of a sequence. So, because you use that... You know, you can say that you're done.

TR: If we proved it, we proved it. Once we proved it, it was established.

Maggie: Right. But I wasn't really sure. I was like, so do I have more to do, you know or not?

TR: But that uncertainty and that struggle is very natural. To me, if a person is in that struggle and feels like feels uncomfortable, feels um, perturbation, then that's a very good sign. Regarding the first point that you made; again, I want you to compare for yourself, your understanding of this relationship before we showed this approach and now. Just compare it for yourself. Which one [do] you feel it's more superior in some sense than the other and why? I mean this is really a kind of a personal thing that you have to compare and say, well before I got to that stage and now I have something else. Did I gain anything?

This interaction epitomizes the struggle Maggie is in at this point. The myriad

forces pulling her in different directions were mentioned only a few class sessions before the stair-like structure problems #5 and 6. Maggie explained that in the past, she has seen RPG as an acceptable standard of evidence in mathematics. The influence of Bernie, William, Derrick, and TR is certainly being felt by her and she is realizing that there is another standard of evidence that she is accountable to because of her responsibility to her students. Even her understanding of what it means to do Algebra is in flux. However, the question remains unanswered whether or not Maggie feels that she has “gained anything” from deductive solutions beyond satisfying TR or conforming to a class norm.

One related question is whether Maggie realizes the dialectical relationship between the creation and usage of the terms even-even, odd-even, odd-odd, and even-odd with her solution strategy. Maggie found that her naming system didn't work for all of her audience and chose to abandon it because one particular member was uncomfortable

with the order Maggie was using (i.e. – odd, even or even, odd). Her solutions strategy and her usage of language constrained her ability to persuade her audience much as it would in a classroom filled with her own students. This is an example of a natural need for communication. As a result, Maggie repeated what she meant rather than use shorthand. This is a valuable lesson for a teacher to realize the struggles that students (or other participants) go through as they try to understand mathematical terminology. This struggle relates to the need for communication which is closely connected with ascertainment and persuasion.

During the stair-like structure problem Maggie demonstrated an affinity for one standard of proof while acknowledging that a different standard of proof is necessary in the PD classroom. However, she also expressed discomfort with deductive proofs. In the first part of the analysis she produced a deductive proof but wasn't sure if it was enough. Still, she was able to pinpoint the shortcomings of her proof. In the second part of the analysis, Maggie expressed a sense of satisfaction with a conjecture as an ending point. As she interacted with William and TR, she accepted a need to produce more but did not know how. It is clear that many external forces pulled on Maggie during this time (e.g. – responsibility to her students, a desire to please TR, and the desire to participate in the perceived socio-mathematical norms of the PD classroom's standards of evidence). Maggie demonstrated two proofs that can be categorized as mainly deductive. According to the duality principle, this is an indication of changes in proof schemes. However, it remains to be seen if these changes are local or global in nature.

Episode 6 - Sara's Conjecture

Question

Sara made the following conjecture about arithmetic progressions. “Choose any item in an arithmetic progression. Take any number of items on its right and the same number of items on its left. The item chosen is the average of the rest of the items.” Is Sara’s generalization valid?

Background and Significant Events

The events of this problem took place on day 26 of the institute (day 4 of the second year). For the first three days (days 23-25), participants worked on the stair-like structure problems. This means that participants had been thinking about arithmetic progressions for 15 hours before Sara’s conjecture was presented.

Synopsis of Maggie’s Solution

Maggie and her group mates represented the arithmetic progression symbolically, labeling the terms $a_1, a_2, a_3, \dots, a_n, a_{n+1}, a_{n+2}, \dots$. She chose x to be the number of terms on either side of a_n and recalled that it had previously proven that in an arithmetic progression with a common difference, d , the n^{th} term can be represented as $a_n = a_1 + d(n-1)$. After declaring the meaning of x , Maggie explained that her strategy was to find the sum of the terms to the right, find the sum of the terms to the left, and divide the sum of the sums by the $2x$.

As she solved the problem and interacted with others, Maggie questioned the applicability of a formula for finding the sum of arithmetic series when the common difference is not 1. She saw her solution as incomplete without a proof that the formula she created was true when d is not 1. Maggie’s realization that her solution was incomplete represented a time when she was aware of her doubt and knew that she needed more to complete her solution. The episode closes with Maggie beginning to

search for a proof, but it was not recorded.

Claims

Representing the complicated situation explained in the problem statement using symbols may not be enough, on its own, to constitute evidence of a solution compatible with a referential symbolic proof scheme. However, the ability to pause midstream and explain the meaning of variables accurately in the context of the problem statement does constitute symbol manipulation characteristic of the Referential Symbolic proof scheme. In this episode, Maggie also demonstrated a very desirable trait of deductive reasoners, the ability to raise a conjecture, set it aside and return to it at a later point in time in order to complete an argument.

Analysis

While attempting to prove Sara's conjecture, Maggie's strategy called for her to find the sum of two arithmetic progressions. However, she had reservations about the applicability of the formula, $S_n = (a_1 + a_n)(n/2)$, in a more general context than the progression $1, 2, 3, \dots, n$. In a conversation with Bernie, Maggie demonstrated her belief that the formula only worked for consecutive positive integers (e.g. $-3, 4, 5, 6, 7$ or $10, 11, 12, 13$). After some prodding from TR, she created the following example where $d=5$ and $a_1=5$: $5, 10, 15$. From this example, she conjectured that the formula holds true for all finite arithmetic series.

TR: Okay, so what is this here? What does it say? [Referring to $S_n = (a_1 + a_n)(n/2)$]

Maggie: Okay, the sum of the sequence of numbers up to...

TR: Not any sequence of, you mean arithmetic sequence.

Maggie: Arithmetic.

TR: Right? Any sequence that you take?

Maggie: No... An arithmetic sequence ... Of consecutive numbers.

TR: The sum of consecutive numbers in arithmetic sequence...

Maggie: Up to n equals... the sum of the first and last term times the number of terms divided by two.

TR: Right? Okay, so, **are you certain that this formula works?**

Maggie: (audio skips) Where as this, **I've only seen this used with consecutive numbers...** Meaning a difference of one... Where you count in order 1, 2, 3, 4, 5.

TR: Oh so you think this formula works only for what you call consecutive? In other words.

Maggie: Oh, I don't know, but I think that's something else.

TR: Let's just play around. Let's try other numbers that are not.

Maggie: Right. So, if I went along with this [writes 5, 10, 15], I would take first and last and then I would multiply by the number of terms over two. So this is 20 times 3. 60 divide by 2. Is 30 and it does equal 30. **In this case, it works. But will it work every time?**

TR: So, we have a conjecture, maybe. Maybe this always works.

Maggie: Okay.

TR: Okay. You have to decide upon your next step now.

Maggie: If I pursue this? [Meaning proving that $S_n = (a_1 + a_n)(n/2)$, for all arithmetic progressions]

TR: You have to make a decision.

Maggie: I'm just going to use it so I can start doing this [giggling].

TR: Okay, but then, but you still, something will be missing. Right?

Maggie: Yeah. Well. Yeah [said hesitantly]

The interaction between Maggie and TR began with the TR's request for Maggie to unpack the meaning of the function she was using to find the sum of an arithmetic progression. Maggie's reply to the question, "What does it say? [Referring to $S_n = (a_1 + a_n)(n/2)$]," indicates that she could pause in the middle of a complex sequence of symbolic manipulations and identify the meaning of the terms in the context of the problem. This characteristic of her problem solving ability is compatible with the referential-symbolic proof scheme.

It is notable that Maggie asked the question, "But will it work every time," spontaneously. This is reminiscent of the instance in problem #5 of the stair-like structure when William posed this question to Maggie after he stated his own conjecture. It is a sign that Maggie doubted the statement, " $S_n = (a_1 + a_n)(n/2)$ yields the sum of the terms in

any arithmetic progress.” She continued using $S_n=(a_1+a_n)(n/2)$, but was reminded by TR that it would have to be proven at a later point in time to complete her solution.

Later TR came back to the group after Maggie had solved the problem. Before he arrived, she was revisiting her notes about the stair-like structure with a page that only said, $S_n=(a_1+a_n)(n/2)$. Her comment to TR provides further evidence that she internalized the need to prove that $S_n=(a_1+a_n)(n/2)$ is valid for all arithmetic progressions.

TR: ... did you conclude eventually about your work?

Maggie: Well actually, it started the same as Bernie, but it ended up pretty different. Um, **but I still have to go back and look at the sum of the sequence or arithmetic progression.**

TR: Right. Before we do that, but let's go back here in your work.

ML: What did I conclude about my work?

TR: Yeah. What... so the generalization is valid or not valid?

ML: It's valid.

Through her conversation with TR, Maggie made it clear that she was setting aside the matter of justifying the usage of the formula within the context of her problem. Twenty minutes later, she began to review her notes, attempting to prove the missing result. However, she did not get to finish her proof. It is notable that she returned to the matter on her own, showing that she had internalized the need to justify this formula in order to complete her proof. This is reminiscent of what was done in the cat and mouse problem. On that occasion, Maggie also made conjectures and set them aside to be proven later. There, TR emphasized that it was a common, and in some cases necessary, part of the proving process.

Synopsis

This episode marks an occasion in which Maggie recognized a missing component of her proof. She realized that a result from a different context might not work

in the current context and attempted to address the generalizability of $S_n = (a_1 + a_n)(n/2)$. Still, she participated in a common mathematical practice that TR emphasized at a different point in the institute. When a conjecture is raised, it is acceptable to set it aside momentarily and come back to it at a later point in time.

Internalizing the proving practice of raising conjectures and setting them aside as statements to be proven later is closely related to the development of a deductive proof scheme in two ways. First, this practice has the potential to help individuals see the connection between intuition and logic. On the one hand, one does not want to pursue an argument too far with a potentially faulty conjecture. Nevertheless, when one feels strongly that the conjecture should be correct, it can still be productive to produce tentative results. Second, this practice can be leveraged to help students produce proofs by contradiction because it allows students to knowingly live in a hypothetical reality temporarily, waiting for evidence that could potentially be used to reject the conjecture.

Another noteworthy aspect of this episode is TR's teaching practices of making it clear when Maggie was proceeding with tentative results ("So, we have a conjecture here") and reminding her that the solution would be incomplete without a proof of the conjecture ("Something will be missing."). These teaching practices help clarify the status of claims and help to convey an expectation about what should be considered a proof. From the observer's perspective, TR's teaching action of reiterating that without a proof of the generalizability of her formula Maggie's solution would be incomplete is TR's way of attempting to negotiate socio-mathematical norms related to standards of evidence for complete solutions in mathematics. Namely, that ideally for a proof to be

complete, the individual must be able to support each part of her argument, especially if it is a linchpin in the proof as a whole.

Episode 7 – Divisibility

Question

Debbie explained a way to determine divisibility by 3 for whole numbers. Namely, that if the sum of the digits of a whole number is divisible by 3, then the number itself is divisible by 3.

Background and Significant Events

This episode occurred in the beginning of the second week in the second summer of the PD. Participants had been discussing arithmetic sequences and series since the second year of the institute began. Several trends emerged relating to past episodes. Once again, Maggie made mention of an imagined proof⁴², what she called a “perfect little proof.” This episode also contained another instance⁴³ in which Maggie ascertained the truth of a result by one standard of evidence, but searched for a persuasive argument fulfilling a different standard. The episode contained direct questioning and self-reporting about what convinced Maggie of the truth of observations.

Synopsis of Maggie’s Solution

Maggie tried to prove Debbie’s claim. In the first scene, Maggie limited herself to proving the result for three digit numbers because Debbie’s claim was in the context of finding the sum of whole numbers greater than 100 and less than 500 that are divisible by

⁴² See the Stair-like structure problem for a previous example.

⁴³ See the Stair-like structure problem for a previous example.

6 and end in 6. However, she questioned the validity of such an approach because of its lack of generality for the number of digits.

Later, Maggie addressed Debbie's claim in a more general context – the number has n -digits. She was able to abstract a process from what had been done in the $n = 3$ case, leaving her with a crucial point to be proven. Maggie used her representation of an n -digit number as $a_1 + 10a_2 + \dots + 10^{n-1}a_n$ to separate the digits in such a way that she could incorporate the hypothesis, “the sum of the digits of a whole number is divisible by

3”, leaving her to prove that $\frac{(10^{n-1} - 1)}{3}$ was a whole number.

$$\begin{aligned} \frac{a_1 + 10a_2 + \dots + 10^{n-1}a_n}{3} &= \frac{a_1 + 9a_2 + a_2 + \dots + (10^{n-1} - 1)a_n + a_n}{3} \\ &= \frac{(a_1 + a_2 + \dots + a_n) + (9a_2 + \dots + (10^{n-1} - 1)a_n)}{3} \\ &= \frac{(a_1 + a_2 + \dots + a_n)}{3} + \frac{(9a_2 + \dots + (10^{n-1} - 1)a_n)}{3} \\ &= \frac{(a_1 + a_2 + \dots + a_n)}{3} + \frac{9a_2}{3} + \dots + \frac{(10^{n-1} - 1)a_n}{3} \end{aligned}$$

Maggie recognized that the $\frac{(10^{n-1} - 1)}{3}$ was a number composed of all 9's, but found it difficult to prove that it was so algebraically.

Claims

This episode is rich in diversity of proof schemes. In fact, all three general categories of proof schemes were demonstrated at one point or another in different contexts. With respect to symbolic manipulation, Maggie demonstrated behavior compatible with both the external conviction and deductive proof schemes at different points in time. In a different instance, Maggie demonstrated evidence of authoritative and

empirical proof schemes as she tried to determine whether or not her proof was complete. The authoritative proof scheme appeared in the context of determining whether or not she had to prove that whole numbers are closed under addition and multiplication. This was an instantiation of Maggie's general statement, "I don't know when it's enough."

Two issues for further investigation are coming into focus: Maggie's image of an acceptable proof and Maggie's relationship to the use of cases as a method of proof. There is evidence that Maggie's vision of proofs requires the use of algebraic expressions, but there is not enough evidence in this episode to determine why. The second issue regarding Maggie's view of the role of cases in proving stems from statements she made during the stair-like structure problem. Recall that during those episodes, Maggie rejected William's RPG-compatible proof because it was based on four examples and commented that generalizations from specific examples were unacceptable during her own presentation. She abstracted a solution method from her investigation of four examples and made them four cases. Then she presented a case-by-case proof. In private, on one particular occasion during the stair-like structure, Maggie explained that she was satisfied by a RPG-compatible argument presented to her by William and during this episode, on two occasions Maggie's ascertainment seems to be compatible with the RPG-proof scheme. These instances beg the questions, "How does Maggie relate cases to examples?" Is there a difference between the two for Maggie? If so, what is the difference? Did she reject generic proof⁴⁴ as a means of persuasion?

Analysis

⁴⁴ During the next day's class session, TR gave a generic proof to support Debbie's result (see 1:11:00 on TR20040727am).

This episode is divided into four scenes. In scenes A and B, Maggie addressed the case $n = 3$, arriving at a tentative conclusion. TR left her to attempt to validate or invalidate her work on her own. When TR returned in scene C, Maggie discussed the source of her uncertainties and decided to move on to the more general case. In scene D, Alfred (a mathematician) discussed Maggie's solution to the general case.

Scene A:

Maggie represented the digits of a three-digit number by x, y, z . She declared that $100x+10y+z = a$ and $(x+y+z)/3 = b$, where b is a whole number. Next, Maggie restated Debbie's claim by saying that if $(x+y+z)/3 = b$, it must be proven that $a/3$ is a whole number.

Maggie: ... **this is any three digit number, I'm not proving that any number.**

TR: Ok well let's, fine I mean you can't conquer the world at once right?

Maggie: Yeah, **but it makes me question whether I should even continue with this or...**

TR: (00:13:47) ... sometimes it happens, we don't have to go to the most general case right away. This is general. You are not talking about a particular whole number. You are talking about any whole number with 3 digits.

Maggie: Mmhmm.

Maggie's reservations were her own. She knew that she wanted to address a more general case, but did not know how to go about producing an answer to her more general question. TR attempted to help her see that investigating smaller cases can often lead to breakthroughs on more general cases or provide one with a generalizable problem-solving approach.

Maggie chose to relate the two expressions, $\frac{100x+10y+z}{3}$ and $\frac{x+y+z}{3}$, as

follows:

$$\begin{aligned}
\frac{100x + 10x + z}{3} &= \frac{99x + x + 9y + y + z}{3} \\
&= \frac{99x + 9y + x + y + z}{3} \\
&= 33x + 3y + \frac{x + y + z}{3} \\
&= 3(11x + y) + \frac{x + y + z}{3}
\end{aligned}$$

She revisited the meaning of x , y , and z to determine if the quantity $3(11x + y)$ was a whole number or not before declaring that it was so, behavior which can be characterized as referential symbolic. Noting that $\frac{x + y + z}{3}$ was a whole number, by assumption, Maggie hesitantly declared that she had proven Debbie's claim. Nevertheless, Maggie was skeptical about her solution because she felt that there were probably gaps somewhere in her logic. She explained that comparing her work to the work of others was a typical way for her to determine if she had produced a convincing argument. TR asked her to go over her argument to insure there are no gaps.

Scene B:

(00:30:00)Maggie: ... **this would be the part where I would ask somebody else what they did so I could compare.**

TR: But I thought what we decided that we are going to do is to find gaps? To see if there are gaps not to find gaps but to see if there are gaps.

Maggie: **Well I think there probably are that I'm not seeing.**

TR: (0:30:37) Which stage do you have doubts?

Maggie: **In the reasoning, in the last step.**

TR: Here whether this $[3(11x + y)]$ is a whole number, so let's go over it again.

...

Maggie: Right, ok x and y are whole numbers.

TR: You have doubt about that?

Maggie: No.

TR: (00:31:13) So that's certain you are saying right?

Maggie: Uh huh.

TR: Ok, and why is that? Why are you so certain they [meaning x and y] are

whole numbers?

Maggie: Because **it's part of how I defined this**, so they must be whole numbers... so if I multiply a whole number by another whole number I get a whole number, I add a whole number to that I get another whole number, and then I multiply a whole number by a whole number my answer should be a whole number.

TR: (00:31:58) So where are the doubts then?

Maggie: Um... I don't know where the doubts are exactly.

TR: Well that means you have no doubts.

Maggie: ... **it just doesn't seem like a sound argument.**

TR: Why.

Maggie: Like **it doesn't seem to be strong.**

TR: Why.

Maggie: (00:32:39) I don't know.

TR: You have a...

Maggie: **I'm going by my feeling more than I'm the math.**

Even after validating that $3(11x + y) + \frac{x + y + z}{3}$ must be a whole number, Maggie

continued to be uneasy with her solution. She had previously explained that she normally relied on the comparison of her work to that of others as a crucial part of the process of dismissing the doubt she had in her solution. It is notable that TR asked her to pinpoint the source of her doubts and her logical argument did not dismiss her doubt. The statements, "... it just doesn't seem like a sound argument," "it doesn't seem to be strong," and, "I'm going by feeling more than I am the math," point to a deep seeded discomfort with her argument, requiring some form of external affirmation from a knowledgeable other. TR did not directly offer his affirmation. Instead, TR handled the matter by asking her to work that alone in order to decide one way or another whether or not her solution was valid.

Scene C:

On her own, Maggie used $x = 1$, $y = 2$, and $z = 3$, comparing the results $\frac{100(1)+10(2)+(3)}{3}$

and $3[11(1)+2]+\frac{1+2+3}{3}$. She determined that their values were equal. Next she used $x = 4$,

$y = 5$, and $z = 6$, arriving at the same conclusion. TR returned to ask Maggie for her conclusions.

TR: (00:41:07) How did you remove the doubts?

Maggie: By looking at specific, only 2 specific cases.

...

TR: How did the specific cases help you to remove doubts.

Maggie: Because I used this form [pointing to $3(11x + y)$] of this [pointing to

$\frac{100x + 10x + z}{3}$] and just in doing that I mean it reinforces that if this is a whole

number which I had to make a whole number by choosing my x , y , and z , there is no way that this [pointing to $3(11x + y)$] is going to be a fraction a decimal, it's going to be a whole number.

TR: (00:41:54) So when you say you used specific cases, you didn't care very much about that the numbers are 1, 2, and 3, it's just that they are whole numbers.

Maggie: Yes.

TR: Ok so just the fact that you saw them there [pointing to the specific cases] to be whole numbers, gave you a little bit more confidence, where as here [pointing to $3(11x + y) + \frac{x + y + z}{3}$] you have to think about it to be whole numbers.

Maggie: Yes, and using it **just doing what I said but with a specific case**, 11 times a whole number plus a whole number, times a whole number again it's going to be a whole number. And then I had to do it again.

TR: Ok and now you feel good about it?

Maggie: Well, yes, better but my answer is still yes. Because I still don't know of a case where it doesn't work, and why it wouldn't be a whole number.

TR: So the whole thing is concentrated here, that's where the source of all the problems, that you are not sure that this is a whole number?

Maggie: No not just that, just like if I think the whole thing through again, is that a sound argument to say that I'm choosing this [pointing to $\frac{x + y + z}{3}$] to be a

whole number, and if it is, then if this is a whole number I can say that this assertion is true. **I don't know but I think it's because I don't have enough experience with proofs. I don't know if this is enough. I don't know when it's enough.**

Maggie: I, well then **maybe a major doubt is, is this even a good or a legitimate starting point...** where if this is a whole number [pointing to

$\frac{x+y+z}{3}$], then this is a whole number [pointing to $\frac{100x+10x+z}{3}$]. ... I had to go back and do that again after I finished this, and I looked if the sum of the digits of a number is divisible by 3, these are the digits, it's being divided by 3. So if it's divisible by 3 this is a whole number. Then the number itself, and I'm representing a 3 digit number, is also divisible by 3... I think I'm content enough to move on.

Maggie pointed out a lack of experience with proof as the source of her uncertainty, saying "I don't know but I think it's because I don't have enough experience with proofs. I don't know if this is enough. I don't know when it's enough." Knowing that some statements can be taken as starting points is crucial in the development of proof expertise. The same is true for knowing which statements can be taken as starting points. Maggie's disequilibrium can be characterized as a difficulty determining valid starting points. When left alone to validate her conclusions, Maggie created two examples. She explained that while the examples gave her a source of confidence in her conclusions, she still wasn't sure if her work constituted a sound argument.

Maggie's statement, "...this would be the part where I would ask somebody else what they did so I could compare," along with her reported certainty generated from the cases when $x = 1, y = 2, z = 3$, and $x = 4, y = 5, z = 6$ supports the interpretation that Maggie's appeal to an authority was due, at least in part, to her discomfort around the determination of acceptable starting points. In the absence of authoritative confirmation, Maggie appealed to empirical reasoning to validate her conjecture. Certainly, proving that whole numbers are closed under addition and multiplication was not an expectation of any fellow PD participant or of TR. For Maggie's development, empirical observations of this fact were her only means of confirming that $3(11x + y)$ was a whole number. Even

as she became comfortable with this fact, Maggie continued to express doubts over other aspects of her proof.

In the end, Maggie explained that she was comfortable enough to move on. This statement expresses a minor amount of doubt, meaning that Maggie was not fully convinced that her argument was acceptable.

Scene D:

With no prompting from others, Maggie later attended to generalizability of her previous claim regarding whole numbers with more than 3 digits. Borrowing from one group mate's WoU place value, Maggie created a way to represent a number with n digits using the sequence of digits, $a_n, a_{n-1}, \dots, a_2, a_1$, to stand for the n digits of a whole number. Expanding on what she did in the $n = 3$ case, Maggie arrived at following derivation:

$$\begin{aligned} \frac{a_1 + 10a_2 + \dots + 10^{n-1}a_n}{3} &= \frac{a_1 + 9a_2 + a_2 + \dots + (10^{n-1} - 1)a_n + a_n}{3} \\ &= \frac{(a_1 + a_2 + \dots + a_n) + (9a_2 + \dots + (10^{n-1} - 1)a_n)}{3} \\ &= \frac{(a_1 + a_2 + \dots + a_n)}{3} + \frac{(9a_2 + \dots + (10^{n-1} - 1)a_n)}{3} \\ &= \frac{(a_1 + a_2 + \dots + a_n)}{3} + \frac{9a_2}{3} + \dots + \frac{(10^{n-1} - 1)a_n}{3} \end{aligned}$$

With an eye on generalizability, Maggie asked if $\frac{(10^{n-1} - 1)}{3}$ is a whole number for all $n >$

1. Generalizing from a few examples, she stated that all the digits of $10^{n-1} - 1$ are 9 and that $10^{n-1} - 1$ has n digits. While Maggie felt personally convinced of her observation, she also felt that convincing another person would require her to use algebra.

Maggie: **10, 100, 1000, they're all, you subtract them by 1 from any of those numbers... You get 9, 99, 999 and it's always divisible by 3.**

Alfred: (01:16:49) Ok, so does that answer your question?

Maggie: If I reason that out it does, and then yes this is a whole number.

Alfred: Ok so when you say if you reason that out what do you mean? What do you need to do?

Maggie: Establish that... I'm going to say greater than 1, ... any power of 10 ... the answer will always be made up of 9's as digits... it would be n-1 9's. Then we'll have N-1 3's... And then this is a whole number... **I mean I'm convinced. I don't know if I can convince anyone else.**

Alfred: ... But what about that isn't convincing, what about that do you think isn't convincing to me?

Maggie: I don't that (inaud.) **I think I need to show it... algebraically ... that there will be, what I know is going to be true, that if you subtract 1 from any power of 10 your answer will be made up of all 9's.**

After citing three examples, Maggie made the observation that one less than any power of ten will be entirely composed of 9's. Although she was personally convinced of the observation, when she considered having an audience to convince, Maggie felt algebra was a necessary means of persuasion.

While she ascertained the result based on empirical evidence, Maggie perceived the need to use algebra for the purposes of convincing others. This instance evidenced that for Maggie, ascertainment and persuasion entailed two different standards. Maggie felt that the statement, $\frac{(10^{n-1} - 1)}{3}$ is a whole number for all $n > 1$, could not be defended with words alone, an indication that she believed proofs necessarily entail the use of symbolic manipulation. If such a claim can be supported, it would indicate that her behavior was compatible with the ritualistic form of the external conviction proof scheme in this instance.

Maggie began by pinpointing the difficulty inherent in the problem from her perspective, finding a representation for a number with n 9's as digits.

Maggie: (01:21:07) How would you even show that? N number of 9's... would I have to do it digit by digit? ... **This one is tricky because you go between numbers and then digits of the number.**

Alfred: Absolutely, but, and if you think about how you add decimal numbers, right I mean the algebra...

Maggie: By place value.

Alfred: By place value, right.

Maggie: (01:22:37) So it would be like, this [drawing many squares to represent digits of a large number above] plus this [drawing one square in the units place and filling it with 1 below] and we're saying this will always be 1, these [pointing to squares above] will be 9's... This is the first [labeling columns], this is the second, this is the third and this is the n^{th} 9 [writing the sum, 10...000, below] The first, the second, the third zero, and then this is the n^{th} zero [labeling zeros].

$$\begin{array}{rccccccc}
 & n^{\text{th}} & & 3^{\text{rd}} & & 2^{\text{nd}} & & 1^{\text{st}} \\
 & \boxed{9} & \dots & \boxed{9} & \boxed{9} & \boxed{9} & & \\
 + & & & & & & & \boxed{1} \\
 \hline
 \boxed{1} & \boxed{0} & \dots & \boxed{0} & \boxed{0} & \boxed{0} & & \\
 & n^{\text{th}} & & 3^{\text{rd}} & & 2^{\text{nd}} & & 1^{\text{st}}
 \end{array}$$

Figure 4.15: Maggie adding a number composed of all 9's and the number 1.

Alfred: That's how you add.

Maggie: Mmmhmm.

Alfred: And then you just, you took, figure out a way to explain that in words.

Maggie: **In words?**

Alfred: **I think so.**

Maggie: **But how about, I think algebraically** (inaud.) I mean... If you add, well if you add the 1's digit it's 10 so you... You get 10, so what we usually do when we add is we carry, so you'll get the same number of zeros as 9's and the last 1 that gets carried is put in front so... so we know that N is the number of 9's and N is the number of zeros.

Maggie attempted to represent the addition of a number composed solely of the repeated

digit, 9, with the number, 1. She invoked the familiar process of carrying used to add. Though she was able to point out that the process of carrying would continue (as indicated by the use of "...") until it ended in the $n + 1^{\text{st}}$ digit, Maggie was unable to describe this process algebraically⁴⁵ by her own admission. When Alfred suggested that a worded description would be an acceptable means of completing her proof, Maggie explained that she felt it would not be enough and continued to search for a way to represent the words algebraically.

Alfred: (01:24:32) ... you know I don't know what isn't convincing about that. Maggie: So it's like this [circling figure 1 above] in itself is... the argument... So because this is true, and if you have any power of 10 it's like being here and you take 1 away what will happen? You will have a number that has n number of digits and all of those digits are 9.

...

I always feel like I'm on shaky ground unless I have some kind of you know that perfect little proof, so I always question, is that enough to convince someone you know? ... So now I can go back here and say this is a whole number, this is a whole number, this is a whole number ...

Alfred: ... you just don't look very convinced...

Maggie's prototypical proof seemed to involve being concise or elegant, involving algebraic representations, and symbolic manipulations, at least it would not be represented in words only. Although she was able to return to her original line of reasoning, Alfred pointed out that she still did not appear convinced regarding whether or not she had proven that $(10^{n-1} - 1)$ is divisible by three. Though she was personally convinced, the use of algebra did not appear to be for the purpose of ascertaining the truth of her claim. Rather, the use of algebra was for the purpose of satisfying a ritual that

⁴⁵ Maggie might have been searching for a representation of $10^n - 1$ like the following:

$$10^n - 1 = [10(10)^{n-1}] - 1 = [(9+1)(10)^{n-1}] - 1 = [9(10)^{n-1} + 1(10)^{n-1}] - 1.$$

would guarantee validity.

Synopsis

This episode demonstrates a dichotomy between Maggie's ascertainment and persuasion. She relied on what others thought to reach a conclusion about whether or not she had a complete proof. Though her reliance on others is a form of the external conviction proof scheme it seems to be more closely related to entering a community of practice and coming to build her own model of what is considered sufficient evidence in the PD classroom. Her ritualistic proof scheme was evidenced by her discomfort with a proof written entirely in words.

By no means can it be discounted that Maggie had the ability to faithfully represent salient aspects of the problem symbolically. At many points she questioned the meaning of the symbols she had written and demonstrated control over them. In particular, Maggie's proving can be characterized as referential symbolic with respect to the meaning of x , y , and z in her proof for the $n = 3$ case. Still, Maggie's relationship to symbolic manipulation was problematic at points (see previous statements about ritualistic proof scheme).

Two trends to watch for include identifying Maggie's prototypical proof and watching for her appreciation of case study in the proving process. The latter trend seems particularly relevant to her potential for distinguishing generic proofs from RPG-like proofs. She has previously commented on two occasions that proofs by example are insufficient at ATI. In fact, earlier on the day of this episode Maggie said, "Well you couldn't prove anything if you only use specific cases." In this episode, Maggie demonstrated a growing need to attend to the generalizability of her conjecture in the $n =$

3 case. Her concern about whether or not to continue proving the result in the $n = 3$ case was spontaneous and she was not sure what could be gained by proving the result in one case. As Maggie rejected the RPG proof scheme for persuasion, she may have come to undervalue the role that case study can play in the proving process.

This episode represents the second occasion of Maggie commenting on needing a “perfect little proof.” She seems to have an image of an ideal proof that she is aiming at. This episode gives evidence that this proof is not something that should be communicated in words alone. At least in the context of this problem, Maggie felt that words alone were insufficient. Much more evidence would be necessary to help bring Maggie’s image of the “perfect proof” into focus. However, what is emerging clearly is that Maggie has some image of a proof which she is using as a target. What can be said about her image is that it does not consist entirely of a finite number of examples. While a set of examples is personally convincing for Maggie, she does not consider a set of examples convincing for others.

Extra Note:

The issue of when to prove and when not to prove was addressed on the following day when Bernie presented his proof. He encountered problems similar to Maggie when he tried to decide how to represent the notions of carrying and borrowing algebraically. TR and Alfred presented rough criteria for making the kinds of decisions Maggie was dealing with in this episode.

TR:... this kind of confusion about do I need to prove, or don't I need to prove is a very natural one. You overcome it the more you are exposed to proofs, the more you read proofs, the more you produce proofs, you understand when to prove and when not to prove. And that is very natural, I mean, it's a confusion that is really necessary... each time we will resolve it specifically, until you will extract when

to prove and when not to prove.

Alfred: But there's another subtlety here, which is in when to prove, and when not to prove for yourself, but in the second case it's when to prove and when not to prove to the people you're talking to, and those can have different answers.

TR: (30:54) Right, any time you do a proof, there are 3 stages. First you prove to yourself, then you prove to your friend, and then you prove to the enemy. There are always 3 stages. You have to convince yourself first, and then friend who is easy on you, and then really somebody who is tough... on being obnoxious and trying to trick you. Right, because that is the nature, I want to be completely convinced.

This conversation is relevant in that it documents the fact that Maggie's problematic situation was addressed in a very straight-forward manner. However, it is an open question whether or not Maggie incorporated this framework for making decisions about when to prove and when not to prove.

Episode 8 – A Special Polygon Problem

Question

A polygon has the following properties: A: it has exactly three acute angles. B: the measures of its angles are all positive integers. C: the measures of its angles form an arithmetic progression, difference equals its first term. How many segments does this polygon have?

Background and Significant Events

Immediately before Maggie presented, a participant had presented a solution to the same problem in which she limited the values of a between 23 and 29, inclusively. Checking each of the possible values of a by plugging them into her formula, she arrived at the conclusion that n must be 9. TR asked the class if that solution constituted a proof.

During the stair-like structure problem, William asked Maggie how they could be sure that an observed pattern continued. Maggie appreciated William's question, asking

how they could be sure that the pattern continued for very large cases. In this episode, Maggie demonstrated similar behavior.

Synopsis of Maggie's Solution

Maggie began by representing the angles in the problem using the sequence: a , $a+a$, $a+2a$, ..., $a+(n-1)a$. She rewrote the sequence as: a , $2a$, $3a$, ..., na . Then she made a table representing the sum of the measures of the angles of polygons for 3, 4, 5, and n sides. She noticed that $a < 30$ degrees by using the $n=3$ case. Maggie explained that for triangles, $180=a+2a+3a$. So, the angles are 30, 60, 90. She noted that this violated constraint A (above). Thus, $a < 30$. Next, she noted that $a \geq 23$ from her previous work (this work was not shown).

Maggie solved the formula as $a = [360(n-2)]/[n(n+1)]$ and showed some selected results in a tabular form (see fig. 1 below).

n	a
4	36
5	36
6	
7	
8	30
9	28
10	26.18
11	24.55
12	23.08
13	21.76

Figure 4.16: Maggie's table showing the value of the first angle, given the number of sides of a polygon.

Knowing that $23 \leq a < 30$, Maggie dismissed possible values of n larger than 12 once she saw that the resulting value a was below 23 in the table. She concluded that the only possible solution was a nonagon.

Claims

This episode demonstrates an instance in which Maggie first demonstrated behavior compatible with the RPG proof scheme and then later rejected an RPG-based solution. She rejected her result-based generalization after TR asked whether or not her solution was a proof. As such, her rejection of the RPG-compatible solution was not spontaneous. However, it does demonstrate that Maggie is in a state of disequilibrium with respect to the RPG proof scheme. Maggie's doubt was rooted in her observation that

the sequence was not strictly decreasing. This eventually led her to doubt her result-based conclusion about the pattern.

Analysis

Once Maggie had concluded her presentation of the solution outlined in the synopsis section above, TR asked whether Maggie's presentation constituted a proof.

This question had been asked previously about a solution earlier in the day.

TR: Question, does Maggie's presentation up to this point constitute a proof? ...

Maggie: Well I said it was but **maybe not** because it's like um, n is really the answer that we're looking for cause we want to know how many line segments are in that polygon and just because this fit [motioning along the “ a ” column] here doesn't mean that I've exhausted [motioning along the “ n ” column] all possibilities, **what if there's some line seg-, number of line segments way beyond that**, that still... Because if you look at this here this, like if you put the number three in “ a ” is thirty, **it's increasing and here it's starting to decrease, what if it doesn't have that cycle again...**

...

TR: Ok, so all this is wonderful thinking and you know, she's in the direction of completing the proof but **Maggie found a problem with her own proof. She's saying how do I know, maybe if I continue here somewhere along the line I'm going to get a good number. It seems like it is increasing but who knows, right? So what, how would you fix this** Maggie, what would you do?

Maggie: I would probably go back and...

TR: No, no, no, no, you're abandoning yours?

Maggie: Well, I mean I, I started doing it out of laziness, I didn't want to...

TR: No, no but how can we fix your, how can we not even fix, how can we complement this? What is necessary, what additional step needs to be done so that we can say, yeah, this is a different approach and it's complete approach different from that one. Yes.

Maggie's response indicates that she had a change of heart. That is, she had at some point been convinced by the results in the table. Because she saw the values of a decreasing, she thought the pattern would continue. However, after TR asked if her presentation constituted a proof, she changed her mind, citing the possibility that the pattern might change from decreasing to increasing for some large value of n . Maggie explained that at

the top of the table she saw the values increasing from $n = 3$ to $n = 4$. Then she saw them decreasing.

At this point, Maggie lost confidence in her argument. It is unlikely that she doubted her final conclusion because it had previously been shown that her conclusion was correct. It is more likely that Maggie merely doubted that her argument was insufficient. The fact the pattern changed from increasing to decreasing played an important role Maggie's decision to reject the RPG-like solution. Still, Maggie did not have a way to explain why the pattern was changing or how she could determine that the pattern was decreasing after $n = 3$. TR pointed out that Maggie's approach did not have to be abandoned. Later, he presented a proof that Maggie's function was decreasing for $n = 3$.

Synopsis

The question, "How do you know the pattern will hold for a large value of n ," is a recurring theme for Maggie. In the stair-like structure problem, it was noted that this question is in Maggie's zone of proximal development. Again, it is noted that TR's question seemed to have been a catalyst for Maggie to raise this question. For this reason, this episode does not constitute an instance in which Maggie rejected an RPG based solution spontaneously. However, it does demonstrate that Maggie is in a form of disequilibrium and that she can point out the source of her doubt.

Another reoccurring theme for Maggie is that asking, "How do you know the pattern will hold for a large value of n ," does not provide her with a way to determine whether or not the pattern will continue. That is, rejecting an RPG based solution does

not mean she is demonstrating behavior compatible with a PPG proof scheme. The source of Maggie's doubt, the seemingly cyclic nature to the function, will be addressed below. TR modeled how to address the question. The latter teaching action is important both mathematically and pedagogically. In terms of Maggie's proof schemes, this episode shows that rejecting the RPG proof scheme is not the same as instantiating the deductive proof scheme. Maggie has shown on several occasions that she is not clear how to turn an empirical solution into a deductive solution, even as she rejects the empirically based solution.

In terms of pedagogy, this is an instance in which TR modeled an important teaching practice, that of completing a student's solution in a deductive manner for the purpose of addressing a need for causality. Observing TR's teaching practice gives Maggie an image of how a teacher instantiates a student-centered approach to instruction in mathematics.

On a final note, it has already been mentioned that Maggie rejected a proof that did not explain why the pattern must hold. However, it is important to revisit the reason she decided that the function might not continue to decrease. For Maggie, it was compelling that the function had previously been increasing. The argument that the function may not always decrease because it has increased in the past is itself an instantiation of the RPG proof scheme.

Episode 9 – A Sequence of Sequences

Part 1 – When the common difference is 1

Question

Take the arithmetic sequence of positive integers, 1, 2, 3, Divide it into the sets, (1), (2, 3), (4, 5, 6), (7, 8, 9, 10), You get a sequence of sets in which the first set consists of one item, the second of two items, the third of three items, and so on.

- How many items are there in the 55th set? What is their sum?
- How many items are there in the n th set? What is their sum?
- Which set contains the number 1000?
- What is the first set the sum of whose items exceeds 1000?

Background and Significant Events

This episode transpired approximately 2 weeks into the second summer. For the entire time, participants worked in the context of a unit on arithmetic series. As in other cases, TR's teaching practices played an integral role in the development of Maggie's proof schemes. TR was observed asking the two familiar questions "Are you satisfied with your answer," and "How do you know that this always happens?" However, in this instance the questions served a different function. Rather than acting as perturbations, these questions functioned primarily to summarize points Maggie had already made. Again, the focus of this analysis remains Maggie's proof schemes rather than TR's teaching actions.

Another reemerging theme is Maggie's justification of the formula,

$$1 + 2 + \dots + (n - 1) = \left(\frac{n-1}{2} \right) (n), \text{ by means of pairing terms.}$$

Synopsis of Maggie's Solution

Scene A: Maggie persuaded Joe that $1 + 2 + \dots + (n - 1) = \left(\frac{n-1}{2} \right) (n)$.

In the process of explaining her answer to the fourth question, “What is the first set the sum of whose items exceeds 1000,” Maggie generalized a way to find the sum of the terms in any set. Her strategy involved applying the formula, $S_n = \left(\frac{n}{2}\right)(a_1 + a_n)$, for finite arithmetic series where S_n is the sum of the terms in the n^{th} set, a_1 is the first term of the set, and a_n is the last term of the set⁴⁶.

In her explanation of how to find a_1 , Maggie told Joe that she could use the last term in the previous set. Maggie used the formula, $1 + 2 + \dots + (n-1) = \left(\frac{n-1}{2}\right)(n)$ to find the last term in the $n-1^{st}$ set. She explained to Joe that $1 + 2 + \dots + (n-1) = \left(\frac{n-1}{2}\right)(n)$ was true because of her ability to pair the terms in such a way that there would be $\left(\frac{n-1}{2}\right)$ pairs each adding up to n . Recall that in the stair-like structure problem, she had previously attended very carefully to the cases where n is odd.

Maggie continued by clarifying that the first term of the n^{th} set is one more than the last term in the $n-1^{st}$ set and that the last term in the n^{th} set is n more than the last term in the $n-1^{st}$.

Claims

In her small group work, Maggie explained how to find the sum of the series $1 + 2 + \dots + (n-1)$. She returned to her previous WoU how to justify a formula by pairing terms in the series. Her persuasive approach was spontaneous, indicating that it was her

⁴⁶ The notation used here is Maggie’s. It is problematic because it does not account for the fact that a_i will take on multiple values for $1 \leq i < n$ whenever $2 \leq n$. Later, Maggie addressed this issue with TR as a source of concern about her proof.

own. Maggie's explanation for why $1 + 2 + \dots + (n - 1) = \left(\frac{n-1}{2}\right)(n)$ demonstrates a global change in her WoU how to compute finite arithmetic series. Entering an equilibrium phase constitutes an indication of a more permanent change in proof schemes.

Further evidence exists that Maggie was developing a stable deductive proof scheme in the context of arithmetic series. Though she demonstrated RPG compatible behavior in her problem solving approach, Maggie only used it to create a conjecture. Her ability to pinpoint the RPG-based result, question it, and eventually justify her result spontaneously in a deductive manner goes beyond mere rejection of the RPG proof scheme. Though TR was initially involved in the reformation of Maggie's WoU the problem, as demonstrated in her comment on Sara's solution, Maggie internalized the content of her discussion with TR, providing evidence that his comments were in her ZPD. Ultimately, her comments were compatible with the deductive proof scheme.

Analysis

Scene A: Maggie proved that $1 + 2 + \dots + (n - 1) = \left(\frac{n-1}{2}\right)(n)$ in a deductive manner.

Maggie noted that the final term in set n is $(n(n+1)/2)$.

Maggie explained to Joe how to answer the question, "What is the first set the sum of whose items exceeds 1000?" Maggie noticed that Joe had solved problems #1 and 2 by plugging numbers in, but did not generalize his thinking to find the sum of the numbers in any set.

She explained how to find the sum of the terms in any set by noting that each set is a finite arithmetic sequence. As such, a familiar formula could be applied. Then she

asked how to get the first term in the n^{th} set. Her description entailed finding the last term of the set before the $n-1^{\text{st}}$ set, and adding one. Maggie re-explained how to add the sum of the terms in an arithmetic sequence. Maggie carefully revisited the meaning of each term in her formulas.

Maggie: ... **If I want to find the sum of all these numbers in order 1, 2, 3, 4, all the way to 54, that you would use, $n+1$ times $n/2$, and what does this mean to you?**

...

Maggie: **And each being, when I look at it, each thing means something to me**, like this is where I started, at number 1 and then I counted up to the number n , and then when I add that, 1, 2, 3... whatever up to n , when I add n , and I add 1, it's going to equal something [drawing a line connecting 1 and n]. And then, when I add the number right before $n-1$, 2, [drawing a line connecting $n-1$ and 2] these are going to equal the same thing. So if I get the sum, how many pairs will I have or how many terms will I have. I have n terms, but I just paired them all up. So I divide it by 2, right. So every time I look at this [meaning the formula $(n/2)(1+n)$], I go, this [pointing to n] is how many terms I have. This is the first number [pointing to 1] I started with and this is the number I ended with [pointing to n].

...

Maggie: So then when I look at what you did right here, **there is a reason why it's in this form... and now I know that this is going to work because we've proven this now in the past couple of weeks.**

She operated in a manner that emphasized the meaning of variables as she worked with Joe, behavior that is compatible with the referential symbolic proof scheme. She attempted to explain why her actions and interpretations were valid and how she knew that what she was doing was correct. In her own language, Maggie repeatedly re-explained the meaning of the formula.

Scene B: Maggie points out one source of her uncertainty about how to defend her WoU how to find the first term in any set.

TR asked Maggie where and why she was not satisfied. Maggie was not entirely satisfied with her answer to the question, “What is the sum of the terms in the 55th set?” TR asked how Maggie knew that for any set, a_1 is always 1 more than the sum of the cardinalities of all the sets before. This was a point of discomfort for Maggie because her method was based on a set of results without understanding why her observed pattern held.

TR:(00:49:30) How are we doing here now? So share with me, now you have a proof that **you are satisfied** with?

Maggie: **No.**

TR: Ok, so let's go over it, where and **why you are not satisfied.**

Maggie: I'm not satisfied with, ok this is the first one on a specific case right.

TR: (00:50:10) ... how did you think about it?

...

Maggie:... I knew that if I added, like I was looking at this case here [pointing at the 5th set], how will I get 11? To begin my 5th set... if I added all of these numbers [circling the position of the first 4 sets]

TR: Right, so then you get 11, because you can calculate and see?

Maggie: Right, because this equals 10 [circling 1,2,3,4] so this [pointing to first term in 5th set] is 1 more than...

TR: **Right, so how do I know that this always happens? That the first term in the nth sequence is the sum of all the terms before it? Was this one of the things you were uncertain about, or this one you were certain about?**

Maggie: (00:52:29) Yes, it was one of the things I was uncertain about, I mean I did everything based on that, but there is still that uncertainty.

TR: What's the source of the uncertainty?

...

Maggie: **Because well I looked at this here and I said well, and then I went and looked at these other cases, for 4, will it start with 6+1, for 3 will it start with... so it's still specific cases, it's not...**

TR: (00:53:05) And you haven't found a more general way?

Maggie: No [hesitantly].

Maggie was able to pinpoint the source of her discomfort. Maggie's comments indicated that she had already anticipated TR's question, “How do you know that this always happens?” She understood that her proof was based on this result and that she could not justify her result beyond citing a few cases in which it held. Furthermore, Maggie was

uncomfortable with this kind of justification and identified it as the source of her uncertainty. This is an indication that Maggie saw the need for the deductive solution TR explained next.

TR presented Maggie with another student's proof that the first term of the n^{th} finite sequence is $1 + [1 + 2 + \dots + (n-1)]$. As TR shared the solution, Maggie commented about why she thought the solution was correct, what was at the heart of the solution.

TR: Ok, I'm going to tell you a general... way, I'm not sure whether it's a good one or not.

Maggie: Ok

...

TR: (00:53:47) So he told me that the way he thought about it is that to get here, ok, I need to pass 1 term, which is I need to pass 1 term, and then I need to pass another 2 terms, I need to pass another 3 terms another, this is the 5th, another 4 terms ok. Um, this is the number of terms I need to pass. Ok to get here, just get the last one here, right, and then to all of this I need to add, 1.

Maggie: Right.

TR: (00:54:31) Because the terms are 1, 2, 3, 4, 5, 6, 7, 8, so I need in order to know what this number is going to be he told me, I need to simply say to myself, well to get here I will go, 1 right, and then another 2 terms, and another 3 terms, another 4 terms and so on. And then to the whole thing I will add 1.

Maggie: (00:55:05) Right, **so is it because you know that these numbers go in order, increasing by 1, that, that they will be the same number in the end?**

...

Maggie: (00:55:43) **They correspond directly with...**

TR: Right, so I just count right, and this number, well what do you think about it, that's what he told me?

Maggie: Yeah, I agree with that.

TR: No I understand you agree, **does it have a different status from your explanation?**

Maggie: **Definitely, yeah.**

TR: **How so?**

Maggie: **Um, because um, because of these conditions given, you know that this increases by 1, it's just basically counting, um, we know that the number in the end, whatever you get after getting this sum will be that last term there because they are going to correspond directly.**

When TR had finished, Maggie explained that she had already thought about the explanation TR gave. For this reason, she claimed that she was comfortable and now felt

confident about that part of her solution. Maggie's later comments about Sara's solution indicate that she fully understood what TR had told her.

Scene C: Maggie's comment on Sara's solution – behavior compatible with the deductive proof scheme.

Sara, another participant was asked to present her solution to the class. When she had finished, Maggie explained that something was missing in Sara's solution. She noted that the formula for determining the first term in any set depended on the fact that when the terms of all sets are considered as terms in an infinite arithmetic sequence, the common difference of that sequence is 1. This meant that the position and the value of each term in the infinite sequence matched. She explained that the fact had to be attended to⁴⁷.

TR: (... was everything that Sara explained and presented things that you felt comfortable with mathematically?

...

Maggie: ... the concern is in this part ... So we have the sequence that we are considering but we also are considering, the position of the set. ... Like this is the first set. This is the second set... This is the 3rd and this is the 4th. If we call this the position of the set, see she changed her words ... the number of terms in the set, one term in the set, 2 terms in the set, 3 terms in the set, and that kind of helped me to see the correlation a little bit better; where I was thinking depending on the position of the set, or set number. I just saw that I could use this, but why can you do that? The way that I resolved it or I got help in resolving it was that um, let's say we're going all the way up to the 4th set for now, ok. In order to get to the 1st term in the 4th set what's happening here? What you had to do was you had to pass one number, and then you had to pass 2 numbers and then you had to pass 3 more numbers which is the sum of all of those numbers, $1+2+3$ and **just because we were lucky enough to get a sequence that increases by 1**, where we're counting in the same way that we're counting. The number of terms that we have passed it also corresponds to what number you land at here. So if I say I passed 1, and then I add 1, I pass 2 more I add that that's 3. I pass 3 more I add that. That's 6. It corresponds with this number here [the last term in each set]. **Whereas the term number here wouldn't necessarily correspond with the**

⁴⁷ This led TR to create a problem in which the terms of the infinite sequence were odd numbers ($d=2$).

value here if we weren't working with a sequence that increased by 1.

It is important to note that in addressing Sara's solution, Maggie used the explanation TR had given her earlier to make a broader point. Maggie was explaining that Sara had not distinguished some aspects of the problem that were coincidentally the same. However, for Maggie, the fact that overall sequence of numbers had a common difference of 1 allowed the position and the value of the last term in any sequence to correspond to the sum of the number of terms passes to get there. Pointing out, and filling in, this gap in Sara's solution is behavior that indicates Maggie internalized her discussion with TR. Such behavior is also consistent with the deductive proof scheme.

Synopsis

Though Maggie created a formula to find the first term in the n^{th} finite arithmetic sequence through means compatible with the RPG proof scheme, when TR asked her what she was uncomfortable about in her proof, Maggie pointed out the formula as a point of discomfort. It was not the formula itself that she was uncomfortable about, meaning that Maggie did not have doubt in the validity of the formula. Rather, she was uncomfortable with her ability to defend it. Though TR showed her how to resolve her uncertainty, Maggie was able to internalize his explanation and use it in a novel situation during her critique of Sara's solution. Maggie saw the relationship between the position and the value of the terms in an overall infinite arithmetic sequence from which the terms in the finite arithmetic sequences were created. Her ability to point out the relationship between the position of terms within the overall sequence and the value of the terms in a case where they were the same was an important contribution to the class discussion. Maggie was also able to coordinate ordinality of sets with cardinality of those sets. This

quantitative reasoning in which Maggie attended to attributes of sets and their terms is indicative of a deductive proof scheme.

Part 2 – When the common difference is 2

Question

Take the arithmetic progression of the odd integers, 1, 3, 5, Divide into the sets, (1), (3,5,7), (9,11,13,15,17),.... You get a sequence of sets in which the first set consists of one term, the second of three terms, the third of five terms, and so on.

- a. How many terms are there in the 178th set? What is their sum?
- b. What is the sum of the terms in the n th set? What is the sum of all the terms in the first n sets?
- c. Do sets with the following property exist:
 1. The sum of the terms in the set is 10,000?
 2. The sum of the terms in the set is 30,071?
 3. The sum of the terms in the set is even?
 4. The sum of the terms in the set is a perfect square?
- d. The sum of the terms in a set exceeds 5,234. What is the position of this set in the sequence?

Background and Significant Events

This analysis focuses on Maggie's solution to part b. Even as she was attempting to solve part a, Maggie addressed the more general question first, noticing later that she had been addressing the more general question simultaneously. The question was an extension of the previous question, based on Maggie's comment on Sara's presentation that the class had gotten lucky that position and value were the same because the

common difference was 1. TR changed the situation so that the common difference was now 2 and the cardinality of sets would also increase by 2.

Maggie had worked with the same participants, Joe and William, to answer the previous question. Though she began working on the new problem the day before, Maggie did not enter the class session with a completed solution. William had arrived at some conclusions and much of the time was spent investigating his conclusions. Still, there is consistent and overwhelming evidence to support the claim that Maggie had internalized the co-constructed solution.

It is an important observation that this problem required Maggie to coordinate three different attributes of terms and two attributes of subsequences in an infinite sequence of subsequences. One attribute of a term is its position – both within the infinite sequence of odd positive integers and within a particular finite subsequence of the infinite sequence. The other attribute of a term is its value. Finite subsequences have ordinality and cardinality. The need to coordinate these attributes of each term (or subsequence) created a need to communicate precisely about which attribute Maggie was referring to at any particular point in time. Furthermore, the ability to coordinate and communicate accurately is essential in the construction of any deductive solution.

Synopsis of Maggie's Solution

Early on, it was established the group's overall strategy would be to view each set as a finite arithmetic sequence. As such, Maggie explained that she could use a familiar formula to find the sum of the terms in any set. Next, the group set the goal of the determining the values of the first term in the n^{th} set. From there they would find the value of the last term in the n^{th} set. Knowing the cardinality of the n^{th} also proved to be a

non-trivial exercise, as the ordinality and cardinality of the n^{th} set did not match, requiring another layer of coordination.

Maggie and her group mates first determined the value of each term as a function of its position within the infinite sequence of odd integers. Next, Maggie determined the position of the first term in the n^{th} subsequence, $a_{n,1}$, within the overall infinite sequence of positive odd integers as a function of n . This was accomplished by finding the sum of $[1 + 3 + 5 + \dots + (2n-3)] + 1$, where $2n-3$ is the cardinality of the $n-1^{\text{st}}$ set. Maggie noted that $[1 + 3 + 5 + \dots + (2n-3)]$ is an arithmetic sequence. Using a previously established formula, she found that $a_{n,1} = 2(n-1)^2 + 1$.

Having determined the value of the $a_{n,1}$, the group found the value of, $a_{n,2n-1}$, the last term in the n^{th} set. Maggie's approach was to subtract 2 from the value of $a_{n,1}$ and add $2(2n-1)$. She explained that she began with $a_{n-1,2n-3}$ so that the number of repeated additions necessary to arrive at $a_{n,2n-1}$, starting at $a_{n-1,2n-3}$, would equal the number of terms in the n^{th} set, arriving at $a_{n,2n-1} = 2(n-1)^2 + 4n - 3$.

Finally, Maggie used all of her results to write a function representing the sum of the terms in the n^{th} set: $S_n = \frac{2n-1}{2}(a_{n,1} + a_{n,2n-1}) = (2n-1)(2n^2 - 2n + 1)$.

Claims

Early on, as William explained his initial approach for finding $a_{n,1}$, TR reminded him and all other members of the group that he would be asked how he knows his pattern will continue to hold. While working with William, Maggie took this possible attack on the solution heart. Though TR had raised a concern about the solution, after he left

Maggie returned to it, asking William what his defense would be. When he offered a defense, Maggie asked him if it was enough. This continues Maggie's trend of asking when is an argument enough to be deemed a proof. Though this is not a claim about proof schemes per se, it is an important characterization of the type of disequilibrium Maggie faced at this point.

Later, Maggie offered her own explanation for how the position of an odd number (within the overall infinite sequence) could be determined given its value. Maggie's explanation can be characterized as deductive in nature. Other instantiations of deductive reasoning occurred when Maggie proved that the position of $a_{n,1}$ is given by $(n-1)^2 + 1$ and when she compared her WoU how to find the value of $a_{n,2n-1}$ to William's.

Additionally, Maggie's overall solution as described in the preceding section can also be characterized as deductive.

Analysis

Early on the William and Maggie devised a problem-solving approach to the first two parts of the question which entailed defining a function that would give the value of $a_{n,1}$ in terms of its position in the overall sequence. However, initially William sought to find the value of $a_{n,1}$ as a function of n directly. William proposed the following:

Let n be the position of a set (e.g. if $n = 3$, referring to (9, 11, 13, 15, 17)) and $a_{n,1}$ be the value of the first term in the n th sequence (e.g. $a_{3,1}=9$). The number of terms in the n^{th} set $= 2n-1$.⁴⁸ If $n = 1$, $a_{1,1} = 1$.

For $n > 1$, $a_{n,1} = 1 + 2[2(2-1)-1] + 2[2(3-1)-1] + \dots + 2[2(n-1)-1]$.

⁴⁸ This statement was offered without justification.

William's function counted the number of times 2 was added to 1 to arrive at the desired term. As the function is read from left to right, it keeps track of twice the number of terms in each set before the n^{th} set, adding this value to the initial value, 1.

Though the function was general, William's explanation to TR made use of specific cases and seemed to be based on a few results, prompting the following comment on TR's part.

“[If] you find $[a_{n,1}]$ based on these particular sets, is this going to be sufficient? Will you be happy to continue? ... I am going to ask you, how do you know this is going to happen all the time?”

Note that William's function gave the value of $a_{n,1}$ as a function of n directly without first computing the position of $a_{n,1}$.

During his attempt to explain his approach, William changed his mental imagery to one in which the position of $a_{n,1}$ could be found first. The position could then be used to find the value of $a_{n,1}$. Maggie seized on this image saying,

“I think an important question will be... How many terms do you have to pass to get the first term [of a given set]? ... How many terms do you have to pass? And then that gives you the position and then you have to decide what to do with the position.”

This question was brought up by TR earlier, but Maggie paid special attention to this question, making it the focus of her investigation when TR left. Recall that in her comment on Sara's solution previously, Maggie mentioned that position and the value of a term were coincidentally the same in the previous question. She said that the class had been “lucky” in that instance because the common difference of the arithmetic sequence was 1.

After the group switched to the new strategy, William pointed out that for all terms in the overall sequence, the value is one more than double the value of the position. Given the group's conversation with TR, Maggie wanted to know whether or not RW's support for his way of computing the value of $a_{n,1}$ could be defended.

Maggie: So, is he saying why is, do you know that's true for sure or and you're saying that because it's the nature of the sequence, that's the way it's gonna be, it's gonna be 2 times that position minus 1.

William: Yeah.

Maggie: So that's not something that can be disputed.

William: (22:16.1) Because that's, yeah that's the definition of an odd number. An even number is $2n$ and one less than that even number is going to be $2n-1$, right.

Maggie tried to anticipate which part of the argument could be disputed and whether or not William believed he had a defense for any attack.

Though the group continued to struggle with the distinction between a term's position and value, was able to coordinate the two successfully and correct William as he made mistakes.

Maggie: Did you say add one or subtract one, if you're finding the, if you know the position and you're finding the value?

William: If I know the position I am finding the value, put the approach we're taking and the approach that we used last time, is find the position prior to it and then add to it. Except, now we're adding 2.

Joe: Yeah.

Maggie: For the position?

William: For the position where any one.

Maggie: You add one.

William: So.

Maggie: So if you have that position already you say. You're saying you just take the position number, if it's in the 15th position or multiply it by 2 and subtract 1 and that gives you the value for that number.

William: Yes, if you're working from the position. If you know the position.

Maggie: So it really looks like you're working with that $2n-1$ in 2 different situations and it's almost like.

William: Yeah, that's, that's that double layer that gives us the...

Maggie: quadratic.

William: Yeah.

...

William: Yeah. So to get the position of the number, of any number, of the last digit of each set, I am doing this. I am just incrementing n by 1. Alright, to get the position of the next number, I've gotta add 1, that will give me the position, that will give the value, I've gotta take that position times 2, so everything in the brackets right here, gets multiplied times 2 and **I am gonna add 1**.

Joe: We add one or we add 2?

Maggie: We subtract 1.

William: This gave me the position of the number I want to get.

Joe: But, but,

William: So now to get the value, the **position here is 10 times minus 1**, so the only thing that's gonna change in this picture as we go for the next set is I am gonna add another set of brackets inside. So to get the next set, I am gonna do this one more time. **I am gonna add in 2 times 4 minus 1** in there, right, would you agree.

Maggie: **Uhuh.**

Maggie's ability to connect position and value of a term at the same time as she considered the cardinality of sets allowed her to construct a reason why she believed the value of the term should be a quadratic function. While Maggie did not cite a reason to believe the relationship between the two "situations" was multiplicative, this instance does highlight her ability to effectively coordinate the meanings of multiple variables as support for her claims, demonstrating behavior compatible with the referential symbolic proof scheme.

Maggie also provided an explanation of how to find the position of the first term in the n^{th} subsequence by comparing the current problem with the previous problem. Though the situation changed, she was able to apply the same methods in a new situation. When questioned by William, she explained that she had already addressed the problem of generalizing her formula for computing the sum of a finite arithmetic sequence to sequences having a common difference different than 1.

Maggie: You have to look at the same thing like yesterday. This position right here [pointing to $a_{n,1}$] is going to be basically depend on how much many numbers

you had in the past... You pass one number, you pass 3, 5, 7, 9, all the way.
Until... the last term of the $n-1$ set.

TR: Ok.

Maggie: So I had to put it in different terms and say like if this is position x [pointing to $a_{n,1}$], I don't know if you want to bring in more variables. This is position $x-1$ [pointing to $a_{n-1,2n-3}$], cause this is totally in order. Like the counting numbers. 1, 2, 3, so this is one number before this number and to know what this number is, we have to know.

TR: We don't want to know what this number is, we want to know the position.

Maggie: We want to know.

TR: So this, this, all the time concentrate on what we want to do, because we have so many things going on.

Maggie: Right.

TR: We only want the position.

Maggie: The x position, then, this is as far as I've gotten. What we need to do is we need to find, how many number we've past to get to this.

TR: Ok.

Maggie: To get to this x position and we're gonna know because the way the set works is that in each set increases by 2 from the set before, so its going to be starting with 1, 3, 5, we need to find the sum of basically the odd integers up to what ever, what ever number this $n-1$ set contains.

...

TR: How many terms do we have in the $n-1$ set?

...

Maggie: We have two less than the n th set, but that's what I am working on, I mean if we can say what this is [referring to cardinality of the $n-1^{\text{st}}$ set].

...

TR: How many terms do we have in the n th set?

Maggie: n terms.

TR: n terms in the n th set. So for example in the third set.

Maggie: No, I am sorry. we have $2n-1$.

TR: we have $2n-1$ terms, right, so let's establish that, $2n-1$ terms... How many terms do we have? ...

Maggie: So it would be $2n-3$.

TR: How did you do that?

Maggie: Because we know that the progression is by $2s$, so it contains 2 less.

...

TR: So here, we are saying how many, what the addend here would be.

Maggie: So, how many terms do we have here [pointing inside the parentheses of $1+3+5+\dots+()$]? ...

...

Maggie: $2n-3$ terms.

The previous transcript demonstrates the difficulty Maggie had to overcome in order to distinguish the position and value of a term, as well as the ordinality and cardinality of a set. Once she was clear about the cardinality of the $n-1^{\text{st}}$ set, Maggie was able to write a function identifying the position of $a_{n,1}$ as a function of n . She found that position of $a_{n,1}$ can be represented by

$$[1 + 3 + 5 + \dots + (2n - 3)] = \frac{n-1}{2}(1 + (2n - 3)) = (n-1)^2.$$

From previous analysis, it has been shown that Maggie was not simply using a memorized formula without understanding.

Rather, she understood in a deep way why the sum $[1 + 3 + 5 + \dots + (2n - 3)]$ could be represented as $\frac{n-1}{2}(1 + (2n - 3))$.

Later, Maggie explained that there must be $n-1$ terms in the sum because each term represents the cardinality of a set from the first to the $n-1^{\text{st}}$ set. That is, Maggie described a one-to-one correspondence between the terms in the sum and the cardinalities of the sets. This was done because Maggie did not know how many terms there were initially. Her behavior highlights the depth of WoU the formula she had created for computing finite arithmetic series.

In order to truly appreciate Maggie's explanation of why the number of terms in the sum is $n-1$, her behavior can be contrasted with Joe's. Joe encountered difficulty on several occasions because he used n for the number of terms at all times. The formula written by Maggie and others was written in the following form: $S_n = \frac{n}{2}(a_1 + a_n)$. Joe had consistent difficulty reinterpreting this formula as it was used in different ways throughout the problem while Maggie consistently considered the meaning of the n as the

number of terms being added. In several instances Maggie used x rather than n to maintain consistency.

As the group returned to the overall question of finding the sum of all terms in the n^{th} set, Maggie continued to demonstrate deductive reasoning, maintaining attention on the meaning of the variables she was manipulating as she found the value of $a_{n, 2n-1}$.

Maggie: So my thinking was just, I am gonna take 2 away from this [referring to $a_{n,1}$], because I know that's gonna leave me at the value of the last term of the set before.

William: Ok.

Maggie: Cause if I take two away and then I am just gonna add on the number of terms [in the n^{th} set] times 2 which is $4n-2$. So, it all worked out to the same, you're thinking was, I am gonna take the value of that first term and then I am going to add, 2 times the number of terms, but then wait, I've gotta subtract 2.

William: Yeah, ok.

Maggie: Because, or you said I have to subtract 1, from the number of terms, so it comes out the same, that thinking is the same I think for the last term.

When William explained that they did not have the same expression, Maggie was able to support her contention that the values of their expressions were the same although the forms of their expressions differed. To support her claims she compared the mental images that guided her and William to their respective symbolic representations of the problem. Harel and Sowder (2007) have explained how this is a valuable WoT in mathematics. In this case, Maggie used this algebraic invariance WoT to support her claim, evidencing a deductive proof scheme.

While considering the second part of question b, "What is the sum of the terms in the first n sets," Maggie addressed the following question. Given the value of a term, how do you find its position in the overall sequence of positive odd integers? Initially, Maggie had difficulty determining the inverse function she needed to find position as a function of value though she understood how to value as a function of position. She was unclear

whether she should add or subtract 1 during the process of inverting her previously known function. Finally, Maggie explained that if you add 1 to the value of a term and then divide by two you can find the position of the term in the overall sequence. TR asked why? She gave the following explanation:

Maggie: Because for odd numbers, I am counting by 2. So I know I'll eventually divide by 2, but I have to decide how to make up for that odd number. Do I take away one or do I add one. I should add one, because I want to go to the next higher number so that I can make sure and include that odd, rather than exclude it as one of my terms. So I want to add 1, to finish off the pairs, you know, like 2, 4, 6, 8, they're all paired. So I want to finish off the pair and then divided by two, so I want to add 1 and then I divide by 2.

To explain why her procedure worked, Maggie envisioned a correspondence between the infinite sequence of positive even integers and their odd counterparts. She understood that the value of even integers could be used to find their position when they are viewed in terms of repeated addition. Division by 2 returns the number of repeated additions that took place. Thus the position of a positive even integer can be determined easily. Next Maggie considered two ways to transform any given odd integer into an even integer and she explained why she would choose one and not the other. Finishing off a pair by adding 1 entails mental imagery in which each odd number is tied to an even number in a set whose position was shared by both the selected odd number and the next higher even number. Therefore, knowing the position of the next higher even number reveals the position of the odd number in question. This explanation satisfies the criteria of transformational proof schemes: generality, operational thought, and logical inference.

Synopsis

One final note must be mentioned which can not be conveyed by analyzing the parts of Maggie's argument. Analysis of the argument in its totality reveals two striking

features of her argument. First, Maggie's argument maintained the meaning of the symbols and operations used throughout. Given the circumstances of the problem (i.e. – need for coordination) Maggie's ability to operate with meaning throughout is remarkable. While a deductive solution to the problem does call for this kind of behavior, there are alternatives. Joe's behavior represents a different possibility. It was possible to not create a solution or to rely so heavily on formulas (and on others) that the solution is not your own. Though she struggled to coordinate attributes of various quantities, Maggie maintained the meaning of variables in the context of the problem at all times. She exhibited control over the symbols, operations, and transformations of forms, carefully maintaining the values of expressions as she changed forms.

Maggie was also able to return to her strategy even after periods of intense work on particular parts of her solution. She was guided by an approach which entailed a rich understanding of her formula $S_n = \frac{n}{2}(a_1 + a_n)$. Maggie reinterpreted the meaning of the variables in the context of finding the sum of cardinalities of sets and the sum of a particular set. The ability to reinterpret the formula and return to an overall plan, after periods of intense investigation into smaller parts of the problem, and unite several results is itself an integral part of deductive reasoning.

What is clear from the first two weeks of the second summer is that Maggie has learned a valuable tool, the formula $S_n = \frac{n}{2}(a_1 + a_n)$. However, how she has come to understand the usefulness of the tool, the context in which this tool can be used, and the valuable experience she has gained in learning how much depth is necessary for an argument to convince participants at the PD are important parts of the development.

Finally, it appears that Maggie was in a state of relative equilibrium in this episode. There was only one part of the solution that she seemed uncomfortable with momentarily. Her justification for the validity of the function that finds position of a term as a function of its value seemed to fill that small whole in a meaningful way since it would have worked in both directions.

Episode 10 - Compound Interest Problem

Question

The following 8 questions were assigned in close proximity. Questions 12-16 were assigned all at one time in the morning, while questions 17-19 were given together in the afternoon of the same day after presentations on questions 12-16.

12. Suppose you have a principal of \$12,000 invested at 11% interest rate per year. Determine the amount in the account after 10 years if the compounding period is: (a) annually, (b) quarterly, (c) monthly.
13. Suppose that you will need \$23,000 to pay for a year of college for your child 18 years in the future, and you can buy a certificate deposit whose interest rate of 10% compounded quarterly is guaranteed for that period. How much do you need to deposit?
14. Suppose that the monthly statement from a fund you have in a Mutual Fund reports a beginning balance at \$17,396.17 and a closing balance of \$21,034.25 for 29 days. Using this rate compounded daily for a year, what will be the total amount in your account at the end of one year?

15. Mary has two accounts, A and B. In January 1, 1993, Mary invested the same amount in each account. Account A pays simple interest at 10% per year. Account B pays 10% per year compounded yearly.
- a. How would you help Mary compare the amounts in each account at the end of each of the first 10 years?
- b. In December 30, 2003, Mary had in one account 3000 more than in the other account. What was Mary's investment?
16. You are to determine the amount in an account after a certain compounding period. What information would you seek? How would you use this information to determine the amount?
17. For 12 months beginning January 1st an individual saves \$120 per month, deposited directly into her account, on payday the last day of the month. The account earns 6.5% compounded daily, what is the total earning at the end of the year, 30 days in a month, 360 days in a year.
18. Suppose that a couple wishes to save for the college education of their child. They want to begin saving a fixed amount per month after their child is born so that their child will have \$100,000 available when she turns 18. How much do they have to save each month, if their account earns 6.8% per year, compounded daily?
19. Sally buys a house for \$275,000 with a loan that she will pay off over 30 years in equal monthly installments. Suppose that the interest rate for her loan is 8.5%. What is Sally's monthly payment?

Background and Significant Events

The events of this analysis took place over the course of three class sessions (days 33 and 34 of the institute). They follow two weeks of problems focusing on arithmetic sequences and series. Though these problems focus on geometric sequences, no formulas were introduced and students began immediately working on the questions once they were read aloud. The following analysis will focus on the development in Maggie's proof schemes as she attempted to solve problems in her small groups and presented solutions to the class. As in other episodes, problems 12, 16, and 17 were selected because the associated data collected contained Maggie's clear articulations of her problem-solving approaches and the kinds of uncertainties she had about her proofs. Several synopses follow for reference.

Over the course of two days, Maggie created and validated a function for computing the value in an account given the initial investment, interest rate, compounding period, and length of time an amount would earn interest. The creation of this formula is the focus of the first part of this analysis. In the second part of the analysis, Maggie expressed some doubt about whether or not she had proven that her formula was valid. TR asked how she was sure that her formula worked and what makes interest grow in the way her formula predicts. A conversation ensued in which Maggie expressed the source of her doubt. In the third part of the analysis, Maggie demonstrated the ability to remain faithful to a recursive process in order to generate a function in a closed form.

As was the case in previous episodes, TR's teaching practices can not be separated entirely from how Maggie demonstrated her W_soU. In this episode, TR was

explicit about what he thought a proof consists of. One of Maggie's criteria for determining whether or not an argument constitutes a proof was revealed.

Focus questions:

12. Suppose you have a principal of \$12,000 invested at 11% interest rate per year.

Determine the amount in the account after 10 years if the compounding period is: (a) annually, (b) quarterly, (c) monthly.

16. You are to determine the amount in an account after a certain compounding period. What information would you seek? How would you use this information to determine the amount?

17. For 12 months beginning January 1st an individual saves \$120 per month, deposited directly into her account, on payday the last day of the month. The account earns 6.5% compounded daily, what is the total earning at the end of the year, 30 days in a month, 360 days in a year.

Synopses of Maggie's Solutions

As Maggie began solving problem 12, she kept track of her work in a table. Her table represented an attempt to shorten the amount of information she would have to write. After line 1, Maggie's table kept track of interest for the first two years.

Table 4.3: Maggie's WoU Problem 12

	Year
12000	
+ 12000(.11)	1 st
+ [12000+12000(.11)](.11)	2 nd

Maggie asked her group mates how to add in what was missing. Although her conversation with Derrick and Peggy, she realized that the last row of the table reading, “ $[12000+12000(.11)](.11)$ ”, only represented the interest for on the money in the 2nd year. She came to see that the length of the terms was quickly getting unwieldy. When the use of 1.11 was suggested as a way to get around this problem, Maggie asked where she would introduce that notation instead and struggled to view the problem using this . Maggie explained that she was trying to see the pattern. From the context, her statement means that she was trying to leave expressions in a form she could visually inspect for a pattern. She eventually wrote the following sequence of calculations to replace what she had written in her original table.

12000

$12000 + 12000(.11)$

$[12000 + 12000(.11)] + [12000 + 12000(.11)](.11)$

$[12000 + 12000(.11)] + [12000 + 12000(.11)](.11) + [[12000 + 12000(.11)] + [12000 + 12000(.11)](.11)](.11)$

...

After her interactions with group mates, Maggie changed her notation from what she had written in her first table, using exponential notation.

Table 4.4: Maggie's new WoU Problem 12

	Year
12000	
$12000(1.11)$	1 st
$12000(1.11)(.11)$	2 nd
$12000(1.11)^3$	3 rd

Note that the two previous ways of writing these computations can be compared with the following in which one distributes at each stage of computation.

Table 4.5: Maggie's final WoU Problem 12

	Year
12000	
$12000 + 12000(.11) = 12000(1.11)$	1 st
$12000(1.11) + 12000(1.11)(.11) = 12000(1.11)(1.11) = 12000(1.11)^2$	2 nd
$12000(1.11)^2 + 12000(1.11)^2(.11) = 12000(1.11)^2(1.11) = 12000(1.11)^3$	3 rd

Claims

During this episode, Maggie demonstrated that she was in a state of disequilibrium about what constitutes a proof. The primary value of this episode came in the form of a conversation between TR and Maggie in which Maggie explains that for her, a proof is an argument that convinces another person of the validity of statement. For TR, one form of proof was explained to be a form of argumentation that relies on an

agreed upon set of definitions of a process as starting points. Moving logically from these premises, possibly using algebra, one moves toward a conclusion. This was a case when the TR told Maggie directly what he was thinking rather than refer her to her group mates.

Analysis

Analysis of problem 12

Maggie considered the situation where \$12,000 is invested at an annual percentage rate. Though her group members explained that computing the rate could be done quickly by multiplying 1.11 by 12,000 taking the answer and multiplying by 1.11 repeatedly, Maggie's WoU was recursive. She computed the interest on \$12000 for one year and added that quantity to the original amount, $12000 + 12000(.11)$. Next she wrote $(12000 + 12000(.11))(.11)$ to compute the interest on in the second year and realized that she would still have to add something to it. Again, group mates tried to reiterate the value of a shorter approach.

Maggie: Tell me if this thinking is right. I'm just going by years cause I don't know how to use the formula. So you, you start with twelve thousand (ina) first year you're going to have twelve thousand times eleven percent (ina) you're going to have those two added together times...eleven percent.

Derrick: Right.

Maggie: But then you have to add back...

Peggy: **That's why you use one point one one and you don't have to add that back.**

Maggie: Ok so where, I mean I see that I can put it in right here [pointing to $12000 + 12000(.11)$], one point one one, but how do, how am I adding the interest to the total that I have before that? Now (ina)...

Derrick: You just did it right here, you have your original twelve thousand...

Maggie: Right.

Derrick: ...and then you're adding the interest, you're adding these two things together...

Maggie: Combining those two.

Derrick: ...and then you're multiplying by...

Maggie: Point eleven.

Derrick: Point one one.

Maggie: So I'll get the interest for this total amount...

Peggy: Right, the interest and then you have add that to twelve...

Maggie: But how do I add it back to my...

...

Penny: (ina) you got to add it to the total.

Peggy: Yeah that, (ina) next.

Maggie: To the one before.

Derrick: Back to the, back to the sum of these two.

Peggy: Right.

Maggie: So I'm not done with this the second year because I've only come up with the interest...

Peggy: Right.

Maggie: ...I have to also add for the second year...

Peggy: Back to that guy.

Maggie: ...this thing right here.

Peggy: Yeah.

Maggie: Is that right?

Peggy: Yeah.

Maggie: **So if I do the third I have to take this whole thing...**

Peggy: Right.

Maggie: ...and multiply it by (ina).

Peggy: That's why (ina)...

Penny: And then add it (ina).

Maggie: **And then add it to this at the same time. That's going to be like this long.**

Peggy: That's why like when he said, well yeah, **that's why when he said multiply by one point one one then you don't have to add it back.**

Maggie: Ok.

Peggy: It comes out that way.

Maggie: Oh, oh...

Penny: You see cause the whole one...

Maggie: **...instead of, ok, so where would I put that in up here?**

Peggy: Start at the very beginning with...

Derrick: If you multiply by one point one one right here you're going to get...

Maggie: **Ok so instead of saying plus...**

Maggie asked, "I see that I could put it in right here. 1.11. [pointing to

[12000+12000(.11)] ... How am I adding interest to the total that I have before that? Now

I'll have a new total..." The group explained that if Maggie added [12000+12000(.11)] to

[12000+12000(.11)](.11), she would get the new account balance. She identified

$[12000 + 12000(.11)](.11)$ as the interest earned in the second year, but only wrote $12000(.11)$ as the account value of the previous year. A group member explained that by using 1.11 she could avoid having to add the interest to the amount of principle in the previous year. Eventually, Maggie was persuaded to multiply 1.11 repeatedly rather than operating recursively. She realized that such a process was cumbersome, but later explained that she was doing it because she thought she could find a pattern that way.

Maggie's recursive WoU was a sign that she understood the definition of compounded interest. However, she realized that her WoU required a shorter way of writing her computations. When group mates offered her a different way to express her computations Maggie had difficulty appropriating their WoU. Maggie's difficulty was evidenced by the three times Derrick and Peggy had to reiterate their use of distribution and exponentiation to write their results in a shorter way. Maggie's solutions to problems 16 and 17 show that eventually she internalized Derrick and Peggy's WoU compounded interest.

To find out how much money was in the account when compounded quarterly (part b of problem 12), she computed $12000(1.0275)^{40}$. When answering the monthly question, she computed $12000(23/12)^{120}$. The group had been talking about what to do in the quarterly case, but had not discussed the monthly case. She tried to confirm her answers with Derrick. Derrick explained that using $1\frac{11}{12}$ was not the same as taking 11 percent and dividing by 12.

After explaining that he used to work in banking and considering the amount and kinds of questions she was asking, Derrick acted as an authority in this domain for Maggie. Still, this did not constitute evidence of an authoritative proof scheme for

Maggie because she did not take his word without asking further questions. Also, considering her sources of conviction in the forthcoming analysis of problems 16 and 17, it is clear that Maggie may have been in disequilibrium, but she was not relying on Derrick to convince her though his explanations were deductive in nature.

Analysis of problem 16

In this section, Maggie explains her answer to problem 16 in a private conversation with TR.

16. You are to determine the amount in an account after a certain compounding period.

What information would you seek? How would you use this information to determine the amount?

Before TR arrived, Maggie had declared the following variables and written expressed the relationship between them as follows:

(T) Total amount in an account

(r) rate

(t) time (in years)

(x) number of times the interest is compounded per year

(t*x) number of times the interest is compounded over the total compounding period

$$T = p(1+(r/x))^{t*x}$$

After Maggie generated a formula for computing the total amount (T) in an account for the given constraints TR came to her group and began to ask her questions about her solution. Eventually, the subject turned to Maggie's conviction about her

solution. Maggie said, “The question is, so how am I sure of this? Or what makes this happen?” TR said, “Right.” Maggie showed TR the following table that she had made.⁴⁹

Table 4.6: Maggie’s WoU Problem 16

Year/period ⁵⁰	[Total amount in the account after t years] ⁵¹
1 st	$p(1+r)^1$
2 nd	$[p(1+r)] (1+r)$
3 rd	$p(1+r) (1+r) (1+r)$
4 th	$p(1+r) (1+r) (1+r) (1+r)$
5 th	$p(1+r) (1+r) (1+r) (1+r) (1+r)$
...	
10 th	$p(1+r)^{10}$
...	
t	$p(1+r)^t$

Maggie began by explaining that what was written for the first year was a simplification of some sort. The power was included purposefully in anticipation of the need for exponentiation. For the second year, Maggie said that you should take the amount in the first year after the value had grown and multiply it by $(1+r)$. Her explanation was procedural in nature with the idea of growth guiding her mental imagery.

TR asked Maggie why she would multiply $(1+r)$ by $p(1+r)^1$. Maggie responded by declaring $[p(1+r)]$ a new unknown at the beginning of the year. She renamed $[p(1+r)]$

⁴⁹ Maggie’s table did not label the second column, but from her statements it is clear that the second column could have been labeled as the total amount in the account after t years.

⁵⁰ Period was added later, after TR asked if that column stood for year or period.

⁵¹ This was not included by Maggie.

using the variable x . Then she explained that you would have to add that same amount times the rate because of the second compounding. Then Maggie used distribution to support her claim, replacing x as the amount that was being compounded. She explained that every year the same process would be employed to calculate the new total in the account.

Maggie's explanation is both PPG and Referential Symbolic. But she did not know if it was enough. TR asked, "So, where is the problem? You said that you have a problem." Maggie replied, "...The problem is... Is this a proof? And if it isn't, how do I take the next step?" TR asked why she wasn't convinced that it was a proof. Maggie explained that she wasn't sure, "...if it's enough to convince somebody else that this is going to happen every time." TR asked her to anticipate a possible argument that could "debase [her] conclusion".

Maggie pointed to the side of the paper where she had written, " $x+x(r)=x(1+r)$ ". She said that if " $x+x(r)=x(1+r)$ " was understood and accepted, the rest of the proof should be accepted. TR said that he did accept it. She also related $x+x(r)=x(1+r)$ to her table, explaining that $[p(1+r)](1+r)$ must be seen as being based on $x+x(r)=x(1+r)$. TR repeated that such an assertion was reasonable.

TR explained that the process of accumulation of interest must be an agreed upon process. Maggie made the connection that this agreement was the source of, " $x+x(r)=x(1+r)$ ". TR pointed to x , referring to it as the balance. Then he pointed to $x(r)$, referring to it as the interest to be added every year. Maggie proposed that if she could

explain what TR said, using one pair in her table⁵², her proof would be more complete. She proposed letting x stand for $p(1+r)^1$.

TR said that what was written in Maggie's table was a direct application of the agreed upon process for computing a new total expressed in the equation, $x+x(r)=x(1+r)$. He asked Maggie again if she had any doubt. She explained that she had no doubt personally. Next, Maggie proposed a standard of proof. She said, "Is it just that if I have no doubt and I feel like this is strong enough to present to someone else and I know they won't have any doubt, then I know it's a proof?" TR said that was necessary, but not sufficient. He explained that one needs to ask, "What are we relying on?" Then he told Maggie that using (1) the agreed upon process for computing interest – an agreed upon definition - coupled with (2) the true algebraic manipulation one could produce a proof. He confirmed that Maggie had met that criterion. Later, TR reiterated, "So, I hope that have resolved some important things here. When we have doubts whether our proof is correct or not one of the questions we can ask ourselves [is] what are we relying on?... In this case, in each case it has to be answered separately and uh individually based on the circumstances, but the main question is what are we relying on? In this case, we are relying on the definition of the process and we are relying on Algebra."

Analysis of problem 17

Problems #17 – 19, were handed out the day before for homework.

17. For 12 months beginning January 1st an individual saves \$120 per month, deposited directly into her account, on payday the last day of the month. The account

⁵² She pointed to $p(1+r)^1$ and $[p(1+r)](1+r)$ as she said this.

earns 6.5% compounded daily, what is the total earning at the end of the year, 30 days in a month, 360 days in a year.

Maggie began her presentation by explaining her approach in small group work. Maggie: I didn't know what to do and I wanted to see the pattern with the numbers so I just went month by month. So January, I thought of it as there was no money in the account until the last day. So when I put the month here, I'm thinking of the last day of the month when I can look at the total that's in the account, at the end of the month. So in January there is \$120, no interest was earned. Ok, so for February my thinking is that at the end of the month **in February there is the 120 that gets deposited and then something happened to this money and what happened to it was it earned interest according to the formula we used yesterday which was [writing Feb: $120 + 120(1 + (.065)/360)^{30}$], um I took the rate divided it by 360 since it's compounded daily and we're considering 30 days [writing 30 in the exponent] in each month, so the number of times this [pointing to $(1 + (.065)/360)$] happened was 30 times on this [pointing to 120] amount of money. So, in February there should be whatever this is 120 something [pointing to $120 + 120(1 + (.065)/360)^{30}$] plus that 120. So um, by March and what I did on the first side of my paper that got crossed out was this started getting, really, really intense.**

TR: That's ok we will do that is really nice, important and nice to see the whole thing here yeah.

Maggie: Oh you want me to write that also?

TR: Absolutely.

Maggie conveyed her early attempt in which she wrote work similar to what she had done in problem 12. She stayed true to the definition of the process for computing the value of the account at the end of each month (see bolded transcript), explaining how daily interest was compounded by referring back to knowledge she learned in problem 12. Though she understood and communicated the process of compounding interest, Maggie's inscriptions became unwieldy. She explained, "...what I did on the first side of my paper that got crossed out was this started getting, really, really intense." She reported that by the third month (March), she decided to name the previous month's interest by name of the month rather than write it out in an expanded form.

TR pointed out that in trying to write her expressions in a shorter way, Maggie created an unintended confusion between the names of the months and the amount of money in the account at the end of that month.

Maggie: ... I made it nice and neat by saying, you know I said February [underlining $120+120(1+(.065)/360)^{30}$] times this thing [pointing to $(1+(.065)/360)^{30}$], so I guess, I thought I was going to be able to do that.

TR: Oh no you called that expression February [pointing to Feb: $120+120(1+(.065)/360)^{30}$].

Maggie: This whole thing [pointing to $120+120(1+(.065)/360)^{30}$] is February now.

TR: **Oh right so you called it February, so February became a quantity.**

Maggie: **February became, instead of using a variable I just said I need to see this so I understand it, so this is what was deposited at the end of March.** So now this whole quantity [pointing to $120+120(1+(.065)/360)^{30}$] is going to be multiplied by that right [pointing to 120] there and that's where it gets really long so I just said...

TR: So **notice that February represents a month, but the same time here for Maggie, the way she treated it, it represents also a quantity right.**

After TR clarified to the class that the word February played two roles in Maggie's solution, Maggie continued to write out her results, showing her final calculations for two months.

Jan: 120

Feb: $120 + 120(1 + .065/360)^{30} \approx 240.6517$

March: $120 + \text{Feb}(1 + .065/360)^{30}$

April: $120 + \text{Mar}(1 + .065/360)^{30}$

May: $120 + \text{Apr}(1 + .065/360)^{30}$

...

Dec: $120 + \text{Nov}(1 + .065/360)^{30} \approx 1490.48$

Maggie went on to explain her recursive routine again and explained that she calculated at each stage. She was happy with her result, but problem 18 caused her to see the need for a formula.

Maggie: ... I'm sure you can see what's happening here, **it just looks like I take the month before and I do the same thing with it every single time** ... Um, and at the same time ... I was saying oh this equals about 240 and I wrote everything out and I left it in my calculator and then I did this calculation.... So by the time I got to December that 120 was deposited at the end of the month and then I took the month before and multiplied it by the same thing and got this, and I got a total of about, 1490.48, which didn't match anybody at my table. So you might have something different also but that's what I got from these numbers. So I was happy with that, and then I went to the second problem and then I realized...

It is interesting to note that when Maggie's solution did not match those of her group mates, she was not perturbed. Instead, it was her need for a formula that could help her solve the next problem that prompted her change her WoU.

Maggie: ... I realized in the second problem that ... I needed to find basically this amount that is being deposited every month [referring to problem 18]. So I need some kind of formula to be able to solve for that.

TR: Because in the second problem you would have had this many, many times.

Maggie: Yes, like 200 and something times, and 12 times was already too much, you know writing all of that out, so...

Though Maggie was willing to write down the end result of what she had learned, TR asked her to try to show the class all of her thinking, even if it was messy. In particular, he asked her to explain how she took what she had written and expanded it into a form that she found more useful.

TR: ... show us what you did.

Maggie: Ok, what I realized in the pattern when I started to expand it was that, um, I would have, I need to start from the middle, that I would have... So I'll show you what June looks like, ok, because December is really long, ... this is February right here [writing $120 + 120(1 + .065/360)^{30}$], ... Ok so this is the way I saw it in my head, when I saw the pattern, here is January [underlining 120], here is February [underlining $120(1 + .065/360)^{30}$], **ok and for the next month what is going to happen to this is um, this whole thing [bracketing the quantity $120 +$**

$120(1 + .065/360)^{30}$], needs to be multiplied by this [writing $(1 + .065/360)^{30}$], and 120 needs to be added to it [writing “120 +” in front of the entire quantity], right so now...Now I'm on March... because March we'll have all of this [pointing to Feb: $120 + 120(1 + .065/360)^{30}$], multiplied by this and 120 will be added to it, right?

...

Bernie: What's in the middle brackets is the principal and then what is in the outside brackets is the principal a month later? Yes, no? [referring to $[120+[120+[120+120(1 + .065/360)^{30}]](1 + .065/360)^{30}](1 + .065/360)^{30}$],

Maggie: (01:56:39) The outside brackets...

Bob: Starting from the center, that's the principal...

Maggie: So this is, this is the total in the account in January [pointing to 120 in the inner brackets], and then the total in the account in February is all of this [pointing to $[120+120(1 + .065/360)^{30}]$], in these brackets, and the total in March is all of this, the total in the account...

...

TR: ... How did you handle this expression?

Maggie: I decided if you look at it a little bit better, if I just made um, **X equal to what's in parentheses** [writing $X=(1 + .065/360)^{30}$].

TR: Ok.

Maggie: So **I could see what the pattern was**, so I rewrote it for um, well I'll just do it for June, right so the same thing again... and then this whole thing, and this whole thing...

When asked by Bernie to explain the meaning of what was written, Maggie was able to operate referentially with respect the symbol manipulation. That is, she could unpack the meaning of the symbols in the context of the problem, explaining to Bernie what was being computed and how. Again, later Maggie made the decision to rename the quantity, $(1 + .065/360)^{30}$, using a variable. These actions are compatible with the referential-symbolic proof scheme.

After substituting for x , rewriting her representation for the amount of money in the account for June, and distributing the multiplication from the inside-out, Maggie wrote the following expression: $120(x^0 + x^{30} + x^{60} + x^{120} + x^{150})$. This expression represents the result of a search for a pattern for writing the account value succinctly. Although she computed in the beginning, Maggie explained that it was because problem 12 allowed her

to do so that she did not see a need for a symbolic representation initially. Problem 18 prompted her to search for the pattern. However, in order to observe some kind of structure, she explained that she wrote out June's value as an example. It is implicit in her argument that the expression can be modified for any month.

By her own reporting, Maggie showed that as she anticipated problem 18 she foresaw the need to change the form of what was written and utilize a more compact notation. Maggie's substitution of x for $(1 + .065/360)^{30}$ and her reference of a definition or process of compounding interest throughout provides evidence that Maggie's pattern generalization was compatible with a PPG proof scheme.

As a final note, it is worth observing that in two different settings (small group and whole-class) Maggie demonstrated W_oU compatible with a PPG proof scheme which speaks to her potential for being in a state of equilibrium with respect to the deductive proof scheme (PPG) at this point in her PD experience.

Synopsis

Initially Maggie admitted a weakness with her W_oU compound interest problems. Originally, her approach was to compute the interest by multiplying principle by interest rate and adding it to the principle in that order. Writing the expressions in an expanded form became too cumbersome for her. However, her faithfulness to the process governing a definition of compound interest is compatible with a PPG proof scheme. Though a creative use of distribution was necessary to understand how to collapse what Maggie had written in problems 12 and 16, through her communication with group mates and TR Maggie came to understand that distribution was at the heart of her ability to represent this kind of growth using exponentiation.

The group understood the power of distribution, that it can help minimize computations and provides a more compact way of writing down expressions. Rather than simply accept the group's WoU, Maggie tried to understand how it could be incorporated into her own WoU. Eventually, she did assimilate the group's WoU into her own. Evidence for this statement was provided in the analysis of problems 16 and 17.

It was notable that Maggie wondered if her solutions to problem 12 and 16 were complete. In problem 16 she mentioned explicitly that she wanted to know when she could be confident that her proof was complete asking, "...[is it] enough to convince somebody else that this is going to happen every time?"

TR was explicit in telling her that her solution constituted a proof because it made repeated use of an agreed upon definition for computing compound interest and it used Algebra correctly. He explained that to be an acceptable proof (at the PD), one must ask, "What am I relying on?" He was explicit in telling her that in this case, an acceptable proof needed to make use of an agreed upon definition and correct Algebra as hers did.

About the TR's teaching practice

This episode marks a case that can be beneficial in chapter 5 whose goal is to find connections between Maggie's proof schemes, teaching practices, and experiences with TR's teaching practices. For this reason, a brief analysis of TR's handling of Maggie's solutions follows.

When Maggie finished her presentation TR analyzed Maggie's solution in the whole class setting. He explained that it was unmanageable for Maggie in the second problem to follow her original strategy of computing at every stage. Instead, she saw a need to maintain a structure. He pointed out that she was demonstrating a WoT in

mathematics, that one should tolerate open form expressions in order to see a structure that emerges. TR pointed out that she wrote x^0 rather than 1 as evidence that she searched for a structure⁵³.

The second WoT TR pointed out was that in a pursuit of a structure Maggie replaced something complex with something simpler. This allowed her to see how that quantity could be manipulated to form the expression Maggie found in the end.

Episode 11 - Jack and Jill's Speed Problem

Question

Jill travels at a constant speed of 20 miles a day. Jack, starting from the same point exactly three days later to overtake Jill, travels at a constant speed of 15 miles the first day, 19 miles the second day, 23 miles the third day, and so on in arithmetic progression.

- a. How long does Jack travel from the time he starts until he overtakes Jill?
- b. What is the accumulated distance that each of them has traveled each day until they met?

Background and Significant Events

Five students presented solutions to the problem. Each solution related number of days running to accumulated distance run by the runners using a variety of representations (e.g. – graphs, equations, and tables). After they had presented solutions, TR offered a partial solution using a graph which related the speed of each runner to the number of days run (see Maggie's diagram below). TR asked the class find a way to use

⁵³ This analysis has added that in her search for structure Maggie substituted something complex for something simple, $x = (1 + .065/360)$, to help her see the emerging (emerging to her, that is) structure more clearly.

his time versus speed graphical approach to solve the problem. That is, TR asked participants to use their $W_s o U$ his $W o U$ to solve the problem. Students were asked to work in small groups to complete the solution.

In a whole class discussion about the solutions, but before the TR's presentation, Maggie said,

"It took a lot of help to see that what I based my formula on for Jack was not on a strong foundation. I didn't see the constraints of using the sum of the arithmetic sequence formula. I would have given up, but I had some good help."

This indicates that Maggie originally had a solution that did not account for "...the constraints of using the sum of the arithmetic sequence formula." She explained that it was difficult to consider and reassess her assumptions about the problem.

It should be kept in mind that the following analysis of Maggie's work understanding TR's approach exists with the aforementioned backdrop. Maggie had already thought about the problem, arrived at her own conclusion, and seen that her solution was incorrect before seeing the presentations. These experiences should be considered in order to form a backdrop to her work completing TR's solution and all results should be considered in this light.

A note about TR's teaching practices. After solutions were presented, TR explicitly mentioning the following goals for this problem to the participants. The problem was selected to advance certain $W_s o T$. First, one should always evaluate solutions against the storyline of the problem, not necessarily against their own algebraic representations of the storyline because the functions may have limited domains. Second, one should never stop thinking about the meaning of the variables as defined in the

context of the problem. This was an example of how pedagogical issues arose out of the mathematical content at the PD.

Synopsis of Maggie's Solution

Maggie used the following diagram, given by TR originally, to guide her solution. Her graph had a gray shaded rectangular region below the horizontal line at 20 (Jill's speed) and bounded by the x and y axes. For Jack, she drew a sequence of red rectangles whose heights were Jack's speed on each day.

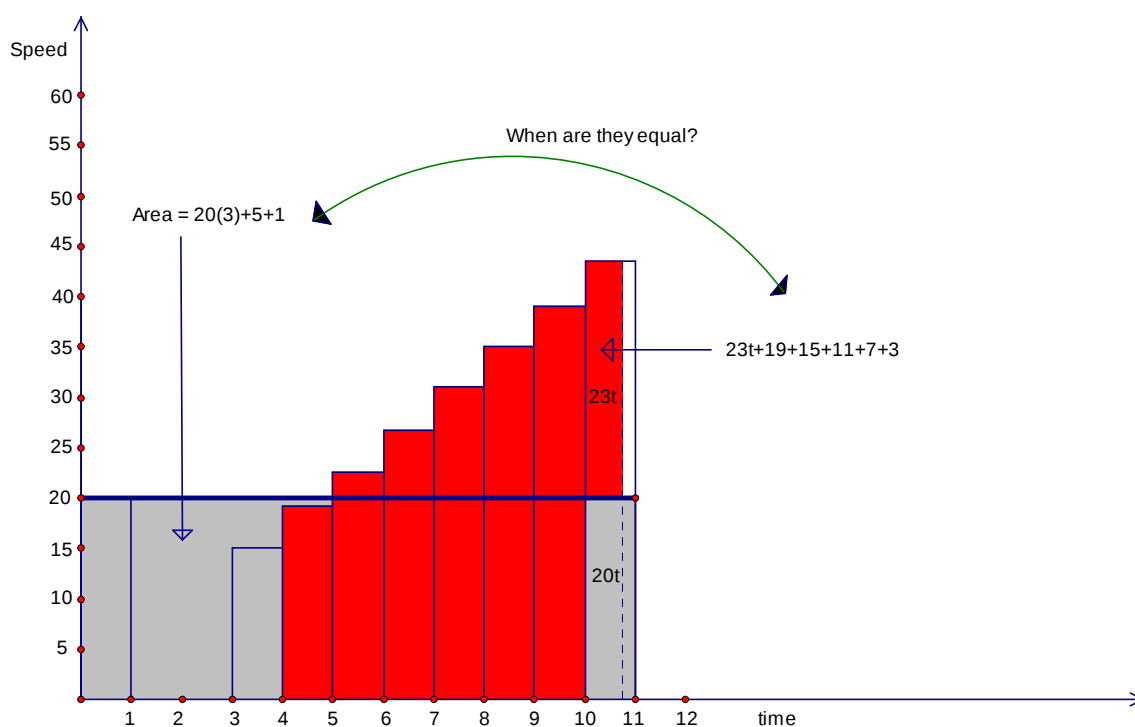


Figure 4.17: Maggie's inscriptions on her copy of TR's graph.

Maggie explained that the areas in regions represented accumulated distances of runners. She explained that one could transfer the area in Jack's rectangles above the horizontal line into the gray space where the two graphs did not overlap. She represented each non-overlapping area by an expression, $20(3)+5+1$ and $23t+19+15+11+7+3$, and set them

equal. Solving the equation, Maggie concluded that the runners met $11/13$ of the way through the 10th day. When asked to explain the meaning of t , Maggie mentioned that it was a portion of the 10th day and that this was what was sought. By using Jill's speed, she computed the distance run by each runner.

Claims

Maggie's final explanation was consistent with the transformational form of the deductive proof scheme. Her solution considered the meaning of the variable t , as well as the interpretation of area explained by TR. She thought carefully about why a general solution could not be found for the problem without first using a different approach. By imagining the area above the horizontal line fitting inside the non-overlapping grey region to the left of the rectangles under the horizontal line and explaining the meaning of the question in terms of distance, Maggie manifested operational thought (in the sense of Harel & Sowder, 1998). She also used logic, in the form of algebra, to write an equation expressing her question, "When are [the areas] equal?"

Analysis

Maggie carefully described her solution to the problem to a facilitator. In the process of doing so, she tried to convince him of the validity of her solution.

Maggie: In this case, this [pointing to gray area] would be $20 \times 3 \text{ days} + 5 + 1$, right? And then we want it to equal this area above this line, right?

Facilitator: I'm just watching now.

Maggie: Yeah, because [with excitement] everything that's overlapped, we know is equal, so we want to know this excess right here [pointing to gray shaded area], when does it equal what's above the line over here? Because area is representing distance, then that would be when the distance is the same. You have your poker face on. So this is, but you see.

As she explained her solution, Maggie interpreted area in the diagram as distance. She

also explained why she would ignore the contribution of the overlapping region.

On a social level, Maggie tried to get approval from the facilitator for her solution as evidenced by her questions, “In this case, this [pointing to gray area] would be 20×3 days + 5 + 1, right? And then we want it to equal this area above this line, right?” The facilitator’s reaction, “I’m just watching now,” confirms that he also believed Maggie was looking at him for confirmation. Additionally, Maggie’s comment, “You have your poker face on,” also provides evidence that Maggie was watching the facilitator for cues that her solution was correct.

Though Maggie looked to the facilitator to validate her argument, there is no indication that it was her primary or ultimate means of conviction. As she continued to solve the problem, Maggie interpreted the meaning of t , using quantitative reasoning and logic to arrive at a solution. Maggie noted that only the last rectangle was variable, in her mind, while the others had fixed values.

Maggie: [off camera] We don't know what this is going to be. It's going to be a fraction of this [meaning a fraction of the way between 10 and 11]. It might go up to here, it might go up to here. So that's why this variable t , because it's part of the data here.

Facilitator: In fact, you should have gone all the way to here. If you notice on his, it doesn't go to the 11, it goes a little bit shorter than the 11, do you see that? So that's why.

Maggie: So I have to say somewhere, I don't know where, so dotted line. [camera returns] But this is not according to a variable, this is what it is. There are 19 square units here [pointing to completed rectangles], so it's $19 + 15 + 11 + 7 + 3$.

Maggie: See this needs to be equal to this, and the t when I solve for that, it will tell me what portion of the day on the 11th day, how far I go on the x 's, basically. So it'll give me the time when that distance is equal. Right?

Facilitator: See how it comes out, I don't know, I'm just watching.

Maggie: It's logical, so it should work out. Isn't it logical?

Facilitator: Sounds logical to me, you explained it pretty good.

Maggie: So when are they equal? That one and that one. (inaud)

...

Maggie: [Turning to a group mate] Did you hear any of that?

Maggie noted that the area in the last rectangle was variable, while the areas of the first few rectangles (in red above the line) were fixed. Next, she explained that she wanted to know what value of t would allow for $20(3) + 5 + 1 = 23t + 19 + 15 + 11 + 7 + 3$. She anticipated that solving her equation for t would tell her what portion of the day would have passed when Jack and Jill met. Again, she looked to Facilitator for confirmation. When he refused to confirm what she had said, Maggie replied, “It’s logical. So, it should work out. Isn’t it logical?” The last question was spoken rhetorically. Given Maggie’s previous explanations that proof is an argument convincing to others, it is likely that she was trying to gather evidence that she had a proof. The question she raised was rhetorical in nature, “Isn’t it logical?”

When the facilitator left, Maggie used her solution to the time spent running in order to answer the part of the question. She turned to a neighbor and explained her solution.

Synopsis

As she interaction with the facilitator, Maggie created a solution that made use of TR’s graphical representation as an expression of his WoU the problem. Maggie’s interactions with the facilitator demonstrated a weak appeal to his authority. However, her comfort with her own solution and subsequent interactions with group mates indicate that she was convinced, not by the word of an authority, but by her ability to envision the transformation of one area into another, and the fact that she saw the last rectangle as “variable” rather than fixed.

Question

A group of workers working together at the same rate can finish a job in 45 hours. However, the workers report to work singly at equal intervals over a period of time. Once on the job, each worker stays until the job is finished. If the 1st worker works 5 times as many hours as the last worker, find the number of hours the 1st worker works? What is the number of workers?

Background and Significant Events

Maggie's work took place in both small group and whole class settings. There were no contradictions or significant differences to report. Other than her use of the formula for computing the sum of finite arithmetic series, there are no significant connections to previous problems. Also, Maggie delivered her presentation with no notes.

Synopsis of Maggie's Solution

Maggie began by representing the amount of time each worker worked. She let x be the amount of time the first worker worked, i be the length of the time interval at which workers entered, and a_n be the amount of time the n^{th} worker worked. For each worker she wrote the following expression:

Workers

$$1^{\text{st}} \text{ w} \rightarrow x = a_1$$

$$2^{\text{nd}} \text{ w} \rightarrow x - 1i = a_2$$

$$3^{\text{rd}} \text{ w} \rightarrow x - 2i = a_3$$

$$4^{\text{th}} \text{ w} \rightarrow x - 3i = a_4$$

...

$$n^{\text{th}} \text{ w} \rightarrow x - (n-1)i = a_n$$

Maggie recognized that the sequence, $a_1, a_2, a_3, \dots, a_n$, was arithmetic. Applying what she already knew about arithmetic sequences, she wrote the following function: $45 = (n/2)(a_1 + a_n)$. Applying the constraint that the first worker worked 5 times as long as the last worker, Maggie wrote: $45 = (n/2)(5a_n + a_n)$ or $15 = na_n$.

Maggie explained that she knew the case $n = 1$ would not work. In the small group data she proved this was so by substituting $n = 1$ into $15 = na_n$. In her whole class presentation, she began by considering the case when $n = 3$. First she found that $a_3 = 5$. Then she solved $25 - (3-1)i = 5$, concluding that $i = 10$. This meant the workers worked, 25, 15, and 5 hours respectively.

Maggie noted that knowing $15 = na_n$, n is a positive integer, and $n > 1$ meant that given an n , you can find the length of time each worker worked. She explained that she had used $n = 3$ as an example. Generalizing, Maggie explained that, "...if you think about this, it could be any number [tapping her knuckle on n in $15 = na_n$], any positive integer [circling motion around n] 2 or greater, and this [point at a_n] will just compensate so the product always equals 15."

Claims

Harel and Sowder (1998) describe a generic proof scheme as follows. "In a generic proof scheme, conjectures are interpreted in generic terms but their proof is expressed in a particular context." In this episode it is argued that the previous instance of Maggie's proving can be characterized as both referential symbolic and generic.

Analysis

In her presentation, Maggie carefully attended to the meaning of the variables in the context of the problem. Not only did she carefully label the variables. As she

presented, Maggie provided insight into her mental imagery, describing how she knew that certain operations could be performed.

TR: Maggie, can I ask you what w_1 , w_2 represent?

Maggie: Nothing, they're not variables.

TR: What are they?

Maggie: They're workers [sheepishly].

TR: What does it mean?

Maggie: Like the 1st worker, the 2nd worker [laughing slightly]... these are names of workers [class members giggling]...

...

Maggie: So I recognize this as an arithmetic sequence. So I thought I could just use a formula for that. And I made it easier by using Paula's notation, well actually not exactly. I said a_1 , and this is a_2 , this is a_3 . So it's still an amount, or a_n .

...

So this is still a number of hours basically is what it is for each of the workers. So I can just use the formula. I know I'm going to have n workers. And for me [the] total hours is 45. And then the 1st worker works a_1 hours. And the last worker works a_n hours. And I put this in terms of the last worker and said that the 1st worker worked 5 times the number of the last worker. And so I get, if I multiply both sides by 2, I get $90 = na_n$. Now this I can write, I did that already, I can add these and I get $6A_n$. And I divide both sides by 6 and I get $15 = na_n$. So at this point I just said I know the number of workers has to be positive integer that's going to equal 2 or more. So I did specific cases, and for example, when n equals let's say 3. When $n=3$, then a_n has to equal 5. Which is the number of hours the last worker worked. Which means that a_1 has to equal 25. And then to find the interval, so I could find the number of hours they worked, I just used this part right here [circling $x-(n-1)i$]. Because I had a_n . So $5 = x$ is 25 minus number of workers is 3 times i . So I have $i = 25 - 2i$. $2i = 20$, so i is 10. So the interval is 10, they went in every 10 hours. So it went $25 - 10$ is 15, -10 is 5. So there are 3 workers, it adds up to 45, and that's how many hours each of them worked. So if you think about this, it could be any number [knuckle on n], any positive integer 2 or greater, and this [knuckle on a_n] will just compensate so the product always equals 15.

After TR's question about the meaning of w_i , Maggie was very careful to explain the difference between a shorthand name or label versus a variable that stood for a quantity. Her grin and the class' laughter were possible indications of a broken socio-mathematical norm. Specifically, TR had negotiated that an element of the didactical

contract that entailed students clarifying the meaning of quantities in the context of the problem. w_i looked like a variable.

When questioned, Maggie's reaction indicates that she realized it was an expectation at the PD that a presenter should anticipate possible confusion created by the use of pseudo-mathematical notation. Communicating the difference between a label and a quantity was previously mentioned in the compound interest problem when TR explained to the class that Maggie had used the names of months as the amount of interest accrued on an account over that month. Implicit in TR's comments was the previously mentioned message about the need for clear communication.

Synopsis

Maggie's proof was generic in the sense that she used the $n = 3$ case to demonstrate that both the criteria (1) that the number of hours worked must be 45 in every case and (2) the product of the number of workers and the length of time the last worker worked must be 15. Maggie explained clearly that since n must be a whole number, there can be a result for a_n in every case greater than 1. Maggie saw a_n as a compensation factor for the changing values of n . In this way, getting 15 was possible for all n and all other constraints were met because her algebraic representations faithfully represented the problem statement.

Results

Research Question 1: What changes were observed in Maggie's proving and proof schemes as she participated in ATI?

This portion of chapter 4 weaves together a final response to the research question from evidence presented throughout the chapter. Tables 4.3 and 4.4 show the results for

research question 1 at a glance. The story of change in Maggie's proof schemes was characterized by increased attention to issues of mathematical rigor. During her time at the PD, Maggie entered disequilibrium with respect to what constitutes a proof. She learned to question when a WoU of understanding is complete.

Though the chart is a useful tool for structuring the large data set, by no means does this chart tell the entire story of the observed changes in Maggie's proof schemes during the two summers of PD. To describe the change in both Maggie's proving and proof schemes a description of each episode is provided. These summaries contain valuable conclusions that go beyond the information that labeling proof schemes provides, but stops far short of redelivering the entire analysis. Finally, a brief overview section entitled, "Synthesis of Results" ties together key moments that contributed to Maggie's observed increased attention to mathematical detail. This section is motivated by the question, "What commonalities and striking differences were observed in Maggie's proving over time?"

X = External Conviction, E = Empirical, D = Deductive

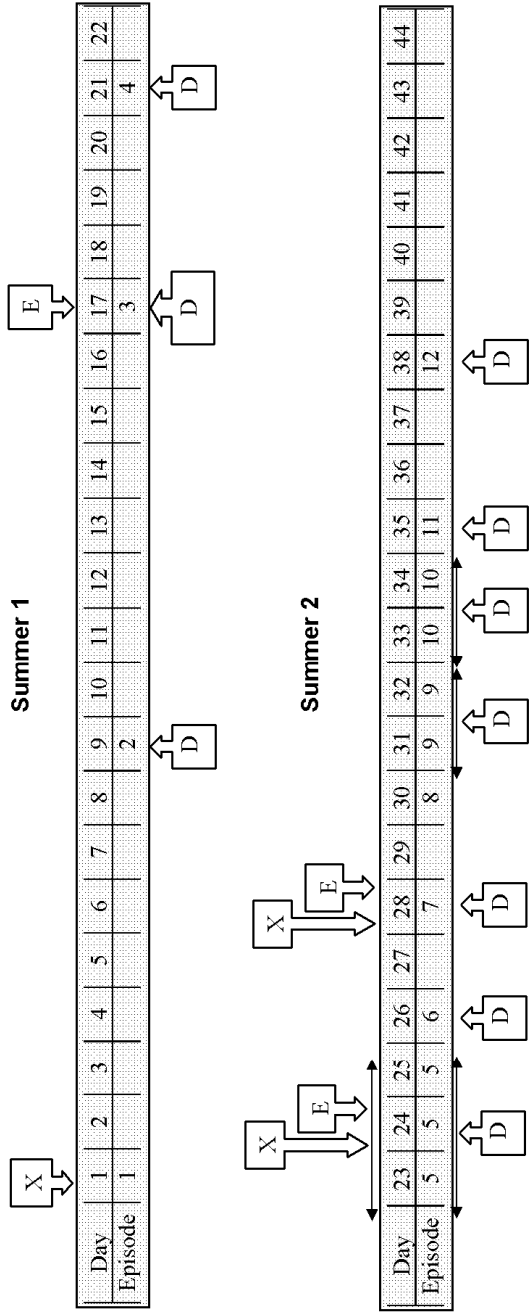


Table 4.7: Proof Schemes Timeline

Table 4.8: Proof Schemes Timeline

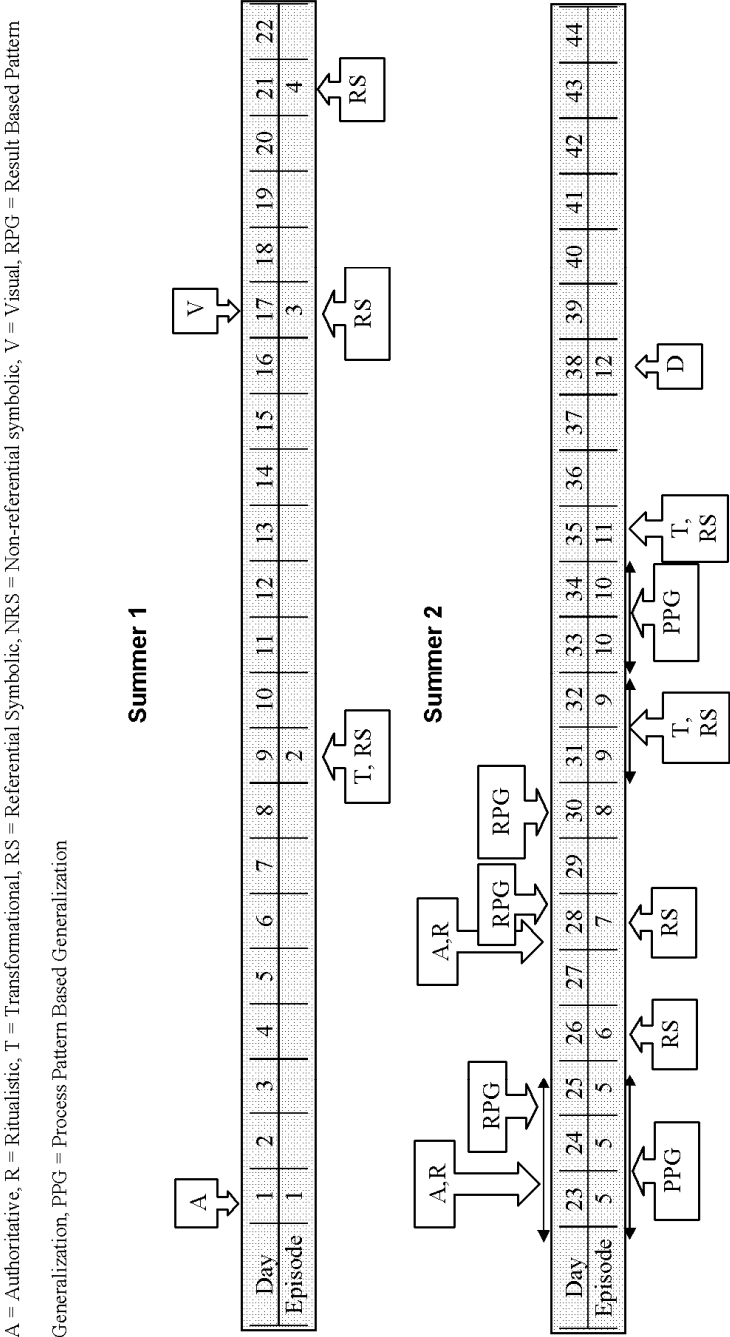


Table 4.8: Proof Schemes Timeline

Findings in Episode 1 – The TV Rating Problem

When solving the TV rating problem, Maggie evidenced $W_s o U$ compatible with the external conviction proof scheme of the authoritative kind in two instances, both when dealing with TR and during interactions with a knowledgeable peer. Despite demonstrating an accurate $W o U$ rating and share, Maggie generated an inaccurate formula and became convinced about its validity by authoritative means. Although her conviction lay primarily in a group member's approval, she initially searched for TR's confirmation. Both sources of conviction can be characterized as instantiations of the authoritative form of the external conviction proof scheme.

Though Maggie's concept definition (in the sense of Tall and Vinner, 1981) was correct, she did not apply it correctly in the context of the TV rating problem. Alcock and Weber (2004) noted that undergraduate mathematics students' prototypical representations of a concept, a subset of students' concept images, influence their proof production. In fact, Alcock and Weber explain that it is both useful and desirable to make use of one's concept image when producing a proof rather than try to produce formal proofs by merely unfolding a definition.

A part/whole $W o U$ fractions (and percents) can account for Maggie's difficulties when seen through the lens of the external conviction proof scheme. If Maggie had encountered many problems that could be solved by using the $W o U$ that $(\text{percent})/100 = (\text{part}/\text{whole})$, she may have been trying to apply the given information in a formulaic manner. Following this interpretation of events allows for a full account of why she was unable to fully incorporate the definitions she struggled to understand. Simply speaking, an overdeveloped schema caused Maggie to disregard less familiar information and

continue acting as she had in similar previously encountered instances involving percents. Because this data was observed on the first day of the PD, it allows a unique glimpse into Maggie's sources of conviction and past experiences before her interaction with TR.

Findings in Episode 2 - Train Station Problem

In the analysis of the Train Station Problem it was shown that Maggie's decision to reject her initial conjecture was guided by a transformational proof scheme and that the manner in which she questioned the validity of algebraic expressions provided evidence of a referential symbolic proof scheme. What is most noteworthy about the events of the Train Station Problem is that Maggie was able to conceptualize the problem, for a time, without variables. This is significant because it speaks to her ability to conceptualize the problem statement and assign meaning to quantities in the context of the problem. These are the characteristics of the transformational proof scheme.

Though Maggie did reach a final conclusion, two characteristics of her proving did become increasingly evident. First, Maggie demonstrated the use of mental imagery rooted in repeated attempts to understand the problem statement as a primary source of conviction. Second, Maggie did not place full faith in a result generated by symbolic representation and manipulation as a person demonstrating the non-referential symbolic proof scheme would. Instead, Maggie relied on a personally meaningful interpretation of the problem involving the coordination of a set of actors together with their movements in space and time. Maggie's recognition that the problem could be solved without the use of the rate, time, distance formula constitutes evidence that for her the rate, time, distance schema is an internalized proof scheme because of her selective application of it.

Findings in Episode 3 - Cat and Mouse Problem

The Cat and Mouse Problem presented Maggie with mathematical content (i.e. – geometry) she was uncomfortable with, as demonstrated in her self-reports and her proving acts. A defining feature of this episode was a learned problem-solving strategy relating to proving. Namely, Maggie learned that when solving a problem or attempting to prove a result it can be productive to make plausible assumptions, set them aside for a time before proving them later, and continuing with what seems right for her at the time. Though she explained that this strategy left her with a sense of insecurity about her overall solution, she realized that without it she would not have arrived at a conclusion, no matter how tentative.

The geometric nature of the problem combined with this important problem-solving strategy allowed Maggie to convey her thinking about the problem. Her thinking ranged from stating conjectures, to making assumptions, to attempting to prove small results along the way. As Maggie presented her WoU, she began to participate in a classroom socio-mathematical norm wherein she clarified what she felt she could prove and what she would take as an assumption. In one particular instance (see scene A), when Maggie felt she could prove a particular result, her sources of conviction could be characterized as empirically based. In another instance (see scene C), Maggie's symbolic manipulation was consistent with the deductive proof scheme (referential symbolic).

From a DNR perspective, changes in proof schemes are claims about learning. According to the necessity principle, learning is conditioned by the intellectual need of the individual. From a more social perspective on learning, it was clear that Maggie's patterns of participation were changing as she interacted with the instructor and her

classmates. What remains to be demonstrated, with respect to supporting the claim that Maggie's proving became more deductive, is whether or not Maggie understood the reasons her classmates and TR did not consider Maggie's sources of evidence convincing (referring specifically to the end of scene A as a case in point).

This episode demonstrates that Maggie has been perturbed about what constitutes an acceptable proof for the PD classroom community. She entered a state of disequilibrium and demonstrated some changes. It remains to be seen whether or not she has internalized the needs for communication and clarification in proving.

Findings in Episode 4 - The Quarterback Problem

Maggie's presentation of the Quarterback Problem constitutes evidence of a referential symbolic proof scheme because of her way of interpreting the variables using the task she was given. Maggie used a tabular representation to order all points satisfying the constraints of the problem in a systematic way. When Maggie checked run/pass combinations she did so by confirming that each point satisfied both time and yardage constraints, exhausting all possibilities.

Maggie interpreted the meaning of the equations she had written in terms of the problem statement and considered the goal of understanding the sense in which a given point was optimal in light of the problem's constraints. In order to do so, Maggie considered the meaning of variables in their context and considered in what sense her final answer was optimal.

Findings in Episode 5 - Stair-Like Structure Problem

The events of the Stair-Like Structure Problem transpired over the course of a 15 hour period and represents intricate evidence of Maggie's proof schemes development.

For this reason, a relatively large section of the results chapter is dedicated to this rich and pivotal portion of the data.

In the process of solving the stair-like structure problem, sharing her results with group mates, watching the solutions of TR and other students, and presenting her own solution Maggie became increasingly aware of her discomfort with the notion of what constitutes a proof. She explained that she was not sure when she was done with a proof and she attributed this uncertainty to repeated experiences of believing she had produced proofs when she had not⁵⁴. This form of disequilibrium was also captured in the analysis of her interactions with class mates during her presentation of the Cat and Mouse problem (see year one data). Though the class had previously pointed out shortcomings in an argument during the Cat and Mouse problem, Maggie pointed to the TR's teaching practices as the primary source of her disequilibrium.

By the end of the episode, she had temporarily resolved her conflict. However, her presentation – while primarily compatible with the PPG proof scheme – demonstrated simultaneously her reliance on an unproven fact and a growing ability to pinpoint a linchpin in her solution. While she was able to do so first with the help of a group mate and later with the help of TR, Maggie did not demonstrate the ability to isolate the pivotal unproven part of her solution spontaneously.

In a conversation with a group mate Maggie explored the difference between relying on multiple examples as a final source of conviction in a proof versus using examples as confirmation. This episode also demonstrates instances when Maggie

⁵⁴ See the TV Rating Problem and Cat and Mouse Problem for instances Maggie may have been referring to.

explicitly rejected proofs relying on RPG and behaved in a manner that can be characterized as PPG primarily. This kind of evidence points to Maggie's increasing awareness of what she was relying on as sources of evidence in her proofs.

During the presentations of the stair-like structure problem part 3 Maggie pointed out that a presenter had generalized from four examples and that this type of generalization did not support the claim that the formula would hold in all instances. Maggie had also generated four examples when she created her solution. However, her solution only used those four examples as a means of verifying a conclusion that was not based on specific instances entirely. Instead, Maggie's solution relied on a counting technique that was grounded in a process used to construct the stair-like structure. As such, this solution was primarily guided by the PPG proof scheme.

Manifestations of Maggie's disequilibrium at this point in her development took were observed in multiple settings. Maggie was not initially convinced that she had a solution until a fellow participant told her so. Later when she presented her solution to the class, Maggie presented a slightly different WoU that made use of more a formula. When TR pointed out that something was missing in her proof, Maggie was able to pinpoint the unproven fact in her argument immediately. With the exception of proving that this formula held, Maggie's public solution is consistent with PPG. Interestingly, her original unaltered WoU, as explained to her group mate, would have been complete as it did not rely on the fact she took for granted.

The events documented in scene D constitute evidence of a RPG proof scheme and an Authoritative proof scheme because Maggie was persuaded by a finite set of results but felt the need to do more with her solution primarily to please TR. However,

scene E shows that when William pointed out a need for causality, Maggie engaged immediately and meaningfully in problem-solving attempts aimed at providing a reason why the function should be correct beyond the fact that it matched a finite number of results. This is another manifestation of Maggie's disequilibrium, demonstrating behavior compatible with RPG and Authoritative proof schemes soon after rejecting an RPG-compatible WoU and providing a PPG-compatible WoU of her own.

During scene F, Maggie explained that she had (at least in part) continued because she felt that TR would be uncomfortable with her solution, consistent with the analysis of scene D. In scene F, Maggie showed another instantiation of the RPG proof scheme in the first part of her presentation. However, she continued by demonstrating a deductive proof and explaining why she felt her previous solution was incomplete. Therefore, it is claimed that scene F is consistent with forms of the deductive proof scheme. Finally, in scene G Maggie autonomously and spontaneously used a WoU arithmetic sequences developed in her solution to problem 5 (scenes D, E, and F). Thus there is evidence that the PPG proof scheme became internalized in the context of arithmetic sequences and series.

This episode shows Maggie living in two worlds simultaneously. While she was personally convinced about the generality of William's conjecture. She did not know how to provide a deductive solution without help, but understood that she should do so. Maggie could visualize some kind of prototypical proof as a target, but she reported that she was initially unable to proceed without TR's help. However, investigation into her work with William in scene E showed fledgling attempts at deductive solutions without TR's help. The collision of these two worlds – the world of personal conviction and the

expectations of the ATI classroom – provides a way to understand the source of Maggie’s disequilibrium at this point in time and implications for professional developers if further research shows consistency in this trend.

Findings in Episode 6 - Sara's Conjecture

In this episode, Maggie demonstrated W_{soU} consistent with this characterization of the symbolic representation. Representing the complicated situation explained in the problem statement using symbols may not be enough, on its own, to constitute evidence of a solution compatible with a referential symbolic proof scheme. However, the ability to pause midstream and explain the meaning of variables accurately in the context of the problem statement does constitute symbol manipulation characteristic of the Referential Symbolic proof scheme.

Maggie also demonstrated a very desirable trait of deductive reasoners, the ability to raise a conjecture, set it aside and return to it at a later point in time in order to complete an argument. This trait was first observed in the Cat and Mouse problem. Observing this trait again in summer two is notable because it contributes to the argument that it is becoming a sustained practice.

Sara’s Conjecture Problem also marks an occasion in which Maggie recognized a missing component of her proof. She realized that a result from a different context might not work in the current context and attempted to address the generalizability of $S_n = (a_1 + a_n)(n/2)$. Still, she raised the conjecture, set it aside momentarily, and came back to it at a later point in time.

Internalizing the proving practice of raising conjectures and setting them aside as statements to be proven later is closely related to the development of a deductive proof

scheme in two ways. First, this practice has the potential to help individuals see the connection between intuition and logic. On the one hand, one does not want to pursue an argument too far with a potentially faulty conjecture. Still, when one feels strongly that the conjecture should be correct, it can still be productive to produce tentative results. Second, this practice can be leveraged to help participants produce proofs by contradiction because it allows participants to knowingly live in a hypothetical reality temporarily, waiting for evidence that could potentially be used to reject the conjecture.

Findings in Episode 7 – Divisibility

This episode is rich in diversity of proof schemes. In fact, all three general categories of proof schemes were demonstrated at one point or another in different contexts. With respect to symbolic manipulation, Maggie demonstrated behavior compatible with both referential and non-referential proof schemes at different points in time. In a different instance, Maggie demonstrated evidence of authoritative and empirical proof schemes as she tried to determine whether or not her proof was complete. The authoritative proof scheme appeared in the context of determining whether or not she had to prove that whole numbers are closed under addition and multiplication. This was an instantiation of Maggie's general statement, "I don't know when it's enough."

Two issues for further investigation began to emerge: Maggie's image of an acceptable proof and Maggie's relationship to the use of cases as a method of proof. There is evidence that Maggie's vision of proofs requires the use of algebraic expressions, but there is not enough evidence in this episode to determine why. The second issue regarding Maggie's view of the role of cases in proving stems from statements she made during the stair-like structure problem. During this episode, on two

occasions Maggie's ascertainment was compatible with the RPG-proof scheme. These instances beg the questions, "How does Maggie relate cases to examples?" Is there a difference between the two for Maggie? If so, what is the difference? Did she reject generic proof⁵⁵ as a means of persuasion?

This episode demonstrates a dichotomy between Maggie's ascertainment and persuasion. She relied on what others thought to reach a conclusion about whether or not she had a complete proof. Though her reliance on others is a form of the external conviction proof scheme it is closely related to entering a community of practice and coming to build her own model of what is considered sufficient evidence in the PD classroom. Her ritualistic proof scheme was evidenced by her discomfort with a proof written entirely in words. In the Train Problem, Maggie was persuaded by group mates to reject her WoU that Jill walked for 60 minutes when she presented it with no symbols.

By no means can it be discounted that Maggie had the ability to faithfully represent salient aspects of the problem symbolically. At many points she questioned the meaning of the symbols she had written and demonstrated control over them. In particular, Maggie's proving can be characterized as referential symbolic with respect to the meaning of x , y , and z in her proof for the $n = 3$ case. Still, Maggie's relationship to symbolic manipulation was problematic at points (see previous statements about ritualistic proof scheme).

The difference between RPG and generic proofs is subtle, but substantial. With respect to RPG proofs, Maggie has previously commented on two occasions that proofs

⁵⁵ During the next day's class session, TR gave a generic proof to support Debbie's result (see 1:11:00 on TR20040727am).

by example are insufficient at the PD. In fact, Maggie said, “Well you couldn't prove anything if you only use specific cases.” Still, she demonstrated a need to attend to the generalizability of her conjecture in the $n = 3$ case. Her concern about whether or not to continue proving the result in the $n = 3$ case was spontaneous and she was not sure what could be gained by proving the result in one case. As Maggie rejected the RPG proof scheme for persuasion, she came to undervalue the role that case study plays in the proving process. That is, in rejecting RPG proofs, Maggie also rejected generic proofs.

Findings in Episode 8 – A Special Polygon Problem

This episode demonstrates an instance in which Maggie first demonstrated behavior compatible with the RPG proof scheme and then later rejected an RPG-based solution. She immediately rejected her result-based generalization after TR asked whether or not her solution was a proof. Once the question was asked, Maggie pointed out how she knew the WoU was incomplete. As such, her rejection of the RPG-compatible solution was not spontaneous. However, it does demonstrate that Maggie is in a state of disequilibrium with respect to the RPG proof scheme and that she can point out which part of her argument needed more support. Maggie's doubt was rooted in her observation that the sequence was not strictly decreasing.

It should be noted that rejecting an RPG based solution does not mean Maggie was demonstrating behavior compatible with a PPG proof scheme. In summer 2 of the PD, Maggie showed on several occasions that she was not clear how to turn an empirical solution into a deductive solution, even as she rejected the completeness of empirically based WoU.

Findings in Episode 9 – A Sequence of Sequences

This episode provides further indications of a deepening in the level of mathematical rigor Maggie believes she must demonstrate in the PD classroom in order for a WoU to be deemed complete. In her small group work, Maggie explained how to find the sum of the series $1 + 2 + \dots + (n - 1)$. She returned to her previous WoU how to justify a formula by pairing terms in the series. Her persuasive approach was spontaneous, indicating that it was her own. Maggie's explanation for why

$1 + 2 + \dots + (n - 1) = \left(\frac{n-1}{2}\right)(n)$ demonstrates a change in her WoU how to compute finite arithmetic series. Entering an equilibrium phase constitutes an indication of a more permanent change in proof schemes.

Further evidence exists that Maggie was developing a stable deductive proof scheme in the context of arithmetic series. Though she demonstrated RPG compatible behavior in her problem solving approach, Maggie only used it to create a conjecture. Her ability to pinpoint the RPG-based result, question it, and eventually justify her result spontaneously in a deductive manner goes beyond mere rejection of the RPG proof scheme. Though TR was initially involved in the reformation of Maggie's WoU the problem, as demonstrated in her comment on a classmate's solution, Maggie internalized the content of her discussion with TR, providing evidence that his comments were in her ZPD. Ultimately, her comments were compatible with the deductive proof scheme.

Maggie's attention to mathematical detail also manifested itself in her ability to use quantitative reasoning flexibly. She saw the relationship between the position and the value of the terms in an overall infinite arithmetic sequence from which the terms in the

finite arithmetic sequences were created. Maggie was also able to coordinate ordinality of sets with cardinality of those sets. Her ability to point out the relationship between the position of terms within the overall sequence and the value of the terms in a case where they were the same was an important contribution to the class discussion and inspired TR to create a problem in which the differences were more pronounced. Maggie's attention to multiple attributes of elements of sets and of finite sets as elements of an infinite set is indicative of a deductive proof scheme.

Findings in Episode 9- Part 2

Maggie offered her own explanation for how the position of an odd number (within the overall infinite sequence) could be determined given its value. Maggie's explanation can be characterized as deductive in nature. Other instantiations of deductive reasoning occurred when Maggie proved that the position of $a_{n,1}$ is given by $(n-1)^2 + 1$ and when she compared her WoU how to find the value of $a_{n,2n-1}$ to William's.

Additionally, Maggie's overall solution as described in the preceding section can also be characterized as deductive.

Analysis of the argument in its totality reveals two striking features of her WoU. First, Maggie's WoU maintained the meaning of the symbols and operations used throughout. Given the circumstances of the problem (i.e. – intense need for coordination) Maggie's ability to operate with meaning throughout was remarkable, reaffirming that her symbolic manipulation can be characterized as Referential-Symbolic.

While a deductive solution to the problem does call for this kind of behavior, there are alternatives. Joe's behavior represents a different possibility. It was possible to

not create a solution or to rely so heavily on formulas (and on others) that the solution is not your own.

As she solved this long and complex problem over the course of several hours, Maggie was also able to return to her strategy even after periods of intense work on particular parts of her solution. She was guided by an approach which entailed a rich understanding of her formula $S_n = \frac{n}{2}(a_1 + a_n)$. Maggie reinterpreted the meaning of the variables in the context of finding the sum of cardinalities of sets and the sum of a particular set. The ability to reinterpret the formula and return to an overall plan, after periods of intense investigation into smaller parts of the problem, and unite several results is itself an integral part of deductive reasoning.

In the first two weeks of the second summer it is clear that Maggie has learned a valuable tool, the formula $S_n = \frac{n}{2}(a_1 + a_n)$. Maggie came to understand the usefulness of the tool in a series of problems that necessitated. While the tool itself may not be directly useful in her instruction of 7th and 8th graders, the valuable experience she has gained in learning how much depth is necessary for an argument to convince participants at the PD is an important part of the development.

Findings in Episode 10 - Compound Interest Problem

During this episode, Maggie continued to demonstrate a state of disequilibrium about what constitutes a proof, a recurring theme in the data. The primary value of this episode came in the form of a conversation between TR and Maggie in which Maggie explains that for her, a proof is an argument that convinces another person of the validity of statement. For TR, one form of proof was explained to be a form of argumentation that

relies on an agreed upon set of definitions of a process as starting points. Moving logically from these premises, possibly using algebra, one moves toward a conclusion. This was a case when the TR told Maggie directly what he was thinking rather than refer her to her group mates.

Initially Maggie admitted a weakness with her WoU compound interest problems. Originally, her approach was to compute the interest by multiplying principle by interest rate and adding it to the principle in that order. Writing the expressions in an expanded form became too cumbersome for her. However, her faithfulness to the process governing a definition of compound interest was compatible with a PPG proof scheme. Though a creative use of distribution was necessary to understand how to collapse what Maggie had written in problems 12 and 16, through her communication with group mates and TR Maggie came to understand that distribution was at the heart of her ability to represent this kind of growth using exponentiation.

TR was explicit in telling her that her solution constituted a proof because it made repeated use of an agreed upon definition for computing compound interest and it used Algebra correctly. He explained that to be an acceptable proof (at the PD), one must ask, “What am I relying on?” He was explicit in telling her that in this case, an acceptable proof needed to make use of an agreed upon definition and correct Algebra as hers did.

Findings in Episode 11 - Jack and Jill's Speed Problem

As she interacted with the facilitator, Maggie created a solution that made use of TR's graphical representation as an expression of his WoU the problem. Maggie's interactions with the facilitator demonstrated a weak appeal to his authority. However, her comfort with her own solution and subsequent interactions with group mates indicate

that she was convinced, not by the word of an authority, but by her ability to envision the transformation of one area into another, and the fact that she saw the last rectangle as “variable” rather than fixed.

Findings in Episode 12 – Workers at Intervals

In this episode it is argued that Maggie’s proving can be characterized as both referential symbolic and generic. Maggie’s proof was generic in the sense that she used the $n = 3$ case to demonstrate that both the criteria (1) that the number of hours worked must be 45 in every case and (2) the product of the number of workers and the length of time the last worker worked must be 15. Maggie explained clearly that since n must be a whole number, there can be a result for a_n in every case greater than 1. Maggie saw a_n as compensation factor for the changing values of n . In this way, getting 15 was possible for all n and all other constraints were met because her algebraic representations faithfully represented the problem statement.

Synthesis of Results

The most robust proof scheme observed was the referential symbolic proof scheme observed in episodes 2, 3, 4, 6, 7, 9, 11. The presence of this proof scheme throughout the data set indicates that Maggie was consistently able to manipulate symbols with the ability to unpack their meanings in the context of the problem at any time if necessary. Therefore, there is no claim of change in Maggie’s proof schemes with respect to symbol manipulation.

Maggie’s ability to conceptualize the problem fully, coupled with TR’s teaching practices, allowed her to realize when a proof was incomplete. This was a common theme in between episode 5 and 10. In those episodes, as time went on, Maggie was able to

internalize the practice of scrutinizing an argument and observing weaknesses in it. When talking to a group mate in a small group setting, Maggie explained that consistently when she thought she had a complete proof TR's questions would cause her to reconsider. As a result she entered a state of disequilibrium with regard to what constituted a complete proof.

In episode 5, Maggie explicitly rejected $W_s o U$ compatible with RPG. However, episodes 5, 7, and 8 were the only manifestations of RPG observed in the data set. These contradictions were indicative of the disequilibrium Maggie experienced. Though she began to recognize incomplete $W_s o U$, she continued to produce them for some time. Simultaneously, she began producing a PPG compatible $W o U$. These episodes taken together represent the heart of the evidence that Maggie was in a state of transition.

It was noted in episode 1, that Maggie simultaneously understood the definitions of rating and share, but appealing to authorities as her ultimate sources of conviction when solving the problem. Therefore, there are two cases where contradictions seem to arise, both are striking. Contradictory actions are indicators of change.

Episode 8 exemplifies how Maggie was able to recognize immediately from the problem situation why her $W o U$ was incomplete. She explained that upon further inspection, she realized the pattern she was relying on changed at some point in the past and it might change again at some point in the future. Past episode 8 when Maggie rejected her own RPG approach, Maggie's $W_s o U$ can be characterized as deductive in nature. This indicates stability in her transformational proof scheme.

Though her $W_s o U$ were all characterized deductively past episode 8, episode 11 marks an important point of development for Maggie. Episode 11 was an instance when

she demonstrated that she had internalized the practice of finding weaknesses in an argument. Though her argument was sound, she was uneasy about a sound solution. She instigated the discussion about her solution with TR and he answered her directly about how she could know that an argument is a proof without relying on the approval of others. The point is that Maggie began to question her own criterion for determining when an argument is complete without prompting from TR or a group mate to do so.

This investigation into Maggie's proof schemes during her two summers of PD encompassed a large amount of data covering several contexts including percentages, geometry, proportional reasoning through rate-time-distance problems, sequences and series, and much more. No doubt Maggie's proof schemes may have been influenced by her comfort level with the mathematics she was solving and this should be kept in mind. However, in no way does this diminish the finds that were observed. Maggie went through a period of disequilibrium with respect to pattern generalization. Eventually, she showed stability in her proof schemes with respect to transformational proof schemes. It is an important part of the story that Maggie's work was compatible with the referential symbolic proof scheme throughout, but the last four episodes documented her proving as transformational with respect to other proof related activities.

Therefore, the major result of this chapter is that Maggie began to question the completeness of her W_{soU} and gained a greater awareness of the need for attention to mathematical detail in her time at the PD.

CHAPTER 5:

ANALYSIS II – Connecting Evolution of Maggie’s Teaching Practice to PD Experiences

Introduction

What connections can be found between Maggie’s experiences at the PD and the evolution of her teaching practices in a whole class setting?

1. What is the evolution of the $W_s o U / W_s o T$ Maggie promoted in whole class discussions during her two years of instruction?
 - a. What $W_s o U$, and their corresponding $W_s o T$, emerged?
 - b. When they were presented, how did Maggie attend to students’ $W_s o U$, and their corresponding $W_s o T$?
 - c. Which $W_s o U$, and their corresponding $W_s o T$, did Maggie promote?

[Answers to question 1 provide a characterization of a set of her TPs. Harel, Manaster, Fuller, and Soto (in prep) have already discussed the teaching practices Maggie experienced in PD.

2. To what extent do Maggie’s teaching practices reflect the PD teaching practices?
3. What relationships can be drawn between the changes observed in Maggie's proof schemes during the PD period and the observed evolution in her teaching practices during the period of time she was teaching?

Figures 5.1 and 5.2 helps to contextualize the goals of the sub-questions within research question 2.

Structure of the chapter

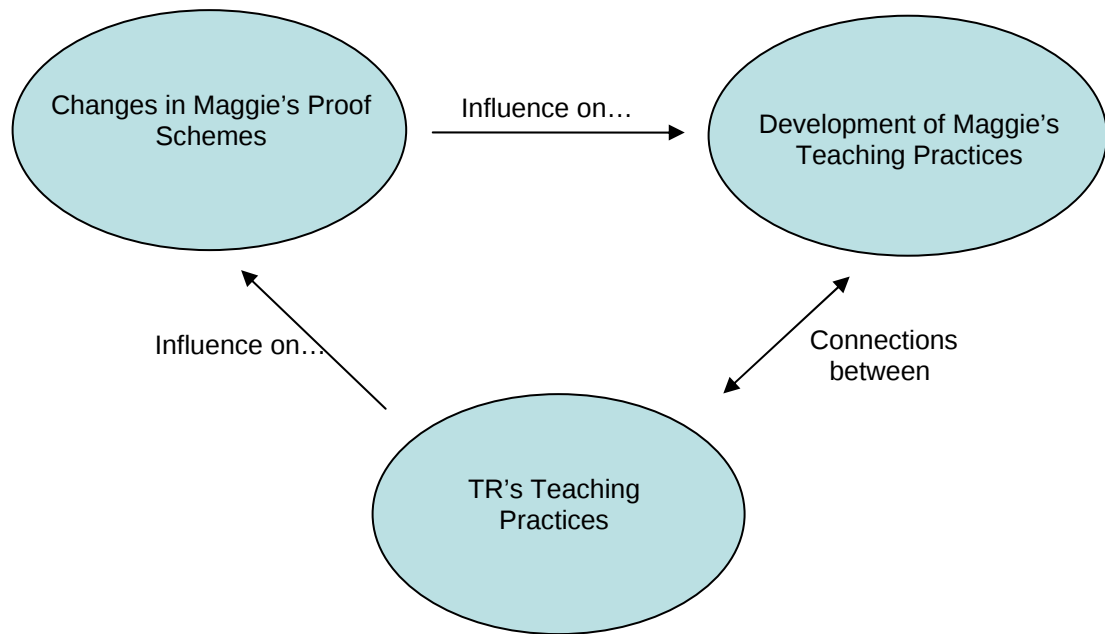


Figure 5.1: Variables of interest in sub-question 3.

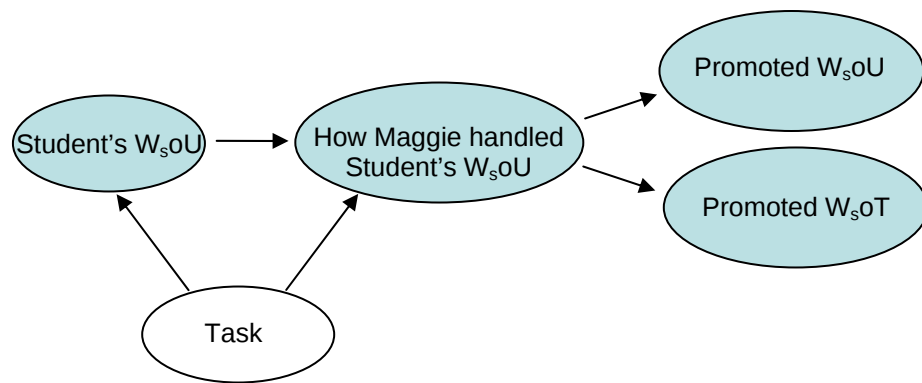


Figure 5.2: Variables in of interest in sub-question 1

In the sections that follow, Maggie's teaching practices are analyzed in the context of whole class discussions with an eye on their evolution over the two years of observations. Considering the goal of identifying connections between developments in Maggie's teaching practices, her proof schemes, and TR's teaching practices this analysis focuses on how Maggie handled students' $W_s o U$ and on the $W_s o U/W_s o T$ Maggie

promoted in whole class discussions. The analysis of each lesson begins with an explanation of background to the lesson. Next, answers to each of the three sub-questions pertaining to Maggie's teaching practices are provided with evidence including transcript.

Following the analysis of the teaching practices in each episode there is a section entitled "Observed Teaching Practices, Connections to the PD, and Connections to Proof Schemes". One goal of this section is to make connections between Maggie's observed teaching practices and those teaching practices observed at PD. Another goal of the section is to find connections within the lesson between Maggie's teaching practices and her proof schemes.

The chapter closes with a section titled "Results," addressing each of sub-questions 1 through 3 explicitly. This section is dedicated to answering the research question holistically. Trends are pointed out in how Maggie handled student's $W_s o U$, which proof schemes were promoted, and what connections have been observed in between the triad of variables represented in sub-question 3.

Claims for readers to watch for

The first part of research question 2 focuses on how Maggie handled students' $W_s o U$ and which proof schemes were promoted. Observations about Maggie's handling of students' $W_s o U$ in class discussions are discussed within three major categories: attention to mathematical details, extending the locus of authority, attention to students' mental images. When viewed by subcategories, these three categories were considered as follows⁵⁶:

⁵⁶ Precise meanings of the subcategories are offered in Chapter 3 and at the end of this chapter in the "Results" section.

4. Attention to overlooked mathematical details
5. Extending the locus of authority
 - a. Encouraging Student to Student Talk
 - b. Encouraged students to state conjectures
 - c. Encouraged students to prove their conjectures
 - d. Allowing an error to persist
6. Attending to students' mental images
 - a. Assigning pattern problems
 - b. Ask students to communicate their thinking about their solutions
 - c. Gathering Distinct $W_s o U$
 - d. Pointing out differences between $W_s o U$ and mental images (either within the individual or across individuals)
 - e. Asking for alternative $W_s o U$ in the presence of correct solutions.

When an instance of a subcategory was observed, it was noted in terms of presence or absence. Qualities of the observations were noted in the descriptive analyses. Over the course of the two years of observations Maggie exhibited a greater range of usage of these teaching practices, with the most teaching practices observed at the end of each year and a higher average number of teaching practices (from within this set) observed in year 2 than in year 1. Maggie engaged in the practice of encouraging student to student talk and encouraging students to prove their conjectures with more frequency in year 2 than in year 1. When viewing the data as a continuous timeline, it is seen that from lesson 8 to lesson 17, Maggie began to allow student errors to persist with greater frequency. Finally, though Maggie attended to students' mental images consistently

throughout, she exhibited the practice of asking for alternative $W_s o U$ in the presence of correct solutions more frequently in year 2 than in year 1.

The determination of which proof scheme is being promoted in a classroom can be a difficult one because a teacher's words can be taken at face value, but her actions can be contradictory. Several instances of this were observed. Nevertheless, the analysis of promoted $W_s o U$ and $W_s o T$ was viewed from the perspective of the observer. Given a classroom discussion, Maggie's promoted argument was the unit of analysis. Under this interpretation of events, no substantial difference was found in proof schemes Maggie promoted between or within years.

There were strong connections observed between Maggie's teaching and her experiences at the institute. In several instances, Maggie made the geometric-algebraic-physical connection mentioned several times at the PD. There were also cases in which Maggie emphasized personal meaning of terms over institutionalized meanings. Another connection to the institute was the practice of completing a student's errant solution rather than abandoning it. Though it occurred too infrequently to become a part of the coding scheme, it was illustrative of Maggie's ability to enter the students' mathematical world and hold up the value of what they had done to other students. Other connections are cited throughout the lessons in the end of each section.

Year 1

Lesson 1

Task

The price of a CD is \$12. Tax is 8% in the city where you bought it. What is the final price of the CD including tax?

Topic of lesson

Percent in the context of tax.

Summary of Events

The problem was given at the beginning of class as a warm-up problem. Maggie asked two students to share their solutions with the class. When both students had finished, Maggie asked each student to explain his/her thinking to the class. Maggie used the second student's thinking to advance her mathematical agenda which included reasoning proportionally to compute the tax on the CD.

Claims

Maggie's message to the class as a whole, and the second presenter in particular, was that she wanted students to communicate their thinking more clearly. The WoU she targeted were: (1) $x\%$ means $x/100$, (2) the fraction $x/100$ can be viewed as a ratio between two quantities⁵⁷ (i.e., for every \$100 paid, \$ x must be spent in taxes), (3) the ratio in (2) can be reduced to determine how many dollars are paid in tax for every one dollar, (4) using (3) one can determine how much is spent in taxes for \$12.

In this lesson Maggie went beyond emphasizing mere procedural knowledge of how to compute the tax on an item. In her choice to highlight the particular WoU, Maggie

⁵⁷ Quantities is used in the sense of Thompson (year??).

conveyed mathematical values consistent with one underlying function of deductive reasoning; the need to communicate clearly one's mental imagery guiding the solution. Ultimately, her choice to focus on student 2's solution instead of student 1's "rule-based" solution is taken as evidence of an attempt to advance a transformational proof scheme via proportional reasoning among students rather than the external conviction proof scheme consistent with student 1's WoU.

Analysis

Maggie asked two students to come to the board simultaneously and write their solutions.

Summary of Student 1's Solution

Student 1 - The first solution was done by changing 8% to .08. Next, the student multiplied by 12 and added the result to the original priced.

Summary of Student 2's Solution

Maggie began by asking student 2 for an explanation of her solution. The student

12		12	12.00
wrote $\times 96$	;	explaining that 8 times 12 gave 96. She changed $\times 96$	to $\frac{\times 96}{12.96}$;
12.96		12.96	12.96

explaining that she had simply written 12 in front of 96; inserting a decimal point.

Handling of Student 1's Solution

After the presenter finished explaining how he solved the problem, Maggie asked him, "How did you know that you could turn a percent into a decimal?" The student responded with an explanation of what he did, "You just move the decimal point 2 places to the left." Maggie responded,

Maggie: Ok. So you memorized that. You know that rule, right? And he just used the rule and then he multiplied. Ok, excellent. Good.

Maggie's response, "So you memorized that. You know the rule, right," helped to clarify what student 1's source of conviction is; a memorized rule which the student could not unpack at the time. Maggie dismissed the student and moved on.

Handling of Student 2's Solution

After hearing the student's explanation Maggie asked,

Maggie: So, you're saying that when you multiply 12 times 96 you get 12.96? Is that correct?

The student clarified what she had meant. When Maggie felt she understood, she reported her understanding to the class and asked the presenter a question; asking for justification and returning to the context of the question.

Maggie: When you had 12 times 8, she got an answer. Her answer ... was 96. This is a whole number. How come all of a sudden you knew that that was going to become 0.96, like it was going to become cents?

As with student 1, the student was unable to shed further light on the reasons for her choices. Maggie pointed out that the answer was correct, but decided to stay with student 2's response in order to explain how she had arrived at the right answer.

Maggie: ... Did she come up with the right answer? ... She did right? She came up with the right answer. We're going to talk about what she did here, alright? ... What did she really do? What do you think she did? ... What could we help her with so that next time she makes it a little more clear?

Maggie chose to highlight the fact that the answer was correct, but the explanation was unclear. She chose to complete the second student's WoU by soliciting and filtering student WoU from the class, choosing one particular WoU percent that made use of proportional reasoning.

Maggie: Ok ... he says 8% means ... 8 out of 100 [writing $8/100$ on the board]. This is what it means to him. Ok? If your CD costs \$100, how much tax would you pay [pointing to 8]? Out of \$100 you would pay how much [pointing to 8]?
 Student: 8%.

Maggie: 8. Ok? 8%. Now what if we made this [pointing to denominator] into a 10. What if we just had \$10. How much tax would you pay on it? If I just divided this [pointing to $8/100$] by 10 and I said now I have \$10, how much tax am I going to pay? What's going to happen to this number? Is it still \$8 that I pay for 10?

Students: No.

Maggie: Is it going to get bigger or smaller?

Students: Smaller. Bigger. [Mixed responses]

...

Maggie: So [a student] is saying that for every dollar of her CD she is going to multiply the 12 by what?

Student: 8.

Maggie: She multiplied by 8 instead of 0.08 but in the end she said, I know that for every dollar, well I don't know if you were thinking this or not, but for every dollar, she added 8. Ok, so 8 how many times?

Maggie began by asking students the question, "What does 8% mean to you?" One student said that, that 8% percent means $8/100$. Maggie followed up on the student's WoU using proportional reasoning. She asked students the following series of questions and chose an answer she could use to advance her WoU.

Maggie: ... What if we just had 10 dollars? How much tax would you pay on it? What's going to happen to this number [pointing to 8]? ... Does it get bigger or smaller?

Through these questions and the incorporation of selected student WoU, Maggie made the point that $8/100 = .8/10 = .08/1$. In each case she interpreted the values in context of money.

Maggie explained that a tax of 8% meant that for every hundred dollars \$8 must be paid in tax; as represented by the ratio $8/100$. Next, she concluded that for every \$10 spent, \$.8 or 80 cents must be spent in taxes; as represented by the ratio $.8/10$. Finally, she concluded that this meant for every \$1 spent, \$.08 or 8 cents must be spent in taxes;

as represented by the ratio $.08/1$. Since the CD cost \$12 and she must spend \$.08 for every dollar, multiplying 12 by $.08$ meant one would spend a total of \$.96 in tax. In effect, after scaling $8/100$ down to $.08/1$, Maggie scaled $.08/1$ up to $.96/12$ using the “for every” relationship. Finally, she added the tax to the price to get the final price of \$12.96.

The aforementioned solution is deductive in nature because of its reference to and use of an explicit definition of percent as a starting point, as well as its use of proportional reasoning to transform the values of quantities while preserving the “for every” – multiplicative – relationship between numerator and denominator. Maggie’s constant attention to meaning of quantities, in terms of dollars and cents, was coupled with her use of proportional reasoning to transform $8/100$ into $.08/1$, and later, to scale $.08/1$ up to $.96/12$.

In her explanation, Maggie tried to help students anticipate the effects of her transformations of the denominator on the numerator; adjusting the numerator of her fractions for desired changes in the denominators proportionally. Thus, the highlighted solution is consistent with the transformational proof scheme that made use of proportional reasoning motivated by a need to clarify what the student had done and how she could communicate better in the future.

Maggie made two points through her treatment of student 2’s solution to the problem: (1) computing tax could be accomplished through proportional reasoning based on the student’s WoU 8% as $8/100$, and (2) she pointed out the need for clarity when writing solutions. Though Maggie use of equivalent fractions as she handled and filled in student 2’s solution appealed to the students’ need to communicate their thinking clearly. She also relied heavily on the context of money/taxes.

Maggie stated these mathematical values clearly by asking, “What could we help her with so that next time she makes it a little more clear? She really did two steps.” Maggie went on to recapitulate the two steps as follows. When finding the cost of an item, one can begin by expressing the tax as a decimal and multiplying the price by the tax rate to find the amount to be paid in tax. Then add the taxed amount to the original price in order to determine the price after tax. The WoU Maggie promoted did not propose how to express the percent as a decimal in general terms. However, the justification, “Hers worked out because it’s 8 cents for every dollar,” reemphasizes a conceptualization of how to express 8% as a decimal. That is, $.08/1$ or $.08$. It is notable that while student 1 explained that he moved the decimal point two places to the left because it was a rule, Maggie did not.

Observed Teaching Practices, Connections to the PD, and Connections to Proof Schemes

It is important to note that Maggie’s solution was given for the purpose of clarifying or filling in a student’s solution. This teaching practice was observed at PD on several occasions. One such event was witnessed when Maggie presented her solution the arithmetic polygon problem. On that occasion, TR asked if Maggie’s solution represented a proof. She realized that her solution did not represent a proof and explained that something was unaccounted for. Though she thought the approach should be abandoned, TR used it as an opportunity to introduce an important mathematical concept (the notion of increasing function). He used his concept to complete Maggie’s proof. It was often pointed at the PD

Maggie used proportional reasoning to fill in what student 2 left unsaid. It is interesting to note that student 1 had a similar solution, but justified it in a manner that

demonstrated an inability to unpack his WoU. On the other hand, student 2's solution was simply unclear. Maggie seized on the opportunity to make it her own and advance a WoU based in proportional reasoning which ultimately justified student 1's WoU in a manner consistent with the transformational proof scheme.

Maggie handled both students' solutions in a similar manner; clarifying the student's WoU and asking them to communicate (and justify) their thinking. The primary form of interaction was that the teacher raised questions and the students attempted to answer them as exemplified below:

Maggie: ... What did she really do right here? ... When I'm asking that, I'm asking about the operations. What do you think she did? Add, subtract, multiply, or divide? ... There are only 4 things you can do in math. You can add, subtract, multiply, or divide. What do you think she did?

In this interaction pattern, the teacher assumed the role of skeptic, filtered student W_soU as they offered them, and took on the responsibility of proving assertions. This interaction pattern does not extend the locus of authority because it increasingly narrows the choices of correct responses.

The task is directly related to Maggie's experiences at the PD. In the TV rating problem, Maggie was specifically asked what her WoU percent was. She explained that $x\%$ means x out of 100 to her. The TR explained a different WoU. This sense of personal meaning for a term came across in her way of handling the student's WoU that percent was 8 out of 100. She said, "This is his way of thinking about percent."

Maggie learned a valuable lesson at the PD. Namely, while teachers may wish students could appropriate definitions for terms like percent, there are often several equivalent W_soU a term. This is related to the belief about mathematics that it is a human

creation. This belief is helpful in diminishing the effects of the external conviction proof scheme because it can produce a sense of autonomy in the individual. Maggie's ability to fluidly compare $W_s o U$ at the PD and her statements in this lesson indicates that she held the aforementioned belief.

Lesson 2

Tasks

1. Simplify $5 \cdot 4 + 3 - 2 \cdot 6 + \frac{10}{2}$.
2. Evaluate $(7-3)^2 + (12/4)$.
3. Simplify $2 \cdot 5^2 + 4(-3)^3$.

Topic of Lesson

Order of operations

Background Events

The lesson is self-contained and does not make reference to previous events.

Claims

In the case of the first task, simplifying $5 \cdot 4 + 3 - 2 \cdot 6 + \frac{10}{2}$, Maggie emphasized the WoU order of operations that is compatible with that of the mathematical community and her textbook. This WoU was promoted through a process of filtering students' solicited W_soU. When Maggie heard what she wanted, she promoted that WoU. The order of operations was deemed correct because it conformed to the textbook WoU. Two W_soU $5 \cdot 4 + 3 - 2 \cdot 6 + \frac{10}{2}$ were presented and Maggie completed a third solution emphasizing that it was correct because it used the textbook WoU order of operations. These ways of handling students' W_soU promote correct mathematics, but also strengthen the effect of the authoritative proof scheme.

In the second task, the teacher asked two students to present W_soU. The teacher asked students to communicate their thinking. When she analyzed their work, Maggie

asked them to write their work more clearly because others may not understand their thinking by what they had written. That is, Maggie used the difference between what was said and what was written to emphasize a need for accurate communication. Though her comments were regarding syntax, the reason for the comments emerged from a legitimate need not a mere request on the part of the teacher. Asking students to write solutions by appealing to a need for accurate communication as a reason for doing so is a teaching practice with potential to encourage the referential symbolic proof scheme.

Analysis:

Task 1: Simplify $5 \cdot 4 + 3 - 2 \cdot 6 + \frac{10}{2}$

The first two students computed in the same order. Both arrived at the same conclusion. Here is the work of student 1:

$$5 \cdot 4 = 20$$

$$20 + 3$$

$$23 - 2$$

$$21 \times 6$$

$$\frac{252}{2} + \frac{10}{2} = \frac{262}{2} = 131.$$

Maggie was asking about “the order of operations”.

Maggie: ... Here are some operations here [pointing to +, -, x, /]. He's telling us, he knows this already, ok so he's gonna share his knowledge with us. P-E-M-D-A-S. Ok, P stands for?

Student: Parentheses. Exponents.

Maggie: Do you know what "exponents" means?

Students: Yes.

Maggie: Ok.

Student: Multiplication. Division. Addition. Subtraction.

Maggie: Subtraction, ok. Ok, good wow. You just did my whole lesson for me.

...

Maggie: Ok, so I guess do I need to ask which one do you think is correct?

Number 1, number 2, or number 3?

Students: Number 3.

Maggie: So it looks like number 3 is correct 'cause now we know what the order is.

Through her line of inquiry in which Maggie filtered student responses, she pointed out the mathematically acceptable order of operations (parentheses, exponents, multiplication, division, addition, subtraction) found in the students' text. It was deemed correct because it was in the text. Maggie completed a partial solution already present on the board by using that order of operations; arriving at the conclusion that

$$5 \cdot 4 + 3 - 2 \cdot 6 + \frac{10}{2} = 16.$$

She emphasized that the third solution⁵⁸ was the correct answer

because it followed the aforementioned order of operations.

Maggie's handled the first two student's solutions by introducing her own WoU through a process of filtering students' responses to solicitations. While this pushed forward her mathematical agenda, this way of handling students' W_soU has the effect of strengthening students' authoritative proof schemes⁵⁹. The promotion of one particular WoU over others is unavoidable in this case if the teacher wishes to participate in the broader practices of the mathematical community. However, the question is how she attempted to justify the reason for existence of the promoted WoU. This was a case of an appeal to authority rather than a promotion of the value of the tool she intended for students to use.

Task 2: Evaluate $(7-3)^2 + (12/4)$.

⁵⁸ Recall that the first two students had arrived at 131 as a solution.

⁵⁹ An alternative approach might have been to ask students to compute on their own at their desks, observe students' W_soU searching for different solutions, and soliciting those different solutions at the board, emphasizing a need for agreement.

Three students came to the board. Maggie asked each to show their work and communicate their thinking. When the students finished, Maggie analyzed the solutions one-by-one asking if they used the correct order of operations. Analyzing the students' W_soU from the perspective of order of operations, Maggie showed the correct solution and made points about how mathematics should be written.

The students W_soU were:

$$\begin{array}{l} (7-3)^2 + \frac{12}{4} \\ 4 + 3 \\ = 7 \end{array}$$

$$\begin{array}{l} (7-3)^2 + \frac{12}{4} \\ 4^2 = 16 + \frac{12}{4} \\ \frac{12}{4} = 3 \quad 16 + 3 = 19 \end{array}$$

$$\begin{array}{l} (7-3)^2 + \frac{12}{4} \\ 4 \times 4 = 16 + 3 \\ = 19 \\ \begin{array}{r} 3 \\ 4 \overline{)12} \\ \underline{12} \\ 0 \end{array} \end{array}$$

Figure 5.3: Three students' W_soU $(7-3)^2 + \frac{12}{4}$.

One student subtracted, divided, and then added. He replaced the square at the end by writing 7^2 as his final answer. The other two arrived at the correct solution, but wrote mathematically incorrect statements. Still, their presentations indicated that they had used the promoted order of operations to arrive at their final answers.

Maggie allowed each presenter to share his/her WoU without correcting errors immediately. That is, she allowed the W_soU to exist side-by-side without immediately correcting errors. Next, Maggie analyzed all three using the order of operations.

In her analysis of the first students' WoU, Maggie pointed out that he might not have known what exponents were, but that she appreciated the clarity of what he had written.

T: So um, order of operations. First thing is parentheses. Did they all do parenthesis first? Yeah, they all found out that 7 minus 2 is 4. Ok, second thing that you need to do is what? Exponents. ... I like that way that Ace put this [pointing to the lines written in the first solution] and then he put the next step underneath it, and then he put his final step at the bottom. That makes it really clear.

Maggie's way of handling the first students' WoU is a case of identifying the role of communication in mathematics by incorporating the students' WoU even though there was an error in it. She promoted the mathematical practice of writing in a line-by-line manner with a minimal number of computations between lines.

Before she analyzed the next two WoU, Maggie presented her own as shown in figure 5.4.

$$\begin{aligned} &(7-3)^2 + \frac{12}{4} \\ &= 4^2 + \frac{12}{4} \\ &= 16 + \frac{12}{4} \\ &= 16 + 3 \\ &= 19 \end{aligned}$$

Figure 5.4: Maggie's WoU $(7-3)^2 + \frac{12}{4}$.

After making her point from the first student's WoU, Maggie explained every step as she solved the problem, modeling what a thorough way of recording her process of

simplification. She said, “I showed you everything I was doing, everything I was thinking. Ok, so I would like you to also do the same thing.”

Next, Maggie returned to her point about writing mathematics in a manner that is clear for the purposes of communicating one’s thinking well. She did by pointing to the third students’ WoU.

Maggie: I showed you everything I was doing, everything I was thinking. Ok, so I would like you to also do the same thing. We know that she got the right answer because she did a lot of stuff in her head. She knew already 4 times 4 is 16 and she added this little 3 at the end. Does 4 times 4, Ana, equal 16 plus 3? Is that true? No, that's not true. She kinda combined everything together. So let's try when we do these problems, ok, to do every single step because I want to know that you understand, ok, the order that it goes in and that you understand how to do all of these operations.

Maggie used the third solution to show how a WoU can be clear to the person who wrote it, but not clear to the reader. Again, she emphasized the need for the students to show all their thinking by writing statements that are correct and do not need further explanation. That is, she promoted the use of a common practice when writing mathematical work, but she did so by incorporating the students’ W_soU.

With respect to symbolic manipulation, this attention to clear communication is compatible with the referential symbolic proof scheme because it aids in the development of the transformational way of thinking that the form of an expression may change, but its value can not. Otherwise, the two expressions are not equivalent. Maggie emphasized the need to communicate clearly to each other (particularly from student to teacher), but her actions also communicated the former lesson.

A stark contrast exists between Maggie’s treatment of task 1 and task 2. In the former case Maggie’s justification for the knowledge she intended students to learn was based on the text as a source of authority rather than a need for agreement on the meaning

of an expression. In the latter case, Maggie emphasized the need for everyone who reads mathematical text to understand what is written. The second way of handling students' $W_s o U$ allowed Maggie to promote a common mathematical practice by appealing to an intellectual need for clear communication.

Observed Teaching Practices, Connections to the PD, and Connections to Proof Schemes

In this lesson Maggie solicited multiple $W_s o U$ for task 1 and task 2. In task 1, Maggie handled the students' $W_s o U$ by soliciting an alternative $W o U$ through a process of filtering student answers to guiding questions. This $W o U$ was promoted over the first two students' $W_s o U$ because it conformed to the text's $W o U$ order of operations. This treatment is seen as strengthening students' authoritative proof schemes.

In contrast, in task 2, Maggie also solicited multiple $W_s o U$. However, in this case she handled the students' $W_s o U$ by acknowledging them and raising a further issue for discussion. Though she introduced her own $W o U$, she compared $W_s o U$ in a way that addressed a need for clear communication. Therefore, while Maggie was telling students to write their work in a particular way, she had justified the need to do so using their own thinking as a platform. This treatment of students' $W_s o U$ is seen as a contributing factor in the promotion of the referential symbolic proof scheme.

There were many instances in which similar points were made at the PD in the first year. In one case TR told Maggie, "One more line is all it takes to make it clear for students." In another case, TR brought up the difference between approximation and accuracy, at a time when Maggie was using the equal sign when she should have been using the symbol for approximation. This level of attention to detail is related to the development of the deductive proof scheme.

In chapter 4, it was shown that attention to detail was an important component in the development of Maggie's proof schemes. There were several instances during the PD when Maggie made mental errors in computation. The TR explained that it is a common practice of his to compute twice. Additionally, in her experiences at the PD, Maggie worked in small groups daily, presented to the class five times during the first summer, and watched presentations daily. In each case, the need for clear communication was evident. Therefore, her treatment of task 2 is seen as closely compatible with, and possibly inspired by, those of TR.

Lesson 3

Tasks

Write the expression using exponents.

1. $5*5*s*s*t*t*t*u*u$

2. $(3*3*3*a*a*b)/(b*b*a)$

3. $d*d*d*d/d*d$

Reduce:

4. $\frac{t^3}{t^6}$

5. $\frac{t^4}{t^2}$

Topic of lesson

Simplifying expressions involving a quotient of two monomials with given exponents.

Background Events

In this lesson, data collection began after task 3, but the work from tasks 1 to 3 was still present on the board. This analysis focuses on how Maggie promoted a particular WoU in the tasks 4 and 5. From the data, it is not possible to tell where the WoU that one can expand rational exponents and cancel them originated. So, it is not possible to say with confidence whether or not Maggie was incorporating a student's WoU or simply promoting her own. However, it is possible to tell which WoU Maggie was promoting and what proof scheme this WoU was compatible with.

Claims

When finding a rule for simplifying rational expressions, Maggie emphasized a process in the generalization of a pattern. Through her examples, she pointed out that variables raised to an exponent whose value is a whole number can be written in an expanded form. Because any number⁶⁰ divided by itself is one, cancellation can take place whenever the same number (or variable) is present in the numerator and denominator of a fraction. Maggie's repeated examples created a situation in which she asked students to look for a pattern. Namely, that subtracting exponents is a shorter way to reduce rational expressions, as opposed to the method of expansion and cancellation. Maggie's emphasis of the process that validates the generalization through repeatedly reasoning with the process promoted the PPG proof scheme. The shortcut was not elicited from the assigned mathematical task, as evidenced by students' repeated misunderstanding of the task to create a shortcut.

This lesson also exemplifies a time when Maggie took responsibility for justifying students' conjectures.

Analysis

The first three tasks asked students to write expressions using exponents and cancel common terms in the numerator and denominator whenever possible. These three problems served as examples illustrating a problem the solving strategy of cancelling or combining like terms.

Task 4

Task 4 was presented by Maggie. To create a solution, she asked students to answer questions while she selected responses to incorporate into her solution from

⁶⁰ It was not mentioned that this is not true for zero.

among those offered. By filtering answers to her questions, Maggie demonstrated the

following solution, modeling a the WoU that one can expand and cancel $\frac{t^3}{t^6}$ to simplify.

$$\frac{t^3}{t^6} = \frac{t \cdot t \cdot t}{t \cdot t \cdot t \cdot t \cdot t \cdot t} = \frac{1}{t^3}$$

Figure 5.5: Maggie's WoU $\frac{t^3}{t^6}$

Maggie: (student name) what did you get on the top?

Student: I got t times t times t.

Maggie: Ok, so he had t times t times t. Anybody disagree with that? So far so good, ok how about on the bottom (same name).

Student: t times t times t times t times t times t.

Maggie: Ok. How many people got this so far? Beautiful, ok you got all of that. Now what do you do when you have this?

It is evident that Maggie was modeling a procedure for handling the situation that was known to the students before Maggie began simplifying $\frac{t^3}{t^6}$ by repeating a WoU.

Before the student was asked to expand, Maggie had already drawn a fraction bar. Her question, "Anybody disagree," could be taken as rhetorical in nature. It is not possible to tell from the data who introduced this WoU, but it is clear that Maggie expected this WoU to be institutionalized by the end of her presentation.

After expanding the numerator and the denominator, Maggie asked a student to help her cancel. As she explained how to find the simplification for $\frac{t^3}{t^6}$, Maggie

reminded students that $\frac{t}{t} = 1$ and that $\frac{1}{t}$ was irreducible. That is, Maggie promoted a

WoU that was useful for her to advance her mathematical point about cancellation. When

the student could not answer, Maggie reminded her how to continue cancelling and eventually pointed out the final answer $\frac{1}{t^3}$.

Identifying a Pattern

After tasks 1 to 4 were completed, Maggie asked students to find a pattern using the examples she had presented. She explained that she had already taught them that, “If the base is the same, you add the exponents. The answer has the same base and the sum of the exponents $[a^x \cdot a^y = a^{x+y}]$.” Maggie emphasized that she was asking them for a quick rule that could be used when dividing.

Maggie: So this is the part where we need to answer this ok. How do I divide powers? So you just saw this one, you saw this one, you saw that one over here, ok, we've done it about four times. Do you think you can come up with rule yet, when you divide powers, what is the rule?

When Maggie asked for a shortcut a student repeated the longer way of simplifying the expressions by expansion and cancelation which was demonstrated in the first four problems. This is an indication that this student did not see the need for a shortcut at this point.

Maggie: Ok we came up with our rule for multiplying, right. We said if the base is the same, then you can add the exponents. The answer has the same base and then it has the sum of the exponents. We came up with a rule in words, right? So what is our quick way of doing it with the rule? So here we have the same base we add the exponents, right. So when we're dividing have you seen a pattern yet? Everything that [the last student] explained is true about this one right here, but every time you divide...do we need to do another one? Should we do one more to see if we can pattern?

When Maggie asked if she needed to provide more examples, students mentioned the shortcut Maggie was looking for. Namely, that in the division situations they had explored one could simply subtract exponents. Maggie seized on the opportunity and

clarified what the student said by solving task 5 using the longer WoU, emphasizing that she could have used a shorter way.

Maggie: You subtract this exponent right here. Ok, let's see if that's true. On the top you're going to have how many t's?

Student: Four.

Maggie: On the bottom you're going to have how many t's? ... You're going to have two, so really in the end, **after you do all that canceling**. Cancel cancel cancel. You can only cancel up to how many? ... Whatever the smallest number is, right ... he said I'm going to have two left over because four minus two. So what did he do, he said t to the fourth minus two equals what? ... t to the second power.

By examining task 5 for confirmation, Maggie assumed responsibility for justifying the conjecture that powers could be subtracted. Though she solicited the WoU from students and followed up on it by showing how it could be used to solve a problem, Maggie did not charge the student with proving the conjecture.

After showing an example of how students could use a shortcut, Maggie gave students time to write a sentence to answer how they “divide powers”. Maggie recorded both $W_s o U$ that students offered.

Maggie: ... write the powers in expanded form then ... cancel the variables ... if possible... write the answer in fraction or simplest form. If you can avoid writing it [as a] fraction then that's the simplest form... this is our long way.

...

Maggie: ... When you divide powers, subtract... exponent from denominator from exponent from the numerator... So we're just subtracting exponents right.

Maggie promoted two $W_s o U$. First, to simplify an expression involving division of exponential monomials with common bases, one can expand and cancelation like terms before regrouping like terms using exponential notation. A shortcut for the aforementioned procedure entails keeping the common base and subtracting the exponent of the denominator from the exponent of the numerator. The shortcut was valued over the

longer solution method though the lesson centered on expansion and cancelation as the causes of the desired WoU. Thus, the promoted WoU is compatible with PPG because cancelation was explicitly mentioned as the cause for the rule.

The concept of subtracting exponents was not elicited from the problems. Students could simplify using expansion and cancelation. Their responses to the question repeated the longer way of solving the problem. This behavior indicates that, at least some, did not see the need for the shorter WoU. Through the teaching practice of soliciting and handling students' $W_s oU$ by filtering, Maggie identified and recorded the shortcut from one student response, but assumed responsibility for justifying the existence of the shortcut.

A broader point can be taken from this lesson. When the necessity principle is implemented, students provide $W_s oU$ that are their own. In this case, it is more reasonable for the teacher to ask students to take responsibility for explaining why their solutions are reasonable than when the teacher asks students to create shortcuts for reasons students may not understand or appreciate, simply to satisfy the teacher. This is related to Brousseau's notion of didactical situations.

Observed Teaching Practices, Connections to the PD, and Connections to Proof Schemes

A pedagogical point presents itself through this lesson, one that was mentioned many times at the PD. Even though Maggie had the question, "How do I divide powers," on the board for all students to see, and she asked them for a shortcut, many continued to use the longer WoU that had been developed. Being clear about her objectives did not necessarily cause students to change their $W_s oU$. This is not to say that several students did not mention the shortcut after the discussion. Rather, Maggie's approach of using

examples helped students develop the WoU that expansion and cancellation are valuable tools for simplifying rational expressions. However, there was no intellectual need for the shortcut as a tool useful for solving problems.

Maggie assumed responsibility for justifying her claims that she wished to promote from among students' W_soU. She did so in a manner that was compatible with PPG.

Lesson 4

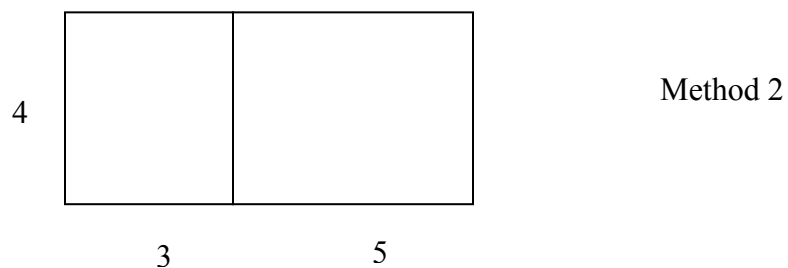
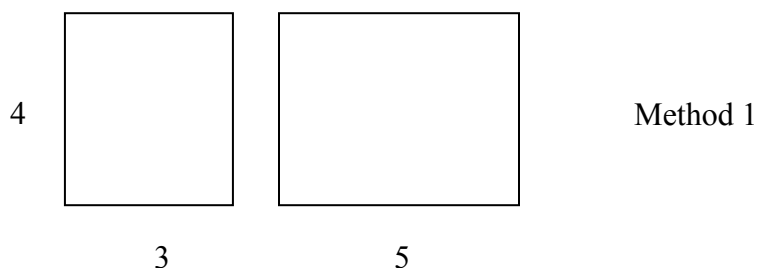
Tasks

1. Multiply mentally [meaning with no paper].

a. $6(52)$

b. $(104)9$

2. Find the total area of the two rectangles using two methods:



3. On a piece of paper, draw and label the dimensions of three rectangles with the same width and different lengths. Repeat method 1 and method 2 with each pair of rectangles.

What do you notice about your results?

Focus of lesson

The distributive property.

Claims

Maggie's treatment of distribution involved a cycle of contextualizing, decontextualizing, and then recontextualizing distribution for her students. In her approach, Maggie initially asked students to multiply numbers mentally. From their $W_s o U$, she attended to one in which a student used the distributive property intuitively. Maggie compared this $W o U$ to a symbolic form of the distributive property found in the students' textbook, decontextualizing it. In tasks 2 and 3, she provided a geometric context for distribution.

Her teaching practices paralleled those at the PD in several ways. She handled students' $W_s o U$ by soliciting multiple $W_s o U$, requesting alternative $W_s o U$, and incorporating students' $W_s o U$. Her task choice made use of the G-P-A model (described in the analysis section and in the hill problem in chapter 4).

The episode also contains a valuable example of a missed opportunity which can be contrasted to an important theme of the PD – handling student errors. When she corrected a student's solution to task 2 publically, an opportunity was missed that may otherwise have been helpful in diminishing the effects of the authoritative proof scheme among students. An important tension is highlighted here between several roles a teacher plays in the classroom including: attending to students' psychological needs and attending to mathematical accuracy.

Finally, this episode demonstrates evidence that the $W o U$ Maggie promoted is compatible with the transformational proof scheme. During the PD Maggie consistently demonstrated a transformational proof scheme (referential symbolic) with respect to symbolic manipulation. Thus, the observed events of this episode are consistent with observations about Maggie's proof schemes witnessed in the PD data.

Analysis of Events

Students' W_soU task 1

To begin, Maggie asked students to multiply $6(52)$ mentally. She solicited students' W_soU, writing them on the board: 612, 312, 312, 312, 302, 312. After asking students what they were thinking, she identified two W_soU. In one WoU, students imagined the traditional algorithm for multiplying two numbers (see figure 5.5). In this method, students envisioned carrying.

$$\begin{array}{r} 52 \\ \times 6 \\ \hline 312 \end{array}$$

Figure 5.6: Some students' WoU mental multiplication

Sunny's WoU $6(52)$ mentally is shown in figure 5.5. Later she explained the error, changing her WoU to figure 5.6.

$$6(100) + 6(2)$$

Figure 5.7: Sunny's initial WoU $6(52)$

$$6(50) + 6(2)$$

Figure 5.8: Sunny's corrected WoU $6(52)$

In part b of task 1, Maggie heard the identical W_soU from students.

Maggie's Handling of Students' W_soU in Task 1

After gathering students' W_soU task 1 (part a) Maggie labeled the first WoU, using pictures in your head. Maggie explained operating with these "pictures" can be the source of errors for students. "What happens when you think like this? You have a big picture in your head and things like this happen. He said he forgot to borrow, or he forgot to carry the 1. Okay, so he got 302." Next, she solicited alternative W_soU by

saying, ““I thought there would be an easier way... Let’s do another one and see if you guys come up with some other things.” Using the motivation of finding an “easier way” became a theme for Maggie throughout the episode.

After assigning task b and gathering students’ W_soU, Maggie showed the class the distributive property in a symbolic form, $a(b+c) = ab + ac$. She compared Sunny’s WoU to the symbolic form. Using an Sunny’s WoU task 1 (problem 2 – multiply $9(104)$ mentally), Maggie promoted the WoU, that $9(104) = 9(100+4) = 9(100)+9(4) = 900+36 = 936$. In doing so, Maggie promoted the WoT that one can change the form of an expression without changing its value a form of the transformational proof scheme.

In another example, Maggie asked which way looks more like distribution, $6(80) + 6(3)$ or direct multiplication using the algorithm? When students said $6(80) + 6(3)$ she asked why. She praised one student’s answer, “She distributed the 6 to the 80 and then the 6 to the 3.” Maggie promoted the WoU that it is valid (and desirable) to “split” 83 into 80 and 3 before multiplying, then adding.

Throughout her lesson Maggie motivated the change of form by repeatedly asking her students which way of computing was “easier”, direct multiplication or using the distributive property.

“[Sunny] said I don't have to multiply ... 9 times 104. I can pull that number apart and make it **easier** for me to multiply in my head. ‘Cause 9 times 100 is **easy**, that's 900, and then 9 times 4 is pretty easy, that's 36. And then I can put them together in the end, right?...”

Maggie’s interpretation of what the student said was that it was easier to solve the problem using distribution. The student did not actually use that word.

Students’ W_soU Task 2

Two students were asked to present their solutions to task 2, one for method 1 and one for method 2. The first student solved method 2. He concluded that the area could be found by multiplying 4, 3, and 5. His final answer was 60. Maggie stepped in and fixed the solution with the student, changing the WoU to $4(5+3)=4(8)=32$. The second student's WoU was that the areas of the two rectangles could be computed separately than added: $4(3) + 4(5) = 12 + 20 = 32$.

Maggie's Handling of the Students' WoU Task 2

When the two students came to the board to offer their WoU, Maggie pointed out an error in the first student's solution.

Maggie: Arturo ... what does area mean to you? How do you find an area?

Arturo: Multiply?

Maggie: You multiply what?

Arturo: The length and width.

Maggie: Okay, so what length do you see there, on that rectangle?

Arturo: 4.

Maggie: And then on the other side, what's the total length that you see?

Arturo: 8.

Maggie: 8, right? If you multiply those together, ... One side is 4, right? And you're gonna multiply that by the other side. What is the other side? 8. What's your final answer?

Arturo: 32.

Maggie guided Arturo in the correction of his solution rather than allow a potential contradiction to arise between his solution and the second student's.

Maggie eventually made the mathematical point that the solutions were the same regardless of the order in which one computes (adding before multiplying or distributing the multiplication before adding).

Maggie: [Arturo] didn't distribute that, right? He just said, I'm gonna add this side up first, get the whole thing, and then do my area formula, which is length times width. So he went 4 times 8 and he got 32, okay? So what does the Distributive Property tell us? If we, okay, take the whole length and we just multiply two

numbers together, we get an answer. What happens if we take that 8 and we separate it into two numbers that equal 8, 3 plus 5, okay? If we, if we take it separately, um, will we still get the same answer? What do you think? Yes or no?

...

Maggie: Yes.

This is a case when Maggie chose not to let the student's error persist. From this data it cannot be determined which of these reasons outweighed the others in Maggie's decision-making process. However, this instance raises the question of timing. When should a teacher attempt to correct a student's WoU and when should she let an error persist? This is an issue that was discussed at the PD. There Maggie pointed out that students have the capability to correct each others' errors and learn from each other. Therefore, it is clear that she was aware of the issue at the PD.

Were the promoted W_soU elicited from the assigned mathematical tasks

The promoted W_soU were rooted in Sunny's WoU how to multiply mentally. Although she initially responded incorrectly, the error was corrected and a new WoU emerged consistent with the distributive property. Maggie incorporated Sunny's WoU by comparing it to the algebraic representation she had prepared and clarifying the step where Sunny split the two digit number into a sum of numbers. Furthermore, the point that the solution is the same through direct calculation or through distribution was made later in the episode. Maggie's attention to students' need for efficiency was apparent in her task selection.

Observed Teaching Practices, Connections to the PD, and Connections to Proof Schemes

In handling students' W_soU, Maggie incorporated Sunny's. She corrected an errant WoU and compared the corrected solution to the second student's WoU task 2. In

both cases, Maggie made her mathematical points via comparisons between $W_s o U$. In the former case, Maggie compared Sunny's $W o U$ to an algebraic $W o U$ distribution. In the latter case, Maggie compared the order in which operations were carried out, noting that the results were the same. The use of students' $W_s o U$ to make the teacher's mathematical points by comparing them was a teaching practice routinely observed at the PD.

Handling student solutions is a teaching action that was discussed at the PD. There it was noted that allowing a student error to persist for some time is a legitimate teaching behavior. In this case Maggie missed an opportunity to allow students to fix an error. Allowing students to fix the error could have been a way to make her point more strongly and help renegotiate the didactical contract regarding who is responsible for errors during presentations. This is a way of handling student solutions in a public forum that is noteworthy because of its potential to increase the power of the authoritative proof scheme.

Nevertheless, the situation is complicated by other factors. For example, had Maggie let the error persist Arturo might have been embarrassed when other students pointed out the error. As a result he may have become more reluctant to share his $W o U$ in the future. Also, Maggie may have been concerned that students wouldn't have seen her mathematical point that the solutions are the same given either way of computing. Given the nature of the data, it is not possible to tell why Maggie chose to intervene. However, what is clear is that she did intervene. Given the discussion about this teaching practice at the institute, the instance is noteworthy.

Maggie's treatment of distribution involved a cycle of contextualizing, decontextualizing, and then recontextualizing distribution. Mental math served to

contextualize the concept initially. Showing the algebraic representation of distribution served to decontextualized distribution. The use of rectangles served to recontextualize distribution.

This cycle of instruction, is a teaching practice compatible with the TR's teaching practices at the PD. The teaching practice included an implementation of the necessity principle and is related to the G-P-A (Geometry, Physical world, Algebra) model presented at the PD. The case of the Hill problem (year 1) included an explicit public conversation regarding the G-P-A model between TR and Maggie as she presented her solution.

TR: But right now, really you are focusing on the geometric reality. You are trying to derive as much information from [it]... You are connecting it here [referring to the graph] because you are saying that's where they meet [referring to the point where two lines segments meet], and this point represents the meeting point, right? And that connection is really making a connection between the geometry and the physical reality. ... And now you are expressing that algebraically. So you are combining the three in some ways.

In this example, TR explained how Maggie used three representations (graphical/geometric, algebraic/symbolic, and visualization of movements between runners) in her solution of the Hill problem. This connection between representations was a vital component of the TR's teaching practices at the PD.

In chapter 4 it was found that during her two years at the PD, Maggie consistently manifested $W_s o U$ compatible with the referential symbolic proof scheme. Maggie's choice to provide a geometric context for the distributive property is an example of a time

when she created a setting for her students to understand the meaning of the distributive property. This is an essential teaching practice for students to develop the referential symbolic proof scheme for themselves. It is also an example of what Ball and Bass (2003) call mathematical knowledge for teaching (MKT).

Lesson 5

Tasks

1. Use the map of Washington, D.C. to answer the following questions:
 - a. How would you describe the locations of George Washington University, DuPont Circle, the White House, and Union Station?
 - b. How would you describe the directions of Massachusetts Ave, New York Ave, and Vermont Ave?
 - c. How would find the distance form Union Station to DuPont Circle?
2. Use the following map to answer questions related to distance

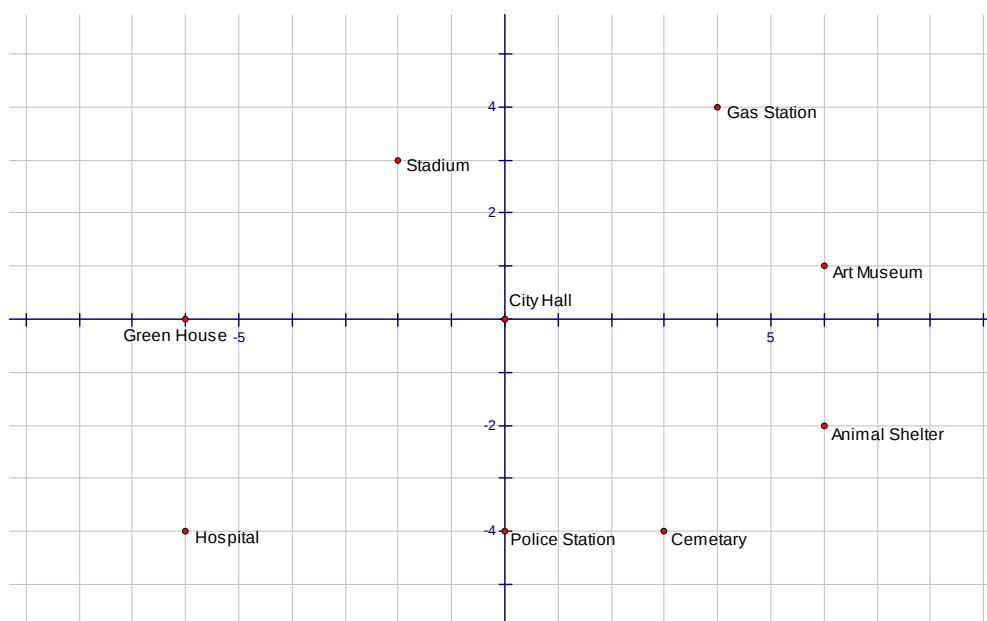


Figure 5.9: Map of a fictitious city

- a. Write the locations of all landmarks.
- b. How far [shortest taxi-cab distance] from hospital to cemetery? City Hall to Police Station? Art Museum to Gas Station?

- c. In part b, how can you tell the distance from one place to another using only the coordinates... hospital to cemetery? City Hall to Police Station? Art Museum to Gas Station?

3. If the Ice Cream shop is 3 blocks from City Hall, give 3 or 4 possible locations for it. The library is 3 blocks from the stadium. Where might it be?

Topic of lesson

(1) Writing coordinates

(2) Taxi-cab distance

Background Events

This analysis focuses on parts b and c of task 2. In task 1, Maggie introduced students to a map of Washington D.C.. Task 1 asked questions about how to: describe the location of a place on a map, describe the direction of streets, and determine distance between locations. Maggie used the real world situation having diagonal streets and uneven block lengths to reintroduce the Cartesian coordinate system to students in a real world context. Student responses provided evidence that some students did not recall how to represent a point in the Cartesian coordinate system. So, Maggie reminded them how to do so twice – first in task 1 and again in task 2a.

As a final note about the context of this problem, it is important to note that absolute values were not mentioned at any time in the less. However, one student, Rogelio provided a WoU length of a horizontal line segment that made use of zero as a reference point. That is, Rogelio had a geometric WoU absolute value. During the hill problem analysis from the PD Maggie evidenced this WoU absolute value, but made no

mention of absolute values explicitly, while she did seize on Rogelio's WoU to solve problem 2c.

Claims

In the analysis of task 2b, as she solicited students' W_soU Maggie encountered a WoU horizontal (and vertical) distance as counting on by pairing 1 with no movement. This produced lengths that were consistently too large. She handled this WoU by soliciting alternative W_soU first, then sharing her own WoU in the absence of alternative W_soU. Maggie identified the students' mental imagery that guided their WoU and told students to count as she had. The WoU Maggie promoted is consistent with the deductive proof scheme.

When points were not on the same horizontal or vertical line (e.g. – the distance from the art museum to the gas station), Maggie claimed that the shortest driving distance is the sum of the lengths of the horizontal and vertical line segments along an eligible path drawn between the two points⁶¹. Her justification for this claim was compatible with the RPG proof scheme.

In the analysis of task 2c, students were asked to find a way to compute distances using only coordinates. That is, to use arithmetic to do so. A student offered a WoU horizontal distance that entailed adding absolute values in the case when one location is positive and another is negative. Maggie incorporated this WoU to provide an answer when one location is zero and the other positive. When both locations have positive (but different) ordinates, Maggie encountered the WoU that the answer can still be found

⁶¹ Though Maggie did not explicitly say so, her work indicated that the shortest path could only be drawn by moving north and west, never south or east. This subject came up in a conversation which is not included here.

using addition. She promoted the WoU that in this case horizontal distance can be found using subtraction. Again her justification for the claim was compatible with the RPG proof scheme.

In the results section it is also explained that task 2c, in particular, is a case in which the necessity principle was not implemented. It is also noted that Maggie's task selection was compatible with the G-P-A model advocated at the PD. A pattern of handling student $W_s o U$ emerged with rich connections to Maggie's own proof schemes as demonstrated at the PD. Finally, sources of student errors as they are connected to Maggie's teaching practices in this episode are also explored briefly.

Analysis

Task 2 centers on helping students learn to find the shortest driving distance from one point to another within a Cartesian coordinate plane. The approach is first to ensure that students know how to name points on the plane. This will not be discussed in the analysis, but student responses in task 1 and task 2a indicated this was not a well-developed skill for students who were not called upon. Maggie conducted a brief review of the topic by telling students how to name points and then asking some questions from the board that students answered from their seats.

Maggie used the setting of maps and direction to motivate and provide a context for computing vertical, horizontal, and diagonal distance. As she asked students for answers to tasks 2b and 2c, she encountered undesirable student $W_s o U$ horizontal and vertical distance computation in several forms. Maggie handled these $W_s o U$ by soliciting alternative $W_s o U$, identifying and correcting errors herself by telling them her WoU, justifying her $W_s o U$, and identifying mental imagery guiding the $W_s o U$. In so doing, she

assumed primary responsibility for verifying and justifying results. In task 2c, Maggie incorporated a student's WoU. It is shown that this incorporation was the source of students' errors in the final part of 2c.

Analysis of task 2b

Maggie began the whole-class discussion of task 2b⁶² by asking a student to tell her how many blocks it was from the hospital to the cemetery. The student responded, ten blocks. When asked to find the distance, this student (and possibly others) counted before moving. Ultimately, Maggie assumed responsibility for correcting the error and justifying the new WoU. She did this by introducing a WoU that coordinated movement along a graph with counting by starting at zero rather than one, using the context of the problem in her justification.

Maggie: We need to find locations of how far the distance is from the Hospital going all the way to Cemetery, okay? If I were to travel this distance, and we're considering each of these squares to be a block, how far did I go? Manny.

Student: 10 blocks.

Maggie: 10 blocks. Okay, how far did I go, um, Miguel?

Student: 10.

Maggie: 10 blocks. Okay, how far did I go, Miriam?

Student: 10 blocks.

When Maggie encountered the initial response of 10 blocks she solicited three more W_soU. No student contradicted the original solution. Her attempt to locate a different of understanding led her to handle the WoU by offering her own.

Maggie: Okay, if I start at the hospital, have I traveled at all yet?

Student: No.

Maggie: I'm at what distance?

Student: Zero.

Maggie: I'm at zero. I haven't traveled at all. I'm gonna walk along here [motioning horizontally on graph]. When I hit this first intersection, that's a

⁶² Recall that in problem 2b the hospital is located at (-6, -4) and the cemetery is located at (3,-4).

block. 1. So count with me everybody. 2 ... 3...4...5...6...7...8...9.

Maggie began by reminding students of the context of the problem. Each square on the map is one block. After two attempts to solicit a solution other than 10, Maggie asked two leading questions, “Okay, if I start at the hospital, have I traveled at all yet,” and, “I’m at what distance?” In her questions Maggie conveyed the message that counting should begin only when movement happens. A lack of physical movement should be paired with the number zero.

Maggie continued by speculating about what Manny did to create his answer and telling him to count only after moving.

Maggie: Okay, that's 9, right? 9 blocks. So, Manny, how did you get 10? Did you count 1 right here? Is that what happened?

Student: Yeah.

Maggie: Okay, so you have to remember, until you finish traveling that block you can't count it. Okay.

Immediately after this interaction, Maggie encountered other students counting in the same way. Her treatment was the same as before. In the presence of an undesirable solution, she solicited an alternative WoU. When no other solution was offered, she counted for the students by coordinating her movement with the numbers again, repeating the previous teaching behavior of handling students’ undesirable W_soU by first soliciting alternative W_soU and then demonstrating a different WoU.

To find the distance between two points that do not lie along the same horizontal or vertical line (e.g. – the art museum and the gas station), Maggie emphasized the use of a taxi-cab metric together with directions in terms of north, south, east, and west. When a student offered his desirable WoU, Maggie solicited an alternative WoU.

Arturo: Go 2 blocks West.

Maggie: Okay, West is this direction and he says 2 blocks.

Arturo: Then, 3 blocks North.

Maggie: 3 blocks North. So, total of how many blocks?

Student: 5.

Maggie: Okay, so our distance was 5 blocks. Okay? Can anybody tell me a different way to get there from Arturo's way?

After the next student offered a WoU compatible with Arturo's, Maggie introduced a third WoU, pointing out that all three W_{soU} yield the same result.

Maggie: Now, what if I decide to go North... and then I go West, I go North again, West, and then North. Is that longer or shorter...or the same?

Student: Same.

Maggie: Okay, let's see. 1...2...3...4...5. It's the same thing. So, probably anything that I do within this rectangle is going to be 5 blocks. Let me make sure. North 2 blocks, West 1 block, North 1 block, West 1 block. 1...2...3...4...5. It's still the same thing.

A student pointed out that within the rectangle with vertices at (4,1), (6,1), (6,4), and (4,4), there are longer paths by pointing out a counter-example. Maggie reframed her claim, saying that 5 was the shortest number of blocks. She justified her claim by recomputing the length of her path.

Summary of Analysis of task 2b

Maggie's treatments of part b emphasized the meaning of quantities within their context. Counting blocks by coordinating movements in the horizontal/vertical case and returning to the setting of the problem in the diagonal case are examples of how Maggie's justification appealed to her mental images of the problem.

A general pattern emerged in the handling of student solutions. When an undesirable WoU was offered by a student, Maggie solicited alternative W_{soU} . When alternatives were not present, Maggie offered her own WoU and justified it. In the last part of task 2b, Maggie solicited an alternative WoU and provided her own, but used

them for different reasons than she had in the first two parts. In the first two questions all W_soU offered were in agreement. Therefore, none acted as perturbations to other students. In the last part of the question, the alternative W_soU served as confirmation of a result. That is, when Maggie justified the assertion that the shortest taxi-cab distance from the Art Museum to the Gas station was 5 blocks, she did so in a manner compatible with the RPG proof scheme.

Analysis of task 2c

In her introduction to task 2c⁶³, Maggie reminded the students that by using the graph, they had already determined the distance from the hospital to the cemetery was 9 blocks and city hall to the police station was 4 blocks. The task asked for a way that used only the coordinates without the graph.

After explaining the question Maggie solicited Rogelio's WoU.

Maggie: Rogelio, what do you think?

Rogelio: Since the two last numbers are -4 (inaudible)

Maggie: Okay, he said the two last numbers are the same number, so what does that mean?

Rogelio: It's on the same part.

Maggie: Okay, he's saying that means it's on the same, on the same line, the same part, okay? So...

Rogelio: And then with the other ones, the -6 and the 3, you just go from -6 to 0 and then 0 to 3 and that's it.

Maggie: Okay, he's saying that if you walked from -6 to 0, he knows that this would be 6 blocks, and then you walked to 3, which would start at 0 to 3, okay, and that would be another 3 blocks, and?

Rogelio: That's 9.

Maggie: And that's 9 blocks, and he would add those together. So you're saying that, even though that's a negative, it's still 6 blocks, right? And this is positive it's 3 blocks that you walk. So he's saying: If I just look at my x's I can add up all my x's and I can get the number of blocks that we walked, right? In this case it seems like it's true, okay, so we're gonna try one more. The next one was City

⁶³ How can you tell the distance from one place to another using only the coordinates... hospital to cemetery (6,-4) to (3,-4)?

Hall to the Police Station. Ron, can you give me those two? I'm gonna see if Rogelio's idea works for this other one.

As Rogelio explained his WoU, Maggie rephrased it by referencing which parts of the graph he was referring to. In so doing, Maggie provided justification for Rogelio's WoU. She explained that his WoU was that when the abscissa of the two points in question match, to find their distance apart one can apply the WoU used to solve problems in part b. She said, "So he's saying: If I just look at my x 's I can add up all my x 's and I can get the number of blocks that we walked, right?" It is important to note that Maggie used the phrase, "I can add up all my x 's." This phrasing was unfortunate as it meant one thing for Maggie and another for students.

Maggie recognized that Rogelio's WoU represented an opportunity to define a procedure for computing horizontal (or vertical) distance. She gave Rogelio's WoU the status of a conjecture by saying, "In this case it seems like it's true, okay, so we're gonna try one more... I'm gonna see if Rogelio's idea works for this other one."

Maggie continued by applying what Rogelio had done to a vertical case (i.e., finding the distance from city hall to the police station). As she explained the computations, Maggie and a student miscommunicated. Each used addition to describe different WoU. While Maggie described a WoU involving counting distance between the abscissa, the student described a WoU involving addition of the abscissa.

Maggie: He just looked at the x 's, so 0 and 0. That's 0 blocks on the X-axis, right? Okay, so his way worked because over here he had different numbers. What about the y ? What should I do with the y ? Now I have different numbers.

...

Student: just leave it. (inaudible)

Maggie: When you go 0 to 4, is there any movement at all?

Student: Yeah, 4.

Maggie: There's 4. So, we're calling them blocks, right? So, when you go from 0

to 4...

Student: You add them.

Maggie: Okay, he's saying you can add them up. And you go a total of what?

Student: 4.

Maggie: 4 blocks on the Y-axis. So, altogether, how many blocks were traveled?

Student: 4.

Maggie: Okay, 4 blocks altogether. Is this the answer we got for number 2 on part b?

Student: Yeah.

Maggie: So, let's put that together, because over here we have everything is in the same place on the Y, so we're just counting the X. Over here everything is in the same place on the X, so we're just counting the Y. What about when you have some movement on the X and some movement on the Y?

While Rogelio's WoU entailed adding both of the abscissa, Maggie's WoU entailed counting. Their uses of the word add meant different things. Rogelio's was literal while Maggie's was figurative. Still, Maggie tried to clarify her WoU later in the conversation when she said, "We're just counting the X," and, "We're just counting the Y." Maggie did not address the student's WoU that distance is the sum of the abscissa in this case beyond focusing on the fact that it also yields the correct answer.

Subtly, Maggie altered the previous WoU, adding elements of her own WoU. She began by saying, "He just looked at the x's, so 0 and 0. That's 0 blocks on the X-axis, right?" This WoU differed from Rogelio's because it considered the horizontal movement as adding no blocks to the final computation rather than disregarding it altogether. Therefore, Maggie provided justification for the procedure of disregarding the ordinates of the two points by introducing the notion that one should add horizontal and vertical components of movement from one point to another. She returned to this point as she concluded saying,

"So, let's put that together. Because over here we have everything is in the same place on the Y. So, we're just counting the X. Over here everything is in the same

place on the X. So, we're just counting the Y. What about when you have some movement on the X and some movement on the Y?"

As Maggie completed the last question, "What is distance from the Art Museum to the Gas Station," she guided the solution from start to finish, asking which mathematical operation could be used to find the answer. Several students had the WoU that finding horizontal (or vertical) distances is accomplished by addition. Indeed, this had been the case in both of the previous examples ($6 + 3$ and $4 + 0$). Maggie handled this WoU directly by telling students her WoU, emphasizing a result without emphasis on the process that causes the result to hold.

Maggie: Okay, on the X-axis, from 6 to 4, how many blocks did we walk? Should we add these numbers? Should we subtract these numbers? What can we do? Manny?

Student:(inaudible)

Maggie: Nelly, what do you think? If you go from 4 to 6, how far did you walk?

Student: 2 blocks?

Maggie: You walked 2 blocks. Does that make sense to everybody? So what did we do? Did we add the numbers or subtract them?

Student: Add.

Maggie: In this case we subtracted them. We said: 6 minus 4 equals 2. She said we walked 2 blocks. Those are our Xs. Okay?

Maggie's focus was on which operations that could be used to find result, 2, from the given values, 6 and 4, without attention to why the operation is appropriate beyond using the result as confirming evidence for the result itself. This treatment can be contrasted to the treatment given to divisibility of fractions in year 2 of this data set.

The scene continued, first by finding the correct answer to the question among student responses, and then by identifying an operation that could be used to combine the values to get the desired result.

Maggie: Now let's see how far we went on the Y-axis. If we went from 1 to 4, 1 to 4, how far did we walk? Rafael?

Student: 5 blocks?

Maggie: Okay, so if you went from 1 all the way to 4, you walked 5 blocks? Evangeline, what do you think? If you went from 1 to 4, how many blocks is that?

Student: 5.

Maggie: Okay, 5 blocks. These people are adding them. If I go from 1 to 4, okay? If I'm at, on the Y-axis, I'm at 1 and I go up to 4, is that 5 blocks?

Maggie appealed to another student's W_{soU} initially when she did not get a desirable WoU through her first solicitation. Next, Maggie made it explicit how she believed the students were arriving at their solution, 5 blocks. Along the way, Maggie hinted that the answer was not 5. She said, "So, if you went from 1 *all the way*⁶⁴ to 4..." Then she explained what was done without validating the WoU . In contrast, when the desired WoU was offered, Maggie validated the result immediately.

Student: No. 3.

Maggie: It's 3 blocks, right? ... So, is that addition? Do I want to add 4 and 1?

Student: Subtract.

Maggie: I want to subtract. 4 minus 1 is...

Student: 3.

Maggie: 3 blocks, okay? Now, all together, how far did I walk? Juan?

Student: 5 blocks.

Maggie: Okay, so in the end, tell me what you did in the end.

Student: Added two numbers, added 2 and 3.

Maggie: Okay, you need to add the distance from the X and the distance that you walked from the Y, and then you put it all together and it's 5 blocks, right?

A pattern of dealing with students' W_{soU} can be observed in this interaction.

When students did not provide a desirable WoU , Maggie solicited alternative W_{soU} and ultimately told them her own WoU by first validating a desirable WoU .

Were the W_{soU} elicited from the assigned mathematical tasks

This kind of a problem does not implement the necessity principle. There was no intellectual need for a shortcut in part 2c. The problem had small enough numbers and

⁶⁴ Italics added for emphasis.

students already had a solution that had been confirmed by Maggie's previous counting strategy. In the data, there is evidence (not presented here for the sake of brevity) that the only reason for finding a different way of computing the distance in 2c was social in nature – to please the teacher.

Observed teaching practices, Connections to the PD, and Connections to Proof Schemes

Maggie posed a problem to students in the context of mapping. Maggie referenced the context repeatedly as she attempted to fix student errors related to counting on in task 2b. She also referred to the context as she described the taxi-cab metric used to find diagonal distances. Providing a task with a physical context and promoting its use to justify $W_s o U$ are teaching practices that promote the referential symbolic $W o T$, a $W o T$ Maggie demonstrated evidence of consistently at the PD.

A pattern of soliciting alternate $W_s o U$ has been observed in this episode. The pattern of solicitation was used in two different ways. In the more common form, Maggie asked for alternative $W_s o U$ when students did not offer a desirable $W o U$. After several solicitations, Maggie explained her own $W o U$ and asked students to change theirs to hers. Then she explained what she thought the students' mental imageries guiding their $W_s o U$ were. The teacher, not the students, was responsible for justification. The focus was on soliciting answers without reasoning. With one exception, Rogelio's $W o U$, Maggie supplied the explanation for students' undesirable responses. As a matter of contrast, it is noteworthy that Maggie did not emphasize the need for students to share the mental imagery that guided their solutions as she had in past episodes.

This episode contains data compatible with Maggie's self-reported teaching practices at the PD. In the second summer of the PD Maggie mentioned that she was

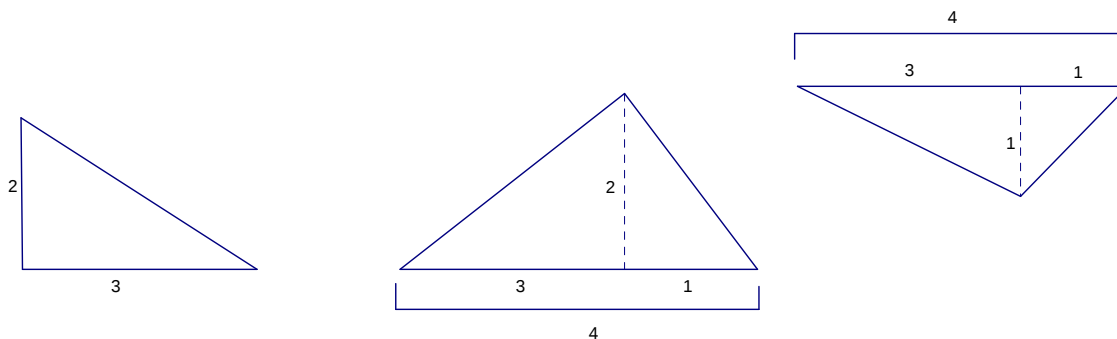
aware of times when she stopped short of proving certain results, unaware that what she had provided was not a complete proof. In this episode Maggie's WoU the shortest taxi-cab distance between two points and her justification of the choice of operations for computing the length of horizontal and vertical segments using only coordinates were examples consistent with her reported practice. This analysis pointed out that the promoted $W_s o U$ were compatible with the RPG proof scheme.

In task 2c intellectual need was not used to elicit concepts. The students were asked to find a shortcut in a situation that did not require one. While the WoU promoted was RPG-compatible and Maggie referred to the context with regularity in her own explanations, this abrupt introduction of task 2c, "How can you tell the distance from one place to another using only the coordinates," without an intellectual need to do so promoted the non-referential and authoritative proof schemes.

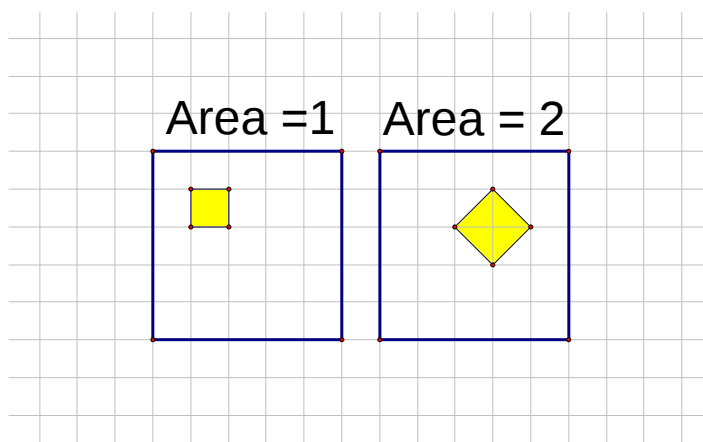
Lesson 6

Task

Task #1: Find the area of the following triangles:



Task #2: Find the other six squares with unique areas within a 5x5 grid: (This question included finding the areas of the other six squares)



Task #3: (Question raised in class discussion)

What is the length of the side of a tilted square (see diagram above) whose area is 2? 8?

5? 10?

Topic of lesson

Area, finding diagonal lengths, square roots

Background Events

This lesson focused on reminding students how to find the area of a triangle⁶⁵, computing the area of upright and tilted squares in a manner compatible with a known proof of the Pythagorean Theorem, and learning how to find the length of a side of a square, given its area. The analysis of the lesson pertains to all three tasks assigned.

Claims

During this lesson Maggie regularly solicited students' W_soU each of the tasks, but did not solicit alternative W_soU. In task 1 there were two kinds of student W_soU offered. When the WoU was desirable, Maggie asked the student to explain her solution. The student's WoU was validated. When the other two students did not know how to begin, Maggie handled the situation by explaining a WoU that highlighted the process by which a triangle's area can be found. Eventually, Maggie generalized a pattern by supporting it with the process used repeatedly to find the area of a triangle.

To construct squares, Maggie promoted a WoU compatible with the empirical proof scheme. Though she justified the fact that the sides were congruent, Maggie did not attempt an explanation of why her method would produce right angles⁶⁶. The WoU Maggie promoted for finding the area of the tilted squares was compatible with the PPG proof scheme. The WoU Maggie promoted for determining how to find the length of a side of a square given its area was compatible with the RPG proof scheme.

⁶⁵ In all examples, the altitude to the selected vertex stayed inside the triangle.

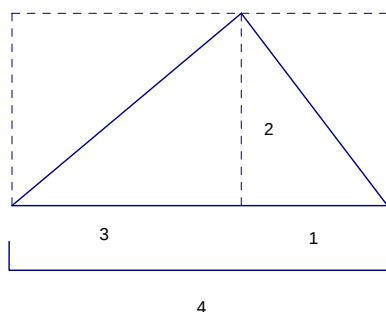
⁶⁶ It is not possible to determine whether or not such a discussion was even appropriate for these students.

Throughout the lesson, Maggie asked students to provide explanations and asked students to question the presenter. These teaching behaviors associated with the teaching action of handling students' public presentations are seen as helping to diminish the effects of students' authoritative proof schemes. A connection is made to Maggie's experiences with an emphasis on repeated reasoning at the PD.

Analysis

Task 1

The lesson began with Maggie calling three students to the board to solve the three questions in task #1. The first student concluded that the area of the triangle was three by comparing it to the area of a 2×3 rectangle. The second and third students were unable to solve their problems. Maggie solved these problems with the students at the board for the class to see.



Maggie's solutions repeated the following reasoning. A triangle can be cut into two triangles (using the altitude)⁶⁷. Then each of the two right triangles can be seen as halves of two different rectangles. The sum of the areas of the two triangles is the area of the entire triangle.

⁶⁷ Maggie did not bring up the possibility that the given altitude might fall outside of the triangle.

When the three parts of task 1 had been solved with the individual students at the board, Maggie reviewed the second and the third problems with the class. After her review of all three parts she made a generalization. She pointed out that in general when finding the area of a triangle, a rectangle can be used to enclose the triangle and the area of the triangle will be half the area of the rectangle saying, “So does this work every single time for a triangle?” In this way, Maggie generalized a process to create the area formula for a triangle. From these two cases, Maggie generalized the process by writing the formula,

$$\begin{aligned} \text{Area of triangle} &= \frac{1}{2} \times l \times w \\ &= \frac{1}{2} \times b \times h \end{aligned}$$

where l is the length and w is the width of the rectangle. She explained how the length of the rectangle was the same as the base of the triangle – as were the width of the rectangle and the height of the triangle. This approach is compatible with the PPG proof scheme.

Maggie handled the first student’s WoU by asking her why and by asking her peers if they agreed. When no resistance was offered, Maggie validated the solution and the class moved on. Neither of the next two students could solve the problem nor offer a solution. In the presence of a solution, Maggie demonstrated a WoU she said had already been explained. That is, she repeated the aforementioned WoU to students.

Task 2

To introduced task #2, Maggie drew the 1x1 upright square; labeling it as $A = 1$ (area equal to 1). She did the same for the tilted square which she called $A = 2$ (area equal to 2). After drawing a few examples, Maggie asked a few students to draw the remaining

tilted squares. In each case, Maggie asked students to explain how they found the area of the tilted square. One student created a rhombus rather than a square. Consequently, Maggie reminded students that squares must have four right angles. She pointed out visually that the object drawn did not have right angles, but offered no justification either that the method students were using guaranteed right angles or that the student's method had not produced right angles. It was visually obvious to Maggie and the class that the object did not have right angles.

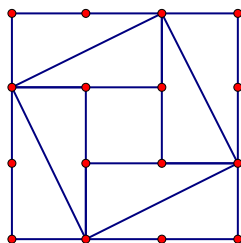
Maggie emphasized repeatedly that each side of the square students were drawing is diagonal of a rectangle saying,

Maggie: It's the diagonal of a one-by-two. Is yours a diagonal of a one-by-two?

Student: Yup.

Maggie: So it's the same length, isn't it?

This WoU was useful in determining not only the area of the triangles, but also provided justification that the sides were all congruent.



Four students presented tilted squares. In each case, Maggie asked students to explain how they found the areas of the tilted squares. After the second student presented his solution for the area = 5 tilted square, Maggie fleshed out the students' solution for

other students and validated it. As she did so, she provided a deductive explanation for why the areas of the tilted square⁶⁸ was 5.

Maggie: Let's go over this. You can see that this goes across as the diagonal. It's the diagonal of what kind of rectangle? A one by? Two. So, how is he getting these areas? If it's a diagonal of a one by two units across, then what would the area of a one by two be? The area of a one by two would be two. He sees that the diagonal cuts it in half, so he knows that the area then is half of two, which is?

S: One.

Maggie: One, right? So he knows this is one. These are all the same triangles, but they're just pointing in different directions, right? So he knows this is also one, one, one, of course the square unit in the middle is one, you add that up, it should be an area of five. So this is absolutely right.

The WoU students presented, and Maggie promoted, for finding the area of the tilted square was that the area can be found by dividing the tilted squares into four or five regions including four right triangles and one smaller upright square. As discussed in task 1, the area of a triangle is half the area of the rectangle enclosing it. The area of the upright squares was found by counting. The sum of the areas of the four or five regions is the area of the tilted square. The WoU presented by students and promoted by Maggie was deductive in nature and would play an important part in determining a way to find diagonal distances on the plane task 3 and in the next lesson (lesson 7) on the Pythagorean Theorem. Thus, this problem sequence constitutes the implementation of the necessity principle.

The teaching practice of handling students' W_soU by soliciting not just answers to questions, but also mental imageries, is both noteworthy and of interest primarily because it has potential to diminish the effects of students' authoritative proof schemes. This practice was often observed at the PD. In that setting, TR asked teachers to compare and

⁶⁸ It was not proven that the angles were right angles, but the area is proven to be 5. So the proof is only deductive as it pertains to the area of the figure not the type of figure.

contrast different WoU. The emphasis was on what the guiding imageries were behind the solutions. In small groups Maggie demonstrated the ability to compare her solutions to those of her group mates. In chapter 4, it is documented that Maggie was aware that often different people see the same problem in different ways and that these images are important in understanding individuals' final conclusions.

Task 3

Maggie began considering the question of how to find the length of the sides of squares. She drew and wrote the areas of the four upright squares. Then, through a question and answer process, she wrote the lengths of the sides of each square. A student offered the WoU that the length of each side of a square is the square root of the area of the square. Maggie recorded it on the board and treated it as fact from this point forward.

Next, Maggie began drawing the tilted squares, writing their areas, and asking for the length of each slanted side. Students guessed at the length of a side of the tilted squares, but had difficulty finding them. Maggie pointed out to the students that the WoU she had previously highlighted also works for tilted squares. She said, "Over here [pointing to a tilted square with area = 2], can I still follow that same pattern?... Yeah. I take my area and I take the square root [writing square root of 2]..." Maggie appealed to the pattern of the first four examples.

Maggie defined square root as the number when multiplied by itself gives the radicand. From student answers to questions, Maggie ascertained that students' WoU square root is to take half the value of the radicand. She handled this WoU by checking student responses against her WoU square root; multiplying each student's guess by itself.

In addressing the problem of finding the lengths of sides of squares, Maggie encouraged the RPG proof scheme. The generalization was made before the definition of square root was discussed. Implicit, but not explicit, in the conversation was the idea that the area of a square can be found by squaring a side of the square and that square root was the appropriate operation to be used for finding a side given the area. Maggie used the WoU that if square rooting can be used to find the length of sides in the upright cases, it will continue to work in the tilted cases. This too was assumed but not discussed.

Observed teaching practices, Connections to the PD, and Connections to Proof Schemes

It was noted in the analysis section that the teaching practices of asking students to explain their thinking and the W_soU Maggie promoted had potential to diminish students' authoritative proof schemes. In the analysis, it was also noted that Maggie promoted W_soU compatible with RPG and PPG.

This episode provides evidence of a time when the necessity principle was implemented. At the PD, it was a common practice for TR to assign several interrelated tasks in which concepts were elicited from the tasks. Each task used results from the previous task as a tool for solving that task. This can be contrasted with the tasks assigned in lesson 5 in which an abrupt abstraction was requested and students had no intellectual need to create a tool for solving problems because the numbers were not large enough to do so.

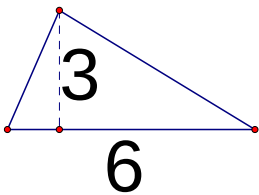
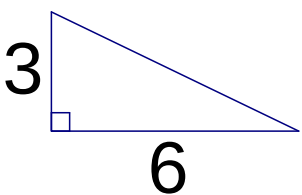
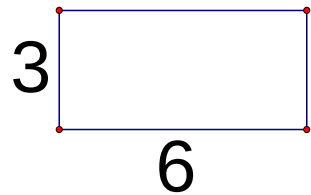
The PD also provided evidence in a geometric context (the Hill Problem and the Cat and Mouse Problem, Year 1) of a time when Maggie's conviction rested on visual form of the empirical proof scheme. In this lesson, the same was true of Maggie's

handling of the proof that the angles of the tilted object were right angles. It was assumed as visually obvious.

Lesson 7

Tasks

1. Find the area of the following figures.



2. For each row, draw a right triangle with the given leg lengths on dot paper. Then draw a square on each side of the triangle.

Length of leg #1	Length of leg #2	Area of square on leg 1	Area of square on leg 2	Area of square on hypotenuse	Length of hypotenuse
1	1	1	1	2	
1	2				
2	2				
1	3				
2	3				
3	3				
3	4				

3. For each triangle, find the areas of the squares on the legs and on the hypotenuse.

Record your results.

4. Look for a pattern in the relationship among the areas of the three squares. Use the pattern you discover to make a conjecture about the relationship among the areas.
5. Draw a right triangle with side lengths that are different from those in the table. Use your triangle to test your conjecture from part C.

Topic of lesson

The Pythagorean Theorem

Background Events

Lesson 7 is a continuation of the material from episodes 5 and 6. In episode 5, Maggie used a taxi-cab metric to find distances between points. In episode 6, Maggie focused on a way to find the area of squares (upright and tilted) and the relationship between the length of the sides of a square and the area of the square. Maggie's instructional approach continues along the same path, relying on previously learned material as tools useful finding a pattern in the relationship between legs of a right triangle and its hypotenuse (the Pythagorean Theorem).

Zaslavsky, Harel, and Manaster (2006) have examined Maggie's choice of examples in this episode, conducted a stimulated recall interview, and made claims about her content knowledge based on these data sources. They found her knowledge in this area to be, "rather deep". Still, they noted "...some missed or misleading" opportunities in this lesson. While their points were focused primarily on the treatment of examples, this analysis concludes that there were missed opportunities in the handling of students' W₃oU in terms of who has the responsibility to justify claims.

Claims

As pre-algebra students, Maggie's students may not have the geometric background to complete a proof of the theorem. Still, this analysis shows that Maggie promoted $W_s o U$ that encouraged students to reason in a manner consistent with the PPG proof scheme. Maggie handled students' $W_s o U$ by asking them to predict and confirm by recalculating, but she did not ask students to prove their conjecture. That is, from a mathematical perspective, the teacher's attention and explanation was consistent with the PPG proof scheme.

The analysis also points out that Maggie was primarily responsible for justifying claims, which she did in a manner indicating rich and flexible knowledge of the subject matter. However, Maggie did not require (nor ask) students to prove their conjecture or even to attempt to prove the conjecture. Though it may well have been beyond them, the episode makes salient an important question. As teachers use their mathematical knowledge to make mathematically accurate and deep points in their instruction, how do they begin to extend the locus of authority to students? For if this is not accomplished, no matter how desirable the presentation the teacher inadvertently continues to encourage the authoritative proof scheme.

Analysis:

Task 1

Three students came to the board to solve task 1. Two students were unable to find the areas of the triangles independently. Maggie handled their $W_s o U$ by repeating the reasoning she had previously demonstrated in episode 6, enclosing triangle within a rectangle using the base of the triangle as the base of the rectangle in both cases. For the

right triangle, Maggie explained why the area of a right triangle is half of its completed rectangle by appealing to the notion of congruent triangles. For the second triangle, Maggie explained that given the entire base only she could not split the triangle along the altitude. Students suggested that the base could be split in two segments of lengths 2 and 4. Maggie acknowledged and used the students' suggestion by completing the problem in a manner consistent with her WoU demonstrated in episode 6, emphasizing that the problem should be solved either using the formula ($\frac{1}{2}$ length x width) or by splitting the triangle's base visually and finding the sum of the two areas.

In their review of this lesson, Zaslavsky, Harel, and Manaster (2006) pointed out that "trying out other and realizing that the area remains the same could have been more supportive." They also noted that Maggie's careful wording explaining that the split along the base (recall students suggested splitting the base into 4 and 2) could have happened in other ways and that students probably did not realize this point. This point is related to the need to have students involved in the justification process, rather than just the teacher. While the point may have been clear to Maggie, Zaslavsky, Harel, and Manaster noted that it most likely was missed by students.

In her recapitulation of the problem, Maggie justified the formula ($\text{Area} = \frac{1}{2}$ length x width), repeatedly asking for the reason that the area of the triangle is half of the area of the rectangle. Her WoU was that the diagonal of a rectangle creates two congruent triangles because the two triangles have the same length and width. Next, Maggie claimed that this formula ($\text{Area} = \frac{1}{2}$ base x height), held for all triangles⁶⁹ and she used

⁶⁹ As was the case in episode 6, Maggie did not address the possibility that the altitude of a triangle could fall outside of the triangle.

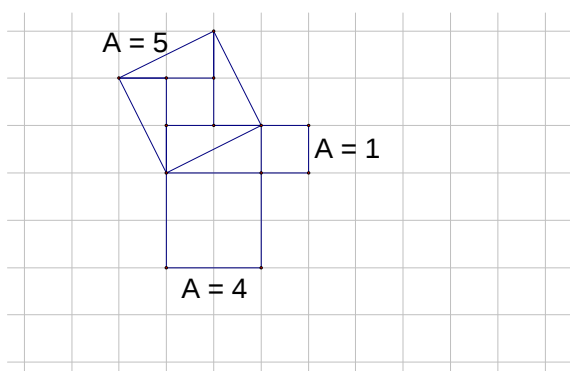
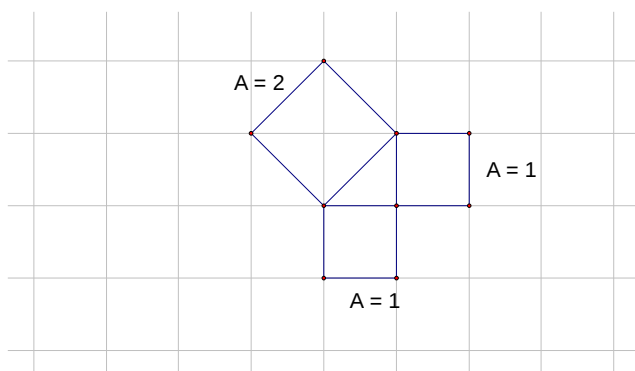
it in the case of the second triangle. After investigating the result for the second triangle in task 1, she returned to the point that the total area is the same either way you compute. Maggie emphasized the point that students need to do what makes sense to them personally.

With respect to the kinds of triangles Maggie was referring to in her examples, the WoU Maggie promoted was deductive in nature. Her repeated reasoning about the situation by comparing triangles (or parts of triangles) to rectangles and her ability to incorporate student's thinking into her presentation on the fly are signs of rich, flexible knowledge about the subject. The level of depth with which she raised the point that a diagonal of a rectangle creates two congruent right triangles was appropriate for her students.

Tasks 2-4⁷⁰

After raising the first question, "What is the relationship between the legs and the hypotenuse of a right triangle," Maggie reviewed the terms right triangle, legs, and hypotenuse. Maggie explained that in order to answer the question students would be asked to draw squares from each leg and the hypotenuse. She referred to the table in question 2 as showed students how to draw the triangles with appropriate squares along legs and diagonals.

⁷⁰ For a more detailed account of how Maggie computed in the four cases she demonstrated, see Zaslavsky, Harel, and Manaster (2006).



Maggie recapitulated how to find the areas of the squares formed along the legs and hypotenuse in by repeating the reasoning she used in episode 6. In the case of each right triangle, as she had done in task 1, Maggie used the WoU that a right triangle has half the area of a rectangle with dimensions equal to the lengths of its legs. Two cases are shown, but Maggie demonstrated the first four cases in the table before asking students to try the rest on their own.

Between the second and third demonstrations, Maggie asked students to predict the area of the hypotenuse before she found it. Several students could not, but many had a prediction. When the class resumed the whole class discussion after working alone, Maggie once again asked students to make predictions about the size of the square along the hypotenuse by observing a pattern in the data.

Maggie asked students to state the relationship they observed between the legs and the hypotenuse. A student said, “If you add the areas of the squares on the legs, it will equal the area of the square on the hypotenuse.” She recorded this observation and pointed out that this was the answer to question #1. Regarding the pattern, Maggie commented, “Isn’t it kind of weird that it worked every time?” She continued by using the observation to fill in the last two lines of the chart and encouraged students to stop “using the drawings” and start using the observation.

Observed teaching practices, Connections to the PD, and Connections to Proof Schemes

Maggie’s approach to instruction through pattern generalization is compatible with TR’s approach at the PD. On many occasions, TR assigned problems that encouraged pattern generalization as a way to make conjectures. Differences between this Maggie’s teaching practices and TR’s may be attributed to several factors. Maggie’s students do not have the level of sophistication that PD participants had. Still, Maggie commented in the first week of year 2 of the PD that she often did not complete proofs or know how much is enough to say one has provided a proof.

Maggie: ...about the difference between these two derivations... I thought, well, if we aren't even motivated, I should speak for myself, to find, the proof for the first one where there is a conjecture, how are we going to get our students motivated to do that. I guess it starts with us first. Secondly, this has been my problem the whole time. I thought I knew algebra. I thought I could do it well. I can find patterns very easily, I can come up with formulas, but I don't truly know the difference between when I've derived something and when I just have a conjecture. Sometimes I think I'm finished and you come along and you say, well what about this and I didn't think about it. I don't really know the difference between when I'm actually done and when I still have some work to do.

Maggie’s reflection on her practice referred to both her own mathematical knowledge and her teaching practice over year 1. She explained that she could find

patterns, but didn't know the difference between a derivation and a proof, nor did she know how to motivate students to go beyond conjecturing to the point of proving results. She credited TR's teaching practices with perturbing her to the point of discomfort, realizing that she did not know the difference. In chapter 4, it was explained that for Maggie this discomfort with proof became a theme.

Certainly, this episode brings to light important issues brought out in this case study raised, in part, by Maggie herself. When is the right time for a teacher to take a conjecture and push students to prove it? How does a teacher establish appropriate levels of rigor for her students when asking them to provide a proof? To what extent should a teacher be happy that she has provided a mathematically compelling argument when it may be the case that students have missed the point? Essentially, what teaching practices help extend the locus of authority beyond the teacher to the students at different grade-levels?

Here I speak of more than knowledge of mathematics, but a blending of knowledge of mathematics with knowledge of student epistemology. Nor is this knowledge necessarily a form of pedagogical content knowledge. In their construct of mathematical knowledge for teaching, Ball and Bass (2003) asked, "How is mathematical knowledge intertwined with other knowledge and sensibilities in the course of that work?" The questions raised here fall into this domain.

Lesson 8

Tasks

1. Solve each equation for y .

a. $y - 2x - 1 = 0$

b. $3x + y = 3$

c. $3x - y = 3$

d. graph $y - 2x - 1 = 0$

2. Toothpick problem: For toothpick patterns #1-3:

Write a sentence that describes the pattern.

Draw the first four figures of the pattern.

Find how many toothpicks are in the 10th figure, 100th figure, and the n^{th} figure.

Create your own pattern then answer all the same questions above.

Pattern #1

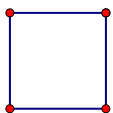


Figure 1

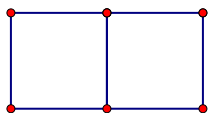


Figure 2

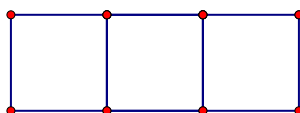


Figure 3

Pattern #2

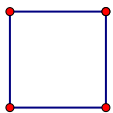


Figure 1

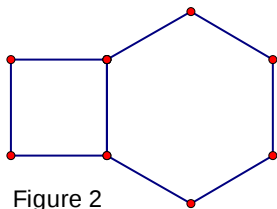


Figure 2

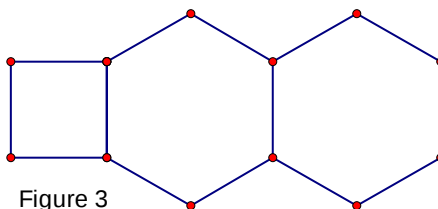


Figure 3

Pattern #3

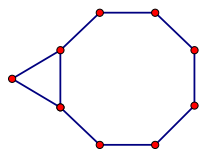


Figure 1

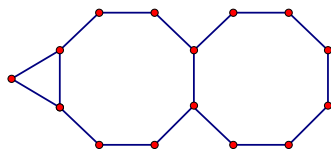


Figure 2

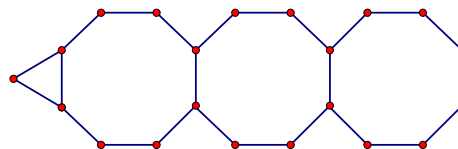


Figure 3

Topics of the lesson:

- Rules of symbolic manipulation – solving equations in two variables for one variable.
- Writing linear functions.
- Graphing linear functions.
- Slope.

Background Events

This analysis focuses on Maggie's treatment in task 2 of pattern #1. The treatment of patterns 2 and 3 was similar to her treatment of pattern 1, including how she handled students' $W_s o U$. Since task 1 was not an integral part of the main lesson, it will not be analyzed. Nevertheless, there is some information about task 1 that might be beneficial for the reader. During the warm-up Maggie discussed what it means to solve an equation in two variables, how to isolate the desired variable, and the symmetric property of equality indirectly. She emphasized that there are alternative ways to solve an equation – the symmetric property of equality allows this – and that students should solve in a manner that makes sense to them personally. This is significant because it opens the door to exploring connections between alternative $W_s o U$.

Claims

In this lesson Maggie encouraged students to make and defend conjectures about patterns. In her treatment of student $W_s o U$, Maggie encouraged students to observe patterns, make conjectures, and examine the cause for the pattern their conjectures. It is this last part, asking students to examine the cause for the pattern to hold that the lesson stands out. Maggie extended the responsibility for justification to her students by asking why.

The problems necessitated student thinking about writing expressions. Maggie promoted $W_s o U$ that were compatible with the referential symbolic proof scheme to validate students' conjectures about which expressions could be written to model the pattern. Though the validity of the function was checked, a process rather than the results formed the primary source of justification. Thus, the promoted $W_s o U$ were compatible with PPG.

Throughout the lesson, Maggie highlighted the source of the constant growth feature in each pattern as the figure number changes using both the diagrams and the table. She pointed out that this constant growth can be thought of as the slope of the linear function graphed, but did not highlight the difference between the discrete function representing the patterns and the continuous linear functions she drew. That is, the observed patterns were sequences. As such, the domains of functions representing them were the positive integers not the positive real numbers. Yet she drew continuous lines without mentioning the important distinction.

In summer 2 (episode 13), Maggie pointed out the difference between continuous and discrete functions for a class of weaker students. There is a strong connection to the

PD here. In the Jack and Jill speed problem (year 2) the difference between discrete and continuous functions was discussed. In that context, Maggie reported that she came to see the difference as valuable because of her error.

Analysis

Maggie began task 1 by drawing the first four figures of pattern #1. She asked a student to state the pattern in words and questioned why he saw the number 3 as valuable.

Maggie: Before we go on to see how many are in the 10th, let's just see if we can write a sentence that says more or less what we think the pattern is. What do we think the pattern is, in words? Andre? Describe the, the pattern for me. Just tell me what you see... I heard you say the number 3. Why is 3 important to you?

After the student answered, Maggie continued helping students connect the geometric shape with the observations that every figure grows by 3 toothpicks.

Maggie: Okay so, keep adding 3 sticks. Does this mean for every figure you're gonna add three sticks, right? What shape are they in?

Student: Square.

Maggie: Okay, 3 sticks in the shape of a square, right? Okay. Anybody want to add anything else? Does this sound like the right pattern to you? Start with 4 sticks in the shape of a square. For every figure, you're going to add 3 sticks, right?

Then she recorded a student's description of what he thought the pattern was on the front board. The student's observations led to a recursive understanding of the pattern. "So I'm always adding 3 sticks to the figure that I had right before this one, right?"

For pattern #1, Maggie suggested making a chart and filling it in up to 10, encouraging students to use the recursive observation about the pattern. She introduced variables, n and t , to stand for the figure number and number of toothpicks respectively.

Figure number (n)	# of Toothpicks (t)
1	4
2	7
3	10
4	13
5	16
6	19
7	22
8	25
9	28
10	31

Figure 5.10: Number of toothpicks

Once the pattern was established, Maggie used it to fill-in the number of toothpicks from 5 to 10 without drawing the figure.

Next, Maggie subtly began to introduce the need for an explicit WoU the pattern saying, “Okay. So, here's the part where we can, either keep going until we get to 10, or if you have a way of finding out how many are in the 10th figure, you can do that.” Later, she said, “Now, do we wanna keep doing this on our graph until 100? So, let's try to think about this in a smarter way.” Maggie transitioned students from the geometric representation toward more abstract ways of dealing with the pattern. “We don't need to draw it anymore, because we can kind of see a pattern, right?”

For the 100th figure, Maggie asked students to state their conjectures. In each case, she asked that students to explain why they gave their respective conjectures. For example,

Maggie: Vincent says for Figure 100, we're gonna take 31, and we're gonna multiply it by what? Take 31 and multiply it by 3. Okay, cause you think if there's 31, in the 10th Figure, you should multiply it by...

Vincent: No, cause... [After some time lapsed and Vincent did not answer]

Maggie: What do you mean? Cause why? Okay, so we'll leave it at that for now, and we'll come back to this. How many is that, Vincent? Okay, this is 93, okay. Is there another way that you can look at this, Juan?

Without deeming Vincent's response correct or incorrect, Maggie recorded his prediction and solicited an alternative WoU and asked that student why as well.

Juan: Um, you multiply 99 times 3?

Maggie: Okay, he says we're gonna multiply 99 times 3.

Juan: And that equals 297.

Maggie: And it equals 297. Done? So you think there's 297, 297 toothpicks? Okay, good. Are there any other ways?

Student: Yeah.

Maggie: No, no, no, wait. Let me first ask him why. Why are you doing this, Jose?

Eventually, Maggie encountered the correct answer, but chose to continue soliciting alternative WoU.

Maggie: Anybody else? Okay, Henry.

Henry: 99 times 3 plus 4.

Maggie: Okay, you have to tell me why you're doing this, Hector.

Henry: Because the, the um, the 99 for the, um 3 for the (inaudible).

Maggie: Okay, he said that it's 99 because that's how many squares there are. And then, he's multiplying it by 3 cause you're adding 3 toothpicks for each of those. Where is the 4?

Henry: From Figure 1.

Maggie: From Figure 1? So he's saying I can add Figure 1 into that. I only counted these squares that have 3. So, the first one had how many toothpicks?

Henry: Four.

Maggie: Had four. So he's just gonna add those at the end. So, let's see what the math looks like. 99 times 3, well, Jose did that for us, it's what? It's 297. And when I add 4, what does this equal?

Henry: 301.

Maggie: It equals 301. So he's saying there are 301 toothpicks. Okay, are there any other explanations for how you can get this without doing the table all the way to 100.

Maggie continued soliciting alternative $W_s o U$ after her interaction with Henry. She encountered another $W o U$. For every figure there are three toothpicks. However, one is uncounted in the beginning. Maggie asked the student explain his thinking.

As she continued, Maggie asked students to turn their worded $W_s o U$ into a function expressing t in terms of n . She solicited students' algebraic representations of the connection between t and n . When she used a student's $W o U$, $t = 3n + 1$, she repeated the reasoning given by the student. By repeatedly connecting the students' $W_s o U$ to the diagram, her treatment of students' $W_s o U$ encouraged development of the referential symbolic proof scheme. Another $W o U$ was written on the board, $t = 4 + (n - 1)3$. Maggie showed algebraically, that it was equivalent to $t = 3n + 1$. Thus, the need for symbolic manipulation arose out of a need to compare $W_s o U$.

Next, Maggie asked students to graph the equation, $t = 3n + 1$. She drew a graph by plotting points from the table and connecting them to create a line. Maggie tried to emphasize connections between the algebraic and geometric representations asking questions like, "Is there anything on the graph that you can see or on the t chart that you can see that pattern with? Is there any pattern between our points that we have and how our line was? What was the pattern?"

By filtering student comments, incorporating the correct answers, Maggie pointed out that the left side of the table goes up by 1 and the right side goes up by 3. A student explained that the y values rise by 3 on the graph. Maggie emphasized that the x values

increase by 1. Maggie emphasized the proportional nature of the change between x and y values. For example, for every 2 figures n increases, the number of toothpicks will increase by 6. A student connected the points by making congruent right triangles. Maggie noted that the triangle would remain the same shape throughout the entire line. This kind of explanation has the potential to explain why the equation $t = 3n + 1$ generates a line on the plane with one exception. It does not account for values between points – the difference between discrete and continuous situations.

Her WoU slope was vertical change/horizontal change. She connected the t-chart for pattern #2 to the graphical representation of the function, $t = 5n - 1$. Finally, Maggie assigned pattern #3 for students to: draw the fourth figure, write a description of the pattern, create a t-chart for the first 10 figures, find the 100th figure, write a function, graph the function, and find the slope of the function.

Maggie's treatment of the two problems elicited the concept of a linear function as a tool for expressing a relationship between quantities in an observed pattern. She used multiple representations (written descriptions of the pattern, t-chart, algebraic expressions, and graphs) to demonstrate the concept. Maggie pointed out the recursive nature of the pattern before discussing slope and connected the proportional change to the geometric properties of the graph. Here I am referring to the "slope triangles" a student noticed together with her follow-up comment that the triangles would always be the "same shape". It was not explicit, but this could have been used to explain why the algebraic representation, $t = 3n + 1$, and geometric representation of the line on the plane are related. The name, slope, was given to the amount of toothpicks the pattern increased by for every figure.

The treatment described is an implementation of the necessity principle and an example of a time when Maggie's way of handling students' $W_s o U$ combined with the $W_s o U$ Maggie promoted, helped encourage the referential symbolic and PPG proof schemes. The most striking event was when Maggie compared the two algebraic representations of the pattern, $t = 3n + 1$ and $t = 4 + 3(n - 1)$, showing that they were the same. The need to do so came out of a natural curiosity to know if the two students had produced different representations of the pattern or the same one. It is also a time when Maggie came very close providing a complete proof of a result in the classroom setting.

Observed teaching practices, Connections to the PD, and Connections to Proof Schemes

Observed teaching practices and Connections to the PD

In this lesson, Maggie solicited alternative $W_s o U$ even after correct answers were stated. Her tolerance for incorrect answers, allowing them to exist without judgment as right or wrong, allowed her to come across multiple desirable $W_s o U$. This marks a significant teaching behavior associated with handling students' $W_s o U$. Maggie's way of handling students' $W_s o U$ demonstrated a value for not only the correct solution, but also for the mental images that guided the solution and the reasoning used to create the solution. By repeatedly using the students' reasoning, the source of conviction moved from the teacher to the students. This is an example of how Maggie extended the locus of authority to include her students.

It is also significant that Maggie solicited alternative $W_s o U$ in the presence of a desirable $W_o U$. By doing so, she was eventually able to solicit another desirable $W_o U$. If she had stopped at the first, she may never have seen the second. Furthermore, her statements that students should solve problems in ways that make sense to them would

have been just empty words. When multiple valid $W_s o U$ are present in the classroom and the teacher questions students' $W_s o U$ by asking for reasoning including mental imagery examples are given of different problem solving approaches that students might connect with.

Nevertheless, at the PD (summer 2), Maggie expressed concern over the presence of multiple $W_s o U$ in the classroom. Namely, she explained that students might get confused and not know what to take away from classroom lessons.

“... my concern would be do you know if the concept that you want to teach is going to come up? I mean you don't have any control over that and I think as teachers that's what we do when we lesson plan, we control exactly what we're going to teach in the classroom and if we just kind of let it flow the way it flows in here we don't know exactly what will come up... And then my other concern with that is with my students, you know, if we're along one path and something comes up and I go along another path I have some students that would just fall apart if I did that.”

Even after having had an experience in which students created multiple desirable $W_s o U$, Maggie noted reservations about relinquishing control over lessons to students despite the events of this lesson.

Again, with exception of task 1, the lesson contains a tight connection to the G-P-A model discussed at the institute. Task 2 and the multiple representations Maggie used were ideal for promoting connections between a physical situation, the geometry (as seen in a graph), and algebraic representations of patterns.

Connections to Proof Schemes

One finding of the summer institute data in chapter 4 was that Maggie became increasingly aware that her proofs were incomplete, both at the PD and in her teaching. This episode demonstrates, in a small way, one example of this incompleteness. In year 2

(episode 13), Maggie pointed out the difference between continuous and discrete functions for a class of weaker students. However, in this lesson she did not. In the Jack and Jill speed problem (year 2) the difference between discrete and continuous functions was discussed. In that context, Maggie reported that she came to see the difference as valuable because of her error. This connection between Maggie's mathematical knowledge and her teaching has to do with the development of the deductive proof scheme in this context. Attending to the domain of a function at all times was an important aspect of the PD and an emphasis of TR's teaching.

Year 2

Lesson 9

Task

Task 1: Parts of a whole

You have $\frac{3}{5}$ of a whole unit. You also have $\frac{1}{4}$ of a whole unit. What is the total amount?

Task 2: Life Goes On

Paula spent $\frac{1}{2}$ of her life in Miami. She spent $\frac{1}{3}$ of her life in Dallas. Then she moved to San Diego. She lived in San Diego for 3 years. How old is Paula?

Topic of lesson

Fractions and proportional reasoning

Background Events

The focus of this analysis is task 1. In this situation, Maggie asked two different students to share their W_soU. The analysis of each is conducted separately.

Claims

In handling student 1's WoU, Maggie targeted students' visually perceptive proof schemes and attempting to advance students' referential symbolic proof scheme. Through the W_soU Maggie promoted, she also targeted the WoT that changing the form of an object does not necessarily change its size.

Maggie's attention to the visual difference in the parts of figure 5.10 coupled with her questions, are reminiscent of her experience during the PD, including the Cat and Mouse problem and the Jack and Jill Hill Problems (both in year 1). In those cases, she accepted visually obvious results and learned to challenge them. In year 2 of the PD, on numerous occasions she was able to pin point a weakness in her arguments, but was

unable to address the weaknesses without help. Still, her awareness of weaknesses was an indication of an appreciation for deductive WsoT, as it is here.

The teaching practices documented here were compatible with those observed at the PD. Both Maggie and the TR added on to students' W_soU. Even when there was something incorrect about a WoU, both found ways of using their students' W_soU productively to advance their own mathematical agendas. It is also noted that in Maggie's teaching practice she asked students to share both their errant approaches and their corrected approach. This allowed the class to participate more fully in the student's problem solving approach.

Analysis

This analysis will consist of two smaller analyses. Each part focuses on one student's W_soU, how Maggie handled them, and the WoU/WoT Maggie promoted from each. As in other analyses, the final section connects Maggie's teaching practices and her experiences at the PD.

Student 1

Maggie asked student 1 to present his WoU the problem. The student used two rectangles; dividing the first into 4 parts and shading 1 part. In the second, he drew five parts; shading 3 (see figure 5.9).

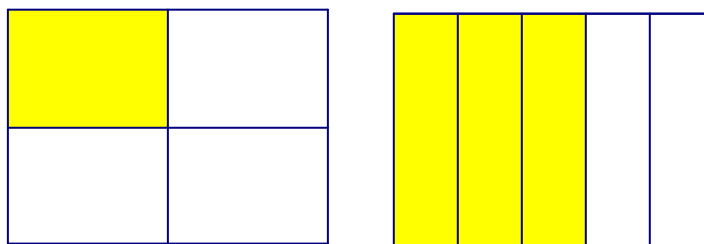


Figure 5.11: Student 1's first illustration of $\frac{1}{4} + \frac{3}{5}$

Maggie asked the student to share his original answer which was $\frac{4}{9}$. She pointed out that concluding the answer was $\frac{4}{9}$ from figure 5.9 was not consistent with the notion of fair sharing. So, student 1 changed his representation (see figure 5.10 below).

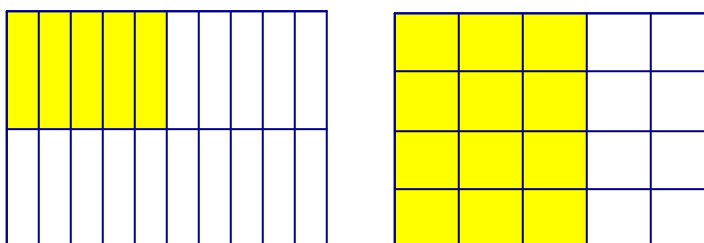


Figure 5.12: Student 1's illustration of $\frac{4}{20} + \frac{12}{20}$

Maggie asked, "Does everybody agree?... Why is it 12 out of 20?" Student 1 explained that it was because he had divided both rectangles into 20 parts.

Noticing that the individual pieces were different shapes, Maggie asked how students know that each shape represents $\frac{1}{20}$.

Maggie: Is everybody happy with this? Do you feel good about this piece [pointing to 1 rectangle in the figure on the right] being $\frac{1}{20}$ and this piece [pointing to a skinnier rectangle in the figure on the left] being $\frac{1}{20}$? Think about that for a second. Do you feel like everything's fine here because he's calling this

1 out of 20. Correct? And he's calling this piece right here 1 out of 20, do you agree?

Class: Yes.

Maggie: So you guys are fine with that. Eric, why do you agree with that?

...

How do I know that it's the same $1/20$ that it is over here? To me it looks like the shape is totally different. How do I know that it's the same $1/20$ that it is over here?

A student responded citing a procedural WoU. Maggie reported the student's point saying,

Maggie: He says it's okay because [student 1] multiplied 1×5 and got 5. He multiplied 4×5 to get 20. So $1/4$ is $5/20$. And then he said he multiplied 3×4 to get 12 and 5×12 to get 20, so $3/5$ equals $12/20$. But does that answer my question which is ... is this shape and size the same as having this shape and size? Is the amount the same in both of those?

Maggie stayed with the visual interpretation; continuing to ask why the value of each skinny rectangular part on the left must be the same as the values of the each part on the right. She acknowledged the WoU that was procedural in nature, but did not accept it because it did not account for the geometric difference between the two shapes. Maggie's reply indicated a profound awareness of the geometry involved. She said, "If there's any difference at all, there's a difference, though. Is there any difference at all? There is a difference."

Eventually, a student said, "They're equal units." Maggie replied, "They're equal units. Yeah, I'm so happy." From that point forward, Maggie treated the pieces as equivalent. Maggie's acceptance this WoU doesn't address the geometric question posed.

It is important to note that is compatible with Maggie's proving experiences at the PD. In year 2 she was often able to point out a missing link in a proof, but unable to fill the gap with out help. It was also shown that in the few geometric contexts observed,

Maggie's proofs were empirical (visual). In this case, it can be seen that Maggie's handling of the student's WoU were inspired by her experiences at the PD.

Student 2

Another student explained that she had an alternative WoU the problem. The student subdivided the rectangle into 20 congruent rectangles by drawing 5 vertical lines and four horizontal lines as shown in figure 5.11. Then she claimed that the left-most column of four rectangles is $\frac{1}{4}$ of the large rectangle and the lower row with 3 shaded rectangles is $\frac{3}{5}$ of the large rectangle. Eventually, Maggie realized that student 2 was not consistent about which unit she was referring to in her fractions. The overlap did not concern student 2.

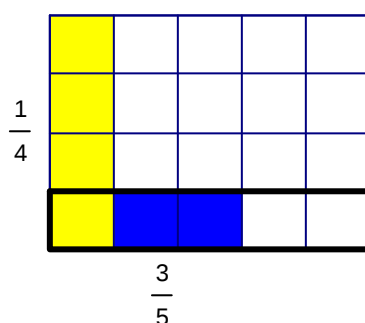


Figure 5.13: Student 2's WoU $\frac{1}{4} + \frac{3}{5}$

Though student 2 was unclear about her approach, Maggie clarified it, explained it to the class, and compared it to student 1's WoU.

Maggie: She wants to put $\frac{1}{4}$ and $\frac{3}{5}$ on the same rectangle ... Whereas [student 1] said, 'I'm going to draw what $\frac{1}{4}$ looks like, and then I'll draw what $\frac{3}{5}$ looks like, and then I'll see what I can do with those two things.'

She claimed that it differed from the first WoU because the student 2 was attempting to compute her result using only one rectangle.

Maggie held up student 2's WoU and tried to fix it by addressing errors in it saying,

"This is a really good drawing. So why shouldn't we understand this in a different way... So let's help her make it fit on here. So where is $\frac{1}{4}$? We'll start from scratch, okay? ... I like your idea. We're going to tweak it just a little bit."

That is, Maggie found something in student 2's solution she could work with to incorporate the student's WoU, rather than abandon it completely. This is a teaching practice Maggie experienced first-hand at the PD during the Arithmetic Polygon problem. When she presented an incomplete proof, TR held her solution up as an example in which a teacher can use a student's WoU to advance his own mathematical agenda.

Maggie used student 2's WoU to promote her own WoU in which the lower horizontal rectangle outlined is $\frac{1}{4}$ of the entire figure because it contains 5 out of 20 parts. To understand $\frac{3}{5}$ of the figure Maggie promoted the WoU that each row contains 5 parts. If she would shade 3 parts of every 5 contained in a row, she would have $\frac{3}{5}$ of that row. By repeating this procedure for each of the 4 rows she could guarantee that she had $\frac{3}{5}$ of the entire figure. This created an overlap for 3 pieces which would throw off the count. Following a student's suggestion, Maggie moved the 3 overlapping rectangles (see figure 5.12), concluding that a total of 17 out of 20 rectangles must be shaded. Maggie also converted the fractions individually and added in a more conventional manner.

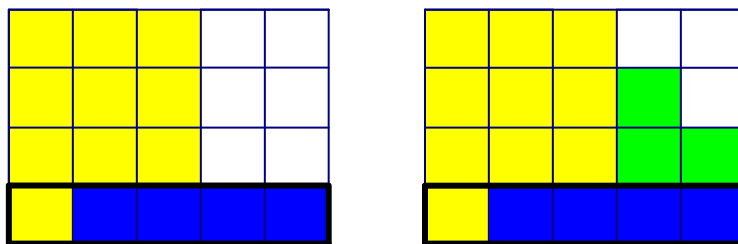


Figure 5.14: Maggie's WoU incorporating student 2's WoU $\frac{1}{4} + \frac{3}{5}$

Observed teaching practices, Connections to the PD, and Connections to Proof Schemes

Maggie handled the students' W_soU by correcting errors immediately when they were observed, asking students to communicate their thinking, comparing them, and incorporating them. She asked students to communicate their thinking and even raised the important point that the parts drawn in student 1's diagrams were not congruent, but were "equal units". Bringing up this point is an example of addressing students' visually empirical W_soT in a manner that could encourage the deductive proof scheme.

In the analysis, it was mentioned that Maggie's handling of students W_soU was similar to TR's handling of her own W_soU at the PD. Each characteristic mentioned was observed at the PD. Additionally, Maggie demonstrated the ability to pinpoint a missing a lynchpin in her arguments at the PD. In geometric settings, it was observed twice that Maggie's W_soU were compatible with visually empirical W_soT. In this case, while Maggie pointed out a shortcoming in student 1's explanation, she was unable to explain why despite the geometric differences in the pieces, they must be congruent in a geometric manner⁷¹.

⁷¹ One way might have been to subdivide each of the squares into 40th's by cutting the left square's rows in half horizontally and the right square's columns in half vertically. This would have allowed her to recompose pieces geometrically in order to address the visual difference between the shapes.

Asking student 1 to go back and share his error, as well as, his completed solution is another teaching practice witnessed at the PD. Often TR asked students, and Maggie in particular, to “tell the whole story”. This teaching practice encouraged her to share the mental imagery guiding her WoU so that other participants could understand, follow, and ask questions about the problem solving approach. These teaching practices were witnessed in this lesson in a very similar way. Student 2 was told that the class could not understand her thinking if she didn’t share it. The teaching practice of asking for a student’s solution together with mental imageries may not be unique to the PD or Maggie’s classroom, but it was a commonality observed in this case.

Lesson 10

Task

Task 1

1. $8 - 3\frac{3}{4}$

2. $8\frac{1}{3} - 2\frac{5}{6}$

3. $8\frac{4}{7} - 1\frac{1}{2}$

4. $3\frac{2}{3} - 1\frac{8}{15}$

Task 2: To Grandmother's House

Grandmother lives $\frac{3}{4}$ mile from the grocery store. You can walk $\frac{1}{8}$ mile in one minute.

How long will it take you to walk from the grocery store to grandmother's house?

Task 3: All Aboard

A train is traveling from San Diego to Santa Barbara. It takes an hour to travel $\frac{1}{4}$ of the way there. The conductor stops $\frac{7}{8}$ of the way there to let some passengers board the train. At this point, how long have the passengers who boarded in San Diego been on the train?

Topic of lesson

Arithmetic of fractions, proportional reasoning.

Background Events

Maggie began the lesson by referring to lesson 9, "Parts of a Whole" problem. In that instance, students drew rectangles and partitioned them to represent fractions. In that lesson, Maggie highlighted the WoU that the denominator of a fractions tells how many

pieces a whole unit is cut into and the numerator tells how many pieces of that whole one has. Two students demonstrated $W_s o U \frac{1}{4} + \frac{3}{5}$.

Claims

In task 1, as she handled students' procedural $W_s o U$ the subtraction of mixed numbers, Maggie's teaching practices promoted a referential symbolic proof scheme. She did so by identifying times when students were in doubt, or didn't understand, a presented $W_o U$. In these instances she repeated the $W_o U$ demonstrated in lesson 9 (Parts of a whole), providing a geometric context to justify the arithmetic in question.

In task 2, three $W_s o U$ were presented. Maggie followed up the three presentations by asking, "What kind of operation do you think you just did? So I'm talking adding, subtracting, multiplying, dividing." Evidence from this lesson, a later lesson, and the PD indicates that Maggie was hoping students would see that the problem was about dividing fractions. That is, this lesson constitutes a strong instantiation of an attempt to use the necessity principle to help students see the need to divide fractions in a context where multiple solutions strategies would work. The lesson is also reminiscent of the PD in the sense that, according to Maggie, tasks 2 and 3 were designed as interrelated and could both be solved using multiple solution strategies.

Analysis

Task 1

Maggie began her class discussion by illustrating $\frac{1}{4} + \frac{1}{3}$, repeating the reasoning of lesson 9 (Parts of a Whole). She gave a meaning for the term denominator in the context of the drawing she had created (see figure 5.13). Maggie said, "In the picture, I don't know what a denominator is. So, what do you call it?" A student answered, "The

parts?” Maggie accepted this term and noted that the parts are not the same size in the two representations of the whole.

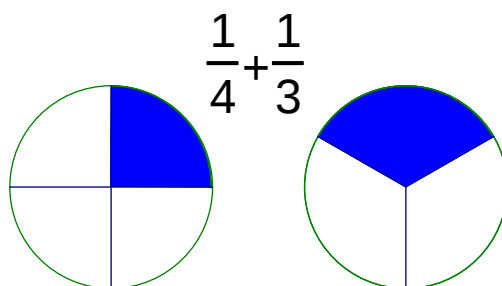


Figure 5.15: Maggie’s representation of one-fourth and one-third.

Maggie promoted the WoU that, in the case of fractions, addition requires having the same size parts in both diagrams. So, each of the quarters must be split into 3 parts while each of the thirds must be split into 4 parts. Reasoning proportionally and referentially, she converted both fractions explaining how and what she was counting.

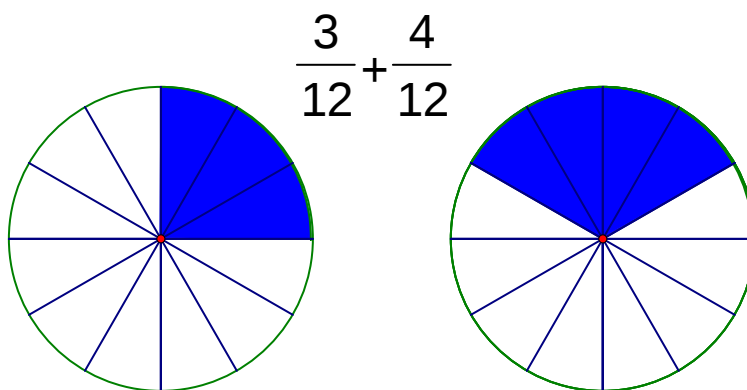


Figure 5.16: Maggie’s representation of $\frac{3}{12}$ and $\frac{4}{12}$

Maggie moved on to the first problem of the warm-up; asking students to help her draw an illustration. She accepted, $\frac{8}{1} - 3\frac{3}{4}$, as a representation. A student converted $\frac{8}{1}$ to

$\frac{32}{4}$. Using 8 circles cut into quarters, Maggie explained why it was so for a student who did not understand. This was another instantiation of creating a context when students indicated a lack of understanding of the procedures being used. Maggie assumed the responsibility for the explanation which was also a form of justification of for the procedure of converting $\frac{8}{1}$ to $\frac{32}{4}$. The same kinds of behaviors were observed in part 4 of the task.

A student used a standard procedure to convert $3\frac{3}{4}$ to $\frac{15}{4}$, but Maggie used proportional reasoning to explain why the mixed number could be converted to the improper fraction. Maggie added the two improper fractions to get the new answer and converted the answer to a mixed number using a standard procedure.

For part 2 of task 1, a student came to the board to solve the second warm-up problem. Maggie allowed the student's error to persist for some time. By asking him to explain what he had done, eventually another student pointed out the error. Eventually, Maggie promoted the WoU that $8 = 7\frac{6}{6}$ and that $8\frac{2}{6} = 7\frac{6}{6} + \frac{2}{6} = 7\frac{8}{6}$. Because the student was not communicating everything he was thinking, Maggie explained this WoU directly to help correct and complete the problem.

After problem #4 was presented, Maggie spent 15 minutes trying to help a student understand it. Maggie emphasized the use of a visual representation to explain the conversion of the $\frac{5}{3}$ into $\frac{25}{15}$. Maggie's WoU was that for every one-third she could make five parts. This would produce fifteenths. Since she had five-thirds originally, after

cutting the pieces she would have twenty-five fifteenths. Hence the WoU Maggie promoted was guided by proportional reasoning. She handled the initial presenter's WoU by identifying a student who didn't understand it and pursuing that student's suggestions, adding her own WoU to produce a solution that made use of circular units as referents.

In task 1 as a whole, a particular pattern of handling students' WoU can be observed. Maggie searched for places where students would have doubts or where there was an error. When these points of the solution were identified, she stepped in to promote a WoU compatible with the deductive proof scheme (proportional reasoning) and the referential symbolic proof scheme.

The concepts were not elicited from the problem. So the necessity principle was not implemented in this case.

Task 2

From "To Grandmother's House," Maggie incorporated four students' WoU. In the first she incorporated a student's WoU applying proportional reasoning by dividing a mile into 4 equal parts and subdividing each part into 2 equal parts, creating eighths of the mile. Counting the eighths up to six-eighths, it was concluded that Grandmother walked for 6 minutes because she walks at a speed of one-eighth of a mile per minute.

Maggie promoted this WoU by calling Lucy, a student, to present. She knew in advance that Lucy had drawn a line segment and divided it into quarters. Lucy used one-quarter in two different ways referring to one-quarter of a mile and one-quarter of one-quarter of a mile. When Lucy cut each quarter into quarters and then cut each sixteenth in half, Maggie stepped in to remind her about the meaning of an eighth. Maggie erased the thirty-seconds Lucy drew. That is, she corrected Lucy's solution mid-stream. This is an

example of not allowing an error to persist. Rather, Maggie fixed Lucy's WoU (from her perspective) and took control of the presentation, completing the solution. From her seat, Lucy explained that the answer was 6 minutes because there were six eighths in six eighths.

The second WoU was to apply subtraction by beginning with $\frac{3}{4}$ of a mile and subtracting eighths until she reached 0. By counting the number of eighths she had to subtract, she would know the number of minutes grandmother walked using her speed. Maggie had seen this approach developing on a student's paper. She tried to explain it to the class, but quickly realized that students did not understand it saying, "I think I'm losing a lot of people here, okay? So I'm going to abandon this one, because this is Victoria's and this is how she started it, but nobody seems to understand it."

Finally, Maggie highlighted Lucy's second WoU the problem by using repeated addition. She explained that one can add one-eighth repeatedly to get six-eighths or $\frac{3}{4}$. The number of times you add is also the number of minutes grandmother walked.

Maggie attempted to take the three solutions and draw a bigger mathematical point from them. She asked her students what they thought they were doing when they solved "To Grandmother's House." She said, "Are you adding, subtracting, multiplying, or dividing? Maybe if we do another problem together, we can find out what this is all about. I don't want to tell you because it doesn't seem like you're sure. So, I'm going to give you a similar problem." In the end Maggie explained that the next problem was designed to help them find out what these problems were about, a clear attempt at implementing the necessity principle.

Observed teaching practices, Connections to the PD, and Connections to Proof Schemes

At the PD Maggie reported encountering such struggles trying to think of how necessitate the concept of division of fractions. Maggie experienced the necessity principle in action first-hand at the PD.

Maggie: It's one thing to know how to do the math but to reverse that and come up with the word problem that's going to bring all of this out... It seems like it would be a whole different class, just learning how to do that. 'Cause I know in the interview I know how to divide by fractions but I couldn't come up with a word problem that would lead a student to do that ... that's another...

TR: That's true, coming, building the curriculum around problems, to implement something that we talked about many times, the necessity principle as we have seen here. This question that we have on the blackboard here was necessitated of your approach Maggie. Creating problems of this situation is very hard, but I can tell you, here I'm going just share with you my own personal experience, it makes teaching so interesting and so fascinating because you always think about how can I bring students to see the necessity for a concept?

Maggie's choice to assign problems "problems that allowed for multiple solution strategies in interrelated series" is a practice that Zack and Reed (in press) had observed in the teaching of some reform-oriented teachers.

Finally, this was the first time Maggie was observed abandoning an attempt to promote a WoU mid-stream when she realized that students did not understand it. Lesson 14 contains the only other instance of this behavior.

Lesson 11

Task

- a. The table shows the amounts of water and fruit concentrate needed to obtain the given amounts of punch. Fill in the two missing entries and explain or show how you determined these numbers.

Amount of Punch		
Water	Juice	Punch
12	2	14
24	4	28
		42
48	8	56

- b. Another row of the table shows 7 for juice. What are the other two numbers in this row? Explain or show how you determine these numbers.

	7	
--	---	--

- c. Extension: Use the letters W, J, and P to stand for the amount of Water, Juice, and Punch. Use these letters to write (two) equations to express patterns you have observed.

Topic of lesson:

Pattern generalization, writing equations, proportional reasoning,

Background Events:

None are necessary.

Claims

In this lesson, Maggie actively pursued alternative $W_s o U$, even when a $W o U$ had been presented. Though four different⁷² correct $W_s o U$ were presented, Maggie promoted two different $W_s o U$ by asking students to solve part b of the task using these $W_s o U$.

Searching for alternative $W_s o U$ in the presence of an error, is different from searching for alternative $W_s o U$ when a solution is correct. In the latter case, a question arises about how Maggie handles students' $W_s o U$ in the presence of multiple correct $W_s o U$.

While there were many times Maggie solicited alternative $W_s o U$, a trend is emerging through lessons 8, 9, and 10 (and now 11) in Maggie's teaching practices. This trend entails handling students' $W_s o U$ by searching for different approaches rather than searching for different answers alone.

The role of assigned tasks should not be ignored as it is crucial in the development of these teaching practices. In these three lessons, tasks included situations that allowed for multiple solution strategies. Maggie could not engage in the aforementioned teaching practices without the existence of multiple $W_s o U$ the tasks. Unfortunately, it is not known whether or not Maggie anticipated the multiple possibilities before the tasks were assigned.

Analysis

When Maggie gathered the class for discussion, she began by asking for student volunteers to fill in the chart, explain their thinking, and write down equations. Maggie's teaching practices were focused on gathering a variety of student $W_s o U$. She explained to

⁷² Different in the sense that each was guided by mental imagery unique from the others.

students that there were several $W_s o U$ in the room. Consequently, she expected students to explain their $W_s o U$ clearly so that others could understand.

Student 1's $W_o U$ Task 1a and Maggie's Handling

Student 1 viewed the first two columns of the table additively and recursively. He concluded that when the amount of Punch was 42 units, the amount of Water would be 36 units.

Amount of Punch		
Water	Juice	Punch
12	2	14
$12+12=24$	$2+2=4$	28
$24+12=36$	$4+2=6$	42
$36+12=48$	$6+2=8$	56

Figure 5.17: Student 1's additive $W_o U$ the pattern for water and juice

Early on Maggie established that the focus of the lesson would be on how students thought about the patterns in the table rather than on the answers alone. After student 1 presented his $W_o U$, she commented, "Is this a surprise to anybody?" Her teaching practice of asking students to explain their thinking was justified by the fact that there were multiple correct $W_s o U$ the question. In this way, Maggie attended to the students' need for communication.

Maggie solicited an alternative $W_o U$ from the same student by asking him if he could describe the pattern in a multiplicative manner. Student 1 demonstrated the

alternative WoU by noting that the numbers in the Juice column could be multiplied by 6 to get the numbers in the Water column.

Amount of Punch		
Water	Juice	Punch
$2 \times 6 = 12$	2	14
$4 \times 6 = 24$	4	28
$6 \times 6 = 36$	6	42
$8 \times 6 = 48$	8	56

Figure 5.18: Student 1's multiplicative WoU the pattern of water and juice

Maggie rephrased his comments by replacing the particular numbers he used with the more abstract label, Juice. She said, "He said Juice times 6 equals water".

Maggie introduced a multiplicative WoU by asking student 1 to reason in a particular way. Using multiplication in this situation promoted the development of an explicit function rather to compliment the recursive WoU student 1 described. In this way, Maggie promoted a particular WoU using the student as a spokesman.

Student 2's WoU Task 1a and Maggie's Handling

Immediately after student 1's presentation Maggie solicited another alternative WoU. She asked, "Can anyone else tell why they got 36 and 6 in a different way that we haven't seen yet? ... How did you see it?" This way of handling students' W_soU was also observed in lesson 8 when students demonstrated W_soU the toothpick problem. In that case, Maggie followed up by using symbolic manipulation to show that the W_soU were algebraically equivalent.

Maggie called on a student with another multiplicative WoU.

Amount of Punch		
Water	Juice	Punch
$12*1=12$	$2*1=2$	14
$12*2=24$	$2*2=4$	28
$12*3=36$	$2*3=6$	42
$12*4=48$	$2*4=8$	56

Figure 5.19: Student 2's multiplicative WoU the pattern of water and juice

She handled student 2's solution by recording it on the board, but moved on immediately by soliciting an alternative WoU. "Anybody have a *different*⁷³ way?"

Student 3's WoU Task 1a and Maggie's Handling

Another student pointed out that Water + Juice = Punch. Maggie made a mathematical point by highlighting the facts that this relationship alone could not be used to fill in the missing values of the table, but that it was helpful for checking the values (i.e., $36 + 6 = 42$). Handling the student's solution in this manner is a form of comparison to other W_soU because it shows that the solution can serve an important purpose in relation to the others.

Maggie continued by soliciting another WoU. This continues the pattern for the lesson of soliciting alternative W_soU in the presence of multiple correct W_soU.

Student 4's WoU Task 1a and Maggie's Handling

⁷³ Italics added to express emphasis by Maggie.

Maggie recorded student 4's WoU as, "He divides water by 6 and he will get the amount of juice." Maggie confirmed the student's statement saying, "He said it, but we gotta check it." That is, Maggie handled the WoU by checking for all cases present.

Summary of Maggie's ways of handling students' W_soU in task 1a

Maggie's handling of students' W_soU task 1a can be characterized by repeated solicitations for alternative W_soU. This differs from multiple observations in year 1 where she focused on the final product (the answers). In this lesson, the answer (36, 6) was endorsed in the beginning and there was no question about its accuracy. Maggie chose instead to focus on how students arrived at their response.

Some comparisons were made between W_soU. However, Maggie primarily checked the accuracy of student's patterns by checking values on the table. In this problem, there is no other way to tell if the pattern holds. That is, the task does not allow for a distinction between RPG and PPG.

Maggie promoted a WoU functions that was explicit rather than recursive by asking student 1 to think about the pattern multiplicatively. When student 2 was reluctant to share her WoU, Maggie called on her and encouraged her to share her WoU which was also multiplicative in nature. Mathematically, multiplicative W_soU are related to proportional reasoning which was a major theme of the course as seen in lessons 9, 10, and 11.

Student's W_soU Task 2b and Maggie's Handling

For task 2b, Maggie asked students to stay with two of the four W_soU presented in task 1a to deduce the amount of water when the amount of juice is 7. She began by saying, "Before we use [student 4's] way, which is a really helpful way, how about those

people that never thought of this [meaning that $6 \cdot J = W$]?” This shows that Maggie intended to return to student 4’s WoU and hold it up as an example because it was “helpful”.

Visualizing the amounts of water and juice as sequences, Maggie chose to stay with a WoU that related the value of a variable to its position in the column. She asked, “Where does this row really belong in the chart if we put it in order?” Maggie emphasized the WoU that when the amount of juice is 7, the students should create a row between the third and fourth rows .

Amount of Punch			
	Water	Juice	Punch
12 x 1	12	2	14
12 x 2	24	4	28
12 x 3	36	6	42
12 x 3.5	42	7	49
12 x 4	48	8	56

Figure 5.20: An application of student 2’s multiplicative WoU the pattern of water and juice

Maggie promoted the following WoU. “...Since 7 is in the middle of 6 and 8, then I’m going to pick 12 times the number that’s in the middle of 3 and 4. And what number is that? ... Three and a half.” She wrote 12×3.5 . Maggie emphasized that this approach was intended for students who did not use student 4’s solution and credited it to one student in particular. Maggie continued by asking the students to solve in a manner consistent with student 4.

Maggie verified the responses using multiple $W_s oU$, pointing out that when several $W_s oU$ return the same response, one can be more certain about it. When she finished with task 2b, Maggie formalized the relationships students had stated in words by asking them to put those worded relationships into equations using the given variables. She recorded the system of equations on the board.

Maggie did not explain what was “helpful” about student 4’s WoU . However, given the context of other solutions like student 1’s additive approach, student 4’s solution can be seen as more efficient comparatively in the sense that fewer computations would be needed to arrive at an answer.

Observed teaching practices, Connections to the PD, and Connections to Proof Schemes

Maggie’s ways of handling students’ $W_s oU$ by soliciting alternative $W_s oU$ in the presence of correct $W_s oU$ stood out in this instance as it helped make an emerging trend salient. Over the course of the last four lessons Maggie solicited more than just alternative answers to questions. She has also begun soliciting different mental imageries that guide solutions. This way of handling students’ $W_s oU$ has potential to help her promote certain $W_s oU$ (and corresponding proof schemes) in a manner that might help diminish students’ external conviction proof schemes as she points to the need for one WoU over another.

Goldsmith & Schifter (1997) referred to aimless solicitations of multiple $W_s oU$ in a classroom as “mathematical show and tell,” indicating that solicitations of multiple or even alternative $W_s oU$ alone do not constitute a desirable teaching practice. In this lesson Maggie solicited several $W_s oU$. She highlighted student 4’s solution, calling it “helpful”. Maggie also pointed out the need for students to communicate how they saw the pattern.

She said that knowing several ways to solve a problem can help raise one's level of certainty in his conclusions. While she stopped short of explaining why she was pointing out that student 4's multiplicative WoU was useful, Maggie's action of asking student 1 to also point out a multiplicative pattern shows that for some reason she felt it was useful to think multiplicatively in this context. Indeed, thinking proportionally by coordinating two multiplicative patterns is useful for turning recursive functions into explicit functions and explicit functions are useful because they can help predict the value of a pattern for much larger examples in an efficient way. For these reasons, Maggie's teaching practices were not a case of what Goldsmith & Schifter called "mathematical show and tell". Rather, it was directed by the mathematical goal of showing that thinking multiplicatively is an efficient way of thinking.

Similar teaching practices to those observed in this lesson were observed at the PD. For almost every problem at the PD, the TR solicited multiple W_soU. As it was done at the PD, Maggie made her point using students' W_soU. This lesson also contains a case of pattern generalization in which more than one pattern was observable. This is another commonality with Maggie's PD experience. TR often assigned patterning problems.

In this case, there were two related problems (task 1a and 1b) that required students to reason with the tools they had invented through pattern generalization (i.e. task 1b). However, at the PD tasks were designed to give students multiple opportunities to confirm their results by applying a known process rather than only by checking results (e.g. – The stair-like structure, investment, and arithmetic polygon: year 2⁷⁴). While Maggie did recognize several W_soU and promote those that used multiplicative

⁷⁴ I focus on year 2 because it was a part of Maggie's more recent experiences at the PD.

reasoning, the table and process for making punch were the same. Therefore, she had limited opportunities to promote PPG. Nevertheless, Maggie did promote a WoU that was helpful to produce explicit functions.

Lesson 12

Task

1. When the Ocean and History Bike Tour reached Williamsburg, the tired riders packed their bikes and gear in the van and headed back toward Philadelphia. They travel by interstate highway, and they average a steady 55 miles per hour for the 310-mile trip. You've seen that making a table or graph can help you understand how the time and distance traveled are related.

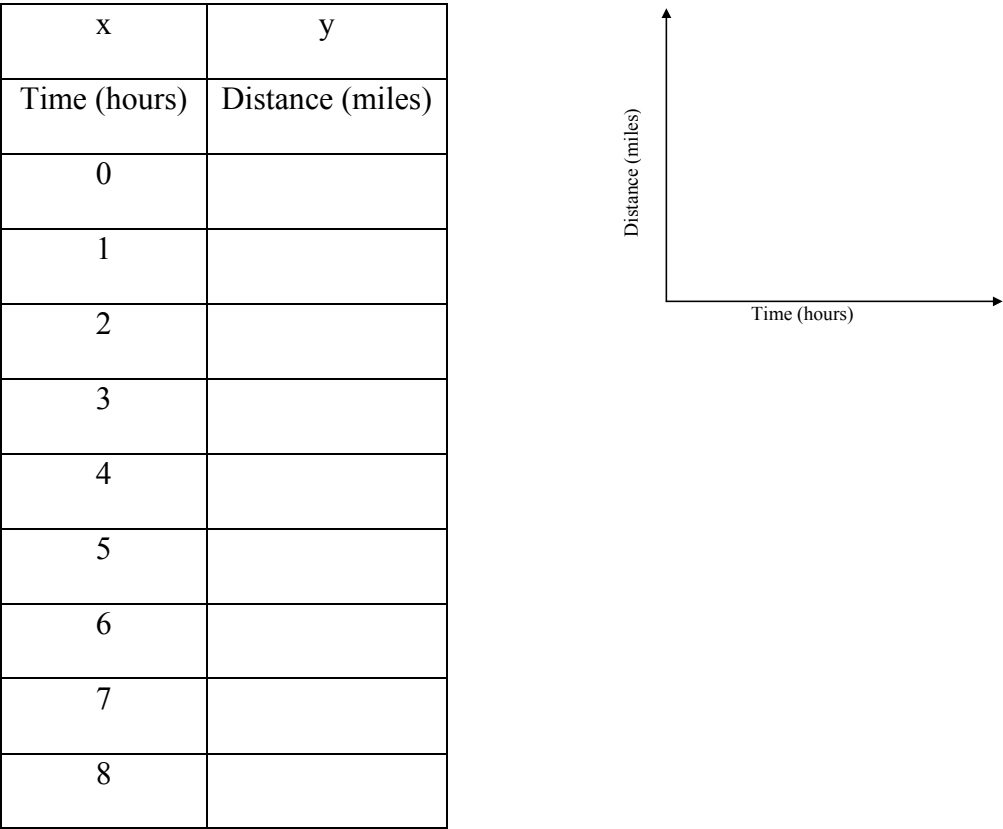


Figure 5.21: The table and graph used in task 1, lesson 12

Topic of lesson

Converting verbal representations of rules for patterns to a symbolic form, plotting points on a graph.

Background Events

No background events are necessary for this analysis.

Claims

During this lesson, Maggie juggled four representations for a situation to help students express their $W_s o U$: verbally expressed patterns, functions, tables, and graphs. She explained that the graph was useful for estimating, while the table was useful for finding exact values. Rather than promote the use of the function for computing the distance for fractional times (i.e., $d = 4\frac{1}{2} \cdot 55$ and $d = 5\frac{1}{4} \cdot 55$), Maggie promoted a proportional WoU compatible with PPG.

The lesson demonstrates is interplay between repeated addition and multiplication, a theme that has been mentioned in lesson 11. Maggie had previously mentioned that multiplication [of whole numbers] can be viewed as repeated addition. While she generated data for the table using repeated addition, she praised the use of multiplication and ultimately wrote an equation using this WoU saying, “If we're going to add by 55, how do I show that?” This is an example of handling students’ $W_s o U$ by promoting one over another for notational reasons or conventional reasons, but not for the purposes of promoting the efficiency of a functional WoU .

Maggie motivated changes in $W_s o U$ by requesting estimations and exact answers separately and in that order. She attended to students’ visually empirical $W_s o U$ by acknowledging their value in estimating distances using the graph. Maggie’s way of

handling students' visually empirical $W_s o U$ included asking questions that exposed the visual approach's shortcomings in finding exact solutions. At the same time, she explained that using the graph is valuable for estimation purposes. Focusing on the meaning of the rate, 55 miles per hour, she pointed out how the table could be coupled with proportional reasoning to find exact solutions.

Maggie also attended to the continuous nature of the graph, explaining the reason why the points should be connected by referring to the situation and the meaning of rate. This represents a difference from the events of the toothpick problems (lesson 8) of her first year in which she represented a discrete situation continuously without attention to the difference.

Analysis

Maggie began the lesson by reminding students of the meaning of the rate, 55 miles per hour. Namely, that it means for every hour the van was traveling, it was displaced 55 miles. She asked students to make a table and a graph documenting the distances the van had traveled each hour from 0 to 8 hours. The table and graph were then filled in for all to see.

Students used repeated addition as they told Maggie what the distances were for given times. In lesson 11 Maggie promoted the $W o U$ that multiplication is a way of representing repeated addition of the same number. When one student offered a multiplicative $W o U$ rather than an additive $W o U$, she praised his approach saying, "I like what he said. He said I multiplied 3 hours times 55 miles. That's pretty good, because a lot of us aren't doing that... You could just be adding 55 each time..."

Maggie promoted the multiplicative $W o U$ distance traveled by endorsing it as a

way of representing repeated addition and by writing distances traveled using multiplication.

Maggie: I want you to tell me the equation, that's our pattern. What was the equation that worked for this?

Student 1: Adding by 55.

Maggie: How do I show that, though? If we're going to add by 55, how do I show that? If it's 1 hour?

Student 2: You add 55 once.

Maggie: You add 55 one time, that's the distance.

Student 2: You multiply by 1.

Maggie: Okay, so you can multiply by 1. Now that I'm writing this, you can see it. 1 times 55 [$1(55)=55$].

The repeated question, “How do I show [repeated addition of 55],” and her use of the notation $1(55)=55$ indicate that Maggie did not ignore students’ additive W_{soU} , but tried to show them a structure within the pattern. She also tried to justify her way of seeing the structure, $t(55)$, by reminding students that repeated addition is her WoU multiplication.

After writing the function, $d = 55t$, Maggie asked students to find how far the van traveled in $4\frac{1}{2}$ hours. She asked students which representation would be best. A student wanted to use the graph and Maggie followed the student’s lead. Since the graph was still discrete at this point and $4\frac{1}{2}$ was between 4 and 5 hours, when Maggie moved up the graph to find the distance there was no line segment connecting them. The student told her where to stop by imagining a straight line between the points, but Maggie challenged the assumption that the points could be connected.

Maggie: Why is this a good place to stop? ... So there's not really a line here, is there? ...so before going further, the question is, should we connect these points or not?

By appealing to the problem statement, Maggie explained that the van can not magically appear and reappear in different places. Therefore, a line segment connecting the points is warranted.

Maggie then asked students to find the exact distance the van had traveled when $4\frac{1}{2}$ hours had passed. Early on Maggie pointed out that for every hour, the van moved 55 miles. A student extended her WoU the rate, explaining that in $\frac{1}{2}$ hour the van travels half of 55 miles. Maggie incorporated this WoU into her explanation.

Maggie: Okay, so he's saying about 260 for $4\frac{1}{2}$ [from the graph]. So we look over here and he's saying 260. Does it seem like that could be right that for $4\frac{1}{2}$ hours, it's around 260 miles? That's his estimate. It seems like at least it falls somewhere in between here. How could we find out from the table exactly what the distance is?

Oscar: 55 divided by a half and add it to 220?

Maggie: This is what Omar is saying, he's saying 55, where does the 55 come from? Because from the 4th hour to the 5th hour, they went 55 miles, right? So 55 miles, they didn't go a full 55 miles because they only traveled for half an hour. So he wants to take how much of 55? He's going to take half of it. And what's half of 55?

Note that the aforementioned WoU does not use the function, $d = 55t$, algorithmically⁷⁵.

Instead Maggie promoted the WoU that the distance could be found as a sum of the distance previously traveled with the portion of the distance traveled in $\frac{1}{2}$ hour, $220 + \frac{1}{2}$ (55). This is an example of using an agreed upon WoU rate to justify the solution by reasoning repeatedly and referring explicitly to a mental image of the problem statement rather than reasoning symbolically or procedurally with the function. It is important to note that Maggie had to choose to reason either way.

The instance raises the question of what purpose the function served for Maggie in this lesson. Students used the function to express a pattern preserving a structure of

⁷⁵ Meaning that Maggie could have computed $55(4.5)$ directly.

repeated addition in the form of multiplication, but it's value was not explained in the explicit way that the graph and table's values were.

In the beginning, when Maggie asked a student to graph the contents of the table. The graph was drawn discretely. Maggie asked a student to estimate the distance traveled in $4\frac{1}{2}$ hours using the discrete graph showing only the points found in the table. She pointed out that a line containing the points (3, 165) and (4, 220) would contain the desired point, but also asked why it was acceptable to connect the points.

Maggie: Okay, Joe, why is it a good idea to connect the points? ... sometimes it is, and sometimes it isn't. What do you have to think about to figure out, should I connect these points or not, Jon?

Jon: Because it doesn't continue?

Maggie: Because it doesn't continue? So what's the situation that we're in right now? The people are driving in a car, and time is passing by, and as time passes by, they're going a certain distance, right? So as time passes by, for example, between the 4th and the 5th hour, is their van still moving? Or does their van reappear in different spots? No, they actually move along the road that many miles as time passes by. Does it seem like that's continuous or not? [Mixed student answers, yes and no]. Okay, so is it a good idea to connect the points?

In contrast with the events of the toothpick problem (lesson 8), students were asked to attend to the reason why the situation was continuous. This demonstrates an increased sensitivity to the domain of functions in Maggie's instruction.

The W_{soU} Maggie promoted were elicited from the task except for the function. Maggie explained the values of the graph and the table using proportional reasoning, but did not point out the efficiency of using a functional approach in fractional situations, nor did she provide a context in which using the function was necessary⁷⁶.

Considering the domain of the function, Maggie's approach covers cases when the time is a rational number, but not when the time is irrational. Clearly this is beyond

⁷⁶ Maggie briefly commented that when the van traveled 20 hours, students could 20 by 55.

her students' capabilities and/or her time constraints. At some point in their mathematical instruction students might return to the question of how to justify that when the time is $\sqrt{2}$, the distance traveled would have been $55\sqrt{2}$. Their repeated addition approaches and proportional reasoning would be useful, but fall short in that situation. What is most valuable for students is that they have been exposed to the perspective that certain WoU can be deficient for answering given questions and there are times when another WoU must be invented to solve a particular problem. These leaps are necessitated when students encounter legitimately problematic situations from them personally.

Observed teaching practices, Connections to the PD, and Connections to Proof Schemes

Both lessons 11 and 12 offer students an opportunity to generalize from patterns. The task in lesson 12 differs from the task in lesson 11 in the sense that in lesson 12 process pattern generalizations can take place. In the fruit punch problem, the only way to confirm the pattern was by checking the table. In contrast, the task in lesson 12 allows students to use an agreed upon process that is independent of a set of time/distance values as the basis for justification.

Maggie pointed out that both representations served a purpose. She explained that the table is helpful for finding exact values and the graph is useful for estimating.

Maggie: What do you think? From the table can you find out exactly how many miles we traveled? From the graph can you tell exactly how many miles? [A student answers "No"]. No, it's harder on the graph, isn't it? So you can see some value in the table, but there's also value in the graph, and we'll see why.

This marks a time when Maggie highlighted the mathematical values of multiple representations. She did so by contrasting their usefulness for answering different kinds of questions. It is a significant teaching practice to value flawed approaches and point out

their shortcomings to necessitate more efficient or more complete $W_s o U$. This was a teaching practice observed at the PD. TR explained on several occasions that participants need to do exactly this. That is, hold up empirical $W_s o U$ as valuable, encouraging students to continue using them, but to also pursue more complete solutions because the empirical $W_s o U$ do not answer valuable mathematical questions (e.g. – find the exact distance the van traveled in $5 \frac{1}{4}$ hours).

Issues about the need to attend to the domain of the function had been discussed at the PD when participants attended to the domains of functions (see Jack and Jill speed problem, year 2). In the case of the Jack and Jill speed problem, Maggie did not attend carefully to the domain of the problem and had made an important mistake because she saw a discrete situation as continuous.

“It took a lot of help to see that what I based my formula on for Jack was not on a strong foundation. I didn’t see the constraints of using the sum of the arithmetic sequence formula. I would have given up, but I had some good help.”

Maggie reported that the instructional intervention had taught her to look more closely at the constraints of the problem. Lesson 12 is an example of a time when Maggie’s teaching practice encouraged students to do the same.

The G-P-A model was also used by Maggie in this case. It is a connection to the PD.

Lesson 13

Task

1. Solve for x:

1. $2x = 5 + (-5)$

2. $2x > 5 + (-5)$

3. $12 = b/3$

4. $12 < b/3$

2. Write an inequality and graph for each situation.

1. Fewer than 45 people attended a show. Let n be the number of people.

2. High fiber foods have at least 5 grams of fiber per serving. Let f be the number of grams of fiber per serving.

3. A student pays for 3 movie tickets with a twenty-dollar bill and gets change back.

Let t be the cost of the movie ticket.

3. Bicycle Savings

You are saving to buy a bicycle that will cost you at least \$120. Your parents give you \$45 toward the bicycle. Write an inequality to find out how much money you will save.

4. Solutions to Inequalities

Solve the equation and inequality. Graph each solution.

a. $n + 8 \geq 19$ and $n + 8 = 19$

b. $-26 > y + 14$ and $-26 = y + 14$

c. Extension: Make up a word problem that matches the inequality in a or b (above).

Topic of lesson

Solving an equation in one variable. Solving an inequality. Translating verbal representations into symbolic representations (writing inequalities). Equivalent inequalities.

Claims

From her examples Maggie emphasized the points that: the solution processes used to solve equations and inequalities were the same, the number of solutions varies between equations and inequalities, and solutions should be checked against the original problem. Still, the emphasis of the lesson was primarily on symbolic manipulation from a procedural perspective. Non-referential symbolic $W_s o U$ were promoted when there was no context and in a few cases referential symbolic $W_s o U$ were promoted when there was a context.

The necessity principle was implemented in one instance, the bicycle savings problem, not in any other. Maggie handled student $W_s o U$ by allowing student errors to persist in two notable instances, comparing representations algebraically, and encouraging student to student talk.

Analysis

In the warm-up, Maggie emphasized two points: the fact that the inequality could be solved in the same way as the equation and the difference between the number of solutions in equations and inequalities (equations have one solution and inequalities have infinite solutions).

Maggie: So what do we notice that's the same about these 2 solutions?

The whole process of solving. What's the same?

Steve: The process.

Maggie: The process is the same. The process maybe meaning all these steps that we took to get to the end. The process is the same. Does it look the same? Are the solutions the same?

Class: No.

Maggie: They are not the same. We can see a big difference already. Here there's one solution. It's 0. That's the only thing that works. Here there is a ton of them, we can never, we would die still saying the solutions if we had to say all of them, because it's infinite, it can go on forever and ever and ever.

In the problems that Maggie assigned, the symbolic manipulation was the same in both cases. However, the reasons that each equivalent equation or inequality could be written were different. For example, $12 < \frac{b}{3}$ implies that $36 < b$ because tripling a smaller quantity and a larger quantity creates an even larger gap between the quantities, magnifying the difference in values between the quantities. It is also true that tripling both sides of an equation preserves the relation of equality, but for a different reason due to the nature of the relation. This point is important because of the complexity involved in explaining why the inequality reverses when multiplying or dividing with negative values. Handling students' W_soU by emphasizing that the process was the same in both cases without explaining what was different beyond the number of solutions promotes a procedural WoU that is compatible with the non-referential symbolic proof scheme.

Next, Maggie introduced task 2. She wrote students' W_soU the statements and checked them by plugging in values, emphasizing the meaning of the symbols in the context of the problems. Then, Maggie graphed the solutions on a number line.

In the third part of task 2, "A student pays for 3 movie tickets with a twenty-dollar bill and gets change back," students' offered the WoU that $t < 20$. Maggie let the error persist for a substantial period of time.

Maggie: Jerry? Give me the inequality. T is going to be the price of a ticket. T is how much 1 ticket costs.
 Jerry: It has to be lower than 20.
 Maggie: Something has to be lower than, whatever this something is is less than 20. What do you want me to put for the something?
 Jerry: T ?
 Maggie: T ? So a ticket is less than \$20, yes?
 Jerry: Yes.
 Maggie: So does it match the situation?
 Class: Yes.
 Maggie: Yes? Okay. Everyone's happy with that?
 Class: Yes.
 Maggie: We can move on? So the student bought 3 movie tickets, and it was less than \$20, and this represents that.
 Jerry: Yes.
 Maggie: So the price of a ticket could be \$19?
 Al: Yes.
 Maggie: It could be \$19.50? The price of a ticket could be \$19.99?
 Class: Yes.
 Maggie: So all we have left to do is graph. How do we graph this, Juan?

In this case, Maggie handled the popular WoU that $t < 20$ by allowing the error to persist. She graphed the solution, but when nobody pointed out the error she eventually returned students to reasoning about the situation.

Maggie: So everybody's happy with that? The student came with a 20-dollar bill, he bought 3 movie tickets for 19.99 and still got some change back? Really?
 Larry: Not really, because if it's \$19.
 Maggie: Because he has \$20 and it's 19.99 and he has to buy 3 tickets?
 Class: That's \$6. That's like.
 Maggie: And still get some change back? It sounds like he's in debt more than anything. So how do we make up for that in our inequality? What did he really buy? Did he buy a ticket? So the price of the ticket is actually multiplied by what?
 Class: By 3.
 Maggie: By 3. He bought 3 tickets and then got some change back. So whatever the cost of these 3 tickets was, it was less than \$20. Does this make sense? I think I'm a little happier with this now but did we solve for t ?

Maggie rewrote the inequality from $t < 20$ to $3t < 20$, solved and graphed it. Then she attended further to the situation and mentioned that the price could not go below 0. This is an example of a time when Maggie handled students' W_soU by allowing an error to persist and hoping that a student would catch the error. When nobody did, she had to exercise her right as a member of the community to express her dissatisfaction with the solution. Additionally, her attention to fact that t could not be negative due to the context of the problem provides further evidence that she is sensitive to the domain of functions.

In the "Bicycle Savings Problem", Maggie solicited and gathered four different student W_soU how to write the inequality (a. $45 < 120$, b. $120 < x + 45$, c. $x \geq 75$, d. $x + 45 = 120$). As she addressed the students' W_soU, Maggie asked the question, "Let's see if these make sense." "Does it make sense," and "Does it match this [situation]", were two other common questions Maggie asked. The first WoU was dismissed because it did not provide information about the savings (x). The second was adjusted to reflect the possibility that the quantities could be equal, $120 \leq x + 45$. Maggie pointed out that in an inequality a goal is to isolate the variable. She pointed out that $120 \leq x + 45$, $120 - 45 \leq x$, and $x \geq 75$ are all equivalent statements algebraically. This was an example of a time when Maggie handled students' W_soU by using symbolic manipulation for the purpose of comparing W_soU.

Maggie asked two students to solve one problem each ($n + 8 \geq 19$ and $n + 8 = 19$) in task 4. The first student solved $n + 8 \geq 19$ by subtracting 8 from both sides. He wrote, $8 - 8 \geq 19 - 8$. Then he wrote $n \geq 11$. Maggie pointed out that it is important to write n in the second statement. Maggie drew the graph and checked the solution set by trying

values on the number line. She checked $n = 4$, explaining that since that didn't work it meant other values in that region of the number line would not work either.

Maggie reviewed the second question. Students solved $-26 > y + 14$ and $-26 = y + 14$ on the board. The first student solved $-26 > y + 14$ incorrectly and graphed it incorrectly, but Maggie allowed the errors to persist. The student concluded that $-12 > y$ after computing $-26 - 14$ incorrectly. Maggie asked if class members agreed or disagreed. After a few disagreed, she asked students to find what went wrong. Maggie also explained that she also disagreed. Eventually, she pointed out that $-26 - 14$ means you start on the number line at -26 and move further left.

Symbolic manipulation was a theme throughout the lesson. The concepts were not elicited from the problems, meaning that the necessity principle was not implemented with one exception. During the bicycle savings problem, Maggie examined several students' $W_s o U$ and determined that 3 were closely related.

Observed teaching practices, Connections to the PD, and Connections to Proof Schemes

In this lesson Maggie handled students' $W_s o U$ by emphasizing writing solutions mathematically accurately, allowing student errors to persist on two occasions, comparing $W_s o U$ algebraically and checked against situation, and encouraged student-student talk. Maggie assumed responsibility for justifying solutions by checking answers rather than asking students to do so. She also consistently asked students to explain their thinking, though she was in charge of verifying it.

A trend of assigning patterning problems had been observed previously, but it was not the case in this lesson. The emphasis of the lesson was symbolic manipulation. There was one instance in which there was an intellectual need for symbol manipulation (see

the bicycle savings problem). In that instance, Maggie compared the students' $W_s o U$ by referring to the situation and manipulating the symbols in the students' representations and comparing the meanings of the symbols using the problem statement. This was a case when Maggie promoted the referential symbolic proof scheme through the G-P-A model expressed at the PD, by focusing on the physical reality represented in the problem statement and the algebraic representations offered by students.

Lesson 14

Tasks

1. List all the factors of:

- a. 56
- b. 72
- c. 81

2. Group Work

There are 126 people in a workshop. The leader wants to put people into groups. There must be at least 5 groups but not more than 20 groups. List all possible ways people can be grouped.

3. Bake Sale

John made oatmeal cookies for a bake sale. The cookies need to be distributed evenly on 2 or more plates. If each plate gets at least 7 cookies, what are the possible combinations for 60 cookies?

4. Sum of Fractions

Which of the following best estimates the sum $\frac{1}{2} + \frac{1}{3}$?

- a. 0.4 b. 0.5 c. 0.8 d. 1.1

5. Divisibility by Multiples

- a. If a is divisible by 2, what can you conclude about $a + 1$? Justify your answer.
- b. If a number is divisible by 9, is it also divisible by 3? Justify your answer.

Topic of lesson

Factoring, Compound Inequalities, Solving systems of inequalities, Converting fractions to decimals

Claims

Students' $W_s o U$ task 1 required checking every number less than the number being factored or every number up to and including half the number. Maggie handled their $W_s o U$ by suggesting a more efficient way. She explained why her method worked in a deductive manner, but left many details out in her explanation. Through her handling of students' $W_s o U$ in task 1, Maggie became aware that her $W o U$ was not in the students' ZPD. She continued using her method to find factors, but did not push her attempts to get students to understand it or to prove it any further.

This episode demonstrates an instance in which Maggie handled students' $W_s o U$ by pointing out a shortcoming in them. Namely, that they were not efficient. When she offered them a more efficient $W o U$ and they did not embrace it, Maggie abandoned her approach. She assumed the responsibility for determining when a list of factors was complete from that point forward by using her $W o U$.

Another significant teaching practice observed was encouraging students to state conjectures and prove them. In other lessons Maggie asked students to explain their thinking, but did not challenge them to prove their conjectures. This teaching practice (asking students to prove their conjectures) was observed throughout the lesson.

Consistently throughout the lesson, Maggie asks students to take more responsibility for explaining results than she has in the past. She also handled student presentations by encouraging student to student talk. Only briefly did the students talk directly to each other during the sum of fractions problem, but it was a situation that was not seen during the first 13 lessons analyzed and it is significant.

Analysis

Task 1a

Maggie asked for volunteers to list factors for each of the three numbers in task 1. Three students came to the board to share their $W_s o U$. For task 1a, the student listed: 1, 2, 7, 8, 4. Before offering her own $W o U$ factoring, Maggie asked students for their personal $W_s o U$ the word factor. “For you the word factors means what?” One student explained that for him factors are the numbers that are multiplied to equal the given number.

Maggie solicited alternative $W_s o U$ by asking two other students what the term “factors” means to him. The students said, “It means fractions”, “It means you multiply something”, and “What could be multiplied... the multiples of that number.” Eventually, Maggie shared her own $W o U$ by emphasizing that a factor of a number is a number that divides into a given number, leaving no remainder.

After adding 14, 28, and 56 to the list in task 1a Maggie asked, “How do we know that we have all the multiples? How do we know that there’s nothing left out that we haven’t already put down?” Addressing a need for certainty and efficiency became a theme. Maggie explained that one way would be to check every number up to the given number by dividing and checking for a remainder.

Maggie tried to point out that students could do less work, compared to the method of checking all numbers less than or equal to 56, and still be sure that all factors were listed. Her shortcut involved checking all numbers in order until the “partner” of a given factor appeared on the list. This number signifies a “turning point”. She denoted the “turning point” drawing a dashed line between 7 and 8. At this point, numbers less than the first value with a partner (e.g. – 7) on the list could be divided into the given number.

Listing the corresponding quotients would complete the list of factors. In this way, one could be certain that all factors were present without having to check every number less than or equal to the given number.

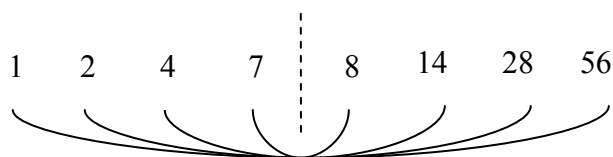


Figure 5.22: Maggie's pairing of factors for 56

Maggie appealed to a need for certainty. She said, "So do we know for sure now that everything that's up here are all the factors of 56?" In response, students offered alternative W_{soU} . One student said that you need to divide all numbers less than or equal to the number being factored. Another student said that you only had to check all numbers up to and including half of the number's value, but Maggie placed the burden of proof for the student's conjecture on the class. She began by asking the students, "Why could we stop there?"

Maggie: ... he's saying when you get to half of 56 which is 28, if you at least try to 28, then you're done, you don't have to pass the halfway point. Does that make sense to everyone?

Class: Yes.

Maggie: It does, why?

Oscar: Because if it was 29, does 29 go into 56, no, it doesn't.

Maggie: But is that a reason why? He says after 28 you just ask yourself does 29 work? No, it doesn't, so why not try 30 then?

Maggie: Why can we stop there? Mark, why can we stop here?

Mark: Because everything above 28 does not go into 56.

Maggie: Because anything above 28 does not go into 56. In this case, we found out that that's true. So why wouldn't anything above 28 go into 56?

Mark: Because it would be greater than 56.

Maggie: Right, and then are you going to have an even number as an answer?

Class: No.

Maggie: The only thing greater than 28 would be what?

Class: 28, 56.

Maggie: There is one number greater than 28.

Al: 56. The number itself.

Maggie did not get satisfactory responses, but she continued to inquire about why nothing greater than half the number could be a factor (other than the number itself). Maggie tried to explain that for any number greater than 28, 56 divided by that number would not return a whole number. Eventually, she dropped her point and moved on to task 1b, but returned to her attempt to explain why her method works.

Task 1b

The student who factored 72 had written an incomplete list. Maggie returned to the strategy that one could check all numbers between 1 and 72 to be sure. As she checked for factors Maggie asked, “Do we want to do this all the way until 36? ... How can we be sure we’re done with everything? ... So we do have to go through all the numbers?” She continued by asked, “Can someone think of a system that would be a little bit more efficient?” Handling students’ W_soU by asking for a system is a teaching practice that attends to intellectual needs for certainty and efficiency.

Maggie listed the known factors smaller than or equal to 9 in order and suggested matching up factors with other factors. Then she began writing factors underneath each know factor.

1	2	3	4	6	8
72	36	24	18	12	9

Figure 5.23: Paired factors of 72 illustrating Maggie’s method

Maggie shared the WoU that factors greater than 8, must “pair up with” factors smaller than 8 and found a way to represent her observation. She pointed out that numbers between 9 and 12 would not have a pair in the list (see figure 5.21), nor would numbers between 12 and 18. Maggie reemphasized the needs for order and efficiency addressed by her approach when she presented her WoU task 1c. Eventually, she realized that students did not understand her approach. Maggie abandoned her attempts to communicate her more efficient WoU. She began task 2.

Promoted WoU and corresponding WoT

After listing the factors as seen in figure 5.21, Maggie explained why her system worked. Her explanation was consistent with the deductive proof scheme.

Maggie: So 8 is actually multiplied by 9. So if I go to 10, will it be a factor?

Class: No.

Maggie: What about 11?

Class: No.

Maggie: These are already paired up to equal 72 [circling 8, 9 and 6, 12].

At this point if I go to 9, what will 9 pair up with to make 72?

Class: 8.

Maggie: 8. It will go back this way. And then 12, with 6. So will 13, 14, 15, 16 and 17 pair up with anything?

Class: No.

Maggie: Probably not. We know that they won't. So over here, let's see if we can do it this way and be a little bit more efficient about it.

Maggie's explanation is closely related to the following proof of her observation:

Let x and y be factors of 72. By definition, $xy = 72$. Assume that $x \geq 9$ and x is a natural number. Then $xy \geq 9y$. By substitution, $72 \geq 9y$. So, $8 \geq y$. This is consistent with Maggie's statement, “It will go back this way,” meaning that for any factor, x , greater than 9, there is a corresponding “partner” factor, y , smaller than 8. Since all factors less than or equal to 8 are known, the only remaining factors must be their “partner” factors.

Maggie's question, "So, will 13, 14, 15, 16 and 17 pair up with anything," is taken rhetorically though her follow-up seems contradictory, "Probably not. We know that they won't."

Maggie's question and statement, "So will 13, 14, 15, 16 and 17 pair up with anything? ... Probably not. We know that they won't," indicate an acknowledgement that the proof may be incomplete, but the observation should nevertheless be taken as fact. Indeed, her explanation is taken to promote a deductive WoT that one can account for and explain why there are no other possibilities by considering what a factor and "partner factor" are and exhausting possibilities.

Teaching practice and significance (Task 1)

Maggie's way of handling students' $W_s o U$ factoring was by asking them for their $W_s o U$ before offering her own. This teaching practice is significant because it emphasizes personal meaning and calls attention to a need to communicate clearly given the fact that individuals may think of the same phenomenon in different ways. The teaching practice also allows Maggie to investigate and gain sensitivity toward students' $W_s o U$ as she attempts to deliver her instruction.

The students' two $W_s o U$ could guarantee that the factors would all be found, if applied correctly, but were less efficient than Maggie's approach. Maggie acknowledged these $W_s o U$ and raised further questions about them, but continued to press for a more orderly and efficient way. She tried to show her students her $W_o U$ three times. Through an informal question and answer process solving tasks 1b and 1c, it became clear to her that students did not understand and appreciate her $W_o U$. Rather than force her students to use her approach, she abandoned it.

This is a form of handling students' WoU by using them to inform her instruction on the fly. It illustrates the kinds of decisions a teacher needs to make in a student centered approach. Choosing to abandon her attempts at promoting her own WoU is a significant teaching practice because it demonstrates a level of sensitivity on Maggie's part to the notion that the knowledge she hoped to convey was not in her students' ZPD at the current time.

Task 2

Maggie began by pointing out that the problem was about factoring. She wrote the factors in pairs, but did not ask if the solution set was complete. Though she used her own WoU, she did not explain that she was doing so or ask the students to do so. Rather, she simply populating and organized the list of factors in a manner consistent with her WoU. Maggie also pointed out that the solution must also fit the conditions of the problem.

Maggie interpreted the meaning of the factors in the context of the question for students (e.g. - 2 groups of 63 people versus 63 groups of 2 people). She had students repeatedly explain why certain factors would (or would not) work according to the problem statement. The problem statement was used as a standard by which to judge solutions.

Task 4

Maggie selected particular students to demonstrate their solutions before asking for volunteers. The first student drew three circles; shading the fractions $\frac{1}{2}$, $\frac{1}{3}$, and a visually estimated sum (see diagram).

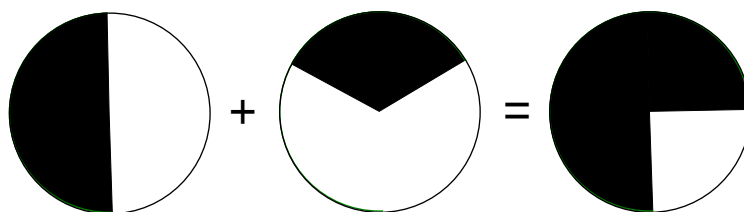


Figure 5.24: A student's WoU Task 4

Maggie asked the student to explain his solution. The student explained by gesturing inside the furthest circle to the right. He ruled out answer choices one by one. First he pointed out that .4 was smaller than this sum saying, “Forty is right here [gesturing inside the third circle].” Then he explained that .5 was clearly smaller than the sum and 1.1 had to be larger than the sum; showing that only choice c was a possibility. This solution is compatible with the visual proof scheme.

Maggie handled the first presenter's WoU by asking the class if they had any questions for the presenter and then later asking him to repeat his explanation. She did not evaluate the student's answer.

Maggie selected another student to present his WoU. The student added $\frac{1}{2}$ and $\frac{1}{3}$ in fraction form; using least common denominators. He arrived at the conclusion that $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, but was uncertain about his result. Maggie ask how the student got .8 from $\frac{5}{6}$ and extracted the WoU that $\frac{5}{6} > \frac{1}{2}$, but smaller than 1. So, .8 must be the answer because it was the only other choice.

Maggie asked a third student to come to the board and explain his thinking. He wrote $.5 + .3 = .8$. Maggie asked how he knew that these were the decimal equivalents. In

particular, she focused on the statement that $1/3 = .3$. Maggie pointed out that the last student saw fractions as division. While Maggie briefly alluded to the discrepancy between .3 and $1/3$, she did not dwell on it. Instead, she chose to promote the way of thinking that was common between the first two presentations.

Maggie handled the solutions by comparing solutions. She did not discuss the final solution because she felt the imagery behind it was clear. For the first two solutions she explained that the students each had different W_sO_U fractions. She distinguished them by indicating that the first was using a fair sharing model and the second was familiar with a set of procedures. However, she explained that they were similar in the sense that both students compared their final results to the choices and eliminated choices in the same way. This teaching practice has been observed repeatedly through out the two summers of the PD. TR often compared participant solutions, highlighting what was common and different between them. He often asked participants to do so as well.

Observed teaching practices, Connections to the PD, and Connections to Proof Schemes

In this lesson Maggie asked students to state and prove conjectures. Reid and Zack (2009) noted this teaching practice in researchers attempting to teach students about proof. This practice was also observed at the PD. TR asked participants to state and prove conjectures on a daily basis and encouraged them to do the same with their students. It is significant that this is the first lesson in which Maggie implemented the practice consistently. Though she eventually assumed the burden of proof there were instances in which students asked the presenter questions following her repeated attempts to get them to do so.

Though it has been mentioned that Maggie handled students' $W_s o U$ factoring by promoting a more efficient way, it must be noted that the data shows no signs of students embracing, understanding, or appreciating her way. What is noteworthy is that Maggie recognized the fact, but did not desist in using this $W_o U$. Furthermore, she gave an explanation that could be completed as a deductive proof. Given the nature of the students as weaker than the students in year 1, it is significant that Maggie attended to mathematical details in the manner in which she did. The proof of her $W_o U$ was challenging for students who did not appear to have developed a number sense to understand the sophisticated level of mathematics Maggie was attempting to promote. Promoting more sophisticated ideas to less mathematically mature students presents increased pedagogical problems for the teacher. This is a question to be investigated in a further study.

An important teaching practice observed in this lesson that has also been observed at the PD is the practice of comparing the mental images that guided the students' $W_s o U$, whether they were correct or in error. Maggie did this task 4. It can only be done when alternative $W_s o U$ are present. Alternative $W_s o U$ can only be present when they are solicited. For a teacher to be aware of alternative $W_s o U$ she must learn to listen carefully to her students. Math education researchers have commented on this very desirable characteristic of reform oriented math teachers; that they are good at listening to and learning from their students. Maggie's ability to solicit and compare these $W_s o U$ is also constrained by the tasks and her own sensitivity to their existence.

Maggie's emphasis on personal meaning for terms and usage of this meaning in the context of the problem is indicative an attempt to promote $W_s o U$ consistent with the

referential symbolic proof scheme. This was most clearly evident in the group work, bake sale, and divisibility word problems. This teaching practice was observed at the PD many times (e.g. – in the TV rating problem TR did the same thing with participants' W_{soU} percent).

Lesson 15

Task

1. Equivalent Fractions

Show two amounts that are equal to $\frac{4}{12}$. Express these amounts as fractions.

Extension: Express all three amounts as a decimal or integer.

2. Simplest Form

You survey your friends about their favorite sandwich and find that 8 out of 12 prefer peanut butter. Write the fraction for 8 out of 12 in simplest form.

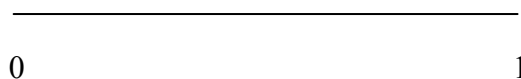
Extension: Write the following in simplest form:

a. $\frac{x}{xy}$ b. $\frac{3ab^2}{12ac}$

3.

Put the following fractions in order from least to greatest:

$$\frac{3}{8}, \frac{5}{6}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$$

*Topic of lesson*

1. Finding equivalent fractions and representing fractions in simplest and decimal forms.
2. Simplifying rational monomials.
3. Comparing values of fractions.

Background Events

Lessons 9 and 10 covered similar topics. In those lessons, students drew representations of fractions and explained how addition and subtraction of fractions works. The analysis of this lesson focuses on task 1 as it is most directly related to the lessons 9 and 10.

Claims

In this lesson Maggie encouraged student to student talk about the mathematics after presentations. She continued to ask students to explain their thinking. Requesting that students ask questions directly to the presenters is a way of extending the locus of authority. Though Maggie stepped into the discussion when needed, she was able to begin to delegate the responsibility of justification to students.

Another important teaching practice observed was pointing out shifts in $W_s o U$. Pointing out shifts in $W_s o U$ is a valuable teaching practice for helping students understand where mathematics comes from. It could not be done unless Maggie asked students to explicate guiding mental imageries behind solution strategies. This was a foundational teaching practice observed at the PD.

Maggie promoted a $W o T$ that is compatible with PPG by pointing out that the process for reducing fractions by repeated divisions had its roots in the students' $W o U$ what a fraction is and his generalization of a pattern. Maggie connected the pattern to the process of repeated cutting. She explained that it was more efficient to repeatedly divide using numbers than to do so by drawing a diagram.

In the last two lessons, 14 and 15, Maggie chose particular students to demonstrate their $W_s o U$ rather than asking for students to volunteer. This allowed her to

make her points through student's WoU, promoting what she considered valuable to make her point about a more efficient WoU and the need the student had to invent it.

Analysis

Task 1

Maggie chose a student to present his solution. This student drew a 2 by 6 rectangle and shaded the first two pieces in the top and bottom rows. Then he cut each piece into two pieces vertically. He recorded the new fraction as $\frac{8}{24}$. Maggie asked if the class had any questions. A question was raised, but Maggie redirected it to the student at the board. The student first asked why the presenter had not done it in another more familiar way. The presenter explained that this was his WoU the question. Next, the same audience member asked how he knew there were now 24 pieces. Maggie explained that each square was split in two; doubling the amount of squares in the numerator and denominator.

Maggie asked the student what he did to arrive at another representation of $\frac{4}{12}$. The student divided both the numerator and denominator by 2 and repeated the divisions; arriving at $\frac{2}{6}$.

Maggie summarized, saying that the point was that the student made one drawing to represent the situation before noticing a pattern – that he could simply divide both numerator and denominator by 2 rather than draw the situation. Maggie pointed out that the student had not explained what he learned. She pointed out that he had seen a pattern in his work (not at the board). She continued by pointing out that the student changed his WoU based on his experiences for the purposes of saving time and doing less work. "... He decided I'm not going to draw pictures anymore and waste my time. He learned from

what he had just did and he said instead if I divide this [pointing to the numerator, 8] by 2, I know that I can divide the bottom by 2, the denominator, and it should be an equal fraction...”

Maggie highlighted the change in mental imagery the student had gone through. She explained that the shift from one WoU to another allowed the student not to “waste time”. Maggie explained that applying division or multiplication to create equivalent fractions came from noticing a pattern. This teaching practice is a connection to the PD. Attending to students’ mental imageries can be used to highlight shifts in them when they occur within the individual. Whether the differences are noted between individuals or within an individual, highlighting differences in WoU helps the teacher explain the purposes of mathematical inventions (or reinventions).

Maggie compared the first student’s original WoU with another student’s WoU that used repeated divisions in the numerator and the denominator from the beginning. She explained that the second student’s WoU made use of the tool (repeated division) the first student had developed. Another student asked why he had chosen to divide numerator and denominator by 2 twice rather than dividing by 4 to find simplest form immediately. The presenter explained that it was because he “...needed two [fractions].” Maggie praised his foresight; seeing it as a sign of expertise. Choosing an expert to demonstrate the first student’s WoU was helpful in demonstrating the value of repeated division because of its efficiency. However, the order in which Maggie structured the presentations allowed for students to see where the tool came from.

A third student presented his WoU the problem. Summarizing his solution, the student explained that he converted three fractions into decimals by dividing numerators by denominators.

Student 3: I got the numerator 4, I divided it by the denominator 12 and got .33 repeating. And I got the same thing, I got the numerator 2 and divided it by the denominator 6 and got .33 repeating. I did the same thing, and got .33 repeating.

$$\begin{array}{r} .33 \\ 6 \overline{)2.00} = .\overline{33} \end{array}$$

Figure 5.25: A student's WoU $\frac{2}{6}$

After his presentation Maggie asked if students were happy with the presentation. She sensed that students were not. So, she waited. Eventually, a student asked if the .33 was the same with or without the bar. Maggie agreed pointed out the difference was indeed significant.

Maggie: ... he pointed out something important. They're not equal, but when he was doing his math, he could have put, as soon as he decided that this was repeating, he could have put his repeating, the bar right over the 3 [on the long division]. But that's a good point that Adam makes. They're not exactly the same number, they are different.

Though Maggie credited Adam for pointing out the difference, her teaching action of waiting for students to examine the solution and pointing out that she felt they were uncomfortable with the solution opened the door for the dialogue. Initially Adam asked, "How come he put .33? It's not the same meaning." Maggie took this point very seriously, dedicating substantial time and effort to his point.

Observed teaching practices, Connections to the PD, and Connections to Proof Schemes

Maggie's experiences at the PD allowed her to value Adam's point. There were similar instances in which TR made the same point about her work. During the hill

problem (PD summer 1), Maggie generated a linear equation by using an approximation for the coordinates of a point. TR drew Maggie's attention to the need for exactness. With the help of fellow participants, Maggie generated the equation again using exact values of coordinates as fractions rather than approximations as rounded decimals.

At the PD Maggie demonstrated the ability to compare her WoU to those of others (e.g. – the stair-like structure and investment problems of year 2). She also experienced the TR's teaching practice of comparing $W_s o U$ by highlighting moments when a shift occurred in the mental imagery that led to a change in WoU and what the student accomplished by changing her WoU in that case. Maggie experienced these shifts in $W_s o U$ in year 2. This was noted in chapter 4 at places when RPG and PPG proof schemes or, more generally, empirical to deductive transitions occurred within one episode.

In this lesson, Maggie pinpointed and held up a transition in one student's $W_s o U$ how to reduce fractions. Students' questions indicate a pedagogical dilemma that teachers may encounter when attempting this approach. Once a student knows how to solve a problem in a more efficient way, he may not value a less efficient way being presented. This was observed when students commented that two repeated divisions⁷⁷ by 2 were equivalent to one division by 4. How to help student understand the shortcomings of more efficient $W_s o U$ is a challenge for teachers that requires a set of unique tasks which may be difficult to come by.

⁷⁷ In the context, it should be clarified that I mean repeated additions in the numerator and denominator of a fraction.

Lesson 16

Task

Pythagorean Theorem

On a graph paper, create right triangles with legs a and b . Measure the length of the third side c with another piece of paper. Copy and complete the table below.

Table 5.1: The Pythagorean Theorem missing values

A	b	c	a^2	b^2	c^2
3	4		9	16	
5	12		25	144	
9	12		81	144	

Extension: Use $<$, $>$, or $=$ to complete the following statement.

$a^2 + b^2$ $\square$ c^2 . Fill in the blank.

Topic of lesson

Pythagorean Theorem

Background Events

In lesson 7 (year 1) Maggie taught students about the Pythagorean theorem.

Zaslavsky, Harel, and Manaster (2006) have investigated these lessons. Through interviews, these researchers found that due in part to institutional constraints Maggie's approach had to be altered in this lesson. They also explain that in this lesson, Maggie promoted a WoU compatible with the RPG proof scheme.

Claims

In this lesson, Maggie extracted a WoU the Pythagorean Theorem that was compatible with the empirical proof scheme of the inductive form; the RPG proof scheme. However, Maggie also demonstrated a willingness to let a student write a conjecture and for a brief period of time allowed students to allow an error to persist. Through a process of guiding students to compare known information to the meaning of what was written, Maggie began to help students make changes to the conjecture, presenting a new WoU the Pythagorean Theorem.

This lesson can be contrasted with Maggie's treatment of the Pythagorean Theorem in year 1 in which the WoU she extracted was compatible with a deductive proof scheme. Zaslavsky, Harel, and Manaster have also analyzed these lessons and arrived at these conclusions.

Analysis:

Maggie began by clarifying terms. She did so by initially asking students for their WoU the terms. Then she pointed out that right triangles are triangles with a right angle. In a similar manner, by asking students for their WoU first, Maggie introduced her WoU the terms legs of a right triangle and hypotenuse of a right triangle. Next, she directed students to create the indicated triangles and gave them strips of graph paper which students used to measure the lengths of the hypotenuses. Note that all triangle side lengths are Pythagorean triples.

Maggie asked students to show her how they drew the triangles. Then she filled in the given table using student comments. She emphasized the WoU c^2 as $c*c$. Next, she asked students to demonstrate examples for the extension problem on the board, filling

values in for a^2 , b^2 , and c^2 and comparing the $a^2 + b^2$ to c^2 . After the first example in which a student concluded that the two quantities were equal, Maggie said that more than one example would be necessary to reach a conclusion. All three examples demonstrated by students confirmed the result that $a^2 + b^2 = c^2$ in all three cases.

Maggie explained that the result only held for right triangles. Then she recorded a student's WoU the Pythagorean Theorem. "If you add the two angles, you'll find the hypotenuse of a right triangle." Maggie asked, "Is this pretty much what happened? Are we happy with this? Does this tell everything that this tells right here? [pointing to something off camera... most likely the table]. Anything we need to add?" Maggie modified the student's statement to read, "If you add the two legs, you'll find the hypotenuse of a right triangle." She showed the students that the implication of $3 + 4 = 5$ being a false statement is that the student's conjecture must be modified. Eventually she changed the statement to the Pythagorean theorem.

Though students brought RPG compatible W_soU, Maggie did not make them deductive. However, she encouraged students to make conjectures and test them against empirical evidence, but not to prove them deductively.

The concepts were not elicited from problems. Rather the concepts were the result of filling in leading questions on the chart. Students' responses demonstrate how it was possible that students could have filled in the table without understanding the significance of the claims. For example, what is the significance of the right angle? How do I know the results will continue to hold? What causes these results to happen?

The instructional approach entailed having students construct examples, filling values into a table, and comparing the squares of the values. A relationship was observed and confirmed using the table. No further confirmation was given.

Maggie: Let's at least check the other two that we have to make sure. Leo, will you do another one, please? And check and see if this triangle has that same relationship? ... So we found out the relationship in all three of them is the same right? ... Every time you have a right triangle, you can use a certain formula. And you guys just found out what the formula is. It's this, $a^2 + b^2 = c^2$. Now this only works with a right triangle. Does it work with any other types of triangles?

Class: No.

Maggie: No.

In contrast to previous lessons, the teacher controlled the presentation with only minimal student presentations verifying results. The pattern was confirmed only because it held for all data in the table. The source of conviction promoted by Maggie was clearly inductive rather than deductive in nature.

In contrast to lesson 7, a process that explains where the theorem comes from was not guiding the creation of examples. Instead, the teacher's rich knowledge of Pythagorean triples allowed for the construction of special examples which avoided square root possibilities and the table was used to observe and confirm the pattern. From this explanation, Maggie had no chance to explain why the pattern held for right triangles and not for other types of triangles.

Observed teaching practices, Connections to the PD, and Connections to Proof Schemes

The teaching practice of encouraging the use of empirical thinking for forming conjectures was observed at the PD. However, TR handled participants' W_soU by helping them complete deductive solutions or completing the deductive solutions himself. There were times when observations were deemed conjectures and set aside to be proven later.

Maggie demonstrated disequilibrium regarding the difference between a derivation and a conjecture during a whole class discussion in the beginning of summer 2 of the PD.

Maggie: I thought of a couple of things while you were talking about the difference between these two, um, derivations... if we aren't even motivated to find the proof for the first one where there is a conjecture, how are we going to get our students motivated to do that. It starts with us first... Um and then secondly, this has been my problem the whole time. I thought I knew algebra. I thought I could do it well, but I can find patterns very easily. I can come up with formulas, but I don't truly know the difference between when I've derived something and when I just have a conjecture. Sometimes I think I'm finished and you come along and you say, well what about this and I didn't think about it. I don't really know the difference between when I'm actually done and when I still have some work to do.

By the end of summer 2, Maggie demonstrated an awareness of when her proofs were incomplete and she could identify places where the argument was lacking (e.g. – Sara's Conjecture and Divisibility by 3 Problems).

Given Maggie's contributions to the aforementioned conversation at the PD and the documented developments in Maggie's proof schemes during summer 2 of the PD, it is likely that some institutional constraints may have been the reason for the difference of approach. Indeed, Zaslavsky, Harel, and Manaster mention the existence of some institutional constraints beyond Maggie's control.

Lesson 17

Task

1. Rest Stops - You are hiking along a trail that is $13\frac{1}{2}$ mi. long. You plan to rest every $2\frac{1}{4}$ mi. How many rest stops will you make?
2. Division patterns - Find each quotient:

$$\frac{1}{2} \div 2$$

$$\frac{1}{2} \div 3$$

$$\frac{1}{2} \div 4$$

$$\frac{1}{2} \div 5$$

Extension: Explain what happens to the quotients as the divisor increases in value.

Remember that in $6 \div 3 = 2$, 6 is the dividend, 3 is the divisor, and 2 is the quotient.

Topic of lesson

Division involving fractions.

Background Events

Lessons 9, 10, and 15 also cover arithmetic of fractions.

Claims

In lesson 17 Maggie focused on students' constructions of rich $W_s o U$ division of mixed numbers. In task 1, she selected students to present a variety of $W_s o U$ grounded in different mental images of the task. In her handling of students' $W_s o U$ Maggie justified the use of the chosen operations in the context of the problem.

In her treatment of task 2 Maggie showed an appreciation for the need for causality, but she could not turn a student's $W_o U$ into a PPG compatible solution.

Allowing students to see the traditional “invert and multiply” WoU in task 1 removed any need for causality on the part of students. This lesson demonstrates how the teacher can see an intellectual need for a WoU, but her students do not. It begs the question, “How can a teacher bring students to see the intellectual need for a concept?”

Analysis

Task 1

The first student was chosen after consultation with the researcher. This shows that Maggie selected the particular student to present first.

Student 1 (Repeated subtraction)

Student 1’s WoU was that you can repeatedly subtract $2\frac{1}{4}$ (or $2\frac{2}{8}$) from the remaining distance to travel. The student’s repeated subtractions were all done by subtracting mixed numbers.

Maggie encouraged student to student talk about the solution by reminding the class to formulate their questions in a useful way saying, “Okay. If you have a question, can you ask Al, but if you say, ‘I don’t get it,’ that’s not really very clear. So, if you have question, ask him a specific question.” This teaching action was aimed at encouraging student to student direct communication, attempting to remove the teacher from direct control over the dialogue.

A student noticed that the presenter had failed to count the last subtraction as a stop. When the presenter attempted to discredit the question as irrelevant, Maggie stepped in to affirm the validity of the question. Eventually, the presenter conceded that he should have counted the last leg of the trip. Maggie asked him why and validated his reasoning. Maggie highlighted the fact that that the student had used repeated subtraction in his

problem solving approach. Other students raised arithmetically oriented questions about the fractional representations the student used.

Student 2 (Drawing Circles)

The second presenter was chosen to come next. Maggie did not ask for a volunteer. The student began by drawing three circles divided into 6 equal parts each. Maggie asked her to explain why she, "...Split it up like that?" The student explained that she wanted to get 12 from the first two circles and that she would use $1\frac{1}{2}$ parts of third circle to represent $13\frac{1}{2}$. Maggie helped clarify that the student using the pieces of each circle to represent miles. The student drew along the edge of the circle making groups of $2\frac{1}{4}$ parts from each circle. Under each circle she wrote $1\frac{1}{2}$. Then she added $1\frac{1}{2} + 1\frac{1}{2} + 1\frac{1}{2} = 4\frac{1}{2}$. She noticed that $2\frac{1}{4} + 2\frac{1}{4} = 4\frac{1}{2}$ and concluded that two more rest stops were used. Finally, she counted the number of rest stops represented in the groupings within the first two circles, 2 each, and added the 2 remaining rest stops, arriving at the conclusion that 6 rest stops had been used.

Maggie decided to "make sure that [students] understand it." She rephrased what had been said, highlighting the meaning of the quantities in the context of the problem. During the presentation Maggie indicated that the significance of the $13\frac{1}{2}$ pieces was that it stood for the total distance to be traveled.

Student 3 (Standard Division of Fractions)

The student silently wrote:

$$\begin{array}{r}
 13\frac{1}{2} \div 2\frac{1}{4} \\
 \frac{27}{2} \div \frac{9}{4} \\
 \frac{27}{2} \times \frac{4}{9} = \frac{6}{1} = 6
 \end{array}$$

Figure 5.26: Student 3's WoU the Rest Stops Problem

After the student explained what she had done without explaining why (see figure 5.22), Maggie began asking questions driven by an attempt to reason quantitatively. She highlighted a WoU division that was partative rather than quotative. Maggie tried to help students justify the choice of division as an operation. She pointed out that the goal was to find how many rest stops were spaced $2\frac{1}{4}$ miles apart within the total distance of $13\frac{1}{2}$ miles and that grouping equal parts together is division.

Maggie: So basically you just realized that you could just divide the two numbers? ... **But why dividing? What is she trying to find here?**

Student 2: The answer?

Maggie: **Yeah, but when you're doing a division problem, what are you really doing? You're finding?** Yeah, you're finding the answer, that's one way of saying it.

Student3: The denominator.

Maggie: Let's be a little bit more descriptive. What are you finding, Ed?

Student5: The rests?

Student6: The rest stops?

Student5: The rest stops, yeah.

Maggie: Here you're finding how many rest stops, what?

Student7: Are in the $13\frac{1}{2}$ miles?

Maggie: **How many rest stops that are spaced at $2\frac{1}{4}$ miles are in $13\frac{1}{2}$ mile? So she decided that a division problem would be fine.** And she just explained all the steps to a division of fractions. Does anyone have questions for Jessie, she was kind of fast?

Maggie attended repeatedly to the meaning of the quantities in the context of the problem. This helped her distinguish what was being grouped from the number of groups.

When the presenter finished, Maggie demonstrated an adding up strategy, comparing it to student 1's subtraction strategy. Maggie pointed out that the problem had been solved by different individuals using either addition, subtraction, multiplication, division, or simply using an illustration in combination with addition. Her way of handling students' W_soU was to expect a justification for the operation students used by referring to the quantities in the problem and how they were being manipulated. Making groupings of a common size – specifically $2\frac{1}{4}$ – was a mathematical thread throughout her recapitulation of the presentations. Maggie's questions, "Why are you dividing" and "What are you really doing," highlight her approach.

Reid and Zack (2009) explained that establishing "community standards" of communication is a teaching practice that "seems important to the success of developing children's proving." Reid and Zack's assertion seems particularly salient in the current lesson. Maggie's expectation that students would justify the operation used in the solution by using the context functioned as her way of renegotiating the didactical contract in a manner consistent with Reid and Zack's observation.

Task 2

Maggie introduced the Division patterns problem. She began by explaining some of the terms needed to communicate accurately (i.e., quotient, divisor, etc). Maggie pointed out that each of the presentations represented a different way to solve the problem.

Student 1

Student 1 was asked to present his solution. He drew a circle divided into quarters, attempting to explain to the teacher directly. She redirected him to address the class. This was another attempt to encourage student to student talk and extend the locus of authority to the class.

When pressed, the presenter drew a circle cut in half with an arrow connecting it to the original circle. Maggie asked, “What did you start with?” The student pointed to the circle indicating $\frac{1}{2}$. The presenter explained that he had cut each piece into 2 pieces, pointing to the divisor. Maggie pointed out that the presenter had divided each half into 2 pieces. The student left two quarters (not $\frac{1}{4}$) shaded throughout his explanation and Maggie did not ask him to make a change.

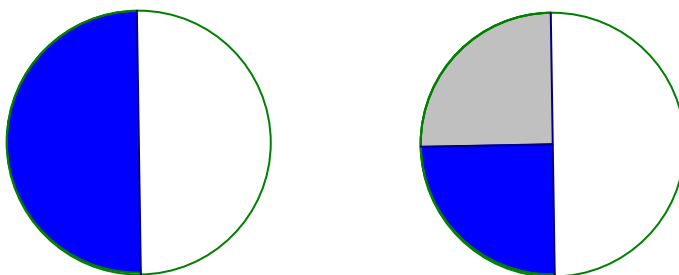


Figure 5.27: Student 1's WoU $\frac{1}{2} \div 2$

Maggie asked students to use student 1's WoU to solve the next problem three problems saying, “We’re going to follow [student 1’s] way of thinking.” Maggie’s WoU student 1’s WoU was to cut the dividend into the number of parts indicated by the divisor. For the purpose of “fair sharing”, Maggie explained that this should be done to both halves of the circle. Maggie kept one of the new parts for each of the next three questions.

Maggie wrote $\frac{1}{2} \div 5$ on the board, but did not draw an example. After a student accurately predicted that $\frac{1}{2} \div 5 = \frac{1}{10}$ Maggie asked, “How did he know that? Is there a pattern there?” This was an example of encouraging a student to anticipate a result. The act of anticipating is closely related to acts of generalizing and conjecture. Given that Maggie had repeated student 1’s reasoning twice,

Student 2 converted $\frac{1}{2}$ to .5 and did long division in each case, arriving at his solutions. Maggie pointed out the issue of approximation in the student’s solutions, but did not ask him for the exact answers. She merely asked him to round correctly.

Student 3 used a traditional invert and multiply strategy in each case.

$$\begin{aligned}\frac{1}{2} \div 2 &= \frac{1}{2} \div \frac{2}{1} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \\ \frac{1}{2} \div 3 &= \frac{1}{2} \div \frac{3}{1} = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} \\ \frac{1}{2} \div 4 &= \frac{1}{2} \div \frac{4}{1} = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8} \\ \frac{1}{2} \div 5 &= \frac{1}{2} \div \frac{5}{1} = \frac{1}{2} \times \frac{1}{5} = \frac{1}{10}\end{aligned}$$

Figure 5.28: Student 3’s WoU the Division Patterns task

Maggie asked the third presenter to provide a justification, “Why can you do that?” His response was that it was “more understandable”. Maggie repeated the question and class members strongly answered things like, “Because that’s the rule,” “That’s what you do when you divide,” and “It’s just a rule.” Maggie pointed out that students were taught to do this. She pointed out that the results were correct, but continued to ask about the cause.

Maggie continued clarifying her question by asking why does $\frac{1}{2}$ divided by 2 become $\frac{1}{2}$ times $\frac{1}{2}$? Eventually, Maggie gave up saying, “Okay. This is something we’re just going to have to do more problems of so you can discover it on your own.” When students tried to claim that there was nothing to understand, Maggie made up a problem... $\frac{1}{2}$ divided by $\frac{1}{2}$. A student said you could multiply the reciprocal, but Maggie replied, “... but what I’m trying to find out – and you guys aren’t helping me with at all is why, why is this okay?” The students seemed frustrated with the claim that they needed to do more problems to understand why.

Maggie’s ultimately handled student 1 and student 3’s $W_s o U$ by searching for a cause. She asked how student 1 knew he could use his $W_o U$ and why student 3’s $W_o U$ worked, but did not provide a justification herself. Rather than take it upon herself to provide justification, Maggie extended the locus of authority saying, “Okay. This is something we’re just going to have to do more problems of so you can discover it on your own.” This marks a lone event observed in the seventeen lessons in which Maggie did not assume the responsibility for proving an observation.

Observed teaching practices, Connections to the PD, and Connections to Proof Schemes

In this lesson, Maggie handled students’ $W_s o U$ by encouraged student to student talk about presentations. By assigned a patterning problem and selecting presenters with different $W_s o U$ to present, Maggie encouraged students to make conjectures and prove them. Given the variety of $W_s o U$ presented in task 1 and the teacher’s focus on causality in task 2, there were non-trivial issues to discuss in the first task.

Maggie handled student 1’s $W_o U$ by asking the class to use it to create other solutions. However, this was a case in which Maggie solved the problem at the board. In

that situation, she was able to filter students' answers to her questions to create the solutions using student 1's WoU. Despite her attempt to bring students to recognize a WoU that was causal in nature, students did not see the need to justify a procedure they were already familiar with for dividing fractions.

At the PD, Maggie specifically mentioned that she thought about how to contextualize division of fractions and "how to make a word problem" providing a context for division of fractions entailed deep knowledge of the content, but requires more.

Maggie: ...it's one thing to know how to do the math but to reverse that and come up with the word problem that's going to bring all of this out is, is, it seems like it would be a whole different class. 'Cause I know in the interview I know how to divide by fractions but I couldn't come up with a word problem that would, you know, lead a student to do that so.

TR: That's true, coming, building the curriculum around problems, to implement something that we talked about many times. The necessity principle as we have seen here ... this question that we have on the blackboard here was necessitated of your approach Maggie. Creating problems of this situation is very hard.

Maggie realized that knowing mathematics and thinking about how to necessitate the mathematics are two different, but related things. She pointed out that, "... Com[ing] up with the word problem that's going to bring all of this out," requires a different kind of thinking. That said, it is notable that she was able to recognize the issue and provide a very powerful example of necessitating a concept. It is notable that once the students recognized that division could be used to solve the problem, they saw no need to justify the use of the operation, but Maggie did.

In task 2, student 1's WoU is quite sophisticated and generalizes as follows.

Given a fraction $\frac{a}{b}$, dividing by $\frac{c}{d}$ means to momentarily consider $\frac{a}{b}$ as the initial unit,

cut it into a parts. For every a parts, one should keep b parts. In these examples where $a > b$ and $b = 1$, this WoU is uncomplicated. However, in other examples it may not be. Bringing an example where $a < b$ might be a next step when students are comfortable with this WoU. This point is made to highlight how Maggie's teaching practice could lead to an implementation of the necessity principle.

Results

1. Attention to mathematical details

When Maggie points out flaws/shortcomings in an argument though the final answer may have been correct.

Examples:

- attending to syntax in a written presentation with an otherwise correct answer,
- dealing with rounding errors in an otherwise correct solution,
- highlighting visual differences,
- highlighting a need for further explanation (usually in a situation where students don't see the need for it).

2. Extending the locus of authority

Handling students' $W_s o U$ by encouraging student to student talk, asking students to state a conjecture, asking students to prove a conjecture, or allowing an error to persist in the final answer are teaching practices that promote students' ownership over $W_s o U$ in the classroom. Though it was often the case that students did not follow Maggie's lead, the focus of this category is the teacher's practice rather than the students' compliance.

3. Attending to students' mental images

Attending to students' mental images entails requesting more information than a simple answer to a task. Tasks that explicitly asked students to complete a pattern can create opportunities for students to use a variety of $W_s o U$ guided by different mental images. Asking students to communicate their thinking can act as a gateway to examining the mental images that guided the production of their $W_s o U$. These mental images may include arithmetic only or they may include more. There are times when the $W_s o U$

students convey are distinct. When this occurs, Maggie may choose to make it explicit that there was a shift in an individual's WoU or she may decide to compare two distinct W_soU previously presented. Soliciting alternative W_soU in the presence of correct solutions is a situation in which Maggie asks for another approach after an approach has been presented that yielded a correct solution.

Codes

1. Attention to mathematical details
2. Extending the locus of authority
 - a. Encouraging Student to Student Talk
 - b. Encouraged students to state conjectures
 - c. Encouraged students to prove their conjectures
 - d. Allowing an error to persist
3. Attending to students' mental images
 - a. Assigning pattern problems
 - b. Ask students to communicate their thinking about their solutions
 - c. Gathering Distinct W_soU
 - d. Pointing out differences between W_soU and mental images (either within the individual or across individuals)
 - e. Asking for alternative W_soU in the presence of correct solutions.

Table 5.2: Observed codes by lesson

Lesson	Code	1	2a	2b	2c	2d	3a	3b	3c	3d	3e	Total	Promoted Proof Schemes
1		Y						Y	Y	Y		4	D
2		Y				Y		Y	Y	Y		5	X,D
3				Y			Y		Y	Y		4	D
4								Y	Y	Y	Y	4	D
5				Y					Y		Y	3	E
6		Y	Y				Y	Y	Y	Y		6	E,D
7				Y			Y					2	D
8				Y	Y	Y	Y	Y	Y	Y	Y	8	D
9		Y						Y	Y	Y	Y	5	E,D
10						Y		Y	Y	Y	Y	5	D
11				Y			Y	Y	Y		Y	5	E,D
12		Y		Y	Y	Y	Y	Y	Y	Y		8	D
13		Y	Y						Y	Y		4	X,D
14			Y	Y	Y			Y	Y	Y	Y	7	D
15		Y	Y					Y	Y	Y	Y	6	D
16				Y		Y	Y					3	E
17		Y	Y	Y	Y		Y	Y	Y	Y	Y	9	D

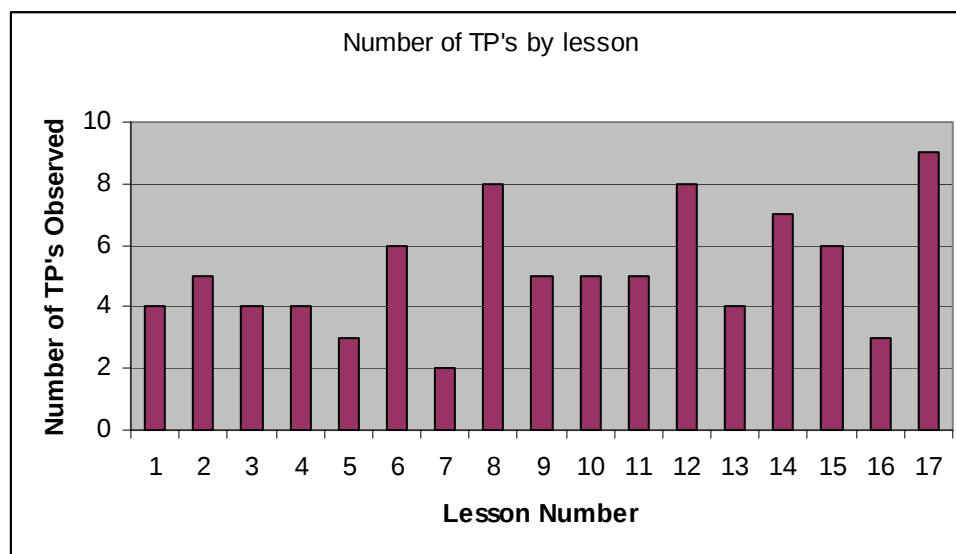


Figure 5.29: Number of teaching practices observed by lesson

Connections to the PD teaching practices

Table 5.2 documents the presence or absence of the selected set of teaching practices. Since this set of teaching practices was observed consistently at the PD and is compatible with DNR, the number of categories of teaching practices observed per lesson tells a story of Maggie's range of flexibility with these DNR-compatible practices. A higher average indicates greater flexibility in using the practices. Analysis of the table can be used to answer parts 1 and 2 of research question 2.

To answer research question 2 the data can be seen in several ways. In particular, one can see the data as two disjoint aggregates (year 1 versus year 2) or as a fluid timeline from lesson 1 to lesson 17. In the latter case, lines of demarcation can be set at particular points to observe trends before and after that point in the timeline. However, this decision backgrounds differences between classes that may have afforded or constrained Maggie's teaching practices. Therefore, when this choice is made, it will be explained why the differences in the classes would not have mattered.

Average number of teaching practices by yearly comparison:

Average number of TP's observed in Year 1 = 4.5

Average number of TP's observed in Year 2 = 5.8

Average number of teaching practices before and after lesson 7:

Lesson 1 to 7 average = 4

Lesson 8 to 17 average = 6

In either way of viewing the data, more elements of the set of teaching practices were observed, on average, as time progressed. This indicates a general trend of greater

flexibility in Maggie's teaching practice as time went on regardless of the student population.

Table 5.3: Percent of lessons with five or more teaching practices by year

	Year 1	Year 2
$T \geq 5$	37.5%	77.8%
$T \leq 4$	62.5%	22.2%

Another way to view the data entails considering the number of lessons in which 5 or more teaching practices were observed. In year 2, 5 or more teaching practices from the indicated set were observed in 7 out of 9 lessons.

Considering that the population of students in year 2 was weaker than the population in year 1⁷⁸ the development of Maggie's teaching practices in the categories of attention to mathematical detail, extending the locus of authority, and attention to students' mental imagery are striking. The forthcoming categorical analyses proceed by major categories, followed by particular subcategories, and then by selected combinations of subcategories. Subcategories 2a, 2c, 2d, and 3e stood out individually, and the combination (1, 3e) were also noteworthy from a developmental perspective.

Promoted W_5oT

It is clear from the table that the deductive proof scheme was most consistent with the W_5oU Maggie promoted throughout both years, the most notable exception being lesson 16. Though it might be possible to speculate why this is so, there is no way to determine from the data why this result was observed.

⁷⁸ Students in year 1 were in 8th grade Pre-Algebra, while students in year 2 were in 7th grade. Students in year 2 were enrolled in a course designed to prepare them for the 8th grade Pre-Algebra course.

Attention to mathematical detail:

There were 3 instances in year 1 when Maggie attended to mathematical details and 5 instances in year 2. Proportionally, this category shows more frequency of occurrences in the second year. Because of the strong relationship between this category and subcategory 3e, a more substantive discussion of its significance is left to the forthcoming analyzing of the combination (1, 3e).

This practice was common at the PD. On multiple occasions, TR pointed out instances when Maggie could write one more line to make something clearer, needed to attend to the difference between approximate and exact solutions, and left something out of a proof. Maggie commented that this left her with a feeling of uneasiness, not knowing when a proof was complete or not. This sense of uneasiness led to greater attention to missing parts of her proofs over time (as reported in chapter 4).

Though it was not the norm, there were instances in which Maggie asked more of her students than what they were comfortable giving. Lessons 1, 9, and 17 provide excellent examples times when Maggie's teaching practices were similar to her experiences at the PD and reflected the changes that were noted in her proof schemes.

Extending the locus of authority:

Instances in all subcategories of category 2 occurred more frequently in year 2 than in year 1, though only slightly for 2b (student to student talk) and 2d (proving conjectures). When viewed as a timeline, development in 2d as seen from lesson 8 forward – far outweighing the number of instances observed in lessons 1 to 7. Instances in which students were asked to state and prove conjectures were observed once in year 1 and 3 times in year 2.

Student to student interactions during whole class discussions, as a way of handling students' WoU, were observed in the first year and 4 times in the second year. This is seen as an indicator that Maggie began extending the locus of authority to students as time went on. She directed students to talk to the presenter on several occasions, attempting to remove herself from the conversation.

Early in summer 1 of the PD, during the TV rating problem, TR began establishing the expectation that students can and should take ownership over the validity of a solution by communicating directly with the presenter in a whole class discussion whenever possible. Maggie came to see a proof as an argument that convinces another person. However, in the Investment Problem (summer 2), she found this WoU proof problematic. At that point, TR explained a more deductive WoU proof.

Maggie's development in categories 2a, 2c, and 2d are reflective of her experiences at the PD, both in terms of her possible observations and proof scheme development. TR's initial teaching practices were geared toward encouraging a shift in Maggie's sources of conviction from reliance on an authority figure only toward a more social definition. Eventually, TR began to emphasize the need for causality as a next step and the need to convince others as a beginning point rather than an ending point in proving.

Attending to students' mental images

It should be noted that category three was robust throughout, with the exception of subcategory 3e. Maggie's constant attention to students' mental imageries was a reason she was chosen. Not only was she herself articulate as a participant at the PD, but she also demanded that students explain their thinking and she recapitulated their

thinking with great regularity. The data also shows that in the presence of distinct $W_s o U$ Maggie regularly pointed out differences them, but these differences were mainly pointed out when there was an error.

In category 3e, handling asking for alternative $W_s o U$ in the presence of other correct solutions, there was development. This subcategory was observed 3 times in the first year and 6 times in the second year. The significance of this difference relates to Maggie's abilities to point out differences, more productive approaches, and more complete approaches. For example, in lesson 8, by continuing to solicit alternative $W_s o U$ and letting an error persist, Maggie was able to point out that two $W_s o U$ which looked symbolically different could be shown to be the same algebraically. This provided students with a powerful example of a time when symbolic manipulation was used to confirm the belief that the students' representations of a pattern were the same, though they looked different.

The best of example of 1 and 3e occurring together can be found in lesson 17. Maggie asked three students to demonstrate $W_s o U$ the Division Patterns problem. The first student used a $W_o U$ that may have been unfamiliar to students, but Maggie appropriated it and used it solve two more problems before asking the class to predict the answer to the fourth problem. The second student solved the problems by converting to decimals. The third student found the reciprocal and multiplied. Though all three $W_s o U$ were correct, Maggie asked why the third student's $W_o U$ was valid. It was a question that class did not appreciate. One student replied, "It's just a rule."

These kinds of situations indicate times when the teacher may be more rigorous than her students can see the need for. However, they are important moments in the sense

that they indicate an attempt on the part of the teacher to renegotiate the didactical contract regarding what constitutes a complete solution in mathematics. In the presence of alternative $W_s o U$ that all lead to a correct answer, students can see the difference between what is complete and correct versus what is correct, but possibly incomplete. The teacher may call to students' attention what one $W o U$ can convey that another can not.

This is an important observation because of its connection to the necessity principle. According to this principle of DNR, in order for students to learn they must see the intellectual need for what the teacher wants to convey. One important time when students have an opportunity to experience intellectual necessity is when they realize that a $W o U$ is incomplete and require a different $W o U$ to solve a problem. Consequently, continuing to draw students' attention to what makes an argument complete is of paramount importance in DNR-based instruction.

An important reason for choosing Maggie initially has to do with her ability to understand and appreciate other participant's $W_s o U$. For example, in the stair-like structure problem, she worked with Bernie, a neighbor. Though she was struggling to complete her own solution, she listened patiently to his approach and engaged in a substantive conversation about it. Another example occurred as she solved Sara's conjecture. In that case, she worked closely with Debbie to produce a solution. Though she ultimately chose her own route, she was able to listen and offer suggestions to Debbie.

Overview

The development of Maggie's teaching practices across the two years she was observed can be connected to her experiences at the PD and the development in her proof schemes. The greatest development was observed in the practices of handling students' W_{soU} by encouraging student to student talk, asking students to prove conjectures, soliciting alternative W_{soU} in the presence of correct solutions, and attending to mathematical detail at times when something had been overlooked by a student, but the solution was correct.

Though it was not a part of the research question, it is important to note that there were two instances in the second year only when Maggie abandoned an approach because she felt students did not understand it. Though there were times when Maggie recognized that her level of mathematical rigor may have been too deep for her students' current stage of mathematical development (e.g., lesson 9, 10, 14 and lesson 17), it is seen as a positive sign that Maggie maintained an awareness of her students' needs even as her own mathematical knowledge deepened.

Though Begel (1976) and Monk (1994) have called into question the importance of teachers' mathematical knowledge in determining their effectiveness. Mathematics educators (cf. Shulman, 1986; Ball, 1991; Harel, 1994; Fennema et al, 1996; Schifter, 1998; Ma, 1999; Sowder, 2007), have been reluctant to dismiss the intuitive argument that teachers need to know their subject matter well to be effective. This case shows how Maggie's ability to make decisions on the fly about what points to bring up, when to pull back, and which W_{soU} to promote rests squarely on her mathematical knowledge (W_{soU} and W_{soT}).

Furthermore, this chapter illustrates how Maggie's experiences afforded opportunities for her to change her pedagogical moves by allowing errors to persist or pointing them out when something important had been overlooked. The case also shows evidence that students' proof schemes mattered to Maggie. The example of lesson 9, when Maggie pointed out the visual difference between parts of the rectangles in each case illustrates how rich understanding of mathematical knowledge can inform the way teachers handle their students' $W_s o U$ in their efforts to change students' $W_s o T$.

CHAPTER 6: CONCLUSIONS

In this chapter, I will summarize the results of this study, discuss contributions of this research to the literature, discuss the limitations of the study, and explain possible next steps in this line of research.

The quality of a nation's teaching corps is an important factor in student achievement. There exists a recognized need for reform of mathematics instruction. However, little is known about how teachers go about developing conceptions and beliefs consistent with the results of recent national studies in mathematics education that could lead to desirable changes in instruction. Though there exists a lack of consensus regarding the roll that content knowledge plays in teachers' instruction of mathematics, many education researchers contend that teachers need to know their content well in order to be effective (e.g. Shulman, 1986; Fennema et al, 1996; Schifter, 1998). There exists a need for greater understanding about what teachers know, how they know it, and how to characterize the kind of knowledge teachers need to meet their students' needs (Ball & Bass, 2003).

Since proving is a central activity within the field of mathematics, there is agreement that teachers' knowledge of proof and its means of instruction deserve the attention of researchers (Knuth, 2002; Nathan and Knuth, 2003). It was argued in chapters 1 and 2 that relatively little is known about teachers' knowledge and beliefs about proof and how teacher's enact their knowledge and beliefs in their teaching practice. Additionally, only a handful of studies have documented longitudinally how changes in teachers' knowledge and beliefs can be connected to development in their teaching practices. In the opening chapter it was noted that this study has the potential to

provide empirical evidence demonstrating how the DNR-theoretical perspective can account for the development of participants' proof schemes and teaching practices.

What is new about these questions is an emphasis on the role of proof schemes in the investigation of change in teaching practices. Connections that are found between Maggie's proof schemes and teaching practices might suggest refinements of the DNR theoretical perspective about how teachers' proof schemes influence their teaching practices and how professional developers can influence teacher's proof schemes. In the future, these refinements can be tested against case studies of other teachers who participated in DNR-based PD.

Summary of Results

Research Question 1: What changes were observed in Maggie's proving and proof schemes as she participated in ATI?

The most robust proof scheme observed was the referential symbolic proof scheme. The presence of this proof scheme throughout the data set indicates that Maggie was consistently able to manipulate symbols with the ability to unpack their meanings in the context of the problem when necessary. Ball and Bass (2000) have noted the importance of the ability to "unpack" content in the everyday work that teachers do. Therefore, there is no claim of change in Maggie's proof schemes with respect to the referential-symbolic proof scheme. Nevertheless, it should be noted that an important feature of PD was attention to mental imagery that guided participant solutions. On multiple occasions, TR asked participants how their $W_s o U$ were similar or different. He also, highlighted instances when a participant demonstrated a shift in mental imagery pivotal in the solution process.

During the two summers Maggie was observed, she learned to scrutinize proofs and observe strengths and weaknesses in them. A pivotal moment in summer 1 was episode 3 (The Cat and Mouse Problem). In her presentation, Maggie came to realize that she was making several assumptions visually without realizing that she could not prove them or that they needed to be proved at all. This was the first documented instance in which she was explicitly confronted with a difference between her expectations for a complete proof and the class's (including TR).

Maggie's disequilibrium, with regard to what constitutes a complete proof, was most pronounced during episodes 5⁷⁹, 7, and 8⁸⁰. In episode 5 she said very clearly, "I don't really know what proves and what doesn't prove." Maggie attributed her disequilibrium to TR's teaching practice, saying, "Sometimes I think I'm finished and you come along and you say, well what about this and I didn't think about it."

Episode 8 demonstrated an instance in which Maggie first demonstrated behavior compatible with the RPG proof scheme and later rejected it after TR asked whether or not her solution was a proof. The fact that TR had previously asked the same question about a participant's solution that used an exhaustive approach, which was considered complete, minimized the possibility that his question was a clue that her solution was incomplete. Rather, Maggie explained why she thought her solution was incomplete.

What was surprising about the events of episode 5 was that even after Maggie rejected a RPG-compatible proof, she evidenced an affinity for it in her own work with a group mate. On the other hand, it was not surprising that episodes 5, 7, and 8 together

⁷⁹ Note that episode 5 alone spanned a total of 15 hours (or three days).

⁸⁰ The elapsed time period between episodes 5 and 8 accounts for 40 hours of class instruction.

demonstrate how for Maggie there was a lag between her ability to recognize the incompleteness of the RPG proof scheme in the proofs of others and her ability to produce PPG proofs. Though she produced a PPG-compatible proof by the end of episode 5 and then again once more in episode 10, evidence of her work with William and TR demonstrates their role in her coming to understand that she needed to attend to a process in order to guarantee the pattern would continue rather than rely solely on a pattern of results.

Maggie began to question her own criterion for determining when an argument is complete without prompting from TR or a group mate to do so. Though her $W_s o U$ were all characterized as deductive after episode 8, episode 11 marks an important point of development for Maggie. Episode 11 was an instance when she demonstrated that she had internalized the practice of finding weaknesses in an argument. Her argument was sound, but she was still uneasy about her solution. She instigated the discussion about her solution with TR and he answered her directly about how she could know that an argument is a proof without relying on the approval of others.

No doubt Maggie's proof schemes may have been influenced by her comfort level with the mathematics she was doing and this should be kept in mind. However, in no way does this diminish the results of the study. It has been shown that Maggie went through a period of disequilibrium with respect to pattern generalization. Eventually, she showed stability in her proof schemes with respect to transformational proof schemes.

Therefore, the major result of this chapter is that Maggie began to question the completeness of her $W_s o U$ and gained a greater awareness of the need for attention to mathematical detail in her time at the PD. Chapter 5 explored the development of

Maggie's teaching practices in the years following summer 1 and summer 2 of the PD, attempting to make connections between the observed development in her teaching practices and her experiences at the PD.

Research Question 2: What connections can be found between Maggie's experiences at the PD and the evolution of her teaching practices in a whole class setting?

- A. What is the evolution of the W_{soU}/W_{soT} Maggie promoted in whole class discussions during her two years of instruction?
 - i. What W_{soU} , and their corresponding W_{soT} , emerged?
 - ii. When they were presented, how did Maggie attend to students' W_{soU} , and their corresponding W_{soT} ?
 - iii. Which W_{soU} , and their corresponding W_{soT} , did Maggie promote?

[Answers to question 1 provide a characterization of a set of her TPs. Harel, Manaster, Fuller, and Soto (in prep) have already discussed the teaching practices Maggie experienced in PD.].

- B. To what extent do Maggie's teaching practices reflect the PD teaching practices?
- C. What relationships can be drawn between the changes observed in Maggie's proof schemes during the PD period and the observed evolution in her teaching practices during the period of time she was teaching?

The research question 2A focused on how Maggie handled students' W_{soU} and which proof schemes were promoted. A major part of answering the question entailed using a grounded approach to discover how she handled students' W_{soU} . Relying on Reid and Zack's (2009) observations of acknowledged expert math educators' teaching,

observations of TR's teaching practices at the PD, and Maggie's teaching, a set of teaching practices were selected for thorough investigation.

Since the set of teaching practices were observed at the PD, their presence in Maggie's teaching provides empirical evidence demonstrating the ability of the DNR-theoretical perspective to account for the development of participants' mathematical knowledge and teaching practices. Observations about Maggie's handling of students' $W_s o U$ in class discussions are discussed within three major categories: attention to mathematical details, extending the locus of authority, attention to students' mental images. The teaching practices chosen for investigation were:

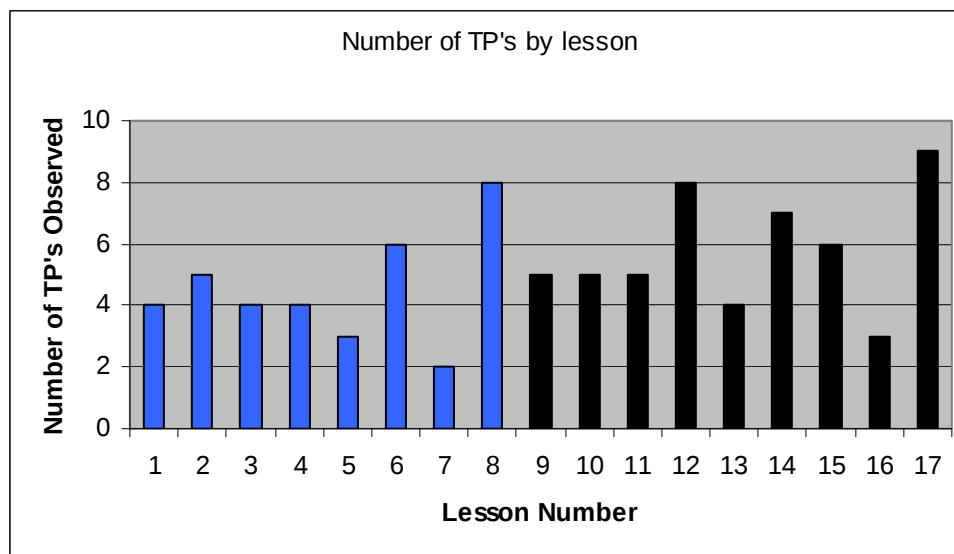
1. Attention to mathematical details
2. Extending the locus of authority
 - a. Encouraging Student to Student Talk
 - b. Encouraged students to state conjectures
 - c. Encouraged students to prove their conjectures
 - d. Allowing an error to persist
3. Attending to students' mental images
 - a. Assigning pattern problems
 - b. Ask students to communicate their thinking about their solutions
 - c. Gathering Distinct $W_s o U$
 - d. Pointing out differences between $W_s o U$ and mental images (either within the individual or across individuals)
 - e. Asking for alternative $W_s o U$ in the presence of correct solutions.

When an instance of a subcategory was observed, it was noted in terms of presence or absence.

Table 6.1: Maggie's Teaching Practices

Code Lesson	1 Detail	2a S-S Talk	2b State Conjectures	2c Prove Conjectures	2d Allow Error to Persist	3a Pattern Problem	3b Ask Students to Communicate	3c Gather	3d Diff.	3e Alt. W ₂ OU	Total	Promoted Proof Schemes
1	Y						Y	Y	Y		4	D
2	Y				Y		Y	Y	Y		5	X,D
3			Y			Y		Y	Y		4	D
4							Y	Y	Y	Y	4	D
5			Y					Y		Y	3	E
6	Y	Y				Y	Y	Y	Y		6	E,D
7			Y			Y					2	D
8			Y	Y	Y	Y	Y	Y	Y	Y	8	D
9	Y						Y	Y	Y	Y	5	E,D
10					Y		Y	Y	Y	Y	5	D
11			Y			Y	Y	Y		Y	5	E,D
12	Y		Y	Y	Y	Y	Y	Y	Y		8	D
13	Y	Y						Y	Y		4	X,D
14		Y	Y	Y			Y	Y	Y	Y	7	D
15	Y	Y					Y	Y	Y	Y	6	D
16			Y		Y	Y					3	E
17	Y	Y	Y	Y		Y	Y	Y	Y	Y	9	D

Table 6.2: Number of Teaching Practices per lesson



Over the course of the two years of observations (lessons 1 – 8 were from year 1 and 9 – 17 from year 2) Maggie exhibited a greater range of usage of these teaching practices, with the most teaching practices per lesson observed at the end of each year and a higher average number of teaching practices (from within this set) observed in year 2 than in year 1.

Table 6.3: Percent of lessons with five or more teaching practices by year

	Year 1	Year 2
$T \geq 5$	37.5%	77.8%
$T \leq 4$	62.5%	22.2%

Maggie engaged in the practice of encouraging student to student talk and encouraging students to prove their conjectures with more frequency in year 2 than in year 1. When viewing the data as a continuous timeline, it is seen that from lesson 8 to lesson 17, Maggie began to allow student errors to persist with greater frequency. Finally, as represented by the though Maggie attended to students' mental images consistently throughout, she exhibited the practice of asking for alternative $W_s o U$ in the presence of correct solutions more frequently in year 2 than in year 1.

The strongest connection observed between the developments in Maggie's teaching practices, the changes in her proof schemes, and the TR's teaching practices is related to mathematical rigor. In Maggie's teaching it was observed that while she attended to student's mental images throughout the two years, her practice showed development in extending the locus of authority and soliciting alternative $W_s o U$ in the presence of correct solutions. At the PD, her exposure to these teaching practices began

on day 1 when TR said, “Don’t look at me. Convince your friends.” It was a consistent practice at the PD to solicit multiple correct $W_s o U$ and for TR to use student solutions to point toward a need for more powerful tools that could solve problems participants could not solve with their existing $W_s o U$.

At the PD, Maggie entered a state of disequilibrium during the first week of summer 2 with respect to what constitutes a proof. She attributed her disequilibrium to TR’s practice of pointing out that something was missing from her arguments. Though she promoted deductive $W_s o T$ in both years of her teaching, there was a difference observed in the quality of this promotion. These differences indicate greater attention to mathematical rigor, not only when there was an error presented, but also when correct answers were present. The most notable instances were in lesson 9 and lesson 17. In these lessons, Maggie attended to her students’ $W_s o T$ rather than simply attending to their $W_s o U$.

One finding of the summer institute data in chapter 4 was that Maggie became increasingly aware that her proofs were incomplete, both at the PD and in her teaching. It was observed in the analysis of attention to mathematical detail that Maggie did so with the same frequency in both years. However, the quality of the instances in year 1 and year 2 were different. The most powerful instances were observed in year 2 because they came in the presence of correct solutions.

Lesson 8 demonstrates, in a small way, one example of a context in which Maggie demonstrated a more complete proof to her class. In year 2 (episode 13), Maggie pointed out to a class of weaker students the difference between a continuous and discrete function. However, in this lesson 8 she did not. Instead, she treated a discrete case as

continuous. In the Jack and Jill speed problem (summer 2 of the PD) the difference between discrete and continuous functions was discussed. In that context, Maggie reported that she came to see the difference as valuable because of an error she made while solving the problem. This connection between Maggie's mathematical knowledge and her teaching has to do with the development of the deductive proof within her. Attending to the domain of a function at all times was an important aspect of the PD and an emphasis of TR's teaching.

When multiple valid $W_s o U$ are present in the classroom and the teacher questions students' $W_s o U$ by asking for reasoning, including mental imagery, examples of different problem solving approaches emerge that students might connect with. At the PD (summer 2), Maggie expressed concern over the presence of multiple $W_s o U$ in the classroom. Namely, she explained that students might get confused and not know what to take away from classroom lessons.

“... my concern would be do you know if the concept that you want to teach is going to come up? I mean you don't have any control over that and I think as teachers that's what we do when we lesson plan, we control exactly what we're going to teach in the classroom and if we just kind of let it flow the way it flows in here we don't know exactly what will come up... And then my other concern with that is with my students, you know, if we're along one path and something comes up and I go along another path I have some students that would just fall apart if I did that.”

It is noteworthy that despite the events of lesson 8, in which she experienced a powerful example of students creating multiple desirable $W_s o U$, she still expressed reservations about relinquishing control over lessons to her students. This speaks to the difficulties researchers have explained regarding their experiences attempting to document teacher change. This case shows that one powerful experience does not change a teacher's

practice. Instead, in Maggie's case she was still concerned that she would lose control or that students would not be able to follow the flow of the lesson.

Contributions of this Research to the Literature

This research informs the bodies of literature about teacher change and professional development. This study puts under a microscope the changes in proof schemes experienced by one participant in a proof-centered PD. It demonstrates the kinds of connections that exist between a teacher's content knowledge and her teaching practice. Specifically, this dissertation demonstrates a case of a teacher whose proof schemes were compatible with RPG for a time, but transitioned toward transformational proving. In her classroom, she was observed asking students to talk to each other directly and asking them to state and prove conjectures. These teaching practices can be connected to her experiences learning to question the completeness of proofs in the context of repeated attention to mathematical detail through a focus on causality on the part of TR.

When reviewing the literature, I found no longitudinal detailed studies of teacher's understanding of proof documenting change in their teaching practices. Stylianides and Silver (2009) explained that these findings might contribute to filling in a gap in the literature on professional development.

“...the existing research knowledge base provides insufficient guidance about the ways in which proficiency with reasoning and proving might be developed over time and how to organize effective professional development for teachers in the domain of reasoning and proving” (pp. 23,24).

This dissertation contributes directly to the DNR-theoretical framework, but also to the literature in the field as a whole. In order to connect Maggie's teaching practices and

proof schemes it was important that the PD was guided by a theoretical framework. Without this framework, it would not have been possible to offer possible connections between changes in proof schemes and teaching practices.

It is acknowledged that addressing Stylianides and Silver's (2009) concerns would have required more focus on TR's teaching practices during the analyses of Maggie's proof schemes. However, analysis of Maggie's proof schemes did offer some byproducts with respect to investigating TR's teaching practices. First, in dealing with Maggie TR's focus on causality (see Cat and Mouse problem and Arithmetic Polygon Problem as good examples of this) was a central feature of his attempts to help her transition from empirical $W_s o T$ toward more deductive $W_s o T$ – specifically from RPG to PPG proof schemes. Second, the kinds of mathematical tasks TR presented at the PD were informed by what he knew about her $W_s o U$ and $W_s o T$ (e.g., episode 9 – A sequence of sequences). In almost every episode, Maggie's fellow participants demonstrated a variety of $W_s o U$ ⁸¹, guided by different mental imageries, and often different proof schemes, to solve the tasks. TR's approach to handling Maggie's proofs entailed pointing out that not all proofs are created equally. Some proofs do not explain why a result will continue to hold (e.g., Episodes 8 and 12). Deductive proofs, such as PPG-compatible proofs, have an advantage over RPG-compatible proofs because they explain why a result must continue to hold. This was one form of TR's attempts to implement of the necessity principle.

⁸¹ In several cases, Maggie herself demonstrated multiple $W_s o U$ a task. One good example was the Train Station Problem in year 1. Though the solutions were incomplete, the $W_s o U$ Maggie demonstrated were substantively different.

Schifter (1998), Shulman (1986), Putnam & Borko (2000), Hill, Rowan, & Ball (2005) and others have expressed the need for case studies of teacher change that help illuminate the processes by which teachers learn mathematics in a qualitatively different way and change their practice accordingly. The detailed information about a teacher's proof schemes provided in this dissertation contributes to the literature because it focused specifically on a teacher's knowledge of mathematics and how it was in turn used to investigate how this knowledge was used in teaching.

For example, a direct connection was made between Maggie's knowledge of continuous and discrete functions and her promoted W_soU. Not all findings in this dissertation were so direct. Most were broader and intended as such. Attention to mathematical details was investigated. Results showed little change in frequency (in terms of lessons per year in which Maggie demonstrated the behavior), but there were powerful examples of Maggie attending to students' proof schemes in the presence of correct solutions. This observation instantiates Shulman's (1986) notion of pedagogical content knowledge, affirming and providing further evidence for why teachers' proof schemes matter and in what ways their proof schemes come into play when doing the mathematical work of teaching. Hill, Rowan, & Ball (2005) referred to this kind of knowledge more specifically as mathematical knowledge for teaching (MKT).

A major contribution of this study to the DNR-theoretical perspective is that it identified teaching practices that TR and Maggie had in common. For example, in lesson 9, Maggie attended not only to students' W_soU $\frac{1}{4} + \frac{3}{5}$, but she also attended to their visually empirical proof schemes regarding the shape and size of the pieces students used to make their arguments. A natural question is whether or not other participants

demonstrated the same teaching practices and to what extent. Was Maggie unique among these teachers? If so, what was it about her that set her apart? Starting with an investigation into the proof schemes of other teachers may reveal similarities in the robustness of the referential-symbolic proof scheme as was observed in Maggie. Using a developmental approach, as Schifter has, one can ask about the trajectories of change for different teachers starting in different states as they enter a proof-centered professional development.

Limitations of the Study and Future Research

One limitation of the study is its generalizability. While the study focuses on important issues in the learning of mathematics for an in-service teacher and the obstacles she faced as she attempted to enact her beliefs about teaching, learning, and mathematics, Maggie was selected because she possessed important traits that would help raise important question by providing insight into the process by which she came to develop her practice and gain new mathematical knowledge.

Maggie's growth with respect to these teaching practices and her proof schemes demonstrates one case in which development of one teacher's practice can be connected to the PD. But what traits stood out in the selection of Maggie as an individual that made this research possible? It is important to identify them, as these special characteristics are important for framing the results in terms of limitations.

In hindsight, Maggie was chosen, in part because of how articulate and willing she was to share her experiences as a teacher and a participant. Maggie's consistent attention to mathematical detail throughout both years of her teaching constitutes one instantiation of these characteristics. Additionally, when I reviewed footage of Maggie's

teaching initially, she stood out because she was able to create a controlled learning environment. This was important because it meant results might be more readily found.

Other personal traits related to disposition stood out. Upon further investigation, what was initially interpreted as a general kindness toward students may be closely related to the manifestations observed as attending to students' mental images. Treating students and their ideas with respect at all times seems like it would be important in all cases, but is it not a part of the theoretical framework.

Another limitation of the study (and a point for future research) is the effect of the tasks on Maggie's ability to gather distinct correct W_{soU} . Maggie alluded to this dilemma at the PD. She explained that designing tasks that implement the necessity principle is a monumental task that she felt would require another course of its own. In her entrance and exit interviews and at the PD, she explained that designing a word problem requiring division of fractions was an extremely challenging task.

Early on, I set out to investigate the question of whether or not the concepts Maggie taught were elicited from the tasks she assigned. Very little could be said with certainty because this data set did not include information about the teacher's plans for lessons and there was no way to ask Maggie why she had chose particular problems. Without the teacher's background knowledge and/or student interviews, it was impossible to tell if a concept had been necessitated with an acceptable degree of certainty. Though there remain traces of this question in the analysis, it would require another study designed in a different way to attend to the implementation of the necessity principle.

The most important implication for further research from this study is that although tasks matter in instruction, a teacher's willingness to focus simultaneously on mathematical detail and students' thinking is constrained by the way she knows mathematics. That is, what teachers expect from students is constrained by how teachers know mathematics, not only how much mathematics they know. For this reason, it is essential that teachers undergo thorough and continuing professional development including opportunities to discuss student thinking and mathematical content simultaneously.

Goldsmith and Schifter (1997) explained that in order for teachers to change their practice, it is crucial that they, "understand the mathematical terrain that they are making available to their students" (p. 32). Maggie's fear about losing control in the class may have related to this point. This study points to a need for teachers to have access to a community in which they can create or find good tasks that are designed to elicit multiple correct distinct W_{soU} and explore and clarify for themselves what the "mathematical terrain" is that they want to explore with their students.

APPENDIX

Summer Institute Problems

Q1: Jill and Jack's Investment Problem 1

Jill and Jack invested money in a mutual fund for one year. Jill invested \$23000 and Jack \$22950. Their broker deducted a 5.5% commission before turning the rest of the money over to the mutual fund. During the year, the value of each share of the mutual fund increased by 11.85%.

What percentage return on their investment did Jill and Jack realized?

Q2: TV Rating Problem

During the Evening Prime Time (Between 7:00PM and 9:00PM) 70% of all TVs in Greenville are turned on, whereas during the Morning Prime Time (between 6:00 AM and 8:00 AM) only 60% of all TVs in Greenville are turned on. Greenville has only four local TV stations: Star Trek Enterprise (STE), Fair and Balance News (FBN), Public Broadcasting Service (PBS), and Quality Family Broadcasting (QFB). John, a local news reporter, obtained from an advertisement firm the "Rating" and "Share" of each station for each of the Prime Times. The "Rating" is the percentage of all TVs that are tuned to a particular show. The "Share" is the percentages of all TVs that are turned on that are turned to this show.

John intended to publish his data in an article about the four local TV stations. He chose a table like the following to report his data:

	STE		FBN		PBS		QFB	
	Morning Prime Time	Evening Prime Time	Morning Prime Time	Evening Prime Time	Morning Prime Time	Evening Prime Time	Morning Prime Time	Evening Prime Time
Rating								
Share								

Unfortunately for John, he lost some of the data he obtained from the advertisement firm. Here is the data he has left:

	STE		FBN		PBS		QFB	
	Morning Prime Time	Evening Prime Time	Morning Prime Time	Evening Prime Time	Morning Prime Time	Evening Prime Time	Morning Prime Time	Evening Prime Time
Rating	25%			14%		16.75%		
Share		30%			27.5%		11.2%	

1. The advertisement firm closed down before John completed his article. Can John derive the missing information from what he has left? Help John fill in as many of the other entries in the table as possible.

Q3: TV Rating Problem

John published his article on the basis of whatever data he was able to derive. However, in his report he intended, but forgot, to mention the percentage of all TVs in Greenville that is turned on during the Prime Times. Kim, a local junior high school student, read the article and was wondering if she could determine the percentage of all TVs in Greenville that are turned on during the Prime Times. Help Kim by either determining those percentages or explaining to her why you can't.

The Flooded Basement Problem

My basement flooded and there is 2.5 inches of water in it. Last time when it flooded there was $\frac{3}{8}$ inch of water and it took my pump 45 minutes to pump it out.

- 1.) How long will it take this time?
- 2.) When I started my pump, I realized that the pump has enough gasoline for one hour. According to the pump manual, the capacity of the gasoline tank is 0.5 gallon, which is sufficient for 5 hours work. What is the least amount of gasoline I need to add in my pump to ensure that it would pump out all water in my basement?
- 3.) Unfortunately, I have only $\frac{1}{5}$ gallon of gasoline to add to my pump. My neighbor has a portable pump with the same capacity as mine. It has $\frac{1}{4}$ gallon of gasoline. Would the two pumps be able to pump 100% of the water in my basement? If the two pumps work together simultaneously, how long would it take them to pump 100% of the water in my basement?

Question 1:

Pump 2 is only $\frac{4}{5}$ capacity of pump 1. While working together how long does it take to pump out the basement?

*Capacity means that in the same amount of time pumping, Pump 2 can only do $\frac{4}{5}$ the amount as Pump 1.

Question 2:

While working together how many inches are pumped at any given moment from the start of their working together?

Q1: Hill Problem

Jack and Jill run 10 kilometers. They start at the same point, run 5 kilometers up a hill, and return to the starting point by the same route. Jack has a 10-minute head start and runs at the rate of 15 km/hr uphill and 20 km/hr downhill. Jill runs 16 km/hr uphill and 22 km/hr downhill.

How far from the top of the hill are they when they pass going in opposite direction?

What is the distance between Jack and Jill at any given moment from the time Jill leaves until Jack arrives?

Question 1:

At what time(s) from Jill's departure that the distance between Jill and Jack is no more than $\frac{3}{7}$ km and no less than $\frac{2}{7}$ km.

$$|x+2| - \left| 2x - \frac{2}{3} \right| \leq 1$$

$$1) \quad -\frac{1}{7} \left| \frac{2}{3} - x \right| - 2x < 3$$

$$2) \quad \left| -x + \frac{3}{2} \right| - \left| \frac{2}{5} - 2x \right| + x \geq 0$$

Pharmacist Problem 1: A pharmacist is to prepare 15 millimeters of special eye drops for a glaucoma patient. The eye-drop solution must have a 2% active ingredient, but the pharmacist only has 10% solution and 1% solution in stock. Can the pharmacist use the solutions she has in stock to fill the prescription?

Pharmacist Problem 2: The same pharmacist receives a large number of special eye drops for glaucoma patients. The prescriptions vary in volume but each requires a 2% active ingredient. Help the pharmacist find a convenient way to determine the exact amounts of 10% solution and 1% solution needed for a given volume of eye drops?

Constant Slope: how do I know that these points with a constant rate form a straight line? What is the cause for them to be on a straight line?

Train Problem

Assume the trains between Los Angeles and San Diego leave each city every hour on the hour with no stops. One-way trip in each direction takes four hours. How many trains going from San Diego to Los Angeles will a train pass on its run from Los Angeles to San Diego?

Clock Problem:

The time now is 12:00 PM. When is the next time the clock hands will be on top of each other?

*'s extension: Find all possible times when they hands are on top of each other.

#'s extension: When will all three be on top of each other?

Homework: Part # 1: 1-2 pages representing your solution and understanding of this problem.

Part #2: The train from LA to SD takes r hours SD to LA takes v hours and the trains leave every f hours. How many meetings will a train from LA to SD have from the time of its departure to its arrival (inclusive)?

Question: Let the distance between the clock hands be the angle between them, measured clockwise from the small hand to the big hand. Find the distance between the clock hands at any moment from 12:00 PM until the next time they are on top of each other. When is the distance between the hands 180° ?

HW:

Solve the clock problem in as many ways as possible. The idea is to reproduce what we have seen. Being able to reproduce it, one can then call it their own.

$$2x + \left| -\frac{4}{3}x + 5 \right| - \frac{1}{6}|6x| - \frac{1}{8}|3x - 5| < 2$$

Illustrate your solution graphically.

Husband and Wife Problem

Jill, Jack's wife, commutes by train from her work to her hometown. She always arrives from work to the train station at 5:00 PM. Jack always leaves home by car at a time so that he too arrives at the train station at 5:00, and without delay he and Jill drive home together. One day Jill took an earlier train—without notifying Jack—and arrived at the train station at 4:00. Upon her arrival at the train station she began walking towards home, and when she met Jack, who was on his way to pick her up, they immediately turned and drove home. They arrived at their home 20 minutes earlier than the time that they usually arrive.

Jill wonders if she can determine how long she walked.

Question 2:

Jill and Jack start running at the same time from a point A on a circular track in opposite directions. They run in constant speed and finish running when they first meet at A. Jill runs at 9 feet per second and Jack at 5 feet per second. How many times do they meet from the time they start running to the time they meet at A?

Power Line Problem-

The state wants to build a power station along an east-west highway to supply power to four small towns, A, B, C, and D, located along the highway. Town C is 100 miles west of D and 85 miles east of B. Town A is 30 miles west of B. The cost of connecting the power station to any town is \$1,000 per mile. The total budget cannot

exceed \$230,000. Where can you locate the power station? You must have a separate power line from the power station to each town.

Carmen's Store Problem

Carmen, a store owner, sells her product according to the following strategy. The percentage she gains on any product she sells is twice what she pays for the product. What she pays for the product does not include shipping and handling charge, which is \$5 per product.

- The most expensive item in her store sells for \$1020. How much did she pay for this item?
- She paid \$270 for one of the items in her store. What is the price tag of this item?
- What is the possible range of the prices in Carmen's store?

Question: why is the solution to

$$Ax^2 + Bx + C = 0 \text{ where } A \neq 0$$

$$x = \frac{-B + \sqrt{B^2 - 4ac}}{2A}$$

OR

$$x = \frac{-B - \sqrt{B^2 - 4ac}}{2A}$$

Budget Deviation Problem

"Budget Deviation" is defined as $|a - b|$ where a is the amount of money allocated and b is the actual amount of money spent. The total budget of a certain institution is divided into three categories: A, B, C and D. $\frac{1}{7}$ of the budget was allocated to Category A, $\frac{2}{9}$ to

Category B, $\frac{5}{14}$ to Category C, and the rest to Category D. The actual amount of money

spent in each category is as follows: 250K for Category A, 340.5K for category B, 17.75K for Category D. What is the exact budget if you know that the sum of the four deviations is 600K?

$$\text{Let } f(x) = x^2 + \left| \frac{2}{3}|x| - 1 \right| + x$$

a.) when is $f(x) = 3$

b.) when is $f(x) \geq 2$

- c.) What does the function look like?
- d.) Where does the graph of this function intersect the x-axis?
- e.) Does this function have maximum or minimum?

Quilt Problem

A company makes square quilts. Each quilt is made out of small congruent squares, where the squares on the main diagonals of the quilt are black and the rest of the squares are white. The cost of a quilt is calculated as follows:

Materials: \$1 for each black square, \$0.50 for each white square.

Labor: \$0.25 for each square

To order a quilt, one must specify the number of black squares, or the number of white squares, or the total number of squares on an order form like the one shown below:

April, Bonnie, and Chad ordered three identical quilts. Each of the three filled out a different order form. April entered the number of black squares in the Black Cell. The other two entered the same number as April's, but accidentally Bonnie entered her number in the Whites Cell, and Chad entered his number in the Total Cell. April was charged \$139.25. How much money were Bonnie and Chad charged?

The Rabbit Problem

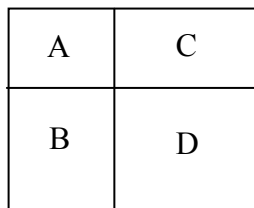
A farmer has some rabbits and some cages. When he puts 2 rabbits in each cage, there are 2 rabbits left over. When he puts 3 rabbits in each cage, there are 16 cages (but no rabbits) left over. How many rabbits and how many cages are there?

The Brick Layer Problem

A mason is building a wall around a square garden. There are 18 bricks for each layer in a wall. If he completes 3 of the walls, he has 22 bricks left over. OR...he could build all four of the walls with 3 fewer layers than the total needed and 4 bricks left over. How many bricks does he have altogether and how many layers are needed to build a wall?

Rectangular Land Problem

A man owns a rectangular piece of land. The land is divided into four rectangular pieces, known as Region A, Region B, Region C, and Region D (See figure).



One day his daughter, Nancy, asked him, what is the area of our land? His son, Ron, asked, what are the dimensions (length and width) of our land? The father replied:

I will only tell you that the area of Region B is 200 ft^2 larger than the area of Region A; the area of Region C is 400 ft^2 larger than the area of Region B; and the area of Region D is 800 ft^2 larger than area of Region C.

- (a) What answer to her question will Nancy derive from her dad's statement?
- (b) What answer to his question will Ron derive from his dad's statement?

The Coin Problem

Ann, Ben, Cari, and Dale collect old coins. Ann has less than twice as many coins as Ben, Ben has less than 3 times as many coins as Cari, and Cari has less than 4 times as many coins as Dale.

Dale has less than 100 coins.

What is the maximum number of coins Ann can have?

Coins Problem #2

Ann, Ben, and Cari collect old coins. The number of Ann's coins is less than 9 times the number of Ben's coins. Furthermore, 9 times the combined number of coins owned by Ann and Ben is less than 4 times the number of Cari's coins. Cari has fewer than 50 coins. What is the maximum number of coins that Ann can have?

Cat and Mouse Problem—Happy Ending

A cat and a mouse were standing on two adjacent vertices of a rhombus, ABCD—the cat on A and the mouse on B. The mouse's safe house is located at O, the intersection point of the rhombus' diagonals. A fence surrounds the rhombus along its sides with only three holes through which the mouse can get to his safe house: one hole at B, one hole at E, which is the midpoint of the route BC, and one hole at C.

The moment the cat and the mouse notice each other, the cat starts running toward B along route AB, and the mouse toward C along route BC. Had the mouse chosen route BO, he would have reached his safe house when the cat is halfway between A and B. Out of excitement, the mouse misses the hole at E, and continues instead toward C. Had he taken route EO, he would have reached his safe house at the same time the cat reaches B. The mouse turns at C and continues running along route CO. Since the cat cannot see the mouse, she enters the hole at E, and runs along the shortest way toward route CO. The moment she reaches route CO, the mouse is six yards away from his safe house. Upon reaching route CO, the cat realizes that with her current speed, she is not going to reach the mouse, and so she increases her speed by 1 minute per yard. The mouse, however, reaches his safe house 15 seconds before the cat does.

- 1) How far did the cat and mouse run?
- 2) What was the speed of each?

Sliding Circle Problem

The sides of triangle ABC have lengths 6, 8 and 10. A circle with center P and radius 1 rolls around the inside of triangle ABC, always remains tangent to at least one side of the triangle. When P first returns to its original position, through what distance has P traveled?

Father and Son Problem

A father and his son are workers, and they walk from home to the plant. The father covers the distance in 40 minutes, the son in 30 minutes. In how many minutes will the son overtake the latter if the latter leaves home 5 minutes earlier than the son?

Father/Son Ages Problem-

A father is 35 years old, and his son is 2. In how many years will the father be four times as old as his son?

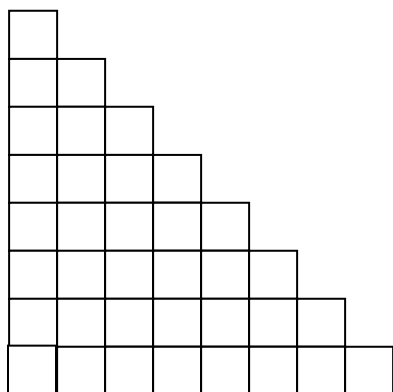
Train Problem-

A train passes through a tunnel 450 m long in d minutes, and goes past a switchman in 15 seconds. What is the train's length and its speed?

Summer 04

Stair-Like Structure Problem

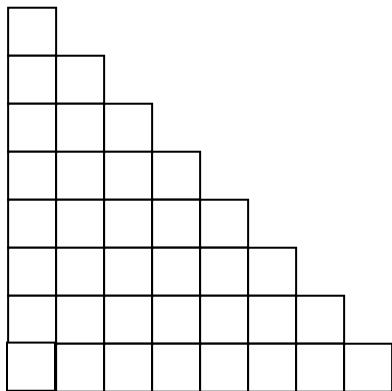
A figure such as the one below is called a *stair-like structure*.



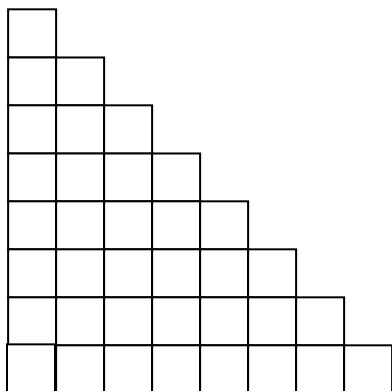
1. You have 1176 identical square pieces. Can you use all the pieces to construct a stair-like structure?

Another Stair-Like Structure Problem

3. You want to build a stair-like structure out of toothpicks. Is it possible to use a total of 2628 identical toothpicks to create a stair-like structure?



3. You have constructed a stair-like structure with a base of 98 pieces (squares). How many squares does your structure have?



4. The figure above has eight steps. You want to use between 2345 and 8789 toothpicks to build a stair-like structure. What is the minimum number of steps your structure can have? What is the maximum number of steps your structure can have?

5. Make up a new problem on “stair-like structure.”

6. In a stair-like structure the number of squares, starting from the top, is 1, 2, 3, ... This sequence is an example of an arithmetic progression. Other examples of arithmetic progressions are:

- 1, 6, 11, 16, ...
- 1, -5, -12, -19, ...

- c. 8, 6.5, 5, 3.5,
- 7. In the first arithmetic progression, find the number in the 88th place, in the second find the number in the 100th place, and in the third find the number in the 73rd place.
 - a. Is it possible that the first two terms of an arithmetic progression are integers, but all consequent terms are not?
- 7. In the first arithmetic progression, find the number in the 88th place, in the second find the number in the 100th place, and in the third find the number in the 73rd place.
 - a. Is it possible that the first two terms of an arithmetic progression are integers, but all consequent terms are not?
 - b. You run a session entitled: “What Do You Notice About Arithmetic Progression?” Ron, a student in your class, claims: “Any item in an arithmetic progression is the average of its two neighboring items—the one on the right and the one on the left.” Is he right?
 - c. Sara, another student in your class, followed up on Ron’s conjecture: “Choose any item in an arithmetic progression. Take any number of items on its right and the same number of items on its left. The item chosen is the average of the rest of the items.” Is Sara’s generalization valid?
- 8. As you have just defined, in an arithmetic progression, each term is the sum of the preceding one and a fixed number. The fixed number is called a *difference* of the arithmetic progression.
 - a. What is the 33rd term of the arithmetic progressions whose first term is 2 and difference is -4?
 - b. If the 119th term of an arithmetic progression is 1175 and its difference is 4, what is its first term?
 - c. 234 is a term in an arithmetic progression whose first term is -12 and difference is 3. In what place in the progression is 234?
 - d. The first and the 99th terms of an arithmetic progression are 10 and 500, respectively. What is its difference?
 - e. List all the formulas you have developed so far in this lesson.
 - f. How many terms does the arithmetic progression, 1,3,5,7,...,997,999 have?
- 7d. Write your definition of *arithmetic progression*.
Is a term in an arithmetic sequence always the average of a term k positions greater than the position of the term and k positions less than the position of the term?
- 9.
 - a. What is the sum of all positive integers between 100 and 500 that are divisible by 6 and their last digit is 6?

- b. What is the sum of all positive integers between 100 and 500 that are divisible by 6 and their last digit is not 6?
- c. What is the sum of all positive integers between 100 and 500 that are not divisible by 6 and their last digit is 6?
5. When the digits of a three-digit number A are reversed, a new number B is obtained. The difference, $B-A$, is 99. What are the numbers A and B?
10. A polygon has the following properties: (a) It has exactly 3 acute angles; (b) the measures of its angles are all positive integers; (c) the measures of its angles form an arithmetic progression whose difference equals its first term. How many segments does this polygon have?
11. Take the arithmetic sequence of positive integers, 1, 2, 3, Divide it into the sets, (1), (2, 3), (4, 5, 6), (7, 8, 9, 10), You get a sequence of sets in which the first set consists of one item, the second of two items, the third of three items, and so on.
- How many items are there in the 55^{th} set? What is their sum?
 - How many items are there in the n^{th} set? What is their sum?
 - Which set contains the number 1000?
 - What is the first set the sum of whose items exceeds 1000?
12. Take the arithmetic progression of the odd integers, 1, 3, 5, Divide it into the sets, (1), (3, 5, 7), (9, 11, 13, 15, 17), You get a sequence of sets in which the first set consists of one term, the second of three terms, the third of five terms, and so on.
- i. How many terms are there in the 178^{th} set? What is their sum?
 - ii. What is the sum of the terms in the n^{th} set?
 - iii. What is the sum of all terms in the first n sets?
 - iv. Do sets with the following property exist?
 - a. The sum of the terms in the set is 10000?
 - b. The sum of the terms in the set is 30071?
 - c. The sum of the terms in the set is even?
 - d. The sum of the terms in the set is a perfect square?
 - v. The sum of the terms in a set exceeds 5234. What is the position of this set in the sequence?
12. Suppose you have a principal of \$12,000 invested at 11% interest rate per year. Determine the amount in the account after 10 years if the compounding period is: (a) annually, (b) quarterly, (c) monthly

13. Suppose that you will need \$23,000 to pay for a year of college for your child 18 years in the future, and you can buy a certificate deposit whose interest rate of 10% compounded quarterly is guaranteed for that period. How much do you need to deposit?
14. Suppose that the monthly statement from a fund you have in a Mutual Fund reports a beginning balance at \$17,396.17 and a closing balance of \$21,034.25 for 29 days. Using this rate compounded daily for a year, what will be the total amount in your account at the end of one year?
15. Mary has two accounts, A and B. In January 1, 1993, Mary invested the same amount in each account. Account A pays simple interest at 10% per year. Account B pays 10% per year compounded yearly.
 - a. How would you help Mary compare the amounts in each account at the end of each of the first 10 years?
 - b. In December 30, 2003, Mary had in one account 3000 more than in the other account. What was Mary's investment?
16. You are to determine the amount in an account after a certain compounding period. What information would you seek? How would you use this information to determine the amount?
17. For 12 months, beginning January 1, an individual saves \$120 per month, deposited directly into her account in payday, the last day of the month. The account earns 6.5% per year, compounded daily. What is the total earning at the end of the year? [You can calculate as if each month has 30 days and the year has 360 days].
18. Suppose that a couple wishes to save for the college education of their child. They want to begin saving a fixed amount per month after the child is born so that their child will have \$100,000 available when she turns 18. How much do they have to save each month, if their account earns 6.8% interest per year, compounded daily?
19. Sally buys a house for \$275,000 with a loan that she will pay off over 30 years in equal monthly installments. Suppose that the interest rate for her loan is 8.5%. What is Sally's monthly payment?
21. Jill travels at a constant speed of 20 miles a day. Jack, starting from the same point exactly three days later to overtake Jill, travels at a constant speed of 15 miles the first day, 19 miles the second day, 23 miles the third day, and so on in arithmetic progression.
 - a. How long does Jack travel from the time he starts until he overtakes Jill?
 - b. What is the accumulated distance that each of them has traveled each day until they met?
13. n arithmetic sequences are defined as follows:
 The first terms of the sequences are 1, 3, 5, ..., respectively.
 The differences for the sequences are 1, 2, 3, ..., respectively.

Compute the sum of all the terms in these sequences.

12. Build a sequence that is both arithmetic and geometric. How many such sequences can you build?

A group of workers working together at the same rate can finish a job in 45 hours. However, the workers report to work singly at equal intervals over a period of time. Once on the job, each worker stays until the job is finished. If the first worker works five times as many hours as the last worker, find the number of hours the first worker works? What is the number of workers?

Each term of an arithmetic sequence is a positive integer. The third term of the sequence is 25. In the sequence there is a pair of consecutive terms whose squares differ by 399. What is the largest term of the sequence?

You offer your class the following game:

The students work in small groups for the purpose of creating interesting sequences. You also take part in the game by forming your own group.

The rules are as follows:

- Each group reveals a property of the sequence they have created.
- A group who offers a property for which there is no sequence that satisfies it loses 5 points.
- A group who does not offer all the sequences that satisfy its own property (i.e., the property it has proposed) loses the game.
- For each proposed property, a group who discovers all the sequences that satisfy the property wins 10 points.
- A group who discovers some, but not all, of the sequences that satisfy a proposed property wins 2 points.

Here are the properties proposed by each group:

Group 1:

Our sequence is an arithmetic progression for which the sum of any number of consecutive terms, starting with the first term, is three times the square of the number of the terms.

Group 2:

Our sequence is an arithmetic progression for which the sum of the first m terms equals the sum of the first n terms, where n and m are different.

Group 3:

Our sequence is a geometric progression for which the sum of the first m terms is equal to the sum of the first n terms, where n and m are different.

Group 4:

Our sequence is a geometric progression. There are three integer terms in the sequence, call them f , g , and h , where g is exactly in the middle between f and h , and the three represent the sides of a right triangle.

Group 5 (your group):

My sequence is so that its second differences are constant and its first differences form a geometric sequence.

Find out your score at the end of our discussion of the game.

A circle is tangent to the two sides of an acute angle; a second circle is tangent to the first circle and to the two sides of the angle; a third circle is tangent to the second circle and to the two sides of the angle; and so on. Select any three consecutive circles, and call the smallest one C_1 , the middle one C_2 , and the largest one C_3 . Let the radii of C_1 and C_2 be r and R , respectively.

What is the sum of the areas of C_1 , C_2 , and C_3 ?

Show that the ratio of the areas of C_1 and C_2 is 1:9 if and only if the measure of the angle is 60° .

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