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The 3-dimensional steady gradient Ricci soliton

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The 3-dimensional Steady Gradient Ricci Soliton

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by

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2008

The dissertation of Hongxin Guo is approved, and it is acceptable in quality and form for publication on microfilm:

Chair

University of California, San Diego

2008

DEDICATION

To

Armstrong, the senior and the junior.

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VITA

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ABSTRACT OF THE DISSERTATION

The 3-dimensional Steady Gradient Ricci Soliton

by

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Doctor of Philosophy in Mathematics

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In the first three chapters, we study the steady gradient soliton, especially the 3-dimensional soliton with positive sectional curvatures and which is κ-noncollapsed on all scales. In Chapter 1, we discuss the background of our project, and introduce the basic definitions. In Chapter 2, we get some geometric properties of the soliton. The asymptotic behaviors are studied. In Chapter 3, we introduce another approach looking at the difference between the two principle curvatures, and prove the soliton is rotationally symmetric out of a compact set under an additional assumption.

In Chapter 4, we generalize Perelman's $\mathcal{L}$-length to high dimensions. We define a new energy in a natural way, and derive the first variation of the energy. When the dimension is reduced to one, our first variation formula is exactly Perelman's $\mathcal{L}$-geodesic equation.

In Chapter 5 we study the mean curvature flow inside the Ricci flow. The interesting result is that in the evolution equation of the second fundamental form, many bad terms are canceled mysteriously due to the evolution of the ambient manifold.

Chapter 1

Introduction

In 2002 and 2003 Grisha Perelman posted three preprints [18], [19] and [20] showing how to use the Ricci flow, introduced and studied by Richard Hamilton, to prove Thurston's Geometrization conjecture, and consequently the famous Poincaré conjecture, which is one of the Clay Mathematics Institute's seven Millennium Prize Problems.

In 1982 in his seminal paper [9] Hamilton introduced his Ricci flow:

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Rc} \tag{1.1}$$

which is now central to our understanding of the geometry and topology of manifolds. The Ricci flow equation has much in common with the heat equation. It evolves an initial metric into ever nicer ones.

The Ricci solitons, which are the self-similar solutions evolving only by scalings and diffeomorphisms, have been extensively studied. These special solutions motivate the general analysis of the Ricci flow through monotonicity formulas and their applications. In general, the Ricci soliton is defined as:

Definition 1. *A solution $g(t)$ of the Ricci flow on $\mathcal{M}^n$ is a **Ricci soliton** if there exist a positive function $\sigma(t)$ and a 1-parameter family of diffeomorphisms $\varphi(t)$:*

$\mathcal{M} \to \mathcal{M}$ such that

$$g(t) = \sigma(t)\varphi(t)^* g(0).$$

*We say that $g(t)$ is **expanding**, **steady**, or **shrinking** at a time t_0 if $\dot{\sigma}(t_0)$ is $> 0, = 0,$ or < 0 respectively.*

In the Ricci flow, the soliton structure can actually be characterized at the initial time $t = 0$ by

$$2R_{ab} + \nabla_a X_b + \nabla_b X_a + \epsilon g_{ab} = 0,$$

where $X(p) = \frac{d}{dt}|_{t=0}(\varphi(t)(p))$, and $0 \le a, b \le n-1$. Note in the above equation, all the quantities are calculated at time 0. The soliton is expanding, steady or shrinking when $\epsilon > 0, = 0$, or < 0 respectively.

If there is a function f (called the **potential function**) generating the vector field X, that is to say $X = \nabla f$, we call $\mathcal{M}$ a **gradient soliton**. In this case, the soliton equation becomes:

$$R_{ab} + \nabla_a \nabla_b f + \frac{\epsilon}{2} g_{ab} = 0. \tag{1.2}$$

Beginning with a complete gradient Ricci soliton structure, we can construct a solution to the Ricci flow, which is a gradient Ricci soliton in canonical form for the associated time-dependent version. The following is standard and one can find a detailed discussion in [6].

Proposition 2. *If $(\mathcal{M}^n, g_0, f_0, \epsilon)$ is a complete gradient Ricci soliton, then there exist a solution $g(t)$ of the Ricci flow with $g(0) = g_0$, diffeomorphisms $\varphi(t)$ with $\varphi(0) = \mathrm{id}_{\mathcal{M}}$, function $f(t)$ with $f(0) = f_0$ defined for all t with $\tau(t) \doteqdot \epsilon t + 1 > 0$, such that the following hold.*

1. *$\varphi(t) : \mathcal{M} \to \mathcal{M}$ is the 1-parameter family of diffeomorphisms generated by $X(t) \doteqdot \frac{1}{\tau}\nabla^0 f_0$; that is*

$$\frac{\partial}{\partial t}\varphi(t)(x) = \frac{1}{\tau}\nabla^0 f_0(\varphi(t)(x)),$$

where we use ∇^0 *to denote the covariant derivative with respect to the metric* g_0.

2. $g(t)$ *is the pull-back by* $\varphi(t)$ *of* g_0 *up to the scale factor* τ,

$$g(t) = \tau\varphi(t)^* g_0,$$

3. $f(t)$ *is the pull-back by* $\varphi(t)$ *of* f_0 :

$$f(t) = \varphi(t)^* f_0.$$

Moreover,

$$\mathrm{Rc}(t) + \nabla^t\nabla^t f(t) + \frac{\epsilon}{2\tau} g(t) = 0, \tag{1.3}$$

where ∇^t *is the covariant derivative with respect to the metric* $g(t)$. *And*

$$\frac{\partial f}{\partial t}(t) = |\nabla^t f(t)|^2_{g(t)}. \tag{1.4}$$

In the present work, we mainly focus on the gradient steady soliton, in which $\epsilon = 0$:

$$R_{ab} + \nabla_a\nabla_b f = 0 \tag{1.5}$$

especially on the case when the dimension is 3 and the sectional curvatures are positive.

We introduce an important definition:

Definition 3 (κ-noncollapsed)**.** *Given* $\rho \in (0, \infty]$ *and* $\kappa > 0$, *we say that a Riemannian manifold* $(\mathcal{M}^n, g)$ *is* κ***-noncollapsed below the scale*** ρ *if for any metric ball* $B(x, r)$ *with* $r < \rho$ *satisfying* $|\mathrm{Rm}| \le r^{-2}$ *for all* $y \in B(x, r)$, *we have*

$$\frac{\mathrm{Vol}(B(x,r))}{r^n} \ge \kappa.$$

We say $(\mathcal{M}, g)$ *is* κ***-noncollapsed on all scales*** *if it is* κ*-noncollapsed below the scale* ρ *for all* $\rho < \infty$.

There is the following claim of Perelman (see Remark 11.9 on page 32 of his remarkable paper [18]).

Claim 4. *I believe that there is only one (up to scaling) noncompact* 3*-dimensional* κ*-noncollapsed ancient solution with bounded positive curvature – the rotationally symmetric gradient steady soliton, studied by R. Bryant.*

While we are still unable to confirm Perelman's claim, we get some geometric properties of the steady soliton, and under one additional assumption we get the soliton is rotationally symmetric outside of a compact set.

Chapter 2

The scalar curvature R and the potential function f

2.1 Fundamentals on a steady soliton

In this section we discuss the basic formulas, which are known but for completeness we present our proofs here.

Suppose that $(\mathcal{M}^n, g, f)$ satisfies Equation (1.5). From this section,unless stated otherwise, all the calculations are understood to be on a gradient steady soliton for a fixed time. Throughout the thesis we use Einstein's convention that repeated indices means taking summation with respect to the metric.

Proposition 5. *We have the following formulas:*

1. $R + \Delta f = 0$,

2. $\nabla R = 2\,\mathrm{Rc}\,(\nabla f)\,$,

3. $R + |\nabla f|^2 = \text{constant}$,

4. $\nabla\nabla R = 2\nabla_{\nabla f}\,\mathrm{Rc} - 2\,\mathrm{Rc}^2 - 2\,\mathrm{Rm}(\cdot, \nabla f, \nabla f, \cdot)$.

Proof. 1. Simply taking trace of Equation (1.5) with respect to the metric we get this formula.

2. Taking once covariant derivative of Equation (1.5) and contract the indices we get

$$\nabla_a R_{ab} + \nabla_a \nabla_a \nabla_b f = 0.$$

By the contracted second Bianchi identity we have $\nabla_a R_{ab} = \frac{1}{2}\nabla_b R$, and by the Bochner formula we get

$$\nabla_a \nabla_a \nabla_b f = \nabla_b \Delta f + R_{ab} \nabla_b f = -\frac{1}{2}\nabla_a R + R_{ab}\nabla_b f.$$

Plug in we get $-\frac{1}{2}\nabla_a R + R_{ab}\nabla_b f = 0$, or $\nabla R = 2\,\mathrm{Rc}(\nabla f)$ as a coordinate-free version.

3. Since

$$\begin{aligned}\nabla(R + |\nabla f|^2) &= \nabla R + 2\langle \nabla\nabla f, \nabla f\rangle \\ &= 2\,\mathrm{Rc}(\nabla f) + 2\nabla\nabla f(\nabla f) \\ &= 0,\end{aligned}$$

we get $R + |\nabla f|^2 = \text{constant on } \mathcal{M}$.

4. Taking once more covariant derivative of the second formula, and by the Bochner formulas we have

$$\begin{aligned}\nabla\nabla R &= 2(\nabla\,\mathrm{Rc})(\nabla f) + 2\,\mathrm{Rc}(\nabla\nabla f) \\ &= 2\nabla_{\nabla f}\,\mathrm{Rc} - 2\,\mathrm{Rm}(\cdot, \nabla f, \nabla f, \cdot) - 2\,\mathrm{Rc}^2.\end{aligned}$$

□

2.2 Characterizing R and f sharing a same level surface

In this section, we discuss the case when the scalar curvature and the potential function share a same level surface. At a point x where $\nabla f \neq 0$, we let $\nu \doteqdot -\frac{\nabla f}{|\nabla f|}$. Then ν is an outward normal vector of the level surface Σ which contains x. The level surface is defined as follows:

$$\Sigma(\sigma) \doteqdot \{x | f(x) = \sigma\}.$$

The second fundamental form h of Σ is defined by

$$h(X, Y) \doteqdot \langle \nabla_X \nu, Y \rangle,$$

for any $X, Y \in T(\Sigma)$. h_{ij} is understood to be $h(e_i, e_j)$, and the mean curvature $H \doteqdot h_{ii}$.

We consider Σ which contains no critical point of f. Plug in the formula for ν we get

$$h = -\frac{\nabla\nabla f}{|\nabla f|} = \frac{\mathrm{Rc}}{|\nabla f|}.$$

Below is a formula which we shall use for a few times in this note. To the best of the author's acknowledge it hasn't shown up in any known literature. The advantage of this formula is that it shows a connection with the geodesic equation.

Lemma 6. *Assume $\mathcal{M}^n$ is a gradient steady Ricci soliton. For any vector $Y \in T_x\mathcal{M}$ we have*

$$\mathrm{Rc}(\nu, \nabla_Y \nu) = \mathrm{Rc}(Y, \nabla_\nu \nu). \tag{2.1}$$

Proof. Since both sides are linear in terms of Y, we only need to verify the formula for an orthonormal basis $\{\nu, e_i\}$. Since $\nabla_{e_i} \nu = h_{ij} e_j$, we have

$$\mathrm{Rc}(\nu, \nabla_{e_i} \nu) = h_{ij} R_{0j},$$

where $R_{0j} \doteqdot \mathrm{Rc}(\nu, e_j)$.

On the other hand we have

$$\mathrm{Rc}(e_i, \nabla_\nu \nu) = \mathrm{Rc}(e_i, \langle \nabla_\nu \nu, e_j \rangle e_j) = R_{ij} \langle \nabla_\nu \nu, e_j \rangle.$$

Furthermore, plugging in the formula for ν we get

$$\begin{aligned} \nabla_\nu \nu &= \frac{1}{|\nabla f|} \nabla_{\nabla f} \left(\frac{\nabla f}{|\nabla f|}\right) \\ &= \frac{1}{|\nabla f|^2} \nabla\nabla f(\nabla f) + \frac{1}{|\nabla f|} \nabla f\left(\frac{1}{|\nabla f|}\right) \nabla f. \end{aligned}$$

Since $\nabla f \perp e_i$ and $\nu = -\frac{\nabla f}{|\nabla f|}$ we have

$$\langle \nabla_\nu \nu, e_j \rangle = \frac{1}{|\nabla f|^2} \nabla\nabla f(\nabla f, e_j) = -\frac{1}{|\nabla f|} \nabla\nabla f(\nu, e_j) = \frac{1}{|\nabla f|} R_{0j}.$$

Finally we get

$$\mathrm{Rc}(e_i, \nabla_\nu \nu) = \frac{1}{|\nabla f|} R_{ij} R_{0j} = h_{ij} R_{0j} = \mathrm{Rc}(\nu, \nabla_{e_i} \nu),$$

and this completes the proof. □

Using this formula, we characterize the scalar curvature and the potential function sharing a same level surface:

Proposition 7. *Suppose Σ is a level surface of f. If Σ is also a level surface of the scalar curvature R, then we have $\nabla_\nu \nu(x) = 0$ for any $x \in \Sigma$.*

Proof. For any $X \in T(\Sigma)$ we have $0 = X(R) = 2\,\mathrm{Rc}(\nabla f, X) = -2|\nabla f|\,\mathrm{Rc}(\nu, X)$, since R is constant on Σ. Furthermore

$$\begin{aligned} \langle \nabla_\nu \nu, X \rangle &= -\langle \nabla_\nu \left(\frac{\nabla f}{|\nabla f|}\right), X \rangle \\ &= -\frac{1}{|\nabla f|} \langle \nabla_\nu \nabla f, X \rangle - \nu\left(\frac{1}{|\nabla f|}\right) \langle \nabla f, X \rangle \\ &= \frac{1}{|\nabla f|} \mathrm{Rc}(\nu, X) - 0 \\ &= 0 \end{aligned}$$

Moreover it's obvious that $\langle \nabla_\nu \nu, \nu \rangle = 0$. Hence $\nabla_\nu \nu = 0$ since its inner product with any vector is 0. □

For the opposite direction of Proposition (7) we have the following:

Proposition 8. *Suppose Σ is a level surface of f and on Σ, $\nabla_\nu \nu = 0$. Furthermore, assume the second fundamental form h is nondegenerate on Σ. Then Σ is a level surface of R as well.*

Proof. It will be sufficient to show that for any X tangent to Σ, $X(R) = 0$.

By our assumption on h, there is an $Y \in T(\Sigma)$ such that $Y = h^{-1}(X)$. Since for any $Z \in T(\Sigma)$,

$$\langle \nabla_Y \nu, Z \rangle = h(Y, Z) = h(h^{-1}(X), Z) = \langle X, Z \rangle,$$

we know that $X = \nabla_Y \nu$. Now we have

$$X(R) = \langle \nabla R, X \rangle = 2 \operatorname{Rc}(\nabla f, X) = -2|\nabla f| \operatorname{Rc}(\nu, \nabla_Y \nu) = -2|\nabla f| \operatorname{Rc}(Y, \nabla_\nu \nu).$$

In the last equality of the above line we used Equation (2.1). Since on Σ, $\nabla_\nu \nu$ is always 0, we are done. □

2.3 R vanishes at spatial infinity

In this section, we focus on a 3-dimensional steady soliton $\mathcal{M}^3$ with bounded positive sectional curvature $0 < \operatorname{sect}(g) \leq C$. We also suppose $\mathcal{M}^3$ is κ-noncollapsed on all scales. The κ-noncollapsing assumption guarantees the injectivity radius is bounded away from 0 for any time. Without loss of generality we assume $R_{\sup} = 1$ at time $t = 0$.

We have the following:

Theorem 9 (Scalar curvature tends to zero at spatial infinity)**.** $\lim_{x \to \infty} R(x) = 0$.

Proof. If the claim is false, then there exists $x_i \to \infty$ such that $R(x_i) \geq c > 0$. By the injectivity radius estimate for complete noncompact manifolds with bounded positive sectional curvature and the compactness theorem, there exists a subsequence such that $(\mathcal{M}^3, g(t), x_i)$ converges to a complete nonflat eternal solution of the Ricci flow $(\mathcal{M}^3_\infty, g_\infty(t), x_\infty)$, $t \in (-\infty, \infty)$, with bounded nonnegative sectional curvature and which is κ-noncollapsed on all scales. Applying the 'finite number of curvature bumps' theorem to the sequence of closed balls $\bar{B}_{g(0)}(x_i, 1)$ in $(\mathcal{M}^3, g(0))$ (see §21 of [11]), we have that for a subsequence,

$$\min_{x \in \bar{B}_{g(0)}(x_i,1)} \operatorname{sect}(g(0)) \to 0,$$

and hence $(\mathcal{M}^3_\infty, g_\infty(0))$ has a zero sectional curvature somewhere in the ball $\bar{B}_{g_\infty(0)}(x_\infty, 1)$. By the strong maximum principle and since we are in dimension 3, the universal covering solution $\left(\widetilde{\mathcal{M}}^3_\infty, \tilde{g}_\infty(t)\right)$, $t \in (-\infty, 0]$, splits as the product of a simply-connected ancient κ-solution $(\mathcal{N}^2_\infty, h_\infty(t))$ with $\mathbb{R}$. By Chen and Zhu's [3] uniqueness result, $\left(\widetilde{\mathcal{M}}^3_\infty, \tilde{g}_\infty(t)\right) = (\mathcal{N}^2_\infty, h_\infty(t)) \times \mathbb{R}$ for all $t \in (-\infty, \infty)$, in particular, $h_\infty(t)$ extends forward in time to an eternal solution. This contradicts Hamilton's result that $h_\infty(t)$ is isometric to a shrinking round 2-sphere. $\square$

Corollary 10. *There is a unique point O where both the scalar curvature R and the potential function f attain their maximums. Moreover,*

$$R(O) = 1, \quad \text{and} \quad \lim_{x\to\infty} |\nabla f| = 1.$$

Proof. By $R > 0$ and $R(x) \to 0$ as $x \to \infty$, we get R must attain its maximum at some point O. At O we have $\nabla R(O) = 0$, then we have $\mathrm{Rc}(\nabla f, \nabla f) = \frac{1}{2}\langle \nabla R, \nabla f\rangle = 0$. By the strictly positivity of Rc, we see $\nabla f(O) = 0$. Combining with $\nabla\nabla f < 0$ we see O is a maximum point of f as well. Also since $\nabla\nabla f$ is strictly negative, we see O is unique.

By the assumption at the beginning of this section, $R_{\sup} = 1$ we have $R(O) = 1$. Since $R + |\nabla f|^2 =$ constant, and evaluate the constant at O we get $R + |\nabla f|^2 = 1$. Let $x \to \infty$, we get $\lim_{x\to\infty} |\nabla f| = 1$. $\square$

Remark 11. *Since $\nabla\nabla f < 0$ everywhere and $\nabla f(O) = 0$, we know $\mathcal{M}^3$ is diffeomorphic to $\mathbb{R}^3$ by Morse theory.*

2.4 $-f$ grows linearly

From section (2.3), we know there is a unique point of maximum O for both R and f. We call O as origin. Without loss of generality, we assume $f(O) = 0$. For any $x \in \mathcal{M}^3$, let $r(x) \doteqdot \text{dist}(O, x)$, and $\gamma(s)$ denote the shortest geodesic from O to x, where s is the arclength. An easy observation is that $\gamma(0) = O$, and $\gamma(r(x)) = x$.

Use the same notation as in section (2.2), $\Sigma(\sigma)$ and ν are the level surface and the outward unit normal vector, respectively. Let $\beta(\sigma)$ denote the integral curve of $\frac{\nabla f}{|\nabla f|^2}(= -\frac{\nu}{|\nabla f|})$. We observe that $\frac{d}{d\sigma} f(\beta(\sigma)) = \langle \nabla f, \dot{\beta} \rangle = 1$. It's not hard to see that as $x \to \infty$, $\sigma \to -\infty$. When σ increases, β goes toward the origin and $\beta(0) = O$. Note we can parameterize the curves by their arclength s or level value σ or distance function r accordingly.

Assume $x = \gamma(r) = \beta(\sigma_0)$. We can estimate $f(x)$ along different curves. Along the minimal geodesic γ, since

$$f(x) = f(x) - f(O) = \int_0^{\sigma_0} \frac{d}{d\sigma} f(\gamma(\sigma)) d\sigma = \int_0^r \frac{d}{ds} f(\gamma(s)) ds,$$

and $|\frac{d}{ds} f(\gamma(s))| \leq |\nabla f| < 1$, we get

$$f(x) > -r.$$

Now we estimate $f(x)$ along the integral curve β. In this situation we can only get good estimate when the point x is faraway from the origin. The reason is because $|\nabla f| \to 1$, we can compare $f(x)$ and $r(x)$ as $x \to \infty$. Precisely, given any positive ϵ, there is a $\bar{\sigma}$ such that when $\sigma \geq \bar{\sigma}$, $|\nabla f(\beta(\sigma))| > (1 + \epsilon)^{-1}$. Let $\bar{x} \doteqdot \beta(\bar{\sigma})$ and

$\bar{r} \doteqdot \mathrm{dist}(O, \bar{x})$, then in β, the length of the portion from $\bar{x}$ to x can be estimated as

$$\begin{aligned}\int_{\bar{\sigma}}^{\sigma_0} |\dot{\beta}(\sigma) d\sigma| &= \int_{\sigma_0}^{\bar{\sigma}} \frac{1}{|\nabla f|} d\sigma \\ &< (1+\epsilon)(\bar{\sigma} - \sigma_0) \\ &= (1+\epsilon)(f(\bar{x}) - f(x))\end{aligned}$$

On the other hand,

$$\int_{\sigma_0}^{\bar{\sigma}} |\dot{\beta}(\sigma)| d\sigma \geq \mathrm{dist}(\bar{x}, x) = r - \bar{r}.$$

So we have

$$(1+\epsilon)(f(\bar{x}) - f(x)) > r - \bar{r},$$

or

$$f(x) < f(\bar{x}) - (1+\epsilon)^{-1}(r - \bar{r}).$$

When r is big enough both $f(\bar{x})$ and $\bar{r}$ are relatively small comparing to r, so they can be absorbed by $r(x)$ by slightly changing the coefficient. We can rewrite the inequality in the following way:

$$f(x) < -(1 - \frac{\epsilon}{2})r.$$

Briefly, we have derived the following lemma:

Lemma 12. *For any $\epsilon > 0$, there is r_ϵ such that when $r(x) \geq r_\epsilon$ we have*

$$1 - \epsilon < \frac{-f(x)}{r} < 1. \tag{2.2}$$

Now we consider the difference between the geodesic and the integral curve. Let $\theta(x)$ be the angle between $\dot{\gamma}(x)$ and $\nu(x)$, we have:

Corollary 13. $\theta(x) \to 0$ *as* $x \to \infty$.

Proof. Since $-f$ is geodesically convex we see

$$\frac{d}{ds}(-f(\gamma(s))) \geq \frac{-f(x)+f(O)}{r} = \frac{-f(x)}{r}.$$

On the other hand,

$$\frac{d}{ds}(-f(\gamma(s))) = \langle \dot{\gamma}, -\nabla f \rangle = \frac{1}{|\nabla f|}\langle \dot{\gamma}, \nu \rangle = \frac{1}{|\nabla f|}\cos\theta.$$

So we have $\cos\theta \geq |\nabla f| \cdot \frac{-f(x)}{r}$. Let $x \to \infty$, by Lemma 12 we get

$$\lim_{x\to\infty} \cos\theta(x) = 1,$$

and this completes the proof. □

On any long stable geodesic, it is well known that $\int_\gamma \mathrm{Rc}(\dot{\gamma},\dot{\gamma})ds$ is bounded. On the steady soliton, we can indeed get an exact number for the integral. This is the following:

Corollary 14. *For any long geodesic $\gamma(s)$ starting from O we have*

$$\int_\gamma \mathrm{Rc}(\dot{\gamma},\dot{\gamma})ds = 1 \tag{2.3}$$

Notice we integrate from 0 to ∞.

Proof.

$$\begin{aligned}\frac{d}{ds}(|\nabla f|\cos\theta) &= \dot{\gamma}\langle -\nabla f, \dot{\gamma}\rangle \\ &= -\langle \nabla_{\dot{\gamma}}\nabla f, \dot{\gamma}\rangle \\ &= \mathrm{Rc}(\dot{\gamma},\dot{\gamma})\end{aligned}$$

Notice that when $s \to \infty$, $|\nabla f| \to 1$, and $\theta \to 0$; and at 0, $\nabla f = 0$. Taking integral from 0 to ∞ along γ we get Equation (2.3). □

Remark 15. *In the above proof, if we replace $|\nabla f|\cos\theta$ by $|\nabla f|$ we will see that*

$$\int_\gamma \mathrm{Rc}(\dot{\gamma},\nu)ds = 1.$$

2.5 R decays linearly

In this section, all the assumptions on $\mathcal{M}$ are the same as in section (2.3). We have seen from Lemma 12 that $-f$ grows linearly. In this section we will see that the scalar curvature R decays linearly. Before proceeding to the result, we first introduce a preliminary lemma proved by Perelman.

Lemma 16 (Canonical neighborhood theorem for ancient κ-solutions)**.** *For every $\varepsilon > 0$ there exists a compact set K_ε in $\mathcal{M}^3$ such that every point $x \in \mathcal{M} - K_\varepsilon$ is the center of an ε-neck. Then for any $\delta > 0$, there exists $\varepsilon > 0$ such that $|\Delta R| \leq \delta R^2$ for $x \in \mathcal{M} - K_\varepsilon$.*

In the above, in particular we choose $\delta = 1$, then there is a compact set out of which we have

$$\langle \nabla R, \nabla f \rangle \left(= \frac{\partial R}{\partial t} \right) = \Delta R + 2 \left| \mathrm{Rc} \right|^2 \geq |\, \mathrm{Rc} \,|^2 \geq \frac{R^2}{3}. \tag{2.4}$$

Theorem 17. *For $r(x)$ big enough, there are positive constants C_1, C_2 such that*

$$\frac{C_1}{r(x)} \leq R(x) \leq \frac{C_2}{r(x)}. \tag{2.5}$$

Proof. We prove the two inequalities separately.

step 1. R **decays at most linearly.**

By Perelman's derivative estimate we have $|\nabla R| \leq CR^2$. Apply this along a geodesic γ emanating from O. $\langle \nabla R, \dot{\gamma} \rangle \geq -CR^2$ so that $\dot{\gamma}\left(\frac{1}{R}\right) \leq C$. Thus

$$\frac{1}{R\left(\gamma\left(r\right)\right)} \leq \frac{1}{R\left(\gamma\left(0\right)\right)} + Cr = 1 + Cr,$$

so that

$$R\left(\gamma\left(r\right)\right) \geq \frac{1}{1 + Cr}.$$

For r big enough, by choosing an appropriate constant C_1 we can rewrite the above as

$$R(x) \geq \frac{C_1}{r}.$$

This shows R decays **at most** linearly.

step 2. R **decays at least linearly.**

Now we prove R decays **at least** linearly. By inequality 2.4, we have out of a compact set $\mathcal{U}_1$

$$\langle \nabla f, \nabla R\rangle = \Delta R + 2|\operatorname{Rc}|^2 \geq |\operatorname{Rc}|^2 \geq \frac{1}{3}R^2.$$

We introduce the integral curve $\bar{\beta}(\bar{\sigma})$ of $-\frac{\nabla f}{|\nabla f|^2}$. Since $|\nabla f| \to 1$ we have out of a compact set $\mathcal{U}_2$

$$\frac{dR}{d\bar{\sigma}} = -\frac{1}{|\nabla f|^2}\langle \nabla f, \nabla R\rangle \leq -\frac{1}{2}\langle \nabla f, \nabla R\rangle \leq -\frac{1}{6}R^2. \tag{2.6}$$

From this differential inequality we get get an upper estimate of $R(\bar{\beta}(\bar{\sigma}))$ in terms of $\bar{\sigma}$, of the order of $1/\bar{\sigma}$. Since $\bar{\sigma}$ indeed takes the value of $-f$ and is comparable with the distance r, we can get a same kind of upper bound of $R(x)$ in terms of r. With this intuition in mind, below we give a rigid argument. The reader may skip this technical part.

Let $\mathcal{S}_0$ be a big sphere out of $\mathcal{U} \doteqdot \mathcal{U}_1 \cup \mathcal{U}_2$. For any x, there is a maximal integral curve $\bar{\beta}$ passing through x with $\bar{\beta}(\bar{\sigma}) = x$. Let x_0 be a point of intersection of $\bar{\beta}$ and $\mathcal{S}_0$ and assume $\bar{\beta}(\bar{\sigma}_0) = x_0$. From Inequality (2.6) we have

$$\frac{1}{R(x)} - \frac{1}{R(x_0)} \geq \frac{\bar{\sigma} - \bar{\sigma}_0}{6}.$$

Let $c_0 \doteqdot \max\{R(\bar{x}) : \bar{x} \in \mathcal{S}_0\}$ then

$$\frac{1}{R(x)} \geq \frac{\bar{\sigma} - \bar{\sigma}_0}{6} + \frac{1}{c_0}. \tag{2.7}$$

Keep in mind that $\bar{\beta}$ is an integral curve of $-\frac{\nabla f}{|\nabla f|^2}$. So that

$$-f(x) - (-f(x_0)) = \int_{\bar{\sigma}_0}^{\bar{\sigma}} \frac{d}{d\eta}(-f(\bar{\beta}(\eta)))d\eta = \int_{\bar{\sigma}_0}^{\bar{\sigma}} 1 d\eta = \bar{\sigma} - \bar{\sigma}_0.$$

Let $c_1 \doteqdot \max\{-f(\bar{x}) : \bar{x} \in \mathcal{S}_0\}$ then

$$\bar{\sigma} - \bar{\sigma}_0 \geq -f(x) - c_1. \tag{2.8}$$

Combining Inequalities (2.7) and (2.8) we get

$$\frac{1}{R(x)} \geq \frac{-f(x)}{6} - \frac{c_1}{6} + \frac{1}{c_0}.$$

Since $-f$ grows linearly, there exist constants $c > 0$ and $c' \doteqdot -c_1/6 + 1/c_0$ so that

$$\frac{1}{R(x)} \geq cr(x) + c'.$$

For r large enough, we can choose an appropriate constant $C_2 > 0$ such that

$$R(x) \leq \frac{C_2}{r(x)}.$$

This shows R decays at least linearly. □

As easy consequences, below we see the asymptotic behaviors of ∇R and ∇f.

Corollary 18. *For $r(x)$ big enough, there is a constant c such that*

$$|\nabla R(x)| \leq \frac{c}{r^2}, \tag{2.9}$$

and

$$1 - \frac{c}{r} \leq |\nabla f| < 1. \tag{2.10}$$

Proof. Inequality (2.9) is directly from $|\nabla R| \leq CR^2$ and Inequality (2.6).

Since $R + |\nabla f|^2 = 1$, for $r(x)$ big enough we have

$$|\nabla f|^2 = 1 - R \geq 1 - \frac{c}{r} \geq (1 - \frac{c}{r})^2,$$

and we get inequalities in (2.10).

□

2.6 Round cylinder limit at spatial infinity (I)

In this section, we shall show that by appropriate point picking and rescaling, there is a round cylinder limit at spatial infinity. We prove this by using the properties of the scalar curvature R. The results in this section are mainly taken from [6]. Since they are very important in the understanding of the steady soliton at infinity, and also they offer a comparison with the next section where we prove the round cylinder limit by looking at the potential function, we include this section in our thesis.

We first introduce an important definition in the studies of Ricci flow.

Definition 19 (ASCR)**.** *The* ***asymptotic scalar curvature ratio*** *of a complete noncompact Riemannian manifold* $(\mathcal{M}^n, g)$ *is defined by*

$$\mathrm{ASCR}(g) \doteqdot \limsup_{d(x,O)\to\infty} R(x)d(x,O)^2, \tag{2.11}$$

where $O \in (M)$ *is any choice of origin.*

ASCR(g) is well-defined because of the following proposition.

Proposition 20. ASCR(g) *is independent the choice of origin* O.

Proof. We pick another point p and let

$$\mathrm{ASCR}_p(g) = \limsup_{d(x,p)\to\infty} R(x)d(x,p)^2.$$

We shall show that $\mathrm{ASCR}_p(g) = \mathrm{ASCR}(g)$ in 3 cases.

Case 1, $\mathrm{ASCR}(g) = +\infty$.

In this case there is a sequence of points x_i such that $d(x_i, O) \to \infty$ and $R(x_i)d(x_i, O)^2 \to +\infty$. When x_i is far enough from O we have

$$d(x_i, O) - d(p, O) \geq d(x_i, O)/2 > 1$$

and

$$R(x_i)d(x_i, p)^2 \geq R(x_i)(d(x_i, O) - d(p, O))^2 \geq \frac{R(x_i)d(x_i, O)^2}{4}.$$

We see $\mathrm{ASCR}_p(g) = +\infty$ as well.

Case 2, $\mathrm{ASCR}(g) = -\infty$.

In this case, for **any** sequence $x_i \to \infty$ we have

$$\lim R(x_i)d(x_i, O)^2 = -\infty.$$

For x_i far enough we have $R(x_i) < 0$ and

$$R(x_i)d(x_i, p)^2 \le R(x_i)(d(x_i, O) - d(p, O))^2 \le \frac{R(x_i)d(x_i, O)^2}{4}.$$

Taking limits of both sides we see that $\mathrm{ASCR}_p(g) = -\infty$.

Case 3, $\mathrm{ASCR}(g) \doteqdot C$ **is finite.**

In this case, since when x is far enough from both p and O

$$\frac{d(x, O)}{2} \le d(x, p) \le 2d(x, O),$$

we see $\mathrm{ASCR}_p(g) \doteqdot C_p$ is finite as well. Assume that $\{x_i\}$ is such a sequence of points such that $R(x_i)d(x_i, p)^2 \to C_p$. Obviously $R(x_i)d(x_i, p) \to 0$ and moreover

$$|R(x_i)d(x_i, p)^2 - R(x_i)d(x_i, O)^2| \le |R(x_i)(d(x_i, p) + d(x_i, O)|d(p, O) \to 0.$$

We now see that $\lim R(x_i)d(x_i, O)^2 = C_p$, and consequently $C \ge C_p$.

On the other hand, by the same reasoning we can also see that $C_p \ge C$. So we have $C_p = C$ and we are done. □

It is known that ASCR is independent of time on a complete noncompact ancient solution with bounded nonnegative curvature operator(See Theorem 8.32 of [6]). The following is Theorem 9.44 in [6].

Lemma 21. *Let* $(\mathcal{M}^n, g, f)$ *be a complete steady gradient Ricci soliton such that* $\mathrm{sect}(g) \ge 0$, $\mathrm{Rc} > 0$, *and* R *attains its maximum at some point. If* $n \ge 3$, *then* $\mathrm{ASCR}(g) = +\infty$.

The following is Theorem 9.66 of [6]. We sketch the proof here.

Theorem 22. *Let $\mathcal{M}^3$ be a steady gradient soliton with bounded positive sectional curvature. If further $\mathcal{M}^3$ is κ-noncollapsed on all scales then there exists a sequence of points $x_i \to \infty$ and a sequence of radii $\{r_i\}$ with $r_i^2 R(x_i, 0) \to \infty$ such that the pointed sequence of solutions*

$$\left(B_{g(0)}(x_i, r_i), g_i(t), x_i\right), t \in (-\infty, 0]$$

converges to a round shrinking cylinder. Here $g_i(t) \doteqdot R(x_i)\, g\left(R(x_i)^{-1} t\right)$ and $R(x_i)$ is evaluated with respect to the metric $g(0)$.

Proof. We have $\mathrm{ASCR}(g) = \infty$. By point picking, there exists a subsequence $x_i \to \infty$, $r_i \in (0, \infty)$, and $\varepsilon_i \to 0$ such that

$$\begin{aligned}
&R(x_i)\, r_i^2 \to \infty, \\
&B(x_i, r_i) \text{ are disjoint}, \\
&\frac{d(x_i, O)}{r_i} \to \infty, \\
&\max_{\bar{B}(x_i, r_i)} R \leq (1 + \varepsilon_i)\, R(x_i).
\end{aligned}$$

Consider the scaled sequence

$$\left(\mathcal{M}^3, g_i(t), x_i\right),$$

where $g_i(t) \doteqdot R(x_i)\, g\left(R(x_i)^{-1} t\right)$. We have

$$\max_{\bar{B}_{g_i(t)}\left(x_i, R(x_i)^{1/2} r_i\right)} R_{g_i(t)} \leq 1 + \varepsilon_i$$

and $R(x_i)^{1/2} r_i \to \infty$. By the curvature bumps theorem of Hamilton, we have

$$\min_{\bar{B}_{g(0)}\left(x_i, R(x_i)^{-1/2}\right)} R(x_i)^{-1} \operatorname{sect}(g(0)) \to 0$$

for a subsequence $i \to \infty$. By the compactness theorem, there is a subsequence of $(\mathcal{M}^3, g_i(t), x_i)$ converges to $(\mathcal{M}^3_\infty, g_\infty(t), x_\infty)$ with $\min_{\bar{B}_{g_\infty(0)}(x_i, 1)} \operatorname{sect}(g_\infty(0)) = 0$, so that $\mathcal{M}^3_\infty$ is a shrinking round cylinder (for topological reasons, a quotient cannot occur; also we used the classification of ancient κ-solutions in dimension 2). □

2.7 Round cylinder limit at spatial infinity (II)

In last section, we have seen there exists a sequence going to infinity converges to a round cylinder by appropriate rescaling. We proved this by investigating the scalar curvature. In this section, we will see that for any sequence going to infinity, there is a subsequence converges to a round cylinder by the same rescaling. We prove this by looking at the properties of the potential function.

Given any sequence $\{x_i\}$ with $x_i \to \infty$, we consider $(\mathcal{M}, g_i(t), x_i, F_i(x))$, where $g_i(t) \doteqdot R(x_i)g(t/R(x_i))$ and $F_i(x) \doteqdot \sqrt{-f(x)} - \sqrt{-f_i(x)}$.

First we process a technical point picking. The goal is to get a subsequence of points, still denoting by $\{x_i\}$, and a sequence of radii $\{r_i\}$ such that the balls $B_{g(0)}(x_i, r_i)$ keep growing unbounded under the rescaled metric meanwhile we still have nice control inside the balls. The reader is notified this technical point picking is not unique, and what we do here is just an option.

We begin with considering the points out of a compact set such that the linear behaviors hold:

$$\frac{C_1}{r(x)} \leq R(x) \leq \frac{C_2}{r(x)}, \quad (1-\varepsilon)r(x) \leq -f(x) \leq r(x).$$

Now let $r_i \doteqdot \sqrt{r(x_i)}/\left(R(x_i)\right)^{\frac{1}{4}}$, remember by our notation, $r(x) \doteqdot d_{g(0)}(x, O)$. Since $x_i \to \infty$, we can pick a subsequence such that $B_{g(0)}(x_i, r_i)$ are disjoint. (This requirement is not essential for our purposes but will make the picture cleaner and clearer.) We notice that $B_{g(0)}(x_i, r_i)$ are the same as $B_{g_i(0)}(x_i, \sqrt{R(x_i)}r_i)$ and

$$\sqrt{R(x_i)}r_i = \sqrt{r(x_i)}\left(R(x_i)\right)^{\frac{1}{4}} \geq C\left(r(x_i)\right)^{\frac{1}{4}} \to \infty,$$

which says that under the (smaller) metric $g_i(t)$, the radii of the balls $B_{g(0)}(x_i, r_i)$ go to infinity.

For any $x \in B_{g(0)}(x_i, r_i)$, we have

$$r(x_i) - \frac{\sqrt{r(x_i)}}{R(x_i)^{\frac{1}{4}}} \leq r(x) \leq r(x_i) + \frac{\sqrt{r(x_i)}}{R(x_i)^{\frac{1}{4}}},$$

or

$$1 - \frac{1}{\sqrt{r(x_i)}R(x_i)^{\frac{1}{4}}} \leq \frac{r(x)}{r(x_i)} \leq 1 + \frac{1}{\sqrt{r(x_i)}R(x_i)^{\frac{1}{4}}}.$$

Since $x_i \to \infty$ and $\frac{1}{\sqrt{r(x_i)}R(x_i)^{\frac{1}{4}}} \to 0$ we may assume x_i far enough such that

$$\frac{1}{2} \leq \frac{r(x)}{r(x_i)} \leq \frac{3}{2}.$$

Now it's not hard to see that within the balls $B_{g(0)}(x_i, r_i)$ there are positive constants C_1 and C_2 which are independent of i such that

$$C_1 \leq \frac{R(x)}{R(x_i)} \leq C_2, \quad C_1 \leq \frac{f(x)}{f(x_i)} \leq C_2 \tag{2.12}$$

and

$$C_1 \leq R(x) \cdot (-f(x_i)) \leq C_2, \quad C_1 \leq R(x_i) \cdot (-f(x)) \leq C_2. \tag{2.13}$$

By now we have finished the point picking and are ready to derive our main theorem of this section:

Theorem 23. *For any sequence $\{x_i\}$ with $x_i \to \infty$ there is a subsequence still denoting by $\{x_i\}$ with associated radii $\{\bar{r}_i\}$ such that the rescaled pointed sequence of solutions*

$$\{\big(B_{g_i(0)}(x_i, \bar{r}_i), g_i(t), x_i, F_i(x)\big)\}, t \in (-\infty, 0]$$

converges in the C^∞ pointed Cheeger-Gromov sense to the round cylinder

$$\big(S^2 \times \mathbb{R}, g_\infty(t), x_\infty, F_\infty(x)\big)$$

where $g_\infty(t)$ is the standard Ricci flow solution and F_∞ satisfies

$$|\nabla^\infty F_\infty|_{g_\infty(0)} = 1, \quad \nabla^\infty \nabla^\infty F_\infty = 0.$$

Proof. Consider the ball $B_{g(0)}(x_i, r_i)$, where x_i and r_i are picked as we described before the theorem. Note the ball is the same as $B_{g_i(0)}(x_i, \bar{r}_i)$ where $\bar{r}_i \doteqdot \sqrt{R(x_i)}r_i \to$

∞ as $x_i \to \infty$. For any $x \in B_{g(0)}(x_i, r_i)$ by Equation (2.13) and the fact that $|\nabla f| \to 1$ we have

$$2\left|\nabla_{g_i(0)} F_i(x)\right|_{g_i(0)} = R(x_i)^{-\frac{1}{2}}(-f(x))^{-\frac{1}{2}}\left|\nabla f(x)\right|_{g(0)} \geq C > 0.$$

Note we use C to denote different constants if there is no need to distinguish them.

On the other hand, we also have

$$2\left|\nabla_{g_i(0)} F_i(x)\right|_{g_i(0)} = R(x_i)^{-\frac{1}{2}}(-f(x))^{-\frac{1}{2}}\left|\nabla f(x)\right|_{g(0)} \leq C.$$

Now we calculate the second derivatives

$$\begin{aligned} 2\left|\nabla_{g_i(0)}\nabla_{g_i(0)} F_i(x)\right|_{g_i(0)} &= R\left(x_i\right)^{-1}\left|-(-f)^{-1/2}\nabla\nabla f + \frac{1}{2}(-f)^{-3/2}\nabla f \otimes \nabla f\right|_{g(0)} \\ &\leq C\sqrt{r\left(x_i\right)}\left|\mathrm{Rc}\right|_{g(0)} + Cr\left(x\right)^{-1/2} \\ &\leq C\sqrt{r(x_i)}R(x) + Cr(x_i)^{-\frac{1}{2}} \\ &\leq C\left(r(x_i)\right)^{-\frac{1}{2}}, \end{aligned}$$

and now it's obvious that

$$\lim_{x_i\to\infty}\left|\nabla_{g_i(0)}\nabla_{g_i(0)} F_i(x)\right|_{g_i(0)} = 0.$$

Furthermore we calculate the third derivatives and we have

$$\begin{aligned} &2\left|\nabla_{g_i(0)}\nabla_{g_i(0)}\nabla_{g_i(0)} F_i(x)\right|_{g_i(0)} \\ &= 2\left(R(x_i)\right)^{-\frac{3}{2}}\left|\nabla_l\nabla_j\nabla_k F_i(x)\right|_{g(0)} \\ &= \left(R(x_i)\right)^{-\frac{3}{2}}\left|\frac{\nabla_l\nabla_j\nabla_k f}{\sqrt{-f}} + \frac{\nabla_j\nabla_k f\nabla_l f + \nabla_l\nabla_k f\nabla_j f + \nabla_l\nabla_j f\nabla_k f}{2\left(\sqrt{-f}\right)^3} + \frac{3}{4}\frac{\nabla_l f\nabla_j f\nabla_k f}{\left(\sqrt{-f}\right)^5}\right| \\ &\leq Cr(x_i)\left|\nabla\nabla\nabla f\right| + C\left|\mathrm{Rc}\right| + Cr(x_i)^{-1}. \qquad (2.14) \end{aligned}$$

By Perelman's derivative estimate we have

$$\left|\nabla\nabla\nabla f\right| = \left|\nabla \mathrm{Rc}\right| \leq CR^{\frac{3}{2}} \leq Cr(x_i)^{-\frac{3}{2}},$$

and now we see all of the three terms in Inequality (2.14) go to 0 as $x_i \to \infty$.

Summarizing we have proved the following:

1. $C_1 \leq \left|\nabla_{g_i(0)} F_i(x)\right|_{g_i(0)} \leq C_2$,

2. $\lim_{x_i \to \infty} \left|\nabla_{g_i(0)} \nabla_{g_i(0)} F_i(x)\right|_{g_i(0)} = 0$,

3. $\lim_{x_i \to \infty} \left|\nabla_{g_i(0)} \nabla_{g_i(0)} \nabla_{g_i(0)} F_i(x)\right|_{g_i(0)} = 0$.

By passing to a subsequence, we conclude in the Cheeger-Gromov convergence of $\left(B_{g_i(0)}(x_i, \bar{r}_i), g_i(t), x_i\right)$ to $(\mathcal{M}_\infty, g_\infty(t), x_\infty)$ that the functions F_i converge to a C^∞ function F_∞ on M_∞ with

$$\nabla\nabla F_\infty = 0 \quad \text{and} \quad |\nabla F_\infty| = C.$$

By rescaling, we can assume $C = 1$. Since ∇F_∞ is a parallel vector field, we get $(M_\infty, g_\infty(t))$ are cylinders and F_∞ is a radial function. $\square$

2.8 Characterizing the rotational symmetry by R and f

The same assumptions as before, $\mathcal{M}^3$ is a steady soliton with positive sectional curvatures and which is κ-noncollapsed on all scales. And adopt all notations from the previous sections.

Theorem 24. *The following conditions are equivalent.*

1. $|\nabla f|^2$ *is constant on the level sets of* f.

2. R *is constant on the level sets of* f.

3. *The integral curves to* ν *are geodesics.*

4. $(\mathcal{M}, g)$ *is rotationally symmetric, that is the level surfaces* $\Sigma(\sigma)$ *are round spheres.*

Proof. (1)$\Leftrightarrow$(2): this is directly from $R + \nabla f|^2 = 1$.

(2)$\Leftrightarrow$(3): except for the origin O, on all level surfaces of f, the second fundamental form h is strictly positive. Then by Propositions (7) and (8), we get this equivalence.

(1),(2),(3)$\Rightarrow$(4): we are going to show all level surfaces $\Sigma(\sigma)$ are round spheres. We split our proof into several steps so that it is easy to read.

step 1. $\mathrm{Rc}(\nu, \nu)$ **is constant on** Σ.

For any vector field X which is tangent to Σ, we extend it to a neigborhood so that it is always tangent to $\Sigma(\sigma)$. We have

$$\mathrm{Rc}(X, \nu) = -\frac{1}{|\nabla f|} \mathrm{Rc}(X, \nabla f) = -\frac{1}{2|\nabla f|} X(R) = 0,$$

since R is constant on Σ.

Furthermore,

$$\begin{aligned} X(\mathrm{Rc}(\nu, \nu)) &= (\nabla_X \mathrm{Rc})(\nu, \nu) + 2\,\mathrm{Rc}(\nabla_X \nu, \nu) \\ &= (\nabla_X \mathrm{Rc})(\nu, \nu) + 2\,\mathrm{Rc}(X, \nabla_\nu \nu) \\ &= (\nabla_X \mathrm{Rc})(\nu, \nu), \end{aligned}$$

where in the second equality we used Equation (2.1). And

$$\begin{aligned} 0 = \nu(\mathrm{Rc}(X, \nu)) &= (\nabla_\nu \mathrm{Rc})(X, \nu) + \mathrm{Rc}(\nabla_\nu X, \nu) + \mathrm{Rc}(X, \nabla_\nu \nu) \\ &= (\nabla_\nu \mathrm{Rc})(X, \nu) \end{aligned}$$

where in the last equality we used the fact that $\nabla_\nu \nu = 0$, and because of that, $\nabla_\nu X \perp \nu$, so $\mathrm{Rc}(\nabla_\nu X, \nu) = 0$.

Since $\nabla f = -|\nabla f|\nu$, we have

$$(\nabla_X \mathrm{Rc})(\nu, \nu) - (\nabla_\nu \mathrm{Rc})(X, \nu) = -\mathrm{Rm}(X, \nu, \nu, \nu)|\nabla f| = 0.$$

Our conclusion is $X(\mathrm{Rc}(\nu, \nu)) = 0$, which means $\mathrm{Rc}(\nu, \nu)$ is constant on Σ.

step 2. The mean curvature H **depends only on** σ.

This is because

$$H = h_{ii} = \frac{R_{ii}}{|\nabla f|} = \frac{R - \mathrm{Rc}(\nu,\nu)}{|\nabla f|},$$

and all the quantities in the last term depends only on σ.

step 3. The norm of the second fundamental form $|h|^2$ is constant.

The level surfaces of f is evolving by the geometric flow

$$\frac{\partial X}{\partial \sigma} = \frac{\nabla f}{|\nabla f|^2} = -\frac{1}{|\nabla f|}\nu.$$

The evolution equation of the mean curvature is

$$\frac{\partial H}{\partial \sigma} = \tilde{\Delta}(\frac{1}{|\nabla f|}) + \frac{1}{|\nabla f|}(|h|^2 + \mathrm{Rc}(\nu,\nu)),$$

where $\tilde{\Delta}$ is the Laplace operator on Σ. We have

$$|h|^2 = |\nabla f| \cdot \frac{\partial H}{\partial \sigma} - \mathrm{Rc}(\nu,\nu),$$

and now it's clear $|h|^2$ depends **only** on σ because all the terms on the RHS do.

step 4. The Gauss curvature $\mathcal{K}_\sigma$ of Σ is constant.

By the Gauss equation we know

$$\mathcal{K}_\sigma = \mathrm{sect}(e_1 \wedge e_2) + \det(h),$$

where e_1 and e_2 are two orthonormal tangent vector fields on Σ, $\mathrm{sect}(e_1 \wedge e_2)$ is the sectional curvature of the plane spanned by $\{e_1, e_2\}$, and $\det(h)$ is the determinant of the second fundamental form h. We'll show both of the two terms on the RHS are constants on Σ.

We have

$$\begin{aligned} 2\,\mathrm{sect}(e_1 \wedge e_2) &= 2R_{1221} \\ &= (R_{0110} + R_{2112}) + (R_{1221} + R_{0220}) - (R_{0110} + R_{0220}) \\ &= R_{11} + R_{22} - R_{00} \\ &= R - 2\,\mathrm{Rc}(\nu,\nu), \end{aligned}$$

where 0 means along ν direction, for example $R_{0110} \doteqdot \mathrm{Rm}(\nu, e_1, e_1, \nu)$, and $R_{00} \doteqdot \mathrm{Rc}(\nu, \nu)$. It's now clear that $\mathrm{sect}(e_1 \wedge e_2)$ is constant on Σ.

Furthermore, since

$$2\det(h) = H^2 - |h|^2$$

is also constant on Σ, we see $\mathcal{K}_\sigma$ is constant for each fixed σ. This shows intrinsically $\Sigma(\sigma)$ is a round sphere with constant Gauss curvature $\mathcal{K}_\sigma$.

(4)⇒(1),(2),(3). If $\mathcal{M}^3$ is rotationally symmetric, then by a standard argument, $\mathcal{M}$ must be a Bryant solion. All the conclusions follow. □

Chapter 3

The second fundamental form

In this chapter, we investigate another quantity to measure the rotational symmetry of the level surfaces. We look at the integral of $(\lambda_2 - \lambda_2)^2$, where λ_1, λ_2 are the principle curvatures of the level surface.

We first derive the evolution equation of $\det(h)$ under general geometric flow. The formula is interesting itself. Then in dimension 3, we get the evolution equation of the isoperimetric quantity $\int (\lambda_1 - \lambda_2)^2 d\mu$, which is $\int H^2 - 4\det(h) d\mu$. Applying this equation into the situation where the flow is the natural flow of the level surfaces of the potential function on a steady soliton, we get a nice evolution equation, and then we get a monotonicity formula out of a compact set. If we assume the quantity goes to 0 at ∞ we can get $\lambda_1 = \lambda_2$. Some other applications are also discussed at the end.

3.1 The evolution equation under general geometric flows

In this section we derive the evolution equation in general geometric flows. Let $(\mathcal{M}^{n+1}, g)$ be a complete Riemannian manifold, and $\Sigma(t)$ be a hypersurface evolving by the flow equation:

$$\frac{\partial X}{\partial t} = -\phi\nu \tag{3.1}$$

where ϕ is a smooth function and ν is the outward unit normal vector of $\Sigma(t)$. We will further assume as an n-dimensional manifold, Σ has no boundary. This is satisfied in all of our applications.

The inverse of the second fundamental form h^{-1} is the usual matrix inverse when h is nonzero. Whenever h is degenerate at some point, we can still define h^{-1} by the following: assume the eigenvalues of h are $\{\lambda_1, \lambda_2, \cdots, \lambda_m, 0, \cdots, 0\}$, then we define h^{-1} to be $\{\lambda_1^{-1}, \lambda_2^{-1} \cdots, \lambda_m^{-1}, 0, \cdots, 0\}$. We will use the notation $h^{ij} \doteqdot (h^{-1})_{ij}$ throughout this chapter. Note the upper indices are not lifted by the metric.

There is the Gauss-Codazzi equation, which we shall use later:

$$(\tilde{\nabla}_k h)_{ij} - (\tilde{\nabla}_j h)_{ik} = -R_{0ijk} \tag{3.2}$$

where we use tilde to denote the quantities on Σ and 0 means the direction along ν. However, we don't bother to use tilde on the first and second fundamental forms because they won't cause any confusion.

The determinant of the second fundamental form is defined to be

$$\det(h) = \frac{\det(h_{ij})}{\det(g_{ij})},$$

which is independent of the choice of coordinates.

In this section we derive the following formula:

Theorem 25. *If $\Sigma(t)$ satisfies Equation (3.1), then we have*

$$\frac{d}{dt}\int_\Sigma \det(h)d\mu = \int_\Sigma \det(h)[\phi h^{ij}R_{0ij0} - e_i(\phi)h^{ik}h^{lj}R_{0lkj}]d\mu, \tag{3.3}$$

The rest part of this section is devoted to proving Theorem 25. First we list the evolution equations which we shall use:

1. $\frac{\partial g_{ij}}{\partial t} = -2\phi h_{ij}$,

2. $\frac{\partial}{\partial t} d\mu_t = -\phi H d\mu_t$,

3. $\frac{\partial h_{ij}}{\partial t} = \tilde{\nabla}_i \tilde{\nabla}_j \phi - \phi g^{kl} h_{ik} h_{jl} + \phi R_{0ij0}$.

Then we derive several lemmas.

Lemma 26. *We have the following evolution equation for* $\det(h)$:

$$\frac{\partial}{\partial t} \det(h) = \det(h) \cdot (h^{ij} \tilde{\nabla}_i \tilde{\nabla}_j \phi + \phi h^{ij} R_{0ij0} + \phi H) \tag{3.4}$$

Proof. By direct computations we have

$$\begin{aligned} \frac{\partial}{\partial t} \det(h_{ij}) &= \det(h_{ij}) h^{ij} \cdot \frac{\partial h_{ij}}{\partial t} \\ &= \det(h_{ij}) h^{ij} \cdot (\tilde{\nabla}_i \tilde{\nabla}_j \phi - \phi g^{kl} h_{ik} h_{jl} + \phi R_{0ij0}) \\ &= \det(h_{ij}) (h^{ij} \tilde{\nabla}_i \tilde{\nabla}_j \phi - \phi H + \phi h^{ij} R_{0ij0}) \end{aligned}$$

and

$$\frac{\partial}{\partial t} \det(g_{ij}) = -2\phi H \det(g_{ij}).$$

Then by the quotient rule in calculus we have

$$\begin{aligned} \frac{\partial}{\partial t} \det(h) &= \frac{\det(g_{ij}) \cdot \frac{\partial}{\partial t} \det(h_{ij}) - \det(h_{ij}) \cdot \frac{\partial}{\partial t} \det(g_{ij})}{(\det(g_{ij}))^2} \\ &= \det(h) \cdot (h^{ij} \tilde{\nabla}_i \tilde{\nabla}_j \phi - \phi H + \phi h^{ij} R_{0ij0}) + 2 \det(h) \cdot \phi H \\ &= \det(h) \cdot (h^{ij} \tilde{\nabla}_i \tilde{\nabla}_j \phi + \phi h^{ij} R_{0ij0} + \phi H). \end{aligned}$$

□

Immediately we have

Corollary 27.

$$\frac{d}{dt} \int_{\Sigma} \det(h) d\mu = \int_{\Sigma} \det(h) \cdot (h^{ij} \tilde{\nabla}_i \tilde{\nabla}_j \phi + \phi h^{ij} R_{0ij0}) d\mu. \tag{3.5}$$

Proof.

$$\frac{d}{dt}\int_\Sigma \det(h)d\mu = \int_\Sigma \frac{\partial}{\partial t}\det(h)d\mu + \int_\Sigma \det(h)\frac{\partial}{\partial t}(d\mu)$$

Combining $\frac{\partial}{\partial t}d\mu_t = -\phi H d\mu_t$ and Equation (3.4) we get Equation (3.5). □

Now we are going to do integration by parts. We will show the first term in the RHS of Equation (3.5)

$$\det(h)\cdot h^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\phi = \mathrm{div}_\Sigma[\det(h)h^{-1}(\tilde{\nabla}\phi)] - \det(h)e_i(\phi)h^{ik}h^{lj}R_{0lkj},$$

where div_Σ is the divergence on Σ. Then Equation (3.3) follows immediately from the divergence theorem. To see that we need a few technical lemmas.

Lemma 28. *Under an orthonormal frame* $\{e_i\}$*:*

$$\mathrm{div}_\Sigma[h^{-1}(\tilde{\nabla}\phi)] = (\tilde{\nabla}_j(h^{-1}))_{ij}e_i(\phi) + h^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\phi. \tag{3.6}$$

Proof.

$$\begin{aligned}
\mathrm{div}_\Sigma[h^{-1}(\tilde{\nabla}\phi)] &= \langle\tilde{\nabla}_k[h^{ij}e_i(\phi)e_j], e_k\rangle \\
&= e_j(h^{ij}e_i(\phi)) + h^{ij}e_i(\phi)\langle\tilde{\nabla}_k e_j, e_k\rangle \\
&= e_j(h^{ij})e_i(\phi) + h^{ij}e_je_i(\phi) + h^{ij}e_i(\phi)\tilde{\Gamma}^k_{kj} \\
&= (\tilde{\nabla}_j(h^{-1}))_{ij}e_i(\phi) + h^{-1}(\tilde{\nabla}_j e_i, e_j)e_i(\phi) \\
&\quad + h^{-1}(\tilde{\nabla}_j e_j, e_i)e_i(\phi) + h^{ij}e_je_i(\phi) + h^{ij}e_i(\phi)\tilde{\Gamma}^k_{kj} \\
&= (\tilde{\nabla}_j(h^{-1}))_{ij}e_i(\phi) + h^{kj}\tilde{\Gamma}^k_{ji}e_i(\phi) + h^{ki}\tilde{\Gamma}^k_{jj}e_i(\phi) \\
&\quad + h^{ij}e_je_i(\phi) + h^{ij}e_i(\phi)\tilde{\Gamma}^k_{kj}
\end{aligned}$$

Since $\{e_i\}$ are orthonormal we have $\tilde{\Gamma}^k_{ij} = -\tilde{\Gamma}^j_{ik}$, and the last line in the above calculations becomes

$$(\tilde{\nabla}_j(h^{-1}))_{ij}e_i(\phi) + h^{ij}[e_je_i(\phi) - \tilde{\Gamma}^k_{ji}e_k(\phi)] = (\tilde{\nabla}_j(h^{-1}))_{ij}e_i(\phi) + h^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\phi.$$

This completes the proof of the lemma. □

Remark 29. *From the proof we see the formula is true when replacing h^{-1} by any general 2-tensor. Indeed the next lemma is also true for general 2-tensors.*

It is not hard to see that

Lemma 30.

$$(\tilde{\nabla}_m h^{-1})_{ij} = -h^{ik}h^{lj}(\tilde{\nabla}_m h)_{kl}.$$

Proof. Firstly we know

$$\tilde{\nabla}_m(h^{ij}) = -h^{ik}h^{lj}\tilde{\nabla}_m(h_{kl}),$$

so we have

$$\begin{aligned}(\tilde{\nabla}_m h^{-1})_{ij} &= \tilde{\nabla}_m(h^{ij}) - h^{-1}(\tilde{\nabla}_m e_i, e_j) - h^{-1}(\tilde{\nabla}_m e_j, e_i) \\ &= -h^{ik}h^{lj}\tilde{\nabla}_m(h_{kl}) - \tilde{\Gamma}^k_{mi}h^{kj} - \tilde{\Gamma}^k_{mj}h^{ki} \\ &= -h^{ik}h^{lj}[(\tilde{\nabla}_m h)_{kl} + h(\tilde{\nabla}_m e_k, e_l) + h(\tilde{\nabla}_m e_l, e_k)] \\ &\quad - \tilde{\Gamma}^k_{mi}h^{kj} - \tilde{\Gamma}^k_{mj}h^{ki} \\ &= -h^{ik}h^{lj}[(\tilde{\nabla}_m h)_{kl} + \tilde{\Gamma}^r_{mk}h_{rl} + \tilde{\Gamma}^r_{ml}h_{rk}] - \tilde{\Gamma}^k_{mi}h^{kj} - \tilde{\Gamma}^k_{mj}h^{ki} \\ &= -h^{ik}h^{lj}(\tilde{\nabla}_m h)_{kl} - \tilde{\Gamma}^j_{mk}h^{ik} - \tilde{\Gamma}^i_{ml}h^{lj} - \tilde{\Gamma}^k_{mi}h^{kj} - \tilde{\Gamma}^k_{mj}h^{ki}\end{aligned}$$

Then again by $\tilde{\Gamma}^k_{ij} = -\tilde{\Gamma}^j_{ik}$, we get our formula as claimed. □

Now we are ready to derive our last formula of this section:

Lemma 31.

$$\mathrm{div}_\Sigma[\det(h)\cdot h^{-1}(\tilde{\nabla}\phi)] = \det(h)\cdot[h^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\phi + e_i(\phi)h^{ik}h^{lj}R_{0lkj}] \tag{3.7}$$

Proof.

$$\begin{aligned}\mathrm{div}_\Sigma[\det(h)\cdot h^{-1}(\tilde{\nabla}\phi)] &= \det(h)\cdot \mathrm{div}_\Sigma[h^{-1}(\tilde{\nabla}\phi)] + \langle\tilde{\nabla}\det(h), h^{-1}(\tilde{\nabla}\phi)\rangle \\ &= \det(h)\cdot h^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\phi + \det(h)\cdot(\tilde{\nabla}_j(h^{-1}))_{ij}e_i(\phi) \\ &\quad + h^{-1}(\tilde{\nabla}\det(h), \tilde{\nabla}\phi).\end{aligned}$$

It's not hard to see that

$$h^{-1}(\tilde{\nabla}\det(h), \tilde{\nabla}\phi) = \det(h)\cdot e_i(\phi)h^{ik}h^{jl}e_k(h_{jl})$$

Then we have

$$\begin{aligned}
&\det(h)\cdot(\tilde{\nabla}_j(h^{-1}))_{ij}e_i(\phi) + h^{-1}(\tilde{\nabla}\det(h), \tilde{\nabla}\phi)\\
&\quad= -\det(h)e_i(\phi)h^{ik}h^{lj}\cdot[(\tilde{\nabla}_j h)_{kl} - e_k(h_{jl})]\\
&\quad= -\det(h)e_i(\phi)h^{ik}h^{lj}\cdot[(\tilde{\nabla}_j h)_{kl} - (\tilde{\nabla}_k h)_{jl} - h(\tilde{\nabla}_k e_j, e_l) - h(\tilde{\nabla}_k e_l, e_j)]\\
&\quad= -\det(h)e_i(\phi)h^{ik}h^{lj}\cdot[-R_{0lkj} - h(\tilde{\nabla}_k e_j, e_l) - h(\tilde{\nabla}_k e_l, e_j)]\\
&\quad= \det(h)e_i(\phi)h^{ik}h^{lj}R_{0lkj}
\end{aligned}$$

The last equality is because $h^{lj}h(\tilde{\nabla}_k e_j, e_l) = \tilde{\Gamma}^m_{kj}h_{ml}h^{lj} = \tilde{\Gamma}^j_{kj} = 0$. □

Finally we give *a proof of Theorem 25* to finish this section.

Proof.

$$\begin{aligned}
\frac{d}{dt}\int_\Sigma \det(h)d\mu &= \int_\Sigma \det(h)\cdot(h^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\phi + \phi h^{ij}R_{0ij0})d\mu\\
&= \int_\Sigma \mathrm{div}_\Sigma[\det(h)h^{-1}(\tilde{\nabla}\phi)] - \det(h)e_i(\phi)h^{ik}h^{lj}R_{0lkj}d\mu\\
&+ \int_\Sigma \det(h)\phi h^{ij}R_{0ij0}d\mu\\
&= \int_\Sigma \det(h)(\phi h^{ij}R_{0ij0} - e_i(\phi)h^{ik}h^{lj}R_{0lkj})d\mu.
\end{aligned}$$

In the last equality we used the divergence theorem, and the fact Σ has no boundary.

□

3.2 An isoperimetric quantity in a 3-dimensional manifold

In this section, we assume $n = 2$, which means the dimension of the ambient manifold is 3. In this case, the Riemann curvature can be replaced by the Ricci curvature in the following way:

$$R_{abcd} = R_{ad}g_{bc} + R_{bc}g_{ad} - R_{ac}g_{bd} - R_{bd}g_{ac} - \frac{R}{2}(g_{ad}g_{bc} - g_{ac}g_{bd}),$$

where $0 \le a, b, c, d \le 2$. Let $a = 0, b = i, c = j, d = 0$ we have

$$\begin{aligned} R_{0ij0} &= R_{00}g_{ij} + R_{ij} - \frac{R}{2}g_{ij} \\ &= \frac{R_{00} - g^{kl}R_{kl}}{2}g_{ij} + R_{ij} \end{aligned}$$

Let $a = 0, b = l, c = k, d = j$ we get

$$R_{0lkj} = R_{0j}g_{lk} - R_{0k}g_{lj}.$$

We have

Lemma 32. *When $n = 2$, assume the principle curvatures of h are λ_1, λ_2, and the corresponding eigenvectors are e_1, e_2 respectively. Under the flow Equation (3.1) We have*

$$\frac{d}{dt}\int_\Sigma \det(h)d\mu = \int_\Sigma \frac{\phi H R_{00}}{2} - \frac{\phi(R_{11} - R_{22})(\lambda_1 - \lambda_2)}{2} + \mathrm{Rc}(\nu, \tilde{\nabla}\phi)d\mu. \qquad (3.8)$$

Proof. By Equation (3.3) and the formulae of the Riemann curvatures in terms of the Ricci curvatures in dimension 3, we have

$$\begin{aligned} \frac{d}{dt}\int_\Sigma \det(h)d\mu = \int & \det(h)\phi h^{ij}(\frac{R_{00} - g^{kl}R_{kl}}{2}g_{ij} + R_{ij}) \\ & - \det(h)e_i(\phi)h^{ik}h^{lj}(R_{0j}g_{lk} - R_{0k}g_{lj})d\mu \end{aligned}$$

The first term of the integrand

$$\begin{aligned} \det(h)\phi h^{ij}(\frac{R_{00} - g^{kl}R_{kl}}{2}g_{ij} + R_{ij}) &= \phi H\frac{R_{00} - R_{11} - R_{22}}{2} + \phi(\lambda_2 R_{11} + \lambda_1 R_{22}) \\ &= \frac{\phi H R_{00}}{2} - \frac{\phi(R_{11} - R_{22})(\lambda_1 - \lambda_2)}{2} \end{aligned}$$

While the second term

$$
\begin{aligned}
-\det(h)e_i(\phi)h^{ik}h^{lj}(R_{0j}g_{lk}-R_{0k}g_{lj}) &= -\det(h)e_i(\phi)h^{ik}h^{lj}R_{0j}g_{lk} \\
&\quad + \det(h)e_i(\phi)h^{ik}h^{lj}R_{0k}g_{lj} \\
&= -\frac{\lambda_2}{\lambda_1}e_1(\phi)R_{01} - \frac{\lambda_1}{\lambda_2}e_2(\phi)R_{02} \\
&\quad + H(\frac{1}{\lambda_1}e_1(\phi)R_{01} + \frac{1}{\lambda_2}e_2(\phi)R_{02}) \\
&= e_1(\phi)R_{01} + e_2(\phi)R_{02} \\
&= \mathrm{Rc}(\nu, \tilde{\nabla}\phi)
\end{aligned}
$$

Plug in these two terms we get Equation (32). □

We aim to find the evolution equation of $\int(\lambda_1-\lambda_2)^2 d\mu$. We first have the evolution equation of the mean curvature:

$$\frac{\partial H}{\partial t} = \tilde{\Delta}\phi + \phi(|h|^2 + R_{00}). \tag{3.9}$$

It's easy to see that:

$$
\begin{aligned}
\frac{d}{dt}\int H^2 d\mu &= \int 2H\frac{\partial H}{\partial t}d\mu + \int H^2\frac{\partial}{\partial t}(d\mu) \\
&= \int 2H\tilde{\Delta}\phi + 2\phi H R_{00} + \phi H(2|h|^2 - H^2)d\mu.
\end{aligned}
$$

Note that

$$2|h|^2 - H^2 = 2(\lambda_1^2 + \lambda_2^2) - (\lambda_1 + \lambda_2)^2 = (\lambda_1 - \lambda_2)^2,$$

we have, by the divergence theorem:

$$\frac{d}{dt}\int H^2 d\mu = \int -2\langle\tilde{\nabla}H, \tilde{\nabla}\phi\rangle + 2\phi H R_{00} + \phi H(\lambda_1 - \lambda_2)^2 d\mu. \tag{3.10}$$

Theorem 33. *When $n = 2$ we have the evolution equation*

$$
\begin{aligned}
\frac{d}{dt}\int(\lambda_1 - \lambda_2)^2 d\mu = \int & -2\langle\tilde{\nabla}H, \tilde{\nabla}\phi\rangle - 4Rc(\nu, \tilde{\nabla}\phi) \\
& + \phi H(\lambda_1 - \lambda_2)^2 + 2\phi(R_{11} - R_{22})(\lambda_1 - \lambda_2)d\mu
\end{aligned} \tag{3.11}
$$

Proof. Notice that $(\lambda_1 - \lambda_2)^2 = H^2 - 4\det(h)$ and combining Equations (32) and (3.10) we get Equation (3.11). $\square$

3.3 Rotational symmetry out of a compact set

In this section we discuss the application in a 3-d steady soliton. We now assume $\mathcal{M}$ is a 3-d gradient steady soliton, with soliton equation:

$$R_{ab} + \nabla_a \nabla_b f = 0, 0 \le a, b \le 2.$$

Note now we are working on a soliton at a fixed time. To distinguish with the real time t, we'll adopt notation σ for the value of f from last chapter. We let $\Sigma(\sigma) = f^{-1}(\sigma)$ be the level surfaces. We already knew $\nu = -\nabla f/|\nabla f|$ is an outward unit normal vector and the flow

$$\frac{\partial X}{\partial \sigma} = -\frac{1}{|\nabla f|}\nu$$

evolves the level surfaces into themselves. We have $h_{ij} = R_{ij}/|\nabla f|$, and

$$e_i(\frac{1}{|\nabla f|}) = -\frac{1}{|\nabla f|^3}\langle \nabla_i \nabla f, \nabla f\rangle = -\frac{R_{0i}}{|\nabla f|^2}.$$

Theorem 34. *On a 3-dimensional steady soliton we have*

$$\begin{aligned} \frac{d}{d\sigma}\int (\lambda_1 - \lambda_2)^2 d\mu &= \int (\frac{H}{|\nabla f|} + 2)(\lambda_1 - \lambda_2)^2 d\mu \\ &- \int \frac{2}{|\nabla f|^3}[2\lambda_1 R_{01}^2 + 2\lambda_2 R_{02}^2 + H(R_{01}^2 + R_{02}^2)] d\mu \end{aligned} \tag{3.12}$$

Proof. In Equation (3.11),

$$\phi = \frac{1}{|\nabla f|}, \tilde{\nabla}\phi = -\frac{1}{|\nabla f|^2}(R_{01}e_1 + R_{02}e_2), R_{ii} = \lambda_i |\nabla f|.$$

Taking trace of the Gauss-Codazzi equation we get:

$$\tilde{\nabla}_i H = -R_{0i} + \tilde{\nabla}_j h_{ij}$$

Furthermore,

$$\begin{aligned}
-\langle\tilde{\nabla}H,\tilde{\nabla}\phi\rangle &= -e_i(\frac{1}{|\nabla f|})e_i(H) \\
&= \frac{R_{0i}}{|\nabla f|^2}(-R_{0i}+\tilde{\nabla}_j h_{ij}) \\
&= -\frac{1}{|\nabla f|^2}(R_{01}^2+R_{02}^2)+\frac{1}{|\nabla f|^2}R_{0i}\tilde{\nabla}_j h_{ij}
\end{aligned}$$

Since $h_{ij} = R_{ij}/|\nabla f|$, we have

$$\tilde{\nabla}_j h_{ij} = \frac{1}{|\nabla f|}\tilde{\nabla}_j R_{ij} + e_j(\frac{1}{|\nabla f|})R_{ij}$$

By $\tilde{\nabla}_i e_j = \nabla_i e_j + h_{ij}\nu$ we get

$$\begin{aligned}
\tilde{\nabla}_j R_{ij} &= e_j(R_{ij}) - Rc(\tilde{\nabla}_j e_i, e_j) - Rc(e_i, \tilde{\nabla}_j e_j) \\
&= e_j(R_{ij}) - Rc(\nabla_j e_i + h_{ij}\nu, e_j) - Rc(e_i, \nabla_j e_j + H\nu) \\
&= \nabla_j R_{ij} - h_{ij}R_{0j} - HR_{0i} \\
&= \frac{1}{2}\nabla_i R - h_{ij}R_{0j} - HR_{0i}
\end{aligned}$$

Plug everything into Equation (3.11) we get Equation (3.12). □

Corollary 35. *For any $\delta > 0$ there is a compact set out of which we have*

$$\frac{d}{d\sigma}e^{-(2+\delta)\sigma}\int_{\Sigma(\sigma)}(\lambda_1-\lambda_2)^2\,d\mu \le 0. \tag{3.13}$$

Proof. Since $0 \le H/|\nabla f| \le R/|\nabla f| \to 0$ as $\sigma \to -\infty$, there is a compact set such that out of this set $H/|\nabla f| \le \delta$. Since the Ricci curvature is nonnegative, by Equation (3.12) we have

$$\frac{d}{d\sigma}\int(\lambda_1-\lambda_2)^2 d\mu \le (2+\delta)\int(\lambda_1-\lambda_2)^2 d\mu$$

or

$$\frac{d}{d\sigma}e^{-(2+\delta)\sigma}\int(\lambda_1-\lambda_2)^2 d\mu \le 0.$$

□

Now we introduce a definition to describe the shrinking rate of a surface converging to a sphere.

Definition 36. *Assume $\Sigma(t)$ is evolving by Equation (3.1). We call a surface $\Sigma(t)$ converges to a sphere of degree α if α is the biggest nonnegative number such that there is a constant such that*

$$\int_{\Sigma(t)} (\lambda_1 - \lambda_2)^2 d\mu \leq Ce^{-\alpha t}, \quad \text{if} \quad t \to \infty$$

or

$$\int_{\Sigma(t)} (\lambda_1 - \lambda_2)^2 d\mu \leq Ce^{\alpha t}, \quad \text{if} \quad t \to -\infty$$

Note by our definition, the higher is the degree, the faster the surface shrinks to a sphere. A family of distance spheres, as we considered in Theorem (39) has a convergence degree of $+\infty$.

Now we give a nontrivial example: we consider a modified inverse mean curvature flow:

$$\frac{\partial X}{\partial t} = \frac{\beta}{H}\nu$$

then we have

$$\frac{d}{dt}\int (\lambda_1 - \lambda_2)^2 d\mu = -\int \frac{2\beta|\tilde{\nabla} H|^2}{H^2} + \beta(\lambda_1 - \lambda_2)^2 d\mu.$$

So we have

$$\int (\lambda_1 - \lambda_2)^2 d\mu \leq Ce^{-\beta t}.$$

That is to say, the surface under the modified inverse mean curvature flow converges to a sphere of degree no less than β.

Corollary 37. *On a 3-d noncompact gradient steady soliton with nonnegative Ricci curvature, if the level surfaces of f converge to a sphere of degree > 2 when $\sigma \to -\infty$. Then the level surfaces are indeed spheres out of a compact set, which depends only on the degree.*

Proof. By the monotonicity formula (3.13) we have

$$e^{-(2+\delta)\sigma}\int(\lambda_1-\lambda_2)^2 d\mu \leq \lim_{\sigma\to-\infty} e^{-(2+\delta)\sigma}\int(\lambda_1-\lambda_2)^2 d\mu$$
$$\leq \lim_{\sigma\to-\infty} Ce^{\delta\sigma}=0$$

by our decay assumption on the level surfaces, we get $\int(\lambda_1-\lambda_2)^2 d\mu\equiv 0$, or $\lambda_1\equiv\lambda_2$ out of the compact set. This completes the proof. □

Remark 38. *There is a claim by Perelman (see Remark 11.9 of [18]): Any 3-dimensional steady gradient Ricci soliton with bounded positive sectional curvature and which is κ-noncollapsed on all scales must be rotationally symmetric.*

People have been trying to prove this conjecture since Perelman stated it in his seminal paper [18] in 2002. While we are still unable to confirm Perelman's claim completely, we get under one additional assumption the soliton is rotationally symmetric outside of a compact set.

3.4 Other applications

In this section we discuss a few applications of our formulas.

Example 1. Assume $M=\mathbb{R}^3$, and the surface evolves by the inverse mean curvature flow, meaning that $\phi=-1/H$. We have

$$\frac{d}{dt}\int(\lambda_1-\lambda_2)^2 d\mu=-\int\frac{2|\tilde{\nabla}H|^2}{H^2}+(\lambda_1-\lambda_2)^2 d\mu.$$

Consequently we have

$$\int(\lambda_1-\lambda_2)^2 d\mu\leq Ce^{-t}.$$

This shows the surface turns to round "exponentially fast" in the above sense.

Example 2. As an easy application of our Equation (3.3), we consider the classical case when the manifold is flat, namely when $M=\mathcal{R}^n$. In this case, the curvature is 0 we get under any flow, $\int\det(h)d\mu$ is a constant. Especially for any star shaped domain, on the boundary Σ, we have $\int_\Sigma \det(h)d\mu = C_n$, where C_n depends only on the dimension and is evaluated on a unit sphere, since such a hypersurface can evolve into a round sphere along certain flow.

Example 3. As another application of the evolution equation for $\det(h)$ we give a proof of the following theorem, which is originally due to Elerath [8].

Theorem 39. *Let M be a 3-dimensional noncompact complete manifold with non-negative sectional curvature, and with a point P at which the sectional curvature is strictly positive. Then M is not flat out of any compact set.*

Proof. By the Soul conjecture, we know the soul of M is a point. Assume Q is the soul, there is a diffeomorphism $T_Q(M) \to M$. Let $S(t)$ denote the distance sphere centered at Q with radius t, $0 < t < \infty$. The distance sphere is convex with $h_{ij} > 0$. Then $S(t)$ evolves by

$$\frac{\partial X}{\partial t} = \nu.$$

The existence of the diffeomorphism guarantees the solution exists for all time. Applying $\phi = -1$ in Equation (3.3) we get

$$\frac{d}{dt}\int_{S(t)} \det(h)d\mu = -\int_{S(t)} \det(h)h^{ij}R_{0ij0}d\mu \leq 0.$$

Let $K(p,t)$ be the sectional curvature of the tangent plane of $S(t)$ at point p, and $\mathcal{K}(p,t)$ be the intrinsic curvature of $S(t)$, then $K = \mathcal{K} - \det(h)$. By the Gauss-Bonnet theorem we have

$$\int_{S(t)} K d\mu = 4\pi - \int_{S(t)} \det(h)d\mu.$$

Consequently we have

$$\frac{d}{dt}\int_{S(t)} K d\mu \geq 0.$$

Let t_0 denote the distance from P to Q, which can be 0 if these two points are the same. Since at P all the sectional curvatures are positive, so when $t > t_0$ we have $\int_{S(t)} K d\mu > 0$ by the monotonicity formula. This shows that outside of any compact set, there must be a point at which the sectional curvature is strictly positive. $\square$

Chapter 4

The $\mathcal{L}$-minimal submanifold

4.1 Motivation

In this section we show the background calculations suggesting us how to define a new energy in the space-time setting. We only work on the 2-dimensional case, which will be enough to show the ideas.

Assume for every $\tau \in [a, b]$,

$$\gamma_\tau : [c(\tau), d(\tau)] \to \mathcal{M}^m$$

is a curve. Then $\gamma_\tau(\theta)$ is a family of 1-parameter curves. The metric g of $\mathcal{M}$ evolves by the backward Ricci flow: $\frac{\partial}{\partial\tau} g = 2\,\mathrm{Rc}$.

We define

$$F : \Sigma \to \mathcal{M} \times [a, b]$$

$$(\theta, \tau) \mapsto (\gamma_\tau(\theta), \tau),$$

where $\Sigma = \{(\theta, \tau) : \tau \in [a, b], \theta \in [c(\tau), d(\tau)]\}$, and $\mathcal{M} \times [a, b]$ is endowed with the metric $\tilde{g} = g \oplus (R + \frac{N}{2\tau})d\tau^2$.

The area of F is given by

$$\tilde{\mathcal{A}} = \int_a^b \int_{c(\tau)}^{d(\tau)} \sqrt{\det(F^*\tilde{g})} d\theta d\tau,$$

where $\det(F^*\tilde{g})$, depending on the choice of θ, is the determinant of $F^*\tilde{g}$. Notice that $\frac{\partial F}{\partial \theta} = (\frac{\partial \gamma}{\partial \theta}, 0)$ and $\frac{\partial F}{\partial \tau} = (\frac{\partial \gamma}{\partial \tau}, 1)$, we get

$$\tilde{g}_{\theta\theta} = |\frac{\partial \gamma}{\partial \theta}|^2, \tilde{g}_{\theta\tau} = \langle \frac{\partial \gamma}{\partial \theta}, \frac{\partial \gamma}{\partial \tau} \rangle, \tilde{g}_{\tau\tau} = |\frac{\partial \gamma}{\partial \tau}|^2 + (R + \frac{N}{2\tau}),$$

and

$$\det(F^*\tilde{g}) = |\frac{\partial \gamma}{\partial \tau}|^2 |\frac{\partial \gamma}{\partial \theta}|^2 + (R + \frac{N}{2\tau})|\frac{\partial \gamma}{\partial \theta}|^2 - \langle \frac{\partial \gamma}{\partial \theta}, \frac{\partial \gamma}{\partial \tau} \rangle^2.$$

So the area is

$$\tilde{\mathcal{A}} = \iint_\Sigma \sqrt{|\frac{\partial \gamma}{\partial \tau}|^2 |\frac{\partial \gamma}{\partial \theta}|^2 - \langle \frac{\partial \gamma}{\partial \theta}, \frac{\partial \gamma}{\partial \tau} \rangle^2 + (R + \frac{N}{2\tau})|\frac{\partial \gamma}{\partial \theta}|^2} d\theta d\tau.$$

We expand this expression in powers of N to get:

$$\begin{aligned} \tilde{\mathcal{A}} &= \sqrt{N} \iint_\Sigma \frac{1}{\sqrt{2\tau}} |\frac{\partial \gamma}{\partial \theta}| d\theta d\tau \\ &+ \frac{1}{\sqrt{2N}} \iint_\Sigma \sqrt{\tau} (|\frac{\partial \gamma}{\partial \tau}|^2 |\frac{\partial \gamma}{\partial \theta}| - \frac{1}{|\frac{\partial \gamma}{\partial \theta}|} \langle \frac{\partial \gamma}{\partial \theta}, \frac{\partial \gamma}{\partial \tau} \rangle^2 + R|\frac{\partial \gamma}{\partial \theta}|) d\theta d\tau \\ &+ O(N^{-3/2}). \end{aligned}$$

We consider the highest order (in N) nontrivial term and define the $\mathcal{L} - area$:

$$\mathcal{L}\mathcal{A} = \int_a^b \int_{c(\tau)}^{d(\tau)} \sqrt{\tau} (|\frac{\partial \gamma}{\partial \tau}|^2 |\frac{\partial \gamma}{\partial \theta}| - \frac{1}{|\frac{\partial \gamma}{\partial \theta}|} \langle \frac{\partial \gamma}{\partial \theta}, \frac{\partial \gamma}{\partial \tau} \rangle^2 + R|\frac{\partial \gamma}{\partial \theta}|) d\theta d\tau. \quad (4.1)$$

Note:The definition is independent of the choice of θ. If we choose θ to be the arclength, the $\mathcal{L} - area$ then can be written as:

$$\mathcal{L}\mathcal{A} = \int_a^b \int_0^{L(\tau)} \sqrt{\tau} (|\frac{\partial \gamma}{\partial \tau}|^2 - \langle \frac{\partial \gamma}{\partial \theta}, \frac{\partial \gamma}{\partial \tau} \rangle^2 + R) d\theta d\tau,$$

where $L(\tau) = \int_{c(\tau)}^{d(\tau)} |\frac{\partial \gamma}{\partial \theta}| d\theta$ is the length of γ_τ. We also note that the perpendicular projection $(\frac{\partial \gamma}{\partial \tau})^\perp$ of $\frac{\partial \gamma}{\partial \tau}$ is $\frac{\partial \gamma}{\partial \tau} - \langle \frac{\partial \gamma}{\partial \tau}, \frac{\partial \gamma}{\partial \theta} \rangle \frac{\partial \gamma}{\partial \theta}$. The $\mathcal{L} - area$ can be written again as:

$$\mathcal{L}\mathcal{A} = \int_a^b \int_0^{L(\tau)} \sqrt{\tau} (|(\frac{\partial \gamma}{\partial \tau})^\perp|^2 + R) d\theta d\tau. \quad (4.2)$$

4.2 Definition and the first variation

In general, assume

$$F : \Sigma^n \times [0, T] \to \mathcal{M}^m$$

is a smooth map and for any $\tau \in [0, T]$ $F_\tau(\cdot) \doteqdot F(\cdot, \tau)$ is an embedding. Further assume $\{u^1, \cdots, u^n\}$ is a coordinate system on Σ^n. Let $U_i \doteqdot \frac{\partial F}{\partial u^i}, 1 \leq i \leq n$. Then $\{U_i\}$ is a base of $T(F_\tau(\Sigma^n))$. For any vector field $V \in T(M)$, let

$$V^\top \doteqdot g^{ij}\langle V, U_i\rangle U_j,$$

and

$$V^\perp \doteqdot V - V^\top.$$

The metric $g(\tau)$ on $\mathcal{M}$ satisfies the backward Ricci flow:

$$\frac{\partial}{\partial \tau} g = 2\,\mathrm{Rc}\,. \tag{4.3}$$

Definition 40. *The $\mathcal{L}$-volume of Σ^n is defined to be*

$$\mathcal{V} \doteqdot \int_0^T \int_\Sigma \sqrt{\tau}(|(\frac{\partial F}{\partial \tau})^\perp|^2 + R) d\mu_{\Sigma(\tau)} d\tau, \tag{4.4}$$

where $d\mu_{\Sigma(\tau)}$ is the volume element of $F_\tau(\Sigma)$ and we'll use a short notation $d\mu$ without causing confusion.

Theorem 41. *Assume the variation field is W, then the first variation formula of the $\mathcal{L}$-volume is*

$$\begin{aligned} \delta_W \mathcal{V} = {} & 2 \int_\Sigma \sqrt{\tau}\langle W, X^\perp\rangle d\mu]_0^T \\ & + 2\int_0^T \int_{\partial\Sigma} \langle \nu_{\partial\Sigma}, \sqrt{\tau}\langle W, X^\perp\rangle X^\top - \frac{\sqrt{\tau}(|X^\perp|^2 + R)}{2} W^\top\rangle d\mu_{\partial\Sigma} d\tau \\ & + \int_0^T \int_\Sigma \langle W, -\frac{X^\perp}{\sqrt{\tau}} - 2\sqrt{\tau} g^{ij} R_{ij} X^\perp + 2\sqrt{\tau}(\operatorname{div} X^\top) X^\perp \\ & \qquad - 4\sqrt{\tau}\,\mathrm{Rc}(X^\perp, \cdot) - 2\sqrt{\tau}\nabla_{X^\perp} X^\perp + \sqrt{\tau}\nabla R \\ & \qquad - \sqrt{\tau}\nabla^\Sigma(|X^\perp|^2 + R) - \sqrt{\tau}(|X^\perp|^2 + R)\vec{H}\rangle d\mu d\tau, \end{aligned} \tag{4.5}$$

where $X \doteqdot \frac{\partial F}{\partial \tau}$, $\nu_{\partial\Sigma}$ *is the outward normal vector of* $\partial\Sigma$ *in* Σ, $d\mu_{\partial\Sigma}$ *is the induced volume element of* $\partial\Sigma$, div *is the divergence on* $F_\tau(\Sigma)$, ∇^Σ *is the covariant derivative on* $F_\tau(\Sigma)$, *and* $\vec{H}$ *is the mean curvature vector of* $F_\tau(\Sigma)$.

Proof. Let

$$\tilde{F} : \Sigma^n \times [0, T] \times (-\varepsilon, \varepsilon) \to \mathcal{M}^m$$

$$(p, \tau, s) \mapsto \tilde{F}_s(p, \tau)$$

be a variation of F with $\tilde{F}_0 = F$. We have the following notations:

$$W \doteqdot \frac{\partial \tilde{F}}{\partial s}|_{s=0}, \tilde{U}_i \doteqdot \frac{\partial \tilde{F}}{\partial u^i}, \tilde{X} \doteqdot \frac{\partial \tilde{F}}{\partial \tau},$$

and $d\tilde{\mu}$ denote the volume element of $\tilde{F}_{\tau,s}(\Sigma)$.

The $\mathcal{L}$-volume of $\tilde{F}$ is

$$\mathcal{V}(s) = \int_0^T \int_\Sigma \sqrt{\tau}(|\tilde{X}^\perp|^2 + \tilde{R}) d\tilde{\mu} d\tau.$$

The first variation is

$$\begin{aligned} \delta_W \mathcal{V} = \frac{d}{ds}|_{s=0} \mathcal{V}(s) &= \int_0^T \int_\Sigma \sqrt{\tau}(\frac{d}{ds}|\tilde{X}^\perp|^2) d\mu d\tau \\ &+ \int_0^T \int_\Sigma \sqrt{\tau}(\frac{d}{ds}\tilde{R}) d\mu d\tau \\ &+ \int_0^T \int_\Sigma \sqrt{\tau}(|X^\perp|^2 + R)\frac{d}{ds}(d\tilde{\mu}) d\tau \\ &\doteqdot A + B + C \end{aligned} \tag{4.6}$$

Now we calculate the three terms A, B and C. We first note that the easiest term

$$B = \int_0^T \int_\Sigma \sqrt{\tau}(\frac{d}{ds}\tilde{R}) d\mu d\tau = \int_0^T \int_\Sigma \langle W, \sqrt{\tau}\nabla R \rangle d\mu d\tau. \tag{4.7}$$

Now we compute A and C.

step 1. Calculating A.

Since the integrand of A

$$\begin{aligned}
\frac{d}{ds}|\tilde{X}^{\perp}|^2 &= 2\langle \nabla_W \tilde{X}^{\perp}, \tilde{X}^{\perp}\rangle \\
&= 2\langle \nabla_W (\tilde{X} - \tilde{g}^{ij}\langle \tilde{X}, \tilde{U}_i\rangle \tilde{U}_j), \tilde{X}^{\perp}\rangle \\
&= 2\langle \nabla_W \tilde{X}, X^{\perp}\rangle - 2g^{ij}\langle X, U_i\rangle\langle \nabla_W \tilde{U}_j, X^{\perp}\rangle \\
&= 2\langle \nabla_X W, X^{\perp}\rangle - 2g^{ij}\langle X, U_i\rangle\langle \nabla_{U_j} W, X^{\perp}\rangle \\
&= 2\frac{d}{d\tau}\langle W, X^{\perp}\rangle - 4\,\mathrm{Rc}(W, X^{\perp}) - 2\langle W, \nabla_X X^{\perp}\rangle \\
&\quad - 2X^{\top}(\langle W, X^{\perp}\rangle) + 2\langle W, \nabla_{X^{\top}} X^{\perp}\rangle \\
&= 2\frac{d}{d\tau}\langle W, X^{\perp}\rangle - 2X^{\top}(\langle W, X^{\perp}\rangle) - 4\,\mathrm{Rc}(W, X^{\perp}) - 2\langle W, \nabla_{X^{\perp}} X^{\perp}\rangle,
\end{aligned}$$

we have

$$\begin{aligned}
A &= \int_0^T \int_\Sigma 2\sqrt{\tau}\frac{d}{d\tau}\langle W, X^{\perp}\rangle d\mu d\tau + \int_0^T \int_\Sigma -2\sqrt{\tau} X^{\top}(\langle W, X^{\perp}\rangle) d\mu d\tau \\
&\quad + \int_0^T \int_\Sigma \sqrt{\tau}(-4\,\mathrm{Rc}(W, X^{\perp}) - 2\langle W, \nabla_{X^{\perp}} X^{\perp}\rangle) d\mu d\tau \\
&\doteqdot A_1 + A_2 + A_3
\end{aligned} \tag{4.8}$$

Now we do integration by parts to get

$$\begin{aligned}
A_1 &= \int_0^T \int_\Sigma 2\sqrt{\tau}\frac{d}{d\tau}\langle W, X^{\perp}\rangle d\mu d\tau \\
&= 2\int_0^T \{\frac{d}{d\tau}(\int_\Sigma \sqrt{\tau}\langle W, X^{\perp}\rangle d\mu) \\
&\qquad - \frac{1}{2\sqrt{\tau}}\int_\Sigma \langle W, X^{\perp}\rangle d\mu - \int_\Sigma \sqrt{\tau}\langle W, X^{\perp}\rangle \frac{d}{d\tau}(d\mu)\} d\tau \\
&= 2\int_\Sigma \sqrt{\tau}\langle W, X^{\perp}\rangle d\mu]_0^T \\
&\qquad - \frac{1}{\sqrt{\tau}}\int_\Sigma \langle W, X^{\perp}\rangle d\mu - 2\int_\Sigma \sqrt{\tau}\langle W, X^{\perp}\rangle (g^{ij}R_{ij}) d\mu d\tau.
\end{aligned}$$

Moreover

$$
\begin{aligned}
A_2 &= \int_0^T \int_\Sigma -2\sqrt{\tau} X^\top(\langle W, X^\perp\rangle) d\mu d\tau \\
&= \int_0^T (-2\sqrt{\tau}) \int_\Sigma \{\operatorname{div}(\langle W, X^\perp\rangle X^\top) - \langle W, X^\perp\rangle \operatorname{div} X^\top\} d\mu d\tau \\
&= 2\int_0^T \sqrt{\tau} \int_{\partial\Sigma} \langle \mu_{\partial\Sigma}, X^\top\rangle \langle W, X^\perp\rangle d\mu_{\partial\Sigma} d\tau \\
&\quad + 2\int_0^T \sqrt{\tau} \int_\Sigma \langle W, \operatorname{div} X^\top \cdot X^\perp\rangle d\mu d\tau.
\end{aligned}
$$

Plug A_1 and A_2 into the expression of A we get

$$
\begin{aligned}
A = {} & 2\int_\Sigma \sqrt{\tau}\langle W, X^\perp\rangle d\mu]_0^T + 2\int_0^T \int_{\partial\Sigma} \langle \nu_{\partial\Sigma}, \sqrt{\tau}\langle W, X^\perp\rangle X^\top\rangle d\mu_{\partial\Sigma} d\tau \\
& + \int_0^T \int_\Sigma \langle W, -\frac{X^\perp}{\sqrt{\tau}} - 2\sqrt{\tau} g^{ij} R_{ij} X^\perp + 2\sqrt{\tau} \operatorname{div} X^\top \cdot X^\perp \\
& \qquad - 4\sqrt{\tau}\operatorname{Rc}(X^\perp, \cdot) - 2\sqrt{\tau}\nabla_{X^\perp} X^\perp\rangle d\mu d\tau. \qquad (4.9)
\end{aligned}
$$

step 2. Calculating C.

Since

$$
\frac{d}{ds}(d\tilde{\mu}) = (\operatorname{div} W^\top - \langle \vec{H}, W\rangle) d\mu
$$

we have

$$
\begin{aligned}
C &= \int_0^T \sqrt{\tau} \int_\Sigma (|X^\perp|^2 + R) \frac{d}{ds}(d\tilde{\mu}) d\tau \\
&= \int_0^T \sqrt{\tau} \int_\Sigma (|X^\perp|^2 + R)(\operatorname{div} W^\top - \langle \vec{H}, W\rangle) d\mu d\tau \\
&= \int_0^T \sqrt{\tau} \int_\Sigma \operatorname{div}((|X^\perp|^2 + R) W^\top) - \langle W^\top, \nabla^\Sigma(|X^\perp|^2 + R)\rangle \\
&\qquad - \langle W, (|X^\perp|^2 + R)\vec{H}\rangle d\mu d\tau \\
&= -\int_0^T \sqrt{\tau} \int_{\partial\Sigma} \langle \nu_{\partial\Sigma}, (|X^\perp|^2 + R) W^\top\rangle d\mu_{\partial\Sigma} d\tau \\
&\quad + \int_0^T \int_\Sigma \langle W, -\sqrt{\tau}\nabla^\Sigma(|X^\perp|^2 + R) - \sqrt{\tau}(|X^\perp|^2 + R)\vec{H}\rangle d\mu d\tau. \qquad (4.10)
\end{aligned}
$$

Now, putting Equations (4.7), (4.9) and (4.10) together we get Equation (4.5). □

Corollary 42. *If we fix the variation on the boundaries, that is* $W|_{\partial\Sigma} = 0$, $W|_{\tau=0} = W|_{\tau=T} = 0$. *Further assume* $\frac{\partial F}{\partial \tau}$ *is perpendicular to* Σ, *then the first variation formula is*

$$\delta_W \mathcal{V} = \int_0^T \int_\Sigma \langle W, -\frac{X}{\sqrt{\tau}} - 2\sqrt{\tau} g^{ij} R_{ij} X - 4\sqrt{\tau}\,\mathrm{Rc}(X, \cdot) - 2\sqrt{\tau}\nabla_X X + \sqrt{\tau}\nabla R - \sqrt{\tau}\nabla^\Sigma(|X|^2 + R) - \sqrt{\tau}(|X|^2 + R)\vec{H}\rangle d\mu d\tau \tag{4.11}$$

Since in most cases we are interested in the case where the flow of Σ is perpendicular to itself, we introduce our definition of $\mathcal{L}$-minimal submanifold as follows:

Definition 43. *A pair* (Σ, F_τ) *is called an* $\mathcal{L}$*-minimal submanifold if it satisfies*

$$\nabla_X X + 2\,\mathrm{Rc}(X, \cdot) + \frac{X}{2\tau} - \frac{1}{2}\nabla R + g^{ij} R_{ij} X + \frac{1}{2}\nabla^\Sigma(|X|^2 + R) + \frac{1}{2}(|X|^2 + R)\vec{H} = 0. \tag{4.12}$$

Remark 44. *If we let* Σ *be a point, our Equation (4.12) is exactly Perelman's* $\mathcal{L}$*-geodesic equation since the last three terms vanish.*

4.3 Example

Example 45. *If* $\mathcal{M} = \mathbb{R}^m$ *then the* $\mathcal{L}$*-minimal submanifold equation becomes*

$$\nabla_X X + \frac{X}{2\tau} + \frac{1}{2}\nabla^\Sigma(|X|^2) + \frac{1}{2}|X|^2 \vec{H} = 0. \tag{4.13}$$

We want to find a rotationally symmetric solution. Let $r(\tau)$ *be the radius. Then*

$$X = r'\frac{\partial}{\partial r}, \vec{H} = \frac{m-1}{r}\frac{\partial}{\partial r}.$$

Plug into Equation (4.13) we get

$$2\tau r r'' + (m-1)\tau r'^2 + r r' = 0,$$

and we know

$$r(\tau) = \tau^{\frac{1}{1+m}}$$

is a solution.

Chapter 5

The mean curvature flow inside the Ricci flow

The mean curvature flow and the Ricci flow are two very important geometric flows. There is one question may sound crazy at first: what would happen if we combine them together? In this chapter, we discuss this problem. If a hypersurface evolves along the mean curvature flow, and meanwhile the ambient manifold evolves along the Ricci flow, we shall see the evolution equation of the second fundamental form becomes simpler. Many "bad" terms are canceled mysteriously due to the evolution of the ambient space.

Let $\mathcal{M}^n$ and $\mathcal{N}^{n+1}$ be two smooth Riemannian manifolds, and

$$F_t : \mathcal{M} \hookrightarrow \mathcal{N}$$

be a one parameter family of smooth immersions. The metric $\bar{g}$ of $\mathcal{N}$ satisfies the Ricci flow equation:

$$\frac{\partial \bar{g}}{\partial t} = -2\bar{\mathrm{R}}\mathrm{c}. \tag{5.1}$$

If we mean the metric, connection or curvature on $\mathcal{N}$, this will be indicated by a bar. The metric g of $\mathcal{M}$ is the induced metric, $g = F^*\bar{g}$. For any point $p \in \mathcal{M}$, any vectors $X, Y \in T_p(\mathcal{M})$, we have $F_*X, F_*Y \in T_{F(p)}(F(\mathcal{M})) \subset T_{F(p)}\mathcal{N}$. The second fundamental form $\vec{h} : T_p\mathcal{M} \otimes T_p\mathcal{M} \to T_{F(p)}\mathcal{N}$ is defined by

$$\vec{h}(X, Y) \doteqdot \vec{h}(F_*X, F_*Y) = (\bar{\nabla}_{F_*X} F_*Y)^\perp,$$

where $\perp$ indicates the component in the normal bundle of $T_{F(p)}(F(\mathcal{M}))$. The mean curvature $\vec{H}$ is the trace of $\vec{h}$. The vector field $\frac{\partial F}{\partial t}$ on $\mathcal{N}$ is understood by the following: for any $f \in C^\infty(\mathcal{N})$,

$$\frac{\partial F}{\partial t}(f) = \frac{d(f \circ F)}{dt}.$$

The family of the immersions satisfies the mean curvature flow equation:

$$\frac{\partial F}{\partial t} = \vec{H}. \tag{5.2}$$

In the following Latin indices range from 1 to n, Greek indices range from 0 to n.

Assume $\{x^i\}$ is a local coordinate system of $\mathcal{M}$, then $\frac{\partial F}{\partial x^i} \doteqdot F_*(\frac{\partial}{\partial x^i})$ are tangent to $T_{F(p)}(F(\mathcal{M}))$. Assume the normal vector at $F(p)$ is ν, then $\{\nu, \frac{\partial F}{\partial x^1}, \cdots, \frac{\partial F}{\partial x^n}\}$ span $T_{F(p)}\mathcal{N}$. As usual,

$$\vec{h}(X, Y) \doteqdot -h(X, Y)\nu,$$
$$H \doteqdot g^{ij} h_{ij} \doteqdot g^{ij} h(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}).$$

At $F(p)$, we also assign a local coordinate $\{y^\alpha\}$, such that $\frac{\partial}{\partial y^0} = \nu$ and $\frac{\partial}{\partial y^i} = \frac{\partial F}{\partial x^i}$.

Lemma 46. *The metric g of $\mathcal{M}$ satisfies the evolution equation:*

$$\frac{\partial g_{ij}}{\partial t} = -2\bar{R}_{ij} - 2Hh_{ij}. \tag{5.3}$$

Proof.

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial t} &= \frac{\partial F^*\bar{g}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)}{\partial t} \\
&= \frac{\partial}{\partial t}\left(\bar{g}\left(\frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\right)\right) \\
&= \frac{\partial \bar{g}}{\partial t}\left(\frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\right) + \bar{g}\left(\bar{\nabla}_{\frac{\partial F}{\partial t}} \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\right) + \bar{g}\left(\frac{\partial F}{\partial x^i}, \bar{\nabla}_{\frac{\partial F}{\partial t}} \frac{\partial F}{\partial x^j}\right) \\
&= -2\bar{R}_{ij} + \bar{g}\left(\bar{\nabla}_{\frac{\partial F}{\partial x^i}} \frac{\partial F}{\partial t}, \frac{\partial F}{\partial x^j}\right) + \bar{g}\left(\frac{\partial F}{\partial x^i}, \bar{\nabla}_{\frac{\partial F}{\partial x^j}} \frac{\partial F}{\partial t}\right)
\end{aligned}$$

Notice that

$$\bar{g}\left(\bar{\nabla}_{\frac{\partial F}{\partial x^i}}\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x^j}\right) = \bar{g}\left(\bar{\nabla}_{\frac{\partial F}{\partial x^i}}(-H\nu), \frac{\partial F}{\partial x^j}\right) = -Hh_{ij},$$

and the lemma follows.

□

Lemma 47. *The normal vector ν satisfies the evolution equation:*

$$\frac{\partial \nu}{\partial t} \doteqdot \bar{\nabla}_{\frac{\partial F}{\partial t}}\nu = \nabla H + 2g^{ij}\bar{R}_{0i}\frac{\partial F}{\partial x^j} + \bar{R}_{00}\nu. \tag{5.4}$$

Proof.

$$\begin{aligned}\bar{\nabla}_{\frac{\partial F}{\partial t}}\nu &= \bar{g}\left(\bar{\nabla}_{\frac{\partial F}{\partial t}}\nu, \frac{\partial F}{\partial x^i}\right)\bar{g}^{ij}\frac{\partial F}{\partial x^j} + \bar{g}\left(\bar{\nabla}_{\frac{\partial F}{\partial t}}\nu, \nu\right)\nu \\ &= \left(-\frac{\partial \bar{g}}{\partial t}(\nu, \frac{\partial F}{\partial x^i}) - \bar{g}(\nu, \bar{\nabla}_{\frac{\partial F}{\partial t}}\frac{\partial F}{\partial x^i})\right)g^{ij}\frac{\partial F}{\partial x^j} - \frac{1}{2}\frac{\partial \bar{g}}{\partial t}(\nu,\nu)\nu \\ &= \left(2\bar{R}_{0i} + \frac{\partial H}{\partial x^i}\right)g^{ij}\frac{\partial F}{\partial x^j} + \bar{R}_{00}\nu.\end{aligned}$$

□

Before deriving the evolution equation of the second fundamental form, we write down a formula of the Rieman curvature along the direction of the flow.

Lemma 48.

$$\bar{R}\left(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x^i}\right)\frac{\partial F}{\partial x^j} = \bar{\nabla}_{\frac{\partial F}{\partial t}}\bar{\nabla}_{\frac{\partial F}{\partial x^i}}\frac{\partial F}{\partial x^j} - \bar{\nabla}_{\frac{\partial F}{\partial x^i}}\bar{\nabla}_{\frac{\partial F}{\partial t}}\frac{\partial F}{\partial x^j} - \frac{\partial}{\partial t}(\bar{\nabla})\left(\frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^i}\right). \tag{5.5}$$

Proof. Notice that $\bar{\nabla}$ depends on time, and we compute at a fixed time t_0

$$\bar{\nabla}_{\frac{\partial F}{\partial t}}\left(\bar{\nabla}_{\frac{\partial F}{\partial x^i}}\frac{\partial F}{\partial x^j}\right) = \bar{\nabla}_{\frac{\partial F}{\partial t}}\bar{\nabla}_{\frac{\partial F}{\partial x^i}}\frac{\partial F}{\partial x^j} + \frac{\partial}{\partial t}(\bar{\nabla})\left(\frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^i}\right).$$

Moreover we have

$$\begin{aligned}\bar{R}\left(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x^i}\right)\frac{\partial F}{\partial x^j} &= \bar{\nabla}_{\frac{\partial F}{\partial t}}\bar{\nabla}_{\frac{\partial F}{\partial x^i}}\frac{\partial F}{\partial x^j} - \bar{\nabla}_{\frac{\partial F}{\partial x^i}}\bar{\nabla}_{\frac{\partial F}{\partial t}}\frac{\partial F}{\partial x^j} \\ &= \bar{\nabla}_{\frac{\partial F}{\partial t}}\left(\bar{\nabla}_{\frac{\partial F}{\partial x^i}}\frac{\partial F}{\partial x^j}\right) - \frac{\partial}{\partial t}(\bar{\nabla})\left(\frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^i}\right) - \bar{\nabla}_{\frac{\partial F}{\partial x^i}}\bar{\nabla}_{\frac{\partial F}{\partial t}}\frac{\partial F}{\partial x^j}.\end{aligned}$$

Plug the first equation into the second one and we get Equation (5.5). □

Remark 49. *The curvature formula differs from the standard one since the ambient space is evolving along the Ricci flow. We can also prove this formula in local coordinates. To see this, we assume at $F(p)$,*

$$\bar{\nabla}_{\frac{\partial}{\partial y^\alpha}} \frac{\partial}{\partial y^\beta} = \bar{\Gamma}^\gamma_{\alpha\beta} \left(F_t(p), t\right) \frac{\partial}{\partial y^\gamma}.$$

Then we have

$$\begin{aligned}\bar{\nabla}_{\frac{\partial F}{\partial t}} \bar{\nabla}_{\frac{\partial F}{\partial x^i}} \frac{\partial F}{\partial x^j} &= \bar{\nabla}_{\frac{\partial F}{\partial t}} \left(\bar{\Gamma}^\gamma_{ij} \left(F_t\left(p\right), t\right) \frac{\partial}{\partial y^\gamma}\right) \\ &= \left(\langle \bar{\nabla} \bar{\Gamma}^\gamma_{ij}, \frac{\partial F}{\partial t}\rangle + \frac{\partial}{\partial t}(\bar{\Gamma}^\gamma_{ij})\right) \frac{\partial}{\partial y^\gamma} + \bar{\Gamma}^\gamma_{ij} \bar{\nabla}_{\frac{\partial F}{\partial t}} \frac{\partial}{\partial y^\gamma}.\end{aligned}$$

Now we are ready to derive the evolution equation of the second fundamental form.

Lemma 50.

$$\frac{\partial h_{ij}}{\partial t} = \nabla_i \nabla_j H - H g^{kl} h_{ik} h_{jl} + H R_{0ij0} - \bar{R}_{00} h_{ij} + \bar{\nabla}_i \bar{R}_{0j} + \bar{\nabla}_j \bar{R}_{0i} - \bar{\nabla}_0 \bar{R}_{ij} \quad (5.6)$$

Proof.

$$\begin{aligned}\frac{\partial h_{ij}}{\partial t} &= \frac{\partial}{\partial t}\left(-\langle \bar{\nabla}_{\frac{\partial F}{\partial x^i}} \frac{\partial F}{\partial x^j}, \nu\rangle\right) \\ &= 2\,\bar{\mathrm{R}}\mathrm{c}\left(\bar{\nabla}_{\frac{\partial F}{\partial x^i}} \frac{\partial F}{\partial x^j}, \nu\right) - \langle \bar{\nabla}_{\frac{\partial F}{\partial t}} \bar{\nabla}_{\frac{\partial F}{\partial x^i}} \frac{\partial F}{\partial x^j}, \nu\rangle - \langle \bar{\nabla}_{\frac{\partial F}{\partial x^i}} \frac{\partial F}{\partial x^j}, \bar{\nabla}_{\frac{\partial F}{\partial t}} \nu\rangle \\ &\doteqdot A - B - C,\end{aligned}$$

where

$$A = 2\bar{\Gamma}^0_{ij} \bar{R}_{00} + 2\bar{\Gamma}^k_{ij} \bar{R}_{0k},$$

$$\begin{aligned}B &= \langle \bar{\nabla}_{\frac{\partial F}{\partial t}} \bar{\nabla}_{\frac{\partial F}{\partial x^i}} \frac{\partial F}{\partial x^j}, \nu\rangle \\ &= -H\bar{R}_{0ij0} + \langle \bar{\nabla}_{\frac{\partial F}{\partial x^i}} \bar{\nabla}_{\frac{\partial F}{\partial t}} \frac{\partial F}{\partial x^j}, \nu\rangle + \langle \frac{\partial}{\partial t}\left(\bar{\nabla}\right)\left(\frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\right), \nu\rangle \\ &= -H\bar{R}_{0ij0} + \langle \bar{\nabla}_{\frac{\partial F}{\partial x^i}} \bar{\nabla}_{\frac{\partial F}{\partial x^j}} \left(-H\nu\right), \nu\rangle + \langle \frac{\partial}{\partial t}\left(\bar{\nabla}\right)\left(\frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\right), \nu\rangle \\ &= -H\bar{R}_{0ij0} - \frac{\partial^2 H}{\partial x^i \partial x^j} + H g^{kl} h_{ik} h_{jl} - \bar{\nabla}_i \bar{R}_{0j} - \bar{\nabla}_j \bar{R}_{0i} + \bar{\nabla}_0 \bar{R}_{ij}\end{aligned}$$

and

$$C = \langle \bar{\Gamma}_{ij}^{0}\nu + \bar{\Gamma}_{ij}^{k}\frac{\partial F}{\partial x^k}, \nabla H + 2g^{ij}\bar{R}_{0i}\frac{\partial F}{\partial x^j} + \bar{R}_{00}\nu\rangle$$
$$= \bar{\Gamma}_{ij}^{0}\bar{R}_{00} + 2\bar{\Gamma}_{ij}^{k}\bar{R}_{0k} + \bar{\Gamma}_{ij}^{k}\frac{\partial H}{\partial x^k}.$$

Notice that

$$\bar{\Gamma}_{ij}^{k} = \Gamma_{ij}^{k}, \bar{\Gamma}_{ij}^{0} = -h_{ij}, \frac{\partial^2 H}{\partial x^i \partial x^j} - \Gamma_{ij}^{k}\frac{\partial H}{\partial x^k} = \nabla_i\nabla_j H,$$

and we proved the lemma. □

There is Simons' identity (see [23]):

$$\Delta h_{ij} = \nabla_i\nabla_j H + Hg^{kl}g_{ik}g_{jl} - |h|^2 h_{ij} + H\bar{R}_{0ij0} - h_{ij}\bar{R}_{00} + g^{kl}g^{rs}h_{jl}\bar{R}_{krsi}$$
$$+ g^{kl}g^{rs}h_{il}\bar{R}_{krsj} - 2g^{kl}g^{rs}h_{kr}\bar{R}_{ilsj} + \bar{\nabla}_i\bar{R}_{0j} + \bar{\nabla}_j\bar{R}_{0i} - \bar{\nabla}_0\bar{R}_{ij} + \bar{\nabla}_0\bar{R}_{0ij0}, \quad (5.7)$$

where $|h|^2 \doteqdot g^{ij}g^{kl}h_{ik}h_{jl}$.

Combining the Simon's identity (5.7) and Equation (5.6) we get

Theorem 51.

$$\frac{\partial h_{ij}}{\partial t} = \Delta h_{ij} - 2Hg^{kl}h_{ik}h_{jl} + |h|^2 h_{ij} - g^{kl}g^{rs}h_{jl}\bar{R}_{krsi}$$
$$- g^{kl}g^{rs}h_{il}\bar{R}_{krsj} + 2g^{kl}g^{rs}h_{kr}\bar{R}_{ilsj} - \bar{\nabla}_0\bar{R}_{0ij0}. \quad (5.8)$$

Now let's compare the the evolution equations between in a fixed ambient manifold and in a manifold evolving along the Ricci flow. First we notice

Proposition 52. *In the mean curvature flow in a Riemannian manifold, the evolution equation is*

$$\frac{\partial h_{ij}}{\partial t} = \Delta h_{ij} - 2Hg^{kl}h_{ik}h_{jl} + |h|^2 h_{ij} - g^{kl}g^{rs}h_{jl}\bar{R}_{krsi} - g^{kl}g^{rs}h_{il}\bar{R}_{krsj}$$
$$+ 2g^{kl}g^{rs}h_{kr}\bar{R}_{ilsj} - \bar{\nabla}_0\bar{R}_{0ij0} - \bar{\nabla}_i\bar{R}_{0j} - \bar{\nabla}_j\bar{R}_{0i} + \bar{\nabla}_0\bar{R}_{ij} + h_{ij}g^{kl}\bar{R}_{0kl0}. \quad (5.9)$$

Proof. We carefully copy Huisken's formula here, which is Theorem (3.4) in [13]. Notice that there is a negative sign on the Rieman curvature between his notation and ours.

$$\frac{\partial h_{ij}}{\partial t} = \Delta h_{ij} - 2Hg^{kl}h_{ik}h_{jl} + |h|^2 h_{ij} - g^{kl}g^{rs}h_{jl}\bar{R}_{krsi} - g^{kl}g^{rs}h_{il}\bar{R}_{krsj} + 2g^{kl}g^{rs}h_{kr}\bar{R}_{ilsj} - g^{kl}\bar{\nabla}_j\bar{R}_{0kli} + g^{kl}\bar{\nabla}_k\bar{R}_{0ijl} + h_{ij}g^{kl}\bar{R}_{0kl0}. \tag{5.10}$$

We notice that

$$g^{kl}\bar{\nabla}_j\bar{R}_{0kli} = \bar{\nabla}_j\bar{R}_{0i},$$

and by the second Bianchi identity we get

$$\begin{aligned} g^{kl}\bar{\nabla}_k\bar{R}_{0ijl} &= -g^{kl}\left(\bar{\nabla}_0\bar{R}_{ikjl} + \bar{\nabla}_i R_{k0jl}\right) \\ &= \bar{\nabla}_0\left(\bar{R}_{ij} - \bar{R}_{0ij0}\right) - \bar{\nabla}_i\bar{R}_{0j} \\ &= \bar{\nabla}_0\bar{R}_{ij} - \bar{\nabla}_i\bar{R}_{0j} - \bar{\nabla}_0\bar{R}_{0ij0} \end{aligned}$$

Plug those two terms in Equation (5.10) we get Equation (5.9). □

Remark 53. *Comparing Equations (5.8) and (5.9), we see that the last four terms in (5.9) are canceled in (5.8) because of the evolution of the ambient manifold.*

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