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FRACTIONAL FACTORIAL MOMENTS AND "INTERMITTENT" BEHAVIOR OF MULTIPLICITY DISTRIBUTIONS

by

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With the advent of very high energy nuclear colliders, such as RHIC at Brookhaven and LHC at CERN, renewed attempts will be made to search by means of large acceptance detectors [1] for the — as yet elusive — expected signals for transition to an unconfined (QGP) parton phase. One such signal would be large "local" fluctuations of the produced hadron multiplicity in restricted intervals ("windows") of space phase. This Letter will deal with ways to define in quantitative terms the notion of "large" fluctuations [2] and to present a new tool to be used to this effect.

Ever since it was first recognized [3] that the multiplicity distribution P(n) of secondaries recorded in finite rapidity windows — treated as "mini-events" — could provide more information about the collision dynamics than just the total multiplicity of the event, numerous investigations in a wide variety of reactions have shown, in essence, that:

a) the relative width of $P(n)$ — as measured by its normalized moments and/or cumulants — increases with decreasing width Δy of the interval considered, and

b) this same relative width decreases when the center of the window shifts away from central c.m.s. rapidity, i.e. towards the "fragmentation" region.

Lately, after the observation [4-6] of rare events with large multiplicity "spikes" in narrow rapidity windows, the interpretation of observation a), above, has centered around the idea [7] that the distribution of particles in phase space shows "intermittent" behavior. This would manifest itself through an increase (with decreasing Δy) of the normalized factorial moments F_q (i.e. of order q of $P(n)$:

$$F_q = \frac{f_q}{f_1^{\,q}} \tag{1}$$

where

$$f_q \equiv \sum_{n=0}^{\infty} P(n).n(n-1)(n-2)....(n-q+1) \qquad (2)$$

are the ordinary (*i.e.* un-normalized) factorial moments of order q. Most models based on the intermittency concept predict a power-law dependence of the F_q on Δy.

Irrespective of the model concepts underlying this approach, we suggest here that interesting information may additionally be gained by a detailed study of the dependence of the F_q on the order q of the factorial moment, at fixed Δy . The impulse for this investigation came from questions raised by the results of (fixed-target) experiments on "central" nucleus-nucleus collisions in the CERN SPS heavy ion beams [8] and their projections to experiments at RHIC [1]. The F_q observed in such reactions are — as a rule — small, even for very small Δy . In other words, the behavior of *P(n)* is very Poisson-like.

This quasi-Poisson shape of *P(n)* could arise in several different ways, of which three are of special interest to us:

i) The "true" (i.e. underlying) distribution is indeed Poisson — as would be expected, *e.g.* from a purely coherent source. The small deviations from a Poisson shape (i.e. of the F_q from unity, see below) could then be due to fluctuations of inelasticity and/or the imperfect stationarity of the rapidity distribution [9] over the y -range under consideration.

ii) Nucleus-Nucleus (A-A) collisions really result just in a (trivial) superposition of many independent nucleon-nucleon (n-n)collisions. Since the *P(n)* of the latter are known to be rather well described [10] by negative binomial (NB) distributions, the folding of an increasingly large number of identical NB's will quickly converge towards a shape almost indistinguishable from that of a Poisson distribution.

To drive this point home, we show in Fig.1,a the (hardly discernible) frequency distributions of a Poisson of mean m =2 and of an NB of the same mean m resulting from the superposition of many n-n collisions. Inserts b) and c) in Fig.1 show (in terms of differences and ratios) the tininess of the differences between these two distributions, which are not readily visible in Fig.1 a.

iii) Of real interest would be deviations caused by "real spiking" from the folding described above (which is at least to a certain extent unavoidable). It would be too much to expect that — once a certain energy and/or baryon density is attained — all collisions lead to the QGP phase transition. Rather one might expect the rare occurence of "spiked" events superimposed on a quasi-Poisson background. We proceed now to show one possible way to try to resolve these alternatives.

Factorial moments of order q are conventionally defined [11] through the probability generating function (PGF):

$$G(z) = \sum_{n=0}^{\infty} z^n P(n) \tag{3}$$

by taking its q -th derivatives at z =1:

$$f_q = D_z^q \ G(z) \ \Big|_{z=1} \equiv \frac{d^q}{dz^q} G(z) \ \Big|_{z=1} \tag{4}$$

It is immediately seen that this definition of the f_q coincides with the "intuitive" one [Eq. (2)] from which,incidentally, the name of factorial moments is derived [see below, Eq.(5)].

If one wishes to investigate in detail the dependence of F_q on q , the question arises how to interpolate between integer values of q .

The intuitive definition (2) for the f_q , re-written as:

$$f_q = \langle n(n-1)(n-2)...(n-q+1) \rangle = \left\langle \frac{n!}{(n-q)!} \right\rangle \tag{5}$$

lends itself in the easiest way to such an interpolation via the generalization of the factorials to gamma functions:

$$f_q = \sum_{n=0}^{\infty} P(n) \frac{\Gamma(n+1)}{\Gamma(n-q+1)} \tag{6}$$

The f_q can thus be defined over the whole range of real numbers.

By substituting in Eq.(6) estimates P_n^* for $P(n)$ derived from experimental data:

$$P_n^* \equiv \frac{w_n}{\sum_{i=0}^{\infty} w_i} \tag{7}$$

where w_n are the numbers of events observed yielding n particles in the window Δy, one may empirically obtain estimates f_q^* for f_q over the whole (real, >0) range of q. For integer values of q these will, of course, coincide with the conventional (i.e. integer order) factorial moments.

If one wishes, however, to obtain in closed form the q -dependence of the f_q of known distribution laws (and that of the F_q, as well) then the natural way is to extend the meaning of q to real numbers by applying differentiation of real order q in Eq.(4) , in other words by using the methods of fractional calculus [12-14]. Of the several available definitions of a fractional derivative, the one most useful for our purpose is that [12] which applies if the whole set of ordinary derivatives $G^{(r)}$ (*i.e.* of integer order r) of $G(z)$ are known . This (generalized) differentiation rule is:

$$D_z^q\ G(z) = \sum_{r=0}^{\infty} \frac{z^{r-q} G^{(r)}(0)}{\Gamma(r-q+1)} \tag{8}$$

Applying it to Eq.(4) we obtain:

$$f_q = \frac{G(0)}{\Gamma(1-q)} \sum_{r=0}^{\infty} T_r \tag{9}$$

where

$$T_0 = 1, \tag{10'}$$

and

$$\frac{T_{r+1}}{T_r} = \frac{1}{r-q+1} \frac{G^{(r+1)}(0)}{G^{(r)}(0)} \tag{10''}$$

Summation of the series appearing in Eq.(9) is eased if a convenient recursion rule is also available for the $G^{(r)}(0)$ [15].

In keeping with the nomenclature used in fractional calculus it seems convenient to refer to the f_q as defined via (9) [or (6), when estimated from experiment] and to their generalized normalized form F_q [Eq.(1)] as fractional factorial moments (for short, FFM's) .

We now apply Eqs. (8) and (9) to the case of two multiplicity distribution laws which are both highly relevant to the physical subject discussed here and also happen to have very convenient recursion rules for the $G^{(r)}$, namely:

i) the Poisson distribution of mean m :

$$P_P(n;m) \equiv e^{-m} \frac{m^n}{\Gamma(n+1)} \tag{11}$$

According to Eq.(3) this has the PGF:

$$G(z) = e^{m(z-1)} \qquad (12)$$

It is easily seen that (12) leads to the recursion rule

$$\frac{G^{(r+1)}(0)}{G^{(r)}(0)} = m \qquad (13)$$

Using this in Eq.(10) we find that Eq.(9) turns into a hypergeometric series and we obtain finally the normalized FFM:

$$F_q = \frac{e^{-m} \cdot {}_1F_1(1, 1-q, m)}{m^q \Gamma(1-q)} \qquad (14)$$

where ${}_1F_1(\alpha, \beta, z)$ is the degenerate confluent hypergeometric (or Kummer) function[16].

We show in Fig.2 a few examples of the way the F_q - curves depend on the mean m of the Poisson distribution. As expected, for integer values of q all F_q are equal to unity (this used to be regarded as the conventional "characteristic signature" of the Poisson distribution !) . However, between these "nodes", the F_q exhibit an oscillatory behavior with successive maxima and minima located near semi-integral values of q (thus corresponding to what are known in fractional calculus terminology [12] as semi-derivatives of $G(z)$).The amplitudes of these extrema are strongly dependent on m , as can be seen from the insert 4) in Fig.2

ii) The Negative Binomial (NB) or Planck-Pólya distribution (characterized by its mean m and — in its quantum-statistical interpretation — the number k of "phase-space-cells"):

$$P_{NB}(n; m, k) \equiv \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)} (1-V)^k V^n \qquad (15)$$

where

$$V \equiv \frac{m}{m+k} \qquad (16)$$

The distribution (15) has the PGF [cf. Eq.(3)]:

$$G(z)=\left(\frac{1-V}{1-Vz}\right)^{k} \tag{17}$$

Eq.(17) also leads to a simple recursion rule:

$$\frac{G^{(r+1)}(0)}{G^{(r)}(0)}=(r+1)V \tag{18}$$

When this is used in Eq.(10"), then Eq.(9) turns out to be another hypergeometric series and we obtain finally the normalized FFM:

$$F_q=\frac{\left(\frac{k}{m+k}\right)^{k}}{m^q\,\Gamma(1-q)}\cdot {}_2F_1\left(1,k,1-q,\frac{m}{m+k}\right) \tag{19}$$

where ${}_2F_1(\alpha,\beta,\gamma,t)$ is the generalized confluent hypergeometric function [17].

In Fig.3 we compare a few examples of FFM's computed for NB distributions having the same mean $m=2$ and increasingly large values of the number of "cells" k with the FFM of a Poisson distribution of the same mean m.

As already stated, even at the lowest window widths Δy investigated hitherto, n-n collisions appear [11,6] to be well described by NB distributions with k decreasing when Δy decreases; then the most simplistic model in which A-A interactions reduce to a superposition of, say, ν identical n-n collisions (each of mean m_1 and k_1 cells) will lead to a $P(n)$ which is again a NB distribution with parameters νm_1 and νk_1. One may thus imagine the different NB curves [a through d in Fig.3] as describing collisions with increasingly heavy target nuclei. It is evident from a comparison between Fig.3 and Fig.1 that FFM's allow a much better resolution than the frequencies $P(n)$. It is serendipitous that while the search for "spiky" or "intermittent" behavior of rapidity density fluctuations focuses on very small Δy, hence on phase space regions populated with very small <u>mean</u> multiplicities m, the "oscillation amplitude" of the FFM's and thus the sensitivity of their pattern to the details of the underlying multiplicity distributions happen to be optimal at very low values of m, too [see Fig.2, insert 4)].

The two preceding examples of FFM's applied to the study of multiplicity distributions illustrate the advantage of "mapping" the probabilities $P(n)$ onto the q-dependence of the FFM's (F_q). The real test of the usefulness of this mapping lies in its ability to distinguish deviations from a quasi-Poisson behavior (NB at k >>1) from the rare occurence of "spiky" events due to collisions in which the multiplicity distribution within a given Δy bin is "very wide", leading to unexpectedly frequent large values of n. In order to simulate such a situation we have computed the FFM's for a mixture:

$$P_{mix}(n;m,a,x,k) \equiv (1-a)P_P(n;m) + aP_{NB}(n;xm,k) \quad (20)$$

which assumes that the distribution of n is essentially Poisson, except for a (tiny ?) fraction a of events when it is of negative binomial shape with a mean x times larger than m and with a value of k chosen such as to maximize the relative width of the corresponding $P_{NB}(n)$ (as measured, *e.g.* by its second moment). This maximum width is reached for the NB at k =1, i.e. when $P(n)$ becomes the Bose-Einstein or geometric distribution, the limiting case of a purely chaotic quantum-statistical source (without squeezed quantum states). Incidentally, such a source would be a very good candidate for a hot (QGP ?) spot.

In Fig.4 we compare, again for the same mean multiplicity m =2, the frequencies $P(n)$ and the FFM curves for the following five distributions:

1: the negative binomial with k =200 (the same as in Fig.1);

2...4: three "spiky" mixtures [18] as described by Eq.(20) with parameters as indicated in the inset, and

5: the Poisson distribution of mean m .

For cases 2 through 4 the parameters x and k in Eq.(20) were chosen so as to yield not only the same mean m but also the same relative width of $P(n)$ as the NB (case 1) when measured by the second moment F_2 (in other words curves 1 through 4 were forced to intersect at q =2). Different values of x were tried, since it is a priori hard to exclude that the chaotic component could have a considerably higher (local) mean multiplicity (xm) than the underlying Poisson component (m).

It is seen in Fig.4 that the patterns reflecting different dynamics become much easier to disentangle after the $P(n)$ have been re-mapped onto $F_q \equiv F(q)$. Although curves 3 and 4 — which correspond to "opposite" combinations of x and k — intersect again at q =3 they can still be distinguished through the whole trend of the F_q-curves.

The upper half of Fig.4 shows that one may expect about one order of magnitude more "spikes" with n >12 (if m =2) from the mix described by Eq.(20) than from the NB (of the order of one event expected when statistics get close to 10^6 sampled windows). The lower half of Fig. 4 suggests that the different behavior of the bulk of the "spiky"events (not just the rare fluctuations with n >>1) is reflected in the shape of the F_q -curves.

One (quite intuitive) explanation for this increased sensitivity of the F_q-curves <u>when $q \neq$ integer</u> can be found in the fact that for small m most events will have very low multiplicity (n =0,1,2.3,....) . The contribution of such events to the factorial moments of integer order [Eqs. (2) and (5)] is strictly zero[19] and only the (sparsely populated) high-n tail matters. This loss of information is avoided when Fractional Factorial Moments are analyzed. The interplay of the two components of Eq.(20) is illustrated in Fig.5 .

The obvious practical question of the statistical sample size needed to achieve the resolution offered by the FFM curves will be addressed in a separate paper. Suffice to say that a preliminary Monte Carlo calculation has shown that, with the kind of sample sizes expected from the luminosities prevailing at future colliders such as *e.g.* RHIC [20] , the prospects for adequate resolving power look promising.

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Footnotes & References.

1: see *e.g.* , STAR Collaboration, J.Harris *et al.* , LBL Report 31050 , Berkeley (July 1991)

2: There is little, if any, agreement in the literature as to what to call these "large" fluctuations. "Non-statistical fluctuations" is a misnomer (what one really means are fluctuations much wider than those expected from naively assumed distribution laws (such as the "~ $\sqrt{n}$ " rule of thumb); while "statistical fluctuation" itself is a pleonasm.....

3: G.N.Fowler, E.M.Friedlander and R.M.Weiner, Phys.Lett., **104B**, 239 (1981)

4: JACEE Collaboration, T.H.Burnett *et al.*, Phys.Rev.Lett., **50**, 2062 (1983)

5: E.M.Friedlander, Proc. 1-st Conf. on Local Equilibrium in Strong Interaction Physics (LESIP I) Bad Honnef, (1984) p.105

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7: A.Bialas and R.Peschanski, Nucl.Phys., **B273**, 703, (1986)

8: EMU01 Collaboration, E.Stenlund *et al.*, ibid., **A498**, 541, (1989), M.Adamovic *et al.* Phys.Rev.Lett., **65**, 412 (1990)

9: E.M.Friedlander, Mod. Phys.Lett.

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11: M.Kendall and A.Stuart, The Advanced Theory of Statistics, MacMillan, N.Y. (1977), p.68

12: K.Oldham, The Fractional Calculus Academic Press, Orlando (1974), p.60

13: B.Ross, Fractional Calculus and its Applications, in: Lecture Notes in Mathematics, vol.457 (Ed.A.Dodd and B.Eckmann), Springer, Berlin (1975) p.1

14 J.Lavoie, T.Tremblay and T.Osler, ibid., p.323

15: Eq.(8), seen as a generalized differentiation rule, is true for any analytic function *G(z)* . For the special case, considered here, when *G(z)* is

identified as the PGF of a discrete distribution function $P(n)$, Eq.(3) implies by definition that $G^{(r)}(0)=r!\ P(r)$; thus a recursion rule for $P(n)$ can be used in Eq.(10"), as well, whenever it turns out to be more convenient.

16: M.Abramovitz and I.Stegun, Handbook of Mathematical Functions, Dover Publications, N.Y. (1972), p.556

17: ibid., p.503

18: In all of these the probability a for the distribution to belong to the "wide" NB species is 10^{-2} - 10^{-3}.

19: Incidentally this leads to weird consequences (such as discrete distributions of the f^*; see M.I.Dremin *et al.*, Correlations and Multiparticle Production (CAMP) (LESIP IV), Marburg (1990) p.304

20: With increasing beam energy, the width Y of the rapidity plateau increases; so does the total multiplicity N within this quasi-stationary range of y ($N \sim 10^3$); thus for, say, m =1...3 the number of comparable sampling windows becomes large, too, ($Y/\Delta y \geq 10^2$).This leads then to a significant increase in effective statistics.

The Poisson distribution $P_P(n;m)$ and the Negative Binomial distribution $P_{NB}(n;m,k)$ compared at the same value of m=2; for the NB, the ("cell") parameter k was chosen as k=200 (≈100 n-n collisions, each with k=2, superimposed).
a): Straightforward comparison of probabilities;
b) their difference; c) their ratio

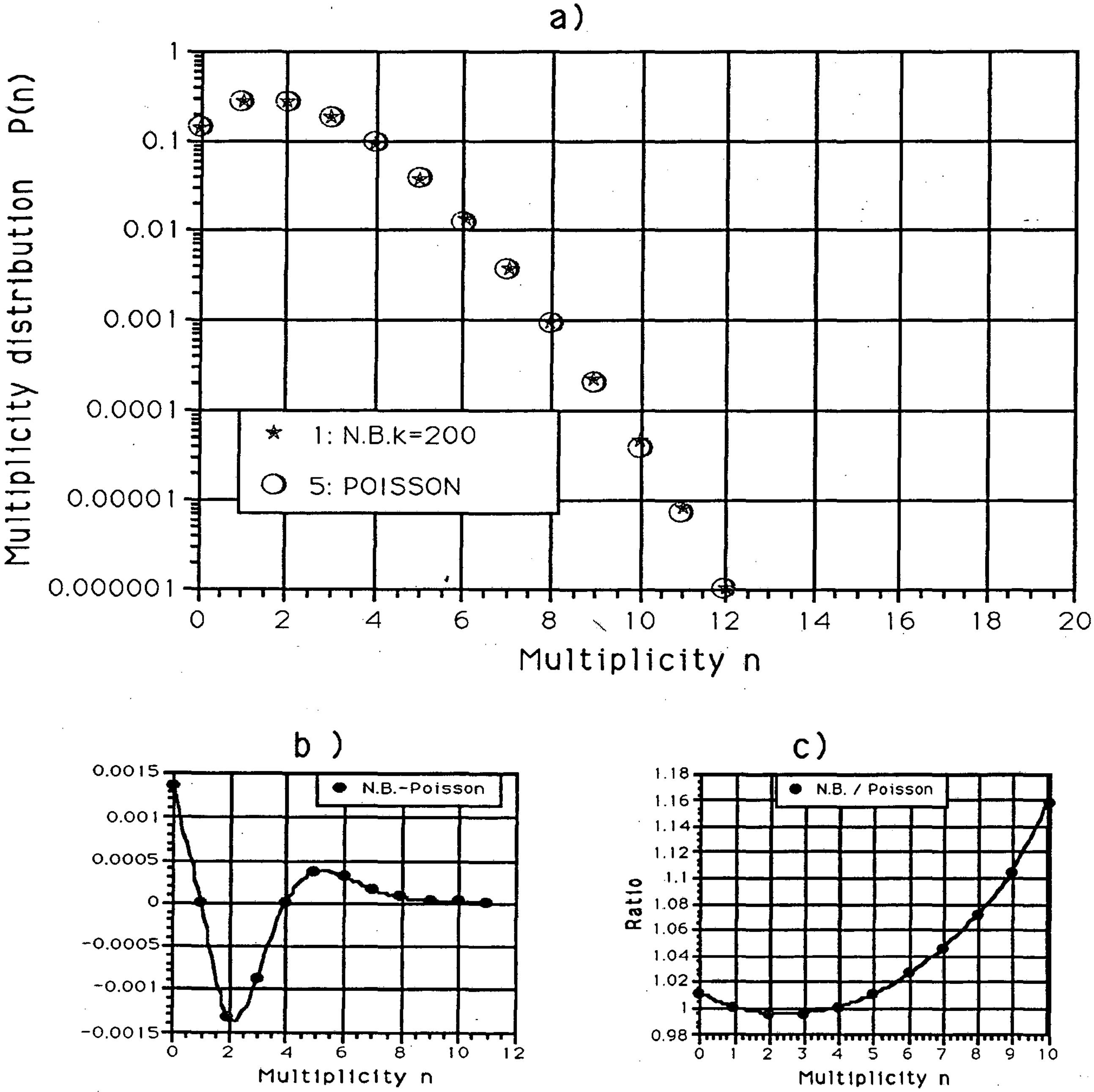
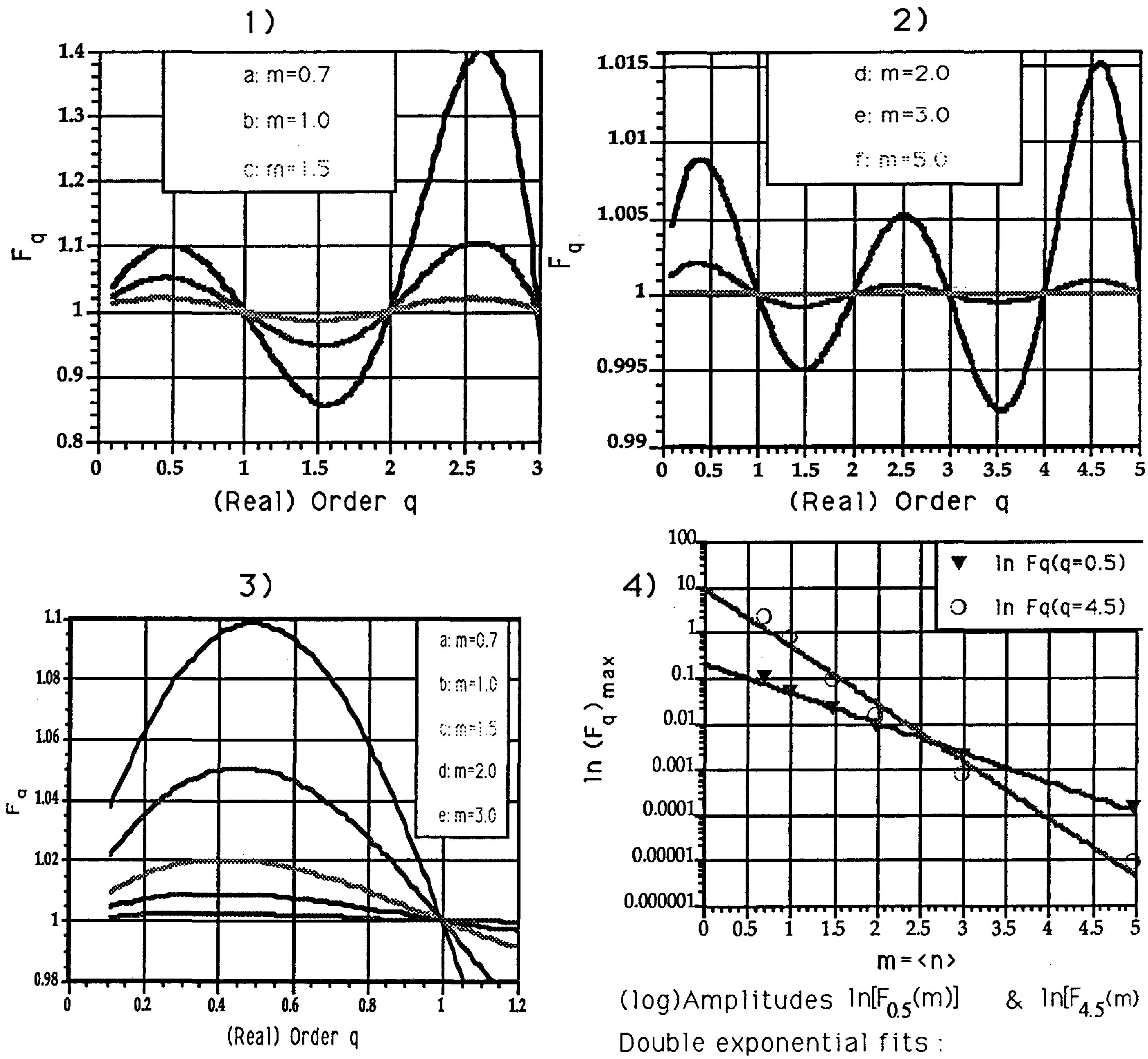


Fig.1

FFM's of Poisson distributions at different m=<n>



q=0.5: ln[F(m)] = 2.069749E-1 * exp(-1.505953E+0*m) R^2 = 9.935645E-1

q=4.5: ln[F(m)] = 9.162908E+0 * exp(-2.905903E+0*m) R^2 = 9.837342E-1

Fig.2

FFM's of Negative Binomials P(n;mk) with the **same** m=<n>=2.0, and **different** k [a ...d], compared to those of a Poisson distribution [e] with the same value of m.

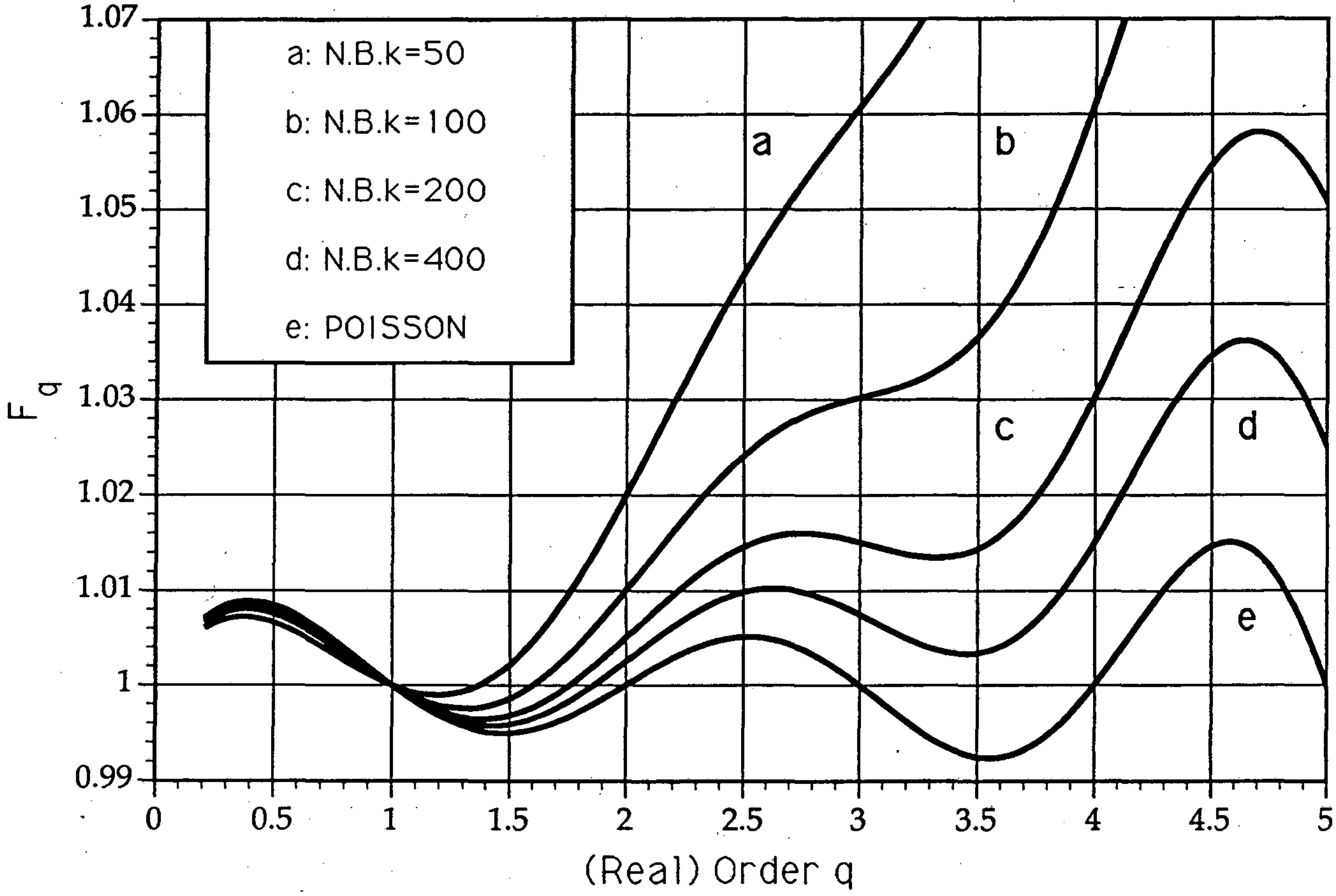


Fig.3

Comparison of Conventional Frequency [P(n)] Representation (a), and Fractional Factorial Moment [*i.e.* FFM] Analysis (b), for five distributions with the same mean m=<n>=2

1: Negative Binomial [NB(m,k)]

2...4 : Mixtures : (1-a) Poisson(m)+a.NB(xm,k)

5: Poisson(m)

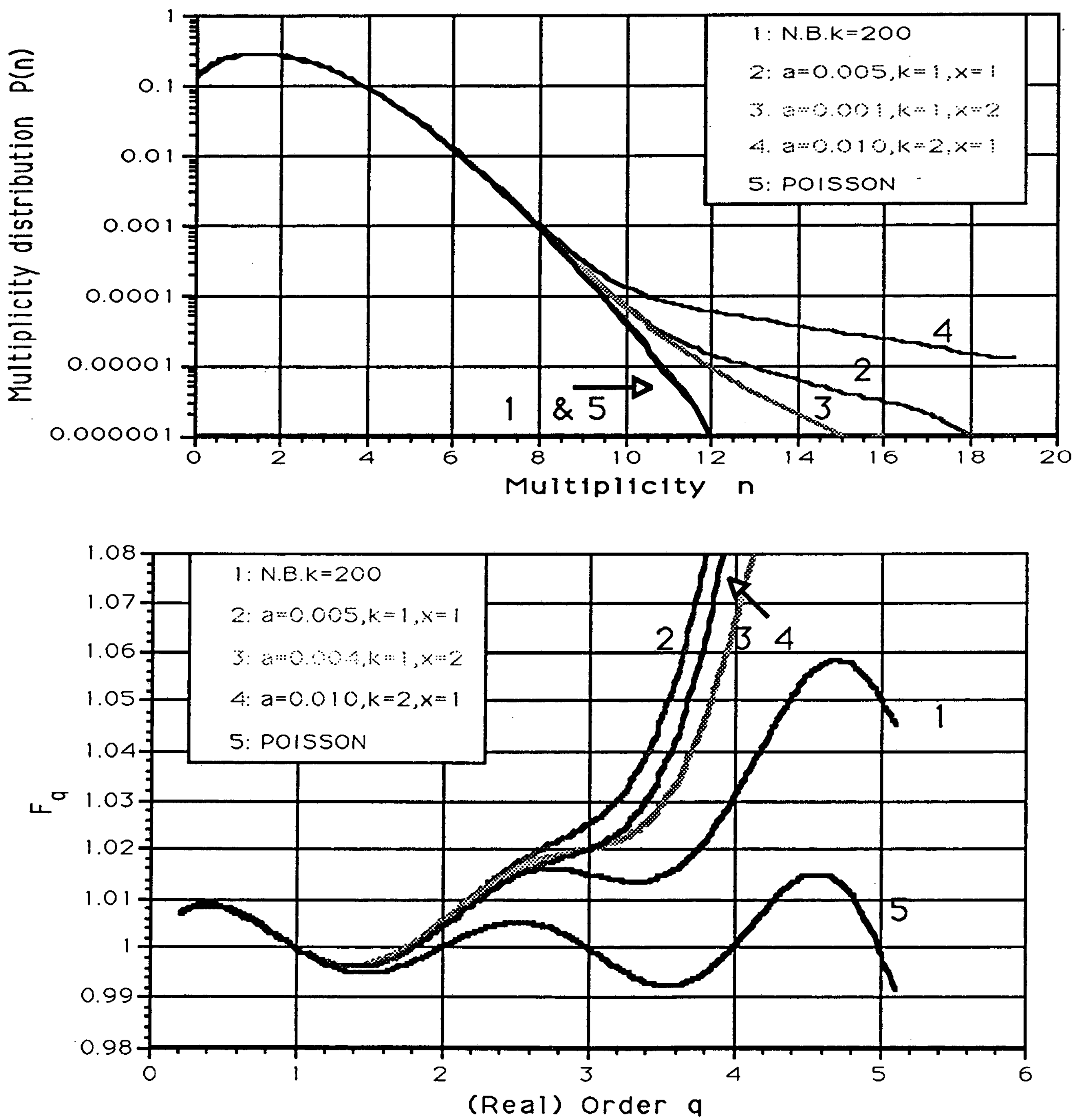
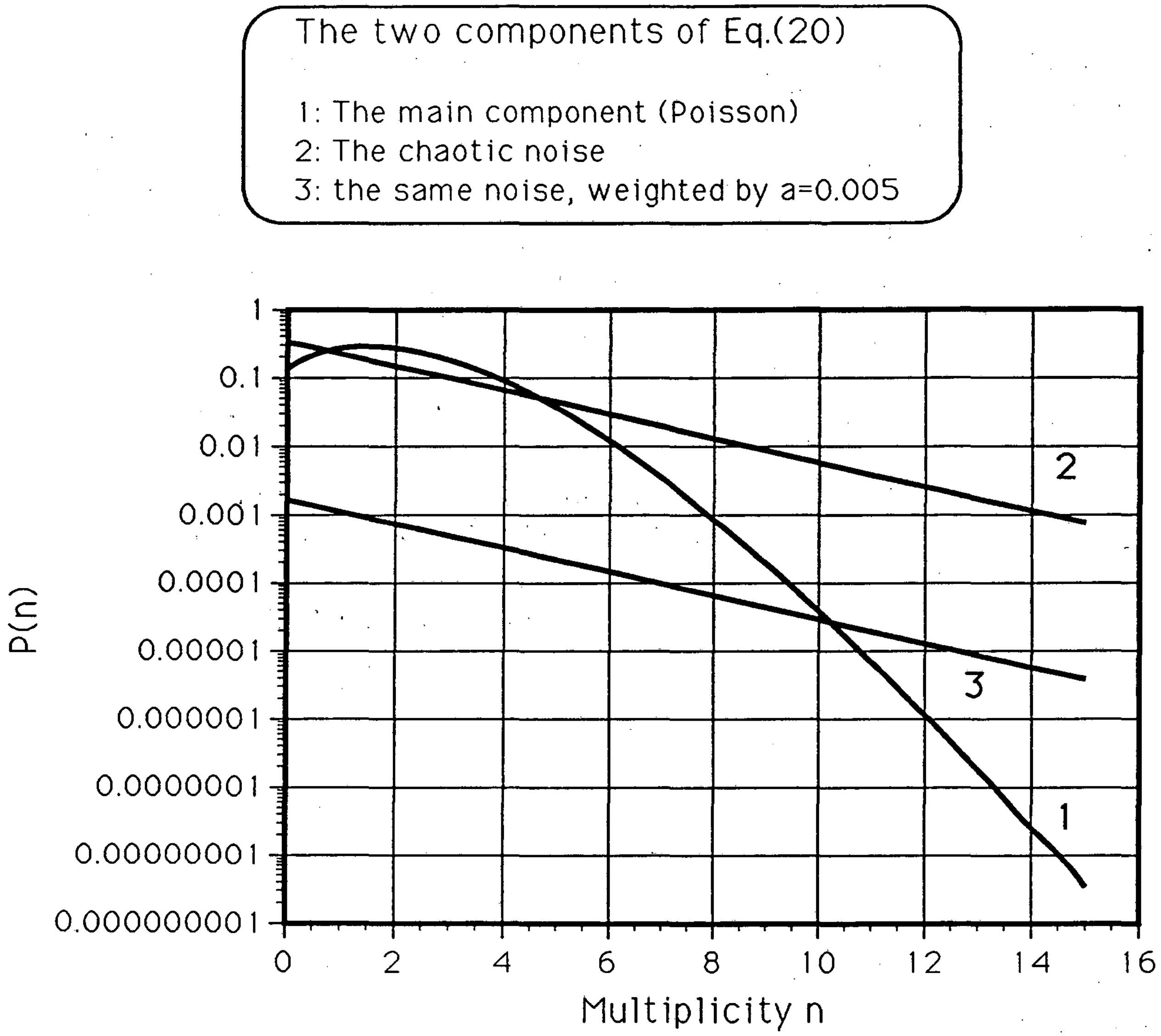


Fig.4



The two components of Eq.(20)
1: The main component (Poisson)
2: The chaotic noise
3: the same noise, weighted by a=0.005
P(n)
1
0.1
0.01
0.001
0.0001
0.00001
0.000001
0.0000001
0.00000001
0.000000001
0
2
4
6
8
10
12
14
16
Multiplicity n
1
2
3

Fig.5

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