SELF-DUAL SOLUTIONS TO EUCLIDEAN GRAVITY

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ABSTRACT

Recent work on Euclidean self-dual gravitational fields is reviewed. We discuss various solutions to the Einstein equations and treat asymptotically locally Euclidean self-dual metrics in detail. These latter solutions have vanishing classical action and nontrivial topological invariants, and so may play a role in quantum gravity resembling that of the Yang-Mills instantons.

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I. INTRODUCTION

The discovery of self-dual instanton solutions in Euclidean Yang-Mills theory [1] has recently stimulated a great deal of interest in self-dual solutions to Einstein's theory of gravitation. One would expect that the relevant instanton-like metrics would be those whose gravitational fields are self-dual, localized in Euclidean spacetime and free of singularities. In fact, solutions have been found which have the additional interesting property that the metric approaches a flat metric at infinity. These solutions are called "asymptotically locally Euclidean" metrics because, in spite of their asymptotically flat local character, their global topology at infinity differs from that of ordinary Euclidean space. Since the Yang-Mills instanton potential approaches a pure gauge at infinity, this class of Einstein solutions closely resembles the Yang-Mills case.

The first examples of asymptotically locally Euclidean metrics were the self-dual solutions given by the authors in ref. [2]. Belinskii, Gibbons, Page and Pope [3] then studied the general class of self-dual Euclidean Bianchi type IX metrics and showed that only metric II of ref. [2] could describe a nonsingular manifold. Gibbons and Hawking [4] have now exhibited an entire series of such metrics. In fact, very general classes of manifolds which could admit self-dual asymptotically locally Euclidean metrics have recently been identified by Hitchin [5].

Asymptotically locally Euclidean self-dual metrics have a number of special properties. For one thing, they have zero action and so must be quite important in the path integral. Secondly, since

the metrics become flat and the gravitational interactions are switched off at infinity, standard asymptotic-state methods can be applied to analyze the quantum effects of such metrics.

For completeness, let us summarize various stages of the search for gravitational instantons which took place before the discovery of asymptotically locally Euclidean metrics. The first step was the identification of the Euler characteristic and Hirzebruch signature of a manifold as the appropriate gravitational analogs of the Yang-Mills topological invariants [6] [7]. A number of standard Riemannian manifolds were of course considered as logical candidates for gravitational instantons. The most remarkable of these, the K3 surface, is the only compact regular four-dimensional manifold without boundary which admits a metric with self-dual curvature [8]; this metric would therefore satisfy Einstein's equations with vanishing cosmological constant. Unfortunately, the explicit form of the K3 metric has so far eluded discovery.

The first known metrics which come to mind are the standard solutions of black hole physics. While all black hole solutions arise in Minkowski spacetime, they can be continued also to the Euclidean regime to produce positive-definite singularity-free metrics [9] [10]. These continued metrics are periodic in the new time variable, which is associated with the thermodynamic temperature, and decay only in the three spatial directions. One example of such a metric is the self-dual Euclidean Taub-NUT solution examined by Hawking [10]. In this case Einstein's equations are satisfied with zero cosmological constant, and the manifold is $\mathbb{R}^4$ with a boundary

which is a twisted three-sphere S^3 possessing a distorted metric. The metric is not asymptotically flat because it does not fall off in all four asymptotic spacetime directions.

Another interesting case is the Fubini-Study metric on $P_2(\mathbb{C})$, two-dimensional complex projective space, studied by Eguchi and Freund [7] [24]. This manifold is compact without boundary and has constant scalar curvature. The metric has self-dual Weyl tensor rather than self-dual curvature, and so solves Einstein's equations with nonzero cosmological term. One drawback is that $P_2(\mathbb{C})$ does not admit well-defined Dirac spinors. Nevertheless, one can construct a more general type of acceptable spin structure on $P_2(\mathbb{C})$ by adding a Maxwell field to the theory [11].

All of the metrics just described are in some sense self-dual, are regular and have finite action, but are not asymptotically flat. The gravitational fields of such metrics persist throughout spacetime and make it difficult to define the asymptotic plane-wave states necessary for ordinary scattering theory. Although these metrics are very interesting, they do not quite coincide with our intuitive picture of instantons as localized excitations in Euclidean spacetime which approach the vacuum at infinity. In contrast, the asymptotically locally Euclidean metrics seem to be very naturally identifiable as gravitational instantons.

The remainder of the paper is organized as follows: Section II contains a complete explanation of the derivation of the regular asymptotically flat self-dual solution presented in ref. [2].

In Section III, we examine the properties of various other metrics which have instanton-like properties. Section IV is devoted to self-dual multicenter metrics and Section V contains concluding remarks.

II. AN ASYMPTOTICALLY FLAT SELF-DUAL SOLUTION OF EUCLIDEAN GRAVITY

We now derive the simplest regular asymptotically flat self-dual solution of Euclidean gravity, which was labeled as metric II in ref. [2]. Let us begin by reviewing a procedure by which one can solve the Yang-Mills equations to obtain the instanton solution [1] and noting possible gravitational parallels. To obtain the instanton, we do the following:

(1) Observe that the Yang-Mills equations

$$\partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, are solved at once due to the Bianchi identities if

$$F_{\mu\nu} = \pm \tilde{F}_{\mu\nu} \equiv \pm \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta} .$$

(2) Choose the Ansatz

$$A_\mu = \rho(r)\, g^{-1} \partial_\mu g$$

for the SU(2) gauge potential, where $r^2 = t^2 + \vec{x}^2$, $g = (t - i\vec{\tau} \cdot \vec{x})/r$, and $\{\vec{\tau}\}$ are the Pauli matrices.

(3) Solve the first-order differential equation

$$\rho'(r) + \frac{2}{r} \rho(\rho - 1) = 0$$

obtained by setting $F_{\mu\nu} = \tilde{F}_{\mu\nu}$; we find

$$\rho = r^2/(r^2 + a^2) .$$

In this way, we find a Euclidean SU(2) Yang-Mills solution with finite action, self-dual $F_{\mu\nu}$ localized at $r = 0$ and falling like $1/r^4$ at infinity, and A_μ asymptotically a pure gauge at infinity.

We wish to find a Euclidean gravity solution with finite action, self-dual curvature localized inside the manifold and falling rapidly at infinity, and with the metric asymptotically locally Euclidean at infinity. We might therefore search for such a solution by undertaking the following gravitational analogs of the Yang-Mills procedure:

(1) Observe that if the spin connection 1-form $\omega^a{}_b$ is self-dual (i. e., $\omega^0{}_i = \pm \frac{1}{2} \epsilon_{ijk}\omega^j{}_k$), the curvature 2-form $R^a{}_b$ is self-dual, so Einstein's equations are satisfied at once due to the cyclic identities.

(2) Choose an Ansatz for $g_{\mu\nu}(x)$ which differs from a flat Euclidean metric by functions of $r^2 = t^2 + \vec{x}^2$ alone.

(3) Solve the first-order differential equations in the metric obtained by requiring $\omega^a{}_b$ to be self-dual.

A. Preliminaries.

First we establish some useful notation and explain more fully the essential concepts appearing in the procedure just outlined. We let the four Euclidean coordinates be $x_\mu = (t,x,y,z)$ so that the flat metric is given by

$$ds^2 = dt^2 + dx^2 + dy^2 + dz^2 . \qquad (2.1)$$

We next change to four-dimensional polar coordinates with $r^2 = t^2 + x^2 + y^2 + z^2$ and define

$$\sigma_x = \frac{1}{r^2}(xdt - tdx + ydz - zdy) = \frac{1}{2}(\sin\psi d\theta - \sin\theta\cos\psi d\phi)$$

$$\sigma_y = \frac{1}{r^2}(ydt - tdy + zdx - xdz) = \frac{1}{2}(-\cos\psi d\theta - \sin\theta\sin\psi d\phi)$$

$$\sigma_z = \frac{1}{r^2}(zdt - tdz + xdy - ydx) = \frac{1}{2}(d\psi + \cos\theta d\phi). \qquad (2.2)$$

The variables θ, ϕ, ψ are Euler angles on the three-sphere S^3 with ranges

$$0 \le \theta \le \pi$$
$$0 \le \phi \le 2\pi$$
$$0 \le \psi \le 4\pi \qquad (2.3)$$

and are related to the Cartesian coordinates by

$$x + iy = r\cos\frac{\theta}{2}\exp\frac{i}{2}(\psi + \phi)$$

$$z + it = r\sin\frac{\theta}{2}\exp\frac{i}{2}(\psi - \phi). \qquad (2.4)$$

The differential 1-forms (2.2) are closely related to the Cartan-Maurer forms for SU(2) and obey the following structure equations under exterior differentiation:

$$d\sigma_x = 2\sigma_y \wedge \sigma_z, \quad \text{cyclic.} \qquad (2.5)$$

The flat metric can now be written in polar coordinates as

$$ds^2 = dr^2 + r^2(\sigma_x^2 + \sigma_y^2 + \sigma_z^2). \qquad (2.6)$$

Next we write an arbitrary metric in terms of the local orthonormal vierbein frame $e^a{}_\mu(x)$,

$$ds^2 = dx^\mu g_{\mu\nu}(x) dx^\nu = \sum_{a=0}^{3} (e^a)^2, \qquad (2.7)$$

where $e^a \equiv e^a{}_\mu dx^\mu$. The spin connection $\omega^a{}_b$ is then a one-form determined uniquely by the structure equations [12]

$$de^a + \omega^a{}_b \wedge e^b = 0$$

$$\omega^a{}_b = -\omega^b{}_a = \omega^a{}_{b\mu} dx^\mu. \qquad (2.8)$$

Greek indices are raised and lowered with $g_{\mu\nu}$, while Latin indices are raised and lowered by the flat metric δ_{ab}. Vierbeins and inverse vierbeins interconvert Latin and Greek indices.

The curvature is now defined as the two-form

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \qquad (2.9)$$

where

$$R^a{}_b = \frac{1}{2} R^a{}_{b\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2} R^a{}_{bcd} e^c \wedge e^d.$$

By exterior differentiation of (2.8), we find the cyclic identity,

$$R^a{}_b \wedge e^b = 0 \rightarrow \varepsilon_{ebcd} R^a{}_{bcd} = 0. \qquad (2.10)$$

We now define the "dual" of the two-form $R^a{}_b$ in its free indices as

$$\tilde{R}^a{}_b = \frac{1}{2} \varepsilon_{abcd} R^c{}_d. \qquad (2.11)$$

Then it is easy to show that Einstein's equations

$$R^{ac}{}_{bc} = \mathcal{R}^\mu{}_\nu e^a{}_\mu e_b{}^\nu = 0, \qquad (2.12a)$$

where $\mathcal{R}^\mu{}_\nu$ is the Ricci tensor, are equivalent to

$$\tilde{R}^a{}_b \wedge e^b = 0. \qquad (2.12b)$$

(One must take appropriate sums and differences of various components to prove the equivalence.) Therefore if $R^a{}_b$ is (anti) self-dual,

$$R^a{}_b = \pm \tilde{R}^a{}_b, \qquad (2.13)$$

the cyclic identity (2.10) implies that the Einstein equations (2.12) are satisfied. This is the analog for gravitation of the fact that self-dual Yang-Mills fields automatically satisfy the equations of motion. However, Eq. (2.13) is still a second-order differential equation in the vierbeins $e^a{}_\mu(x)$. It is remarkable that we can now go one step further and deduce the Einstein equations from a first-order differential equation in the fundamental variables, just as in the Yang-Mills case. We simply observe that Eq. (2.9) can be written

$$R^2{}_3 = d\omega^2{}_3 + \omega^2{}_0 \wedge \omega^0{}_3 + \omega^2{}_1 \wedge \omega^1{}_3 , \quad \text{cyclic},$$
$$R^0{}_1 = d\omega^0{}_1 + \omega^0{}_2 \wedge \omega^2{}_1 + \omega^0{}_3 \wedge \omega^3{}_1 , \quad \text{cyclic}. \tag{2.14}$$

Thus if

$$\omega^0{}_i = \pm \tfrac{1}{2} \epsilon_{ijk} \omega^j{}_k \tag{2.15}$$

is obeyed, then Eq. (2.13) is immediately satisfied. Defining

$$\tilde{\omega}^a{}_b = \tfrac{1}{2} \epsilon_{abcd} \omega^c{}_d, \tag{2.16}$$

we see that the first-order condition on $e^a{}_\mu$,

$$\omega^a{}_b = \pm \tilde{\omega}^a{}_b , \tag{2.17}$$

is a sufficient condition for the self-duality of $R^a{}_b$, and hence for solving the Einstein equations.

In fact, Eq. (2.17) is also necessary for a self-dual $R^a{}_b$ in the following sense: if Eq. (2.13) is satisfied, one can always transform $\omega^a{}_b$ by an $O(4)$ gauge transformation into the form (2.17). To see this, we examine the change in $\omega^a{}_b$ when the orthonormal frame specified by e^a is rotated by an x-dependent orthogonal transformation $\Lambda^a{}_b(x)$:

$$e'^a = (\Lambda^{-1})^a{}_b \, e^b. \tag{2.18}$$

A simple calculation using the structure equation (2.8) shows that the form of the structure equation is preserved if we identify the new spin connection as

$$\omega'^a{}_b = (\Lambda^{-1})^a{}_c \omega^c{}_d \, \Lambda^d{}_b + (\Lambda^{-1})^a{}_c \, d \, \Lambda^c{}_b. \tag{2.19}$$

Thus $\omega^a{}_b$ transforms exactly like an $O(4)$ Yang-Mills gauge potential. Furthermore, the curvature behaves as

$$R'^a{}_b = (\Lambda^{-1})^a{}_c \, R^c{}_d \, \Lambda^d{}_b. \tag{2.20}$$

The conclusion of our argument is as follows: Suppose $R^a{}_b$ is self-dual, but $\omega^a{}_b$ is not. Then split $\omega^a{}_b$ into self-dual and anti-self-dual parts; one can explicitly construct a $\Lambda^a{}_b$ which will gauge transform away the anti-self-dual part. Since self-duality of $R^a{}_b$ is preserved under the orthogonal transformation (2.20), we find that any self-dual curvature comes from a self-dual connection if a "self-dual gauge" is chosen.

In Table 1, we present a summary of these results and compare them with the analogous properties of Yang-Mills theories in differential-form notation. The point is that although the Euler equations of the Einstein and Yang-Mills theories are quite different, they both are automatically solved when the spin connections or field strengths obey the appropriate self-duality conditions. In gravity, the self-duality condition (2.17) is a first-order differential equation in the vierbein $e^a{}_\mu(x)$, while in Yang-Mills the self-duality

condition $F_{\mu\nu} = \pm \tilde{F}_{\mu\nu}$ is first-order in the potentials $A^a_\mu(x)$. We remark that the difference between Yang-Mills theory and Einstein's theory in the orthonormal frame basis is that the gravity $O(4)$ connections $\omega^a{}_b$ follow from the metric and thus guarantee that $R^a{}_b$ obeys the cyclic identity. No such additional restriction occurs in general in an $O(4)$ Yang-Mills theory since the group indices and the spacetime indices are uncorrelated.

B. The Metric Ansatz

We now continue to follow the pattern observed in Yang-Mills theories by choosing a metric Ansatz differing from the flat metric by functions of the radius alone. We choose to examine the axially symmetric Ansatz

$$ds^2 = f^2(r)dr^2 + r^2(\sigma_x^2 + \sigma_y^2 + g^2(r)\sigma_z^2). \qquad (2.21)$$

(This was Ansatz II of ref. [2].) More general Ansätze will be examined in the next Section.

If we decompose the metric (2.21) into the orthonormal vierbein basis

$$e^a = (f(r)dr, r\sigma_x, r\sigma_y, rg(r)\sigma_z), \qquad (2.22)$$

we find that the structure equations (2.8) give the spin connections

$$\omega^1{}_0 = \frac{1}{rf} e^1 \qquad \omega^2{}_3 = \frac{g}{r} e^1$$

$$\omega^2{}_0 = \frac{1}{rf} e^2 \qquad \omega^3{}_1 = \frac{g}{r} e^2$$

$$\omega^3{}_0 = \left[\frac{1}{rf} + \frac{g'}{fg}\right] e^3 \qquad \omega^1{}_2 = \frac{2 - g^2}{rg} e^3. \qquad (2.23)$$

With our choice of orientation, we are led to impose anti-self-duality on the $\omega^a{}_b$, leading to the differential equations

$$fg = 1$$

$$g + rg' = f(2 - g^2). \qquad (2.24)$$

These equations are integrable, with the result

$$g^2(r) = f^{-2}(r) = 1 - (a/r)^4, \qquad (2.25)$$

where a is the integration constant.

Hence we find a new metric [2]

$$ds^2 = [1 - (a/r)^4]^{-1} dr^2 + r^2(\sigma_x^2 + \sigma_y^2) + r^2[1 - (a/r)^4]\sigma_z^2 \qquad (2.26)$$

which satisfies the Euclidean empty space Einstein equations. The spin connections are

$$\omega^1{}_0 = \omega^2{}_3 = [1 - (a/r)^4]^{\frac{1}{2}}\sigma_x = [1 - (a/r)^4]^{\frac{1}{2}}e^1/r$$

$$\omega^2{}_0 = \omega^3{}_1 = [1 - (a/r)^4]^{\frac{1}{2}}\sigma_y = [1 - (a/r)^4]^{\frac{1}{2}}e^2/r \qquad (2.27)$$

$$\omega^3{}_0 = \omega^1{}_2 = [1 + (a/r)^4]\sigma_z = [1 + (a/r)^4]e^3/(r[1 - (a/r)^4]^{\frac{1}{2}}).$$

We easily compute the curvature components to be

$$R^1{}_0 = R^2{}_3 = -\frac{2a^4}{r^6}(e^1 \wedge e^0 + e^2 \wedge e^3)$$

$$R^2{}_0 = R^3{}_1 = -\frac{2a^4}{r^6}(e^2 \wedge e^0 + e^3 \wedge e^1) \qquad (2.28)$$

$$R^3{}_0 = R^1{}_2 = +\frac{4a^4}{r^6}(e^3 \wedge e^0 + e^1 \wedge e^2).$$

It is straightforward to construct also a <u>combined solution</u> of the Maxwell-Einstein equations in the presence of the metric (2.26). Choosing the potential

$$A = \frac{1}{r^2}\sigma_z \qquad (2.29)$$

one finds the field strength

$$F = dA = \frac{2}{r^4}(e^3 \wedge e^0 + e^1 \wedge e^2). \qquad (2.30)$$

Since F is anti-self-dual, it is harmonic and has vanishing (Euclidean) energy-momentum tensor. Thus Einstein's equations retain their empty-space form and the Maxwell-Einstein equations are

automatically satisfied. As we will demonstrate shortly, the coordinate system "origin" occurs at $r = a$ and the manifold is regular there, so F is regular and finite everywhere. (The other two anti-self-dual Maxwell fields that naturally present themselves, with $A_1 = r^2\sigma_x/(r^4 - a^4)$ and $A_2 = r^2\sigma_y/(r^4 - a^4)$, are $F_1 = 2(r^4 - a^4)^{-1}(e^1 \wedge e^0 + e^2 \wedge e^3)$ and $F_2 = 2(r^4 - a^4)^{-1}(e^2 \wedge e^0 + e^3 \wedge e^1)$ and are thus singular.) Suggestively, the $1/r^4$ asymptotic behavior of the Maxwell field (2.30) is <u>the same</u> as that of the Yang-Mills instanton.

C. <u>Properties of the Manifold</u>

We now need to determine whether there are any true singularities in the new metric (2.26) and whether it describes a geodesically complete manifold [3]. We begin by writing the metric in several alternative forms. First, let

$$\rho^4 = r^4 - a^4; \qquad (2.31)$$

then

$$ds^2 = [1 + (a/\rho)^4]^{-\frac{1}{2}}\left\{d\rho^2 + \rho^2\sigma_z{}^2\right\} + [1 + (a/\rho)^4]^{\frac{1}{2}}\left\{\rho^2\sigma_x{}^2 + \rho^2\sigma_y{}^2\right\}. \qquad (2.32)$$

These coordinates are well-adapted to converting the metric into complex form using

$$z_1 = x + iy \qquad z_2 = z + it$$

$$\rho^2 = z_1\bar{z}_1 + z_2\bar{z}_2.$$

One then finds two equivalent ways of writing the metric in terms of a Kahler form [5] [13] on $\mathbb{C}^2 - \{0\}$:

$$1)\quad K = \frac{\rho^2}{[\rho^4 + a^4]^{\frac{1}{2}}}(dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2) + \frac{a^4}{[\rho^4 + a^4]^{\frac{1}{2}}}\partial\bar{\partial}\ell n(\rho^2) \tag{2.33a}$$

$$2)\quad K = \partial\bar{\partial}\ \ell n\ \phi, \qquad \phi = \frac{\rho^2 \exp[\rho^4 + a^4]^{\frac{1}{2}}}{a^2 + [\rho^4 + a^4]^{\frac{1}{2}}}\ . \tag{2.33b}$$

The $\partial\bar{\partial}\ \ell n\ \rho^2$ term in these forms causes problems at $\rho = 0$ (i. e. at $r = a$). However, this apparent singularity can be removed if one identifies opposite points of the manifold,

$$(z_1, z_2) \sim (-z_1, -z_2)$$

or (2.34)

$$(x_\mu) \sim (-x_\mu).$$

We next give a more elementary explanation of this fact.

First, let us change radial variables once again by defining

$$u^2 = r^2[1 - (a/r)^4]. \tag{2.35}$$

Then the metric can be written

$$ds^2 = du^2/[1 + (a/r)^4]^2 + u^2\sigma_z^{\ 2} + r^2(\sigma_x^{\ 2} + \sigma_y^{\ 2}). \tag{2.36}$$

Very near to the apparent singularity at $r = a$, or, equivalently, $u = 0$, we have

$$ds^2 \approx \frac{1}{4}du^2 + \frac{1}{4}u^2(d\psi + \cos\theta\, d\phi)^2 + \frac{a^2}{4}(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.37}$$

For fixed θ and ϕ, we obtain

$$ds^2 \approx \frac{1}{4}(du^2 + u^2 d\psi^2). \tag{2.38}$$

A short exercise tells us whether or not the singularity at $u = 0$ is real or is a removable polar coordinate singularity. We simply note that the apparent $r = 0$ singularity in the $\mathbb{R}^2$ metric

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\Phi^2 \tag{2.39}$$

is removable provided that

$$0 < \Phi < 2\pi. \tag{2.40}$$

We therefore conclude that if the range $0 < \psi < 4\pi$ given by Eq. (2.5) is changed to

$$0 < \psi < 2\pi, \tag{2.41}$$

we can remove the apparent singularity at $r = a$ and obtain a geodesically complete manifold.

The global topology of our manifold is now the following: Near $r = a$, the manifold has the topology $\mathbb{R}^2 \times S^2$ indicated by

Equation (2.37). To be precise, at each point of the two-sphere parametrized by (θ,ϕ), there is attached an $\mathbb{R}^2$ which shrinks to a point as $r \to a$ $(u \to 0)$. The manifold is thus homotopic to S^2 and has the same Euler characteristic as S^2, $\chi = 2$.

For large r, the metric approaches a flat metric. However, because of the altered range (2.41) of ψ, the constant-r hypersurfaces are not three-spheres, but three-spheres with opposite points identified. The boundary as $r \to \infty$ is thus the familiar group manifold of $SO(3) = P_3(\mathbb{R})$, for which $S^3 = SU(2)$ is the double covering. This is an explicit example of a metric whose topology is asymptotically locally Euclidean ($P_3(\mathbb{R}) = S^3/Z_2$), but not globally Euclidean (i. e. not S^3).

It can be shown [5] that the entire manifold M we have just described is in fact the cotangent bundle of the complex plane, $P_1(\mathbb{C}) \simeq S^2$, and so we may write

$$\begin{aligned} M &= T^*(P_1(\mathbb{C})) \\ \partial M &= P_3(\mathbb{R}) \,. \end{aligned} \tag{2.42}$$

In Figure 1, we present a description of the topology of the manifold which we have deduced from the metric (2.26) and the regularity requirements.

D. Action and Topological Invariants

Using the connections and curvatures (2.27) and (2.28) for the metric (2.26) with the ψ-range (2.41), we now calculate the various integrals characterizing the solution. Since our manifold has a non-empty boundary surface, we will repeatedly need the second fundamental form,

$$\theta^a{}_b = \omega^a{}_b - (\omega_0)^a{}_b, \tag{2.43}$$

to compute boundary corrections. If we choose the radial direction as the direction everywhere normal to the boundary, $(\omega_0)^a{}_b$ is the connection of the product metric for fixed r_0,

$$ds^2 = [1 - (a/r_0)^4]^{-1}\, dr^2 + r_0^2(\sigma_x^2 + \sigma_y^2 + [1 - (a/r_0)^4]\sigma_z^2). \tag{2.44}$$

Since our scalar curvature is identically zero, the entire action comes from the surface term [9]. Defining $K^i{}_j$ by [13] $\theta^0{}_i = -K^i{}_j e^j$, we calculate the surface action at large r to be

$$\begin{aligned} -\frac{1}{8\pi}\int_{\partial M(r)} K^i{}_i \, d\Sigma &= \frac{\pi}{8}\left[3r^2 - \frac{a^4}{r^2} - 3r^2(1 - (a/r)^4)^{\frac{1}{2}}\right] \\ &\sim \frac{\pi}{16}\,\frac{a^4}{r^2} \,. \end{aligned} \tag{2.45}$$

Since the surface term falls like $1/r^2$ as $r \to \infty$, we find vanishing action for the metric (2.26),

$$S[g] = 0. \tag{2.46}$$

We have already stated the topological arguments giving our manifold the same Euler characteristic, $\chi = 2$, as S^2. We confirm this fact using Chern's formula [14][15]

$$\chi(M) = \frac{1}{32\pi^2}\left(\int_M \varepsilon_{abcd} R^a{}_b \wedge R^c{}_d - \int_{\partial M(\infty)} \varepsilon_{abcd}\left(2\theta^a{}_b \wedge R^c{}_d - \frac{4}{3}\theta^a{}_b \wedge \theta^c{}_e \wedge \theta^e{}_d\right)\right)$$

$$= \frac{3}{2} - (-\frac{1}{2}) = 2. \qquad (2.47)$$

Had we allowed ψ to range over all of S^3 instead of $P_3(\mathbb{R})$, we would have found twice this answer, $\chi = 4$. The apparent disagreement between the topology and the Chern-Gauss-Bonnet theorem for the wrong ψ range shows that for $0 < \psi < 4\pi$, the manifold would have "cone-tip" singularities at $r = a$; this implies the necessity of cone-tip corrections (such as effective delta-functions in the curvature at $r = a$) in order to adjust the Euler characteristic to its correct topological value. This does not seem to be a very satisfactory physical situation, so that the proper range of ψ must indeed be $0 < \psi < 2\pi$.

To compute the signature τ of our manifold M, we first compute the integral of the first Pontrjagin class,

$$P_1[M] = \int_M p_1 = -\frac{1}{8\pi^2}\int_M \mathrm{Tr}(R \wedge R) \qquad (2.48)$$

$$= -3.$$

The Chern-Simons boundary correction [16] vanishes,

$$-Q_1[\partial M] = \frac{1}{8\pi^2}\int_{\partial M} \mathrm{Tr}(\theta \wedge R) = 0. \qquad (2.49)$$

The signature η-invariant η_s for the canonical metric on $P_3(\mathbb{R})$ has been computed by Atiyah, Patodi and Singer [17] to vanish also:

$$\eta_s(P_3(\mathbb{R})) = \frac{1}{4}\cot^2\left(\frac{\pi}{2}\right) = 0. \qquad (2.50)$$

Thus the signature of M is

$$\tau[M] = \frac{1}{3}(P_1 - Q_1) - \eta_s = -1. \qquad (2.51)$$

By the Atiyah-Patodi-Singer extension [17] of the Hirzebruch signature theorem, there is exactly one anti-self-dual harmonic 2-form with the appropriate boundary conditions.

The index $I_{\frac{1}{2}}$ of the spin $\frac{1}{2}$ Dirac operator in the presence of the metric (2.26) is given by [17] [15],

$$I_{\frac{1}{2}} = -\frac{1}{24}(P_1[M] - Q_1[\partial M]) - \frac{1}{2}(\eta_{\frac{1}{2}} + h_{\frac{1}{2}}). \qquad (2.52)$$

Atiyah [18] has extended the computation of ref. [17] to the Dirac case, with the result

$$\eta_{\frac{1}{2}} = \frac{1}{4}\,\frac{1}{\sin^2\frac{\pi}{2}} = \frac{1}{4}$$

$$h_{\frac{1}{2}} = 0. \qquad (2.53)$$

Thus

$$I_{\frac{1}{2}} = -\frac{1}{24}(-3-0) - \frac{1}{2}(\frac{1}{4}) = 0 , \qquad (2.54)$$

and there is no asymmetry between the right and left chirality zero-frequency modes of the Dirac operator.

For the spin 3/2 Rarita-Schwinger operator, the index is given by

$$I_{3/2} = \frac{21}{24}(P_1[M] - Q_1[\partial M]) - \frac{1}{2}(\eta_{3/2} + h_{3/2}) , \qquad (2.55)$$

where we have corrected the result of ref. [19] to include boundary terms in the obvious way. Hanson and Römer [20] have calculated the expression involving the Atiyah-Patodi-Singer η-invariant with the result

$$\eta_{3/2} + h_{3/2} = -\frac{5}{4} . \qquad (2.56)$$

There are thus two excess negative chirality spin 3/2 fields obeying the Atiyah-Patodi-Singer boundary conditions,

$$I_{3/2} = \frac{21}{24}(-3-0) - \frac{1}{2}(-\frac{5}{4}) = -2. \qquad (2.57)$$

This is in agreement with the explicit construction of Hawking and Pope [21], who build two spin 3/2 wave functions out of two covariant constant spinors and the single anti-self-dual Maxwell

field whose existence is required by Eq. (2.51) for the signature. The indicated spin 3/2 solutions may in fact follow directly from an appropriate supersymmetry transformation.

While the spin 2 index characterizing the number of anti-self-dual perturbations about the metric (2.26) has not been calculated at this time, there is at least one zero-frequency mode, corresponding to a dilatation, which is not a gauge transformation [22].

III. PROPERTIES OF MORE GENERAL METRICS

A. General Bianchi IX Ansatz

One might naturally ask what happens if the Ansatz (2.21) is replaced by the most general Ansatz giving a Bianchi IX metric [3],

$$ds^2 = f^2(r)dr^2 + a^2(r)\sigma_x^2 + b^2(r)\sigma_y^2 + c^2(r)\sigma_z^2. \tag{3.1}$$

For the case

$$a(r) = b(r) = c(r), \tag{3.2}$$

we find that self-duality implies a vanishing curvature and hence a flat metric. The case

$$a(r) = b(r) = 1, \quad c(r) \equiv g(r) \tag{3.3}$$

was our choice II of ref. [2], which we studied in the previous section; the choice

$$a(r) = b(r) \equiv g(r), \quad c(r) = 1 \tag{3.4}$$

was case I of ref. [2]. While self-dual solutions of (3.3) describe the regular manifold of Fig. 1, the self-dual solutions of (3.4) in fact have a singularity at finite proper distance and are therefore unacceptable.

A general solution of the self-duality equations for the Ansatz (3.1) has been given by Belinskii, Gibbons, Page and Pope [3]. They find

$$\begin{aligned} f^2(r) &= F^{-\frac{1}{2}}(r) = r^6 a^{-2}(r) b^{-2}(r) c^{-2}(r) \\ a^2(r) &= r^2 F^{\frac{1}{2}}(r)/(1 - (a_1/r)^4) \\ b^2(r) &= r^2 F^{\frac{1}{2}}(r)/(1 - (a_2/r)^4) \\ c^2(r) &= r^2 F^{\frac{1}{2}}(r)/(1 - (a_3/r)^4) \end{aligned} \tag{3.5}$$

where

$$F(r) = (1 - (a_1/r)^4)(1 - (a_2/r)^4)(1 - (a_3/r)^4) \tag{3.6}$$

and a_1, a_2, a_3 are constants. They find (see also ref. [13]) that for general parameters a_i, these metrics all have singularities at finite proper distance and so describe physically unacceptable manifolds. Only the particular degenerate case (3.3) described in Section II allows a mechanism for "shielding" the naked singularity inside the S^2 at $r = a$ so that geodesics cannot get to it. (This is of course analogous to what happens in the Euclidean continuation of the Schwarzschild and Taub-NUT metrics.)

B. Nuts and Bolts

Given a metric, one of the most important things one must know is whether or not its apparent singularities are removable. The two known types of removable singularities have been christened

"nuts" and "bolts" by Gibbons and Hawking [23]. A "nut" is a four-dimensional $\mathbb{R}^4$ polar coordinate singularity in a metric which is flat at the origin, like the self-dual Euclidean Taub-NUT metric. A "bolt" is a two-dimensional $\mathbb{R}^2$ polar coordinate singularity in a metric looking like $\mathbb{R}^2 \times S^2$ near the origin, like (2.26). Nuts carry one unit of Euler characteristic, while bolts carry two units.

A precise formulation of the concepts of nuts and bolts is as follows [13] [23]: Consider the metric

$$ds^2 = d\tau^2 + a^2(\tau)\sigma_x{}^2 + b^2(\tau)\sigma_y{}^2 + c^2(\tau)\sigma_z{}^2 , \qquad (3.5)$$

where a variable change has been made on (3.1) to convert the coordinate radius r into the proper distance (or proper time) τ. In general, one would require that a,b,c be finite and nonsingular for finite τ to get a regular manifold. (For infinite τ, this restriction can be relaxed if the manifold has a suitable boundary at $\tau = \infty$.) However, the manifold can be regular even in the presence of apparent singularities.

Let us for simplicity consider singularities occurring at $\tau = 0$. A metric has a removable nut singularity provided that

$$\text{near } \tau = 0, \qquad a^2 = b^2 = c^2 = \tau^2. \qquad (3.6)$$

In this case at $\tau = 0$ we have simply a coordinate singularity in the flat polar coordinate system on an $\mathbb{R}^4$ centered at $\tau = 0$. The singularity is removed by changing to a local Cartesian coordinate

system near $\tau = 0$ and adding the point $\tau = 0$ to the manifold. Near $\tau = 0$, the manifold is topologically $\mathbb{R}^4$.

A metric has a removable bolt singularity if

$$\text{near } \tau = 0, \quad \begin{cases} a^2 = b^2 = \text{finite} \\ c^2 = n^2\tau^2, \quad n = \text{integer}. \end{cases} \qquad (3.7)$$

Here $a^2 = b^2$ multiplies the canonical S^2 metric $\frac{1}{4}(d\theta^2 + \sin^2\theta d\phi^2)$, while at constant (θ,ϕ), the $(d\tau^2 + c^2(\tau)\sigma_z{}^2)$ piece of (3.5) looks like

$$d\tau^2 + n^2\tau^2\tfrac{1}{4}\,d\psi^2 . \qquad (3.8)$$

Provided the range of $n\psi/2$ is adjusted to $0 \to 2\pi$, the apparent singularity at $\tau = 0$ is nothing but a coordinate singularity in the flat polar coordinate system on $\mathbb{R}^2$. Again, this singularity can be removed by using Cartesian coordinates. The topology of the manifold is locally $\mathbb{R}^2 \times S^2$ with the $\mathbb{R}^2$ shrinking to a point on S^2 as $\tau \to 0$.

C. The Fundamental Triplet of Self-Dual Metrics

The prototype of a metric with a single bolt is the metric (2.26) introduced in ref. [2]. The removal of this singularity was discussed in detail in Section II. The prototype of a metric with a single nut is the self-dual Euclidean Taub-NUT metric [10]

$$ds^2 = \frac{1}{4}\frac{r+m}{r-m}\,dr^2 + \frac{1}{4}(r^2 - m^2)(d\theta^2 + \sin^2\theta d\phi^2) + m^2\frac{r-m}{r+m}(d\psi + \cos\theta d\phi)^2. \qquad (3.9)$$

To remove the apparent singularity at $r = m$, we first change to the proper distance coordinate

$$d\tau^2 = \frac{1}{4}\left(\frac{r+m}{r-m}\right) dr^2 \tag{3.10}$$

and consider only the region $r = m + \varepsilon$, $\varepsilon \ll m$. Then

$$\tau = \int_m^{m+\varepsilon} \frac{1}{2}\left(\frac{r+m}{r-m}\right)^{\frac{1}{2}} dr \approx (2m\varepsilon)^{\frac{1}{2}}. \tag{3.11}$$

The metric near $\varepsilon = 0$ is thus

$$\begin{aligned} ds^2 &\approx d\tau^2 + \frac{1}{4}\tau^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{1}{4}\tau^2(d\psi + \cos\theta\, d\phi)^2 \\ &\approx d\tau^2 + \tau^2(\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2) \end{aligned} \tag{3.12}$$

and the condition (3.7) for a nut is met.

Both the bolt metric (2.26) and the nut metric (3.9) are noncompact with boundary at ∞. A very instructive compact case is the Fubini-Study metric on $P_2(\mathbb{C})$, which has <u>both</u> a nut and a bolt. To see this, we first write the $P_2(\mathbb{C})$ metric in the form [24]

$$ds^2 = \frac{dr^2 + r^2\sigma_z{}^2}{(1 + \Lambda r^2/6)^2} + \frac{r^2(\sigma_x{}^2 + \sigma_y{}^2)}{1 + \Lambda r^2/6}\,. \tag{3.13}$$

Here Λ is the cosmological constant in Einstein's equations for this metric. As $r \to 0$, we recover the flat metric and thus learn that $r = 0$ is a removable nut singularity. The other interesting region is $r = \infty$, which we examine by changing variables to

$$u = \frac{1}{r} \tag{3.14}$$

so that

$$\begin{aligned} ds^2 = {} & (u^2 + \Lambda/6)^{-2}(du^2 + \tfrac{1}{4}u^2(d\psi + \cos\theta\, d\phi)^2) \\ & + \tfrac{1}{4}(d\theta^2 + \sin^2\theta\, d\phi^2)/(u^2 + \Lambda/6)\,. \end{aligned} \tag{3.15}$$

As $u \to 0$ (or $r \to \infty$), the coefficient of $(d\theta^2 + \sin^2\theta\, d\phi^2)$ stays finite while that of $(d\psi + \cos\theta\, d\phi)^2$ vanishes, so the bolt criterion (3.7) is satisfied. At fixed (θ,ϕ), we have for $u \to 0$

$$ds^2 \approx (\Lambda/6)^{-2}(du^2 + \tfrac{1}{4}u^2\, d\psi^2). \tag{3.16}$$

Thus the singularity at $u = 0$ is removable if

$$0 < \psi < 4\pi\,, \tag{3.17}$$

and the constant-r manifolds in the $P_2(\mathbb{C})$ metric are complete S^3's, unlike those of the metric (2.26), which had $P_3(\mathbb{R})$'s.

Modulo this difference, we are now led to group together the $P_2(\mathbb{C})$ metric (3.13) or (3.15), the Taub-NUT metric (3.9) and our "bolt" metric (2.26) as a "fundamental triplet." We note that both (3.9) and (2.26) have self-dual Riemann curvature tensors and so satisfy Einstein's equations without a cosmological constant. The $P_2(\mathbb{C})$ metric, in contrast, has a self-dual <u>Weyl tensor</u> which is as close as one can get to having self-dual Riemann tensor if there is a nonzero cosmological constant. The Taub-NUT metric and the

$P_2(\mathbb{C})$ metric both have nuts at the origin, but Taub-NUT opens up at infinity while $P_2(\mathbb{C})$ compactifies. On the other hand, our metric (2.26) at the origin looks like the $P_2(\mathbb{C})$ metric at <u>infinity</u> - both have bolts at these locations; furthermore, the flat infinity of (2.26) strongly resembles the flat (but compact) origin of $P_2(\mathbb{C})$. Figure 2 gives a schematic representation of the relationships among the manifolds described by these three metrics.

We next make the remark that all three of the metrics just discussed are derivable from a more general three-parameter Euclidean Taub-NUT-de Sitter metric, although some hindsight is necessary to notice the existence of the appropriate singular limits. If we write the general Taub-NUT-de Sitter metric as

$$ds^2 = \frac{\rho^2 - L^2}{4\Delta}\, d\rho^2 + (\rho^2 - L^2)(\sigma_x^{\,2} + \sigma_y^{\,2}) + \frac{4L^2\Delta}{\rho^2 - L^2}\, \sigma_z^{\,2}, \tag{3.18}$$

where

$$\Delta = \rho^2 - 2M\rho + L^2 + \frac{\Lambda}{4}(L^4 + 2L^2\rho^2 - \frac{1}{3}\rho^4), \tag{3.19}$$

then the choice

$$\Lambda = 0, \quad M = L \tag{3.20}$$

immediately gives the self-dual Taub-NUT metric (3.9). If we set

$$M = L(1 + \tfrac{1}{3}\Lambda L^2) \tag{3.21a}$$

and take the limit [24]

$$\begin{cases} L \to \infty \\ \text{with} \\ \rho^2 - L^2 = r^2/(1 + \frac{1}{3}\Lambda r^2) \ \text{ fixed}, \end{cases} \tag{3.21b}$$

we recover the Fubini-Study $P_2(\mathbb{C})$ metric in the coordinate system (3.13). Finally, our metric (2.26) can be reproduced by setting

$$M = L\left(1 + \frac{a^4}{8L^4} + \frac{\Lambda L^2}{3}\right), \tag{3.22}$$

putting $\Lambda = 0$ and taking the limit

$$\begin{cases} L \to \infty, \\ \text{with} \\ r^2 = \rho^2 - L^2 \ \text{ fixed}. \end{cases} \tag{3.23}$$

This is a rather peculiar limit which has previously escaped attention. If we keep $\Lambda \neq 0$, we find a new metric resembling (2.26) except that it satisfies Einstein's equations with nonzero cosmological term,

$$ds^2 = \frac{dr^2}{1 - (a/r)^4 - \frac{\Lambda r^2}{6}} + r^2(\sigma_x^{\,2} + \sigma_y^{\,2}) + r^2\left[1 - (a/r)^4 - \frac{\Lambda r^2}{6}\right]\sigma_z^{\,2} \tag{3.24}$$

By taking an appropriate limit of this metric, we can eliminate the singularity and obtain a metric on $S^2 \times S^2$ with a twist.

Another amusing comparison which we may make among these three metrics involves their natural self-dual Maxwell fields. For $P_2(\mathbb{C})$, Trautman [11] observed that the metric possessed a natural (anti) self-dual Maxwell field given by the $P_2(\mathbb{C})$ Kahler form,

$$F = 2(e^0 \wedge e^3 - e^1 \wedge e^2) , \qquad (3.25)$$

where $e^a = \left[dr(1 + \Lambda r^2/6)^{-1}, r\sigma_x(1 + \Lambda r^2/6)^{-\frac{1}{2}}, r\sigma_y(1 + \Lambda r^2/6)^{-\frac{1}{2}}, r\sigma_z(1 + \Lambda r^2/6)^{-1}\right]$. Since (anti) self-dual Maxwell fields have vanishing energy-momentum tensor, the Einstein equations are undisturbed and we have an automatic solution of the Einstein-Maxwell equations (see also ref. [24]). For the Taub-NUT metric, it is also easy to find the Maxwell field

$$\begin{aligned} A &= \frac{r - m}{r + m}\sigma_z \\ F &= \frac{1}{(r + m)^2}(e^0 \wedge e^3 - e^1 \wedge e^2) \end{aligned} \qquad (3.26)$$

where

$$e^a = \left[\left(\frac{r + m}{r - m}\right)^{\frac{1}{2}} dr,\ 2(r^2 - m^2)^{\frac{1}{2}}\sigma_x,\ 2(r^2 - m^2)^{\frac{1}{2}}\sigma_y,\ 2m\left(\frac{r - m}{r + m}\right)^{\frac{1}{2}}\sigma_z\right].$$

The Maxwell field for our metric (2.26) was presented earlier in Eq. (2.30). We note for comparison that the Taub-NUT Maxwell field has the characteristic $1/r^2$ behavior of a magnetic monopole, while (2.30) has the $1/r^4$ behavior of the Yang-Mills instanton. The $P_2(\mathbb{C})$ field (3.25), on the other hand, is constant everywhere.

Finally, we give a brief summary of the topological invariants of the fundamental triplet of metrics. $P_2(\mathbb{C})$ is the easiest, since it is a compact manifold without boundary. Because $P_2(\mathbb{C})$ has a bolt and a nut, its Euler characteristic is $\chi = 2 + 1 = 3$, while the signature is $\tau = 1$. If $P_2(\mathbb{C})$ were a spin manifold, the spin $\frac{1}{2}$ index would be

$$I_{\frac{1}{2}} = -\frac{\tau}{8} = -\frac{1}{8} . \qquad (3.27)$$

Since $I_{\frac{1}{2}}$ must be an integer for a manifold admitting well-defined spinor structure, we confirm the fact that $P_2(\mathbb{C})$ has no spin structure. For the Taub-NUT metric, $\chi = 1$ and the spin $\frac{1}{2}$ index with boundary corrections vanishes [15]. For the metric (2.26), the topological invariants were previously given in Eqs. (2.47-54). A tabulation of the properties of the fundamental triplet is presented in Table 2 alongside the properties of the K3 manifold mentioned in the Introduction.

As before, we need the periodicity (4.5) for a regular manifold, but now one finds

$$\left(\varepsilon = 0, \quad n = 1\right) = \text{flat space metric.} \qquad (4.11)$$

The boundary at ∞ for $n = 1$ is just the Euclidean space boundary S^3. For $n = 2$, one finds a two-nut metric with nonzero Riemann tensor. In this case the boundary at ∞ is the same lens space $S^3/Z_2 = P_3(\mathbb{R})$ found for our metric (2.26) and also the curvature invariants are the same; thus it seems certain that

$$\left(\varepsilon = 0, n = 2\right) = \text{the metric (2.26)} \qquad (4.12)$$

up to a singular change of frame. For higher values of n, the boundary at ∞ consists of S^3 with points identified under the action of the cyclic group of order n.

We conclude that in general the multicenter metric (4.1-4.4) with $\varepsilon = 0$ strongly parallels the Jackiw-Nohl-Rebbi multi-instanton solution [25]. In particular, there are $(n + 1)$ positions appearing in the description of the "n-instanton" solution.

The topological invariants for the $\varepsilon = 0$ metric are [21]

$$\begin{aligned} \chi &= n \\ \tau &= \pm (n - 1) \\ I_{1/2} &= 0 \\ I_{3/2} &= 2\tau = \pm 2(n - 1). \end{aligned} \qquad (4.13)$$

There is in fact a general theorem showing that the spin 1/2 index vanishes for asymptotically flat self-dual metrics [13]. We are led to conclude that for gravity, a spin 3/2 axial anomaly replaces the spin 1/2 axial anomaly induced by Yang-Mills instantons. Thus, the roles of gravitational and Yang-Mills instantons in symmetry breaking may be summarized as follows:

$$\begin{aligned} &\text{Yang-Mills solution, Chern class } k \to \text{ Dirac index } = k \\ &\text{Einstein solution, signature } \tau \to \text{ Rarita-Schwinger index } = 2\tau. \end{aligned} \qquad (4.14)$$

We conclude with the remark that we can write down natural self-dual Maxwell fields for the multicenter metrics just as we did for the metrics in previous Sections. One such field is

$$\begin{aligned} A &= V^{-1}(d\Psi + \vec{\omega} \cdot d\vec{x}) \\ F &= dA = V^{-2}\, \partial_i V(e^0 \wedge e^i \pm \tfrac{1}{2} \varepsilon_{ijk}\, e^j \wedge e^k). \end{aligned} \qquad (4.15)$$

C. More General Metrics

We have now seen the natural appearance of higher order lens spaces of S^3 in the multicenter self-dual Einstein metrics (4.1). Hitchin [5] has examined the known complete classification of spherical forms of S^3 and has found regular complex algebraic manifolds with boundaries corresponding to each spherical form. It is conjectured that a unique self-dual metric can be obtained for each of these manifolds using the Penrose twistor construction [27]. Although this subject is not completely understood at this time, let us at least list the

spherical forms of S^3 corresponding to each possible asymptotically locally Euclidean self-dual metric. The spherical forms are classified according to their associated discrete groups as follows [28]:

Series A_k : cyclic group of order k
(= lens spaces $L(k+1,1)$)

Series D_k : dihedral group of order k

T : tetrahedral group

O : octahedral group $\approx$ cubic group

I : icosahedral group $\approx$ dodecahedral group .

We note that our metric (2.26) corresponds to A_1, while the general n-center metric (4.1) corresponds to A_{n-1}.

If we could derive self-dual metrics for the manifolds having each of these spherical forms as boundaries, the problem of finding zero-action solutions of the Euclidean Einstein equations would be essentially solved. We would then have a better understanding of the structure of the vacuum in quantum gravity.

V. CONCLUSIONS

The discovery of the self-dual instanton solutions to Euclidean Yang-Mills theory suggested the possibility that analogous solutions to the Euclidean Einstein equations might be important in quantum gravity. Here we have discussed a number of self-dual solutions to Euclidean gravity and indicated their properties. We have concentrated particularly on the asymptotically locally Euclidean metrics, of which the authors' solution (2.26) is the simplest nontrivial example. These gravity solutions have properties which are strikingly similar to those of the Yang-Mills instanton solutions:

(1) They describe gravitational excitations which are localized in Euclidean spacetime.

(2) Their metrics approach an asymptotically locally Euclidean vacuum metric at infinity.

(3) They have nontrivial topological quantum numbers.

However, there are also some important distinctions between the two sets of solutions:

(1) The gravity solutions contribute only to the spin 3/2 axial anomaly, while the Yang-Mills solutions contribute to the spin 1/2 axial anomaly.

(2) The gravity solutions have zero action, while the Yang-Mills solutions have finite action.

The pairing of the Yang-Mills field with the spin 1/2 anomaly and the pairing of gravity with the spin 3/2 anomaly are very likely

due to the existence of supersymmetry. It would be interesting to see whether supersymmetry gives any further insight into the structure of these systems.

As is well-known, the finite Yang-Mills instanton action implies the suppression of the transition amplitude between topologically inequivalent sectors of the theory. On the other hand, in gravity there appears to be no such suppression. The vanishing action of asymptotically locally Euclidean self-dual metrics implies that in the path integral they have the same weight as the flat vacuum metric. Thus these solutions will presumably be of central importance in understanding quantum gravity.

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FIGURE CAPTIONS

Fig. 1: The manifold $T^*(P_1(\mathbb{C}))$ described by the metric (2.26). For fixed S^2 coordinates (θ, ϕ), the manifold has local topology $\mathbb{R}^2 \times S^2$. Constant radius hypersurfaces have the topology of $P_3(\mathbb{R})$. At ∞, the metric on the boundary is the canonical $P_3(\mathbb{R})$ metric. As $u \to 0$ [Eq. (2.36)] or equivalently, as $r \to a$ [Eq. (2.26)], the manifold shrinks to $S^2 \simeq P_1(\mathbb{C})$.

Fig. 2: Relations among the manifolds of our metric (2.26), the self-dual Taub-NUT metric (3.9) and the Fubini-Study metric (3.13) or (3.15) on $P_2(\mathbb{C})$.

TABLE I: Comparison of the Euclidean Yang-Mills and Einstein equations.

Property	Yang-Mills	Einstein
Metric	------	$ds^2 = \sum_{a=0}^{3} (e^a)^2$
Structure equation	------	$0 = de^a + \omega^a{}_b \wedge e^b$
Connection	$A = A^a_\mu \frac{\tau^a}{2i} dx^\mu$	$\omega^a{}_b = -\omega^b{}_a = \omega^a{}_{b\mu}\, dx^\mu.$
Curvature	$F = dA + A \wedge A$ $= \frac{1}{2} F^a_{\mu\nu} \frac{\tau^a}{2i} dx^\mu \wedge dx^\nu$	$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$ $= \frac{1}{2} R^a{}_{bcd}\, e^c \wedge e^d$
Dual Curvature or Connection:	$\tilde{F}^a_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} F^a_{\alpha\beta}$ $\tilde{F} = \frac{1}{2} \tilde{F}^a_{\mu\nu} \frac{\tau^a}{2i} dx^\mu \wedge dx^\nu$	$\tilde{R}^a{}_{bcd} = \frac{1}{2} \varepsilon_{abef} R^e{}_{fcd}$ $\begin{cases} \tilde{R}^a{}_b = \frac{1}{2} \tilde{R}^a{}_{bcd}\, e^c \wedge e^d \\ \tilde{\omega}^a{}_b = \frac{1}{2} \varepsilon_{abcd}\, \omega^c{}_d \end{cases}$
Bianchi identity	$dF + A \wedge F - F \wedge A = 0$	$dR + \omega \wedge R - R \wedge \omega = 0$
Cyclic identity	-----	$R^a{}_b \wedge e^b = 0$ $\rightarrow \varepsilon_{abcd} R^e{}_{bcd} = 0$
Euler equation	$d\tilde{F} + A \wedge \tilde{F} - \tilde{F} \wedge A = 0$	$\tilde{R}^a{}_b \wedge e^b = 0$ $\rightarrow R^a{}_{bcb} = 0$
Automatic solution	$F = \pm \tilde{F}$	$R^a{}_b = \pm \tilde{R}^a{}_b$
First order automatic solution	$F = \pm \tilde{F}$	$\omega^a{}_b = \pm \tilde{\omega}^a{}_b$
Basic function	$A^a{}_\mu(x)$	$e^a{}_\mu(x)$

Table II: Properties of the fundamental triplet of self-dual metrics compared with the self-dual metric of the K3 manifold.

	ref. [2]	Taub-NUT	Fubini-Study	K3
Metric	Eq. (2.26)	Eq. (3.9)	Eq. (3.13)	explicit form unknown
Cosmological Constant	$\Lambda = 0$	$\Lambda = 0$	$\Lambda \neq 0$	$\Lambda = 0$
Manifold	$T^*(P_1(\mathbb{C})$	$\mathbb{R}^4$	$P_2(\mathbb{C})$	K3
- Origin	S^2(Bolt)	Point (Nut)	Point(Nut)	--
- Infinity	$P_3(\mathbb{R})$	distorted S^3	S^2(Bolt)	--
- Boundary	$P_3(\mathbb{R})$	distorted S^3	none	none
Euler Characteristic	2	1	3	24
Hirzebruch Signature	-1	0	1	-16
Dirac Spin-$\frac{1}{2}$ Index	0	0	(no spinors)	2
Maxwell Field Strength	$\sim 1/r^4$	$\sim 1/r^2$	1	?
Rarita-Schwinger Spin 3/2 Index	-2	?	(no spinors)	-42

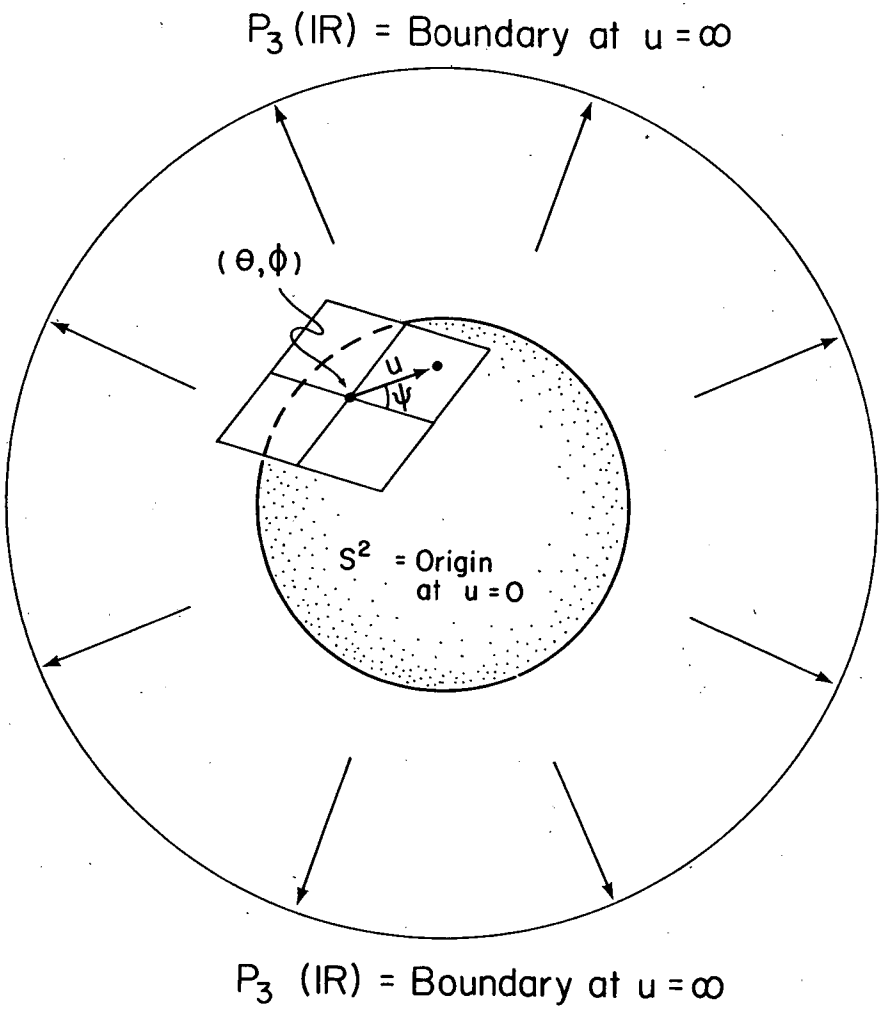

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Fig. 1

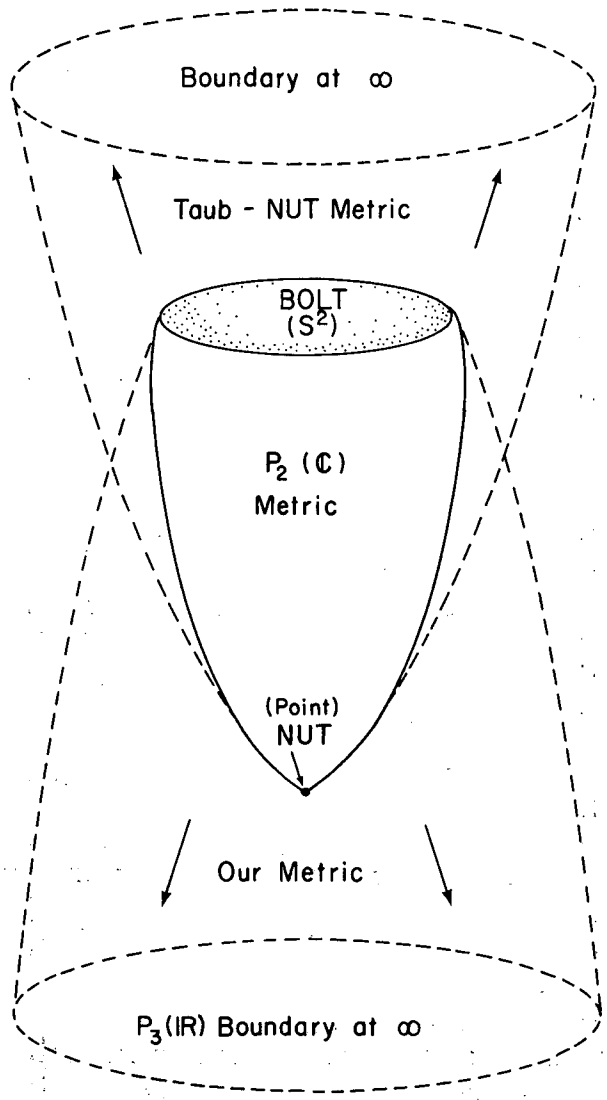

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Fig. 2