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Students' Meaning Making of Algebraic Expressions and Equations through Engaging with Dialogic Mathematics Videos

A Dissertation submitted in partial satisfaction of the requirements

for the degree Doctor of Philosophy

in

Mathematics and Science Education

by

Isabel Alejandra Sorto White

Committee in charge:

University of California San Diego
Professor Chris Halter
Professor Lisa McDonnell
Professor David Quarfoot

San Diego State University
Professor Joanne Lobato, Chair
Professor William Zahner

2025

The Dissertation of Isabel White is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

Chair

University of California San Diego

San Diego State University

2025

DEDICATION

In memory of my father, whose love and encouragement sustained me throughout my life and academic journey. You saw the whole sky in me, even when I didn’t. Thank you for every song, every ride, every act of quiet love. I carry your brilliance, your humility, and your boundless care with me always. This dissertation is a tribute to your legacy, and I hope it makes you proud.

Now I’ve found thee, and the world shall know thy glory.

— Man of La Mancha.

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LIST OF ABBREVIATIONS

IO	Instrumental Orchestration
DCA	Documenting Collective Activity Method
QR	Quantitative Reasoning
RME	Realistic Mathematics Education
TE	Teaching Experiment
VL	Vicarious Learner

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VITA

2018 Bachelor of Arts in Mathematics, Rice University

2020 Master of Arts in Mathematics, San Diego State University

2025 Doctor of Philosophy in Mathematics and Science Education, University of California San Diego/San Diego State University

PUBLICATIONS

Foster, M., White, I., & Lobato, J. (in press, 2025). How do students make sense of dialogic mathematics videos? Journal of Computers in Mathematics and Science Teaching.

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White, I., Foster, M., & Lobato, J. (2023). Making sense of algebraic expressions in context. Mathematics Teacher: Learning and Teaching PK-12, 116(8). https://doi.org/10.5951/MTLT.2022.0196

Tenney, K., Stringer, B. P., LaTona-Tequida, T., & White, I. (2023). Conceptualizations and limitations of STEM literacy across learning theories. Journal of Microbiology & Biology Education, 24(1), 1-5. https://doi.org/10.1128/jmbe.00168-22

O'Neill, C., & White, I. (2021). On minimal presentations of shifted affine semigroups with few generators. Involve, A Journal of Mathematics. https://doi.org/10.2140/involve.2021.14.617

Reinholz, D. L., White, I., & Andrews, T. (2021). Change theory in STEM higher education: a systematic review. International Journal of STEM Education, 8(1), 1-22. https://doi.org/10.1186/s40594-021-00291-2

Reinholz, D. L., Stone-Johnstone, A., White, I., Sianez, L. M., & Shah, N. (2020). A Pandemic Crash Course: Learning to Teach Equitably in Synchronous Online Classes. CBE- Life Sciences Education, 19(4), 1-13. https://doi.org/10.1187/cbe.20-06-0126

ABSTRACT OF THE DISSERTATION

Students' Meaning Making of Algebraic Expressions and Equations through Engaging with Dialogic Mathematics Videos

by

Isabel Alejandra Sorto White

Doctor of Philosophy in Mathematics and Science Education

University of California San Diego, 2025
San Diego State University, 2025

Professor Joanne Lobato, Chair

Algebra continues to act as a gatekeeper, yet many students continue to struggle with foundational concepts related to algebraic expressions and equations. There is a general consensus among mathematics educators that these issues could be remedied by emphasizing meaning making over procedural fluency. Accordingly, algebra education research has

documented persistent challenges and productive efforts in supporting students’ meaning making of algebraic topics. However, scholars have yet to provide a learning trajectory for how secondary students come to understand algebraic expressions and equations in context.

Additionally, in light of the COVID-19 pandemic and astonishing technological advancements, there has been an increasing interest in instructional approaches that leverage technology as a means for supporting algebra learning. A promising yet underexplored approach is one that integrates an innovative video technology, dialogic mathematics videos. Unlike conventional mathematics videos that typically demonstrate procedures, these videos place an explicit focus on meaning making through the unscripted dialogue of two students as they collaboratively work through problems. Although emerging research points to its potential, little is known about how these videos function as instructional supports within classroom settings.

This dissertation addresses these gaps through two research aims: (a) to explore how a classroom community develops meaning of algebraic expressions and equations in context over time and (b) to identify the instructional supports that contributed to students’ development, with particular attention to those related to the use of dialogic mathematics videos. The study followed a classroom teaching experiment methodology with thirteen ninth-grade students over ten one-hour sessions. The learning environment was grounded in a quantitative reasoning approach, which emphasizes meaning making in context, and integrated dialogic mathematics videos as a central instructional tool.

To address (a), I drew on the emergent perspective and used the Documenting Collective Activity method to analyze whole-class argumentation as a means of tracking collective mathematical progress of the classroom community. The result is a five-part learning trajectory that documents the evolution of students' reasoning about the unit’s focal algebraic topics. The findings demonstrate how students' meaning making deepened as they formed connections among interrelated aspects of algebra grounded in context.

To address (b), I drew on the instrumental orchestration framework to examine how the teacher facilitated technology use in support of students' reasoning. Using a combination of a priori codes from the literature and inductive coding, I identified eleven instrumental orchestration types according to the components of the framework. Eight types seemed to support students' reasoning, while three appeared to be less productive. The findings contribute to a nuanced understanding of how dialogic videos can be used in instruction to foster meaningful algebra learning.

Chapter 1: Introduction

Algebra has long been considered a “gatekeeper” in education due to its status as a prerequisite for advanced mathematics courses, college admission, and careers in STEM, making it a key determinant of academic, economic, and life opportunities (Kaput, 1995; Larnell, 2016; Moses & Cobb, 2001). Consequently, algebra continues to be a central focus of educational research and policy (e.g., Closson, 2024; Gates, 2021, 2023; Walkington et al., 2025). Yet, challenges in algebra persist. Algebra I is the single most failed course in U.S. high schools, and in California, nearly one-third of students repeat the course during high school (Rickles et al., 2017; Finkelstein et al., 2012). Many researchers attribute these struggles to a longstanding emphasis on procedural fluency and rote-symbol manipulation rather than meaning making.

In response to these challenges, and especially in light of the COVID-19 pandemic and astonishing technological advancements, there has been an increasing interest in instructional approaches that leverage technology as a means for supporting algebra learning (Haspekian et al., 2023; NCTM, 2020). Recent innovations include instructional environments that integrate dynamic applets, interactive mathematics games, and video lessons designed to connect algebra more directly to students’ everyday experiences (e.g., Paoletti et al., 2024; Rickel et al., 2023; Ottmar et al., 2023). A promising yet underexplored instructional approach is one that integrates an innovative video technololgy, dialogic mathematics videos. Unlike conventional mathematics videos that typically demonstrate procedures, these videos place an explicit focus on meaning making through the unscripted dialogue of two students as they collaboratively work through problems. This dissertation seeks to advance knowledge on students’ meaning making of algebra and on instructional supports for students’ algebraic reasoning, with attention to those related to the use of dialogic mathematics videos.

Although algebra defies any concise definition, it is largely concerned with generalizing and expressing relationships among quantities using symbols (Kieran, 1992). Therefore, making meaning of algebraic expressions and equations is at the core of algebra learning. Algebraic expressions can consist of variables, coefficients, and constants, along with the operation signs of addition, subtraction, multiplication, division, and/or exponentiation (e.g., $2x - 5$), but do not include an equal sign (Haspekian et al., 2023). Algebraic equations consist of two algebraic expressions (or an algebraic expression and an arithmetic expression), with an equal sign between them (e.g., $7x + 5 = 2x - 3$). Algebraic expressions and equations can represent a range of mathematical meanings, including a generalization of an arithmetic pattern, a functional relation, or a general formula. The variety of forms and interpretations associated with expressions and equations brings into focus other important aspects of algebra such as equivalence, meanings of literal symbols, and generalizing arithmetic to algebra, making this topic a fruitful area of study.

Despite its importance, recent empirical studies illustrate that students continue to revert to numerical and procedural thinking over conceptual thinking across a variety of tasks related to understanding algebraic equations and expressions (e.g., Christou et al., 2022; Liang & Moore, 2021; Tondorf & Prediger, 2022; Wilkie, 2022) (see next section for an example). One explanation is that typical instruction often presents algebraic expressions and equations as devoid of context, hindering meaning making. In response, scholars have argued for grounding algebra in realistic contexts—those that are imaginable and meaningful to students (Gravemeijer, 2020). Such contexts allow learners to draw on everyday experiences and engage in informal reasoning. As students move toward more formal reasoning, these contexts can continue to act as anchors. Accordingly, this dissertation is interested in how students reason about algebraic expressions and equations in context as a means of nurturing mathematical meaning-making abilities in algebra.

In sum, this study is guided by two central aims: (a) to explore how students learn to make sense of algebraic expressions and equations in context over time, and (b) to identify the instructional supports for students' reasoning, with particular attention to those related to the use of dialogic mathematics video. The remainder of the chapter is organized into five main sections. First, I analyze the complexity of understanding algebraic equations by presenting a specific example, which will motivate four interrelated algebraic subtopics and introduce a quantitative-reasoning approach to algebra (Thompson, 2011). Second, I provide a brief overview of the relevant literature on algebra and quantitative reasoning and discuss the specific ways in which my study extends the existing literature. Third, I introduce the dialogic mathematics video as a promising means for instructional support of mathematics learning and argue that my study contributes to the field of vicarious learning. Fourth, I motivate and introduce the theoretical framing for the study. Finally, I conclude with a presentation of my research questions and a discussion of the significance of each to the field of mathematics education.

The Complexity of Making Sense of Algebraic Expressions and Equations in Context

To explore the conceptual complexity of understanding algebraic expressions and equations in context, consider a typical equation that students might encounter in a beginning algebra course, namely $5c + 3c = 8c$. Students often experience such an equation as part of the skill of combining like terms, where they are given an expression, such as $5c + 3c$, and are expected to produce $8c$ as a simplified expression. Students are often taught the rule that when one adds or subtracts "like terms" (terms with identical variables; here $5c$ and $3c$), then they add or subtract the coefficients to obtain the combined term (here, $8c$). Teachers commonly draw on a context of apples and oranges and tell students that $5c + 3c = 8c$ because 5 apples and 3 apples is 8 apples (where c is an apple), and $5c + 3d$ cannot be combined because that would be like adding apples and oranges (Hallagan, 2006). While this approach may produce a correct response, it does not generalize to other situations. For example, it does not help one reason that $5c \bullet 3c$ should equal $15c^2$, because c^2 would not have meaning as an "apple squared." At the heart of the problem is the treatment of the literal symbol c as a physical object rather than as a symbol that represents a numerical value that can vary (Lucariello et al., 2014).

One way of addressing the problematic conception of $5c$ as 5 apples and the superficial role of context here serving only as the source for a mnemonic device is to take a quantitative-reasoning approach (Thompson, 2013; Kaput, 2008). Quantities are defined as mental operations involved in conceiving of a situation or context as having measurable attributes (e.g., my height or a car's distance; Thompson, 1994). Quantitative relationships are the connections between those attributes (e.g., how much taller I am than you or a car's speed as a relationship between distance and time). Quantitative reasoning involves analyzing the quantities and relationships among quantities in a situation, creating new quantities, and making inferences about quantities.

Consequently, a context is needed in which each symbol and operation in the equation $5c + 3c = 8c$ represents quantities and relationships in that context. In particular, we need a context in which it makes sense for c to represent varying amounts. Consider a context in which two friends, Sofia and Manuel, buy game apps. Sofia purchases 5 apps, and Manuel buys 3 apps. Each app costs c dollars. Then, for the expression $5c$, 5 is the number of apps Sofia bought, c is the cost per app, and multiplication can be conceived as the quantitative operation of "groups of." Thus, when the cost of each game app is \$1.49, then $5c$ can have the meaning of 5 groups of \$1.49. While grounding instruction in quantitative reasoning can serve to foster comprehension of algebraic expressions and equations (Smith & Thompson, 2007), it can also increase the complexity. Instead of focusing on the procedural skill of combining like terms by carrying out an elementary addition calculation ($5 + 3 = 8$), interpreting the equation $5c + 3c = 8c$ with quantitative meaning involves four related subtopics, each of which I explore next.

First, one goal is for students to see each expression—$5c$, $3c$ and $8c$—as representing a generalization of an arithmetic relationship. For example, students might first encounter an arithmetic task in elementary school of being given a certain number of game apps that one buys (5) and a cost of one app (e.g., \$2) and be asked to find how much the person spent in all. To solve they multiply a 5 • \$2. In later grades of elementary school, they may explore decimal values for the cost of a game app (such as \$2.99 or \$3.05) and calculate the total amount spent on 5 game apps as 5 • \$2.99 or 5 • \$3.05. Eventually, the generalization can be expressed as $5c$, where c is the cost per app.

Second, we want students to understand that the literal symbol c is not a physical object (e.g., an apple). When students begin to grasp that literal symbols can represent numeric values, their dominant conception is often that of an unknown, which represents a limited number of numerical solutions to an equation (e.g., x in the equation $3x – 4 = 11$). Instead, c is being used in $5c + 3c = 8c$ as what Philipp (1992) calls a generalized number. The values of the literal symbol vary but are not limited to just one or two values (as is often the case with an unknown). Algebraic symbols are often used as generalized numbers to express a mathematical relationship, property, or identity, as is the case with $5c + 3c = 8c$. If students first conceive of the equations $5c + 3c = T$ and $8c = T$, where T if the total amount of money spent, before forming the equation $5c + 3c = 8c$, then the literal symbols c and T are functioning as varying quantities (Philipp, 1992). Varying quantities are used to express a dependency relationship between the values of the literal symbols. Thus, an important part of algebraic reasoning with equations and expressions is having facility with multiple uses of literal symbols.

Third, we want students to interpret each expression in the equation from both process and product perspectives (Sfard, 1992). An algebraic expression represents both a computational process for calculating a quantity and a product (or resulting quantity) of that calculation. Specifically, an interpretation of $5c$ from both process and quantitative perspectives is as the number of game apps purchased by Sofia multiplied by the cost per app. In contrast, from a product perspective, $5c$ represents the result of that computation as a whole or entity, rather than as the computation process itself. Specifically, $5c$ is the total amount of money spent by Sofia.

Fourth, we turn to conceptions of equivalence. In the traditional approach to combining like terms introduced at the outset of this section, the equal sign in $5c + 3c = 8c$ has an operational interpretation (Knuth et al., 2006), meaning that the equal sign is a symbol that indicates the need to compute an answer (i.e., "=" is a "do something" signal). Here students are given "$5c + 3c$" and are told to "simplify," thus, producing $8c$. In contrast, researchers advocate for a relational understanding of equivalence, where students see the numeric value of each side of the equal sign as being the same. For example, $5c + 3c$ equals $8c$, because $5 \cdot 2 + 3 \cdot 2 = 8 \cdot 2$ when $c = 2$. In recent work, Cook et al. (2022) presented a new categorization of conceptions of equivalence. In their taxonomy, they include descriptive equivalence, which involves viewing objects as equivalent on the basis that they describe the same quantity or serve the same purpose in a given situation. In the example above, $5c + 3c$ and $8c$ both represent the same quantity of the total amount of money spent by Sofia and Manuel on game apps.

In turn, interpreting the entire equation $5c + 3c = 8c$ with a conception of descriptive equivalence has different flavors depending on whether one is operating from a process or a product perspective. For example, one might interpret the equation as representing two different ways to find the total amount spent by the two friends. One could multiply the number of apps Sofia bought times the amount each app costs and add it to the number of apps Manuel bought times the cost per app. Alternatively, one can find the total number of apps the two friends bought (8) and multiply that by the cost per app. From a product perspective, $5c + 3c = 8c$ can be interpreted as the amount of money spent by Sofia plus the amount of money spent by Manuel equals the total amount of money spent together by the two friends.

Algebraic and Quantitative Reasoning

In the previous section, I illustrated how four algebraic subtopics are involved in making meaning of the algebraic equation $5c + 3c = 8c$ in the context of friends purchasing game apps: (a) equivalence, (b) process and product perspectives, (c) generalizing, and (d) the meaning of literal symbols. However, these four aspects are related beyond this specific example. For any algebraic equation, students must first recognize algebraic expressions as a generalization of an arithmetic relationship. The resultant generalized expression involves literal symbols (e.g., c as a cost per app), which can take on various meanings. Furthermore, a quantitative-reasoning approach compels the learner to interpret subexpressions or terms from both process and product perspectives. Lastly, algebraic equations necessarily require an understanding of equivalence. Next, I briefly review the literature on quantitative reasoning and each of these four domains, motivating a need for a trajectory grounded in QR wherein these aspects interrelate.

Quantitative reasoning has been argued to be a useful basis for algebraic reasoning because it enables students to reason about quantities and their relationships without relying on specific numerical values (Smith & Thompson, 2008; Ellis, 2007). While prior research has shown the value of quantitative reasoning for developing concepts like function, proportional reasoning, and rate of change, fewer studies have examined its potential to support interconnected aspects of algebraic reasoning (Lobato & Siebert, 2002). This study contributes to this body of literature by shifting the focus from isolated concepts to the ways in which quantitative reasoning can serve as a unifying approach across multiple aspects of algebraic reasoning. In doing so, it highlights the potential for quantitative reasoning to not only deepen conceptual understanding, but also to build coherence across ideas, including the four domains of interest in this study.

Within the research area of equivalence, a vast majority of the literature has explored the development of children's understanding of operational and relational conceptions of the equal sign within an arithmetic context (e.g., Falkner et al., 1999; Knuth et al., 2006; McNeil & Alibali, 2005; Ralston & Li, 2022). This literature often highlights the challenges that students face in overcoming an operational view, which views the equal sign as an operator, to develop a relational view, which focuses on the equivalence of two expressions. For instance, in Falkner et al. (1999), students were asked to solve problems like $8 + 4 = \_\_\_ + 5$, where operational thinking might lead them to incorrectly insert 12 or 17 on the left side. The failure to develop a relational understanding has been linked to students' difficulties in solving algebra problems (e.g., Alibali et al., 2007; Knuth et al., 2006). Although Cook et al. (2022) recently emphasized the importance of a descriptive (i.e., quantitative) interpretation of equivalence, research in this area has been limited. Tondorf and Prediger's (2022) recent study examining middle school students' quantitative meaning for equivalent arithmetic expressions in the context of floor plan areas is one exception. To contribute to this body of research, this study aims to investigate high school students' development of descriptive equivalence with algebraic expressions.

With respect to the topic of process and product perspectives in algebra, Sfard's (1991, 1992) conceptualization of these two views within the theory of reification has offered important insights. According to Sfard, these perspectives provide distinct lenses through which algebraic expressions and equations can be understood. For instance, Sfard's and others' work illustrates how students often struggle to see the product perspective (the quantity representing the expression or equation as a whole) when they are focused solely on the process perspective (the steps taken to transform expressions) (e.g., Breidenbach et al., 1992; Sfard, 1992; Sfard & Linchevski, 1994). However, there remains a gap in understanding students' development of process and product interpretations of algebraic expressions and equations over time, especially through the lens of quantitative reasoning, which I aim to address in this study.

In the context of generalizing arithmetic via visual patterns to introduce algebraic expressions and equations, numerous studies have identified different levels of generalizations, students' difficulties, and types of justifications that students generate (Ellis, 2007; Lannin et al., 2006; Markworth, 2010; Walkowiak, 2014). These interventions often utilize visual pattern tasks to facilitate understanding. While these studies are grounded in quantitatively-rich situations (i.e., growing visual patterns), the use of a quantitative-reasoning approach is typically limited to design of the task rather than other aspects of the instructional environment (e.g., instructional goals, teaching actions, use of digital tools). This study extends this literature by deepening the focus on quantitative reasoning, uncovering additional layers of students' reasoning when encountering algebraic expressions and equations in visual pattern context.

Finally, understanding the meanings of literal symbols in algebra is a critical yet challenging aspect of algebra. The literature has extensively documented students' difficulties in grasping the meaning of literal symbols (Lucariello et al., 2014; Knuth et al. 2005; Kuchemann, 1978, 1981; Stacey & MacGregor 1997). These symbols often embody different meanings, including unknowns, generalized numbers, and varying quantities (Philipp, 1992). While some studies have explored students' construction of quantitative meanings for literal symbols, such as interpreting them as varying quantities (e.g., Carlson et al., 2002) and quantitative unknowns (e.g., Hackenberg et al., 2021), research in this area remains limited, particularly regarding how students develop quantitative meanings for literal symbols over time.

In sum, despite a substantial body of research on these four aspects of algebraic reasoning, several gaps remain. First, research in algebra has been prolific at the elementary level (Grades 1-6), with fewer studies situated in middle and secondary levels. This study addresses a need to explore how core algebraic topics develop in later grades, once algebraic contexts are introduced. Second, prior research in these areas has primarily captured students' reasoning at a single moment in time (with the exception of some work on generalizing via visual patterns), often through static assessments or clinical interview studies. Yet recent empirical studies continue to document persistent difficulties in these areas, suggesting a need for more studies that investigate how this reasoning develops over time. Thus, this study explores a qualitative investigation in the form of a learning trajectory. Finally, this study is distinct in its use of a quantitative reasoning approach to serve as a cohesive thread across these interrelated ideas. By addressing these gaps, this work extends the literature in algebraic and quantitative reasoning.

A Promising Instructional Tool: Dialogic Mathematics Videos

Although research has yet to document a learning trajectory of students' evolution of reasoning about algebraic equations and expressions from a quantitative-reasoning perspective, such a trajectory is demonstrated in a unique and promising instructional tool. Project MathTalk (www.mathtalk.org) has developed a unit of videos that feature the unscripted dialogue of two

Grade 9 students—Haleemah and Elijah (who goes by ET)—as they work collaboratively on a set of tasks that involve the four related algebraic subtasks discussed previously: (a) equivalence, (b) process and product perspectives, (c) generalizing, and (d) the meaning of literal symbols (Lobato et al., 2023; White et al., 2023). This video unit, called Meaning and Equivalence of Algebraic Expressions, simultaneously show images of the students and their mathematical inscriptions (see Figure 1). A teacher can be heard guiding the students but not seen, so that the focus remains on the students' mathematical reasoning. Importantly, the videos contain instances in which Haleemah and ET's initial ideas are incomplete or incorrect. Through their collaboration across many video lessons, Haleemah and ET are able to resolve authentic struggles and build on each other's ideas to come to a shared understanding.

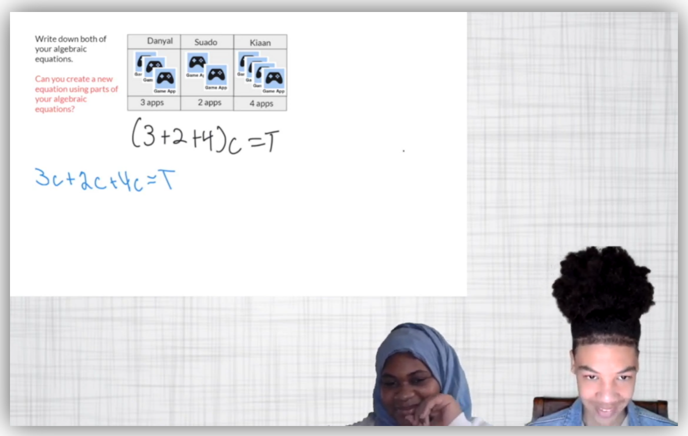


Figure 1: Haleemah and ET in the Meaning and Equivalence of Algebraic Expressions Unit

The Project MathTalk videos were created to respond to the increased interest in using videos in mathematics instruction, as well as to respond to problems with the dominant video model. Specifically, efforts to leverage video technology for meaning making have been limited because most videos available for K-12 student learning show a teacher presenting procedures and worked examples, without explaining why the procedures work or developing mathematical meanings and without including student voices (Bowers et al., 2012; Klinger & Walter, 2022). For example, the topic of combining like terms as it is featured on Khan Academy (see Figure 2) presents the equation $2x + 3x = 5x$ as similar to combining 2 "Chuck Norrises" plus 3 "Chuck Norrises" to give 5 "Chuck Norrises" (Khan, 2012). This meaning is similarly problematic of instruction adding "apples" and "oranges." In contrast, the Project MathTalk videos focus on meaning making, quantitative reasoning, and the importance of showcasing mathematical argumentation and justification.

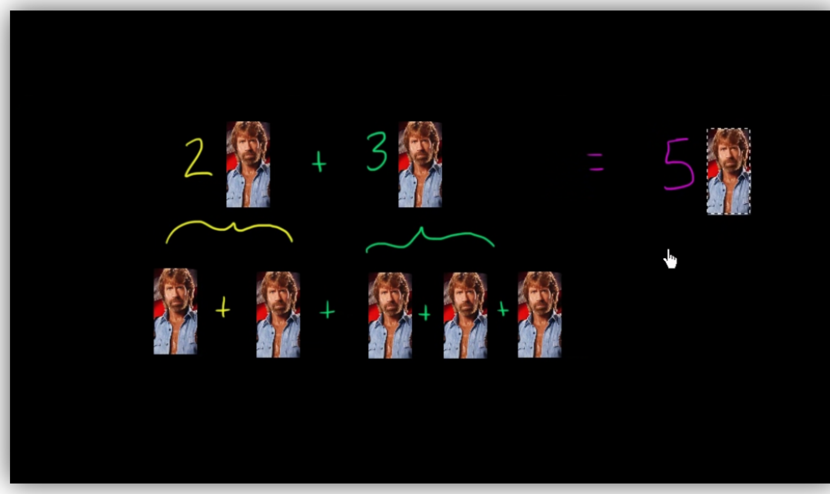


Figure 2: Combining Like Terms as Combining "Chuck Norrises" in a Khan Academy Video

The Project MathTalk videos are one type of dialogic video that are described in an emerging body of research and development projects (see for example Ben-David Kolikant & Broza, 2011; Seethaler et al., 2020; Weinberg et al., 2022). Dialogic videos feature the conversation of participants, but they can vary in terms of the nature of the dialogue, namely whether dialogue is scripted (e.g., Muller, 2008) or unscripted (e.g., Chi et al., 2017). Additionally, most dialogic videos feature dialogue between a tutor and student (e.g., Muller et al., 2007), whereas the Project MathTalk videos feature pairs of students (e.g., Lobato et al., 2019). The dialogic nature of the videos means that viewers can participate indirectly, or vicariously, in the conversations of the students in the video, whom I refer to hereafter as the *video participants*.

Vicarious Learning

According to Chi et al. (2008), vicarious learning refers to learning by observing and engaging with video- or audio-taped presentations of other people learning. The emerging body of literature on vicarious learning in STEM education has documented several benefits of indirect participation in the conversations of others when watching dialogic videos (Boesdorfer et al., 2011; Ben-David Kolikant & Broza, 2011; Lobato & Walker, 2019; Lobato et al., 2023; Muller, 2008). Researchers in the vicarious learning literature have been interested in questions such as: Are dialogic videos more effective than expository videos?, What features of dialogic videos enhance learning outcomes?, Under which conditions is vicarious learning more likely to occur while watching dialogic videos?, What is the nature of engagement of vicarious learners as they engage with dialogic videos?, and How do students appropriate voices they hear when watching dialogic videos? I briefly summarize the empirical findings that were aimed to analytically answer these questions and elaborate in more detail in Chapter 2.

A major focus of vicarious learning studies in science education has been on the relative effectiveness of dialogic videos compared to monologic videos via quantitative studies (Chi et al., 2017; Driscoll et al., 2003; Fox Tree, 1999; Gholson & Craig, 2006; Muldner et al., 2014; Muller et al., 2007). The general design of these studies involves assigning participants a pre-test, then placing them randomly in treatment groups (e.g., monologic conditions vs. dialogic condition), asking them to view a short video (e.g., monologic or dialogic depending on the condition), and then assigning a post-test. In general, the results of these studies indicate that students who view dialogic videos evidence better performance gains on pre/post assessments when compared to those in monologic conditions. However, these studies do not explore the evolution of learning over time.

While the focus on shorter videos may be sufficient in some science contexts (where shifts in conceptual understanding may occur relatively quickly), mathematics learning often develops over longer periods of time. This pattern is well-documented in research on learning trajectories in mathematics education (Lobato & Walters, 2017). To illustrate this contrast, a study with 81 preservice science teachers that found that their explanations about why the seasons change improved and contained fewer misconceptions as a result of watching a dialogic video that featured incorrect explanations of the phenomenon (Boesdorfer et al., 2011). By contrast, a vicarious learning study in mathematics education found that students' explanations of fraction expansion did not improve after viewing a short dialogic video clip. However, when the tutor in the research session provided scaffolding, the students participants were able to make use of the video as a tool.

One way to address this gap in the literature is to create a learning trajectory that documents student mathematics learning as they engage with dialogic mathematics videos across multiple sessions. There have been a few vicarious learning studies in mathematics that have utilized a qualitative approach in investigating the experience of viewers' engagement with videos over time, but these studies have not had an explicit focus on exploring the learning trajectories of the students who view the videos in parallel with the learners featured on screen. Instead, studies have focused on how viewers approach dialogic videos (Lobato et al., 2019), the learning processes involved while using the videos (Lobato et al., 2023), how viewers orient toward the video participants in the videos (Lobato & Walker, 2019; Foster, 2023), and students' experiences of intellectual need as they engage with dialogic videos (Weinberg et al., 2022).

Another pressing need in the literature is to investigate how dialogic mathematics videos may serve as an instructional support in a classroom context. The few existing vicarious learning studies in classroom contexts are at the undergraduate level in physics and biology, often with the primary aim of replicating findings from laboratory studies (Cooper et al., 2018; Craig et al., 2009; Ding et al., 2021). As a result, these studies similarly emphasize the relative effectiveness of dialogic videos compared to monologic videos but within the context of undergraduate courses. Additional work is needed to explore how dialogic mathematics videos can be used meaningfully by a teacher for the purpose of supporting mathematics learning.

Several vicarious learning studies in mathematics have laid the groundwork for classroom use by illustrating that (a) viewers are able find mathematics ideas from the videos as salient and useful (Lobato et al., 2019), (b) viewers develop mathematical meaning through processes of negotiation and interaction with peers (Lobato et al., 2023), and (c) viewers benefit mathematically through teacher scaffolding while watching videos (Ben-David Kolikant & Broza, 2011). Accordingly, I situate this study in an instructional environment by conducting a classroom teaching experiment (TE) (Cobb, 2003). This work is much needed in the literature on vicarious learning and dialogic video use because the goal of utilizing dialogic videos for instruction has not yet been realized.

Instrumental Approach

Situating the study within a classroom context necessitates a theoretical framework that considers the role of the teacher, a group of learners, a digital tool or artifact (in this case, dialogic mathematics videos), and the mathematics involved (in this case, algebra from a quantitative reasoning perspective). One useful framework is Verillon and Rabardel's (1995) instrumental approach, which focuses on the role of artifacts in technology-rich environments. The basic concept of the instrumental approach is that users engage in an activity in which actions are performed upon an object (e.g., mathematics) to achieve a certain goal (e.g., solving a mathematics task), using an artifact (e.g., dynamic geometry software such as GeoGebra). This approach assumes that a mathematical artifact becomes a mathematical instrument (i.e., the artifact becomes valuable and useful) as the learner develops the skill to express mathematical ideas with it. This process is known as *instrumental genesis* (Artigue, 2002), after which the artifact moves from being a human-made object to something meaningful to the learner.

The instrumental approach also accounts for the components of teaching (such as the designing, planning, and managing of artifacts in technology-rich environments) using the notion of *instrumental orchestration* (IO) (Trouche, 2004). IO refers to the intentional and systematic organization of the various artifacts available in a learning environment by the teacher for a given mathematical situation to guide students' instrumental genesis (Drijvers & Trouche, 2008; Trouche, 2004, 2005). IO draws on the metaphor of a teacher functioning as a conductor of an orchestra. Drijvers and Trouche (2008) note that while "orchestration" suggests a professional symphony orchestra, the metaphor is more akin to a high school jazz band that allows for improvisation and includes a mix of novices and more advanced musicians. This conceptualization draws parallels with a technology-rich classroom environment. In addition to the conductor, who serves as a teacher, the students function as musicians who are using technological artifacts, which serve as instruments.

In this study, I use IO as a research lens to explore the teacher's role in managing the artifact of dialogic mathematics videos to support the classroom community's mathematical reasoning in an classroom context. I note that while that the use of instrumental genesis can potentially serve as a way of investigating the relationship between the artifact of dialogic mathematics video and the emergence of students' mathematics reasoning, this research focus is beyond the scope of this study.

Research Questions and Significance

In this section, I present the two research questions that frame this study. I connect each question to aspects of the rationale and then discuss the significance of addressing these questions. The two research questions are:

1. What is the evolution of ways of reasoning of 13 Grade 9 students over the course of a TE as they engage with dialogic mathematics videos about algebraic expressions and equations from a quantitative-reasoning approach?
2. What instrumental orchestrations of the artifact of unscripted dialogic mathematics videos support the emergence of 13 Grade 9 students' ways of reasoning?

Question 1

This question addresses the first aim of this study, which is to explore how students may reason about algebraic expressions and equations from a quantitative-reasoning perspective over time. Specifically, responding to this research question entails creating a learning trajectory of the evolution of students' reasoning about this content over the course of the TE. The motivation for this question emerged from two bodies of literature. The first is in the area of algebraic and quantitative reasoning. In particular, students continue to have difficulty with this mathematical content, and while quantitative reasoning supports deeper conceptual understanding, it also adds significant complexity that requires learning over time. Current literature suggests that a learning trajectory connecting the four interrelated aspects of making meaning of algebraic expressions and equations from a quantitative reasoning approach (i.e., equivalence, generalizing via visual patterns, process and product duality, and meaning of literal symbols) would contribute to the field.

The second motivation for this question stems from the literature on vicarious learning and dialogic videos. The majority of vicarious learning studies share a common research design: students watch short videos under different conditions in controlled laboratory settings, and learning is then measured through pre- and post-assessments. While these studies provide valuable insights into learning gains, they often overlook how students actually develop mathematical reasoning through interaction with the videos and instructor intervention. Therefore, the field would benefit from a qualitative investigation of students' evolution of reasoning as they engage with dialogic videos in a classroom setting. Of course, a claim cannot be made that all learning observed in the TE stems solely from the videos, as instruction also involves teacher intervention and collaboration with peers. The focus on how the dialogic videos may be used as a pedagogical approach is investigated in the next research question.

Question 2

This question addresses the second aim of this dissertation, which is to identify the instructional supports for the algebraic reasoning identified in Research Question 1, specifically supports related to the use of dialogic mathematics videos. Earlier, I argued that literature of how to use dialogic mathematics videos in the classroom is currently non-existent. Accordingly, this question draws on the construct of IO to explore the meaningful and systematic management of the dialogic mathematics video tool. Thus, I advance knowledge in the vicarious learning literature of how dialogic mathematics videos may be used to support mathematics learning. Additionally, in several reviews of researchers utilizing IO (e.g., Drijvers et al., 2020; Song, 2021), no study has explored video as an orchestrable technology. Adapting the theoretical framework of instrumental orchestration to consider video has theoretical implications for how video tools can be managed by a teacher.

Significance

The two research questions of this dissertation advance knowledge about these relationships and contribute to three fields of mathematics education: algebraic and quantitative reasoning, vicarious learning and dialogic video use, and digital tools in mathematics education. I organize this section by the three fields of literature and discuss how answering these research questions contributes to each field.

Significance to the Field of Algebraic and Quantitative Reasoning

Both the first research question (the learning trajectory) and the second research question (instructional supports) contribute to the field of algebraic and quantitative reasoning. For any content area for which a learning trajectory is created, researchers and practitioners are able to gain theoretical and practical insights. Theoretically, it is common for these learning trajectories to contribute to theory about how individual learners or a classroom community comes to understand mathematics concepts. Practically, learning trajectories can be used in professional development for teachers to design more effective instructional sequences (e.g., Sztajn & Wilson, 2019). For the field of algebraic and quantitative reasoning, this is important because of the persistent difficulties that students have learning algebra topics. The field would benefit theoretically from understanding how students develop quantitative meanings for equivalence, generalizing via figural patterns, process and product perspectives, and meanings for literal symbols.

The second research question concerns instructional supports. It is not only important to understand how students may reason about these topics over time, but there is also a need for instructional support in algebra classrooms. While student-centered and inquiry-oriented instruction seems to have taken root at the elementary level, these instructional practices are still not prevalent in algebra classrooms at the secondary level (Litke, 2020). For instance, one large-scale study found that across six districts in the United States, and 108 algebra classrooms at the secondary level, a majority of instruction was largely teacher-directed with limited student participation and engagement with mathematical ideas (Litke, 2020). Thus, an analytical account of instructional supports while using dialogic videos will serve to inform algebra instructors to promote reasoning in these four areas.

Significance to the Vicarious Learning Literature

Answering these research questions has theoretical and practical implications for the literature on vicarious learning and dialogic video use. Theoretically, the field will gain insight into how learning vicariously occurs in conjunction with instructor intervention in a classroom context. The findings may confirm previous findings in studies conducted in laboratory settings and also glean new insights into how students engage and learn from dialogic videos in a different context.

Practically, answering these questions has the potential to promote the use of alternative video formats for classroom use. Empirical studies have demonstrated the potential for dialogic videos to increase learning gains when compared to expository videos, encourage collaborative behaviors, and foster rich mathematical meanings (Chi et al., 2017; Lobato & Walker, 2019; Lobato et al., 2023). Thus, understanding the use of this alternative video format in this classroom setting can offer concrete strategies that teachers may use to integrate dialogic mathematics video into their instruction.

Significance to the Use of Digital Tools in Mathematics Education

Lastly, this dissertation contributes to the knowledge about the use of digital tools in mathematics education. Recently, the theoretical construct of IO been used in mathematics education but has yet to consider video technology as a digital tool (Drijvers et al., 2020). Therefore, this dissertation advances knowledge about IO and how dialogic mathematics videos can act as a digital tool for algebra learning. Second, the dissertation contributes to this field of mathematics education in a practical way because to my knowledge, dialogic mathematics videos are currently not being used in mathematics classrooms. Thus, by investigating how dialogic videos may be used as an instructional support, this dissertation would provide mathematics instructors with knowledge of how to use alternative video formats for a pedagogical purpose.

Chapter 2: Literature Review

The goal of this chapter is to review the bodies of literature related to the two aims of my study. As presented in Chapter 1, the two aims are (a) to explore how students learn to make sense of algebraic expressions and equations in context over time and (b) to identify the instructional supports for that learning, specifically supports related to the use of dialogic mathematics videos.This chapter is organized into four sections. In the first section, I review the literature on algebraic reasoning and quantitative reasoning that is related to the mathematical topics targeted in the TE. Second, I present research on learning from dialogic videos, situated within the vicarious learning literature. In the third section, I present the theoretical framing of the dissertation. Finally, I end this chapter by re-presenting and elaborating my research questions in light of the literature and the theoretical constructs.

Algebraic Reasoning and Quantitative Reasoning

Research in algebraic reasoning has been prolific, specifically at the elementary level (Haspekian et al., 2023; Kieran, 2022). Over time, this work has evolved from a narrow focus on symbolic manipulation to a broader emphasis on “relations, patterns, and structures in numbers, operations, and functions, and on the mathematical processes of seeking structure, conjecturing, generalizing, and justifying” (Kieran, 2018b, p. 437). Despite this progress, pushing this work forward continues to be a challenge because an emphasis on rote symbol manipulation has pervaded algebra curricula historically (Kaput, 1995; Kilpatrick & Iszak, 2008; Litke, 2020). Recent empirical studies continue to document that students often revert to numerical and procedural thinking over conceptual thinking about algebra topics (e.g., Christou et al., 2022; Liang & Moore, 2021; Tondorf & Prediger, 2022; Wilkie, 2022). These findings highlight that reasoning in this way is difficult and requires continual investigation by mathematics education researchers, especially at the middle and secondary levels.

One promising approach to meaning making that has gained traction in mathematics education is quantitative reasoning (QR), which shifts the focus from symbols and numbers to quantities and their releationships (Thompson, 2013; Kaput, 2008). In this study, I use QR in the sense of Thompson (1988, 1990, 1993, 1994, 2011, 2013. QR involves *quantities*, *quantitative relationships*, formed using *quantitative operations*. Quantities refer to the mental operations involved in conceiving of a situation or context as having measurable attributes (e.g. my height or a car's distance). The quantitative relationships are the relationships between those attributes (e.g., how much taller I am than you or a car's speed as a relationship between distance and time). Quantitative operations are defined as the mental actions by which new quantities are formed from two or more quantities that have already been conceptualized. QR as a way of thinking can be characterized by (a) conceptualizing quantities and (b) organizing relationships between quantities with quantitative operations to form a structured mental representation of a situation or context (Carlson et al., 2023).

Furthermore, an important distinction in this approach is the difference between numeric reasoning and QR. In particular, quantities are not numeric and likewise, quantitative operations are not arithmetic operations. In Chapter 1, I illustrated this distinction with the contrasting approaches to making sense of the equation $5c + 3c = 8c$. To use another example, consider the equation $W = fd$ (where W is work, f is force, and d is distance) (Thompson & Saldanha, 2003). A numeric interpretation would conceive of the equation as a way to calculate a particular quantity's measure, where the result of the formula depends on the particular units used. On the other hand, a quantitative meaning for the formula would entail conceiving of $W = fd$ as the following: work, as a quantity, is created by applying a force to an object and thus, moving it some distance. Specifically, the equation does not make reference to measurement units.

This perspective offers several advantages in the context of algebra. First, QR's lack of reliance on symbolic notation can support students' ability to reason across problems without being bound by numeric computation. This crucial in algebra because of the barriers students face in transitioning from arithmetic to algebra. In particular, QR can expose students to mathematical problems earlier, which suggests an early route to understanding algebraic expressions. For instance, students can explore the meaning of 38/3 – 7 rather than automatically writing 5 2/3, which may develop their understanding of an expression like $x/3 - 7$. Second, QR emphasizes making sense of quantities in contexts that draw on everyday experiences. Smith and Thompson (2007) argue that when developed, QR can give meaning and purpose to algebraic expressions and, thus, to algebra as a whole.

Subsequent studies have demonstrated QR's potential to support student understanding when accompanied by instructional supports (Carlson et al., 2023). For example, research has used QR to investigate individuals' meaning for concepts central to algebra, such as rate of change (Byerley & Thompson, 2017; Johnson, 2015a, 2015b) and function (Oehrtman et al., 2008; Paoletti & Moore, 2018). Other studies have focused more specifically on QR middle and high school algebra contexts (e.g., Ellis et al., 2020; Paoletti & Vishbuhotla, 2023; Smith III & Thompson, 2007). The results of these studies suggest the potential of quantitative reasoning to promote sophisticated mathematics understanding in algebra.

Building on this work, I adopt this approach and argue that reasoning with quantities offers a powerful way of developing secondary school students' algebraic reasoning. The focus of this study on algebraic expressions and equations requires that particular attention be paid to four interrelated aspects of algebraic reasoning: (a) equivalence, (b) meaning of literal symbols, (c) process and product perspectives, and (d) generalizing via visual patterns. In Chapter 1, I illustrated how these aspects were related in the example of making sense of the equation $5c + 3c = 8c$ from a QR perspective. However, these four aspects are related beyond this specific example. For any algebraic equation, students must first recognize algebraic expressions as a generalization of an arithmetic relationship. The resultant generalized expression involves literal symbols (e.g., c as a cost per app), which can take on various meanings. Furthermore, a QR approach compels the learner to interpret subexpressions or terms from both process and product perspectives. Lastly, algebraic equations necessarily require an understanding of equivalence. In the sections that follow, I review the literature on each of these four domains and motivate a need for a trajectory grounded in QR wherein these aspects interrelate.

Equivalence

One goal of this study is to gain insight into how secondary school students come to understand the equivalence of algebraic expressions (such as $5c + 3c = 8c$) in context. Consequently, one element of developing this understanding is one's conceptions of equivalence. This area requires attention because of students' difficulty in developing sophisticated interpretations of the equal sign and because equivalence "links numerous mathematical understandings into a coherent whole" (Charles, 2005, p. 10). Although mathematical equivalence is widely considered to be important for later mathematics success (Kieran, 1981; Stacey & MacGregor 1997), this area has predominantly focused on elementary and middle school students' understanding of the equal sign in arithmetic contexts.

Children are introduced to the equal sign at an early age in school. Beginning in elementary formal schooling, students are often asked to operate on numbers to get an answer (e.g., 2 + 2 = ___). This experience encourages the view that the equal sign serves as a "do something signal" that indicates a need to compute an answer, which researchers call an *operational* view. This can be problematic when confronted with more complex or alternate uses of the equal sign. In contrast, a *relational* view takes the equal sign to mean "the same as." For instance, someone with an operational view may see 2 + 2 = 4 + 3 as valid, whereas an individual with a relational view would view 2 + 2 as the same as 1 + 3.

Most research in early algebra focusing on these two views of equivalence has focused on (a) diagnosing students' conceptions of the meaning of the equal sign, (b) connecting this understanding to solving algebra problems, and (c) investigating instructional approaches to developing sophisticated understanding of the equal sign. I will briefly summarize the empirical findings from each. First, the most often documented difficulty is that students often misinterpret the equal sign operationally rather than a relational understanding (Blanton et al., 2015; Carpenter et al., 2003; Falkner et al., 1999; Knuth et al., 2005; Knuth et al., 2006; McNeil & Alibali, 2005; Prediger, 2010; Ralston & Li, 2022; Stephens et al., 2013).

Second, researchers have found that children with a relational understanding of the equals sign can more successfully solve equations and produce equivalent quantities than children with operational conceptions (Alibali et al., 2007; Knuth et al., 2006; Matthews, Rittle-Johnson et al., 2012). For instance, in their longitudinal study across grades 6-8, Alibali et al. (2007) found that holding a relational view of the equals sign was associated with better performance on equivalent equations problems (e.g., Is the number that goes in the "__" the same number? 2 __+ 15 = 31 and 2 __+ 15 – 9 = 31 – 9). Lastly, researchers have suggested that the prevalence of operational views may not be a developmental issue but rather an issue of how the equal sign is presented in early grades (McNeil et al., 2006; McNeil & Alibali, 2004). Consequently, researchers and practitioners recommend exposing students early and often to non-standard equation formats (i.e., 3 = 8 – 5) in order to promote a relational understanding (Powell, 2012; Shoecraft, 1989).

The primary focus of the aforementioned studies has been on the dichotomy between relational and operational views of the equal sign. However, a recent framework developed by Cook and colleagues (2022) moves beyond the relational/operational categories by exploring three ways in which equivalence may be conceptualized across middle, high school, and undergraduate levels and across multiple domains. The three interpretations of equivalence detailed in Cook et al.'s (2022) framework include *common characteristic*, *transformational,* and *descriptive equivalence*. Next, I discuss each interpretation and then argue that the area of descriptive equivalence warrants further empirical investigation.

According to Cook et al.'s (2022) framework, the studies reviewed thus far fall under the first category of *common characteristic*. They define a common characteristic interpretation via two criteria: criteria of sameness (C1) and a justification of why they are the same (C2). The justification is what defines this interpretation of equivalence, which is by identifying some common feature between two mathematical objects. Take, for example, two lines that a student is looking at graphically. They might interpret the equivalence of those two lines by saying they are the same (C1) because they have the same intercept and slope (C2). Another example would be explaining that 8 • 12 + 2 • 4 is the same as 26 • 4 (C1) because they both result in the same number, 104 (this is referred to as a *result equivalence* interpretation in Tondorf and Prediger (2022)). The criterion of sameness (C1) aligns with the relational view of equivalence discussed above.

The second category detailed in Cook et al.'s (2022) framework is that of *transformational equivalence*, which refers to the specific actions that students take to determine equivalence (e.g., multiplying ½ by 3/3 to determine that ½ is equivalent to 3/6). A student evidences a transformational interpretation if they recognize that new forms being generated are equivalent (according to common characteristic or descriptive interpretations). For example, a student might supplement a transformational interpretation with a common characteristic interpretation by subtracting the same number from both sides of an equal sign, while being aware that the subtraction preserves equality. More specifically, a student holds a transformational interpretation supplemented with a common characteristic if they are aware that the forms produced by the transformations she is applying share a common attribute.

The final interpretation identified in Cook et al.'s (2022), descriptive equivalence, is the primary interest of this study. This view is similar to common characteristic in that it shares a criterion of sameness (D1). Where it differs is in the explanation of sameness, which requires a reference to an aspect of the given mathematical situation or task setting (D2). More broadly, a descriptive interpretation involves viewing objects as equivalent on the basis that they describe the same quantity or serve the same purpose in a given situation (e.g., explaining ½ = 3/6 by referencing a rectangular model as in Wong & Evans, 2007). This interpretation is of interest because it aligns with Thompson's (2011) notion of QR in emphasizing quantifying objects in a particular context. Therefore, I claim that descriptive equivalence is a quantitative meaning of equivalence.

Unfortunately, empirical investigations of descriptive equivalence are limited. Most of the examples given in Cook et al. (2022) (e.g., Wong & Evans, 2007; Thompson et al., 2014; Lockwood et al., 2020; Cook, 2012) do not explicitly investigate equivalence and are instead viewed from this lens retrospectively in their framework. One exception is a recent study conducted by Tondorf and Prediger (2022) in arithmetic. Their study investigated the learning progression of students as a consequence of a design intervention to bridge the gap between transformational equivalence and descriptive equivalence with 5th graders. They found that students developed robust descriptive interpretations by comparing possible decompositions of a given figure to ensure that both expressions describe the same figure and are thereby equivalent. While existing research has tended to focus on relational and operational interpretations (i.e., common characteristics), Tondorf and Prediger's study suggests that more studies should investigate the novel construct of descriptive equivalence.

This study seeks to extend the findings reported by Tondorf and Prediger (2022). Similar to their study, the design of the TE will use the context of visual patterns, which has been shown by other researchers to promote descriptive equivalence (Mason et al., 1985; Wilkie & Clarke, 2016; Zwetzschler & Prediger, 2013). This study will differ from Tondorf and Prediger (2022) in its extension of arithmetic to algebra. Extending to an algebra context requires four additional aspects of algebraic reasoning. The first is generalizing arithmetic to algebra via visual patterns. In the process of generalizing, students will connect terms in their arithmetic expressions to substructures in the visual figures, which requires process and product perspectives (Tondorf & Prediger, 2022). Students will do the same with algebraic expressions, which similarly require process and product perspectives. Lastly, students will need to conceive of the literal symbols in their expressions in ways that are consistent with the context. In the subsequent sections, I discuss the remaining aspects as they are situated in the literature.

Meaning of Literal Symbols

Another focus of this study is to understand how students develop productive meanings for the letters that appear in algebraic expressions and equations, what are often referred to as *literal symbols*. While traditional approaches have historically introduced these symbols as stand-ins for specific but unknown numbers, mathematics educators have argued for a more nuanced view. Such a view is necessary because the meanings of literal symbols shift depending on context and are deeply connected to other dimensions of algebraic reasoning. For example, in the transition from arithmetic to algebra, literal symbols are essential tools for expressing general methods (e.g., Blanton et al., 2017). They can also be imbued with quantitative meaning, such as representing a measurable quantity that varies, like h for the changing height of a plant (e.g., Hackenberg et al., 2021). The variety of roles that literal symbols can play suggests that this area warrants continued investigation, particularly into how students develop quantitative meanings of literal symbols over time and in relation to other aspects of algebraic reasoning.

Several large-scale studies haved shown that middle and secondary students often hold faulty conceptions of literal symbols even before formal instruction in algebra begins (Kuchemann, 1978; Knuth et al., 2011; Stacey & MacGregor, 1997; Lucariello et al., 2014). These studies consistently identify three common misconceptions. The first and most persistent misconception is *literal symbol as object*, where the literal symbol is interpreted as a label for an object or as an object itself (e.g., $5a$ is 5 apples). Lucariello et al. (2014), found that 25% of secondary students in their study interpreted the symbol as the label for an object. Second is *ignoring the literal symbol* altogether, a pattern documented by Kuchemann (1978). In their study, 20% of secondary students responded to the question “Add 4 onto $n + 5$” with “9,” indicating that they had ignored the literal symbol. Third, students often incorrectly treat literal symbols as fixed values rather than representing values that vary (Stacey & MacGregor, 1997; Knuth et al., 2011).

Many argue that much of the difficulty students encounter stems from the myriad meanings of literal symbols in mathematics. Consequently, one focus of scholarly work has been to categorize different meanings of literal symbols, depending on the context (Booth, 1984; Kuchemann, 1981; Philipp, 1992; Usiskin, 1988). Three meanings for literal symbols that are particularly important for this study are as *specific unknowns*, as *varying quantities*, and as *generalized numbers*. As Phillip (1992) describes, when a literal symbol is a *specific unknown*, it represents a limited number of numerical solutions to an equation, such x in $4x = 17$. When literal symbols represent *varying quantities*, they express a dependency relationship between two or more quantities, meaning that there is a systematic relationship between the sets of values for the quantities (e.g., x and y in $y = 2x + 1$). Literal symbols can also represent a *generalized number*, where the symbol, when replaced with any other value, will result in a true statement (e.g., a, b in $ab = ba$). While these interpretations are distinct, it is often a challenge for students to flexibly shift between meanings.

Researchers have responded to these challenges by identifying ways to support more sophisticated meanings, particularly for generalized numbers and varying quantities. Across these efforts, several themes emerge. First, researchers recommend bridging the meaning between numbers and literal symbols early in schooling (Ellis & Özgür, 2024). One way to do this is to engage students in "quasi-variable thinking," where they reason with general quantities, instantiated by numbers, prior to their use of literal symbols (Fujii & Stephens, 2008; Carpenter et al., 2003; Dougherty, 2008). For instance, in Fujii and Stephens's (2008) report with children in Grades 2-3, they found that students could explain why an equation such as $78 - 49 + 49 = 78$ was always true by seeing 49 as a "quasi-variable" for a generalized number as opposed to a numeric value. Other studies have shown that with proper instructional support, children as young as six years of age can use literal symbols to express relationships between varying quantities (Blanton et al., 2017; Brizuela et al., 2015; Ventura et al., 2021).

Second, studies suggest that making the roles and meanings of literal symbols explicit during instruction can help students better distinguish between uses of symbols (Driscoll, 1999; Kilhamm et al., 2022; Phillip, 1992). For example, Kilhamm et al. (2022) found that a programming environment where literal symbols in the equation $y = km + b$ appeared explicitly in code offered students opportunities to choose values for x, y (representing varying quantities) and for k and m (representing parameters), highlighting the distinction between the two meanings. Lastly, another set of studies have shown that middle grade students demonstrate success representing functional relationships with literal symbols when tasks are grounded in rich contexts, such as visual patterns (Rivera & Becker, 2008; Ellis, 2011; Ellis et al., 2016; Markworth, 2010). The findings suggest that use of rich contexts, where literal symbols represent varying rather than fixed quantities can support the emergence of robust meanings for symbols. Together, these findings indicate that making sense of algebraic expressions and equations in context, with an explicit focus on quantities, may disrupt common misconceptions of literal symbols.

Despite efforts to provide instructional support, student difficulties persist. In particular, students face challenges when translating between verbal descriptions and algebraic expressions (MacGregor & Stacey, 1993; Molina et al., 2017; Soneira, 2022). For example, Molina et al. (2017) classified and compared errors in translation made by two groups of secondary students: those beginning and completing their studies in school algebra. Among the more advanced students, over half of the errors were related to characteristics of algebraic symbolism. These included misinterpreting the structure of an algebraic statement, an inability to correctly distinguish the use of different letters in a statement, and incorrectly generalizing part of a statement. One student, for instance, translated the statement “a certain number plus the consecutive number, equals another number minus two" as "$x + (x + 1) = x - 2$," using the same literal symbol "x" to represent two different quantities. These errors point to a deeper issue, which is a disconnect between literal symbols and the quantities they represent, which this study aims to address.

In light these challenges, one fruitful avenue of exploration is the use of a QR approach to examine how students come to develop meaning for literal symbols. An emerging body of literature has contributed to this effort (Carlson et al., 2002; Hackenberg et al., 2021; Steffe et al., 2014). For instance, Carlson et al. (2002) and others have used the notion of *varying quantity*, where quantity is understood in the sense of Thompson (2011) (as a measurable attribute of a conceived object). With this lens, varying quantities reflect a dependency relationship between two or more such attributes, such as the relationship between volume and height in a growing container. Complementing this work, Steffe et al. (2014) and Hackenberg et al. (2021) have introduced the construct of *quantitative unknown*, referring to the value of a fixed quantity that is initially unknown but can be determined. Hackenberg et al. (2021) found that when students were encouraged to reason quantitatively about literal symbols, they shifted from relying on numerical values to conceiving of *potential* numerical values, a more general form of reasoning. This study seeks to extend these findings by further examining quantitative meanings for literal symbols.

While studies have documented productive shifts in students' reasoning about literal symbols, developing robust quantitative meanings remains a complex process. This suggests that students would benefit from repeated exposure to targeted instructional supports, and that the field would benefit from a qualtitative investigation into how such reasoning develops across time. Unfortunately, most existing research on how students come to develop meaning for literal symbols over time is concentrated in the elementary grades (e.g., Blanton et al., 2017; Ventura et al., 2021), leaving a gap in our understanding how this reasoning develops in secondary contexts. One partial exception is found in the literature on generalizing via visual patterns. However, these studies typically focus on the types of generalizations students produce rather than the evolving meanings they assign to the symbols themselves. As a result, the field lacks an empirically grounded account of how secondary students develop meaning for literal symbols, particularly quantitative meanings, over time.

Process and Product Duality

Interpreting algebraic expressions in multiple ways is powerful for investigating how students see "through symbols," not only the symbols themselves (Kieran & Sfard, 1999, p. 6). Mathematics education researchers have identified two ways to interpret an algebraic expression, as a process and as a product (e.g., Sfard & Linchevski, 1994). A mathematical expression represents both a computational process for calculating a quantity, and a product (or resulting quantity) of that calculation. For example, the algebraic expression $5c$ from process perspective is a sequence of computational steps: multiply 5 times a number, c. From a product perspective, $5c$ represents the result of that computation as a whole or entity, rather than as the computational process itself.

Researchers have proposed theories for how learners move through the process-product continuum (e.g., Dubinksy, 1991; Gray & Tall, 1994; Sfard, 1991, 1992). The stages of development from process to product have been described as action-process-object (Arnon et al., 2014; Dubinsky, 1991), procedure-process-procept (Gray & Tall, 1994), and interiorization-condensation-reification (Sfard, 1991, 1992). Each of these variations describes a development of growing sophistication, starting with a procedure/process and ending with an object/concept that can be manipulated as an entity in its own right (Tall et al., 1999).

Sfard's theory of reification differs from others in viewing both process and product conceptions as important in mathematical understanding and particularly in algebra (Sfard & Linchevski, 1994). The process perspective, for instance, is essential for determining final answers to mathematics problems. A product perspective, on the other hand, condenses processes to allow for higher-level problem-solving. Sfard's (1991) theory posits that process views typically develop before product views, and that the transition from process-oriented to product-oriented thinking occurs in three stages. The first stage, *interiorization*, is the stage at which one has a process view; for example, at this stage, a student can conceive of an algebraic expression as a process without carrying it out (e.g., $5c$ as 5 times c).

An individual is said to be in the *condensation* stage if they can conceive of processes as a self-contained whole, without needing to think about component steps. A process is said to be condensed if it stands as an "autonomous procedure" (Jankvist & Misfeldt, 2015; p. 2015). For example, a student who has condensed a process may be able to conceive of a combination of algebraic expressions (e.g., $5c + 3c$ as 5 times c plus 3 times c). In this case, the student has condensed the process of 5 times c together with another process of 3 times c. Finally, *reification* refers to the stage were condensed processed are viewed as product-like entities. At this stage, concepts are detached from their related processes. With the example of $5c + 3c$, reification would be evidenced if $5c + 3c$ can be viewed as a single quantity that can be set equal to another equivalence expression, $8c$.

Empirical evidence suggests that developing product-oriented conceptions are difficult for students (Breidenbach et al., 1992; Sfard, 1992; Sfard & Linchevski, 1994). Studies have explored this in the context of fractions (e.g., Pantziara & Philippou, 2012), functions (e.g., Oehrtman et al., 2008; Blanton et al., 2017), and limits (e.g., Gucler, 2014). Gucler's (2014) case study investigated an instructor's and his students' discourses on limit in the context of an undergraduate calculus class. They found that while the instructor moved flexibly between these perspectives, students struggled to develop product views, highlighting the importance making process and product perspectives explicit in instruction. Another study framed first grader's mathematical progressions of variable notation using the process-to-product lens (Blanton et al., 2017), illustrating how children may progress from process views to product views in thinking about functional relationships. Indeed, more sophisticated levels of reasoning were associated with product-oriented perspectives and lower levels with process-oriented perspectives.

On the other hand, exploring students' development of process and product views through the perspective of QR remains underexplored. Thompson and Sfard (1994) contend that, in principle, reification and QR are "complementary rather than mutually exclusive" (p. 28). For instance, consider a student, Sofia, who interprets $5c$ from both a process perspective and quantitatively in the context of buying game apps (where c is the cost per app). She could think of $5c$ as the 5 apps that she purchased times the cost per app. Alternatively, she could interpret $5c$ as total amount of money that she spent, which is an interpretation from a product perspective with quantitative meaning. Having a quantitative referent can motivate taking a product perspective because products have meaning in context (e.g., total amount spent by a friend) as opposed to a computational process. In the other direction, product views are crucial for developing quantitative meaning for other mathematical ideas such as equivalence of expressions.

To illustrate, I consider an example from Tondorf and Prediger (2022), which investigated equivalence of expressions 8 • 12 + 2 • 4 and 26 • 4 in the context of floor plans (see Figure 3). Without using a QR approach, 8 • 12 does not have meaning in context, and is thus likely to be viewed as a process of multiplication. On the other hand, when a context is introduced, suddenly 8 • 12 represents a portion of floor plan that visually depicts 8 groups of 12. Additionally, in order to conceive of the equation 8 • 12 + 2 • 4 = 26 • 4, the learner must connect 8 • 12, 2 • 4, and 26 • 4 to substructures in the visual figures, which requires a product view.

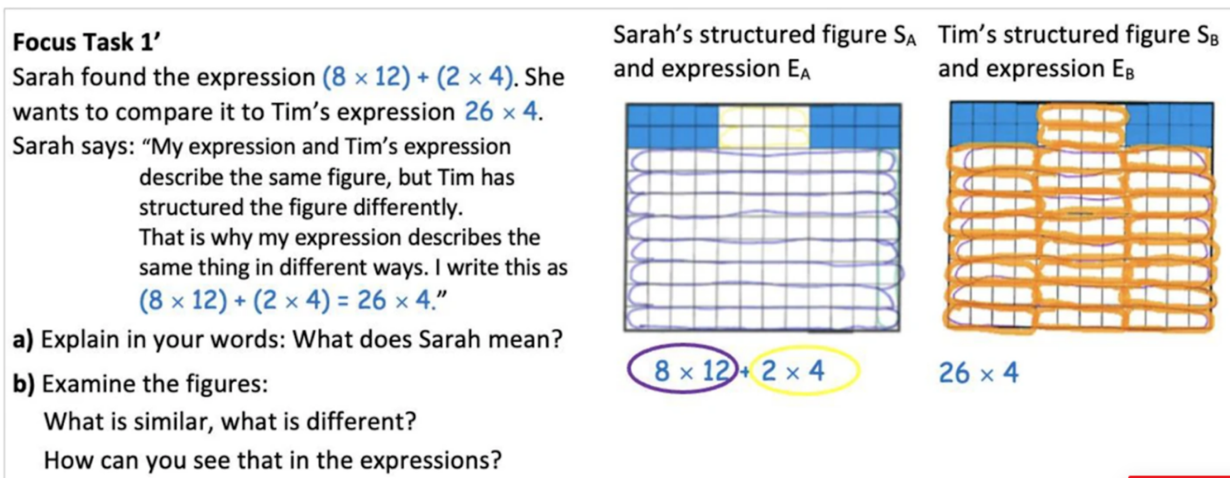


Figure 3: Task about Equivalent Arithmetic Expressions from Tondorf and Prediger (2022)

To my knowledge, Tondorf and Prediger is the only study that has explored students' development of both process and product views using a QR approach. They sought to examine the trajectory of students' meaning of equivalence as they transitioned from process-oriented to product-oriented perspectives. Their study aligns with a QR approach because they asked students to express and describe their arithmetic expressions in terms of their quantities in the context of making equivalent floor plans (Figure 3). Tondorf and Prediger (2022) note that the focus on arithmetic is a limitation of their study, and that a crucial next step is creating a trajectory that includes algebraic expressions with variables. Thus, the present study extends their work from arithmetic reasoning to algebraic reasoning at the secondary level. Moreover, according to Kieran (2018a), moving from process-oriented to product-oriented perspectives is also a precondition for pattern generalization activities, which is elaborated in the next section.

Generalizing via Visual Patterns

Over the last three decades, generalizing from numerical and visual patterns has received much research interest (e.g., Bourke & Stacey, 1988; Mason, 1996; Orton, 1999; Radford, 2000; Rivera & Becker, 2011; Stacey, 1989). Empirical studies have demonstrated the power of using figural over numeric patterning tasks for developing generalizations of linear functions (e.g., Garcia-Cruz & Martinon, 1997; Lannin et al., 2006; Markworth, 2010; Walkowiak, 2014). Researchers have also captured the progression of students' generalizations as participants in instructional interventions (e.g., Rivera & Becker, 2008).

Typically, these figural patterning studies use tasks in which a growing visual pattern is presented for several items in a sequence, along with a brief contextual storyline (Kieran, 2022). As an example, the 'Upside-Down T' plant task (Wilkie & Clarke, 2016) presents a situation where a plant grows each day in the same way. The task presents a plant with four leaves on Day 1, seven leaves on Day 2, and ten leaves on Day 3 and asks the learner to generalize the pattern between the Day Number and the number of leaves and then express that relationship using algebraic symbols (Figure 4).

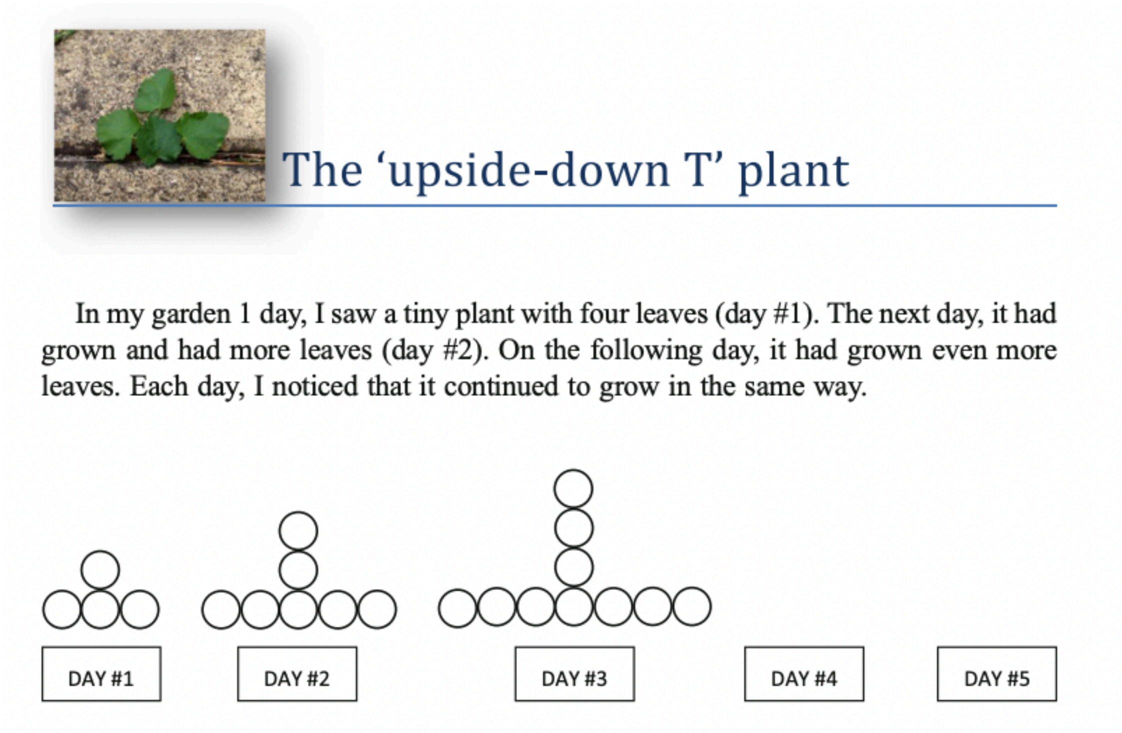


Figure 4: 'Upside-Down T' Plant Task

Researchers advocating for the use of figural patterns in helping students comprehend algebraic expressions and equations point to the importance of grounding generalizations and algebraic symbolization in the contextual referents of visual patterns (e.g., Lannin, 2005). Thus, much of the research on generalizing from visual patterns can be seen as consistent with a QR approach, even if the language of QR is not used. Other researchers are more explicit about the connection to QR. For example, Moore et al. (2022) argue that using figural contexts is organic to students' engagement with QR. They present an example contrasting making sense of a symbol "3" with interpreting a segment representing a difference in heights. According to Moore and colleagues, the symbol "3" does not afford the enactment of QR. A segment, on the other hand, naturally provides figurative information that elicits quantitative meaning (i.e., having a measure of 3 cm).

Despite the promise of grounding the comprehension of algebraic expressions and equations in the generalization of visual patterns, subsequent student difficulties have been identified. First, rather than generalizing a relationship between an independent variable (typically the figure number in a sequence of figures) and a dependent variable (representing the number of items corresponding to that figure number), many students focus on recursive patterns instead and then struggle to generate a correct algebraic representation (Barbosa & Vale, 2015; Ellis, 2007; Lannin et al., 2006; Markworth, 2010; Orton et al., 1999; Rivera, 2006; Rivera and Becker, 2007; Warren, 2006). An example of evidence of a recursive approach would be "when the item number increases by 1, the item increases by 2." The second approach is called correspondence, in which the learner perceives the relationship between two quantities or variables. For instance, a learner might perceive a relationship between the item number and a quantifiable aspect of the item itself, such as the total number of pieces (also known as an *explicit generalization*, *direct*, or *closed form*).

Second, empirical studies have shown that students often rely on numerical strategies even when engaging with figural pattern tasks (Markworth, 2010; Rivera & Becker, 2005). This tendency is problematic, as figural reasoning has been found to support the development of key algebraic concepts (Garcia-Cruz & Martinon, 1997; Lannin et al., 2006; Markworth, 2010; Walkowiak, 2014). For instance, in one study with pre-service teachers, Rivera and Becker (2007) found that 26 of the 42 participants reasoned numerically, while the other 16 used figural reasoning. They found that those who used figural reasoning strategies were more successful on the tasks and had a better understanding of the generalizations they formed. A more recent study with middle school students found that individually, students did not automatically intuit algebraically useful spatial characteristics of figures but required interaction in pairs to help them notice and then reason figurally (Montenegro et al., 2018). These findings suggest visual growth patterns may afford students ways of generalizing functions that non-figural function tasks do not. At the same time, they indicate that students may require additional support in utilizing figural reasoning when generalizing functional relationships.

Third, empirical studies show that students struggle to generate justifications for specific types of generalizations. In their study with 6th graders, Rivera and Becker (2008) identified a *deconstructive* generalization of a linear figural pattern (like the one in Figure 4), which applies in cases in which learners generalize based on initially seeing overlapping sub-configurations in the structure of the cues (e.g., seeing overlapping circles in Figure 4). Justifying a deconstructive generalization was found to be significantly more difficult for middle school students than creating and justifying *constructive* generalizations (e.g., seeing repeatedly adding circles in Figure 4). More formally, these refer to direct or closed degree-1 polynomial forms that learners easily abduce or induce from the available cues as a result of cognitively perceiving figures that structurally consist of non-overlapping constituent parts.

Finally, even when students in this type of study are able to generalize a correspondence relationship between independent and dependent variables and express the generalization using algebraic symbolization, they are seldom asked to reverse the direction and make sense of the quantities involved in their expressions after they have the generalized form. Thus, this research area would benefit theoretically by exploring how students make sense of their generalizations using QR.

Summary and Need for LT-Research

In summary, I reviewed four interrelated subtopics within the field of algebraic reasoning from a QR perspective. These four areas are related in that they serve as fundamental building blocks for making sense of algebraic expressions and equations. Moreover, the relationships between these (when viewed from a QR approach) are complex and develop over time. In this study, I investigate the evolution of students' ways of reasoning about these four aspects of algebraic expressions and equations from a QR approach by generating a learning trajectory. A learning trajectory is broadly defined by "descriptions of the successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic over a broad span of time" (National Research Council, 2007, p. 214).

Prior work has developed learning trajectories in equivalence (Tondorf & Prediger, 2022) and meanings of variable (Blanton et al., 2017) but only in the context of arithmetic. For instance, Tondorf and Prediger (2022) documented a progression of students coming to understand descriptive equivalence of arithmetic expressions such as $3(10+5) = 3 \bullet 10 + 3 \bullet 5$. However, a trajectory of equivalence of expressions in an algebra context is needed. Similarly, the exploration of students' development of process and product perspectives, though essential, has not yet been developed into a learning trajectory. Thus, this study extends existing literature by generating a learning trajectory that traces how students' reasoning develops across four interrelated domains essential to understanding algebraic expressions and equations from a QR perspective. Furthermore, doing so will meet the first aim of this study which is to explore how secondary students learn to make sense of algebraic expressions and equations in context over time.

Dialogic Videos and Vicarious Learning

Dialogic videos are instructional videos that portray two or more people reasoning together about a topic. The learning that occurs through observing and engaging with dialogic videos, or with any video- or audio-taped presentations of others learning, is referred to as *vicarious learning* (Chi et al., 2008). Depending on their design, such as whether the dialogue is scripted or unscripted, who the actors are (e.g., peers, tutors, teachers), and what content is featured, dialogic videos can vary in how they support students' vicarious learning. As a result, understanding the mechanisms through which dialogic videos promote learning has become a growing area of research interest. The roots of vicarious learning trace back to Bandura's foundational studies in the 1960s, which demonstrated that children could acquire new behaviors simply by observing adults, even through mediated forms such as film (Bandura et al., 1961, 1963). While Bandura's work focused on modeling social behaviors, his findings offered early evidence that observation alone could have powerful learning effects, laying the groundwork for subsequent applications in academic contexts.

The notion of vicarious learning entered the education domain with the Scottish Vicarious Learning Project (McKendree et al., 1998), which shifted the focus from social behavior to academic learning. These researchers argued that dialogue was central to learning because it offers opportunities for argumentation and logical reasoning. Building on this premise, many studies in science education have explored the relative effectiveness of dialogic videos compared to monologic videos. These studies are quantitative in nature, with participants randomly assigned to a monologic condition (asked to view a video featuring a single speaker) and others to a dialogic condition (asked to view a video featuring a conversation between two people). For instance, in a comparative study with undergraduate physics students, Muller (2008) found that students who viewed dialogue performed significantly better on a post-test than those who saw the monologic video. Numerous other studies have confirmed these results, indicating that dialogic videos tend to promote higher learning gains than monologic formats (Chi et al., 2017;

Driscoll et al., 2003; Fox Tree, 1999; Gholson & Craig, 2006; Muldner et al., 2014; Muller et al., 2007).

Empirical studies have also identified several features of dialogic videos that are productive for vicarious learning. One such feature is the presence of misconceptions (Boesdorfer et al., 2011; Muller, 2008). In the same study as above with physics undergraduates, participants who viewed monologic videos that did not include misconceptions interpreted the video in ways that reinforced their existing misconceptions (Muller, 2008). On the other hand, while viewers of the dialogic video found the content to be more confusing, participants saw a decrease in their misconceptions. Another study with 81 pre-service teachers enrolled in science methods courses found similarly that teachers' explanations about the causes of the change in seasons improved and contained fewer misconceptions as a result of watching a dialogic video that featured incorrect explanations of the phenomenon (Boesdorfer et al., 2011). These findings suggest that the presence of misconceptions in videos within dialogic exchange within a science context can promote learning.

Another productive feature of dialogic videos is the use of deep-level questioning. In a series of studies, Craig and colleagues found that when dialogic videos included questions probing underlying concepts (e.g., How does CPU use RAM when you are running an application?) learners showed significantly greater learning gains (Craig et al., 2000; Craig et al., 2006; Craig et al., 2012). In related work Gholson et al. (2009) and Sullins et al., (2010) observed that the presence of deep questions not only enhanced performance but also increased metacognitive abilities among viewers. A particularly telling study demonstrated a significant correlation between the number of deep questions posed by a tutee in a dialogic video and the number subsequently asked by student viewers ($r = 0.30$, $p = 0.01$), suggesting that deep questions may have encouraged viewers' ability to ask deep questions themselves.

Vicarious Learning Studies in Mathematics

In contrast to studies conducted in science contexts, vicarious learning studies in mathematics education have predominantly focused on the nature of engagement of learners when watching dialogic videos (Foster, 2023; Lobato & Walters, 2019; Lobato et al., 2023; Weinberg et al., 2022). The studies illustrate several promising findings for the use of dialogic videos as a tool to use in the classroom. In one exploratory study with a pair of high school students, Lobato et al. (2019) found that vicarious learners (VLs) approached the research sessions where they were asked to watch dialogic videos in four distinct ways. Two of the approaches to the research sessions were productive (Definition and the Concept Image). In these instances, VLs used ideas that were presented by the video participants in their own work. Meanwhile, in the other approaches (Video Expert and Procedure), VLs viewed the videos without attending to certain mathematical concepts. Thus, while making use of mathematical ideas from the videos is possible, it is not guaranteed by watching alone.

In another study, Lobato and Walker (2019) examined a pair of VLs' orientations (their stance, behaviors, and display of emotions) toward the video participants when asked to view dialogic mathematics videos. The case study found instances where VLs spoke directly to the video screen, shared strategies that could help the video participants, and acted as if video participants had access to their own work. They also found that the VLs were able to take on the perspective of the video participants, as evidenced by their descriptions of the video participants' mathematical personalities, predictions of their mathematical work, and attention to the details of the video participants' methods and ideas. These behaviors and displays of emotion suggest that vicarious learners take an emotionally-involved, quasi-collaborative stance toward the video participants while engaging with dialogic mathematics videos.

While the above studies suggest several benefits of engaging with dialogic mathematics videos, they fall short of investigating how viewers developed mathematical meaning from viewing the videos. In contrast, Lobato et al. (2023) explored how two secondary school VLs developed meanings about parabolas from dialogic mathematics videos. Their study found that the process by which VLs were able to make meanings was complex. The VLs sometimes resisted voices (words, meaning, or ideas) from the videos or simply mimicked them, before they were able to engage in negotiation with the mathematical meanings and make them their own. For instance, with regard to an important idea in the videos, "*p* as a distance," the VLs initially showed resistance and later engaged in a negotiation process with an incorrect idea presented by the video participants earlier in the session. Through interaction and negotiation, they were able to develop the meaning of "*p* as a distance." In particular, the VLs benefited mathematically from having a partner to negotiate the voices that were presented in the videos.

Together, the literature on vicarious learning in mathematics suggests a need for (a) investigating how students develop mathematics ways of reasoning over time through making regular use of the videos and (b) exploring how dialogic videos may be used for classroom use. I make an argument for each given the literature I have reviewed.

First, it is apparent that many vicarious learning studies in science education follow a similar design, in which participants are asked to watch a short video clip (e.g., a 10-minute video about Newtonian physics in the case of Muller (2008)). Then, their learning is measured via comparisons on pre/post assessments. In the context of learning mathematics, it is important that students wrestle with concepts over time, as is illustrated in the prevalence of learning trajectories in mathematics education (e.g., Blanton et al., 2017; Ellis et al., 2016; Gruver, 2018; Markworth, 2010; Rivera & Becker, 2008;).

A possible explanation for the focus on shorter videos in science education is that shifting conceptions of science topics requires less time. On the other hand, as was demonstrated in several studies in mathematics (e.g., Ben-David Kolikant & Broza, 2011; Lobato et al., 2023), VLs were not able to shift their thinking after viewing one video. For example, Ben-David Kolikant and Broza (2011) found that students' explanations of fraction expansion did not improve after viewing a short dialogic video clip. However, when the tutor in the research session provided scaffolding, the student participants were able to make use of the video as a tool. These findings suggest that investigating students' learning across multiple sessions would be beneficial. While other literature in mathematics has utilized multiple sessions (e.g., Lobato et al., 2019; Lobato & Walker, 2019), these studies do not have an explicit focus on the evolution of mathematics learning. Additional research is needed that identifies learning trajectories of viewers' evolution of reasoning as they engage with multiple dialogic videos over time.

Second, the reviewed literature suggests that there is a need to investigate how dialogic videos may be used in the classroom. While empirical work in vicarious learning seeks to implement dialogic videos as an instructional tool for learning, this goal has not yet been realized in STEM education. Indeed, prior studies in science education have primarily been conducted in the context of laboratories, while studies in mathematics education have had little to no teacher intervention. On the other hand, existing literature, primarily in mathematics, has laid the groundwork for the use of dialogic videos for instructional purposes. For instance, these studies have revealed that VLs can find mathematical ideas from the videos salient and useful (Lobato et al., 2019), develop mathematical meaning through peer negotiation and interaction (Lobato et al., 2023), and benefit from teacher support while viewing videos (Ben-David Kolikant & Broza, 2011). Therefore, this study situates itself within a classroom context by using a classroom TE methodology (Cobb, 2000). This addresses a gap in the literature by realizing the potential of dialogic videos for a pedagogical purpose.

Theoretical Framing

This study explores several important aspects of a technology-rich classroom environment: a group of learners, a technological artifact (dialogic mathematics videos), the teacher, and the mathematics involved (understanding algebraic expressions and equations from a QR perspective). In order to capture the relationships of interest, I draw on two theoretical perspectives: the Instrumental Approach (for Research Question 2) and the Emergent Perspective (for Research Question 1).

This section is organized into three main subsections. Rather than present the frameworks in order of their relevance to the research questions, I will first present the Instrumental Approach as the overarching theoretical framing for Research Question 2. Within this subsection, I define the theoretical construct of instrumental orchestration (which will be used to answer Research Question 2). The Instrumental Approach is presented before the Emergent Perspective because it is the dominant theoretical framework of this study. In the second subsection, I argue for the need for an additional theoretical framework to answer Research Question 1. Thus, I turn to the Emergent Perspective. Lastly, I discuss the relationship between the Instrumental Approach and the Emergent Perspective.

Instrumental Approach

The instrumental approach, as proposed by Verillon and Rabardel (1995), focuses on how learners interact with artifacts to achieve specific goals under the guidance of a teacher. This approach encapsulates various perspectives on artifacts' roles, represented as faces of a tetrahedron (Rezat & Sträßer, 2012), thus forming the foundation of this study's theoretical framework (Figure 5). For instance, an instrumental lens might be used to understand how learners use dynamic geometry software (a technological artifact) to construct geometric shapes with the teacher guiding their actions to achieve the goal of solving geometric mathematics problems.

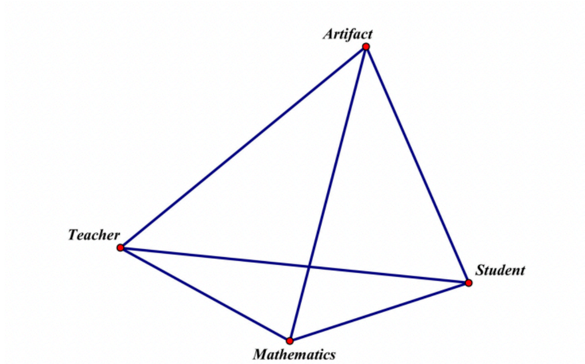


Figure 5: Tetrahedron Model of the Instrumental Approach from Rezat and Sträßer (2012)

Learning mathematics in a technology-rich environment involves both cognitive and social dimensions. Historically, instrumental approaches have theoretical foundations in both cognitive ergonomics (Rabardel, 2002; Verillon & Rabardel, 1995) and the anthropological theory of didactics (Chevallard, 1999). Since then, researchers in mathematics education have taken theoretical departures, some with a cognitive focus on Piagetian schemes and techniques (e.g., Roorda et al., 2016), and others with an emphasis on the Vygotskian notion of mediation of artifacts on learning (e.g., Alqhatani & Powell, 2017).

Using the latter perspective is useful for the purposes of this study for two reasons. First, I argued in a previous section that situating the study in a classroom context was needed in the literature on vicarious learning. Second, using Piagetian schemes to document how students make use of dialogic videos would be unconventional because videos are not a manipulable technology that one can act on in a traditional sense (for more detail, see the section Video as an Artifact). Thus, I follow the theoretical framing that emphasizes the social aspects of learning mathematics with artifacts. With this framing in mind, I turn to the sections of Video as an Artifact and the instrumental orchestration construct.

Video as an Artifact

Central to the theory of instrumental approach is the assumption that subjects may perform actions on objects using an artifact (Verillon & Rabardel, 1995). In the traditional sense, performing an action may mean that learners can directly manipulate and create new objects or relationships. A video is indeed distinct from manipulable technologies explored in other studies. Unlike in manipulable adaptive technologies, the control that learners have over videos is limited to functionalities like pausing, fast-forwarding, and rewinding. Viewers are also limited in that they cannot alter or create new videos within the system.

However, the conception of action can be broadened to include engagement with videos through other means. For instance, learners may act upon a video by using language that was stated in the video, or applying techniques or ideas that they heard in the video to solve their own tasks. The use of verbal utterances, gestures, or other mathematical activity can represent "actions" that enable them to acquire meaning.

In addition, dialogic videos offers unique affordances which may enable viewers to actively engage in learning mathematics. In particular, the videos are edited so they focus on key elements of the video participants' problem-solving, the videos are annotated with labels that highlight the video participants' work, and voiceover summaries are included that summarize what occurs in the video. Thus, these features highlight how dialogic mathematics videos can be experienced as a technology that is used to perform a range of tasks, which aligns with others' conceptualization of an artifact (Drijvers, 2015). By broadening the conception of action to include these types of engagement, this study aims to explore how dialogic videos become meaningful artifacts for learners engaged in mathematics tasks.

Instrumental Orchestration

As a way to account for the role of the teacher within the instrumental approach, Trouche (2004) introduced the notion of *instrumental orchestration* (IO). This construct refers to the systematic organization of artifacts by teachers to guide students' instrumental genesis, which complements instrumental genesis by focusing on teacher-artifact-learners interaction. In particular, instrumental orchestration accounts for the organizational components of teaching, such as the designing, planning, and managing of artifacts in technology-rich environment. Instrumental orchestration draws on the metaphor of a teacher functioning as a conductor of an orchestra. Drijvers and Trouche (2008) note that while "orchestration" suggests a formal symphony, the metaphor is more akin to a jazz band that allows for improvisation and includes a mix of novices and more advanced musicians. This conceptualization draws parallels with a technology-rich classroom environment. In addition to the conductor, who serves as a teacher, the students function as musicians who are using technological artifacts, which serve as instruments. This study employs instrumental orchestration as a lens to explore how the teacher orchestrates or manages the use of unscripted dialogic mathematics videos, along with classroom discourse, to support a classroom community's mathematical reasoning.

Drijvers et al. (2010) created a typology of whole-class instrumental orchestrations by coding the actions of three teachers and 38 mathematics lessons. Since then, other researchers (e.g., Bozkurt & Ruthven, 2018; Eckert & Nilsson, 2022; Savuran & Akkoc, 2023; Trouche & Drijvers, 2014) have validated Drijvers et al.'s (2010) typology and identified additional orchestration types. Table 1 (Tabach, 2013) provides a summary of instrumental orchestrations that have been identified in the literature. These studies provide insight into possible ways for teachers to orchestrate technologies to support mathematics learning that might extend to other settings.

Table 1: Instrumental Orchestration Types

Orchestration types	Didactical configuration	Exploitation mode/Didactical Performance	References
Technical-demo	The whole-class setting, one central screen	The teacher explains the technical details for using the tool.	Drijvers et al. (2010)
Explain-the-screen	The whole-class setting, one central screen	The teacher's explanations go beyond techniques and involve mathematical content.	Drijvers et al. (2010)
Link- screen-board	The whole-class setting, one central screen	The teacher connects representations on the screen to representations of the same mathematical objects that appear either in the book or on the board.	Drijvers et al. (2010)
Sherpa-at-work	The whole-class setting, one central screen	The technology is in the hands of a student, who brings it up to the whole class for discussion.	Trouche (2004)
Not-use-tech	The whole-class setting, one central screen	The technology is available but the teacher chooses not to use it.	Tabach (2011)
Discuss-the-screen	The whole-class setting, one central screen	Whole class discussion guided by the teacher to enhance collective instrumental genesis.	Drijvers et al. (2010)
Spot-and-show	The whole-class setting, one central screen	The teacher brings up previous student work that he/she had stored and identified as relevant for further discussion.	Drijvers et al. (2010)
Work-and-walk-by	Students work individually or in pairs with computers	The teacher walks among the working students, monitors their progress, and provides guidance as the need arises.	Drijvers (2012)

Table 1: Instrumental Orchestration Types (Continued)

Orchestration types	Didactical configuration	Exploitation mode/Didactical Performance	References
Discuss-tech-without-it	Each student has their laptop or laptops are brought to the classroom with wheeled vehicles	The teacher uses a mobile transport system if he/she needs computers in teaching.	Tabach (2013)
Monitor-and-guide	N/A	The teacher uses a learning management system by giving guidance to students.	Tabach (2013)
Predict-and-test	Students work individually or in pairs with computers	The teacher encourages the students to test their ideas in the technology and to confirm or modify them in light of the results.	Bozkurt & Ruthven (2018)

Summary and Need for an Additional Theoretical Framework

Thus far, I have discussed the primary assumptions of the instrumental approach, which include the construct of instrumental orchestration. In this study, I will use this construct to answer Research Question 2, which asks which orchestrations of the video by the teacher supported the emergence of students' reasoning about algebraic expressions and equations from a QR approach.

As stated before, the aim of Research Question 1 is to investigate how students learn about algebraic expressions and equations from a QR perspective over time. Previous research utilizing the instrumental approach has tended to investigate individual students' evolution of instrumentation schemes (in the sense of Piaget) over time (e.g., Roorda et al., 2016). Indeed, a study could be designed in which one generated a learning trajectory consisting of a set of instrumentation schemes for a pair of students across multiple sessions. However, there is a need in the vicarious learning literature for studies that can offer guidance on how dialogic videos can be used in a classroom. Thus, using at least a small classroom of students (in this study, 13 students) is desirable.

A critical reader might note that one could arrange the small classroom of students in pairs, give each pair access to their own computer, and try to study the evolution of instrumentation schemes for each of the four pairs over time. There are three reasons why this is not desirable or feasible. First, this approach would require a larger team of researchers to capture the audio and video-taped data for each pair. Second, the teacher-researcher would likely experience a loss of information regarding the evolution of each pair. While normally one could draw on the literature to account for this loss, existing work in vicarious learning is currently insufficient in providing information because classroom studies are new to the field. Finally, it is hypothesized that students would benefit from hearing the ideas of each other as they engage with dialogic video as a whole class. Therefore, a theoretical framework is needed for studying the ways of reasoning that become accepted by a classroom community as they learn to comprehend algebraic expressions and equations via the use of dialogic mathematics videos. Consequently, I turn to the emergent perspective (Cobb & Yackel, 1996).

The Emergent Perspective

The emergent perspective (Cobb & Yackel, 1996) is a theoretical approach which considers individual constructs as being coordinated with social constructs. A primary assumption of the emergent perspective is that learning occurs both socially and psychologically and that the processes of learning at the social and psychological levels share an indirect yet reflexive relationship (Cobb & Yackel, 1996). It is an important distinction in the emergent perspective that neither the social nor the psychological perspective is given primacy over the other. Rather, each perspective provides a backdrop when considering the other (Yackel & Rasmussen, 2002), making it a flexible lens for research.

The interpretive framework (see Figure 6) associated with the emergent perspective captures the distinct yet interrelated aspects of the learning process in the classroom (Cobb & Yackel, 1996). The psychological (individual) constructs include (1) beliefs about one's own role, other's roles, and the general nature of mathematical activity; (2) mathematical beliefs and values; and (3) mathematical conceptions and activity. The social constructs include (1) social norms, (2) sociomathematical norms, (3) and classroom mathematical practices. The construct of classroom mathematics practices, also known as *collective mathematics practices*, will be the focus of Research Question 1.

Social Perspective	Psychological Perspective
Classroom social norms	Beliefs about own role, others's roles, and the general nature of mathematical activity in school
Sociomathematical norms	Mathematical beliefs and valued
Collective mathematical practices	Mathematical conceptions and activity

Figure 6: Interpretive Framework

Collective mathematics practices refer to ways of operating that have become accepted in the classroom community. By *collective*, researchers refer to a quality of a group rather than individuals' ways of thinking. As such, the classroom community is thought of as its own entity that exists outside of the characteristics of its individual members. However, Cobb and Yackel (1996) maintain that the relationship between collective mathematics practices and its individual counterpart, mathematical conceptions and activity, is indirect and reflexive. In other words, ways of operating that become accepted influence, but do not determine, individual students' conceptions. In the other direction, collective mathematics practices emerge through argumentation that occurs between individuals (Cobb et al., 2003). For this reason, other researchers have referred to collective mathematics practices as *emergent mathematics practices.* Going forward, I use the term collective mathematics practices to emphasize that this construct refers to qualities of the group rather than features of an individual.

Conceptualizations of collective mathematics practices have changed over time. There are three features that distinguish earlier conceptualizations (e.g., Cobb & Yackel, 1996) to more recent work (e.g., Rasmussen et al., 2024). The first is the terminology used to describe how collective mathematics practices become normative in the classroom. According to Cobb and Yackel (1996), ways of operating, arguing, and using tools in the classroom become normative when they are *taken-as-shared* (i.e., they no longer need justification). The phrase "taken-as-shared" is meant to reflect the fact that while individuals' ideas and interpretations cannot be known, they nevertheless can be collectively agreed upon, and thus, create a shared way of communicating in the classroom. However, Rasmussen and Stephan (2008) modified this terminology to talk about ways of reasoning that "function-as-if shared" rather than as ones that are taken-as-shared. The shift in phrasing to include "function" reflects the methodology used to operationalize collective mathematics practices, which involves analyzing how utterances function in an argument and by certain criteria (for more detail, see the Analysis section). Whether these ways of operating are "taken-as-shared" or "function-as-if-shared," in both cases, they are said to form a *normative way of reasoning* (NWR).

The second distinction refers to how normative ways of reasoning relate to collective mathematics practices. Initially, Cobb and Yackel (1996) focused on one normative way of acting per practice, whereas, in more recent work, related ways of reasoning are clustered together to form collective mathematics practices (e.g., Rasmussen & Stephan, 2008; Rasmussen et al., 2015). Finally, the nature of collective mathematics practices and the language used to describe them has changed over time. Earlier work was primarily interested in documenting observable actions or strategies such as how students transformed the packing arrangement of candies (e.g., Bowers et al., 1999). However, other researchers have taken collective mathematics practices to describe how ways of operating can function-as-if-shared using cognitive language of meanings or ideas (Lobato & Walters, 2017). In sum, although conceptualizations of collective mathematics practices have evolved, this study will follow other researchers (e.g., Gruver, 2018; Stephan & Rasmussen, 2002; Tobias, 2009) in defining collective mathematics practices as a cluster of ways of reasoning (which may include meanings, ideas, or interpretations) that function as if shared in the classroom. I refer to these ways of reasoning as normative ways of reasoning.

To answer Research Question 1, I draw on the *collective mathematics practices approach* to learning trajectory, which captures the evolution of collective mathematical practices within a classroom community (Lobato & Walters, 2017). A learning trajectory from this approach consists of a “sequence of [collective] mathematical practices together with conjectures about the means of supporting their evolution from prior practices” (Cobb, 1999, p. 9). This approach to learning trajectories contrasts with others focusing on individual thinking. For example, the *cognitive-levels approach* features a hierarchy of cognitive milestones of increasing sophistication. The *disciplinary-logic approach* generates a trajectory by reflecting on experts’ knowledge and by synthesizing research from studies of student knowledge.

One example of a learning trajectory from this approach is Gruver’s (2018) progression of students’ conceptual understanding of logarithmic relationships. An example from this study of a collective mathematics practice “interpreting logarithms.” This mathematical practice included a cluster of three normative ways of reasoning: “logarithms are exponents,” “the ‘on what day’ interpretation of logarithms,” and “the factor interpretation of logarithms.” The first interpretation was consistent with the desired interpretation of a logarithm as a number. The other interpretations, while tied to a context, were suggested to be important for making sense of logarithm rules. Hence, by assessing the evolution of collective mathematics practices, and specifically interpreting logarithms, Gruver (2018) was able to offer insight into the mathematical development of the classroom community.

As mentioned previously, one way of conceiving mathematical practices involves using cognitive terms to describe ideas, interpretations, or meanings that function-as-if-shared in the classroom. In the above example, the practice of “interpreting logarithms” documented three interpretations of logarithms that emerged in the class. In the context of this study, the nature of collective mathematics practices might similarly fit with a meaning or idea about algebraic expressions and equations from a QR approach. For instance, it may become normative that students shift from a numeric product interpretation of a term like $5c$ to a quantitative product interpretation. Product interpretations is one example of an emergent practice that could arise from individuals’ ways of reasoning (e.g., $5c$ as a numeric product, $5c$ as a quantitative product) that are then discussed, modified, and eventually accepted or rejected by community members.

Relationship between Instrumental Approach and the Emergent Perspective

The instrumental approach has been posited to be connected to other theoretical perspectives that include social and individual dimensions (Drijvers & Gravemeijer, 2013). Within the instrumental approach, individual processes are assumed to emerge from collective processes involving the artifact (Rabardel & Samurcay, 2001). Furthermore, in reference to individual processes that describe the function of an artifact (called utilization schemes), Rabardel and Sacuray (2001) write that these individual processes "concern the coordination of action, not only within the subject but also inter-subjects in collective activities, whether in everyday life, in training or in work" (p. 87). In other words, the individual and social dimensions are coordinated by actions of the learner and other learners in a social context.

Drijvers and Gravemeijer (2013) note that the emergent perspective similarly considers the individual and social dimensions of learning. The relationship between these dimensions entails the coordination of individual, psychological constructs with social constructs (Figure 6). In the case of the emergent perspective, this coordination is elaborated via the indirect and reflexive relationship between social norms, sociomathematical norms, and mathematical practices on the one hand, and individual beliefs and insights on the other. The emergent perspective does not however, consider the role of technology nor the role of the teacher. For this reason, Drijvers and Gravemeijer (2013) suggest that the emergent perspective may add value to an instrumental approach.

Research Questions Revisited

I now return to my research questions and elaborate on them in terms of the literature that I have reviewed. The first research question investigates students' learning of algebraic expressions and equations from a QR perspective over time by means of a collective mathematics practices learning trajectory. The second research question examines the orchestrations of the video artifact that support the emergence of collective mathematics practices.

Research Question 1: Learning Trajectory of Collective Mathematics Practices

RQ: What is the evolution of the collective mathematics practices of 13 Grade 9 students over the course of a TE about algebraic expressions and equations from a QR approach?

Elaboration. This research question is interested in the evolution of mathematical reasoning of 13 Grade 9 students as they engage with dialogic mathematics videos as part of a classroom TE (Cobb, 2000). The TE spanned 10 one-hour research sessions. This research question tracked the progression of the normative ways of reasoning of four subtopics of algebraic expressions and equations: (a) equivalence, (b) generalizations with figural patterns, (c) meanings of literal symbols, and (d) process and product perspectives. Each subtopic was addressed from a QR perspective. The TE made regular and frequent use of the unscripted dialogic mathematics videos created by Project MathTalk (www.mathtalk.org), specifically from the Algebraic Expressions video unit.

To answer this research question, a learning trajectory was constructed as a set of collective mathematical practices, which is grounded in Cobb and Yackel's (1996) emergent perspective. The collective mathematics approach to learning trajectories emerged in response to a need to explain regularities in collective behavior in a classroom community that may be missed by an approach focused on individual thinking. The trajectory consists of a list of mathematics practices (e.g., observable actions, interpretations, or normative ways of reasoning). Collective mathematics practices trajectories often include a narrative description of instructional supports that are influential to the emergence of such practices (Lobato & Walters, 2017). In this study, instructional supports wee investigated analytically in Research Question 2.

Cobb et al. (2001), adapting work by Simon (1995), distinguished between a hypothetical learning trajectory and an actual learning trajectory. In this context, a hypothetical learning trajectory involves conjectures about the classroom community's possible learning trajectory and the means of supporting, organizing, and guiding that development. The hypothetical learning trajectory provides a means of comparing what happens in the classroom TE (the actual learning trajectory) with what was conjectured, which feeds back to inform the instructional design effort (Bowers et al., 1999). It is important to note that the video lessons designed for Project MathTalk were designed according to a hypothetical learning trajectory (HLT). Given that the videos were be a primary source of instruction, the proposed conjectured learning trajectory contained elements from the HLT differed based on choices for the instructional design. The next Hypotheses section describes the conjectured learning trajectory in more detail.

Hypotheses. I hypothesized that six ways of reasoning would become normative in the class: (a) using a figural pattern to arrive at a generalization, (b) using figural reasoning to verify a generalization, (c) developing descriptive equivalence, (d) developing process and product perspectives with quantitative meaning, (e) treating literal symbols as generalized numbers and as varying quantities, and (f) treating literal symbols as being able to take on a range of values, including negative numbers and decimals.

I hypothesized that the first two sessions of the TE would be set in a pool-tiling context and follow the first video lesson. The Pool Task asks students to form a relationship between a dependent and independent variable (the number of tiles on the side of a square pool and the number of tiles in the border). This task uses a non-sequential number of tiles on the side for different square pools (e.g., 10 tiles on the side and 15 tiles on the side) in a figural pattern, which suggests that students will more easily generate generalizations that relate the dependent and independent variables (Kuchemann, 2010). Moreover, Rivera and Becker (2008) identified collective mathematics practices that supported students generalizing visual patterns. One that I expect to emerge for students is using a figural generalization to arrive at an expression for a pattern (i.e., use the pool figure to arrive at an expression).

The next two sessions will continue in the pool context and draw on the second video lesson, which introduces the generalization of a pattern that contains overlapping segments. In these sessions, I expect the practice to become function-as-if-shared that students will be able to use figural reasoning as a way of verifying that their generalization is valid. Rivera and Becker (2008) point out that generalizing a pattern with overlapping segments is difficult for students. However, I hypothesize that an emphasis on QR will allow students to generate this type of generalization and provide a quantitative justification for their generalized expression. I also expect students to go back and forth between the use of numerical and figural reasoning (Rivera & Becker, 2008) but ultimately rely on figural reasoning for their final justifications.

The three subsequent sessions use the third video lesson, which explores equivalent algebraic expressions in the context of friends buying game apps. One goal of the TE will be students' evidencing a descriptive interpretation of equivalence from a quantitative perspective. I hypothesize that students will first justify their equivalence from an operational approach by substituting a value. I also suspect that some students will rely on a symbolic justification (e.g., $5c + 3c = 8c$ because $5+3 = 8$). In order to move from these interpretations to a descriptive equivalence interpretation, students will need to develop process and product perspectives (Sfard, 1992). I hypothesize that transitional interpretations will emerge as a result of the difficulty and complexity of this task. For example, some students may have quantitative meaning for 5 and c (and similarly, 3, 8, and c), but may not evidence a fully quantitative process view (e.g., $5 \cdot c$ is 5 apps times the cost per app). On the other hand, some students may evidence a process view and can express a product view, but do not use a product perspective as a resource for justifying equivalence in context. Thus, conjectured ways of reasoning from these sessions are (c) developing descriptive equivalence and (d) developing process and product perspectives of subexpressions of algebra equations with quantitative meaning.

The last three sessions use the fourth video lesson, which explores the meaning of velocity in the context of a person taking a scooter on a trip. I anticipate that students will first express a natural number bias (Christou et al., 2022). The design of the TE is meant to disrupt problematic views of symbols; thus, I expect students to develop meanings for symbols as varying quantities that can take on negative and decimal values (ways of reasoning (e) and (f)). Additionally, the emphasis on QR will allow students to think about the meaning of symbols in context and correct errors when they occur (Moore & Carlson, 2012).

Research Question 2: Instrumental Orchestration

RQ: What instrumental orchestrations of the artifact of unscripted dialogic mathematics videos support the emergence of 13 Grade 9 students' ways of reasoning?

Elaboration. Whereas Research Question 1 explored the evolution of students' normative ways of reasoning about algebraic expressions, Research Question 2 explored the aspects of instruction that supported the emergence of normative ways of reasoning. These instructional supports were investigated from the perspective of the instrumental orchestrations of the video artifact. The shift in focus to teaching practice in this research question serves as a way of considering how the video artifact can be used for a mathematical pedagogical purpose.

An instrumental orchestration is defined by three components: (a) a didactical configuration, (b) exploitation mode, and (c) didactical performance (Drijvers et al., 2010; Trouche, 2004). These layers are rooted in the corresponding features of an orchestra: (a) the arrangement of musicians, (b) the prior designing and planning of a performance, and (c) the dynamic management of people and activities to achieve certain productive results (Drijvers & Trouche, 2008; Sharples & Anastopoulou, 2012).

A didactical configuration refers to the arrangement of the teaching environment and the artifacts within it, including both digital and non-digital tools (e.g., dialogic videos, calculators), their physical placement (e.g., shared screen vs. individual devices), and aspects of the classroom layour (e.g., seating arrangments conductive to collaboration or discussion). In this study, the didactical configuration involved projecting dialogic videos onto a shared screen in front of the classroom (see Figure 7), with students seated in small groups of 3-4, and working individually on paper worksheets aligned with the video participants' tasks from the video. This configuration was selected for several reasons. First, it was practical for data collection purposes, as a centralized display simplified video capture and observation logistics compared to distributing laptops to each group. Second, having the whole-class as main center of discussion provides the researcher more insight into the evolution of learning of the group. Third, it supported the pedagogical goal of fostering whole-class discussion, a key feature of inquiry-oriented mathematical classrooms needed for the design of the study (Lobato & Walters, 2017).

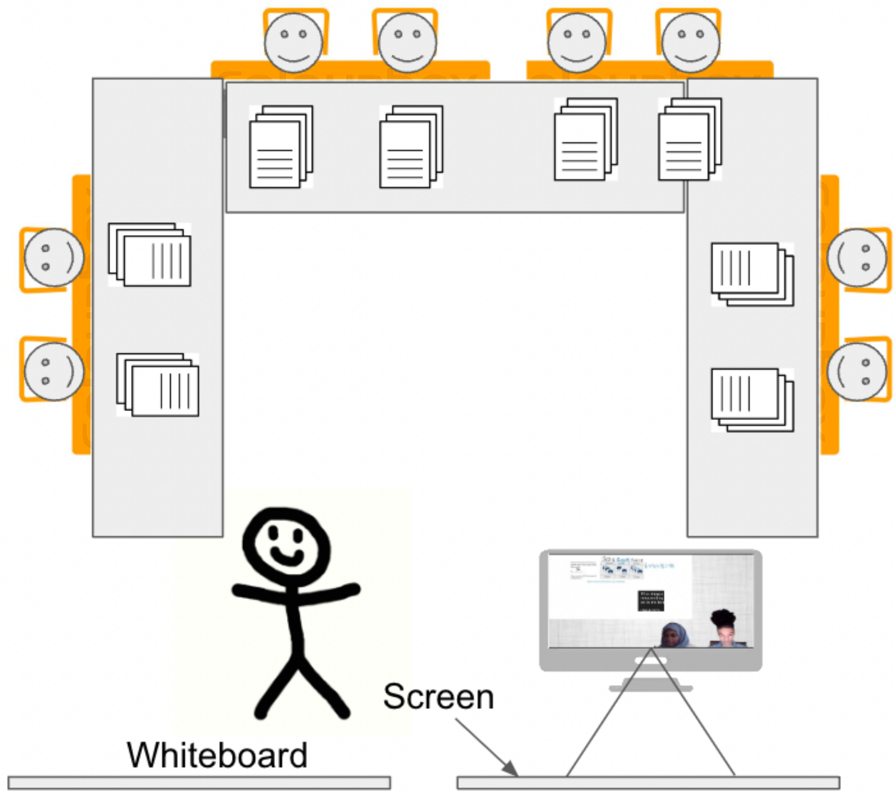


Figure 7: Didactical Configuration

Exploitation mode refers to the design decisions that a teacher makes toward their pedagogical goals. This is analogous to the decisions an orchestra conductor makes in anticipation of expected harmonies. Exploitation mode includes the decisions about how a task is introduced and worked through, possible roles of artifacts to be used, or conjectured ways of reasoning to be established by the students (Drijvers et al., 2010). In fact, Drijvers et al. (2010) liken the exploitation mode to what constitutes a hypothetical learning trajectory in the sense of Simon (1995). Drijvers et al. (2010) note that an exploitation mode can be thought of as the design of curriculum (typically researcher-controlled), the actual teaching actions (teacher-controlled), or some combination. This study conceives of an exploitation mode as the design decisions that occur prior to the TE, including decisions about which video segments to include, when and if the teacher will pause the video, teaching moves affiliated with each video episode, and which tasks will be presented when. Like the didactical configuration layer of IO, the exploitation mode was not subject to analysis and is instead presented in the Methods chapter (see section for Planned Teacher Design Decisions). However, relevant aspects of the exploitation mode influenced the actual teaching actions throughout the TE, which is captured in the third component of didactical performance.

Didactical performance accounts for the "on the fly" choices made by teachers during classroom activity. This component is defined as the ad hoc decisions that teachers make while teaching. These decisions concern how the chosen didactical configuration and exploitation mode are actually performed (e.g., what question to pose now, how to do justice to (or to set aside) any particular student input, how to deal with an unexpected aspect of the mathematical task or the technological tool, or other emerging goals). Because the didactical performance is the only component that is not determined a priori, this was subject of analysis for this research question. More specifically, analysis of this research question examined how the teacher's didactical performance guided students' ideas functioning as if shared.

The didactical performance was guided by the assumptions of the classroom TE methodology (Steffe & Thompson, 2000). Using this methodology required that I, as the teacher-researcher, create hypotheses of student learning in real time and engage in responsive and intuitive interactions with the students. Specifically, this means that the didactical performance differed from what is hypothesized in the exploitation mode.

Hypotheses. I hypothesize that instrumental orchestrations that will support the emergence of secondary school students' normative ways of reasoning include those that have been identified in the literature (i.e., discuss-the-screen, link-screen-paper, and predict-and-test; see Table 1). For instance, in one of the video episodes, one of the video participants, Haleemah, miscounts the number of tiles in the border around sa square pool with 10 tiles on a side as 40 tiles. I might post the question, “What do you think about what Haleemah did? Do you agree that there are 40 tiles in the pool?” This would be an example of discuss-the-screen because, hypothetically, a discussion about phenomena on the screen would follow. This discussion might be crucial to the students’ understanding of overlapping tiles, an idea that is revisited in a later lesson when they have to generate a new method. Other orchestrations, such as link-screen-paper and predict-and-test, may also be essential for the emergence of normative ways of reasoning.

This research question also seeks to identify instrumental orchestrations that are emergent. Because the construct of instrumental orchestration has not been used to explore video technology, it is likely that orchestrations will emerge that are specific to the features of the video. One hypothesized orchestration is (re)watch-and-notice. This orchestration stems from the teaching practice of noticing (Hohensee, 2017), where teachers steer students to attend to the mathematical features of a task environment. In this case, I would plan on asking students to watch or rewatch a video and direct their attention to a specific mathematical idea that is presented in the video. For instance, I might ask students, “Notice how Haleemah and ET write their equation when given only the number 50.” I will also periodically ask students, more generally, what they notice, after watching a video clip. I hypothesize that this orchestration would emerge as one that is important for the evolution of collective reasoning as well as an orchestration that is specific to video technology.

Chapter 3: Methods

In the previous two chapters, I have argued for the need to investigate students' learning to comprehend four intertwined aspects of algebraic expressions and equations from a quantitative-reasoning (QR) approach and as they regularly engage with dialogic mathematics videos over time. I made a case that within a classroom context, there are complex relationships between the classroom community, the video artifact, the teacher, and the mathematical content of interest.

This dissertation investigated two research questions:

1. What is the evolution of the collective mathematics practices of 13 Grade 9 students over the course of a TE about algebraic expressions and equations from a QR approach?
2. What instrumental orchestrations of the artifact of unscripted dialogic mathematics videos support the emergence of 13 Grade 9 students' ways of reasoning?

In order to explore the relationships between a teacher's actions, the classroom community, a digital artifact (unscripted, dialogic videos), and mathematics content, I use a classroom teaching experiment (TE) methodology (Cobb, 2000). This TE consisted of instruction on making meaning of algebraic expressions and equations from a QR approach. For the first research question, it was crucial to capture whole-class argumentation. For the second research question, it was important to capture the ways in which the teaching actions were related to the use of video throughout the TE. Therefore, whole-class observations, whole-class inscriptions, and participants' written work were collected.

The first research question explores the relationship between the teacher, the classroom community, and the mathematical content of interest (understanding of algebraic expressions and equations from a QR perspective).

To answer this question, I adopted a collective mathematics practices learning trajectory approach, which involves identifying clusters of ways of reasoning that had become accepted, or functioned as if shared in the classroom (in the sense of Rasmussen & Stephan, 2008, as described in Chapter 2). In particular, "mathematics practices" in this context do not refer to the Common Core State Standards of Mathematics (CCSSM) mathematics practices. To analyze the data, I followed the Documenting Collective Activity (DCA) method (see Data Analysis section for details).

The second research question explores the relationship between the teacher, the video tool, and the classroom community. My approach to analysis for this research question involved examining ways in which management, or orchestration, of the video tool by the teacher supported normative ways of reasoning that were identified in the first research question. A finer grain-size afforded a detailed analysis of the instrumental orchestrations that guided students' emergence of normative ways of reasoning. Both research questions used whole-class observations and whole-class inscriptions as data sources.

The remainder of this chapter is organized into five sections: (a) participants, (b) the classroom TE, (c) data collection, (d) analysis, and (e) validity. First, I describe the setting and the participants. Next, I describe the following elements of the classroom TE: (a) the general features of TEs, (b) the Realistic Mathematics Education approach to the instruction, and (c) the methodology and the instructional goals of the study. From there, through the lens of instrumental orchestration, I describe the principles guiding how I utilized the dialogic video tool in the TE. This includes a description of the mathematics videos used, the classroom organization (or the didactical configuration), and the teaching decisions for managing the tool to guide student learning (or the exploitation mode). Additionally, I provide an overview of the unit planned and implemented in the TE sessions, and which video lessons were drawn on for each session. Next, the data collection processes and the methods of analysis for each research question are described. Finally, I conclude the chapter with a discussion of validity measures taken to ensure the quality of my findings.

Participants

The research questions guiding this dissertation explore both student learning and teaching practice as they were enacted within the classroom TE. This meant that the participants included both the students and the teacher. In this section, I discuss how I selected students to participate in the study and why I served as a teacher-researcher.

Setting

The site for this study was a charter high school in a large city in the southwestern United States. The school had approximately 600 students, of whom 66% identify as Hispanic, 3% as White, 18% as Black, 5.5% as Asian, and 4.6% as other race or ethnicity. A teacher from this school was selected to assist me in recruiting students and providing a classroom to conduct the TE. This teacher was selected because of her willingness to support reform curricula and her familiarity with Project MathTalk videos.

Additionally, this site was chosen because students had some exposure to non-traditional instruction. After conducting several observations of the teacher’s classroom, it was clear that inquiry-oriented learning was encouraged. This is important because to answer Research Question 1, the classroom environment requires that students give arguments and explain their thinking (Lobato & Walters, 2017). Thus, the fact that students are expected in their mathematics classes to share their thinking and work in groups was beneficial for the success of the classroom TE and important for establishing normative ways of reasoning using the DCA method.

Students

Using criterion sampling (Patton, 1990), I recruited 13 Grade 9 students from an Integrated Math I class to participate in the TE during Fall 2023. Criterion sampling is a method used in qualitative research in which a set of criteria or characteristics important to the nature of the study are identified. Then, participants who fit the criteria are recruited. The first criterion of my study was that participants needed to have been introduced to the use of letters in algebra (e.g., $y = mx + b$) but not have yet mastered the material featured in the TE. To identify students who met this criterion, I asked the teacher to administer a 10-15 minute screening instrument that consisted of five mathematics tasks in mid-September 2023 to students in two of her Integrated Math 1 classes (about 50 students) (Appendix A).

Second, students were selected based on their ability to work well together and their comfort in articulating their thoughts. This second criterion was important because the analysis of collective mathematical practices (needed to answer Research Question 1) was dependent upon a classroom community in which students voice their arguments and respond to the reasoning of other students. Third, a strong attempt was made to ensure racial, ethnic, and gender diversity among the 13 participants. To meet the second and third criteria, I introduced the study to all students who had met the first criterion and sent the students home with a flyer and informed consent forms (parental permission and child assent). From those who returned the consent forms, I sought input from the students' teacher regarding the second and third criteria. Finally, to promote excitement about participating in the study and to honor the students' time (since the study occurred during after-school hours), a financial incentive was given to each participant, $10 per session and an additional $100 for completing all ten sessions. Nine students attended all ten sessions, three students attended nine sessions, and one student attended eight sessions.

Teacher

I served as the teacher-researcher. There are several reasons why it was important that I served as the teacher in the TE, as opposed to another teacher. First, I was involved in the development of the video tool and, thus, knew the material very well. I was involved in the filming and editing as well as in the creation of the voiceover summaries and teacher materials for the video episodes used in the TE. Second, I served as a researcher in a previous study that investigated how pairs of students made sense of the videos for one of the video lessons, without teacher intervention. Thus, the knowledge gleaned from that study of how students may engage with the videos made me well-suited to make informed teaching decisions.

The Classroom TE

Features of TEs

The purpose of a TE is to investigate the process by which students reorganize their mathematical ways of knowing over extended periods of time (Cobb & Steffe, 1983; Steffe & Thompson, 2000). In this dissertation, I was interested in how students reason over time about four intertwined algebra topics related to understanding algebraic expressions and equations from a QR perspective. Thus, a TE was well-suited to meet this goal. Furthermore, the condition of regular engagement with the innovation of dialogic mathematics videos was not met in current classrooms. Thus, a TE allowed me to bring about the conditions that I wish to study.

TEs are characterized by several key features that distinguish it as a research methodology. First, TEs are guided by conjectures or hypotheses about students' mathematical reasoning. As Confrey (2000) explains, a conjecture is a theoretically-grounded idea that emerges from a dissatisfaction with an existing conception and points toward a reconceptualization worth exploring. In TEs, hypotheses often take the form of hypothetical learning trajectories of how students' conceptions might evolve in response to particular tasks and teaching actions. Unlike experimental designs that test whether an intervention was effective, TEs focus on generating explanatory models of students' thinking and adapting instruction to refine those models in real time (Cobb et al., 2003).

Second, TEs are defined by their responsiveness to students' reasoning. Instruction is deliberately opportunistic, designed to build on students' spontaneous ideas rather than follow a predetermined plan. This responsive teaching supports the dual goals of maximizing student learning and deepening researchers' understanding of how that learning evolves. The third essential feature is the presence of an observer who participates in post-session debriefings to reflect on students' conceptual activity (Steffe & Thompson, 2000). In my own TE, I worked closely with a fellow doctoral student who served as an observer. After each session, we debriefed to analyze students' reasoning. I then used these reflections to revise upcoming activities. Through this iterative process, I developed, tested, and refined my models of student reasoning as they engaged with the mathematical tasks and dialogic mathematics video tool.

Classroom TEs represent an extension of the original one-on-one constructivist TEs developed by Steffe and colleagues (Cobb & Steffe, 1983; Steffe, 1983), shifting the focus from individual learning to the emergence of collective mathematics practices. Rather than tracing individual learning trajectories, classroom TEs are designed to investigate how shared ways of reasoning emerge within a community of learners. Grounded in the emergent perspective (Cobb & Yackel, 1996; Cobb 2000), this approach emphasizes the co-construction of normative mathematics practices and classroom social norms. Researchers have advanced this methodology to trace the evolution of classroom conceptual activity, supported by analytic tools like the documenting collective activity method (Rasmussen & Stephan, 2008), which has since been used in other STEM disciplines (Cole et al., 2012). In this way, classroom TEs affords researchers a way of investigating how the collective reasoning of a classroom community becomes increasingly sophisticated through interaction with peers, the teacher, and instructional sequences.

Realistic Mathematics Education

The instructional sequence for the TE was guided by the principles of Realistic Mathematics Education (RME) (Gravemeijer, 2020). RME was chosen not only for its emphasis on supporting rich mathematical reasoning, but also because it underpins the instructional sequences in the Project MathTalk videos. To ensure coherence between classroom instruction and the dialogic video tools, it was important to adopt a similar instructional approach. While the mathematical progression of the classroom community was expected to differ somewhat from that of the video participants, aligning the design principles helped ensure that the video tool complemented and extended the teacher-led instruction.

RME is a domain-specific theory of instructional design in mathematics, grounded in several core principles. For instance, a central tenet of RME is that mathematical learning should begin with problems situated in contexts that are realistic or imaginable to students. “Realistic” in this sense refers not necessarily to everyday experiences but to situations that students can envision and find meaningful (Drijvers et al., 2011). For example, tasks involving visual patterns allow students to imagine how a pattern continues and make meaningful inferences. At this early stage, students typically engage in informal reasoning that remains anchored to the context. Their thinking is captured through a “model of” the situation, an initial representation that reflects their understanding of the specific problem setting.

RME is also grounded in the idea that mathematics is a product of human invention and social activity. This belief shapes instructional designs that encourage students to engage in *guided reinvention*, where they are active creators of mathematical knowledge. It emphasizes the value of whole-class discussions and group work as ways for students to share their strategies and inventions. Within this framework, teachers play a proactive role in shaping students' learning opportunities. Instructional sequences, therefore, should include scenarios that foster these conditions, such as tasks embedded within coherent, long-term trajectories that support students' inventions and socially constructed knowledge.

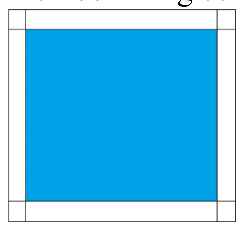
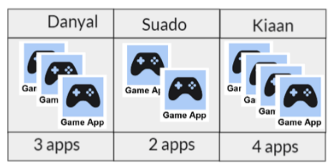
Another principle of RME posits that learning means passing through various levels, bridging informal context-specific reasoning to more formal and abstract reasoning. This progression is supported by the processes of *horizontal mathematization*, in which students organize mathematical information from reality (or an imagined world) to the world of the subject matter, and *vertical mathematization*, where they organize information from one's own mathematical activity to further understand the subject matter (Gravemeijer, 1999). As students move through these levels, their initial “model of” a problem evolves into a “model for,” which represents an organization of formal, general mathematical reasoning that can be applied across different contexts (van den Heuvel-Panhuizen, 2003). This continuous process, known as the *emergent model*, illustrates how learners pass through successive layers of abstraction, with each new level grounded in experientially real phenomena.

Instructional Goals

The TE was also guided by the goal of supporting students in building from situations where they could reason quantitatively about algebraic expressions and equations, rather than numerically or procedurally. My goals for the unit were to support students in the development of (a) generalizing figural patterns with quantitative meaning, (b) forming and making sense of equivalent algebraic expressions with quantitative meaning, (c) developing process and product perspectives with quantitative meaning, and (d) making quantitative meaning of literal symbols.

These goals are related to the hypothesized evolution of ways of reasoning to address Research Question 1, stated in the previous chapter. The TE was originally designed around three contexts: (a) a pool-tiling context, (b) a context in which friends are purchasing game apps, and (c) a scooter context involving locations and velocities, realized in a GeoGebra simulation. However, due to time constraints during implementation, only the first two contexts were enacted (see the Results chapter for further details). These instructional contexts were intentionally chosen to parallel those featured in the dialogic videos to provide continuity between the videos and the tasks. Table 2 summarizes example tasks from each context along with their corresponding instructional goals.

Table 2: Mathematical Contexts and Instructional Goals

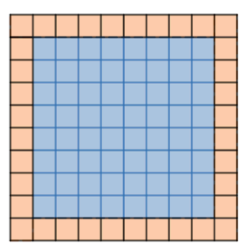
Instructional goal	Mathematical Context
Generalize a relationship between an independent and a dependent variable in a figural pattern context, and then express that generalization using an algebraic equation (generalize figural pattern with quantitative meaning).	The Pool-tiling context  Write an algebra equation that generalizes Haleemah's and ET's method for finding the number of tiles in the border of a pool with any number of tiles on one side.
Understand arithmetic expressions from both process and product perspectives and with quantitative meaning.	The Game App context  $(3\cdot5)+(2\cdot5)+(4\cdot5)=(9\cdot5)$ $15\ +\ 10\ +\ 20\ =45$ Three friends want to buy game apps. The number of apps each will buy are provided in the table. For each part of the equation, write what it means in the game app context.

Dialogic Mathematics Videos

As described in Chapter 2, the dialogic mathematic videos used in this study were taken from Project MathTalk's *Meaning and Equivalence of Algebraic Expressions* video unit (accessible at www.mathtalk.org). This unit is one of several video units on the Project Mathtalk website (e.g., Parabolas unit, Proportional Reasoning unit), each of which serves the purpose of supporting students' meaning making of mathematical topics from a QR approach (Thompson, 2011). This Algebraic Expressions unit was chosen for this study because of its alignment with the dissertation's focus on algebraic expressions and equations, as well as my prior involvement as a co-developer of the materials in an earlier study. For the TE, I used two of the six video lessons from the unit, chosen specifically for their alignement with the instructional goals of the TE.

The Project MathTalk videos were designed as visual enactments of a research-informed learning trajectory. Each video unit is grounded in a hypothetical learning trajectory (HLT) (Clements & Sarama, 2009) developed from the literature on algebra and QR. This design enables each video unit to function as a tangible learning trajectory for the video participants, consisting of 30-40 short videos that trace their evolving mathematical reasoning over time. The conjectured ways of reasoning described in Chapter 2, along with the instructional activities used in the TE, were informed by both the HLT and the observed progression of the video participants. However, these elements were adapted for the classroom context based on the design principles discussed in the section Planned Teacher Design Decisions.

Lesson Number 1: USING ALGEBRAIC SYMBOLS TO EXPRESS A GENERALIZED RELATIONSHIP

Concepts	Elaboration	Video Types & Tasks	CCSSM
• Understand that algebraic symbols (such as x, y, n, t, etc.) can take on a range of numerical values (rather than representing a fixed value, such as x representing one object or 6x meaning 6 apples. • Generalize a relationship between an independent variable and a dependent variable and then express that generalization using an algebraic equation. • Justify algebraic equations by appealing to the quantities in and structure of a visual pattern.	Instruction in algebra often underplays the multiple meanings that literal symbols take on (Philipp, 1992). Instruction can narrow in too heavily on one meaning leading to cognitive conflict when literal symbols take on new forms. Specifically, research indicates that children often think of literal symbols as objects, abbreviations or labels (Stacy & McGregor, 1997). Although there are instances in everyday life in which we use letters as abbreviations (e.g., 3 T in a recipe means 3 tablespoons), in algebra students are expected to cope with statements like 2x + 3x = 5x or y = -6x, where the values of the letters vary. Our goal in this lesson is to develop meaning for literal symbols as varying quantities, which are letters that are used to express a dependency relationship between two or more quantities. We use a variant of the classic Border Problem from Burns and Humphreys (1990).	**Making Sense.** Pose the Swimming Pool Task: *Asha owns a tiling company and was recently hired to tile a pool. Below is a drawing of the swimming pool (shown in blue) surrounded by a border of orange tiles. Each tile measures 1 foot on each side. Each side of this border is 10 ft long. Can you figure out how many tiles are in the border, without counting each one?*  The talent work together to come up with the total number of tiles on the border. Have them share their method, write down an arithmetic statement of the method (e.g., 10 + 10 + 8 + 8 = 36), and circle or highlight each part of the drawing that corresponds with each number (e.g., 10 might be the length of the border on top). Follow up with a prompt for them to come up with another way of counting, with the respective arithmetic statements, explanations, and coloring of the figure.	**CCSS.Math.Content.HSN.Q.2** Define appropriate quantities for the purpose of descriptive modeling. **CCSS.Math.Content.HSF.BF.A.1:** Write a function that describes a relationship between two quantities **CCSS.Math.Content.HSA.SSE.1** Interpret expressions that represent a quantity in terms of its context. **Math Practice 7** (Look for and make use of structure): According to the CCSSM, "Mathematically proficient students look closely to discern a pattern or structure." This lesson provides an opportunity for students to discern a visual pattern in the initial Swimming Pool Task, which will allow them to determine the number of tiles in the border without counting each one.

Lobato, Foster, & Haro NSF Project: *Developing and Investigating Unscripted Dialogic Mathematics Videos* 3

Figure 8: Hypothetical Learning Trajectory for Video Lesson 1, Episode 1

The Project MathTalk videos incorporate several features designed to support students' meaning making. One feature is the simultaneous visual display of the video participants and their written work (see Figure 9). To maintain focus on student thinking, a teacher remains off-screen but provides verbal guidance throughout the video. In addition, the videos are edited and include several post-production elements. Voice-over narration appears at the beginning of each video to introduce the task and again at the end to summarize the students' mathematical progress. Annotations are also used strategically to highlight aspects of the students' work, for example, labeling parts of an equation that correspond with their verbal explanations. Lastly, due to the fact that the videos are unscripted, the videos show the video participants (in this case, Haleemah and ET) engaging in problems where their initial thoughts are often incomplete or incorrect. Through their collaboration, they are shown to be able to resolve authentic struggles and build on each other's ideas to come to a shared understanding.

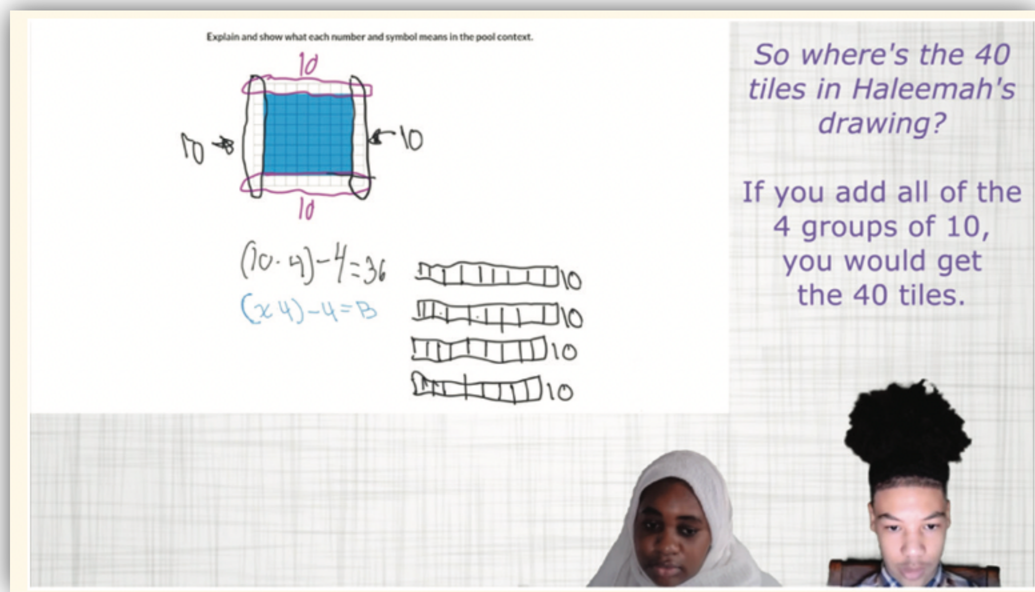


Figure 9: Haleemah and ET Making Sense of Algebraic Expressions and Equations

Planned Lesson Sequence in the TE

The TE consisted of ten one-hour sessions and was taught after school hours. The planned lesson sequence is presented in Table 3, but as stated before, only two contexts were implemented: the Pool-tiling context for sessions 1-5 and the Game app context for sessions 6-10 (see the Results chapter for the actual sequence). The table provides an overview of the planned sequence of activities that students were expected to engage in across the ten sessions. As the actual lesson sequence will show, I followed the experiment tradition, where tasks were modified or replaced depending on the learning progress of students in the moment (Steffe & Thompson, 2000; Cobb, 2000).

Table 3: Overview of Planned Lesson Sequence

TE Session	Video Lesson Used	Context	Class Activities
1	Video Lesson 1: Using Algebra to Express a Generalization (Method 1)	Pool-tiling context	• Make sense of the context of tiling around a swimming pool • Form a relationship between the number of tiles on the side of a square pool and the total number of tiles in the border
2	Video Lesson 1: Using Algebra to Express a Generalization (Method 1)	Pool-tiling context	• Generalize the relationship and express it using algebra • Justify the algebraic equation by making sense of quantities in the pool context
3	Video Lesson 2: Using Algebra to Express a Generalization (Method 2)	Pool-tiling context (Paired)	• Form a different relationship between the number of tiles on the side of a square pool that has overlapping segments • Apply students' method to different sized pools

Table 3: Overview of Planned Lesson Sequence (Continued)

TE Session	Video Lesson Used	Context	Class Activities
4	Video Lesson 2: Using Algebra to Express a Generalization (Method 2)	Pool-tiling context (Paired)	• Generalize the method and express it using algebra • Justify the algebraic equation by making sense of quantities in the figural pattern
5	Video Lesson 3: Simplifying Algebraic Equations	Game app context (Paired)	• Make sense of the game app context • Form a relationship between the cost per app and the total money spent
6	Video Lesson 3: Simplifying Algebraic Equations	Game app context (Paired)	• Form a different relationship between the cost per app and the total money spent • Explore the equivalence between the two methods
7	Video Lesson 3: Simplifying Algebraic Equations	Game app context (Paired)	• Make sense of the equivalent expressions from process and product perspectives
8	Video Lesson 4: Using Negatives and Decimals as Values for Algebraic Symbols	Scooter context	• Make sense of the scooter applet • Form a relationship between the velocity, trip time, starting location, and ending location
9	Video Lesson 4: Using Negatives and Decimals as Values for Algebraic Symbols	Scooter context	• Apply the method to several situations • Generalize the method and express the generalization using algebra

Table 3: Overview of Planned Lesson Sequence (Continued)

TE Session	Video Lesson Used	Context	Class Activities
10	Video Lesson 4: Using Negatives and Decimals as Values for Algebraic Symbols	Scooter context	• Make meaning of the literal symbol representing velocity as it takes on non-positive, non-whole number values • Interpret a given algebraic equation (with a negative sign in front of the literal symbol) in terms of the scooter context

Planned Teacher Design Decisions

The focus of this section is to elaborate one of three components of the instrumental orchestration framework (presented in Chapter 2), the exploitation mode, which refers to the design decisions that a teacher makes prior to teaching. This component is necessary for the analysis of Research Question 2 (i.e., What instrumental orchestrations of the artifact of unscripted dialogic mathematics videos support the emergence of secondary students' normative ways of reasoning?). The other two components (didactical configuration and didactical performance) are detailed elsewhere in this dissertation. Specifically, details of didactical configuration (the physical arrangement of the teaching setting and artifacts) can be found at the end of Chapter 2 and in the Data Collection section below, while the didactical performance (the choices and actions a teacher makes during teaching) is described in Chapter 5.

Three categories of design decisions constitute the exploitation mode for this study. The first category includes routine teaching actions incorporated after watching each video (see Table 4). The choice of teaching actions was made based on conceptual difficulties documented in the literature, information from data previously collected about how students make sense of mathematical ideas presented in the videos, and lastly, from the literature on teaching actions of how to promote QR. More specifically, to facilitate QR, specific scaffolding questions were planned in the design of the sessions, drawing on four scaffolding moves reported in Czocher et al. (2024). These included (a) attending to objects and/or attributes represented by symbols for a variable, (b) using precise and detailed language to refer to referents, (c) supporting participants in quantification, and (d) guiding participants to re-quantify a quantity.

In regard to facilitating engagement with the videos, a prior study (Foster et al., in press) highlighted the importance of asking students what they noticed after each viewing. The importance of this practice is also supported by previous work investigating algebra learning (Hohensee, 2017). Lastly, it was important to make explicit connections to mathematics and the video (e.g., "Is that similar or different to what Haleemah and ET did?"). Making explicit connections to mathematics featured in technology is suggested to support the implementation of the instrumental approach (Moore-Russo, 2015). Table 4 illustrates the exploitation mode related to this category, which corresponds to a video episode in which Haleemah and ET work on developing a deconstructive generalization (an expression that generalizes a figural pattern including overlapping segments, Rivera & Becker, 2008).

Table 4: Exploitation Mode: Routine Teaching Actions Related to Video Watching

Guiding Questions	Example Teacher Follow-Ups
Am I orchestrating the use of the video through teacher moves?	"What do you think about what Haleemah did? Do you agree that there are 40 tiles in the pool?"
	"Could someone describe in your own words H&ET's method?"
	"Is that idea different or the same as Haleemah and ET's?"
Am I facilitating QR through teacher moves?	"Do 8+8 + 10 + 10 = 36 and 8 • 2 + 10 • 2 = 36 mean the same thing? Why?"
	"Could someone describe what their arithmetic equation means in terms of the pool?"
Am I developing student noticing of important ideas in the video?	"Let's rewatch, and I want you notice the difference in the way they formed their arithmetic equation."
	"What did you notice?

The second category that constitutes the exploitation mode is guidelines for task selection. For the videos to serve as a support for collective reasoning, it was important to design tasks that were the same or close to those featured in the videos. I followed a precedent from previous studies utilizing Project MathTalk videos, which selects tasks for participants based on whether a mathematical task in a video episode is particularly challenging. For the Pool-tiling context, I asked participants to work on the identical task, and for the Game app context, I asked them to work on a paired task. A paired task is one that is similar to the task from the video but different in terms of the numbers or the context. Drafts of paired tasks were created prior to the TE. However, following the teacher-experiment tradition, tasks were modified or replaced depending on the learning progress of students in the moment (Steffe & Thompson, 2000; Cobb, 2000). See the Results chapter for the modifications made.

The third category of design decisions involved episode-by-episode planning (see Table 5). For each video episode, detailed notes were taken, such as the episode length, the possible trouble spots for VLs, and a conceptual analysis. Other tentative video-specific decisions were made before the TE began. These decisions included whether to include the entirety of a video episode or voiceovers, when to pause the video for discussion, and whether to watch a video episode first or have students work on a task. For each decision, a rationale was given that draws on literature in algebra, vicarious learning studies, and technology use. These plans were intended as flexible guides, with the understanding that actual choices would depend on students' reasoning during instruction. For the actual decisions made in response to student reasoning, see the Chapter 5.

Table 5: Exploitation Mode: Episode-by-Episode Decisions

Design Decisions	Questions	Decision	Rationale
Cutting video chunks	Is the entire video (excluding the voiceover (VO)'s)) necessary?	Yes	The entirety of the video is important for establishing how Haleemah and ET come up with their method.
	Should the VLs watch the voiceover(s)?	Skip VO1. Have VLs watch the second VO. Skip VO3.	The first VO is the statement of the task, this can be said verbally. The second VO is important for getting VLs to attend towards a relational view of the equals sign ["instead of writing 8 + 8 + 10 + 10 = 36 they wrote (8 • 2) + (10 • 2) = 36, they were able to make it easier because there were two 8's, giving them 8 • 2 and two 10's giving them 10 • 2"]
Continuity of video watching	Should I pause the video and discuss?	Pause at 1:31 to discuss Haleemah & ET's viewing 40 tiles	While it will take more time, pausing here might clarify what is meant by "Haleemah and ET's method." Rivera and Becker (2008) note that overlapping segments is a difficulty student have when figurally reasoning.

Data Collection

The TE spanned ten one-hour sessions over five weeks after school hours. The research sessions took place in the students' mathematics classroom, equipped with a smartboard to project the videos. The 13 participants were placed in groups of 3-4, each with access to blank paper and writing utensils. Throughout the sessions, students engaged with the video tool at the level of the whole class on a shared screen (see Figure 7 above depicting the Didactical Configuration). Groups were asked to work on paired tasks and discuss their reasoning. A single camera was placed in the room and run by an assistant (a fellow doctoral student) who recorded whole-class discussions. She also took notes for the purposes of debriefing after each session. The sources of data for this study consisted of video-taped data of the research sessions, written work from the groups, and whole-class inscriptions. Together, the video-taped data of classroom discussion, whole-class inscriptions, and groups' written work served as the data subject to analysis for the two research questions.

Data Analysis

The two research questions that frame this study have different foci—the evolution of collective mathematics practices and orchestrations of the video artifact that support normative ways of reasoning. As such, each research question requires a distinct analytical process that I detail in this section.

Analysis for Research Question 1

RQ: What is the evolution of the collective mathematics practices of 13 Grade 9 students over the course of a TE about algebraic expressions and equations from a QR perspective?

As I elaborated in the Elaboration of Research Question 1, this question was interested in the evolution of collective mathematical progress of a cassroom community. As such, I drew on the collective mathematics practices approach to construct a learning trajectory. However, recall that a collective mathematics practices trajectory consists of two components: (a) a list of mathematics practices (i.e., normative ways of reasoning) and (b) a description of instructional supports that are influential to the emergence of such practices (Lobato & Walters, 2017). To analyze this question, I focused on the first component, creating a list of collective mathematics practices. Theoretically, a collective mathematics practice is defined as a group of normative ways of reasoning (reasoning that functions as if shared) that develop as students work together in a classroom context. Analytically, collective mathematics practices are established first by identifying argumentations that occur in the classroom, then checking the argumentations against certain criteria to identify normative ways of reasoning, and lastly, grouping these ways of reasoning to form a set of collective mathematics practices. This analytic process is known as the Documenting Collective Activity (DCA) method.

The DCA methodology is an approach developed by Rasmussen and Stephan (2008) as an empirical way of examining collective learning**.** The approach assumes that learning is "created in argumentations" and is thus grounded in Toulmin's (1969) argumentation scheme (Rasmussen & Stephan, 2008, p. 197). Toulmin's model of argumentation characterizes the core of an argument by three components: the *data,* the *claim*, and the *warrant*. Often, a student or a teacher makes a conclusion or *claim* that is supported by evidence or *data*. The data can consist of specific facts or procedures that led to their claim. The *warrant* explains why the evidence connects to the claim and is usually more abstracted from the problem. Once a claim, data, and warrant are offered, *rebuttals* and *qualifiers* can follow. A *rebuttal* challenges the claim with a counterargument, while a *qualifier* specifies conditions under which the claim holds. If needed, the *backing* is the mathematical authority of the argument and can justify why the warrant is valid. Collective reasoning is then said to occur if there are shifts in the way students engage in argumentation with guidance from the teacher (Cole et al., 2012). While it is not uncommon for arguments to include rebuttals, qualifiers, and backing, only the three components of claim, data, and warrant appeared in the arguments of my participants.

The DCA method uses a three-phase approach to document collective mathematics practices. For the first phase, I developed a scheme for argumentation using Toulmin's model.

This involved making an argumentation log, which notes the function of the argumentation elements: claim, data, and warrant. To do this, I transcribed the whole-class observations and then identified arguments in the transcript. I defined an argument as an episode where a participant made a mathematical assertion and then supported their assertion with evidence. From there, I identified the function (warrant, claim, or data) of each element of the argument. Arguments need not be constructed by talking alone (Gruver, 2018). For instance, gesturing and inscriptions also served as elements of an argument.

The second phase aims to see which arguments become normative ways of reasoning for the collective group by checking three criteria: (a) the dropping off of warrants and backings, (b) the shifting function of argumentation elements, (c) repeated use of an idea as the data or warrant for different claims across multiple days (Rasmussen & Stephan, 2008; Cole et al., 2012). One might use the third criterion in a scenario where first two criteria are not satisfied (e.g., if the teacher provides a majority of the warrants). Lastly, the third phase organizes the information gathered from the previous two and identifies the mathematical activity that students engaged in when ideas emerged. I then grouped the normative ways of reasoning to form the set of collective mathematical practices.

Analysis for Research Question 2

RQ: What instrumental orchestrations of the artifact of unscripted dialogic mathematics videos support the emergence of secondary students' normative ways of reasoning?

To analyze this question, I drew on the theoretical framework of instrumental orchestration, which focuses on the teacher's role in a technology rich-environment (Trouche, 2004). The framework comprises three components, which guided my analysis. The first is the didactical configuration, referring to the physical arrangement of tools and participants. Second is the exploitation mode, which describes the teacher's intended instructional use of the technological artifact within the didactical configuration. Third is the didactical performance, which captures the real-time enactment of planning efforts. In the present study, the didactical configuration remained constant across the teaching experiment and was therefore not subject to analysis. Similarly, the exploitation mode, presented earlier in the chapter in the section on Planned Teacher Design Decisions, was not analyzed. Unlike studies that examine the practices of multiple teachers (e.g., Drijvers et al., 2010), this investigation focused on a single instructor (the researcher), and thus did not include data sources like planning documents or cross-teacher comparisons. As a result, only the the didactical performance component of instrumental orchestration was analyzed in response to Research Question 2.

A two-phase analysis was conducted. For the first phase, I identified classroom episodes that were significant for the emergence of each of the ten NWRs reported in Research Question 1. These episodes were analyzed using an a priori coding scheme based on didactical performances characterized in previously established IO types, as detailed in Chapter 2 and summarize in Table 1. Given that dialogic videos constitute a distinct technological tool not previously analyzed within the IO framework, I engaged in a process of adapting these codes to account for the unique affordances of video (e.g., pause, rewind, fast-forward). When an existing code was conceptually applicable but did not fully capture the interaction observed, I adapted the code to reflect the video context. In instances where no existing orchestration type adequately described the teacher's use of video, I developed new codes inductively.

In the second phase of analysis, I identified whether each of the IOs were productive or unproductive in supporting the emergence of normative ways of reasoning. An IO was considered *productive* if there was evidence that it contributed meaningfully to the emergence of a NWR. Conversely, an IO was considered *unproductive* if, immediately following its use, (a) student confusion emerged in ways that were unexpected given their prior engagement, or (b) students did not interact with the video as the teacher had intended. The terminology of productive or unproductive is not meant to serve as a permanent, binary descriptions, but rather as micro-categories that are sensitive to the context and content. Given the scope of the data and the diversity of orchestrations across the ten NWRs, I used the analysis on two representative NWRs: stating a general method (NWR 1.1) and developing descriptive equivalence of algebraic expressions (NWR 5.2). These two were selected because, together, they encompassed the full range of orchestration types identified in the study.

Validity

To ensure the quality and credibility of the study's findings, steps were taken to address validity. In qualitative research, validity refers to the extent to which codes accurately represent the data they are intended to capture (Maxwell, 2005), and more broadly, to the credibility and trustworthiness of analytic interpretations (Yilmaz, 2013). For each research question, I sought expert feedback to evaluate the fit between my analytic claims and the supporting evidence. For Research Question 1, I consulted with Dr. Chris Rasmussen, an expert in the DCA Method. For Research Question 2, I presented my analytic process to Dr. Lobato, whose expertise includes vicarious learning and video technology. Following Maxwell (2005), I also addressed potential validity threats, defined as plausible ways the research might be wrong, by actively seeking disconfirming evidence and considering alternative interpretations throughout the analysis. Additionally, I remained attentive to the possibility of researcher bias, particularly given my dual role as both teacher and researcher in the teaching experiment.

Chapter 4: Collective Mathematics Practices Learning Trajectory

The aim of Research Question 1 was to document the evolution of ways of reasoning about algebraic expressions and equations from a quantitative-reasoning (QR) approach in the form of a learning trajectory of collective mathematics practices. Answering this question is important for advancing knowledge about the evolution of the following ideas over time: generalizing via figural patterns, making meaning of literal symbols, interpreting expressions from process and product perspectives, and reasoning about equivalence. In contrast to the current literature, in which studies tend to focus on just one of these ideas, this study demonstrates their interrelatedness. Additionally, most studies in these areas have investigated reasoning as measured at one point in time, meaning more studies are needed to characterize change over time. Finally, investigating the evolution of ideas from a QR approach offers insight into its potential for providing connective tissue across multiple interrelated ideas.

To answer Research Question 1, I needed to identify ways of reasoning that functioned as if shared in the classroom using the Documenting Collective Activity (DCA) method (Rasmussen & Stephan, 2008). I applied the method in three phases. First, I created an argumentation log consisting of a daily record of the arguments made in the whole class from Day 1 to Day 10. I identified arguments by applying codes from Toulmin's argumentation scheme (i.e., claim, data, warrant, and backing) to transcripts of whole-class data (Toulmin, 1969). Second, I identified ways of reasoning that fulfilled one of three criteria: (a) warrants becoming implicit in arguments, (b) pieces of an argument shifting in a subsequent argument, or (c) ideas repeatedly being used as data or warrant in arguments across multiple days. If a way of reasoning fulfilled one or more of these criteria, they were considered to function as if shared (FAIS) and thus named a normative way of reasoning (NWR). Third, I grouped related NWRs to form collective mathematics practices. [For additional details on the DCA method, see Chapter 3.]

In the following results section, I will provide evidence and analysis to support the claim that five mathematics practices FAIS. Specifically, the learning trajectory comprises five mathematics practices: (a) stating a general method, (b) expressing a general method with algebra, (c) interpreting arithmetic expressions from multiple perspectives, (d) interpreting algebraic expressions from multiple perspectives, and (e) developing descriptive equivalence. I summarize these mathematics practices in Table 6, where the normative ways of reasoning grouped under each practice are also listed. I follow a numbering scheme where "NWR a.b" can be read as a normative way of reasoning in the "a^{th}" mathematics practice and is the "b^{th}" way of reasoning within that mathematics practice.

Table 6: List of Mathematics Practices and Normative Ways of Reasoning (NWRs)

Mathematics Practice 1: Stating a General Method
NWR 1.1: Stating a General Method
Mathematics Practice 2: Expressing a General Method with Algebra
NWR 2.1: Accepting an Equation with Three Literal Symbols as Valid When Only Two Are Needed
NWR 2.2: Re-expressing an Equation with Three Literal Symbols Using Only Two Literal Symbols
NWR 2.3: Expressing a General Method with Literal Symbols that Represent Quantities with Varying Values
Mathematics Practice 3: Interpreting Arithmetic Expressions from Multiple Perspectives
NWR 3.1: Interpreting Arithmetic Expressions from a Quantitative Process View
NWR 3.2: Interpreting Arithmetic Expressions from a Quantitative Product View
Mathematics Practice 4: Interpreting Algebraic Expressions from Multiple Perspectives
NWR 4.1: Interpreting Algebraic Expressions from a Quantitative Process View
NWR 4.2: Interpreting Algebraic Expressions from a Quantitative Product View
Mathematics Practice 5: Developing Descriptive Equivalence
NWR 5.1: Developing Descriptive Equivalence of Arithmetic Expressions
NWR 5.2: Developing Descriptive Equivalence of Algebraic Expressions

Other empirical studies (e.g., Gruver, 2018; Stephan & Akyuz, 2012) have shown a sequential progression of mathematical practices, whereas the nature of my data was different. The mathematics practices documented in this trajectory were overlapping, as illustrated in Figure 10. The evolution of each practice is represented by arrows, where the left node marks the occurrence of the first argument of the first NWR in that practice, and the right node indicates the occurrence of the final argument in the last NWR, marking the day in which the last NWR in that mathematics practice FAIS. The figure shows that mathematical practices emerged concurrently and not sequentially. For instance, Mathematics Practice 1 began on Day 4, with the final NWR becoming normative on Day 7, whereas Mathematics Practice 2 started on Day 2, with the final NWR not becoming normative until Day 8.

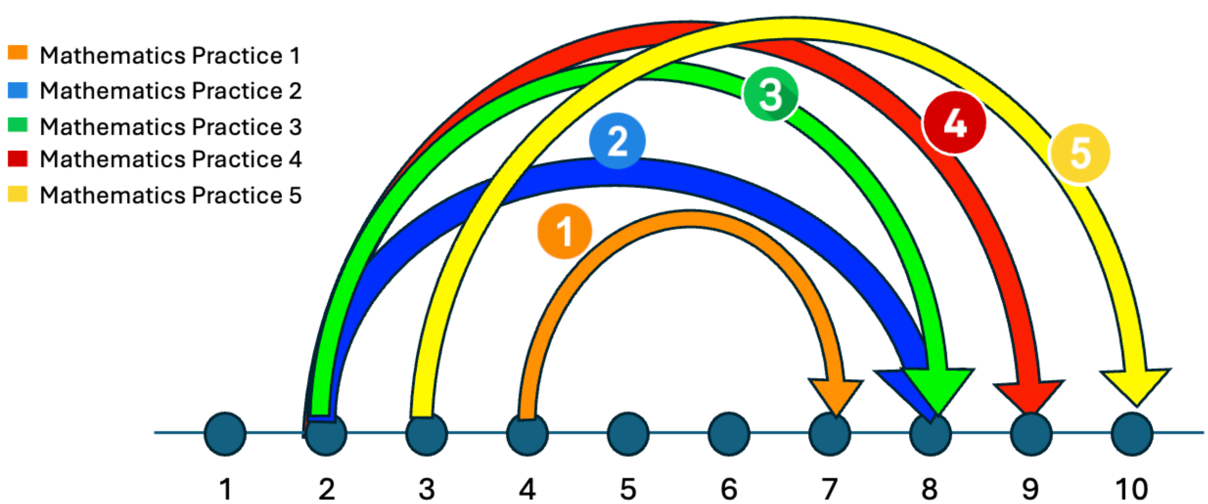


Figure 10: Evolution of Five Mathematics Practices

Despite the overlapping nature of these mathematics practices, I ordered the mathematics practices chronologically based on the day when the last normative way of reasoning FAIS. For example, Mathematics Practice 1 is numbered first because NWR 1.1 functioned as if shared on

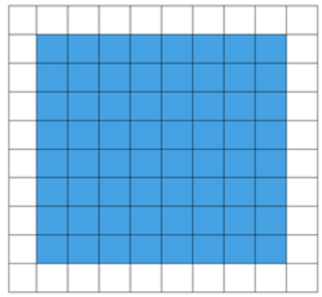
Day 7, before any other normative ways of reasoning. I direct the reader to the discussion section at the end of this chapter to further explore why these mathematics practices overlap and the implications for understanding the progression of students' reasoning.

Mathematics Videos

The mathematics videos used in this TE featured the unscripted dialogue of two Grade 9 students, Haleemah and Elijah (who preferred to be called ET), who I refer to as the video participants. The viewing experience involves viewing Haleemah and ET tackling conceptually rich mathematics problems while simultaneously viewing their work. The teacher's guidance in problem solving can be heard but her presence is kept off-screen (see Figure 1 in Chapter 1). Additionally, the videos are edited and produced, including two elements: voiceovers and annotations. The voiceovers appear at the beginning of each video to introduce Haleemah and ET and their task and at the end to provide a summary. The annotations highlight aspects of Haleemah's and ET's work, such as annotating tiles flying off the screen to demonstrate subtraction.

Two video lessons structured the TE. The first video lesson, used during Days 1-5 of the TE, was set in a pool-tiling context, and focused on developing students' ability to generalize a method from arithmetic to algebra using a visual pattern and connecting expressions to quantities in context. Haleemah and ET were asked to determine the number of tiles in the border of a square pool with varying numbers of tiles on a side: first 10, then 7, 15, 12, and 50 tiles (see Figure 11). To streamline references throughout the chapter, I refer to a pool with n tiles on a side as Size n pool (e.g., a Size 10 pool has 10 tiles on a side). Students in the TE watched Haleemah and ET tackle the same tasks they were assigned. In both cases, the goal was to generalize a method for finding the total number of tiles in the border of square pools of varying sizes.

The picture below shows a tiled swimming pool. The water is shown in blue. The border is shown in white. The border has 10 tiles along one side. In the video, Haleemah and ET will find the number of tiles in the border.



1. What is the total number of tiles in the border of the pool?

   Answer: ________________________

   Circle and label the drawing using Haleemah's and ET's method to show show how they found the number of tiles in the border.

2. **Write an arithmetic** equation that shows their method for finding the number of tiles in the border of the pool.

Figure 11: The Pool Task

The second video lesson, used during Days 6-10 of the TE, was set in the game app context and focused on developing students' meaning for equivalent algebraic expressions. In this video lesson, Haleemah and ET worked on the Three Friends Task, where three friends (Danyal, Suado, and Kiaan) each bought a fixed number of game apps (3, 2, and 4, respectively). The cost of each app was either known (e.g., \$5) for arithmetic tasks or represented by a variable, *c*, for algebraic tasks. The goal of this task to develop quantitative meaning for combining like terms (i.e., to understand the meaning of the equation $3c + 2c + 4c = 9c$ in a context). As shown in Figure 12, Haleemah and ET interpreted the algebraic expressions like $3c$ as the amount of money spent by Danyal, and $9c$ and as the total amount of money spent by all three friends (see their written response below the arrows pointing to the brackets indicating $3c$ and $9c$). In the arithmetic case for a cost per app of \$5, they interpreted $3 \cdot 5$ as the amount of money spent by one of the friends, Danyal.

Unlike the first video lesson, which aligned the task content directly between Haleemah and ET and the students, the second video lesson was used differently. While Haleemah and ET worked on the Three Friends Task, the students in the TE worked the related but distinct Two Friends Task. In this version, two friends (Sofia and Manuel) purchased 5 and 3 games apps, respectively, as shown in Figure 11. The goal for the Two Friends task was to understand the meaning of the equation $5c + 3c = 8c$ in context, where c represented an unknown cost per app.

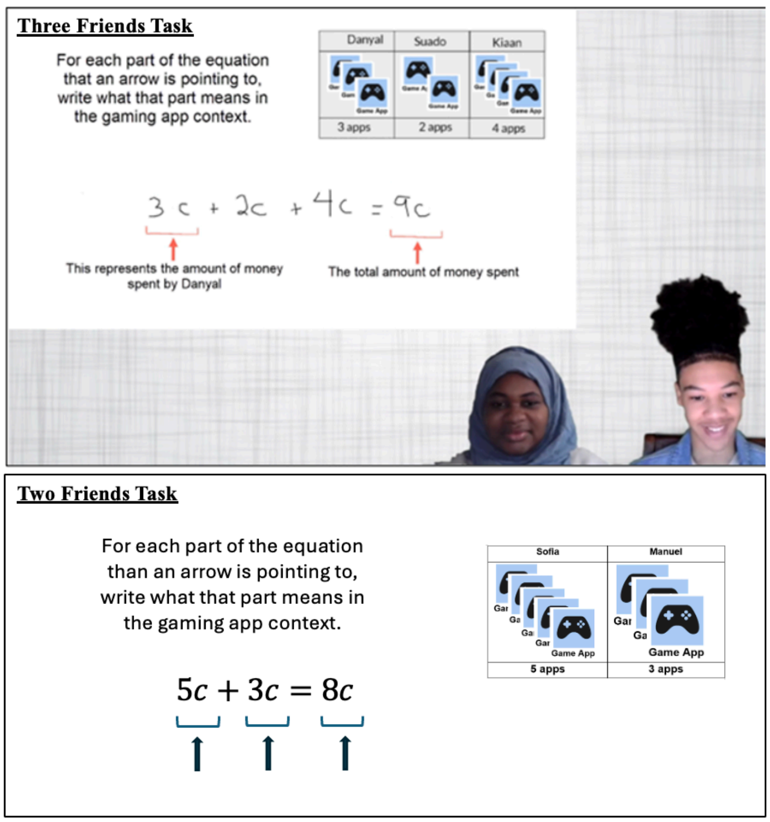


Figure 12: Three Friends Task and Two Friends Task

Instructional Context and Tasks

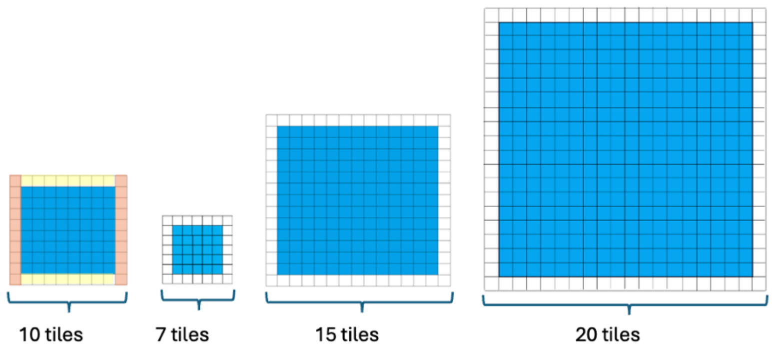
The instructional context was designed to encourage argumentation within small-group and whole-class discussions as the class engaged with the videos. Students sat in groups of three to four, with dialogic videos projected on a shared screen at the front of the classroom (see

Figure 7 in Chapter 2). In a typical session, students began by tackling a challenging task and sharing initial ideas within their group. After small-group discussions, students were expected to share their ideas with the whole class. Teacher prompts typically involved asking students to justify their reasoning and interpret arithmetic and algebraic expressions in terms of quantities in context. Small-group and whole-class discussions were interspersed with video-watching and discussion, where the teacher asked the class to make connections between the videos and their ideas. For more detail about how the videos were used to support students' reasoning, see Chapter 5.

Instruction across the TE followed two main sets of tasks, each aligned with one of the two video lesson contexts: Days 1-5 for the pool-tiling context and Days 6-10 for the game app context. Below, I describe the lesson sequences based on these two phases. During Days 1-5 students began by watching a video where Haleemah and ET developed a method for finding the total number of tiles in the border around a square pool. Their method involved adding the number of tiles in the left and right sides, including the corner tiles, together with the number of tiles in the top and bottom sides. This method, highlighted in orange and yellow in Table 7 (row 1), served as the focus for classroom activities on Days 1-3, where students applied the method to pools of varying sizes and then expressed it with algebra. On Day 4, instruction focused on connecting elements of the method. Students were asked to compare visualizations that depicted the same or a different method and explain which best captured Haleemah's and ET's method. On Day 5, the focus was on connecting each number and symbol of their algebraic equations to the pool context. The full sequence of tasks for Days 1-5 is reflected in Table 7.

Table 7: Lesson Sequence of Days 1-5

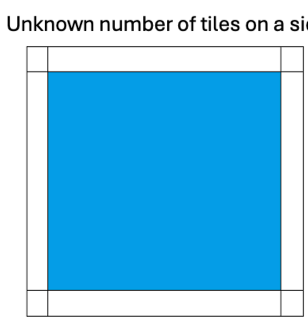
TE Session	Tasks
1-3	**<u>The Pool Task</u>**



For each pool, apply Haleemah's and ET's method to find the total number of tiles in border.

a. Number of tiles in the border: ________________
b. Arithmetic equation: __________________
c. Circle and label the drawing to show how the arithmetic equation is connected to the pool.

<u>Expressing the Method with Algebra Task</u>



1. Write an algebra equation that generalizes Haleemah's and ET's method for finding the number of tiles in the border of a pool with an unknown number of tiles on one side.

2. Define your variables in terms of the pool.

Table 7: Lesson Sequence of Days 1-5 (Continued)

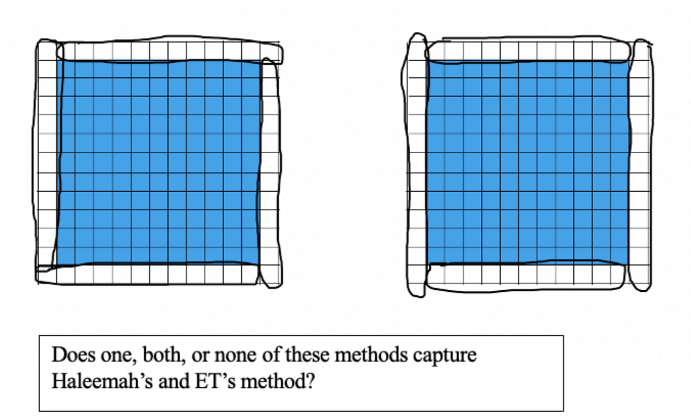
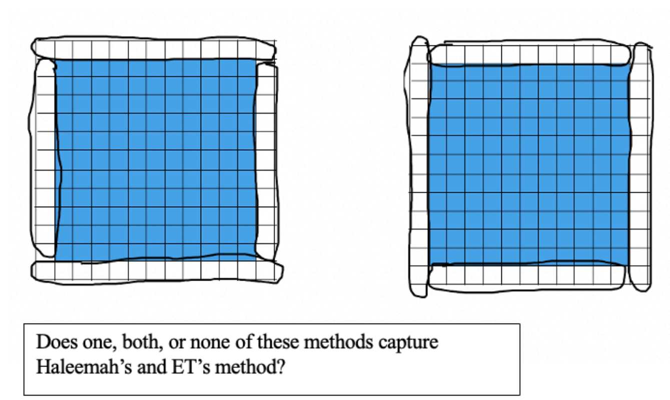
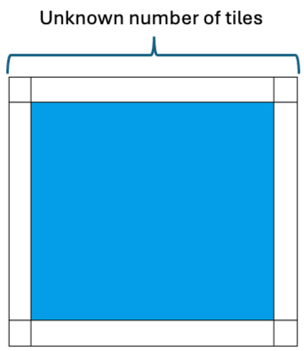
TE Session	Tasks
4	**<u>Comparing Visualizations Task: Same Method</u>**  **<u>Comparing Visualizations Task: Different Method</u>** 

Table 7: Lesson Sequence of Days 1-5 (Continued)

TE Session	Tasks
5	**<u>Interpreting the Method with Algebra Task</u>**



$$x \cdot 2 + (x - 2) \cdot 2 = B$$

Describe what each part of the equation means in terms of the pool.

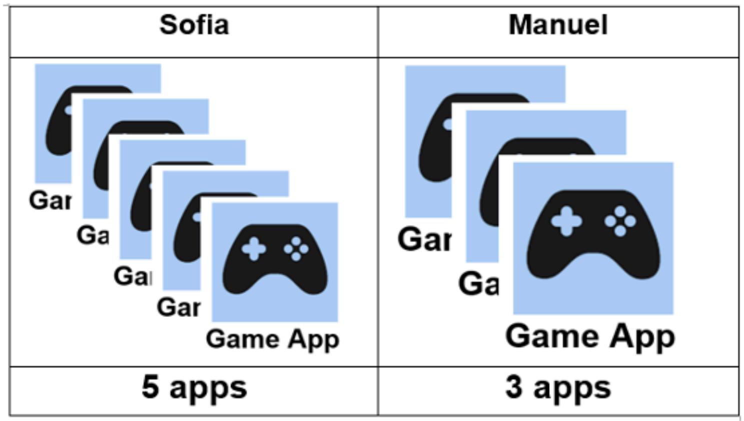
Circle and label the drawing for each.

Explain in your own words why $x \cdot 2 + (x - 2) \cdot 2 = B$ makes sense as a generalization for finding the number of tiles in the border of a pool with an unknown number of tiles on a side.

During Days 6-10, instruction shifted to tasks aligned with the second video lesson in the game app context. While students viewed videos of Haleemah and ET working on the Three Friends Task, they themselves worked on the related but distinct Two Friends Task. Both tasks supported the exploration of equivalent algebraic expressions by asking students to generalize two different methods for determining the total amount spent by friends altogether. The lesson sequence began with the “long method,” which students interpreted and generalized across Days 6 and 7. Instruction then shifted to the “short method” on Days 8 and 9, which they interpreted and generalized (see Mathematics Practice 3 and 4 for more detail). The sequence culminated on Day 10, where students were asked to articulate the equivalence between two algebraic expressions, each representing a method. The lesson sequence of the TE on Days 6-10 is reflected in Table 8.

Table 8: Lesson Sequence of Days 6-10

TE Session	Tasks
6-7	**<u>The Two Friends Task: The Long Method</u>**



Sofia and Manuel want to buy game apps. Each app costs the same.

a. Pick a price for a game app.
b. Determine how much Sofia and Manuel spent together.
c. Write an arithmetic equation that shows your method for finding how much Sofia and Manuel spent together.

<u>Expressing the Long Method with Algebra Task</u>

1. Write an **algebra** equation that shows the relationship between a game app that costs some unknown amount of money and how much Sofia and Manuel spend altogether.
2. Define your **variables** in the game app situation.

<u>Interpreting the Long Method with Algebra Task</u>

Explain what each part of the equation $5c + 3c = T$ means in the game app context.

Table 8: Lesson Sequence of Days 6-10 (Continued)

TE Session	Tasks
8-9	**The Two Friends Task: The Short Method**

Sofia	Manuel
	
5 apps	**3 apps**

Sofia and Manuel want to buy game apps. Each app costs the same.

a. Pick a price for a game app.
b. Determine how much Sofia and Manuel spent together using the **shorter, faster method**.
c. Write an arithmetic equation that shows the **short method** for finding how much Sofia and Manuel spent together.

Expressing the Short Method with Algebra Task:

1. Write an **algebra** equation that shows the relationship between a game app that costs some unknown amount of money and how much Sofia and Manuel spend altogether.
2. Define your **variables** in the game app situation.

Interpreting the Short Method with Algebra Task

Explain what each part of the equation $8c = T$ means in the game app context.

Table 8: Lesson Sequence of Days 6-10 (Continued)

TE Session	Tasks
10	<u>**Interpreting Equivalence of Algebraic Expressions Task**</u> Sofia has 5 apps and Manuel has 3 apps. Each app costs the same. Explain why $5c + 3c$ is **equal to** $8c$ in the game app context.

In the remainder of this chapter, I present evidence and analysis for the five mathematics practices comprising the learning trajectory. For each mathematics practice I present, I will introduce the practice and the normative ways of reasoning within that practice. After that, I present data and analysis in the form of arguments to make the case that these ways of reasoning came to FAIS via criteria using the DCA method. Because the evolution of the mathematics practices was quite complex, it was also important to include initial ways of reasoning related to each normative way of reasoning. However, providing an evidentiary case for these preliminary ways of reasoning was beyond the scope of this thesis, so within each section for an argument, I simply present a narrative account before presenting the results.

Mathematics Practice 1: Stating a General Method

Mathematics Practice 1 documents students' ways of reasoning of stating a general method. Harel and Soto (2017) note that students often rely solely on result pattern generalization without moving to the more sophisticated process pattern generalization (PPG). The PPG moves beyond discerning a number pattern and appeals to a logic that explains why a pattern is valid in general. This construct aligns with students' reasoning about generalizing a method observed in Mathematics Practice 1. Additionally, students can express a generalization in multiple ways, such as a written equation with algebraic notation or as a statement using everyday language. For instance, writing $1+8r$ differs from saying "one more than a multiple of 8" (Zazkis & Liljedahl, 2002). Mathematics Practice 1 focuses on describing a general method verbally or in writing, while Mathematics Practice 2 (see next section) focuses on expressing a general method with algebra.

Mathematics Practice 1 comprises one normative way of reasoning, NWR 1.1, which presents evidence of stating a general method functioning as if shared. The way of reasoning observed in the dataset aligns most closely with the aforementioned process pattern generalization (PPG) and the use of everyday language to state a general method. As I will show, the students' evolution of reasoning in stating a general method was complex for several reasons. First, stating a general method emerged in the pool context and was renegotiated in the game app context, with both contexts posing unique challenges. Second, students' progress toward stating a general method was not straightforward or linear, meaning that they often indicated making progress but did not build on their prior meaning afterward. Still, through repeated opportunities to voice their interpretations and the use of novel tasks, this way of reasoning ultimately became normative.

NWR 1.1 Stating a General Method

This section will provide evidence that stating a general method, as defined below, FAIS within the classroom community by Day 7. I define *stating a general method* using three characteristics. The first characteristic relates to students' structural reasoning, which is putting elements in relationship to each other in a particular way. The second characteristic involves students' ability to reason in generality, entailing reasoning without reference to specific instances. The third characteristic involves appealing to a logic that explains why the relationship is valid.

These characteristics are adapted from Harel and Soto's (2017) process pattern generalization (PPG) construct, which similarly entails structure, generality, and the use of logic with explanatory power. I extend their construct of PPG by drawing on quantitative reasoning (in the sense of Thompson, 1990, as described in Chapter 2) as the basis for the explanatory power. For example, a quantitative interpretation of the general formula $W = fd$ (where W is work, f is force, and d is distance) entails conceiving of work as a quantity, being created by applying a force to an object and thus, moving it some distance (Thompson & Saldanha, 2003). Such an interpretation not only grounds the meaning of the symbols in context but also explains why the formula is true in general. Thus, I consider a student to demonstrate evidence of *stating a general method* if the following three criteria are met:

(a) <u>Structure Criterion:</u> the student mentions multiple elements of their method and how they are related rather than an isolated element.

(b) <u>Generality Criterion:</u> the student describes the method without reference to specific instances.

(c) <u>Quantitative Criterion (or Logic Criterion):</u> the student names or points to the quantities involved in their general method.

I present two arguments from Day 4 and Day 7 in the following sections. I then apply Criterion 2 from DCA (the shift of position of an argument element) and argue stating a general method shifted from the "warrant" position to the "claim" position.

Argument 1

Preliminary Related Reasoning. In this subsection, I provide a narrative account (rather than an evidentiary case) of initial reasoning related to the emergence of stating a general method. In the initial days of the TE, students worked on the Pool Task (Figure 11), where they were asked to generalize Haleemah's and ET's method applied to pools of different sizes. Haleemah's and ET's method entailed adding the number of tiles in the left and right sides, including the corner tiles, together with the number of tiles in the top and bottom sides. Although students eventually stated this general method by Day 4, their path to doing so was neither linear nor straightforward. This journey is best illustrated through five whole-class discussions: two on Day 2 and three on Day 3, which ultimately contributed to students' successful statement of a general method on Day 4. I summarize the five discussions next.

On Day 2, students made progress in identifying isolated components of their method and appeared ready to generalize. For example, in discussing a video where Haleemah and ET articulated their method, Gisela and Esperanza noted that the corner tiles needed to be included on the left and right sides but should be omitted from the remaining sides, a key structural element of the method. However, they had not yet articulated the general method as adding together all of the components (the tiles on the left and right sides and the tiles remaining on the top and bottom after the corner tiles had been removed). Without this coordination, their reasoning fell short of satisfying the structure criterion. Later that day, students struggled to express Haleeemah's and ET's method with algebra. One student wrote an equation focusing on a Size 50 pool, which indicated a potential problem with the the generality criterion, the ability to reason without relying on specific instances.

The students' struggles in reasoning structurally and with generality became more evident in three whole-class discussions on Day 3. In the first discussion, the class reviewed the meaning for structural elements voiced on Day 2. However, rather than building on their prior ideas, students did not restate earlier ideas, nor did they spontaneously offer a general description of the method. The second discussion involved applying the method to a Size 12 pool shown on the board alongside other pool sizes. The exercise was intended to help students see patterns and relationships across pools. Instead, one student focused on the associative property of the addition in the method, and another described a completely different method for finding the total number of tiles in the border. Neither description provided evidence of stating a general method. In the third discussion on Day 3, the class revisited the algebraic equations they had attempted on Day 2. The teacher directly asked them which of their equations best captured Haleemah's and ET's method. In response, several students mentioned that the method need not count the corners twice, which was an important component of the method that was repeated throughout the session. However, this remained a partial description, not a full articulation of the method, and failed to satisfy the structure criterion.

Across these five discussions, students had not yet stated a general method in a way that met the criteria for structure or generality. Their descriptions tended to focus on isolated components (e.g., not counting the corners twice) or specific numerical cases (e.g., a Size 50 pool). They had yet to articulate the method as the tiles on the left and right sides and the tiles remaining on the top and bottom after the corner tiles had been removed and, therefore, did not yet FAIS.

Background. Argument 1 took place during a whole-class discussion on Day 4, following student difficulties on Days 2 and 3 in articulating Haleemah's and ET's general method. To support students, the teacher began by showing a video in which the video participants' first instance of applying their method to a Size 10 pool. After watching and discussing, the teacher introduced a new task, one that had not appeared in the videos (Figure 13). Unlike previous tasks, this one did not ask students to apply Haleemah's and ET's method to a specific pool size (e.g., Size 13). Instead, it asked students to determine whether one, both, or neither of the two visualizations captured Haleemah's and ET's method. The visualizations each depicted a distinct approach to finding the total number of tiles in the border. After working on the task in groups, the teacher asked Jose to present his group's work at the board, where a recreation of the two visualizations with number labels was displayed (Figure 14).

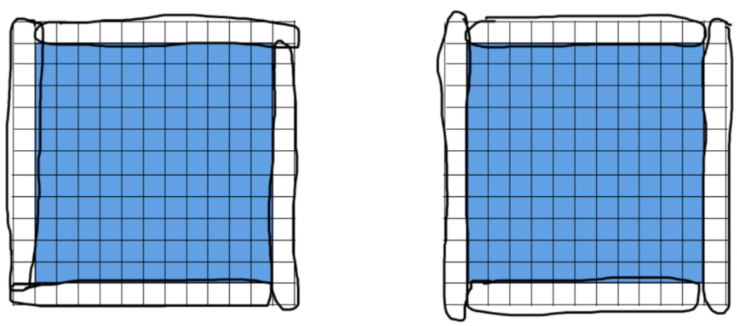


Figure 13: Comparing Visualizations Task: Different Methods

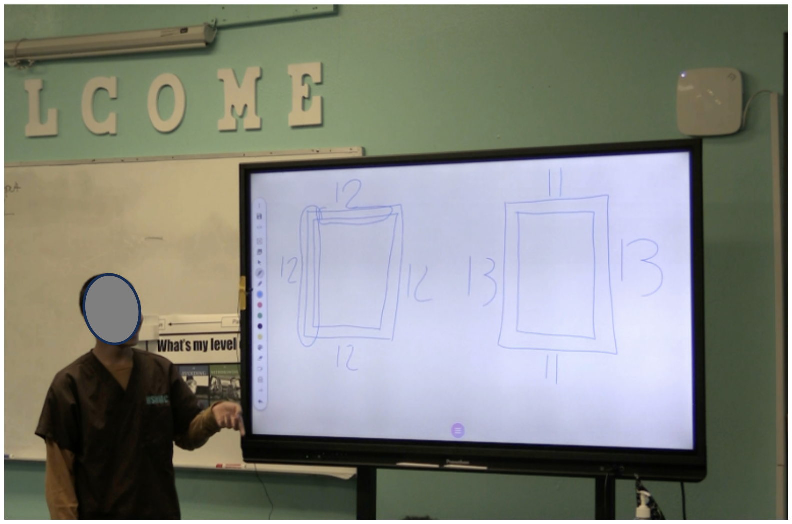


Figure 14: Jose Presenting his Group’s Work on the Comparing Visualizations Task

Toulmin’s Scheme. Jose’s claim is that the visualization on the right (see Figure 14) shows Haleemah’s and ET’s method. His evidence is that this visualization counts the tiles on the left and right sides together with the tiles on the top and bottom, without counting the corners. The warrant, which connects the data to the claim, is that Haleemah’s and ET’s method involves two groups of tiles on the left and right and two groups of tiles on the top and bottom. The transcript in Table 9 reflects the speech patterns typical of children, characterized by informal language. A detailed analysis and interpretation of Jose’s speech will provide further clarification.

Table 9: Coding for Argument 1 of NWR 1.1

Participant	Speech	Actions	Code
Jose	**I said the one on the right was the one Haleemah and ET's method.**[1] I said that because it showed each…**like the method that they counted the ones on the side**[a] **with the corners**[b] **and the ones on the top and bottom**[c] **were counting…. which were the ones that were left out without counting the corners.**[3d]	[a]**Points to the left and right sides of the pool on the right (see Figure 14).** [b]**Points to top corner on the right side of the pool and to the bottom corner on the on the right side of the pool (see Figure 14).** [c]**Points to the sides on the top and bottom of the pool (see Figure 14).** [d]**Gestures to the corners on the bottom side of the pool (see Figure 14).**[2]	[1]Claim [2]Data [3]Warrant
Teacher	Okay, so the one on the right, you're saying that it's counting the tiles on the left and the right-hand side?		
Jose	Yes.		
Teacher	And then they count the tiles on the top on the bottom without counting the corners?		
Jose	Yes.		

When asked whether one, both, or neither of the visualizations captured Haleemah's and ET's method, Jose claimed that only the right visualization accurately represented the method. In support of his claim, Jose articulated Haleemah's and ET's method while pointing and gesturing toward the drawing on the board. These gestures function as data because they suggest that Jose saw the visualization as reflecting a method consistent with Haleemah's and ET's approach. His explanation of the method serves as a warrant because it justifies why the visualization captures Haleemah's and ET's method.

Furthermore, Jose's description and behaviors satisfy the three criteria for stating a general method. The first criterion entails mentioning multiple elements of a method and how they are related. Rather than focusing on a single element of Haleemah's and ET's method (such as the method counts the corners twice), Jose referred to multiple elements and described how these elements related to each other. The first element Jose acknowledged was the number of tiles on the left and right sides. When he said, "They counted the ones on the side" and pointed to the left and right sides of the pool, I interpret this to mean that he viewed Haleemah's and ET's method as accounting for the number of tiles on the left and right sides. He then added that these sides were "with the corners" and pointed to the top and bottom corners on the right side of the pool, indicating that he saw the corners as a salient feature in the method.

The second element Jose mentioned was the number of tiles on the top and bottom sides of the pool. Jose said, "And the ones on the top and bottom," pointing to the sides on the top and bottom of the pool. His statement and gestures suggest that he viewed the method as additionally counting the number of tiles on the top and bottom sides. The third element that Jose acknowledged was the distinction between the tiles on the top and bottom and the tiles on the left and right sides. Jose noted that the top and bottom sides were "the ones that were left out without counting the corners" and gestured to the bottom corners of the pool, implying that he saw the method as leaving out the corner tiles on the top and bottom sides. Altogether, Jose demonstrated facility in putting the elements of Haleemah's and ET's method in relationship to each other in a way consistent with the general pattern they had been applying.

Additionally, Jose's description satisfies the second criterion of stating a general method because he did not refer to specific instances (such as 13 tiles on the left side or 26 tiles on the left and right sides together), suggesting that he was reasoning in general terms. One may argue that these numbers are inscribed on the board during Jose's explanation. However, these labels were written on the board prior to Argument 1, and Jose's verbal and written statements of the method to not rely on these numbers. Finally, his description fulfills the third criterion because he named and pointed to the quantities involved in the method. For example, when he referred to "the ones on the side," he pointed to the tiles on the left and right sides of the pool. His frequent references to "sides" and "corners," combined with his gestures and the teacher's precision in revoicing (e.g., "You're saying that it's counting the tiles on the left and right-hand side?"), provide evidence that Jose drew upon quantitative reasoning to explain why Haleemah's and ET's method was valid.

A critical reader may argue that Jose's language is too informal or imprecise to constitute a mathematically rigorous statement of the general method. However, additional evidence from his written work, shown in Figure 15, further suggests that he was reasoning structurally, in generality, and quantitatively, albeit using everyday language. In sum, I argue that Jose's description qualifies as stating a general method, which, in Argument 1, functions as warrant. In the following argument, stating a general method will function as claim, signaling a shift in argument position.

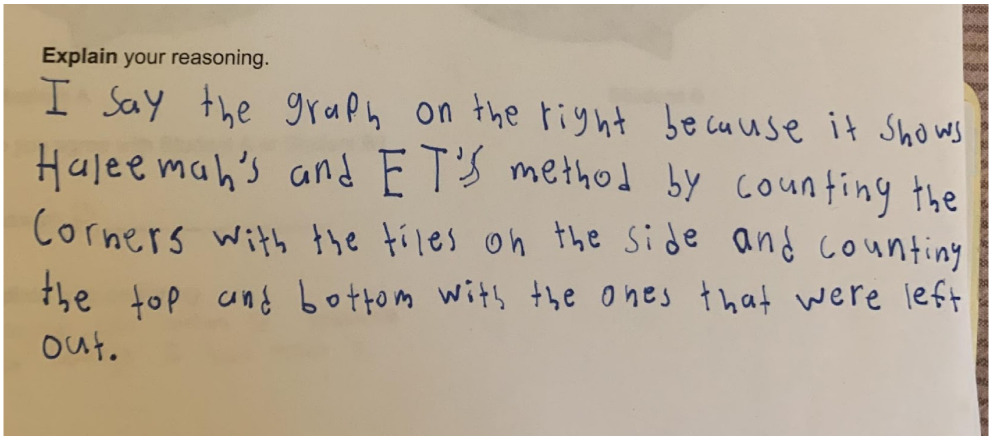

Explain your reasoning.
I Say the graph on the right because it shows Haleemah's and ET's method by counting the Corners with the tiles on the side and counting the top and bottom with the ones that were left out.

Figure 15: Jose's Written Response to the Comparing Visualizations Task

Argument 2

Preliminary Related Reasoning. The preliminary related reasoning discussed in this subsection took place in the game app context, which introduced a new layer of complexity for students. Unlike the pool context, where students were asked to generalize a single method already modeled by the video participants, the game app context required them to make sense of two different general methods demonstrated in the videos and then apply those methods to a new game app task. This shift introduced conceptual and logistical challenges.

Specifically, students watched Haleemah and ET work on the Three Friends Task, which involved finding the total amount of money spent by three friends—Daniel, Kiaan, and Suado—who each purchased a different number of game apps. From this scenario, two general methods emerged. The "long method" involved taking the number of apps each friend buys and multiplying by the cost per app, one at a time, before summing the amounts spent by the three friends. The "short method" involved adding the number of apps together and multiplying by the cost per app. Students were then asked to apply both methods to a new scenario: the Two Friends Task, involving two friends, Sofia and Manuel, who each purchased a different number of apps.

This context, illustrated in Figure 16 (with an example using \$2 cost per app), required students to coordinate four methods in total: both the long and short methods, each applied to both the Three Friends and Two Friends scenarios.

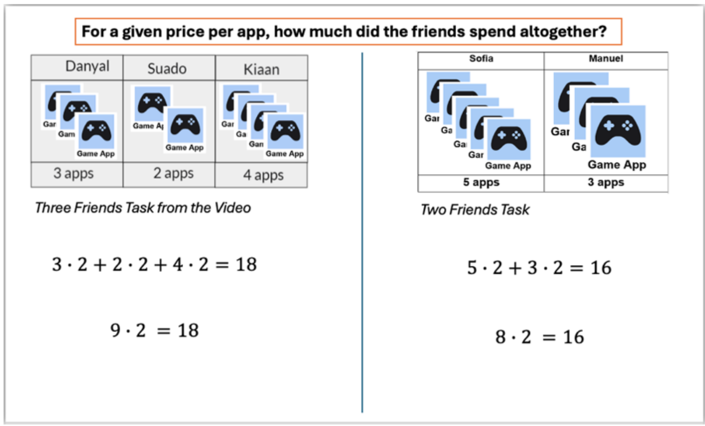


Figure 16: Example of Long and Short Methods Applied to a Price per App of \$2

A sequence of events on Day 6 highlighted students' confusion about which method to generalize. The class began by watching a video of Haleemah and ET using what would later be labeled the "long method" for the Three Friends Task with a cost per app of \$2. When asked to explain the method, one student, Esperanza, described it as "complicated" and proposed an alternative method, the one that would later be called the "short method." The teacher clarified that the method Esperanza had proposed (the short method) would be revisited later, and asked the class to apply the method they had seen in the video (the long method) to a new situation: the Two Friends Task, with a cost per app of \$3. The task was meant to guide students toward applying and ultimately generalizing the long method. However, several students adopted Esperanza's method, resulting in two arithmetic equations: 5 • 3 + 3 • 3 = 24 and 8 • 3 = 24. The use of both methods indicated confusion about which method the students were meant to apply and ultimately generalize.

To direct the students' focus to just one method, the teacher asked the class whether they wanted to generalize Esperanza's shorter method or the original method from the video. The class chose Esperanza's method. The teacher then asked students to apply this method to a new variation of the Two Friends Task, this time using a cost per app of $2.50. However, the confusion persisted. Once again, both the long and short methods appeared in students' responses. When asked to explain the difference between the two, students struggled to articulate a distinction. Although they could apply each method correctly to different numerical inputs, they could not yet articulate the difference between the methods, which was necessary to generalize.

Background. Argument 2 occurred on Day 7, as the class worked to resolve the confusion from Day 6. To help clarify the students' understanding, the teacher introduced a naming convention. The two general methods were now referred to as "long method" and the "short method." She also organized the board accordingly, creating a visual structure (see Figure 16) that distinguished between the four variations: the long method applied to the Three Friends Task, the long method applied to the Two Friends Task, the short method applied to the Three Friends Task, and the short method applied to the Two Friends Task. The teacher directed the focus first on the long method. She asked students to rewatch a video from Day 6, in which Haleemah and ET applied the long method to the Three Friends Task using a cost per app of $2. The teacher then guided a discussion about the video by asking students to explain the meaning of the long method in their own words.

Toulmin's Scheme. In the exchange below (see Table 10), Gisela claims that the long method multiplies the number of apps a person has by the cost per app, one at a time. Her subsequent elaboration that costs are calculated separately depending on how many friends there are functions as data.

Table 10: Coding for Argument 2 of NWR 1.1

Participant	Speech	Actions	Code
Teacher	So could someone, in their own words, say, what is the long method?		
Rodolfo	Extended.		
Teacher	Extended. Gisela?		
Gisela	**The longer method…it means that multiplying the amount of games one person has by the price, one at a time.**[1]		[1]Claim
Teacher	Okay. Multiplying the…can you say that again?		
Gisela	Multiplying the number of games.		
Teacher	Okay. I'll write it up here, what you say. Multiplying the number of games…		
Gisela	By the cost.		
Teacher	Okay. I know. Okay. Games by the cost.		
Gisela	**But the costs separately, depending on how many friends there are.**[2]		[2]Data

When students were asked to describe the long method in their own words, Gisela claimed that it meant "multiplying the amount of games one person has by the price, one at a time." Because the central assertion was her interpretation of the long method, this utterance functions as claim. Gisela's further explanation about the costs being separate ("but the costs separately") depending on how many friends there are functions as data. This is because her elaboration clarified what she meant by "one at a time" in the claim. Her explicit mention of the costs being calculated separately supports why the long method would multiply the amount of games one per has by the cost per app, rather than the approach used in the short method, which adds the number of apps all the friends have together and then multiplies by the cost per app.

Furthermore, Gisela's description of the long method satisfies the three criteria for stating a general method. Her explanation satisfies the first criterion of structure by relating the three elements of the number of apps each friend buys, the cost per app, and the total amount spent by the friends. First, she mentioned that the long method means "multiplying the amount of games one person has by the price," which relates the number of games to the cost per app via multiplication. She additionally mentioned that this process is done "one at a time" and "separately depending on how many friends there are." Her statement refers to another element of the method: total amounts are calculated for each friend. For example, with the three friends, Danyal's total amount is calculated first, followed by Kiaan's and Suado's. For the situation with two friends, Sofia's total amount is calculated first and then Manuel's. Hence, Gisela's description satisfies the structure criterion.

Gisela's explanation satisfies the second criterion (generality) because she did not refer to specific instances, such as the total amount of money spent by the friends when the cost per app is \$3 or \$2. Additionally, Gisela's explanation is grounded in the quantities of the game app context, thus satisfying the third criterion (quantitative). Had Gisela described the method as multiplying two *numbers* together multiple times, she would not have been engaging in quantitative reasoning. Instead, she named quantities by saying "the amount of games one person has" and "the price." She also connected the cost per app to the friends by saying, "the costs separately depending on how many friends there are." Therefore, her explanation satisfies all three criteria for stating a general method.

Fulfilling Criterion 2

For NWR 1.1, I appeal to the second criterion of DCA, which involves a part of a previous argument shifting position in a subsequent related argument. A shift was evidenced in Arguments 1 and 2 because in Argument 1, stating a general method functioned as the warrant, whereas in Argument 2, stating a general method functioned as claim. Gisela's explanation functioned as a claim because it directly responded to the teacher's question, "What is the long method?" By contrast, in the pool context, Jose did not directly state the meaning of the method and instead stated the general method as a justification for answering the task at hand. The shift in stating a general method from "warrant" to "claim" satisfies Criterion 2 of DCA.

Summary of NWR 1.1 and Mathematics Practice 1

Normative way of reasoning 1.1 consisted of students using everyday language to express their general methods, characterized by three characteristics: structure, generality, and quantitative reasoning. This way of reasoning functioned as if shared in the class by Day 7 via Criterion 2 of DCA. First, I presented an argument on Day 4 in the pool context, where stating a general method functioned as warrant. Afterward, I presented an argument on Day 7 and made the case that stating a general method shifted positions from "warrant" to "claim."

Mathematics Practice 1 entailed students stating their general methods (NWR 1.1). In the process of generalizing a method from arithmetic to algebra, students' initial descriptions of their method did not attend to the way that multiple elements in their method were related. While some students could articulate elements in isolation, stating the method as a whole was challenging. By Day 7, they could provide a verbal description of a method in general and exhibited greater facility in attending to relationships between elements involved. Namely, students could directly articulate their general method in response to the teacher's probe, "What did the method mean?" which signaled a shift in their reasoning. Students' challenges in stating a general method in Mathematics Practice 1 are related to their challenges in expressing their methods with algebra, which is explored in Mathematics Practice 2.

Mathematics Practice 2: Expressing a General Method with Algebra

Research on students' meaning-making of literal symbols in algebra has highlighted several difficulties. These include not knowing the appropriate number of symbols to use (Molina et al., 2017), misunderstanding the relationship between symbols and the quantities they represent (Hackenberg et al., 2021), or not knowing that a literal symbol may take on multiple values (Knuth et al., 2005). To tackle these issues, the design of the TE aimed to promote productive meanings of literal symbols, for example, by first establishing a foundation of generalizing arithmetic, forming a general method, and only afterward expressing this method with algebra. If issues arose regarding the use of literal symbols, the teacher planned to show videos where the video participants overcame similar challenges. Therefore, it was anticipated that students would follow the planned hypothetical trajectory.

Contrary to expectations, the TE did not unfold as planned. On Day 2, students appeared to have formed a general method and were prepared to express it algebraically. The teacher, not knowing the extent of their struggles until subsequent analysis, moved forward prematurely by asking students to attempt to express the general method with algebra. The analysis later revealed that at that point, students had only grasped isolated components of their method (for more detail, see NWR 1.1). This led to an unexpected progression of ways of reasoning about literal symbols. Specifically, as I will show, the last and most sophisticated normative way of reasoning in this mathematics practice did not FAIS until Day 8, a day after stating a general method had become normative. Despite these challenges, I will present evidence to claim that students progressed in developing two less sophisticated ways of reasoning, which FAIS by Day 4.

In this section for Mathematics Practice 2, evidence and analysis will be presented to claim that there were three normative ways of reasoning about students' expressions of a general method with literal symbols. The first normative way of reasoning (NWR 2.1) involves students accepting an equation with three symbols (x, y, and B) when only two (x and B, with y defined as $x - 2$) were needed. NWR 2.2 shows students grappling with the quantities associated with y and $x - 2$ and eventually re-expressing the equation with three literal symbols as an equation using only x and B. Lastly, the third and most sophisticated normative way of reasoning in this mathematics practice (NWR 2.3) demonstrates how students built upon the quantitative meaning they had developed to express a general method where their literal symbols represented quantities that vary in value.

NWR 2.1 Accepting an Equation with Three Literal Symbols as Valid When Only Two Are Needed

This section presents evidence of accepting an equation with three literal symbols as valid when only two are needed FAIS in the classroom. While students may use a surplus of symbols in mathematically correct ways, using the fewest number of symbols necessary in algebra is desirable for several reasons. First, simplifying equations by minimizing the number of symbols makes mathematical statements clearer, facilitating easier communication of mathematical ideas. Second, reducing the number of symbols can often reveal important underlying patterns and relationships that provide insight into an algebraic structure.

For example, a student could represent the area of a square using the equation $L \bullet W = A$, where L represents the length of a side, W represents the width, and A represents the area. This representation is not mathematically incorrect, assuming L and W are equal in a square. However, a student's expression of the area using the formula $L^2 = A$ is more sophisticated because it simplifies the equation and reveals the important mathematical structure of A as representing a squared value. As the analysis will show, the use of more symbols than necessary was reflected in the observed dataset. Their re-expression of the equation with the fewest number of symbols needed is tackled in the section for NWR 2.2.

In the following sections, I present two arguments on Days 2 and 4 and apply Criterion 2 from the DCA method (an element of an argument shifting positions in a subsequent argument). I then make the case that the acceptance of an equation using three literal symbols shifted positions from "claim" to "data."

Argument 1

Background. Argument 1 occurred on Day 2, where students worked in the pool-tiling context. Earlier that session, students worked to apply Haleemah's and ET's method to different-sized pools. They also viewed and discussed a video where Haleemah and ET addressed the relationship between the tiles on different sides of the pool as representing a difference of two. Their work and discussions indicated that they had formed a general method and grasped the relationship between the sides of the pool, which could be symbolically represented as $x – 2$. Altogether, they appeared ready to express Haleemah and ET's method as an equation with two literal symbols (i.e., $x \bullet 2 + (x – 2) \bullet 2 = B$).

The students' uncertainty in how to express Haleemah's and ET's general method became more apparent after the teacher asked them to work in groups to write an algebra equation. While she expected variations in their equations, she observed that almost all of the students had, in fact, written equations with three symbols on their sheets (e.g., $x \bullet 2 + y \bullet 2 = T$, $c \bullet 2 + s \bullet 2 = B$).

Anticipating that the video may address the unexpected uniformity in students' responses, the teacher then showed another video of Haleemah and ET, where they wrote an equation using three literal symbols: $x \bullet 2 + y \bullet 2 = B$. Immediately before Argument 1, she paused the video where Figure 17 was shown and initiated a class discussion.

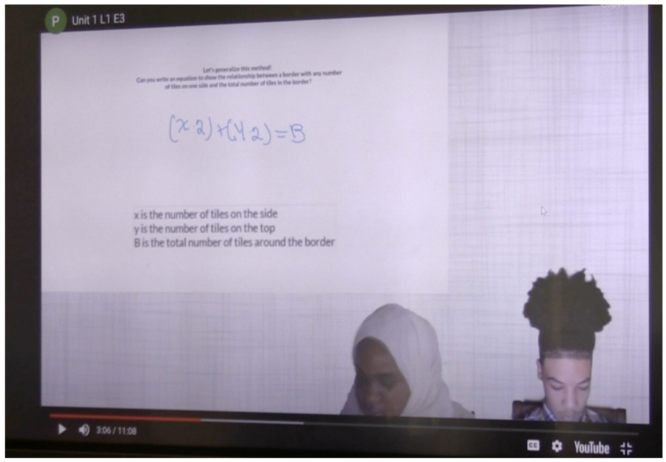


Figure 17: Haleemah's and ET's Equation with Three Literal Symbols

Toulmin's Scheme. In this argument, the core assertion is that $x \bullet 2 + y \bullet 2 = B$ is a valid symbolization for expressing Haleemah's and ET's general method. The evidence for the claim is that the literal symbols (x, y, and B) are not restricted to particular numeric values (or, as Felicia puts it, "those are unknown numbers"), which functions as data.

Table 11: Coding for Argument 1 of NWR 2.1

Participant	Speech	Actions	Code
Teacher	All right. What do we notice?		
Felicia	She has variables.		
Teacher	She has variables?		
Felicia	Yeah, it's basically the same thing. Just without the actual numbers. **Those are unknown numbers, but basically the same forms and everything**.[1]		[1]Data
Teacher	Okay. So, did people agree with this equation? Who had the same equation as Haleemah and ET?[a]	[a]**Eight of the 12 students raise their hand**[2]	[2]Claim
	Who had a different equation?[b]	[b]Two students raise their hand	
	Do you want to talk about…Do you disagree with Haleemah and ET or they convinced you?		
Fatima	No, they convinced me.		

I interpret the students' response to the question, "Did people agree with the equation?" as the claim in this exchange. Their response involved eight of the 12 students raising their hands, suggesting that a majority of students accepted Haleemah's and ET's equation $x \cdot 2 + y \cdot 2 = B$ as valid. Felicia's comment that the equation had "basically the same forms" but with "unknown numbers" functions as data because it implies that she viewed the equation as consistent with their previous work with arithmetic. By "unknown numbers," Felicia meant that the literal symbols x, y, and B were not restricted to particular values. The implicit justification, or warrant, was that if the algebraic equation is in the same form as the arithmetic equations, then the algebraic equation is valid.

Surprisingly, no student disagreed with the equation, suggesting that perhaps the students were struggling to see the relationship between y and x as representing a difference of two, which could be symbolically represented as $x - 2$. I note here that the class's agreement with Haleemah's and ET's equation $x \cdot 2 + y \cdot 2 = B$ functions as the claim where it will later function as data.

Argument 2

Preliminary Related Reasoning. The students' preliminary reasoning related to accepting an equation with three literal symbols as valid when only two were needed is best illustrated by three events on Days 2 and 3. On Day 2, during a discussion following Argument 1, students initially struggled to make appropriate meaning for the symbol y in the equation $x \cdot 2 + y \cdot 2 = B$. For instance, Esperanza and Asma argued that y could represent any number without seeing y as a symbol that represented a value dependent on the value of x. Their reasoning shifted when Diego pointed out that y could not represent any number. He argued that if x represented all the tiles on one side, then y would necessarily exclude the corners. Building on this, Asma amended her initial claim by explicitly stating the dependent relationship between x and y with particular values (e.g., when x is 20, y is 18). This discussion marked progress toward understanding the relationship between x and y as $y = x - 2$, though they had not yet fully articulated the relationship.

Students gained clarity on the dependent relationship between x and y after watching a video in which Haleemah and ET articulated that y was equal to $x - 2$. This insight, however, did not immediately lead to a more sophisticated expression of the equation using two symbols. Following the video, while some students argued that y was unnecessary because $x - 2$ was equal to y, they hesitated to omit y from the equation. For instance, when asked to predict Haleemah's and ET's re-expression for the equation, students ultimately wrote two versions of the equation that included y (i.e., $x \cdot 2 + y \cdot 2 = B$, $(50 \cdot 2) + (y \cdot 2) = 196$). The discussion revealed a resistance to the use of the expression $x - 2$, a challenge and resolution explored further in NWR 2.2.

By Day 3, students had made significant progress. They successfully re-expressed the original equation $(x \cdot 2) + (x - 2) \cdot 2 = B$ by developing quantitative meaning for the symbols y and $x - 2$, which functioned as if shared near the session's end (see NWR 2.2). However, immediately afterwards, some students still defaulted to using the initial equation $x \cdot 2 + y \cdot 2 = B$, but now with the explicit constraint that $y = x - 2$. Their use of the equation with three symbols indicated that while students seemed to have shifted to more sophisticated reasoning about the symbols y and $x - 2$, some felt more comfortable using the equation with three symbols when only two were needed.

Background. Argument 2 occurred on Day 4, which focused on connecting the meaning the students had made for isolated components of Haleemah's and ET's general method on Days 2 and 3. During the session, students tackled two tasks. The first involved comparing visualizations representing different methods, while the second focused on visualizations representing the same method. Before Argument 2, the students worked on the first task, where Jose fully described the general method (as detailed in Argument 1 of NWR 1.1). Argument 2 occurred after group work on the second task, which required students to compare visualizations of the same method (Figure 18). Ahmed shared his group's ideas in this discussion, explaining that both visualizations were equivalent because they showed the same grouping but on different sides. As he explained, he recreated the images from the task on the board and wrote the arithmetic equation $(11 \cdot 2) + (13 \cdot 2) = 48$ beneath the pools (Figure 19). The teacher then prompted Ahmed to elaborate on why the two visualizations captured the same method.

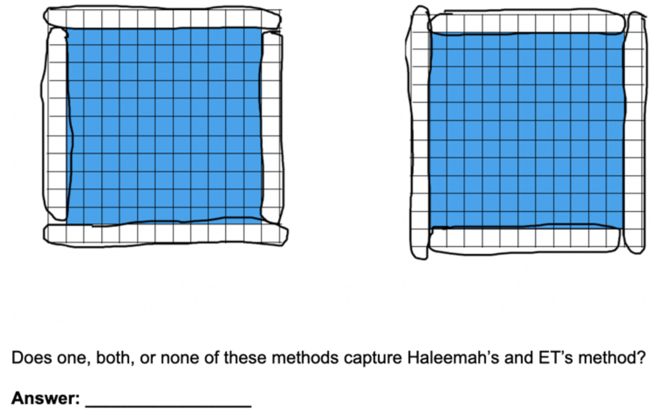


Figure 18: Comparing Visualizations Task: Same Method

Toulmin's Scheme. The argument I present below claims that both visualizations capture Haleemah's and ET's method. Ahmed's evidence for the claim is that both visualizations use the same equation: $x \cdot 2 + y \cdot 2 = B$.

Table 12: Coding for Argument 2 of NWR 2.1

Participant	Speech	Actions	Code
Teacher	Okay. So Ahmed **says both of them are Haleemah's and ET's method**[1], because it's the same circling, just different sides. Okay? And Ahmed mentioned something else in his group about what a method is.		[1]Claim
Ahmed	Oh, so **they were using the same equations. So the equations were…or if I can use…so *x* times two plus *y* times two. So, they both use the same exact one**.[2]	Writes "$x \cdot 2 + y \cdot 2$" on the board (see Figure 19).	[2]Data
Teacher	Whereas the other method…		
Ahmed	While the other method was just 12 times four. That's it.		

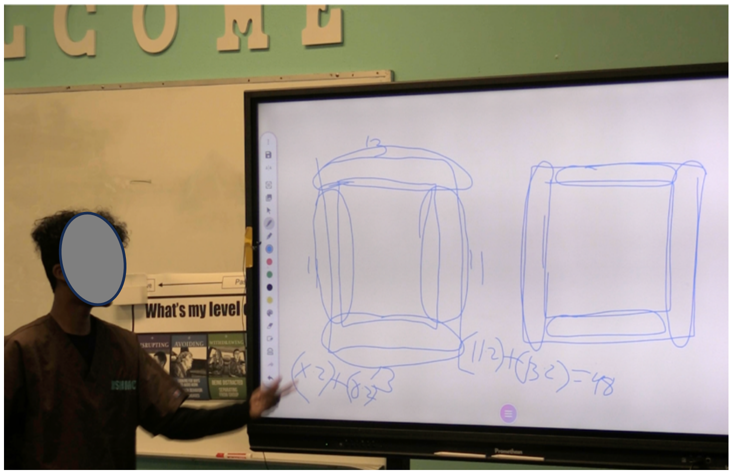


Figure 19: Ahmed Writing an Equation with Three Literal Symbols when Only Two Were Needed

In response to the task, Ahmed claimed that both visualizations captured Haleemah's and ET's method. His initial grounds for this claim was that both visualizations used the same circling but different sides, but when the teacher probed him to elaborate, he provided a different justification. Ahmed articulated that both drawings "were using the same equations," which functions as data. Then he wrote "$x \bullet 2 + y \bullet 2$," suggesting that he viewed both visualizations as capturing Haleemah's and ET's method because they could be expressed using the same algebraic equation. The implicit warrant of Ahmed's argument was that Haleemah's and ET's method can be expressed using the equation $x \bullet 2 + y \bullet 2 = B$. While one may have expected Ahmed to refer to the equation $x \bullet 2 + (x - 2) \bullet 2 = B$, which had been previously formed and discussed, Ahmed instead wrote "$x \bullet 2 + y \bullet 2$."

When Ahmed wrote the expression "$x \bullet 2 + y \bullet 2$," I interpret this to be shorthand for the equation "$x \bullet 2 + y \bullet 2 = B$." One may argue that because Ahmed explicitly uses two literal symbols, the exchange does not provide sufficient evidence that he is referring to an equation with three literal symbols. However, on Days 2 and 3 before Argument 2, students had been viewing videos and discussing the equation $x \cdot 2 + y \cdot 2 = B$, which suggests that when Ahmed said "they were using the same equations," he was referring to $x \cdot 2 + y \cdot 2 = B$. Thus, despite omitting the third literal symbol, I argue that the exchange provides evidence that Ahmed relied on an equation with three literal symbols when only two were needed.

Fulfilling Criterion 2

Criterion 2, which entails an element of an argument shifting positions in Toulmin's scheme, was fulfilled by Arguments 1-2. In Argument 1, accepting an equation with three literal symbols as valid when only two were needed functioned as "claim" because students accepted equation $x \cdot 2 + y \cdot 2 = B$ as valid. Ahmed later used the equation $x \cdot 2 + y \cdot 2 = B$ as grounds for a different claim, which indicated a shift in position from "claim" to "data," therefore satisfying Criterion 2.

Summary of NWR 2.1

This normative way of reasoning focused on students' acceptance of the equation with three literal symbols (x, y, and B) as valid when only two are needed (x and B). This NWR functioned as if shared by Day 4 via Criterion 2 of DCA. The fact that the equation $x \cdot 2 + y \cdot 2 = B$ was accepted as valid FAIS on Day 4 was especially surprising because of the re-expression of the equation as an equation using two literal symbols FAIS the previous day. However, I noted that while the acceptance of an equation with three literal symbols becoming normative was neither intended nor desired, students made progress in making meaning of the symbol y in the equation $x \cdot 2 + y \cdot 2 = B$. Specifically, they shifted from conceiving of y as representing any numeric value to conceiving of y as a quantity that depended on x. A discussion about why the preference for the equation with three literal symbols persisted even after the equation with two symbols functioned as if shared is presented in the Discussion section at the end of this chapter.

NWR 2.2 Re-expressing an Equation with Three Literal Symbols Using Only Two Literal Symbols

Whereas the section for NWR 2.1 presented evidence that students came to accept an equation with three literal symbols, this section for NWR 2.2 will present evidence that students re-expressed their equation with three literal symbols using only two literal symbols. As I will show, re-expressing the equation $x \cdot 2 + y \cdot 2 = B$ as $(x \cdot 2) + (x - 2) \cdot 2 = B$ came to function as if shared in the classroom. In the subsequent sections, I present three arguments: Argument 1.A, Argument 1.B, and Argument 2. Arguments 1.A and 1.B take place within the same whole-class discussion but are presented separately to highlight an important transition related to the eventual acceptance of the re-expression of the equation $x \cdot 2 + y \cdot 2 = B$ as $(x \cdot 2) + (x - 2) \cdot 2 = B$ on Day 3. All three arguments occurred back-to-back on Day 3. Afterward, I apply Criterion 2 from DCA, making the case that one part of Argument 1.B shifted positions in the subsequent argument.

Argument 1.A

Preliminary Related Reasoning. As was discussed in the section for NWR 2.1, on Day 2, more students than expected used three literal symbols rather than two to express Haleemah's and ET's method. After forming the equation $x \cdot 2 + y \cdot 2 = B$, they made progress in seeing that y depended on x, but they initially did not explicitly state that y was equal to $x - 2$ and omitted quantitative meaning for their symbols (for more detail see the Preliminary Related Reasoning subsection for Argument 2 of NWR 2.1). The teacher then showed a video of Haleemah and ET where they made explicit the relationship between x and y as $y = x - 2$ and paused the video just before Haleemah and ET re-expressed their equation as $(x \cdot 2) + (x - 2) \cdot 2 = B$. She then facilitated a discussion about the ideas in the video, in which two students revoiced that y was not needed because $y = x – 2$, suggesting they were ready to re-express their initial equation as $(x \bullet 2) + (x – 2) \bullet 2 = B$.

However, to the teacher's surprise, when asked to predict Haleemah's and ET's new equation, they had difficulty re-expressing the equation as $(x \bullet 2) + (x – 2) \bullet 2 = B$. Maricruz initially wrote "$(x \bullet 2)$" but was unsure how to express the next term. Aislyn attempted to finish the equation and then wrote "$(y \bullet 2)$," crossed it out, and wrote "$(x – 2) + (y \bullet 2)$" in the space above. It was still not clear whether Aislyn was attempting to write the equation $(x – 2) + (y \bullet 2) = B$, the equation $(x \bullet 2) + (x – 2) + (y \bullet 2) = B$, or something else entirely. Maricruz eventually wrote "$(x \bullet 2) + 2(x – 2) = B$." Lastly, Kaelyn wrote another equation with a "y," namely, "$(50 \bullet 2) + (y \bullet 2) = 196$," as shown in Figure 20. In sum, although Maricruz ultimately formed an equation with two literal symbols, the discussion revealed that this way of expressing Haleemah's and ET's method was not yet functioning as if shared in the classroom.

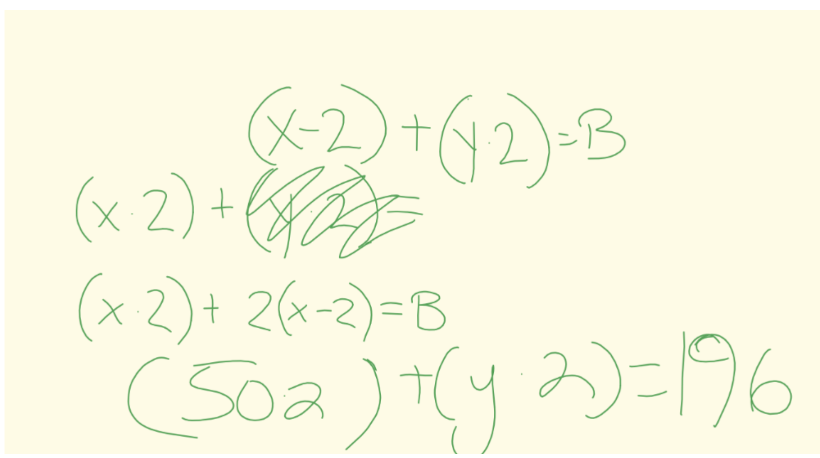


Figure 20: Students' Initial Attempts at Re-expressing an Equation with Three Literal Symbols Using Two Literal Symbols

The students' struggles became more apparent on Day 3 when their descriptions of Haleemah's and ET's general method revealed they had only developed meaning for isolated components (for more detail, see the Preliminary Related Reasoning subsection of Argument 1 for NWR 1.1.). For example, in a discussion revolving around negotiating the meaning of the algebraic equations they had formed on Day 2, one student claimed all of them because they demonstrated not counting the corners twice. This comment indicated that some students viewed $x - 2$ as representing "not counting the corners twice," which failed to consider the meaning of each number and symbol in the pool context. It appeared that without the proper quantitative meaning for $x - 2$, students could not use the idea that y was equal to $x - 2$ to re-express their equation using only two symbols.

Background. Argument 1.A occurred on Day 3 during the whole-class discussion about the meaning of the algebraic equations they had formed on Day 2. Argument 1.A occurred immediately after the teacher asked the class whether the second equation, $x \bullet 2 + 2(x - 2) = B$, captured Haleemah's and ET's method. In the exchange below, Aislyn compares the equation written on the top in Figure 21 to the second equation, $(x \bullet 2) + 2(x - 2) = B$.

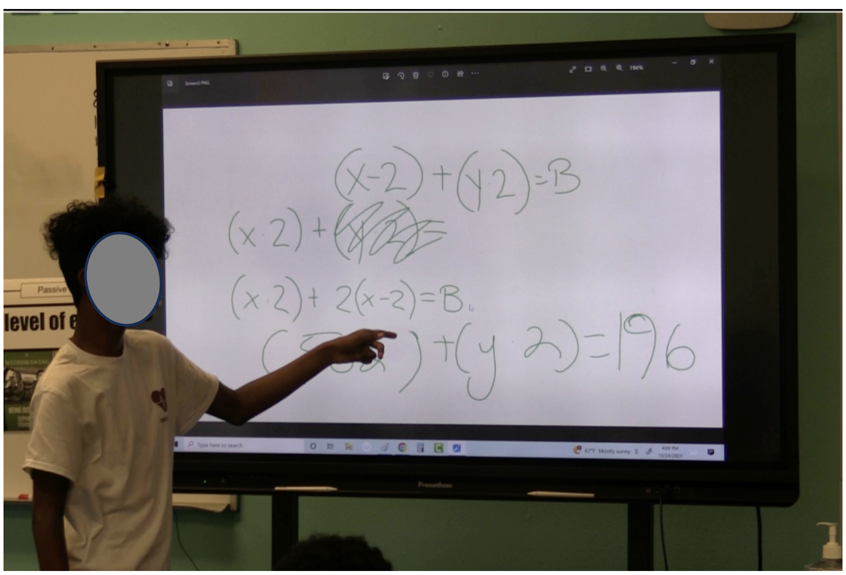


Figure 21: Revisiting Initial Attempts at Re-expressing an Equation with Three Literal Symbols Using Two Literal Symbols

Toulmin's Scheme. The claim in the exchange presented below is that the equation $x \bullet 2 + 2(x - 2) = B$ captures Haleemah's and ET's method. The evidence (data) that students present for this claim is that when $x = 12$, $x \bullet 2 + 2(x - 2) = B$ is a true statement. The warrant is that $(y \bullet 2)$ is equal to $2(x - 2)$.

Table 13: Coding for Argument 1.A of NWR 2.2

Participant	Speech	Actions	Code
Aislyn	The one on the top only shows the equation for one side, like the bottom side and **that one illustrates it more.**[1] **I think the *y* times two is equal to two of *x* minus two.**[2]		[1]Claim [2]Warrant
Teacher	Okay, Aislyn said something really interesting. Who heard what she said? Who wants to repeat what Aislyn just said? Ahmed do you want to go? Could you go?		
Ahmed	So, what she said was, in this equation[a], let's say this[b]… **there's this one side that's 12**[b] **12 times two equals 24, plus 12, *x* minus two. *x* minus two equals 10, which equals answer.**[c] [3] But if you also **just remove two, and just put *y*, *y* equals *x* minus two.**[2]	[a]Points to the equation $(x \bullet 2) + 2(x - 2) = B$ [b]Gestures an imaginary square on the bottom left of the board [c]Points to each number in the equation $(x \bullet 2) + 2(x - 2) = B$	[3]Data [2]Warrant

Aislyn's first utterance includes a claim that $(x \bullet 2) + 2(x - 2) = B$ better illustrates Haleemah's and ET's method as opposed to an equation that includes $(x - 2)$ by itself. As mentioned previously, "the one on the top" could refer to the equation $(x - 2) + (y \bullet 2) = B$, the equation $(x \bullet 2) + (x - 2) + (y \bullet 2) = B$, or something else entirely. However, Aislyn's mention of "equation for one side, like the bottom side" suggests that she was referring to an equation including the expression $(x - 2)$, which represents the number of tiles on the bottom side as opposed to $2(x - 2)$, which accounts for both sides. She followed her comment by stating, "that one illustrates it more," which I interpret as functioning as her claim that the second equation, $(x \cdot 2) + 2(x - 2) = B$, is more representative of Haleemah's and ET's method.

A critical reader may interpret Aislyn's initial utterance about the top equation to function as data. While it may be true that Aislyn is making a separate argument about the comparison between the top equation, including $x - 2$ and the equation $(x \cdot 2) + 2(x - 2) = B$, Aislyn's acknowledgment of $(x \cdot 2) + 2(x - 2) = B$ as valid also functions as the claim for Ahmed's subsequent argument. This argument focuses on negotiating the validity of $(x \cdot 2) + 2(x - 2) = B$ between Aislyn and Ahmed, so I coded Ahmed's utterance as data.

Indeed, when asked to repeat what Aislyn said, Ahmed explained why the second equation held with an example of $x = 12$, which functions as data. His gestures, which include pointing to each number and symbol in the equation $(x \cdot 2) + 2(x - 2) = B$, suggest that he was supporting her claim that $(x \cdot 2) + 2(x - 2) = B$ does in fact capture Haleemah's and ET's method. His empirical evidence demonstrates that the equation yields the correct result with a specific value of 12.

The warrant in the argument is Aislyn's statement about the relationship between $y \cdot 2$ and $2(x\text{-}2)$, which provides a more general justification for why $(x \cdot 2) + 2(x - 2) = B$ is valid. Previously, the students had agreed upon $(x \cdot 2) + y \cdot 2 = B$ a valid expression of Haleemah's and ET's method. They were still negotiating whether $(x \cdot 2) + 2(x - 2) = B$ could be valid, but here, make the connection that $y \cdot 2$ is actually equal to $2(x - 2)$. For example, Aislyn stated, "I think the *y* times two is equal to two of *x* times two," and Ahmed similarly said, "if you just remove two, and just put *y*, *y* equals *x* times two."

This exchange is evidence of re-expressing the equation with three literal symbols as $x \cdot 2 + 2(x—2) = B$. Ahmed and Aislyn acknowledged that this equation was valid by saying, "That one illustrates it more," and by substituting a specific value to check its validity. In the following argument, which occurs immediately after Argument 1, A, the class negotiates specifically why $y = x—2$ by attaching quantitative meaning to their symbols.

Argument 1.B.

Background. The exchange presented in Argument 1.B. occurs immediately after Argument 1.A, where the teacher asks Ahmed to elaborate on why he thought y was equal to $x - 2$.

Toulmin's Scheme. In the following discussion, Ahmed claims that y is equal to $x - 2$. Asma supports this claim by explaining that when x is equal to 50, which is illustrated in the equation $(50 \cdot 2) + (y \cdot 2) = 196$, y is equal to 48, which gives the correct answer. Her empirical evidence functions as data. Ahmed and Maricruz provide a warrant that y represents taking the original side, x, subtracting by two tiles.

Table 14: Coding for Argument 1.B of NWR 2.2

Participant	Speech	Actions	Code
Teacher	Okay, Ahmed said ***y* equals *x* minus two.**		[1]Claim
Kaelyn	Wait, can you repeat that? Why?		
Teacher	Yes, could you explain again?		
Ahmed	***Y* equals *x* minus two**[1] **because you're going to have to subtract two from both sides**[a]**, two times.**[2]	[a]Points to the top and bottom sides of the imaginary square	[1]Claim [2]Warrant
Teacher	So, what is *y*?		
Esperanza	*Y* is 40…		
Maricruz	**The original side, like when you had 12 there, you're just subtracting two from that.**[2] So you really don't need *x* in order to get *y*.		[2]Warrant
Asma	**So, it's 48 even in this equation it's 48.**[3]		[3]Data
Teacher	In which equation?		
Ahmed	The last one.		

This argument concerns the underlying reason that y is equal to $x – 2$. Therefore, y equaling to $x – 2$ functions as the claim. Asma supports this claim using an example with $x = 50$, which functions as data. She said, "It's 48, even in this equation, it's 48." I interpret this to mean that when $x = 50$, then $y = 48$, which supports the claim that the relationship between x and y is $y = x – 2$. She referred to "this equation," which Ahmed later clarified was the equation $(50 \bullet 2) + (y \bullet 2) = 196$. Asma's substitution of y with 48 indicates that she viewed y and $x – 2$ as equal to 48 when x equals 50.

The warrant in the argument is the quantitative meaning attached to y, which Ahmed argued represented taking the original side, x, and subtracting by two tiles. Ahmed said, "y equals x minus two because you're going to have to subtract two from both sides, two times." By "both sides," Ahmed likely meant the top and bottom sides of the pool, which would imply that he was viewing y and $x - 2$ as representing the number of tiles on the top or bottom sides subtracted by two. I interpret his utterance "two times" to refer to the fact that the subtraction occurs twice, once for the top side and once for the bottom side. Algebraically, this action is represented by the multiplication of $x - 2$ by two. Ahmed's grounding of the symbols in quantitative meaning marks a shift from prior reasoning, which focused on the pattern of a numeric difference of 2 (e.g., when x is 50, y is 48; when x is 20, y is 18). Furthermore, the idea that y is equal to $x - 2$ functions as claim in this argument, whereas it will function as warrant in the following argument.

Argument 2

Background. Immediately following the previous argument, the teacher showed a video where Haleemah and ET wrote the equation $x \cdot 2 + (x - 2) \cdot 2 = B$, and the following voiceover was shown, which summarized the video participants' work up until that point:

> Haleemah and ET have discovered a lot. The left-hand side of their first equation, x times two, plus y times two, involves two variables: x, which represents the number of tiles on the left and right side, and y, which was the number of tiles on the top or bottom. The lefthand side of their second equation only uses x. Haleemah and ET realized that they didn't need the other variable, y because y was just x minus two. They solved this using an example of the 50 by 50 pool. Where the number of tiles at the top could be expressed as the number of tiles on the side, subtracted by two to not include the corners twice. Their new equation, x times two, plus the quantity x minus two times two equals B, showed this relationship perfectly. Next, they'll reflect on the meaning of each symbol.

The teacher then asked the students to explain Haleemah's and ET's equation in the context of the video they had just viewed.

Toulmin's Scheme. Maricruz's claim in the following exchange is that $x \cdot 2 + y \bullet 2 = B$ can be rewritten as $x \bullet 2 + (x - 2) \bullet 2 = B$. She asserts that $y = x - 2$ as grounds for supporting her claim, functioning as data.

Table 15: Coding for Argument 2 of NWR 2.2

Participant	Speech	Actions	Code
Teacher	Can someone come up and sort of explain Haleemah's and ET's algebra equation. So, I'm going to have her come up, and someone from your group come up and add on to it, and then your group, and then your group, and then we'll end.		
Maricruz	So they **basically just figured out that *y* isn't really needed in the equation. Cause the *y* is just *x* minus two.**[1] **So, you can just rewrite it as *x* minus two, times two. Like this, I don't know how to say it. Is it like, two times the quantity of *x* minus two.**[2]		[1]Data [2]Claim

Maricruz stated, "so you can just rewrite it as…two times the quantity of *x* minus two," suggesting that she thought that $x \bullet 2 + y \bullet 2 = B$ could be rewritten as $x \cdot 2 + (x - 2) \bullet 2 = \text{B}$. This assertion functions as the claim because it is the primary realization of the video participants, summarized in the voiceover. Her claim drew on the idea that y is equal to $x - 2$ ("cause the *y* is just *x* minus two"), which functions as data. By contrast, students' initial explanations on Day 2 recognized that y was equal to $x - 2$ but failed to use this equivalence to argue how $x \bullet 2 + y \bullet 2 = B$ could be re-expressed as $x \bullet 2 + (x - 2) \bullet 2 = B$.

Fulfilling Criterion 2

Criterion 2 was satisfied on Day 3 because of a shift in position of a part of Argument 1.B changing positions in Argument 2. In Argument 1.B, the claim was that y was equal to $x – 2$. This claim was made as a follow-up argument to Argument 1.A, which discussed the validity of an equation using two literal symbols: $(x \bullet 2) + 2(x – 2) = B$. In Argument 2, the previously expressed idea that $y = x – 2$ functioned as data supports the claim that the equation with three literal symbols could be rewritten as an equation with two literal symbols. Therefore, the shift in argument position indicates that re-expressing an equation with three literal symbols using only two literal symbols became normative in the classroom via Criterion 2.

Summary of NWR 2.2

Normative way of reasoning 2.2 explored students' re-expression of an equation with three literal symbols as an equation using only two literal symbols: $(x \bullet 2) + 2(x – 2) = B$. The NWR functioned as if shared by Day 3 via Criterion 2 of DCA. Students initially struggled to see how the equivalence of y and $x – 2$ would translate into a re-expression of their initial algebraic equation. However, it appeared that their re-expression of the equation with three literal symbols progressed naturally from the emergent idea that y depended on x, which was voiced on Day 2. Additionally, the instructional support of the teacher asking them to explicitly state the quantitative meaning of y and $x – 2$ and watching a video voiceover seemed to help consolidate several complex ideas.

A critical reader may wonder how students could re-express their equation using two symbols on Day 3 before verbally stating their general method, which was first articulated by Jose on Day 4 (Argument 1 for NWR 1.1). Indeed, by Day 3, students had only grasped pieces of the method and had not yet articulated it in its entirety. However, some evidence suggests that students could draw upon emergent ideas before becoming normative. For instance, it seemed that students made progress by drawing on the emergent meaning they had developed for pieces of the method on Day 2 (e.g., the method does not count the corners twice). It may also be the case that stating a general method in its entirety is necessarily a prerequisite to writing a mathematically correct, simplified expression with algebra. It is telling, however, that the last and most sophisticated normative way of reasoning in this mathematics practice did not FAIS until Day 8, a day after stating a general method had become normative, which I explore next.

NWR 2.3 Expressing a General Method with Literal Symbols that Represent Quantities with Varying Values

This section will focus on the third and final NWR in this mathematics practice, which is expressing a general method with literal symbols representing quantities with varying values. As stated in the introduction of this mathematics practice, students struggled to appropriately express their general method using literal symbols without the foundation of stating a general method. Specifically, it was observed in the dataset that one challenge that students faced was distinguishing between fixed quantities and quantities with varying values and later expressing these quantities with literal symbols. However, as I will show, by drawing on the quantitative meaning they had developed throughout the TE, expressing a general method with literal symbols that represent quantities with values that vary came to FAIS in the classroom by Day 8.

I present two arguments on Days 6 and 7 in the following sections. I apply Criterion 1 from the DCA method, which entails a warrant becoming implicit in a student's argument. I argue that expressing a general method with literal symbols that represent quantities with varying values is explicit in Argument 1 and becomes implicit in Argument 2.

Argument 1

Preliminary Related Reasoning. In this subsection, I discuss initial reasoning related to the emergence of NWR 2.3 during Day 6 as they worked in the context of the game apps. As mentioned in the Preliminary Related Reasoning subsection of Argument 2 for NWR 1.1, students applied both the short method (adding the number of apps and multiplying by the cost per app) and the long method (multiplying the number of apps by the cost per app for each friend, one at a time) to costs per app of $2 and $2.50 but did not agree upon which method to generalize. For instance, although students were asked to apply only the short method to a cost per app of $2.50, one student shared their solution as $(5 + 3) = 8 \cdot 2.50 = 20$ (representing the short method). Another shared their solution as $5(2.50) + 3(2.50) = 20$ (representing the long method). Without a statement of these general methods, which would function as if shared by Day 7, the students would have difficulty developing adequate quantitative meaning for their literal symbols.

On Day 6, when asked to express their general method with algebra, several students attached quantitative meaning to their symbols. Still, they did not see the need for their symbols to represent a quantity with varying values. For instance, when asked to share her ideas, Marisa wrote the equation $x + x = x$, explaining that the equation represented $5 + 3 = 8$. Marisa's explanation did not treat x as a symbol that could take on varying values. Instead, she used x to represent the fixed quantities of 5, 3, and 8 and argued that x could represent different quantities because 5, 3, and 8 each represented a number of apps. Thus, while she attached quantitative meaning to the literal symbol x as a number of game apps, she did not yet view her literal symbols as needing to represent a quantity that varied in value.

Background. Argument 1 occurred after Marisa discussed the equation $x + x = x$. She additionally wrote the equation $y \bullet x = B$ and explained that it represented $8 \bullet 2.50 = 20$. (One interpretation of this equation is that y represented a number of apps, x represented a cost per app, and B represented the total amount spent by the friends, although an evidentiary case is beyond the scope of this analysis). The two arithmetic equations, $(5 + 3) = 8 \bullet 2.50 = 20$ and $(5 \bullet 2.50) + (3 \bullet 2.50) = 20$ as well as Marisa's two algebraic equations, $x + x = x$ and $y \cdot x = B$ were shown (Figure 22). The teacher asked Felicia to add onto Marisa's ideas by coming to the board.

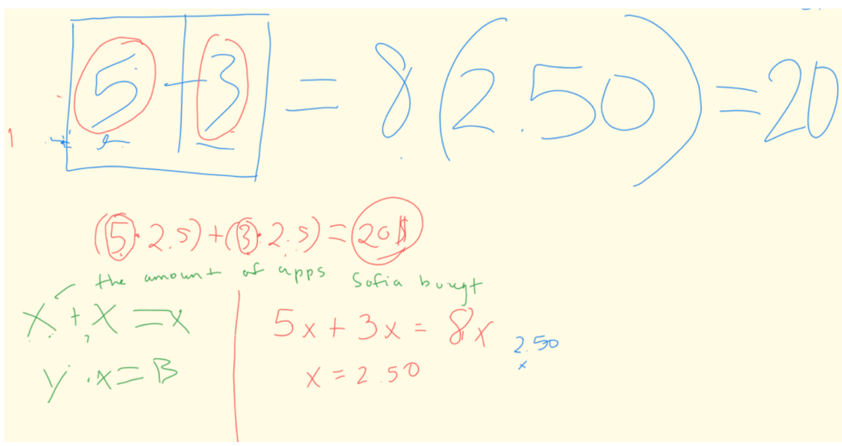


Figure 22: Board Work Before Argument 1

Toulmin's Scheme. Felicia's claim in the exchange below is that the methods could be represented algebraically by $5x + 3x = 8x$. The data is that x represents the cost per app, which is the quantity with varying values. Her warrant is that each time a method is applied, the cost per app is a different value.

Table 16: Coding for Argument 1 of NWR 2.3

Participant	Speech	Actions	Code
Teacher	Wait, wait, wait. Felicia has an idea.		
Felicia	**Could we also put it as five *x* plus three *x* equals eight *x*?**[1] **Does *x* represent the cost? Because the cost kind of changes**[2] **since each one will be a different cost.**[3] Could we do that instead? Or not instead, but is that another way?		[1]Claim [2]Data [3]Warrant
Teacher	Is this another way?	Teacher writes "$5x + 3x = 8x$" on the board.	
Felicia	Yeah.		

While Felicia uses the equation $5x + 3x = 8x$, I do not interpret this moment as evidence that she viewed $5x + 3x$ and $8x$ as equivalent expressions. That interpretation is taken up more fully in Mathematics Practice 5, where I analyze how students developed quantitative meanings for the two equivalent expressions by Day 10. Instead, the focus of the analysis of this argument is Felicia's meaning for the symbol x.

In response to the teacher's question of how to represent their method algebraically, Felicia offered the equation $5x + 3x = 8x$, which functions as her claim. The meaning of the literal symbol x as a cost per app that changes functions as data because it supports her use of x in place of the changing quantity rather than a fixed quantity (e.g., 5, 3, or 8). By contrast, Natalie used x to represent the fixed quantities of 5, 3, and 8 in her equation $x + x = x$. Lastly, Felicia's justification of why the cost per app was differing functions as warrant. Felicia justified why the cost per app varied by stating, "since each one will be a different cost." When Felicia said "each one," I interpret this to mean each time the method is applied. For instance, they had been asked to apply the short method to the cost per app of \$2 and \$2.50.

Felicia's comments provide evidence that she viewed the literal symbol x as a quantity with varying values. She first named the quantity associated with x as the "cost," which I take to mean the cost per app. She then articulated that x "changes since each one will be a different cost." Her use of the word "changes" to refer to the symbol x suggests she sees x as taking on values that vary. This meaning for literal symbols is explicitly stated in the data and warrant. In the next argument, expressing a general method with literal symbols that represent quantities with varying values will become implicit in a student's argument.

Argument 2

Preliminary Related Reasoning. Following Argument 1, students continued to negotiate which method to generalize and how to express their generalizations with algebra, with some continuing to agree with the equations $x + x = x$ and $y \bullet x = B$. Accordingly, on Day 7, the teacher made several adjustments. First, she facilitated a discussion on stating the long method, culminating in this idea FAIS at the outset of Day 7 (see NWR 1.1). Second, she observed that another challenge may have stemmed from students' focus on the difference in the number of apps in the Three Friends Task (i.e., 3, 4, and 2) and the number of apps in the Two Friends Task (i.e., 5 and 3) rather than the changing value of the cost per app (e.g., \$2, \$2.50, \$3). Therefore, the teacher clarified that in the Two Friends Task, 5 and 3 would always stay the same and regularly asked the students questions such as, "What is changing?" to encourage a meaning for the literal symbol x as a quantity with varying values.

Throughout the rest of Day 7, students made progress in their meaning for symbols. For instance, when asked to apply the long method to costs per app of \$2 and \$2.50, students could point to the cost per app as the changing quantity. They worked in groups and ultimately wrote the equation $5x + 3x = B$, where x represented the cost per app, and B represented the total amount spent by the friends. Additionally, when asked "what is changing," students explicitly and frequently responded with the "cost per app."

Background. Argument 2 occurred on Day 8. The students spent the first half of the session interpreting the quantities associated with each number and term in the arithmetic equation (5 • 7) + (3 • 7) = 56 (see Mathematics Practice 3 for more details). In the second half of the session, the teacher asked the students to imagine that she was thinking of an unknown number representing the cost per app, to which several students called out different numbers that it could be, including 8, 26, 7, 10, and 15. The teacher clarified that the number was unknown and asked them to explain how to get the total amount spent by the friends.

Toulmin's Scheme. The claim in the exchange is that the long method involves taking five apps times the cost per app. The data is that the number of apps, 5 and 3, do not change. Lastly, the implicit warrant is that the cost per app is the quantity that varies.

Table 17: Coding for Argument 2 of NWR 2.3

Participant	Speech	Actions	Code
Teacher	How would you describe how to find the total amount spent?		
Marisa	**Person one's apps times cost. So, it's like, person one's apps times cost. And then like, person two.**[1] **Because the apps don't change, right? Or the apps do change?**		[1]Claim
Student	**The apps don't change.**[2]		[2]Data
Teacher	**They don't.**[2]		[2]Data
Marisa	Oh okay, so it's like **five apps times cost.**[1]		[1]Claim
Teacher	Times cost, or cost per app?		
Marisa	**Cost per app.**[1]		[1]Claim

Argument 2 focuses on Marisa's description of the part of the long method that finds Sofia's total amount spent, involving taking the number of apps she purchased multiplied by the cost per app. Marisa initially claimed that the long method took "person one's apps times cost" but later corrected her statement to "five times cost [per app]." I argue that Marisa's amended claim relies on the idea that Sofia's number of apps, 5, does not change, which is evidenced in the back-and-forth exchange between Marisa, the **student**, and the teacher. Marisa stated, "Because the apps don't change, right?" followed by the **student's** statement, "The apps don't change," and the validation by the teacher when she said, "They don't." The exchange serves as grounds for Marisa's use of the number 5 as a fixed, known amount rather than the more general "person's one apps" and, therefore, functions as data.

I argue that Marisa's description of the long method relies on an implicit warrant of the meaning of the cost per app as a quantity whose values vary. The implicit warrant is evidenced by the presence of a fixed quantity meaning (e.g., "the apps don't change") and the absence of a meaning where the cost per app represented a quantity with varying values (e.g., "the cost per app changes"). Additionally, Marisa's statement shifted from "person 1's apps times cost" to "5 apps times cost," suggesting that she viewed the fixed quantity interpretation as applying to the number of apps but not to "cost." The absence of an explicit statement clarifying this meaning likely reflects that they did not feel the need to fully articulate the warrant.

Fulfilling Criterion 1

Criterion 1 was satisfied on Day 8 due to expressing a general method with literal symbols that represent quantities with varying values becoming implicit in Argument 2. In Argument 1, students negotiated which quantity could be represented by an algebraic symbol and needed to explicitly state that the symbol should represent that quantity that is varying.

Following the expression of their method as $5x + 3x = B$ on Day 7, the meaning for the literal symbol as a quantity that varied no longer appeared in their argumentation. In particular, on Day 8, the implicit warrant of Argument 2 of the cost per app as a quantity with varying values did not need to be fully articulated. Hence, Criterion 1 of the DCA method is satisfied.

Summary of NWR 2.3

Normative way of reasoning 2.3 explored students' expression of a general method with literal symbols representing quantities with varying values, which came to function as if shared by Day 8 via Criterion 1. Students' difficulties articulating a general method on Days 2 and 6 (see Mathematics Practice 1, NWR 1.1 for more detail) made it challenging to appropriately express their general method using literal symbols. Students' initial meaning for symbols did not distinguish whether the quantitative meaning for their symbols varied or was fixed (e.g., x represents "5 number of apps"). After repeated questioning (e.g., What is changing?), students no longer needed to fully articulate that the cost per app represented a quantity that was varying. This way of reasoning becoming normative also signaled progress in students' ability to attach appropriate quantitative meaning to their symbols, contrasting previous NWRs where quantitative reasoning was absent (as in NWR 2.1) or emerging (as in NWR 2.2).

Summary of Mathematics Practice 2

Mathematics Practice 2 presented evidence for students' ways of expressing their general methods with algebra. While it was intended for the final normative way of reasoning, expressing a general method with literal symbols that represented quantities with varying values, to become normative earlier in the TE, challenges in students stating a general method along with a misalignment of teaching decisions and students' readiness to symbolize hindered their progress. Nevertheless, two less sophisticated normative ways of reasoning emerged over Days

2-4, including accepting an equation with three literal symbols when only two were needed (NWR 2.1) and re-expressing that equation using only two literal symbols. (NWR 2.2). These normative ways of reasoning set the stage for the eventual acceptance of expressing a general method using literal symbols that represented quantities with varying values (NWR 2.3).

I presented evidence for NWR 2.1 that showed more students than expected using three symbols instead of two to write an equation that expressed their general method. Initially, students saw y as an arbitrary symbol representing any number but later began to see it as dependent on x. In NWR 2.2, students made progress by developing quantitative meaning for the symbols y and $x - 2$. This enabled them to re-express the equation with three symbols using only two symbols, which FAIS on Day 3. Surprisingly, even after forming and discussing the new equation, some students still defaulted to the initial equation using three literal symbols, leading them to accept the equation as valid, which FAIS by Day 4. Finally, in NWR 2.3, they built upon the quantitative meaning they had developed throughout the TE and expressed their general method by grounding their symbols in quantitative meaning. Specifically, they came to distinguish between fixed quantities and quantities with varying values and appropriately related their symbols to the quantities they represented.

Mathematics Practice 3: Interpreting Arithmetic Expressions from Multiple Perspectives

Mathematics Practice 3 focuses on how students interpreted arithmetic expressions from multiple perspectives. According to Sfard and Linchevski (1994), mathematical expressions can be conceived of as both procedures to be carried out (processes) and as entities that can be operated on (products). Expressions conceived of as processes or products can additionally be seen quantitatively, which entails conceptualizing aspects of the expressions as “things that can be measured” (Thompson, 1994, p.6). Students used both quantitative process and quantitative product perspectives to interpret arithmetic expressions. Mathematics Practice 3, therefore, comprises two normative ways of reasoning (NWRs). The first normative way of reasoning (NWR 3.1) demonstrates how students came to adopt a quantitative process interpretation of their arithmetic expressions. In contrast, the second normative way of reasoning (NWR 3.2) demonstrates how students developed a quantitative product interpretation of arithmetic expressions.

Before the emergence of NWR 3.1 (on Days 6 and 7), students initially interpreted arithmetic expressions from a process perspective but omitted the quantitative meaning of the expressions. Later, they could connect the process interpretation with its meaning quantitatively in context. NWR 3.2 emerged from students' initial resistance to directly interpreting arithmetic expressions (5 • 7) as quantitative products. For example, they first needed to calculate 5 • 7 as 35 and interpret the quantitative meaning of 35 in the context before assigning the same quantitative meaning to 5 • 7. However, over time, students could directly interpret arithmetic expressions both as quantitative processes and quantitative products.

The reader may wonder why there is no evidence of interpreting expressions from multiple perspectives from earlier days in the TE. As I will show, students' interpretations appeared to be emerging on Day 2, with these interpretations functioning as if shared on Day 7. Data and analysis will be presented to claim that stronger evidence was identified later in the TE once students' interpretations were more explicit in the whole-class exchange and their language had become more precise. Similarly, I will present data to claim that students' interpretations of arithmetic expressions as quantitative products emerged on Day 2 with these interpretations FAIS by Day 8. However, it may be possible that these interpretations were voiced in small groups rather than during whole-class discussions and, therefore, not well-captured by the DCA method. For a more detailed discussion about the limitations of the DCA method, see the Discussion section at the end of the chapter.

NWR 3.1 Interpreting Arithmetic Expressions from a Quantitative Process View

In this section for NWR 3.1, I will present evidence of the interpretation of arithmetic expressions from a quantitative process view functioning as if shared in the classroom. I consider a student to be employing a *quantitative process* interpretation of arithmetic expressions if both of the following criteria are satisfied:

(d) Process Criterion: the student uses a temporal ordering in speech that corresponds to the ordering of numbers and operations in an expression (from left to right).

(e) Quantitative Criterion: the student names or references the quantity associated with each number or symbol in the expression.

The first criterion is an example of an observable behavior that suggests a procedure being carried out. However, other definitions may include other observable behaviors, such as explicitly mentioning arithmetic operation(s) involved in an expression. For the observed dataset, however, the criterion of verbalizing steps of a procedure in sequential order (typically from left to right) was deemed sufficient for capturing students' quantitative process interpretations. The second criterion is satisfied if the student names the quantities within the expression (e.g., describing 3 • 5 as involving "3 game apps" and "5 dollars"). While other definitions of a quantitative process view may require that students have a quantitative meaning for operation(s) (e.g., having a "groups of" meaning for multiplication), this requirement was considered too restrictive for the dataset. The definition presented here allows for a broader interpretation that captures both the process view and quantitative reasoning.

I present three arguments that occurred on Days 6 and 7 and then apply Criterion 3 from the DCA method (repeated use of an idea as data or warrant). Afterward, I will present a fourth argument that occurred on Day 8 and then apply Criterion 2, making the case that the students' quantitative process interpretation of an arithmetic expression shifted positions in the arguments. As with Mathematics Practice 1 and 2, I include narrative accounts (rather than an evidentiary case) of preliminary reasoning related to each argument.

Argument 1

Preliminary Related Reasoning. At the outset of the TE (Days 1 and 2), students interpreted arithmetic expressions from a process view but did not provide evidence of quantitative reasoning. For example, on Day 1, the students worked in the pool context to find the total number of tiles in the border of a pool with 10 tiles on a side. When explaining their work at the board, Rodolfo and Ahmed described their equations in terms of the numbers and operations involved, such as multiplying 10 by 2 to get 20 and 8 by 2 to get 16 and then adding the results. However, neither Rodolfo nor Ahmed demonstrated awareness that their calculations had relevance to the pool context, indicating that their reasoning was not yet quantitative.

On Day 2, students began to associate numbers with quantities but did not seem to conceive of the quantities as reflecting a procedure, which is consistent with a process view. For instance, when prompted to explain how his equation, $(10 \cdot 2) + (8 \cdot 2) = 36$, related to the pool, Ahmed grounded it in quantities: he stated that 10 represented tiles, while 8 represented one side of the pool. Other students, like Maricruz (Figure 23), later followed suit, associating the numbers with tiles and sides of the pool. While there was some evidence of a quantitative process interpretation, their whole-class descriptions did not acknowledge the ordering of numbers and operations in their expressions, which failed to satisfy the process criterion.

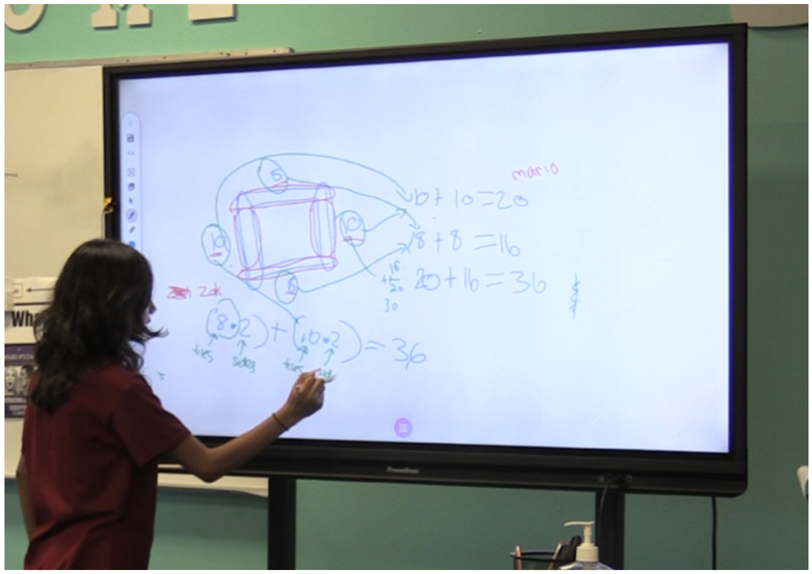


Figure 23: Maricruz Attaching Quantitative Meaning of “Tiles” and “Sides” to an Equation

Background. Argument 1 occurred during a whole-class discussion on Day 6 of the TE. The students viewed a clip of Haleemah and ET explaining their method for finding the total amount spent by the friends on the Three Friends Task with a cost per app of \$2 (see Figure 24). In the clip, Haleemah made a statement resembling a quantitative process view, after which Haleemah was asked to write the arithmetic equation (3 • 2) + (2 • 2) + (4 • 2) = 18. She said, “It costs two dollars, so they bought three games, it would be three times two, and same with all the other purchases as well. And then we just added it all up, which was 18.” After watching the clip, the teacher asked students to come up to the screen and discuss Haleemah’s and ET’s method. Esperanza was asked to discuss the screen, where the equation (3 • 2) + (2 • 2) + (4 • 2) = 18 was shown (Figure 24).

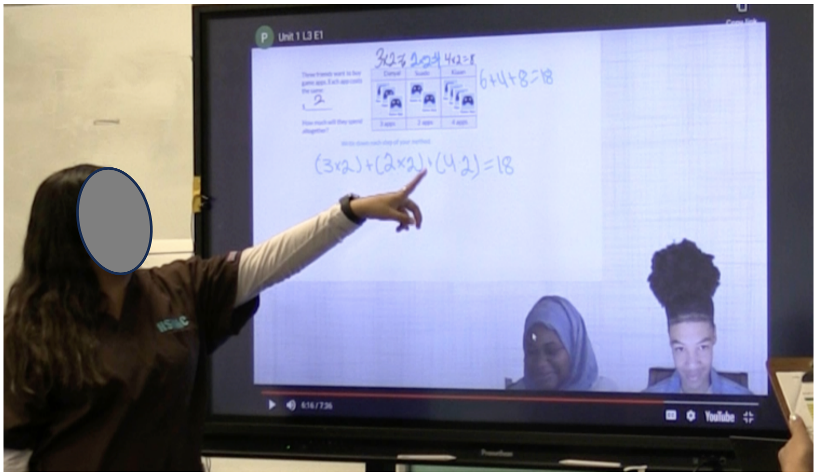


Figure 24: Esperanza Interpreting Arithmetic Expressions in the Equation

$$(3 \bullet 2) + (2 \bullet 2) + (4 \bullet 2) = 18$$

Toulmin's Scheme. In the following discussion of the video clip, Esperanza claims that Haleemah's and ET's method involves getting a total of 18 by multiplying the number of game apps by $2 and adding the results. Esperanza's interpretations of (3 • 2), (2 • 2), and (4 • 2) as quantitative processes function as data to support her claim about Haleemah's and ET's method. I present evidence for Argument 1 in Table 18, which includes transcript along with gestural and other physical actions, and the analytic codes from Toulmin's scheme.

Table 18: Coding for Argument 1 of NWR 3.1

Participant	Speech	Actions	Code
Esperanza	Okay, so what do you want me to do? Explain?		
Teacher	Yeah, explain how they solved the problem, what they were trying to show. Okay. Let's give Esperanza our attention.		
Esperanza	So, the 3 times 2 would be ... Wait, hold on. Oh, okay, okay. The **three times two**[a] **would be three game apps times two dollars**[b] **and then the two times two would be two game apps times the two dollars and then the four times two. It would be the four apps times two dollars**[c1] and **then they just kind of added them all up together and they got 18**[d2]. But I think they kind of made it complicated.	[a]Points to the "3" in "(3 • 2)" and the "2" in "(3 • 2)" (see Figure 24). [b]Points to the game app icon showing "3" apps and the "$2" written in the task statement (see Figure 24). [c]Similar gestures for "(2 • 2)" and "(4 • 2)" (see Figure 24). [d]Makes a sweeping gesture across the equation from left to right when referring to adding together the expressions to get 18 (see Figure 24).	[1]Data [2]Claim

When Esperanza said, "they just kind of added them all up, and they got 18," I interpret this to be a description of Haleemah's and ET's method, where "them" refers to the results of the expressions (3 • 2), (2 • 2), and (4 • 2) and "added them all up" refers to the process of adding the results to obtain the total. A critical reader might argue that her claim reflects her interpretation of the expressions (3 • 2), (2 • 2), and (4 • 2) rather than a description of the method. However, her subsequent comment, "I think they made it kind of complicated" suggests that her main goal was to explain Haleemah's and ET's method rather than to produce interpretations of the arithmetic expressions. Esperanza's interpretations of the expressions (3 • 2), (2 • 2), and (4 • 2) function as data because they support her assertion of how Haleemah and ET found the total amount of money spent by the friends. Lastly, the implicit warrant in her argument is the reasoning for why Haleemah and ET's method is valid. In other words, the implicit warrant is that multiplying the number of game apps by the cost per app for each friend and summing the products gives the total amount spent by the friends.

Esperanza evidenced a quantitative process view in her interpretations of (3 • 2), (2 • 2), and (4 • 2). First, she read the equation (3 • 2) + (2 • 2) + (4 • 2) = 18 in temporal order (from left to right) in a way that mirrored the visual ordering of the equation, beginning with 3 and ending with 18, thereby satisfying the process criterion. Esperanza additionally articulated the meaning of (3 • 2), (2 • 2), and (4 • 2) by naming and pointing to each quantity in the expressions. For example, when describing (3 • 2), she pointed to the three and the 2 in the equation, explicitly stated the meaning as "three game apps times two dollars," and then pointed to the image of three game apps and the $2 in the task statement, suggesting that she was attaching quantitative meaning to both the three and the 2 in (3 • 2). Thus, Esperanza's description of the arithmetic expressions satisfies the definition for a quantitative process interpretation and, in this argument, functions as data. In the next argument, the quantitative process view will be used repeatedly as data but for a different claim.

Argument 2

Background. Argument 2 occurred in a whole-class discussion later on Day 6. This time, the students worked to apply Haleemah's and ET's method to their situation with two friends for the cost per app of $3. After working on the task in groups, Asma was asked to write her group's arithmetic equation and explain what her group thought about the problem.

Toulmin's Scheme. In the exchange presented below, Asma claims that 24 is the total amount spent by the two friends when the cost per app is $3. Asma's interpretations of (5 • 3) and (3 • 3) in the equation (5 • 3) + (3 • 3) = 24 function as data to support her claim that the total is 24.

Table 19: Coding for Argument 2 of NWR 2.1

Participant	Speech	Actions	Code
Teacher	Yeah, you can write the equation you did how you thought about the problem, and then we're going to talk about how it's similar or different to Haleemah and ET's. Do you want to write your equation?		
Asma	I can do it.		
Teacher	You can add on to her idea. Okay. This is Group 2's idea. Go ahead.		
Asma	Like what me and my team said was **five games times three dollars for Sofia's**[a].	[a]Writes "(5 • 3)"	[1]Data [2]Claim
	And then it would be for Manuel, three games times three dollars[b1].	[b]Writes "+ (3 • 3)"	
	And that would equal 24[c2].	[c]Writes "= 24"	

In the above exchange, Asma read the equation (5 • 3) + (3 • 3) = 24 from left to right, suggesting the carrying out of a procedure, which satisfies the process criterion of a quantitative process interpretation. She also articulated the quantities associated with each number in her expressions as measures of the amount of game apps and the cost per app. For instance, Asma wrote (5 • 3) on the board and said, "5 games times three dollars for Sofia's," where five and three represented the number of Sofia's game apps and the cost per app, respectively. She offered a similar description of (3 • 3), representing Manuel's number of game apps times the cost per app of three dollars. Therefore, Asma's description of the arithmetic expressions satisfies the definition for a quantitative process interpretation. Similarly to Argument 1, the quantitative process view functions once more as data in this argument. In the next section, I present evidence of a third instance where the quantitative process interpretation functions as data.

Argument 3

Background. Argument 3 took place on Day 7, where students continued to apply the same method to their situation with the two friends with different costs per app. At this point in the TE, the class had adopted a naming convention of the "long method." In the following back-to-back exchanges, two students, Marisa and Diego, were asked to share their group's work applying the long method to a cost per app of \$2.50. Marisa had written three equations on the board: 2.50 • 5 = 12.50, 2.50 • 3 = 7.50, and 12.50 + 7.50 = 20.

Toulmin's Scheme. Much like Argument 2, Argument 3 is about the students' solution to the Two Friends task but now with a cost per app of \$2.50. Both Marisa and Diego claim that \$20 is the total amount spent by the two friends when the cost per app is \$2.50. Their interpretations of (2.50 • 5) and (2.50 • 3) in the equations (2.50 • 5) = 12.50 and (2.50 • 3) = 7.50 function as data to support their claim that the total is 20.

Table 20: Coding for Argument 3 of NWR 3.1

Participant	Speech	Actions	Code
Teacher	All right, so Marisa's going to talk about how their group was thinking about applying the long method to a price of 2.50. All right. Let's give Marisa our attention. You want to explain your thinking?	Marisa writes "(2.50 • 5) = 12.50", "(2.50 • 3) = 7.50", and "12.50 + 7.50 = 20" on the board	
Marisa	**We did 2.50 times five is the cost times the apps**[1], equals 12.50, and then **2.50 times three equals 7.50**[1]. **And then you add the costs together and then you get the total cost**[2].		[1]Data [2]Claim
Teacher	Okay, and so the 2.50, is that the total cost spent altogether or the cost per app?		
Marisa	**The cost per app**.[1]		[1]Data
Teacher	And then the five, is that—		
Marisa	**The amount of apps Sofia bought**.[1]		[1]Data
Teacher	Sofia bought?		
Marisa	Yeah.		
Teacher	Okay, let's hear from Group 4. You want to add on to their thinking?		
Diego	That's what I got too. **I got the same thing.**[2]		[2]Claim
Teacher	That's what you got too?		
Diego	Yeah.		
Teacher	You want to explain it in your own words? How do you think this is the applying long method?		

Table 20: Coding for Argument 3 of NWR 3.1 (Continued)

Participant	Speech	Actions	Code
Diego	**Well, because the 2.50 times five is the game…the number of games times the price of the game.**[1] And then say for the 2.50 times three, and then they got, or 2.50 times five, that was 12.50. And then three times 2.50, they got 7.50, and then they added those two together to get 20…. **which was the total cost of all they spent for the eight games.**[2]		[1]Data [2]Claim

Marisa's contribution made progress toward a quantitative process interpretation of 2.50 • 5 in the equation 2.50 • 3 = 12.50. With teacher probing, Marisa articulated that 2.50 was the cost per app and 5 was the number of apps Sofia bought, where the expressions were treated as a procedure from left to right (e.g., "we did 2.50 times five is the cost times the apps."). Her utterance suggests that she is seeing the expressions 2.50 • 5 and 2.50 • 3 both quantitatively and as processes, therefore satisfying both criteria.

Her explanation was followed by Diego's more precise description, who articulated the quantities associated with 2.50 and 5 and treated 2.50 • 5 as a procedure from left to right. Diego explained that 2.50 • 5 was "the number of games times the price of the game." The ordering of his speech, which corresponds with the visual ordering of 2.50 • 5 and his attachment of quantitative meaning to each number, serves as evidence of a quantitative process interpretation. Additionally, Diego's description is a more precise description of Marisa's prior articulation of 2.50 • 5 as "cost times the apps" and, therefore more strongly satisfies the quantitative criterion.

Fulfilling Criterion 3

Criterion 3 involves repeatedly using an idea as data or warrant across multiple days for different claims. In the DCA method, fulfilling Criterion 3 signals that a particular idea functions as if shared, meaning that the idea has become a standard way of reasoning in the classroom. This was the case with Arguments 1-3, where the repeated use of the quantitative process interpretation functioned as data for multiple claims across two days, thereby satisfying Criterion 3. In the first argument, the quantitative process view functioned as data in Esperanza's interpretation in her discussion of the video clip. The quantitative process interpretation of arithmetic expressions functioned as data again in the second argument, where students applied what they saw in the video to the Two Friends Task. Lastly, the quantitative process interpretation of arithmetic expressions functioned as data for a third time on Day 7, where students applied the same method to another cost per app.

Next, I will present additional evidence that the interpretation of arithmetic expressions using a quantitative process perspective came to function as if shared in this classroom by appealing to another criterion from the DCA method for a way of reasoning becoming normative—Criterion 2. To satisfy Criterion 2, a part of a previous argument must shift its position in a subsequent related argument. Argument 4 will show how the quantitative process interpretation shifts from the position of "data" to "claim."

Argument 4

Background. On Day 8, students were asked to apply the long method to one more cost per app before generalizing the method and expressing it with algebra. Gisela was asked to share her group's work applying the long method for their situation with a cost per app of $7.

Toulmin's Scheme. Gisela's primary claim is her interpretations of the arithmetic expressions (5 • 7) and (3 • 7) in the equation (5 • 7) + (3 • 7) = 56. The equation itself, (5 • 7) + (3 • 7) = 56, functions as data.

Table 21: Coding for Argument 4 of NWR 3.1

Participant	Speech	Actions	Code
Teacher	Who wants to come up? Gisela? All right, let's give Gisela our attention. You want to explain the equation over here? Okay, Gisela is going to explain the equation.	Gisela writes "(5 • 7) + (3 • 7) = 56"[1] on the board	[1]Data
Gisela	Okay, so for this first part, it would be Sofia's cost……my foot stepped in the wire……**Sofia's amount, including five games and the cost of seven. That would be one thing. And then the amount of games Manuel is buying by seven. And that would be another equation, another answer**.[2] **And the numbers add up to 56**.[1]		[2]Claim [1]Data

The exchange above provides evidence that Gisela was interpreting (5 • 7) and (3 • 7) as quantitative processes. The order in which Gisela interprets the expressions corresponds to the visual ordering in the arithmetic expression. When she says, "five games and the cost of seven," this corresponds to the expression (5 • 7), suggesting the carrying out of a process from left to right. Hence, the process criterion is satisfied. Gisela also describes the numbers in the arithmetic expressions in terms of games and costs (e.g., "five games and the cost of seven"), which indicates that she is additionally attaching quantitative meaning.

Fulfilling Criterion 2

Fulfilling Criterion 2 entails a part of a previous argument shifting position in a subsequent related argument. Argument 4 signaled such a shift from earlier arguments because instead of Gisela's interpretations of (5 • 7) and (3 • 7) as quantitative processes functioning as data, they instead functioned as claim. This was evidenced by the fact that Gisela did not require teacher prompting to interpret these expressions as quantitative processes, and her focus was not on the procedure for obtaining the total. In contrast, earlier arguments (such as Argument 3 on Day 7) included the teacher probing students to interpret the expressions as quantitative processes because the students' primary goal seemed to be to provide the correct numerical answer to the task. Therefore, the shift in position of the interpretation of arithmetic expressions from data to claim, satisfies Criterion 2 of the DCA method.

Summary of NWR 3.1

NWR 3.1 involved students interpreting arithmetic expressions as quantitative processes. The quantitative process interpretation of arithmetic expressions was repeatedly used as data on Day 6 and Day 7 for three different claims. By Criterion 3, this became an accepted form of reasoning in the classroom. Additionally, a shift in argument occurred on Day 8, satisfying Criterion 2.

Some evidence was presented that students' interpretations of arithmetic expressions as quantitative processes were emerging on Days 2 and 3, but there was insufficient evidence to claim that these interpretations FAIS before Day 7. As noted in the introduction of this mathematics practice, it is possible that students did, in fact, interpret arithmetic expressions as quantitative processes earlier in the TE, but in small groups rather than in the whole-class exchange. Although the data did not provide evidence that NWR 3.1 functioned as if shared before Day 7, it appeared as though students' emergent reasoning on Days 2 and 3 laid the groundwork for their interpretations of *algebraic* expressions as quantitative processes becoming normative on Day 3 (see Mathematics Practice 4, NWR 4.1 for more detail).

NWR 3.2 Interpreting Arithmetic Expressions from a Quantitative Product View

In this section for NWR 3.2, I will present evidence of the interpretation of arithmetic expressions from a quantitative product view functioning as if shared in the classroom. In contrast to NWR 3.1, the "product" aspect of a quantitative product view emphasizes the perception of the expression as a whole rather than a focus on the procedural steps (as in the process view). Research suggests that interpreting expressions as products is more challenging for many students than interpreting expressions as processes (e.g., Sfard & Linchevski, 1994). The "quantitative" aspect of a quantitative product view entails the interpretation of the whole or entity as some measurable aspect of a situation (Thompson, 1990). As the analysis will demonstrate, this pattern was reflected in the observed dataset.

Consequently, I consider a student to be employing a *quantitative product* interpretation of arithmetic expressions if both of the following criteria are satisfied:

(a) <u>Product Criterion:</u> the student uses gesture or speech to refer to at least one expression as a single entity or whole.

(b) <u>Quantitative Criterion:</u> the student associates the entity identified in (a) with a quantity in context.

The presence of a process interpretation does not preclude a quantitative product view. For instance, an interpretation of $5 \cdot 7 + 3 \cdot 7$ as the amount spent by Manuel plus the amount spent by Sofia is considered to satisfy a quantitative product view, even with the absence of an interpretation of the entire expression as a quantity (the amount spent by the two friends together).

In the following sections, I will present three arguments: Argument 1.A, Argument 1.B, and Argument 2. Arguments 1.A and 1.B occur within the same whole-class discussion but are presented separately to highlight an important transition related to the eventual acceptance of the quantitative product perspective of arithmetic expressions. The arguments occurred on Days 7 and 8. Afterward, I apply Criterion 2 from DCA, making the case that two parts of Argument 1.B shifted positions in the subsequent argument.

Argument 1.A

Background. Argument 1.A occurred on Day 7. Prior to Argument 1.A, students had worked in groups on the Two Friends Task with costs per app of $2, $2.50, and $5. They had previously discussed their work on costs per app of $2 and $2.50 as a whole class but had moved on to verbal descriptions of their general method (see NWR 2.1). In the last minutes of class, the students were asked to present their work on the Two Friends Task with a cost per app of $5. Aislyn volunteered and wrote the equations shown in Figure 25 on the board [(5 • 5) + (3 • 5) = 40 and 25 + 15 = 40], where the results of (5 • 5) and (3 • 5), namely 25 and 15, were written beneath the two expressions and connected to them via line segments.

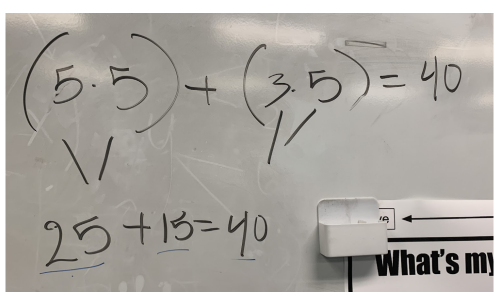


Figure 25: Arithmetic Equations for the Two Friends Task for $5 per App

Toulmin's Scheme. In the following exchange, Gisela's claim is her interpretation of the numbers 25, 15, and 40, shown in Figure 25. She interprets 25 as the total amount spent by Sofia, 15 as the total amount spent by Manuel, and 40 as the total amount spent by all the friends. The visual juxtaposition of the numbers 25 and 15 underneath the expressions (5 • 5) and (3 • 5) functions as data.

Table 22: Coding for Argument 1.B of NWR 2.2

Participant	Speech	Actions	Code
Aislyn		Writes "(5 • 5) + (3 • 5) = 40" and "25 + 15 = 40" on the board with line segments connecting the 25 to (5 • 5) and the 15 to (3 • 5)[1] (see Figure 25)	[1]Data
Teacher	I noticed Aislyn wrote 25, 15, and 40 underneath. Would someone mind explaining what these mean in terms of the game apps? Gisela, do you want to come up?		
Gisela	The **25 is Sofia's cost, it's Sofia's amount, the expense for her games. And 15 is the amount the other person spent, and the 40 is the total cost of all the games**.[2]		[2]Claim
Teacher	Okay. Do we agree with what Gisela said?		
Asma	Yeah.		

I interpret Gisela's response to the teacher's question, "Would someone mind explaining what [25, 15, and 40] mean in terms of the game apps?" as the central assertion of her argument. I note here that Gisela's interpretation of the results of arithmetic expressions (e.g., "25 is Sofia's cost, the expense for her games") functions as a claim. In contrast, it will later function as a warrant in Argument 2. Aislyn's written work on the board functions as data because it illustrates, how Aislyn obtained the result of 25 by multiplying 5 times 5. The implicit warrant in Gisela's argument is that multiplying the number of game apps by the cost per app for a friend gives the total amount spent by that friend. For instance, multiplying five game apps by 5 dollars per app will result in the total amount spent by Sofia.

Gisela's description of 25, 15, and 40, while quantitative, does not yet satisfy the definition of a quantitative product interpretation. For instance, she explained that 25 represented "Sofia's amount, the expense for her games," which indicates that she was interpreting the result of (5 • 5) as a quantity. However, she has not yet evidenced a quantitative product interpretation because she assigned quantitative meaning to a number rather than an expression. In the next argument, Gisela will interpret the expression (5 • 5) directly.

Argument 1.B

Background. The data presented in Argument 1.B occurs immediately after Argument 1.A, where the teacher redirects the focus of the discussion to be about interpreting the expressions (5 • 5) and (3 • 5).

Toulmin's Scheme. The claim in Argument 1.B is Gisela's interpretation of the expression (5 • 5) as the total amount spent by Sofia. The data, like in Argument 1.A, is the visual juxtaposition of the result of 25 underneath (5 • 5). The warrant is Gisela's prior interpretation of the number 25 representing Sofia's total amount spent.

Table 23: Coding for Argument 1.B. of NWR 2.2

Participant	Speech	Actions	Code
Teacher	So, what does five times five mean?		
Esperanza	Five game apps times five dollars.		
Teacher	Gisela said it means something else.		
Esperanza	Oh, what does it mean, Gisela?		
Gisela	**The total…. The cost of games Sofia will have to pay**.[1]		[1]Claim

Unlike Argument 1.A, this argument focuses on the meaning of the expression (5 • 5) rather than the meaning of 25. As such, Gisela's interpretation of (5 • 5) as "the total…the cost of games Sofia will have to pay" was coded as the claim. I argue that the written expression (5 • 5) with the two line segments connecting to 25 provides evidence that the meaning of (5 • 5) is connected to the meaning of 25 and, therefore functions as data to support Gisela's claim about the meaning of (5 • 5). The identification of the warrant is informed by Gisela's utterance in Argument 1.A, where she used an interpretation similar to the interpretation she gave of (5 • 5). Namely, in Argument 1.A she interpreted 25 as "Sofia's cost." In this argument, she interpreted (5 • 5) as "the total…the cost of games Sofia will have to pay." The parallel between the interpretations implies that she relied on the same underlying logic to make a claim about the meaning of (5 • 5). Therefore, the warrant is Gisela's previous interpretation of 25 as "Sofia's cost."

Gisela's interpretation of (5 • 5) "the total…the cost of games Sofia will have to pay" in Argument 1.B marks the first instance of a quantitative product interpretation of an arithmetic expression. Gisela's description satisfies both criteria of a quantitative product interpretation. First, she treated (5 • 5) as a single entity by naming it a "total." Second, she grounded her interpretation in the context of the game apps by specifying that (5 • 5) corresponded to the total cost of games Sofia would have to pay. By interpreting (5 • 5) directly, Gisela evidenced a quantitative product view rather than naming the quantity associated with the calculational result, as she did in Argument 1.A. I note here that the quantitative product interpretation functions as claim and will later function as data in Argument 2.

Argument 2

Background. On Day 8, the students worked on the Two Friends Task with a cost per app of $7, resulting in the equation (5 • 7) + (3 • 7) = 56. The students worked on an activity that entailed labeling descriptions to parts of the equation (5 • 7) + (3 • 7) = 56. Similarly to the discussion in Argument 1.A, students could assign quantitative meaning to the result of products 35 and 21 by labeling them as "total cost." Their labeling suggests that students felt comfortable interpreting the *results* of arithmetic expressions from a quantitative perspective. Still, they felt less inclined to assign the same meaning to expressions like (5 • 7) and (3 • 7). Immediately after the labeling activity, students worked in groups to write precise descriptions of the quantities 5, 7, 3, and 56 in the equation (5 • 7) + (3 • 7) = 56.

Then, the teacher sought to guide students toward a quantitative product interpretation of the expressions (5 • 7) and (3 • 7). Specifically, they were given the task, "Why do you multiply 5 x 7 to find the total amount the friends spent?" (Figure 26). The task was meant to highlight that 5 • 7 represents a single entity, namely, the amount of money spent by Sofia. Just before Argument 2, Ahmed was asked to write the meaning of (5 • 7) on the board.

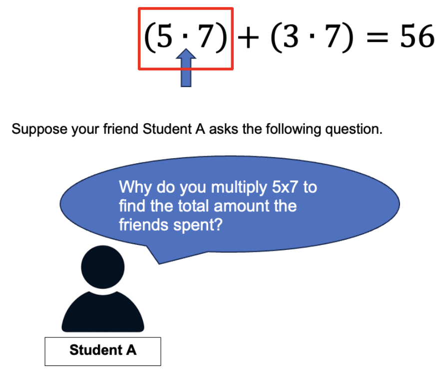


Figure 26: Day 8 Task to Elicit a Quantitative Product Interpretation of (5 • 7)

Toulmin's Scheme. The claim in the following exchange is that (5 • 7) does not give the total amount spent by the friends. The evidence supporting the claim is that (5 • 7) only gives Sofia's total amount, which functions as the data element of the argument. Lastly, the warrant is that the total altogether is 56, and 5 • 7 equals 35, which only gives Sofia's total amount spent.

Table 24: Coding for Argument 2 of NWR 3.2

Participant	Speech	Actions	Code
Teacher	Okay. So, Ahmed wrote, "Sofia has five apps that each cost $7." Why does that give you the total amount spent by all the friends?		
Esperanza	Does... **Why does that give us? It doesn't.**[1]		[1]Claim
Teacher	Does it?		
Esperanza	That... **Just that? The five times seven doesn't give us a total amount spent by all the friends.**[1]		[1]Claim
Teacher	What does it give you?		
Esperanza	**It gives you the amount Sofia spent on games apps.**[2]		[2]Data
Marisa	**The total amount of Sofia.**[2]		[2]Data

Table 24: Coding for Argument 2 of NWR 3.2 (Continued)

Participant	Speech	Actions	Code
Teacher	Could someone repeat Esperanza's idea?		
Gisela	**It only gives you the total amount that Sofia spent**.[2]		[2]Data
Esperanza	Sofia's total amount spent? On game apps?	Writes "5 x 7 only gives you Sofia's total amount spent on game apps" on the board	
Teacher	Everyone agrees with this?		
Ahmed	Yep. Mhmm.		
Kaelyn	I thought they like, added up each other's costs spent altogether, you know?		
Teacher	Yeah. What adds up... What is the cost altogether?		
Kaelyn	**56**.[3]		[3]Warrant
Teacher	56. **And so the 5x7 is?**		
Kaelyn	30...		
Ahmed	**Oh, that's...35. Yeah.**[3]		[3]Warrant
Kaelyn	**How much her games came out to.**[3]		[3]Warrant
Teacher	How much her games came out to? Or?		
Asma	35		[3]Warrant
Teacher	Or 35?		
Esperanza	**Or Sofia's total amount spent on game apps.**[3]		
Teacher	Or Sofia's total amount spent on the game apps. Okay everyone agrees with that?		
Students	Yes.		

The essence of this argument is about whether the quantity (5 • 7) does or does not give the total amount of money spent by all the friends and why. Therefore, Esperanza's statement "the 5 times 7 doesn't give us a total amount spent by all the friends," functions as the claim. The evidence she provided for the claim was that "(5 • 7) only gives you Sofia's total amount spent on game apps." The justification for why (5 • 7) did not give the total amount was expressed by Kaelyn and Ahmed, which was that 5 • 7 equals 35, which only gives Sofia's total amount spent. For example, Kaelyn responded that the cost altogether was "56," and Ahmed replied that (5 • 7) is equal to 35 when he said, "Oh, that's…35. Yeah." Kaelyn additionally attached quantitative meaning to (5 • 7) and 35 by adding that it was "how much her games came out to." I interpret this part of the exchange as Kaelyn and Ahmed, with help from the teacher, justifying Esperanza's initial claim.

The above exchange illustrates how students like Esperanza made a breakthrough in interpreting (5 • 7) as a quantitative product. Previously, Esperanza had only interpreted arithmetic expressions as quantitative processes. For example, in Argument 1.B, she described (5 • 5) as "5 game apps times $5." However, when the teacher asked, "Why does that give you the total amount spent by all the friends?" Esperanza could express a quantitative product perspective, viewing (5 • 7) as "the amount Sofia spent on game apps." She now saw (5 • 7) as both a single entity and having quantitative meaning. Instead of interpreting only the result of (5 • 7) as Sofia's total, she directly interpreted the arithmetic expression as representing Sofia's total amount spent.

Fulfilling Criterion 2

Criterion 2 was satisfied on Day 8 due to two parts of Argument 1.B changing positions in Argument 2. The first shift was the quantitative interpretation of the calculational result of an arithmetic expression shifting from claim to warrant. In the first argument, the claim was that 25 was the total amount spent by Sofia, 15 was the total amount spent by Manuel, and 40 was the total amount spent by all the friends. These are all evidence of a quantitative interpretation of the result of arithmetic expressions. In the second argument, the warrant was that 5 • 7 was equal to 35, the total amount spent by Sofia, whereas the total amount spent by all the friends was 56. Similarly, this was evidence of a quantitative interpretation of the result of an arithmetic expression, which functioned in Argument 2 as warrant.

The second shift was the quantitative product perspective of arithmetic expressions shifting from claim to data. In Argument 1.B, Gisela claimed that 5 • 5 was the total amount spent by Sofia. In the second argument, another student, Esperanza, made an argument where 5 • 7, representing the total amount spent by Sofia, functioned as data. Hence, the two shifts in argument position indicate that interpreting arithmetic expressions as a quantitative product had become normative in the classroom via Criterion 2.

Summary of NWR 3.2

This normative way of reasoning entailed students interpreting arithmetic expressions as quantitative products, which functioned as if shared in the classroom by Day 8 via Criterion 2 of DCA. I presented three arguments that occurred across Days 7 and 8 and made the case that (a) students' quantitative interpretations of the calculational result of arithmetic expressions shifted position from claim to warrant and (b) students' quantitative product interpretation of arithmetic expressions shifted positions in the arguments from claim to data. Although students could interpret arithmetic expressions from a quantitative process view, they initially struggled with interpreting the same expressions as a quantitative product. They made progress by interpreting the result of an arithmetic expression (e.g., the result of 5 • 5). Through targeted questioning (e.g., why do you multiply in this context?), students treated arithmetic expressions directly as whole entities.

Summary of Mathematics Practice 3

Mathematics Practice 3 documented how students interpreted arithmetic expressions from both quantitative process (NWR 3.1) and quantitative product perspectives (NWR 3.2). Students first interpreted arithmetic expressions as a set of procedures with quantitative meaning, which laid the foundation for viewing them as a whole or entity in context. The shift from a quantitative process to a quantitative product interpretation was evident when students explained why multiplication was necessary in context. In the case of the game apps, to find the total amount spent by all the friends, one must first find the amount spent by each friend, reflecting a quantitative product interpretation. Forming these interpretations with arithmetic expressions appears to be an important basis for forming quantitative process and product interpretations of algebraic expressions, which is detailed in Mathematics Practice 4.

Mathematics Practice 4: Interpreting Algebraic Expressions from Multiple Perspectives

Mathematics Practice 4 documents how students interpreted algebraic expressions from multiple perspectives. Like Mathematics Practice 3 with arithmetic expressions, algebraic expressions can be conceived as quantitative processes and products. However, interpreting algebraic expressions (e.g., $5c$) introduces unique challenges. For instance, the underlying operation in $5c$—multiplication—is not immediately apparent, nor is it straightforward to recognize $5c$ as representing a total since the multiplication cannot be directly computed as it would in arithmetic expressions (e.g., $5 \cdot 3 = 15$, representing a total amount spent). Mathematics Practice 4 includes two normative ways of reasoning: interpreting algebraic expressions as quantitative processes (NWR 4.1) and interpreting algebraic expressions from quantitative products (NWR 4.2).

NWR 4.1 Interpreting Algebraic Expressions from a Quantitative Process View

This section provides evidence of the interpretation of algebraic expressions from a quantitative process view functioning as if shared in the classroom. I have adapted the definition of a quantitative process interpretation provided in section NWR 3.1 to the interpretation of algebraic (rather than arithmetic) expressions. This definition includes two criteria:

(a) Process Criterion: the student uses a temporal ordering in speech that corresponds to the ordering of numbers, literal symbols, and operations in an algebraic expression (from left to right).

(b) Quantitative Criterion: the student names or references the quantity associated with each number or literal symbol in the expression.

I will first present two arguments that occurred on Days 2 and 3. Next, I apply Criterion 2 from the DCA method, which entails an element of an argument shifting positions in a subsequent argument. I make the case that the quantitative process interpretation shifts position from “claim” on Day 2 to “warrant” on Day 3.

It is important to address a potential reader's concern regarding the order in which normative ways of reasoning became FAIS. Specifically, I address why the quantitative process interpretation of algebraic expressions seems to precede that of arithmetic expressions, which, as noted at the end of Mathematics Practice 3, is typically foundational for developing such interpretations with algebra. First, it is possible that students interpreted arithmetic expressions as quantitative processes but only in small groups, which is not captured by the DCA method. Second, on Day 2, algebraic equations were introduced by the teacher prematurely (as was discussed in Mathematics Practice 2). This allowed for more opportunities for discussion in the whole-class venue to interpret algebraic expressions, providing sufficient evidence to claim that these interpretations FAIS by Day 3. This contrasts data presented for NWR 3.1, where there was insufficient evidence to argue that it had become normative before Day 7.

However, students also built upon normative ways of reasoning established in previous mathematics practices. Specifically, students appeared to draw on emergent interpretations of arithmetic expressions evidenced on Day 2, which enabled them to interpret algebraic expressions as quantitative processes. Additionally, earlier on Day 3, students' re-expression of an equation with three symbols using only two symbols (NWR 2.2) FAIS. As I will show, their discussion of the algebraic expression $x - 2$ in the context of re-expressing their equation was pivotal to interpreting algebraic expressions as a quantitative process becoming normative in the classroom on Day 3.

Argument 1

Background. As was discussed in previous mathematics practices, on Days 1-2, students were still coming to terms with quantitative reasoning. Argument 1 occurred during a whole-class discussion on Day 2 after almost all the students had formed an equation with three literal symbols (e.g., $(x \cdot 2) + (y \cdot 2) = B$) in the Pool Task context (see Figure 27). They viewed a clip of Haleemah and ET forming and interpreting the equation $(x \cdot 2) + (y \cdot 2) = B$ in the Pool Task context (see Figure 27). The clip featured Haleemah and ET discussing what they thought the expressions $(x \cdot 2)$ and $(y \cdot 2)$ meant. Their meaning was similarly emerging (e.g., regarding x as the "two sides on the side" rather than attributing that meaning to x). However, watching the clip allowed the students in the class to refine the interpretations of the algebraic expressions they heard in subsequent discussions.

For instance, following the clip, the teacher engaged the class in a discussion, asking how the equation (x • 2) + (y • 2) = B made connections across Size 7, Size 10, Size 20, and Size 15 pools. The students first discussed this in their groups, and then Marisa was asked to share how her group saw the equation connecting the different-sized pools. She shared her ideas at the board where the screen in Figure 27 was shown.

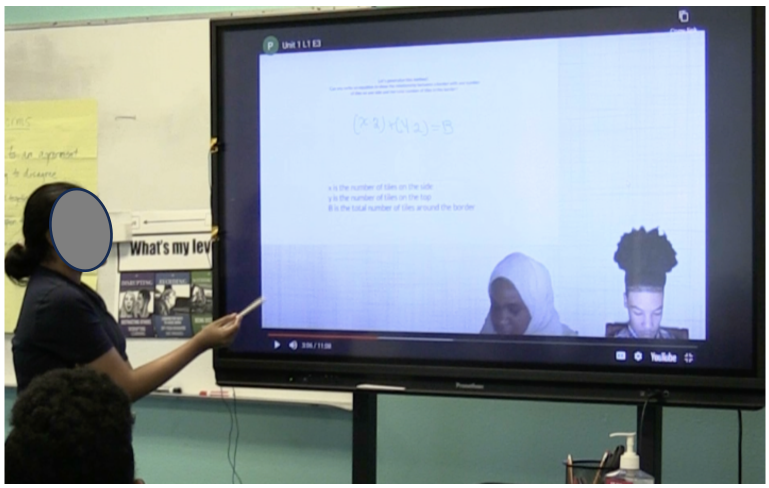


Figure 27: Marisa Interpreting Algebraic Expressions in the Equation (x • 2) + (y • 2) = B

Toulmin's Scheme. In the following back-to-back exchanges between Marisa, the teacher, and Esperanza, the claim is that (x • 2) represents the number of tiles on a certain side multiplied by two sides. Esperanza's example when x = 10 functions as data. In the transcript below, there is another argument focused on interpreting (y • 2). However, Argument 1 focuses only on Marisa's and Esperanza's interpretations of the expression (x • 2) as a quantitative process. The following exchange reflects students' use of informal language at the outset of the TE. I will provide a detailed analysis and interpretation of this speech for further clarification.

Table 25: Coding for Argument 1 of NWR 4.1

Participant	Speech	Actions	Code
Teacher	Okay. Anyone want to share a connection they saw or how this equation shows the relationship across all the pools that you looked at? Yeah. Can you come? Sorry. Or you can point, too. You can point.		
Marisa	Okay. **So for the two, they have the *x* times two, right? So the two are the sides.**[a] **The two would represent the sides for all of the different problems. And the *x* is a certain side.**[b] **So for each, the *x* could represent any amount of tiles for all of the…What is it? For all of the squares on each side**.[1]	[a]Gestures with two hands parallel to each other [b]Gestures with one hand	[1]Claim
Teacher	So is x a side or is x a number of tiles?		
Marisa	Number…oh sorry, **a number of tiles. And then the two are the sides.**[c1]	[c]Points to "2" in (x • 2)	[1]Claim
Teacher	Do people agree with that? Aislyn, you want to say something? Just thumbs up or you want to say something?		
Aislyn	No.		
Esperanza	I'll go.		
Teacher	Esperanza?		
Esperanza	So, basically it's the same thing that Marisa said. The y, since it's a variable it's an unknown number. So, you can put any number of tiles in the y. And then the times two is just the two sides that you're times'ing it by.		
Teacher	Okay. Could you give an example with one of the…[d]	[d]Esperanza goes back to her seat to get her worksheet with Size 10, Size 7, and Size 15 pools shown.	

Table 25: Coding for Argument 1 of NWR 4.1 (Continued)

Participant	Speech	Actions	Code
Esperanza	Okay, **so for the first one,**[5] **it says that there's 10 tiles on the side. So let's say** ***x*** **times two.**[e] **Since it's 10 tiles, you can fill in the variable with the 10. And then the times two, you just leave it there because you times it by the sides.**[2] And then you could do the same thing for the *y* times two.	[5]Points to her worksheet [e]Gestures toward (x • 2) on the board	[2]Data

While the exchange above includes interpretations of both of the expressions (x • 2) and (y • 2), Argument 1 focuses specifically on the argument about the meaning for the algebraic expression (x • 2). Therefore, Marisa's combined utterances function as the central claim because they involve her interpretation of (x • 2), where x was described as the "number of tiles" on a side and two sides ("two are the sides"). This claim is a product of Marisa's initial utterance, where she described x as "a certain side," and 2 as "the sides for all the different problems." A critical reader may argue that Marisa's language is too informal or imprecise to constitute a precise statement of the claim. However, after the teacher asked to clarify, Marisa explained that x was "a number of tiles," suggesting that she was interpreting x as the number of tiles on a certain side.

Esperanza's elaboration with the example of $x = 10$ functions as data because it supports Marisa's interpretation of (x • 2) but with a numeric example. She said, "So for the first one, it says there's 10 tiles on the side." I interpret "first one" as referring to the first pool on Esperanza's task sheet that showed a Size 10 pool, which she pointed to as she explained. She then said, "Since it's 10 tiles, you can fill in the variable with the 10," suggesting that when x = 10, the 10 indeed represented the number of tiles on one side. Lastly, she explained the remainder of the expression (x • 2) by explaining, "The times two, you just leave it there because you times it by the sides." By connecting the meaning of (10 • 2) to (x • 2), Esperanza supported Marisa's assertion of the meaning of the algebraic expression.

Both Marisa and Esperanza provided evidence of a quantitative process interpretation of (x • 2). For example, Marisa, at first, interpreted (x • 2) in reverse order, beginning with the two and then with x, but later read the equation from left to right in a way that matched the visual ordering of the equation. As mentioned earlier in the context of the claim, in her later utterance, she explained that x was "a number of tiles and the two are the sides," thereby satisfying the process criterion. The quantitative criterion was also satisfied because she named the quantities associated with x (as a "number of tiles") and with 2 (as "the sides"). Hence, Marisa's description of the algebraic expression satisfies the definition for a quantitative process interpretation and, in this argument, functions as claim. In the following argument, the quantitative process interpretation will function as warrant.

Argument 2

Background. Argument 2 took place on Day 3, during a review of students' initial ideas for expressing Haleemah's and ET's method with algebra. This argument, previously presented as evidence for the emergence of NWR 2.2 (expressing a general method with two variables), is revisited here, emphasizing the students' interpretation of the algebraic expression $x - 2$ as a quantitative process. The discussion leading up to Argument 2 centered on the validity of the equation $x \bullet 2 + 2(x - 2) = B$ (as shown in Figure 28). Ahmed had just argued that $x \bullet 2 + 2(x - 2) = B$ was valid because y was equal to $x - 2$ and was asked to elaborate on his thinking.

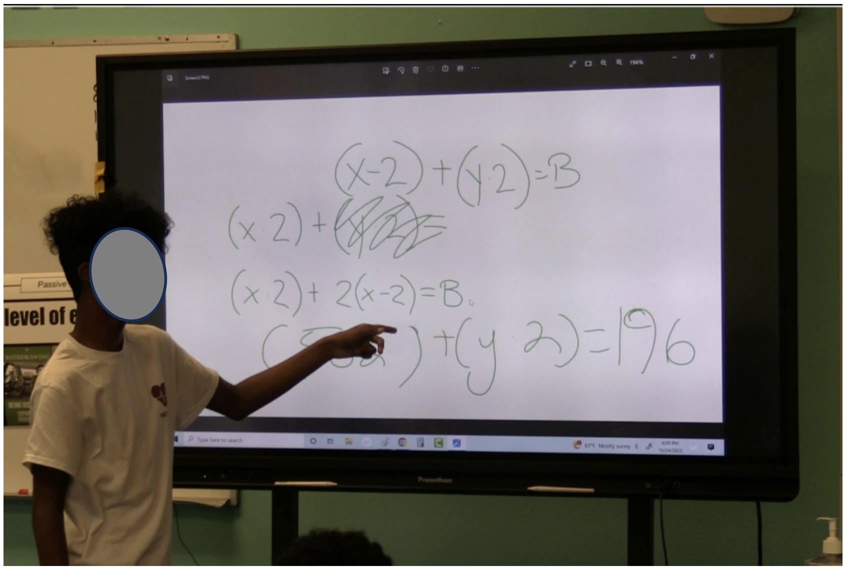


Figure 28: Discussing Ways of Expressing Haleemah's and ET's Method

Toulmin's Scheme. In the following discussion, Ahmed claims that y is equal to $x - 2$. Asma supports this claim by providing an example: when x is equal to 50, as shown in the equation $(50 \cdot 2) + (y \cdot 2) = 196$, y is equal to 48, which gives the correct answer. Her empirical evidence functions as data. Ahmed and Maricruz provide a warrant that y represents taking the original side, x, subtracting by two tiles. Note that this discussion is based on the same table and coding presented in NWR 2.2 (Table 14: Coding for Argument 1.B of NWR 2.2). However, in this subsection, I focus on the *warrant* in the argument in light of a quantitative process interpretation of an algebraic expression and in relationship to Argument 1 (see Table 26).

Table 26: Coding for Argument 2 of NWR 4.1

Participant	Speech	Actions	Code
Teacher	Okay, Ahmed said ***y* equals *x* minus two.**		[1]Claim
Kaelyn	Wait, can you repeat that? Why?		
Teacher	Yes, could you explain again?		
Ahmed	***Y* equals *x* minus two**[1] **because you're going to have to subtract two from both sides**[a]**, two times.**[2]	[a]Points to the top and bottom sides of the imaginary square	[1]Claim [2]Warrant
Teacher	So, what is *y*?		
Esperanza	*Y* is forty…		
Maricruz	**The original side, like when you had 12 there, you're just subtracting two from that.**[2] So you really don't need *x* in order to get *y*.		[2]Warrant
Asma	**So, it's 48 even in this equation it's 48.**[3]		[3]Data
Teacher	In which equation?		
Ahmed	The last one.		

Although students in the exchange presented in Table 26. use informal and imprecise language typical of speech patterns in children, there is additional evidence from their gestures and the context of the task (working with a Size 12 pool) of a quantitative process interpretation of $x – 2$. Both Marisa and Ahmed provided evidence of a quantitative process interpretation of $x – 2$. For instance, Marisa interpreted the expression $x – 2$, first as the "original side" and then as "subtracting two," implying that she saw the expression as carrying out a procedure from left to right. Second, she attached quantitative meaning to the expression. Marisa said, "The original side, like when you had 12 there," where I interpret "the original side" to refer to the number of tiles on a side. By using the example of 12 in place of *x,* Marisa provides additional evidence that she viewed *x* as a number of tiles rather than a side. She additionally explained that you "subtract two," which I interpret to mean two tiles. This speech interpretation at this state of the TE becomes more precise later as students have more opportunities to develop precise language for their quantities.

Fulfilling Criterion 2

Criterion 2 is fulfilled via Arguments 1 and 2 because there was evidence of an element of an argument shifting positions in Toulmin's scheme. The quantitative process interpretation of the expression $x \bullet 2$ functioned as "claim" in Argument 1, while the quantitative process interpretation of $x – 2$ functioned as "warrant" in Argument 2. In Argument 1, the quantitative process interpretation functioned as the central assertion, whereas in Argument 2, the quantitative process interpretation was used as evidence for explaining the equality of $y = x – 2$.

Summary of NWR 4.1

Interpreting algebraic expressions from a quantitative process view became normative in the pool context via a piece of an argument shifting positions, thereby satisfying Criterion 2. Additionally, a warrant became implicit in an argument, satisfying Criterion 1. Students built upon interpretations voiced in the videos and later expressed quantitative process interpretations. For instance, in Argument 1, students viewed a video where Haleemah assigned quantitative meaning to the expression ($x \bullet 2$) and afterwards interpreted ($x \bullet 2$) as a quantitative process. Students then used these interpretations as justification for future claims. In Argument 2, for example, the quantitative process interpretation of $x – 2$ was used to justify the claim that y was equal to $x – 2$.

NWR 4.2 Interpreting Algebraic Expressions from a Quantitative Product View

This section presents evidence of interpreting algebraic expressions from a quantitative product view functioning as if shared in the classroom. I have adapted the definition of a quantitative product interpretation provided in the section for NWR 3.2 to the interpretation of algebraic (rather than arithmetic) expressions. This definition includes two criteria:

(a) <u>Product Criterion:</u> the student uses gesture or speech to refer to at least one algebraic expression as a single entity or whole.

(b) <u>Quantitative Criterion:</u> the student associates the entity identified in (a) with a quantity in context.

This normative way of reasoning draws upon three related normative ways of reasoning discussed in previous mathematics practices. To interpret an expression such as $5c$ as a quantitative product, students first need to express $5c + 3c = T$ as a generalization of an arithmetic relationship (e.g., $5 \cdot 7 + 3 \cdot 7 = 56$). As demonstrated in the section for Mathematics Practice 1 and 2, both ways of reasoning of stating a general method (NWR 1.1) and expressing a general method with literal symbols that represent quantities with varying values (NWR 2.3) functioned as if shared by Day 8. Additionally, since expressing quantitative product interpretations is more challenging than expressing quantitative process interpretations, it is helpful to first interpret arithmetic expressions such as $5 \cdot 7$ as a quantitative product. In the section for NWR 3.2, I showed that this way of reasoning also functioned as if shared by Day 8. These three normative ways of reasoning together became normative on Day 8, establishing a strong foundation for interpreting algebraic expressions as quantitative products to become normative across Days 8 and 9.

In the following sections,, I will present one argument from Day 8 and another from Day 9. Afterward, I will apply Criterion 2 from DCA, arguing that a piece of Argument 1 shifted positions in the subsequent argument.

Argument 1

Preliminary Related Reasoning. As was shown in the section for NWR 4.1 (interpreting algebraic expressions from a quantitative *process* view), students could interpret algebraic expressions like $x - 2$ and $x \bullet 2$ as quantitative processes. However, they struggled to interpret the same expressions as quantitative products. Their difficulties are exemplified by a whole-class discussion on Day 5 concerning the meaning of $x \bullet 2$. During this discussion, Asma claimed that $x \bullet 2$ was a number of tiles multiplied by two sides. Felicia offered a different meaning by describing $x \bullet 2$ as the total amount of tiles on two sides of the pool—a quantitative product interpretation. However, another student, Esperanza, disagreed with Felicia. Her reasoning was that $x \bullet 2$ could not represent a number of tiles because it represented a number of tiles times a number of sides. This argument appeared convincing to the class, with many students voicing agreement with Esperanza's claim. While both Asma's and Esperanza's arguments were reasonable, their claims prevented Felicia's quantitative product interpretation of $x \bullet 2$ from gaining traction in the class. Therefore, by the end of Day 5, the quantitative product interpretation of algebraic expressions had not yet become normative.

Accordingly, in the second half of the TE, the teacher made interpreting arithmetic and algebraic expressions as quantitative products an explicit goal. On Days 6 and 7, students made progress interpreting arithmetic expressions as quantitative processes, which functioned as if shared on Day 7 (see NWR 3.1 for more details). In contrast to the class's work in the pool context, this laid a stronger foundation for interpreting algebraic expressions as a quantitative product to become normative on Days 8 and 9.

Background. Argument 1 occurred on Day 8. Earlier that day, the students continued to work on stating their meaning for the long method, which became normative in the classroom (see Argument 2 of NWR 1.1). Afterward, the students had a breakthrough where they realized that 5 • 7 represented the total amount spent by Sofia explaining the reason for multiplication, leading to the quantitative product interpretation of arithmetic expressions becoming normative in the classroom (see Argument 2 of NWR 3.2). Immediately after that, they successfully expressed the long method as $5c + 3c = T$, where c represented the cost per app, a quantity that varied in value. Similarly, this way of reasoning became normative (see Argument 2 of NWR 2.3).

When it came time to interpret the equation $5c + 3c = T$, students readily interpreted $5c$ as “5 game apps multiplied by the unknown cost of c” and $3c$ as “3 game apps multiplied by the unknown cost of c.” Indeed, the section for NWR 4.1 showed that interpreting algebraic expressions as quantitative processes became normative in the classroom by Day 3 (see Argument 2 of NWR 4.1). Immediately before Argument 1, the teacher asked for an additional meaning of the expressions $5c$ and $3c$, where the descriptions for $5c$ and $3c$ as quantitative processes were written on the board (see Figure 29).

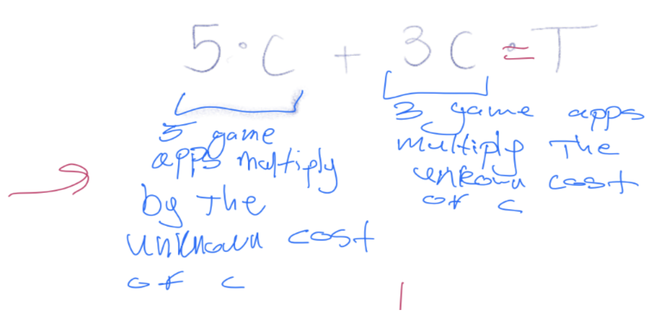


Figure 29: Interpretations 5c and 3c as Quantitative Processes

Toulmin's Scheme. The claim in the following exchange is that 5*c* represents the total amount spent by Sofia on game apps, and 3*c* represents the total amount spent by Manuel on game apps. The data is Maricruz's reference to a previous statement by Esperanza ("Esperanza said it"). The warrant is that the same reasoning used by Esperanza applies to 5*c* and 3*c*. While it is not completely clear what Maricruz refers to in this argument in the data and the warrant, I direct the reader's attention to the statement of a quantitative product interpretation, which is coded as claim.

Table 27: Coding for Argument 1 of NWR 4.2

Participant	Speech	Actions	Code
Teacher	Okay. Alright, alright y'all. Okay. So here's a meaning of 5*c* plus 3*c*. Five game apps…Hey. We have one minute, one minute. Five game apps multiplied by the unknown cost of *c*. Do people agree with this?		
Students	Yes.		
Teacher	Okay. Are there any other ways of thinking of 5*c* plus 3*c*?		
Maricruz	**Esperanza said it?**[1] **Because it's like the way we did it before**.[2] Like the way…Five times…**The total amount of money spent by Sofia**[3]…		[1]Data [2]Warrant [3]Claim
Teacher	The total amount of money spent by Sofia?	Writes "total amount of money spent by Sofia on game apps" on the board	
Maricruz	…on game apps.		
Teacher	Amount of money spent. Do you agree with that, Esperanza? Yeah okay, go ahead. On game apps? Is that the same as 5*c*?		
Esperanza	Oh yeah.		
Teacher	What about 3*c*?		
Esperanza	**The total amount of money spent…**[3]	Writes "total amt of money spent by Manuel" on the board	[3]Claim
Maricruz	**By Manuel.**[3]		[3]Claim
Teacher	…by Manuel.		

In this exchange, Maricruz and Esperanza offer additional interpretations of *5c* and 3*c* when probed, which function as the claim. First, Maricruz spontaneously offered an additional meaning for 5*c* as the "total amount spent by Sofia on game apps." Later, Esperanza and Maricruz claimed that 3*c* meant the total amount spent by Manuel. Maricruz likened her interpretation to a prior way of reasoning that she attributed to Esperanza ("Esperanza said it?"), which functions as data. While Maricruz did not specify which of Esperanza's previous ideas she was referring to, I argue that she likely meant Esperanza's prior interpretation of 5 • 7 as "the amount Sofia spent on game apps" (see line 5 of Table 24, Coding for Argument 2 of NWR 3.2). This is because, of the contributions made by Esperanza on Day 8, her interpretation of 5 • 7 as the total amount spent by Sofia was the most notable. Through her breakthrough, the quantitative product interpretation of arithmetic expressions became normative. I acknowledge, however, that stronger evidence would have articulated that "Esperanza said that 5 • 7 was the total amount Sofia spent on game apps."

I coded "because it's like the way we did it before" as warrant because her utterance connects Esperanza's previously expressed idea to the new situation of interpreting $5c$. That is, Maricruz connected the data to the claim. As before, I argue that she is likely recalling the quantitative product interpretation of 5 • 7 and assigning the same meaning to $5c$ and $3c$.

Moreover, this exchange provides evidence of a quantitative product interpretation of $5c$ and $3c$. For instance, Maricruz interpreted $5c$ as "the total amount of money spent by Sofia on game apps." This description satisfies both the quantitative criterion and product criterion. First, Maricruz quantified $5c$ in the game app context as an amount that Sofia spent. Second, Maricruz interpreted $5c$ as a whole entity by describing it as a "total amount." Thus, her description meets the requirements of a quantitative product interpretation. A similar description was given for $3c$. I note that in this argument, the quantitative product interpretations of $5c$ and $3c$ function as claim where it will function as data in the next argument.

Argument 2

Preliminary Related Reasoning. Before Argument 2 on Day 9, the teacher led an activity to connect quantitative process and quantitative product interpretations—something that was challenging for students in the pool context. The activity involved asking students to draw arrows from five verbal descriptions to quantities in the equation $5c + 3c = T$, as depicted in Figure 30. These descriptions were taken from their work on the previous day, with two corresponding to the quantitative process view ("5 game apps multiplied by the cost per app" and "3 game apps multiplied by the cost per app") and two corresponding to the quantitative product view ("Sofia's total amount spent on game apps" and "Manuel's total amount spent on game apps"). During their whole-class discussion, students drew arrows from the written descriptions to their corresponding symbolic representation in the equation.

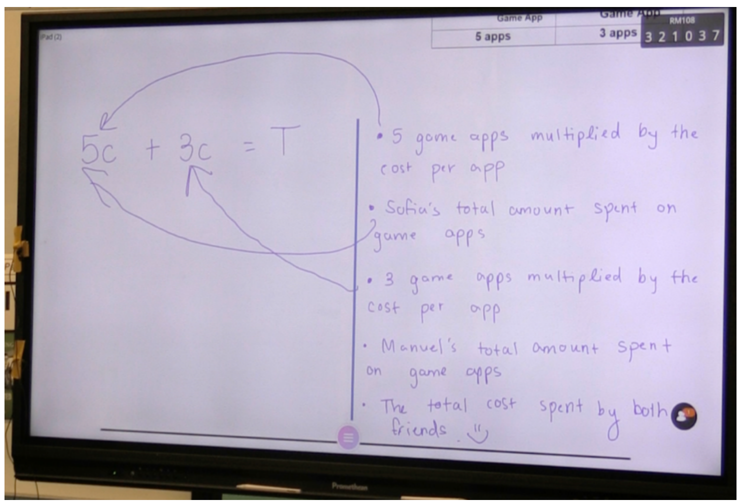


Figure 30: Drawing Arrows from Quantitative Process and Product Descriptions to Algebraic Expressions

The teacher then led a discussion where students explicitly articulated the relationship between the quantitative process and quantitative product interpretations. For instance, when asked why it was okay to have two arrows pointing to $5c$ and two arrows pointing to $3c$, Gisela explained that the quantitative process description was "basically the equation" and the quantitative product description was "whatever the answer of that equation." Two other students also acknowledged that the quantitative product description corresponded to the result of the procedure exemplified by the quantitative process description. The discussion indicated that the students could not only articulate a quantitative product perspective, as in Day 8 but could also explain why this interpretation was valid for any algebraic expression.

Background. Argument 2 occurred on Day 9. Before Argument 2, the teacher asked students to view a clip where Haleemah and ET interpreted $3c$ as the total amount spent by Danyal (one of the friends in the Three Friends Task). Following the video, the students discussed in their groups why Haleemah and ET would give that description for $3c$. The teacher revoiced an idea expressed by Aislyn in groups, which was that $3c$ represented a total because it was three game apps that each cost the same amount, which results in the total amount spent by Danyal. After discussing, the teacher asked the group presenter from Group 1, Maricruz, to come to the board and show how Haleemah and ET would match the verbal descriptions to the equation in their situation with the two friends.

Toulmin's Scheme. The claim in this exchange is that Haleemah and ET would label $5c$ and $3c$ as "Sofia's total amount spent on game apps" and "Manuel's total amount spent on game apps," respectively. The grounds for this claim is that in the video, they similarly described their algebraic expressions as the total amount spent by a friend and the total amount spent by another friend, but for their situation with the three friends ("the other way"). The warrant is that their way of describing the algebraic expressions works for their situation with the two friends ("it works for ours"). The text below reflects the typical speech of children, which includes a casual use of language. Following this, I provide a detailed analysis and interpretation.

Table 28: Coding for Argument 2 of NWR 4.2

Participant	Speech	Actions	Code
Teacher	Okay. So now I'm going to ask you guys how Haleemah and ET would have labeled this. How would Haleemah and ET label this? I'm going to ask your group presenter to come up.[a]	[a]Points to Group 1	
Maricruz	Do I underline?		
Teacher	It's on highlight.		
Maricruz	Oh okay.[b]	**[b]Highlights the description "Sofia's total amount spent on game apps" and "Manuel's total amount spent on game apps"**[1]	[1]Claim
Teacher	Let's pay attention.		
Maricruz	They do…**they would put these two**[1] because **they just said it was the total amount spent by Sofia and the total amount spent by Manuel—Oh…the other way**.[2] Yeah. **Well it works for ours because that's their names**.[c3]	[c]Points to the board	[1]Claim [2]Data [3]Warrant

The central issue in this exchange is how Haleemah and ET would link the verbal descriptions displayed on the board to the quantities in the equation. Therefore, Maricruz's highlighting of the two quantitative product interpretations and her statement, "they would put these two," function as the claim. When Maricruz mentions that "they just said it was the total amount spent by Sofia…the other way," I interpret this in two potential ways. Either Maricruz was referring to a direct quote from the video they watched prior to Argument 2, suggesting she meant "the total amount spent by Danyal," or Maricruz was hypothesizing that Haleemah and ET would describe 5*c* similarly, as "the total amount spent by Sofia." These interpretations are implied from Maricruz's abrupt correction ("Manuel—Oh…the other way"), indicating her realization of the discrepancy between the video's content and the descriptions on the board.

I coded Maricruz's utterance "they just said…the other way" as data because it explains how she views what she heard in the video ("they just said") in the situation with three friends as related to the verbal descriptions written on the board. Namely, the statement suggests she views both the ideas from the video and the written descriptions as quantitative product interpretations, which supports her claim that Haleemah and ET would identify $5c$ and $3c$ as the total amounts spent by Sofia and Manuel on game apps, respectively. Maricruz's statement, "Well it works for our situation because that's their names," suggests that Haleemah's and ET's description of the algebraic expression $3c$ is appropriate for their situation involving Sofia and Manuel. She points out that the difference is merely the names. Her utterance functions as warrant, linking the quantitative product interpretations to her claim about how Haleemah and ET would match the descriptions to the algebraic expressions.

Finally, this exchange is evidence of a quantitative product interpretation. Maricruz said "the total amount spent by Sofia" when referring to $5c$ and "the total amount spent by Manuel" when referring to $3c$. These descriptions satisfy both the quantitative criterion and product criterion. For example, $5c$ was quantified in the game app context as an amount that Sofia spent and as a whole entity by describing it as a "total amount."

Fulfilling Criterion 2

Criterion 2 was satisfied on Day 9 because of the shift in positions of the quantitative product interpretation of algebraic expression from "claim" to "data." On Day 8, Maricruz claimed that $5c$ and $3c$ could be interpreted from a quantitative product perspective as the total amount spent by Sofia and Manuel, respectively. On Day 9, the quantitative product interpretation functioned as data to support the claim that Haleemah and ET would link verbal descriptions to the quantities in the equation $5c + 3c = T$. Hence, Criterion 2 was satisfied.

Summary of NWR 4.2

Because adopting a quantitative product view of algebraic expressions was challenging for students in the pool context, the teacher frequently asked students to voice their interpretations of algebraic expressions as a whole class in the game context. I presented evidence to show that by Day 9, interpreting algebraic expressions as quantitative products became normative in the classroom via Criterion 2 of DCA. Interpreting algebraic expressions from a quantitative product view is crucial for developing a descriptive equivalence interpretation of an equation, which is tackled in NWR 5.2 of Mathematics Practice 5.

Summary of Mathematics Practice 4

Mathematics Practice 4 documented the evolution of students' interpretations of algebraic expressions from both quantitative process (NWR 4.1) and quantitative product perspectives (NWR 4.2). The quantitative process interpretation was salient for students in the pool context. Once students were explicit about the relationship between the quantitative process interpretation and the quantitative product interpretations, they progressed in expressing a quantitative product interpretation.

Mathematics Practice 5: Developing Descriptive Equivalence

The focus of Mathematics Practice 5 is students' ways of reasoning about equivalent expressions from a quantitative reasoning approach known as *descriptive equivalence*. In general, a descriptive interpretation of equivalence treats two objects as equivalent if they describe, represent, or model the same quantity or situation (Cook et al., 2022). I explore descriptive equivalence here as it applies to the equivalence of algebraic expressions (e.g., $5c$

$+3c = 8c$, which treats two expressions as equivalent if they represent the same quantity. By contrast, two expressions are *syntactically equivalent* if they can be rewritten as the same expression by applying properties or theorems that produce simplified or expanded forms (Solares & Kieran, 2013). I note that a syntactic interpretation affords little insight into the equivalence of algebraic expressions because it does not explain why the expressions are equal. On the other hand, a descriptive equivalence interpretation relies on quantities to justify the equivalence of expressions in context.

There are two normative ways of reasoning grouped in Mathematics Practice 5. The first normative way of reasoning (NWR 5.1) documents students' development of descriptive equivalence interpretations of arithmetic equations. The second normative way of reasoning (NWR 5.2) documents students' development of descriptive equivalence interpretations of algebraic equations. The most important result of Mathematics Practice 5 is NWR 5.2 because it was an explicit goal of the TE. However, it is essential to acknowledge that developing descriptive equivalence of arithmetic expressions (NWR 5.1) was also observed in the dataset and functioned as if shared by Day 3. Because NWR 5.1 emerged in the initial days of the TE, the students' language is less precise than their language towards the end of the TE, where their precision of language had improved. For a discussion of how descriptive equivalence of algebraic expressions could become normative before interpreting arithmetic expressions as quantitative products (which FAIS on Day 8), I direct the reader to the Discussion section at the end of the chapter.

Additionally, it is important to note that the method used to establish NWR 5.2 relies on an extension of the DCA method because the whole-class data was insufficient to establish that the reasoning had become normative. Instead, additional analysis at the small group level was needed. Consequently, in the section for NWR 5.2, I present data and analysis to show that descriptive equivalence of algebraic expressions *tentatively* functioned as if shared by following an emerging methodology introduced by Dreyfus et al. (2023).

NWR 5.1 Developing Descriptive Equivalence of Arithmetic Expressions

This section presents evidence that developing descriptive equivalence of arithmetic expressions functioned as if shared. Much of the literature on equivalence has focused on the meaning of the equal sign within an equation (e.g., Falkner et al., 1999; McNeil & Alibali, 2005; Ralston & Li, 2022). However, recently researchers have established an extension, in the sense of Richardson et al. (2024), where they also looked at the sameness of expressions without involving the equal sign. The interpretations observed in the dataset of equivalence of arithmetic expressions are most closely aligned with Richardson et al.'s (2024) extension. Therefore, I associate *descriptive equivalence* with two observable behaviors:

(a) <u>Sameness Criterion:</u> The use of words like *equivalent, equal, same, common, similar, identical, invariant,* or *shared* (or a reasonable synonym) to describe the relationship between the expressions in question.

(b) <u>Quantitative Criterion:</u> The act of pointing to or naming the same quantity when referring to the expressions in question.

This definition includes two criteria: the sameness criterion and the quantitative criterion. An alternative definition could require that the sameness criterion and quantitative criterion be connected by explaining that two expressions are equivalent *because* they represent the same quantity. Such a definition might require observable behaviors, such as using words like "because" or "so" to explain equivalence. By contrast, this definition considers it sufficient to acknowledge that the expressions in question represent the same quantity (e.g., by pointing or naming), leaving the implication of equivalence implicit. Given that descriptive equivalence was particularly challenging for students in the TE, requiring explicit explanatory words like "because" was deemed overly restrictive. A discussion of potential reasons why descriptive equivalence posed such challenges can be found in the discussion section at the end of this chapter.

In the next sections, I present two arguments on Day 2 and Day 3. I argue that the naming of quantities of two equivalent arithmetic expressions repeatedly functioned as data on Day 2 and Day 3, thus fulfilling Criterion 3 of the DCA method. I also present evidence that the exchanges in Arguments 1 and 2 satisfy the given definition of descriptive equivalence of arithmetic expressions, concluding that this idea functioned as if shared in the classroom.

Argument 1

Preliminary Related Reasoning. In this subsection, I provide a narrative account of initial reasoning related to NWR 5.1, which involved syntactic equivalence interpretations. Students' syntactic equivalence interpretations emerged from their work on a task from Day 1 in the pool context. This task came from a video they watched of Haleemah and ET finding the number of tiles in a pool with 10 tiles on a side. In the video, Haleemah initially wrote equations using addition, including $10 + 10 = 20$ and $8 + 8 = 16$, to claim that the total number of tiles was 36. Then, when prompted to combine her addition equations into a single equation, ET wrote an equation using multiplication: $(10 \cdot 2) + (8 \cdot 2) = 36$. The students were then asked to solve the same task as Haleemah and ET, which similarly involved writing an arithmetic equation to find the total number of tiles in a pool with 10 tiles on a side.

When discussing their work, Rodolfo presented his approach using addition, which resembled the initial method from the video. However, Ahmed commented that they were supposed to explain "ET's way" rather than their own, and he wrote an equation with multiplication ((8 • 2) + (10 • 2) = 36; see Figure 31). The approach using multiplication was henceforth referred to as "ET's way," and the approach using addition was referred to as "Rodolfo's way." In the subsequent whole-class discussion, three students noticed similarities and differences between the two approaches, mainly drawing on the idea that (10 • 2) + (8 • 2) = 36 was a simplified version of the approach using addition. Their comments suggested that they appealed to algebraic properties that produced simplified or expanded forms, reflecting a syntactic equivalence interpretation.

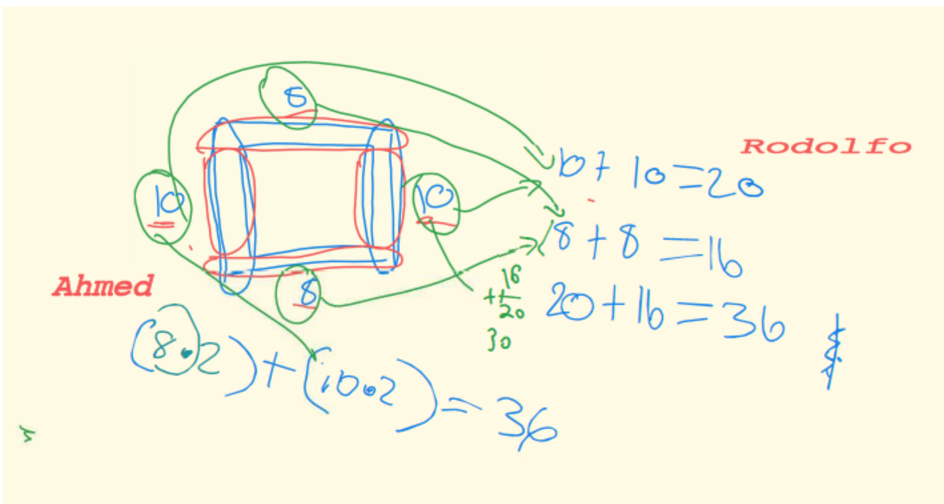


Figure 31: Discussing Rodolfo's Way and ET's Way

Background. Argument 1 occurred on Day 2. Students reviewed their work from the previous day and continued discussing the similarities and differences between Rodolfo's and ET's ways. Several students disagreed, with some claiming that the two ways were different because they represented different quantities. For example, just before Argument 1, Felicia had argued that the two arithmetic expressions 8 + 8 (from Rodolfo's way) and (8 • 2) (from ET's way) were different in the context of the pool. She reasoned that 8 + 8 only added tiles, while in (8 • 2), the "2" represented two sides. While Felicia offered an explanation that appealed to the pool context, she did not view the two expressions as representing the same quantity, which failed to satisfy the definition of descriptive equivalence. The teacher then probed the class to think of the expression (8 • 2) as two groups of eight. Afterward, Maricruz was asked to come to the board to write the meaning of (10 • 2). The following argument compared the meanings of 10 + 10 and (10 • 2).

Toulmin's Scheme. The claim in the following exchange is that 10 + 10 and 10 • 2 represent the same quantity in the pool. The data is that 10 • 2 represents two sides of ten tiles, and 10 + 10 represents two sides of ten tiles, which is indicated by Maricruz's gesturing. I direct the reader to Figure 32, showing Maricruz's board work to follow her inscriptions and gestures.

Table 29: Coding for Argument 1 of NWR 5.1

Participant	Speech	Actions	Code
Maricruz	How would I write it? Like, this represents[a] two groups of eight[b]. And it's the same for that[c] too because two groups of 10…	[a]Draws an arrow pointing to "(8 • 2)" [b]Writes "2 groups of 8" next to the arrow (see Figure 32) **[c]Points to "(10 • 2)"**[1]	[1]Data
Teacher	Is it?		
Maricruz	**Yeah, because there's…[d] two sides. 10.**[1]	**[d]Gestures both hands parallel to each other in front of the two vertical sides in the pool**[1]	[1]Data
Teacher	Of 10 what?		
Maricruz	10 tiles. Okay, do I write that?		
Teacher	Yeah, go ahead and write it.[e]	[e]Adds "tiles" to "2 groups of 8" and **writes "2 groups of 10 tiles" above (10 • 2) with a line segment connecting them**[1]	[1]Data
Maricruz	Okay. And then that[f], I don't know how to explain it well, **but each equation just represents like, the side… like the side of 10 plus 10. The whole equation represents these two sides**.[1g] It's not in the equation how many sides there are, but it just represents the whole thing.	[f]Points to the side of the screen where the equations "10 + 10 = 20" and "8 + 8 = 16" are written **[g]Gestures both hands parallel to each other in front of the two vertical sides in the pool**[1]	[1]Data
Teacher	So even though the two is not in the equation, **it still represents the same thing in the pool**[2]?		[2]Claim
Maricruz	**Yes.**[2]		[2]Claim

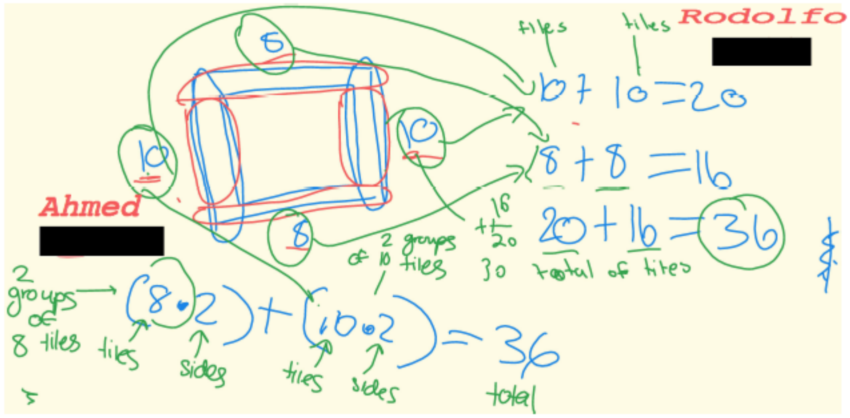


Figure 32: Describing the Meanings of 10 + 10 and 10 • 2

The exchange above features multiple arguments, but I focus the reader's attention on the argument completed by the teacher, which centers on the assertion that 10 + 10 and 10 • 2 represent the same quantity in the pool. Initially, Maricruz argued that 10 • 2 meant “two groups of ten” because there are “two sides” of “10 tiles.” She then separately stated that 10 + 10 also represents the “two sides.” While these arguments could be viewed as distinct, they are interpreted here as collectively supporting the idea that 10 + 10 and 10 • 2 represent the same quantity in the pool. With this perspective, the central claim of Argument 1 is that 10 + 10 and 10 • 2 represent the same quantity in the pool. This is evidenced by Maricruz's affirmative response, “Yes,” to the teacher's question, “It still represents the same thing in the pool?” Maricruz's descriptions of 10 + 10 as “two sides” and 10 • 2 as “two sides” of “10 tiles,” together with her consistent hand gestures, function as data.

A critical reader might argue that Maricruz's statements focus more on distinguishing each expression's meanings rather than asserting their equivalence. However, her use of identical gestures for both expressions (i.e., placing her hands in parallel lines to represent the two vertical sides of the pool) indicates that she aims to demonstrate that the expressions convey the same meaning in this context. Further evidence of Maricruz's focus on the equivalence of the two expressions is her disagreement with Felicia's earlier argument that 8 • 2 and 8 + 8 do not share the same meaning. For example, Maricruz clarified, “It’s not in the equation how many sides there are, but just represents the whole thing,” suggesting that she rejected the notion that including the number of sides in 10 • 2 makes it different from 10 + 10.

This exchange satisfies the two criteria for descriptive equivalence. The first criterion was satisfied by the teacher’s use of the word “same” when referring to the meanings of 10 + 10 and (10 • 2) in the pool. The second criterion was satisfied because Maricruz gestured to and named the same quantity for 10 + 10 and for (10 • 2). Namely, she used the same gesture to describe the two expressions 10 + 10 and (10 • 2) as representing two groups of tiles on the pool’s left and right sides.

Argument 2

Background. Argument 2 occurred on Day 3, during a whole-class discussion in which the students reviewed their work from the previous day. The students were asked again to compare Rodolfo’s and ET’s ways in small groups. Marisa, the presenter for Group 2, was asked to share her group’s ideas.

Toulmin’s Scheme. Marisa’s claim in the following exchange is that ET’s and Rodolfo’s ways are the same. Her evidence for the claim is that the expressions 10 • 2 and 10 + 10 represent two sides of 10 tiles. The warrant is that ET’s way and Rodolfo’s way include the quantity of two groups of 10 tiles.

Table 30: Coding for Argument 2 of NWR 5.1

Participant	Speech	Actions	Code
Marisa	We just agreed and said the same thing, that…I forgot what we decided if they were the same or if they were different…**I think we said the equations were the same thing.**[1]		[1]Claim
Teacher	So Group 2 says that they're the same and Group 1 says that they're different.		
Marisa	I thought Aislyn said that they were different.		
Teacher	Why do you think it's the same?		
Marisa	Because **you get the same answers. So, for here[a] how it has the two sides…The two would be the two 10s.[b] [2]**	**[a]Points to "(10 • 2)"[2]** **[b]Points to "10 + 10"[2]**	[2]Data
Teacher	Two what of 10?		
Marisa	Two groups. So, like for this[c] how we said the "2" would be the two different groups.	[c]Points to "(10 • 2)"	
Teacher	And where do you see the two groups in the pool?		
Marisa	**The sides, so this would be one group[d] and group two.[d] [2]**	**[d]Points to the left side of the pool[2]** **[e]Points to the right side of the pool[2]**	[2]Data
Teacher	Could you point one more time?		
Marisa	Right here[f] and right here.[g][7]	**[f]Points to the left side of the pool[2]** **[g]Points to the right side of the pool[2]**	
Teacher	Are they two groups of eight or two groups of 10?		
Marisa	**Two groups of 10.**		
Teacher	Okay. Do people agree with that? Okay, so you see ET's way the same as Rodolfo's way because why?		

Table 30: Coding for Argument 2 of NWR 5.1 (Continued)

Participant	Speech	Actions	Code
Marisa	**Because they both…you both get the same answers and because that's the two groups.**[3]		[3]Warrant
Teacher	Of?		
Marisa	**Of 10.**[2]		[3]Warrant
Teacher	Of 10 what?		
Marisa	**10 tiles.**[2]		[3]Warrant

Marisa claimed that the "equations were the same thing," where "equations" refers to two expressions—10 + 10 from Rodolfo's way and (10 • 2) from ET's way. Marisa supported this claim by explaining the sameness of 10 + 10 and (10 • 2), which functioned as data. For instance, she first connected the two expressions by saying "for here [points to (10 • 2)]…how it has the two sides…the two would be the two 10s [points to 10 + 10]." She later clarified that the "two 10s" were, "two groups of 10." Marisa further elaborated by grounding the meaning of the two expressions in the pool context. She pointed to the two groups of 10 on the left and right sides of the pool and explained that these were the two groups. Marisa's final statement is that both Rodolfo's and ET's way show the two groups of 10 tiles functions as warrant. Initially, Marisa said "You get the same answers," however, her subsequent explanation appeared to focus more on the quantitative meaning of the two expressions.

Marisa's explanation satisfies the two criteria of descriptive equivalence. A critical reader may argue that Marisa's mention of "same answers" suggests a common characteristic interpretation of equivalence, which requires a justification involving the identification of some common feature between two mathematical objects. However, the rest of Marisa's explanation did not focus on the answers to the expressions 10 + 10 and (10 • 2) as being equal to 20. Instead, she focused on the meaning of 10 + 10 and (10 • 2) in the pool context. Therefore, I interpret "same answers" as evidence of the sameness criterion and her subsequent explanation as evidence of the quantitative criterion for descriptive equivalence. Specifically, she articulated the meaning of 10 + 10 and (10 • 2) as two groups of 10 tiles, which she then associated with the left and right sides of the pool via pointing. Thus, the second criterion of descriptive equivalence was satisfied.

Fulfilling Criterion 3

For NWR 5.1, I apply the third criterion of the DCA method, the repeated use of an idea as data or warrant for more than one argument across multiple days. In Argument 1, Maricruz named the quantities for 10 + 10 and (10 • 2) as two groups of 10 tiles, which functioned as data for the claim that both expressions represented the same quantity in the pool. In Argument 2, Marisa similarly named the quantities for 10 + 10 and (10 • 2) as two groups of 10 tiles, which also functioned as data. In Argument 2, however, the data supported a different claim. The repeated use of the meanings for 10 + 10 and (10 • 2) as data fulfills the third criterion of the DCA method. Additionally, in both arguments, I made the case that the definition for descriptive equivalence of arithmetic expressions was satisfied. Thus, I conclude that descriptive equivalence of arithmetic expressions became normative in this classroom.

Summary of NWR 5.1

On Day 2, students began to view the arithmetic expressions 10 + 10 and (10 • 2) as equivalent in the pool context, each representing two groups of ten tiles. Their understanding deepened on Day 3 when this equivalence of the expressions was used as evidence to support the claim that Rodolfo's way and ET's way were equivalent. Even though developing descriptive equivalence of arithmetic expressions was not an explicit goal of the TE, this normative way of reasoning highlighted students' ability to move from a syntactic equivalence interpretation to a descriptive equivalence interpretation by attaching quantitative meaning to their arithmetic equations. Although NWR 5.1 tackled interpretations of expressions rather than equations, this type of reasoning about expressions is likely an essential foundation for interpreting an arithmetic equation like 10 + 10 = (10 • 2) from a descriptive equivalence perspective.

NWR 5.2 Developing Descriptive Equivalence of Algebraic Expressions

This section on developing descriptive equivalence of algebraic expressions will be distinct from other normative ways of reasoning in this chapter. This case differs because additional analysis of small group interactions was needed to establish that it had become normative on Day 10. I present data and analysis to argue that developing descriptive equivalence of algebraic expressions *tentatively* functioned as if shared by following an emerging methodological extension of the DCA method introduced by Dreyfus et al. (2023). This extended methodology includes an analysis of both whole-class data and small-group data. According to Dreyfus et al. (2023), extending the DCA method to small-group interactions provides valuable insights into how mathematical progress is made.

This approach was chosen because there was not substantial evidence at the whole class level to claim that descriptive equivalence of algebraic expressions FAIS. Specifically, further investigation into this idea was motivated by several factors: (a) time constraints in discussing descriptive equivalence of algebraic expressions at the whole-class level, (b) the expression of a syntactic equivalence interpretation immediately preceding a descriptive equivalence interpretation, (c) the brevity of a descriptive equivalence argument in the whole-class, and (d) the notable progress observed in small groups. The data and analysis presented in the subsequent sections will highlight a complex story of mathematical progress, including descriptive equivalence of algebraic expressions functioning as if shared within one small group and emerging in the three remaining small groups.

This normative way of reasoning entailed students interpreting the algebraic equation $5c + 3c = 8c$ from a descriptive equivalence perspective. Unlike the definition provided in NWR 5.1, in this context, the expressions $5c + 3c$ and $8c$ are presented within an equation where the equal sign denotes equality. To demonstrate evidence of the sameness criterion, a student would use terms such as "equivalent" or "same" to articulate the relationship between the expressions in the equation. Therefore, in this case, the sameness criterion aligns with a *relational* interpretation of the equal sign, where the symbol "=" means "the same as" (Knuth et al., 2006).

The development of descriptive equivalence of algebraic expressions was complex in part because it built upon ideas from the previous four mathematics practices. First, students needed to express $5c + 3c$ and $8c$ as generalizations of an arithmetic relationship. Specifically, $5c + 3c = T$ expressed the "long method," where the number of apps each friend purchased was multiplied by the cost per app (c), and the results were summed to find the total cost (T). In contrast, $8c = T$ expressed the "short method," where the total number of apps was combined first and then multiplied by the cost per app. Second, students worked to interpret c as a symbol that represents a variety of numerical values rather than a physical object, like apps. Third, they interpreted each expression from both quantitative process and quantitative product perspectives. In particular, from a quantitative product perspective, $5c + 3c = 8c$ could be interpreted as the amount of money spent by Sofia and the amount of money spent by Manuel equals the total amount spent altogether by the two friends. Such a description aligns with the criteria of a descriptive equivalence interpretation.

In the following sections, I first present one argument that occurred on Day 10 during a whole-class discussion and explain why additional analysis was warranted at the small-group level. I then shift to the small-group data analysis. First, I detail how I applied the method from Dreyfus et al. (2023). Next, I present evidence to claim that Group 1 negotiated and elaborated the idea of descriptive equivalence of algebraic expressions before the whole class discussion. I then present evidence to make the case that Groups 2-4 were close to establishing the idea. Lastly, in light of the small-group analyses, I apply a variant of Criterion 1 (no member of the classroom issues a challenge) to argue that descriptive equivalence of algebraic expressions was FAIS in Group 1 and was close to FAIS within Groups 2-4 to conclude that this idea tentatively FAIS in the whole-class by the end of the TE.

Argument 1

Background. Argument 1 occurred in the last minutes of Day 10. Just before Argument 1, several students initially appealed to a syntactic interpretation of equivalence. First, Maricruz formed the equation $5c + 3c = 8c$, and the teacher asked the class why $5c + 3c$ was equal to $8c$. Next, rather than naming the quantities associated with $5c$, $3c$, and $8c$ in context, Ahmed and Aislyn provided explanations that appealed to syntactic properties of combining like terms by using words like “simplify” and circling 5 + 3 and 8 in the equation. In response, the teacher asked students to explain why the expressions were equivalent by appealing to the context of Sofia and Manuel. She had the class work on a task in groups meant to elicit a descriptive equivalence interpretation (Figure 33). After working in groups, the teacher asked Fatima to present her group’s ideas.

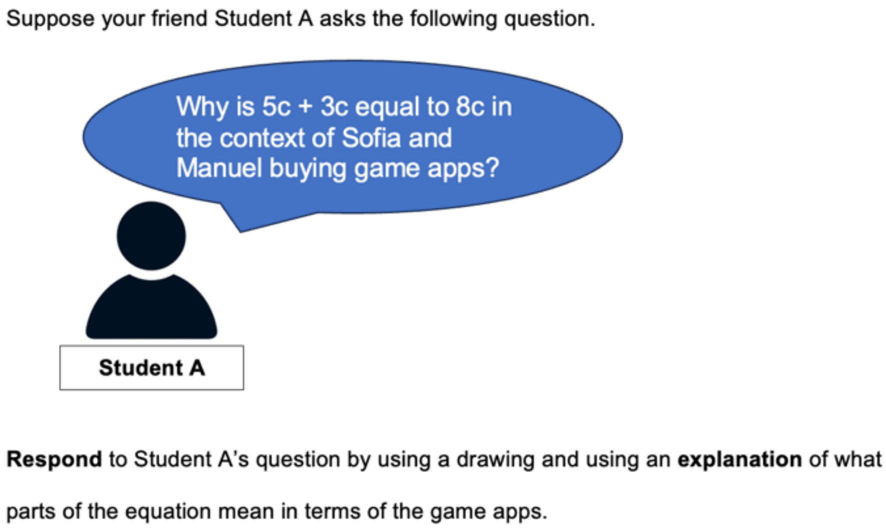


Figure 33: Equivalence Task

Toulmin's Scheme. In the following exchange, the claim is that 5*c* plus 3*c* equals 8*c*. The explanation that 5*c* is Sofia's amount spent, 3*c* is Manuel's spent, and 8*c* is the total amount spent for both friends functions as data.

Table 31: Coding for Argument 1 of NWR 5.2

Participant	Speech	Actions	Code
Teacher	Okay guys. So, do people want to share? Okay, I asked you guys to share, and this group to share. And you guys want to share too? Do you want to do the…with the price per app of two dollars? Okay. Alright, so first **we said five *c* plus three *c* equals eight *c*[1]**, because five plus three equals eight. Then what did I say? What did I ask you to do? Yeah…what did I ask you to do?		[1]Claim
Felicia	Oh, explain the context in that equation…in the context of Sofia and Manuel.		
Teacher	To explain it in the context of Sofia and Manuel. So, let's see how this…Did you want to..? Okay. Let's listen to how Fatima explains it. Alright, Fatima is going to explain.		
Fatima	**Five *c* is Sofia's amount spent. Three *c* is Manuel's amount spent. Eight *c* is the total amount Sofia and Manuel spent for both friends.**[2]		[2]Data
Teacher	Yeah. Do people disagree with that?		
Several students	No.		
Teacher	No?		
Gisela	That's all we were supposed to do?		
Teacher	Does that make sense?		
Gisela	That's all we were supposed to do…		
Teacher	Yeah, Gisela do you want to add onto that?		
Gisela	No.		
Teacher	No? Okay. Did people agree with that?		
Marisa	Yeah.		
Maricruz	I approve.		
Felicia	We approve.		

In this argument, the teacher first repeated the students' claim from before, that "5*c* plus 3*c* equals 8*c*." While Aislyn had previously argued that $5c + 3c$ was equal to $8c$ because 5 + 3 =

8, Fatima shared an alternative explanation in this argument. Her alternative explanation, which functions as data, was that "$5c$ is Sofia's amount spent, $3c$ is Manuel's amount spent, and $8c$ is the total amount spent for both friends."

The teacher's revoicing of "$5c$ plus $3c$ is equal to $8c$" meets the first criterion for descriptive equivalence by using the word "equal" to describe the relationship between $5c + 3c$ and $8c$. Fatima's statement satisfies the second criterion by explicitly linking the quantities associated with $5c$, $3c$, and $8c$ to the game app context. She described $5c$ as "Sofia's amount spent" and $3c$ as "Manuel's amount spent." While Fatima did not directly interpret the element $5c + 3c$, I argue that due to the students' extensive work with the long method (expressed by $5c + 3c = T$), Fatima was likely seeing $5c + 3c$ as equivalent to the total amount spent by both friends. She then described $8c$ as "the total amount spent for both friends," suggesting that Fatima viewed $5c + 3c$ and $8c$ as the same quantity in the context of the game apps. Fatima's explanation therefore satisfies the second criterion for descriptive equivalence.

I acknowledge that Argument 1 was brief and lacked elaboration due to time constraints. Notably, Fatima did not mention sameness or equality and did not explicitly articulate an interpretation of $5c + 3c$ as representing the total amount spent by both friends. Stronger evidence of descriptive equivalence would have articulated that $5c + 3c = 8c$ because $5c + 3c$ represents the total amount spent by Sofia and Manuel together. Similarly, $8c$ represents the total amount spent by both friends. Because the two expressions have the same meaning in the game app context, the two are equivalent. In the next section, I provide data and analysis from small-group discussions that occurred before Argument 1 to bolster my claim that descriptive equivalence of algebraic expressions tentatively functioned as if shared in the classroom.

Small Group Arguments

Small Group DCA Analytic Method

The DCA method has traditionally been applied exclusively to whole-class data. However, recent studies by Dreyfus et al. (2023) and Rasmussen et al. (2024) have expanded its use to include small-group settings. This extension was motivated by the desire to provide a more comprehensive view of how mathematical ideas are coordinated across both classroom contexts. The researchers recognized that the whole-class setting is not the only venue for negotiating mathematical ideas. In many cases, small groups are the primary setting for debating and justifying ideas, while whole-class discussions often serve merely to report ideas that have already been established. By applying DCA to both small-group and whole-class discussions, the method seeks to offer a more complete picture of how knowledge functions as if shared within the classroom.

In the present study, I follow the methodological approach detailed in Dreyfus et al. (2023) and specifically follow their DCA analysis of data related to the idea that "area goes to zero" (p. 703-709). In general, their method involved transcribing video data for six whole-class discussions and five episodes of small-group work for both groups. They then created argumentation logs for both sets of data and identified ways of reasoning that functioned as if shared in the whole class setting *or* in one of the two groups for which they had a full record of discussions. They did this by applying DCA criteria to small-group and whole-class discussions.

Three ways of reasoning were then selected because they were methodologically complex. In the case of one of the ways of reasoning ("area goes to zero"), they found that the idea FAIS in one small group via Criterion 2, and was emerging in another small group. On the other hand, the evidence in the whole-class discussion that the idea functioned as if shared was not strong (the argument was brief, and it was also a conversation between only two members of the class who were in the same group). However, coupled with the evidence from the small groups, the authors elected to apply a variant of Criterion 1 to claim that the idea *tentatively* functioned as if shared.

The variant of Criterion 1 refers to a weaker version of the traditional Criterion 1, which requires multiple arguments to show that a warrant or backing has become implicit in an argument (Rasmussen and Stephan, 2008). Because Dreyfus et al. (2023) only had evidence for one argument at the whole class, they could not claim that a warrant had dropped off. However, they felt there was enough evidence to claim that no member of the classroom community offered a rebuttal in response to the argument, which they attributed to the idea being negotiated prior to the whole-class discussion in Group 1, and somewhat negotiated in Group 2. The lack of strong evidence led them to claim that the idea tentatively functioned as if shared. Had the small-group analysis not been coordinated with the whole-class analysis, this idea would likely not have been identified as FAIS.

In general, the use of the variant of Criterion 1, which relies on one argument, is not sufficient to make claims about an idea becoming normative in the classroom. Indeed, students may not issue a rebuttal or question the result for various reasons, including confusion about the mathematics or fatigue. Despite the presence of social norms that would encourage disagreement, it is possible that the use of this variant may not be a strong enough indicator that an idea has been accepted. However, as I will argue, using the extension of the DCA method with and applying the variant of Criterion 1 to the observed dataset sufficiently reveals the complexity of the emergence of descriptive equivalence in this classroom.

Following Dreyfus et al. (2023), I transcribed small-group data and analyzed the transcript using Toulmin's Scheme and the DCA method. Specifically, I transcribed 10 minutes of audio of students working on the Equivalence Task (Figure 33) in small groups. I then identified arguments related to descriptive equivalence in Groups 1, 2, 3, and 4. I applied Criterion 1 to two arguments in Group 1 (their small-group argument and the whole-class argument) and identified emergent ways of reasoning in Groups 2-4. Lastly, I used the small-group data to bolster claims about the descriptive equivalence of algebraic expressions tentatively FAIS in the whole class by applying a variant Criterion 1.

Group 1 Argument

Background. Before forming the equation $5c + 3c = 8c$, the teacher had asked students to interpret the equation $5c + 3c = T$ in their own words. Her goal was to have students become more comfortable with interpreting $5c$ and $3c$ as quantitative products to be able to then interpret the equation $5c + 3c = 8c$ from a descriptive equivalence perspective. In this discussion, Jose described $5c$ as "the amount of money she wasted on apps," which the teacher commended as a creative way of describing the meaning of $5c$. The rest of the session involved forming the equation $5c + 3c = 8c$, where Ahmed and Aislyn initially provided syntactic interpretations. As the class worked on the Equivalence Task in groups, she monitored the group discussion. Just before Group 1's argument, Felicia initially shared a syntactic explanation, drawing on the fact that $5c$ and $3c$ were like terms. In response, the teacher prompted the students in Group 1 to draw upon Jose's idea of the amount of money "wasted" on apps.

Toulmin's Scheme. Group 1's claim is that $5c$ plus $3c$ is the same as $8c$. The data to support this claim is that $5c$ is Sofia's amount "wasted," $3c$ is Manuel's wasted, and $8c$ is the total combined that Sofia and Manuel both wasted. The warrant is that $5c + 3c$ is equivalent to $8c$

because it separates the total amount spent by both friends into Sofia's amount spent and Manuel's amount spent, but adding them together results in the quantity 8*c*, which also represents the total amount spent by both friends.

Table 32: Coding for Group 1's Argument

Participant	Speech	Code
Teacher	Okay, but what about using this idea of wasted?	
Felicia	Is it because it combines the total amount they wasted together?	
Jose	Yeah…	
Felicia	I don't know…	
Jose	Yeah. We should use Felicia's [idea]. She's smarter than me, like way smarter.	
Teacher	I like your idea. What's the meaning of five *c*?	
Jose & Felicia	**The total amount Sofia wasted.**[1]	[1]Data
Teacher	And three *c*?	
Jose & Felicia	**Total amount Manuel wasted.**[1]	[1]Data
Teacher	And eight *c*?	
Jose & Felicia	**The total combined they both wasted.**[1]	[1]Data
Teacher	**So it's the same?**[2]	[2]Claim
Jose	**Yes.**[2]	[2]Claim
Teacher	Why? Yeah. Talk to each other.	
Felicia	**Because if it was five *c* by itself it'd be the total cost Sofia's wasting, but you're putting it together with another person. That's why it's the same[1]… When the equation is separated or like, broken down, five *c* still represents the total Sofia has and the total Manuel has. But when you add it together, that's how you get eight *c*.**[3] **That represents eight *c*. The total amount they're going to spend together.** That kind of make sense?	[3]Warrant

When probed to utilize the quantitative product perspective ("What about using this idea of wasted?"), Group 1 could articulate the quantities associated with 5*c*, 3*c*, and 8*c*, which functioned as data to support their claim that 5*c* + 3*c* was equal to 8*c*. Felicia's final explanation functions as a warrant by explaining that 5*c* + 3*c* was a "separated" or "broken down" way of representing the total amount spent by both friends. She also explained that 8*c* resulted from adding the two quantities and that this expression similarly represented the total amount spent by both friends. For instance, she said, "When the equation is separated or, like, broken down, $5c$ still represents the total Sofia has, and [$3c$ is] the total Manuel has." I interpret her statement to mean that the left-hand side of the equation ($5c + 3c$) is a breakdown of the total amount spent into Sofia's share ($5c$) and Manuel's share ($3c$). Her statement, "When you add it together, that's how you get $8c$," indicates that she views 8c as representing the combined total of Sofia's and Manuel's amounts. Felicia viewed this as an explanation for why $5c$ plus $3c$ is equal to $8c$, as evidenced by her statement, "That's why it's the same." Thus, her explanation functions here as warrant.

Felicia's statement, "That's why it's the same," satisfies the first criterion of descriptive equivalence because she used the word "same" to describe the relationship between $5c + 3c$ and $8c$. Felicia explicitly quantified 8c as "the total amount they're going to spend together," which she explained as representing the same quantity as $5c + 3c$. Hence, Felicia's description satisfies the second criterion.

A critical reader may argue that Felicia is viewing $8c$ as the result of the sum of $5c$ and $3c$ on the left-hand side rather than seeing both expressions $8c$ and $5c + 3c$ as representing the same quantities. However, her use of the words "separated" and "broken down" indicates that she viewed $5c$ and $3c$ as parts of a whole, namely, the whole quantity represented by $5c + 3c$. She then separately interpreted $8c$ as representing the total amount spent by both friends. This is further supported by the class's extensive work interpreting the long method, expressed by $5c + 3c = T$, making it more likely that Felicia saw $5c + 3c$ as representing the total amount spent by both friends. The general statement of the long method already functioned as if shared in the classroom by Day 7, suggesting that Felicia's grasp of this idea was already well-established by

Day 10. I note that Group 1 provided an elaborated explanation of why $5c + 3c$ was equal to $8c$, which will contrast with the meanings provided by Groups 2-4.

Group 2-4 Arguments

Overview of Arguments within Groups 2-4. In general, articulating descriptive equivalence was challenging for Groups 2, 3, and 4. However, with teacher probing, all three groups could name the quantities associated with $5c$, $3c$, and $8c$ as a justification for why $5c + 3c$ was equal to $8c$. Namely, each group argued that $5c + 3c$ was equal to $8c$ by naming $5c$ as Sofia's total amount spent, $3c$ as Manuel's total amount spent, and $8c$ as the total amount spent by everyone. Their arguments, therefore resembled Argument 1 at the whole-class level voiced by Fatima, where the claim was $5c + 3c$ was equal to $8c$, and the data was the quantities associated with the expressions $5c$, $3c$, and $8c$. Below, I present evidence from the small-group discussion within Group 4.

Table 33: Example with Group 4's Argument

Participant	Speech	Code
Gisela	Explain it in a different way…than five plus three equals eight…How do you do it?	
Teacher	What about using the total cost? What does five *c* represent?	
Diego	What? I'm lost.	
Teacher	So what does five *c* represent?	
Gisela	**Sofia's cost.**[1]	[1]Data
Teacher	What does three *c* represent?	
Gisela	**Manuel's cost.**[1]	[1]Data
Teacher	What does eight *c* represent?	
Gisela	**The total amount.**[1]	[1]Data
Teacher	So how would you say it in one sentence? So, Sofia's total cost with Manuel's total cost is the amount spent by both friends? Are those **equal**?	
Gisela	Okay so, **Sofia's cost plus Manuel's cost equals the total amount spent**[2] but worded differently.	[2]Claim

Emergence of Descriptive Equivalence within Small Groups

Group 1's small group discussion provided evidence that descriptive equivalence was negotiated before the public exchange. In contrast to Fatima's final statement in Argument 1, Group 1's discussion of descriptive equivalence was well-elaborated. For instance, Felicia was able to explain why $5c + 3c$ was equal to $8c$ by describing $5c$ and $3c$ as parts of a whole, where $5c$ represented Sofia's total, and $3c$ represented Manuel's total, and together they represented the total amount spent by both friends. She explained that this was equivalent to $8c$, representing the total amount spent by both friends.

In contrast, Groups 2-4 made progress in naming the quantities associated with $5c$, $3c$, and $8c$ but did not provide as strong evidence that descriptive equivalence was functioning as if shared. For instance, in Group 2, Asma struggled to articulate a product perspective of $8c$ (e.g., referring to it as "total times the cost"), suggesting that descriptive equivalence was still emerging within this group before the whole class discussion. In Group 4, Gisela evidenced a descriptive equivalence interpretation. However, Diego's expression of confusion ("I'm lost") and Aislyn's earlier syntactic explanation indicated that descriptive equivalence was likely still emerging within this group. Thus, while Group 1's discussion provided stronger evidence that descriptive equivalence was established before the whole class discussion, it appeared that this understanding was emerging but not yet fully developed in Groups 2 through 4.

Tentatively Fulfilling Criterion 1

I first apply Criterion 1 to Group 1, which was satisfied due to the warrant elaborating the interpretation of $5c + 3c = 8c$ becoming implicit in the whole-class argument. Specifically, the warrant articulated in Group 1's argument was that $5c + 3c$ and $8c$ were equivalent because the left-hand side represents separating the total amount spent by both friends into Sofia's amount spent and Manuel's amount spent, but adding them together results in the right-hand side, which also represents the total amount spent by both friends. In the whole-class argument, no member of Group 1 made the warrant explicit, evidenced by Maricruz's statement, "I approve" and Felicia's statement, "We approve." I infer that because Group 1 negotiated descriptive equivalence prior to the whole-class exchange, they did not feel the need to articulate the warrant in response to Fatima's statement. Hence, Criterion 1 of the DCA method is satisfied.

I apply a variant of Criterion 1 to Argument 1, following Dreyfus et al. (2023). Using this characterization, I coordinate the evidence from small-group analysis together with the whole-class exchange. In small groups, there is evidence to suggest that Criterion 1 was fulfilled in Group 1, and descriptive equivalence was emerging in Groups 2 through 4. In Groups 2-4, I showed that students provided evidence of a descriptive equivalence interpretation. On the other hand, their discussions did not provide evidence that descriptive equivalence had been sufficiently negotiated or elaborated, as was the case with Group 1. I conjecture that members of Groups 2-4 recognize descriptive equivalence as a conclusion from their prior work and hence accept Fatima's statement. However, they did not explicitly elaborate this interpretation of equivalence in their groups.

As noted previously, descriptive equivalence directly builds upon ideas that Groups 2-4 expressed previously and functioned as if shared within the class (as shown in NWR 4.2, interpreting algebraic expressions from a quantitative product perspective). Furthermore, no members of Groups 2-4 challenged Fatima's claim, providing evidence that descriptive equivalence is functioning as if shared for some, if not all, members of Groups 2-4. Furthermore, no student in the class rejected Fatima's claim, despite the presence of social norms, which would make that an expected course of action if someone objected to the statement or had some confusion about the mathematics. This serves as further evidence that the idea was functioning as if shared by some, if not all, classroom members.

In conclusion, the DCA analysis of the whole-class discussion and the four small-group discussions tentatively identified descriptive equivalence of algebraic expressions as an idea that functions as if shared in the classroom. It was tentative because the given arguments are brief, and the arguments within Groups 2-4 are not well elaborated. I follow Dreyfus et al. (2023) in electing to identify the idea as tentatively functioning as if shared in part since no class members issued a rebuttal or questioned the result. It appeared that Group 1 had already established the idea before the public exchange, while Groups 2-4 were close to establishing the idea.

Summary of NWR 5.2

The second normative way of reasoning (NWR 5.2) documented students' development of descriptive equivalence interpretations of algebraic equations. This normative way of reasoning took time to develop, as it drew upon many related ways of reasoning presented in this chapter. The method used to establish NWR 5.2 relied on an extension of the DCA method because the whole class data was insufficient to establish that the reasoning had become normative. However, there was enough evidence to claim that this idea tentatively functioned as if shared in the classroom when the whole-class discussion occurred, guided by teacher probing during small-group work. I acknowledge that students' understanding of descriptive equivalence was likely still emerging at this stage.

Discussion

This section offers reflections on the results of this chapter, which addressed the research question: What is the evolution of the collective mathematics practices of 13 Grade 9 students over the course of a TE about algebraic expressions and equations from a quantitative reasoning approach? I organize this section into three areas of discussion. The first area addresses the discrepancies between an initial ambitious plan for the TE and the messy reality of the results. The second area reflects on how these results extend our knowledge in the field of algebraic and quantitative reasoning. Finally, I reflect on the use of the DCA method in the context of my study.

Ambitious TE Meets Reality

In this TE, my initial plan was more ambitious than what was feasible for this specific classroom community. There were at least two areas where I miscalculated. First, I had planned for students to be ready to form their general method and express it with algebra by Day 2. However, by Day 3, students had only grasped pieces of the general method. As a result, stating a general method (NWR 1.1) did not become normative until Day 7 and expressing a general method with literal symbols that represented quantities with varying values (NWR 2.3) did not become normative until Day 8. I suspect that prematurely introducing algebra on Day 2 before they were ready likely hindered the evolution of a more natural progression of reasoning.

Second, I anticipated that students' interpretations of arithmetic expressions from multiple perspectives (Mathematics Practice 3) would become normative earlier in the TE. Instead, interpreting arithmetic expressions from a quantitative process view (NWR 3.1) did not become normative until Day 7 and interpreting arithmetic expressions from a quantitative product view (NWR 3.2) did not become normative until Day 8. I attribute this phenomenon to one of the challenges of using a quantitative reasoning approach, which is dependent on students' precision of language. Specifically, students struggled to describe the quantities associated with particular numeric values and arithmetic expressions. This may be due in part to the fact that students are not typically exposed to quantitative reasoning in traditional instruction (Thompson

& Thompson, 1995). Consequently, evidence for Mathematics Practice 3 didn't come until later in the TE, despite the early introduction of quantitative reasoning.

These observations highlight a difference between my study and previous research on learning trajectories at the collective level. Prior research uses design-based research methodologies, which often involve multiple iterations (e.g., Tondorf & Prediger, 2022; Walkinton, 2017). Undergoing several iterations can allow for refined instructional sequences based on accumulated insights. In contrast, this TE represents a first attempt, which may explain some of the misalignments observed between my hypothetical learning trajectory and the actual learning trajectory. My reflections on how I may build upon the results for a future iteration are presented in the Chapter 6.

Reflections on the Results

Despite facing numerous conceptual challenges, the students were able to eventually engage in productive algebraic and quantitative reasoning in the TE. Furthermore, each mathematics practice presented in the Results Section contributes to the literature on generalizing via figural patterns, making meaning of literal symbols, interpreting expressions from process and product perspectives, and reasoning about equivalence. I discuss each contribution next.

Many studies on generalizing figural patterns have focused on students deriving general rules, often overlooking the potential for quantitative reasoning to promote meaning making. An exception is Harel and Soto's (2017) concept of process pattern generalization, which emphasizes the use of logic with explanatory power. In this TE, students evidenced statements of their general method that were aligned with this type of generalization, as seen in Jose's description within the pool context (see Argument 1 of NWR 1.1). This finding contributes to the field by illustrating the potential of a quantitative reasoning approach to offer more meaningful descriptions of generalizations, for instance by consistently connecting elements of a general method back to the context. Nevertheless, as mentioned in the previous section, developing this reasoning was challenging. Instructional supports such as the use of novel tasks (see Comparing Visualizations Task: Same Method) and attention to the precision of language seem to have facilitated students' progress in stating a general method.

The literature has documented students' struggles with interpreting literal symbols in algebra, often treating them as objects or placeholders without quantitative meaning (Lucariello et al., 2014; Stacey & MacGregor, 1997). In this TE, an unexpected pattern emerged where many students used three literal symbols to form an equation when only two were needed (see Mathematics Practice 2). This finding is significant as it highlights a previously under-documented use of symbols, extending our understanding of students' use of symbols when implementing a quantitative-reasoning approach to algebra.

Using more symbols than needed may have stemmed from students' reliance on intuition rather than number patterns, leading them to express different quantities in the context with different symbols. For example, suppose students been asked to make sense of the number pattern $10 \bullet 2 + (10 - 2) \bullet 2 = 36$, $7 \bullet 2 + (7-2) \bullet 2 =$ __, $15 \bullet 2 + (15 - 2) \bullet 2 =$ __, etc. and then express the pattern with an algebraic equation. They likely would have formed an equation with only two literal symbols such as $x \bullet 2 + (x - 2) \bullet 2 = B$, where x and B take the place of the changing values. However, the latter approach is devoid of quantitative meaning, suggesting that using a quantitative reasoning approach may foster meaning making, but comes at the cost of an initial surplus of symbols. Additionally, the assumption that fewer literal symbols is more desirable was not made explicit to them. Not making this assumption explicit together with the fact that Haleemah and ET used three literal symbols possibly contributed to this way of expressing the general method functioning as if shared.

Another focus in research on the learning of algebra has been the interpretation of expressions from process and product perspectives (e.g., Sfard, 1992). Yet, researchers have yet to take a quantitative reasoning lens (with a notable exception being Thompson, 2011). In Mathematics Practice 3 and 4, this gap was addressed by illustrating that interpreting expressions with both quantitative process and product perspectives can promote meaning-making. The study confirmed the conjecture that quantitative process interpretations typically precede quantitative product interpretations, though a notable challenge emerged when dealing with algebraic expressions, where students struggled to reconcile these two interpretations.

Specifically, I found that students struggled to view an expression like $x \cdot 2$ as representing both the number of tiles times the number of sides (associated with a quantitative process view) and the total number of tiles in two sides of the border (associated with a quantitative product view). Following Thompson (2011), to conceive a quantity, a person must conceive of a measurement unit and measurement process, where implementing the process produces a value for the quantity. However, with algebraic expressions, it is not straightforward to recognize an expression like $x \cdot 2$ as representing a total, since the multiplication cannot be directly computed as it would in arithmetic expressions (e.g., $8 \cdot 2 = 16$, representing a total number of tiles). In sum, utilizing a quantitative reasoning approach brought richer interpretations but also added complexity in recognizing both process and product interpretations as valid. Progress was made when students were prompted to explain the rationale behind the multiplication (e.g., Why do you multiply?), which helped to elicit a quantitative product interpretation.

Lastly, the vast majority of research on equivalence has focused on meanings for the equal sign as operational (i.e., as a "to do" sign) or relational (i.e., indicating equal amounts on both sides of the sign). A novel construct proposed by Cook and colleagues (2022) introduces descriptive equivalence as a quantitative interpretation of equivalence. This TE aimed to develop an understanding of descriptive equivalence by building upon quantitative product interpretations of algebraic expressions (see Mathematics Practice 5). Initially, students demonstrated a syntactic interpretation of equivalence, which was surprising given their prior development of quantitative reasoning in other practices. The prevalence of syntactic interpretations suggests that students might have benefited from more explicit guidance on identifying and comparing quantities on both sides of the equation in addition to interpreting algebraic expressions as quantitative products. It may also been helpful to have an explicit instructional focus on developing descriptive equivalence with arithmetic (NWR 5.1). Furthermore, the syntactic rule of combining like terms by adding the coefficients appeared to be salient for students, indicating a need for additional support in transitioning from syntactic to descriptive interpretations of equivalence.

Reflections on the DCA Method

This study used the DCA method to document the evolution of students' algebraic and quantitative reasoning over time. There were at least three aspects of the dataset that were not well-captured by the DCA method. First, this TE used a quantitative reasoning approach, which meant that crucial ideas often emerged before students could articulate a precise statement grounded in quantitative meaning. Therefore, it is possible that other methodologies such as a fine-grained discourse analysis may have better captured students' meanings earlier in the TE. Second, while learning progressions using the DCA method typically follow a linear progression, evidence from this TE suggests that students could draw upon emergent ideas before becoming normative. This aspect suggests that DCA may overlook critical moments of reasoning in transition by focusing primarily on moments when ideas become normative. Lastly, as discussed in the section for Mathematics Practice 5, certain stories of mathematical progress may be more complex than the DCA method can capture analytically. For instance, the idea of descriptive equivalence seemed to nearly function as if shared in the whole class due to the rich discussions that occurred in small groups. Recent advancements by Dreyfus et al. (2023) and Rasmussen et al. (2024) have extended the method to include small-group data, which may address some of these analytical challenges.

Chapter 5: Instrumental Orchestrations

The aim of Research Question 2 is to identify the instructional supports for the normative ways of reasoning, specifically supports related to the use of the unscripted dialogic math videos. In the results that I follow, I do not attempt to provide an exhaustive account of all factors that may have influenced students' reasoning, as is common in other studies documenting collective mathematics trajectories (Lobato & Walters, 2017). Rather, this research question homes in on how the dialogic videos, as the primary instructional tool, were used by the teacher to support the emergence of normative ways of reasoning. Given that the videos were not merely a supplement to the unit's design, but a centerpiece of the learning environment, this approach provides insight into how teachers could support reasoning about these topics in general.

To address this question, I drew on the theory of instrumental orchestration (IO), which describes how teachers deliberately organize and manage the use of technological artifacts to support students' mathematical learning (as presented in Chapter 2). First introduced by Trouche (2004), the concept draws on the metaphor of the teacher as a conductor, not of musical instruments, but of classroom activity, where the "instruments" are technological artifacts used in instruction. There are three components of the IO framework: the didactical configuration, which refers to the physical and digital arrangement of tools and participants; the exploitation mode, which describes the teacher's planned approach to using this configuration for instructional purposes; and the didactical performance, which encompasses the real-time enactment of the orchestration, including on-the-fly decisions and interactions. This research question only analyzes didactical performance because it is the only component that is not determined a priori.

IO is grounded in the theory of instrumental genesis, which describes the process by which an artifact moves from being a human-made object to something that is meaningful to the learner. Within a mathematical context, a mathematical artifact becomes a mathematical instrument (i.e., the artifact becomes valuable and useful) as the learner develops the skill to express mathematical ideas with it. While it is beyond the scope of this dissertation to present direct evidence for the instrumentation of the dialogic videos for the study participants, readers can get a sense for how the videos mediated the arguments that became normative (see Chapter 4).

Tackling this research question also addresses two significant gaps in the literature. First, existing studies on dialogic videos have primarily focused on student learning without a focus on the support of learning from the videos as an object of analysis. As a result, more studies are needed to understand how dialogic videos can be used to support learning in a classroom setting. Second, while current literature using an IO framework has focused on teachers' use of manipulable technologies like applets, researchers have not considered the role of videos (Song, 2021). Therefore, this study extends theoretical understandings of IO by investigating how videos can be meaningfully and systematically managed by a teacher.

Answering this question required a means of analyzing and categorizing the ways in which the teacher managed the dialogic videos to support the emergence of students' ideas becoming normative in the classroom. To examine the relationship between teaching actions and the emergence of specific NWRs, I conducted a two-phase analysis. For the first phase, I identified classroom episodes that were significant for the emergence of each of the ten NWRs reported in Research Question 1 and analyzed these moments using an a priori coding scheme. This scheme was based on established IO types from Drijvers et al. (2010), Drijvers (2012), and Tabach (2013), as described in Chapter 2 and summarized in Table 1.

Because a video-based technological environment had not previously been analyzed within the IO framework, I needed to follow an adaptation process when applying the codes a priori. First, I assessed whether existing IO types could be directly applied to my data. In many cases, this was possible because the didactical configuration in my study, technology displayed on a shared screen, aligned with the contexts in which many of these IOs were originally developed. Furthermore, the exploitation modes and didactical performances described in prior literature were often general enough to apply across different technological contexts. As a result, seven IO types were applied without modification.

When a code was conceptually relevant but did not fully align with the video context, I adapted it to fit the observed classroom interactions. This was necessary in the case of Sherpa-at-Work, which originally described a student manipulating a technological tool in front of the class while others observed. In my study, students could not directly interact with the video, so I reinterpreted this orchestration to capture moments when the teacher asked students to explain their work on a digital task at the board during a whole-class discussion, where the screen is displayed (see subsection Sherpa-at-Work for details and evidence). This characterization served as a proxy for student-driven technological work.

Finally, in cases where none of the existing IOs captured the teacher's intentional orchestration of the video, I induced codes. This was anticipated, as dialogic videos introduce distinct affordances, such as the ability to pause, fast-forward, and rewind, which are not inherent in the manipulable technologies considered in prior IO research. Through this process, I identified three novel orchestrations: Rewatch-to-Review, Pause-and-Predict, and Skip-to-Screen. In total, the first phase of analysis yielded a list of 11 IOs, summarized in Table 34.

In the second phase, I identified whether each of the IOs were productive or unproductive in supporting the emergence of normative ways of reasoning. An IO was deemed productive if there was evidence that it contributed meaningfully to the emergence of one of these NWRs. Conversely, an IO was considered unproductive if, immediately following its use, (a) student confusion emerged in ways that were unexpected given their prior engagement, or (b) students did not interact with the video as the teacher had intended. Table 34 indicates which IOs I claim to be productive or unproductive in relation to these criteria.

Table 34: Intrumental Orchestrations

Orchestration types	Productive or Unproductive	Characterization	Origin of Code
Discuss-the-Screen	Productive	The teacher action of guiding a whole-class discussion for the purpose of eliciting a normative way of reasoning	Drijvers et al. (2010)
Pause-and-Predict	Productive	The teacher actions of pausing the video and asking students to predict what will happen next.	Induced Code
Not-Use-Tech	Productive	Instances where technology is available, but the teacher chooses not to use it.	Tabach (2013)
Rewatch-to-Review	Productive	The teacher action of asking students to rewatch a previously viewed video for the purpose of reviewing prior material.	Induced Code
Explain-the-Screen	Productive	A whole-class explanation by the teacher, guided by what happens on-screen.	Drijvers et al. (2010)
Link-Screen-Board	Productive	The teacher action of establishing links between mathematics as it is represented in the technological environment and mathematical representations and techniques on the board.	Drijvers et al. (2010)
Discuss-Tech-Without-It	Productive	The teacher action of discussing the technology without the technology being physically present	Tabach (2011)

Table 34: Intrumental Orchestrations (Continued)

Orchestration types	Productive or Unproductive	Characterization	Origin of Code
Work-and-Walk-By	Productive	The teacher action of walking among working students, monitoring their progress, and providing guidance as the need arises.	Drijvers (2012)
Sherpa-at-Work	Unproductive	The teacher action of asking a student to explain their work on a digital task at the board during a whole-class discussion, where the screen is displayed.	Trouche (2004)
Skip-to-Screen	Unproductive	The teacher action of responding to a students' idea by skipping to a video clip that she identified as relevant for further discussion.	Induced Code
Technical-Demo	Unproductive	The teacher uses a learning management system by giving guidance to students.	Tabach (2013)

Because reporting the orchestrations relevant to all ten normative ways of reasoning was not tractable, I narrowed my focus to two normative ways of reasoning, stating a general method (NWR 1.1) and developing descriptive equivalence of algebraic expressions (NWR 5.2). These two were selected because, together, they accounted for the full range of IOs identified in the study. Figure 34 illustrates the occurrence of the 11 IOs across the ten sessions of the TE in relation to the development of NWR 1.1 and NWR 5.2. Note, that the novel orchestration Rewatch-to-Review appeared on both Day 4 and Day 7. In the figure, productive IOs are outlined in green, while unproductive IOs are outlined in black, providing a visual summary of the relationship between IOs and the emergence of normative ways of reasoning.

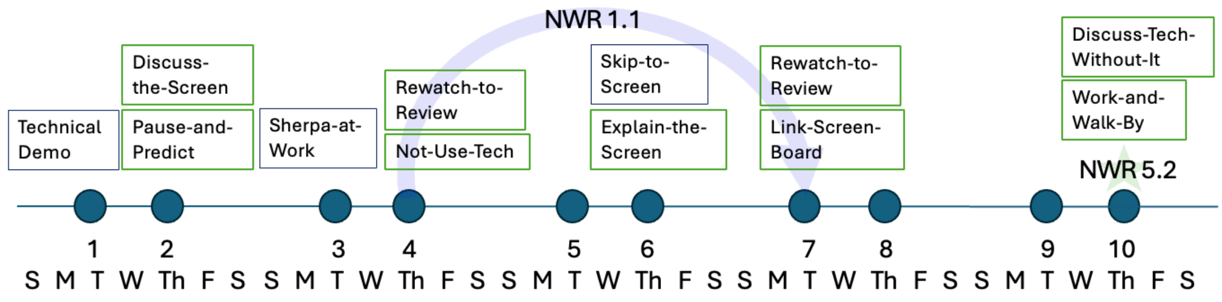


Figure 34: IOs and Evolution of NWR 1.1 and NWR 5.2

In the following results section, I present evidence for the 11 IOs. Within each section, I characterize the IO, describe the background relevant to the use of the IO, present evidence of the teacher's use of the IO, followed by evidence that the use of the IO was productive in supporting (or unproductive in supporting) the emergence of students' normative ways of reasoning. The first six IOs I present are those which I claim to have been productive for supporting the emergence of NWR 1.1, stating a general method. After that, I present two IOs that I claim were productive for supporting the emergence of NWR 5.2, developing descriptive equivalence of algebraic expressions. Next, I present three IOs that I argue were unproductive in supporting the emergence of NWR 1.1, stating a general method.

Discuss-the-Screen IO

In this section, I present evidence to claim that Discuss-the-Screen was a productive IO for supporting the emergence of NWR 1.1—stating a general method. Drijvers et al.'s (2010) original characterization of Discuss-the-Screen entails the teacher guiding a whole-class discussion to enhance collective instrumental genesis, meaning the transformation of a technological tool into a meaningful instrument for learning. Because providing evidence of instrumental genesis is beyond the scope of this thesis, I adapt Drijvers et al.'s (2010) characterization to refer to the teacher action of guiding a whole discussion for the purpose of eliciting a normative way of reasoning. First, I present evidence the teacher employed a Discuss-the-Screen IO, followed by evidence that using Discuss-the-Screen played a role in articulating the relationship between the number of tiles in the top and the side, a key element of their general method in the pool context.

Evidence of a Discuss-the-Screen IO

I begin by summarizing the relevant background for the use of Discuss-the-Screen on Day 2. Prior to the discussion, the students had been working to apply the video participants' (Haleemah and ET) general method for finding the total number of tiles for different-sized pools (e.g., Size 10, Size 7, Size 20). The teacher aimed to guide students towards articulating Haleemah's and ET's general method, which involved adding the tiles on the left and right sides and the tiles remaining on the top and bottom after the corner tiles had been removed. A key element of this method, the consistent difference of two tiles between the tiles on the side and the top, was highlighted in a particular video. In the video, Haleemah and ET demonstrated this relationship using a Size 15 pool, where there were 15 tiles on the side and 13 tiles on the top. They explained how they got 13 from 15 by saying, "After I counted the tiles, assuming that there's 15 on this side and then I subtracted two, which would be the corner tiles, to make sure I had 13." Before playing this video, the teacher directed the students' attention to this relationship, preparing them for a focused discussion afterward.

The subsequent discussion was evidence of a Discuss-the-Screen IO. After showing the video, the teacher initiated the discussion by asking, "What did you guys notice?" Gisela noted that Haleemah and ET "used the same method as the last one," referring to Haleemah's and ET's way of finding the number of tiles for different-sized pools. To guide the class toward an articulation of the method, the teacher asked, "And what was their method?" After hearing from

Gisela, she called on another student, Esperanza, to further develop their ideas. By guiding a targeted discussion about the video and specifically asking students to describe Haleemah's and ET's method, the teacher's actions were evidence for Discuss-the-Screen. Next, I argue that the students' contributions to this discussion supported the emergence of stating a general method (NWR 1.1).

Evidence of Discuss-the-Screen Supporting NWR 1.1

Gisela's and Esperanza's contributions to the whole-class discussion marked progress in clarifying a specific component of their general method. For instance, in response to the teacher question "What was their method?" Gisela acknowledged the distinction between the number of tiles on the side and on the top using the Size 15 pool. She noted that for 15 tiles on each side, "you have to just include the two tiles, meaning the corners as well," indicating that she understood that the method accounted for corner tiles on the left and right sides. Following Gisela's input, the teacher prompted Esperanza, who further clarified that the two corner tiles were being subtracted. She stated, "once they get the [left and right] sides, the middle [of the top and bottom sides], you automatically know that you have to take away two because you're not counting the corners." Together, Gisela's and Esperanza's contributions suggest that they grasped that the corner tiles needed to be included on the left and right sides and omitted from the remaining sides, a key structural element of the method.

This meaning was important for the eventual articulation of Haleemah's and ET's method by Jose on Day 4, as detailed in Argument 1 of NWR 1.1. Specifically, he successfully linked the relationship between the number of tiles on the sides, top, and bottom to other components of the method, resulting in a single verbal statement, "Like the method that they counted the ones on the side with the corners and the ones on the top and bottom were counting…. which were the ones that were left out without counting the corners." As discussed in the section for Argument 1 of NWR 1.1, Jose's statement was interpreted as adding the number of tiles on the left and right sides to the tiles remaining on the top and bottom, while excluding the corner tiles. His reference to "the ones on the side with the corners" and "the ones that were left out without counting the corners" suggests that his explanation was built upon the meaning that Gisela and Esperanza developed during their discussion on Day 2. The evolution of reasoning from Day 2 to Day 4 shows how Discuss-the-Screen on Day 2 helped solidify a key component of Haleemah's and ET's method, thereby, supporting a full articulation of their general method on Day 4.

Pause-and-Predict IO

This section provides evidence and analysis demonstrating that Pause-and-Predict was another productive IO for supporting the emergence of stating a general method. I define Pause-and-Predict as the teacher actions of pausing the video and asking students to predict what will happen next. Pause-and-Predict was identified as a new orchestration type, although it shares similarities with the previously documented Predict-and-Test (Bozkurt & Ruthken, 2018). Predict-and-Test refers to teaching actions of encouraging students to make predictions, test their ideas in the technology, and then confirm or modify them in light of the results. Pause-and-Predict similarly asks students to make predictions, however, in this case, the teacher's goal was not for students to test and adjust their reasoning afterward, but rather, to elicit emergent ideas before continuing the video. Other uses of Pause-and-Predict may include a "test" component, which could involve comparing predictions to the video and modifying them afterward, but this aspect was not evidenced in the dataset. For this reason, Pause-and-Predict was identified as a novel IO rather than a variant.

Evidence of a Pause-and-Predict IO

I begin by providing the relevant background for the use of the Pause-and-Predict IO on Day 2. The video that is the focus of this IO showed Haleemah and ET expressing their method with algebra. In the video, Haleemah and ET initially wrote an equation with three literal symbols ($x \cdot 2 + y \cdot 2 = B$), where x represented the number of tiles on each side, y represented the number of tiles on the top or bottom, and B represented the total number of tiles in the border. However, they later re-expressed it as an equation with two literal symbols ($x \cdot 2 + (x - 2) \cdot 2 = B$), recognizing that the third symbol was not needed. In the case that students might also start with three symbols, the teacher had planned to use the video to guide them toward the algebraic expression using only two symbols. In practice, however, students did not generalize at the same pace as Haleemah and ET. Subsequent analysis revealed that by the time the teacher asked the class to write an algebraic equation, the class had only grasped fragments of Haleemah's and ET's method, leading more students than expected to use three literal symbols. Suspecting that the class were reasoning at a different pace than Haleemah and ET, the teacher adjusted her approach.

The teacher's on-the-spot decision to modify her initial plan, specifically choosing to pause the video early and ask students to predict Haleemah's and ET's new equation, was evidence of a Pause-and-Predict IO. Instead of letting the video play through to the end, she paused the video just before Haleemah and ET re-expressed the equation $x \cdot 2 + (x - 2) \cdot 2 = B$, prompting students to generate their own predictions. She asked, "so what do you think their new equation's going to look like?" and had students write their ideas on the board to bring their emergent ideas into the public exchange (see Figure 35). Hence, her actions satisfy the definition of a Pause-and-Predict IO.

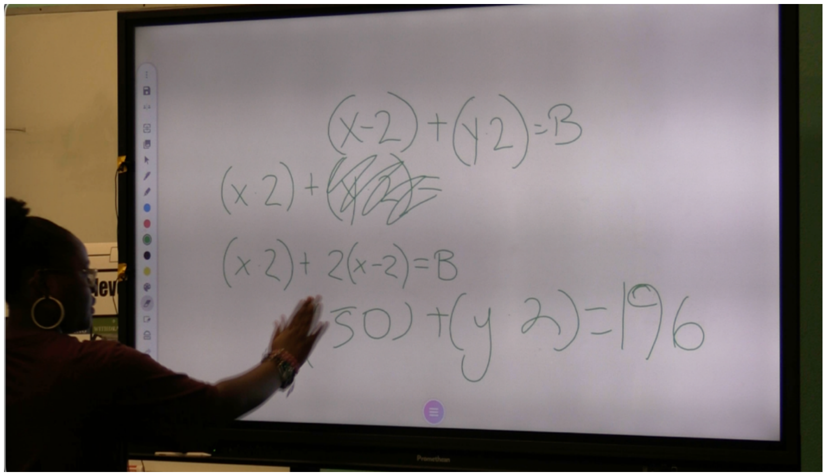


Figure 35: Pause-and-Predict IO

Evidence of Pause-and-Predict Supporting NWR 1.1

By creating an opportunity for students to predict the video participants' equation before watching the resolution, Pause-and-Predict uncovered a key difficulty in articulating Haleemah's and ET's method. Prior to this moment, students had identified some structural elements, such as the consistent difference of two tiles between the top and side (see Discuss-the-Screen) and the need to avoid double-counting corner tiles. However, they had not yet articulated the general method in its entirety. As a result, when asked to predict the equation, several students struggled, as detailed in Argument 1.A of NWR 2.2 (see Figure 20).

First, Maricruz wrote "(x • 2)" to represent the number of tiles in the left and right sides but was unsure how to express the next term representing the number of tiles in the top and bottom. Aislyn similarly did not know how to express the number of tiles in the top and bottom, writing "(x – 2) + (y • 2)" in the space above. Finally, Maricruz wrote the desired equation, "(x • 2) + 2(x – 2) = B," although this was shortly followed by Kaelyn writing another equation focusing on the Size 50 pool ("(50 • 2) + (y • 2) = 196"). Both Maricruz's and Aislyn's uncertainty in how to express the second term in the equation, (x – 2) • 2, appeared to be related to a difficulty putting elements of the general method in relation to one another (the structure criterion for stating a general method; see Chapter 4). Additionally, Kaelyn's reliance on the empirical example of a Size 50 pool indicated a potential problem with generality (the generality criterion).

Exposing the students' emergent ideas was pivotal in initiating a shift in the teacher's approach from closely following the videos to being more responsive to student thinking. Had the teacher let the video play in full, she likely would not have recognized the extent of the students' struggles, as they would have simply observed Haleemah and ET re-express their equation without an opportunity to make their emergent ideas public. Instead, the teacher paused at a critical moment and assessed students' thinking in real-time. The opportunity for students to share their ideas appeared crucial for NWR 1.1 to emerge because it revealed students' difficulties with two key criteria for articulating a general method, enabling the teacher to recalibrate subsequent instruction. In particular, the use of Pause-and-Predict set the stage for the introduction of a novel task on Day 4, which is explored in the next section for Not-Use-Tech.

Not-Use-Tech IO

I present evidence that Not-Use-Tech was a productive IO for supporting the emergence of stating a general method. Not-Use-Tech refers to instances where technology is available, but the teacher chooses not to use it (Tabach, 2011). I begin by presenting evidence that the teacher employed the Not-Use-Tech IO, followed by evidence that this decision supported students in articulating a general method in the pool context.

Evidence of a Not-Use-Tech IO

The use of Not-Use-Tech on Day 4 stemmed from the need to focus students' attention on the structure of Haleemah's and ET's method, rather than on isolated components. Evidence of structure understanding would entail students describing multiple elements of the method and how they relate (e.g., evidence of structure understanding of Haleemah's and ET's method would involve describing how the number of tiles on the left and right sides related to the number of tiles on the top and bottom sides, including the corner tiles). However, through the use of Pause-and-Predict IO at the end of Day 2, the teacher recognized that students had only articulated isolated components rather than the method as a whole.

Despite the teacher's attempts to clarify these ideas on Day 3, students still struggled to grasp the structure of Haleemah's and ET's method. In one instance, the teacher showed students a video voiceover in which the narrator stated Haleemah's and ET's general method verbatim, which resulted in student confusion (see the Sherpa-at-Work section below). The teacher then asked students to apply the method to another pool size and then use their work to identify patterns across different-sized pools on the screen. However, this led one student to focus on the commutative property rather than the elements of the method and how they relate. The teacher's unsuccessful attempts indicated that neither hearing the method stated verbatim in a voiceover nor applying the method to an additional pool size was effective in fostering structural reasoning.

Recognizing these challenges, on Day 4, the teacher chose to forgo technology and instead designed two tasks to focus students' attention on the structure of Haleemah's and ET's method. The first task (Task 1) required students to compare Haleemah's and ET's method to a different method, emphasizing their structural differences (see Figure 36). Students had to select if one, both, or none of the visualizations captured Haleemah's and ET's method. The second task (Task 2) asked students to compare Haleemah's and ET's method to itself but in a different orientation, highlighting structural similarities. Like the first task, students were asked to decide whether one, both, or none of the visualizations captured Haleemah's and ET's method. These tasks were designed to help students discern essential features of the method by engaging in direct comparisons rather than attempting to abstract structure from repeated applications to pools of different sizes. The intentional choice to use non-digital tasks, even when the technology was available, serves as evidence of the Not-Use-Tech IO.

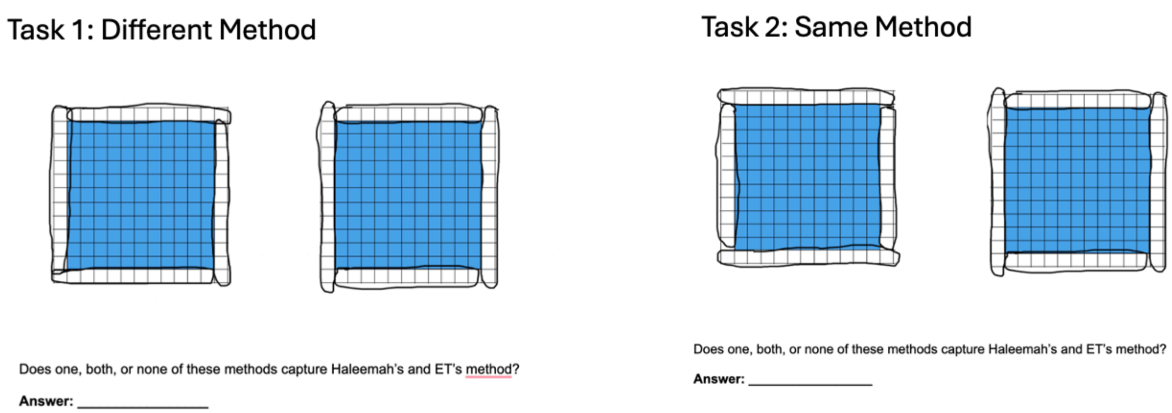


Figure 36: Not-Use-Tech IO

Evidence of Not-Use-Tech Supporting NWR 1.1

I argue that Not-Use-Tech played a crucial role in supporting the emergence of stating a general method. Specifically, it enabled students such as Jose to focus on the overall structure of Haleemah's and ET's method rather than on isolated components. For example, in his response to Task 1, Jose articulated Haleemah's and ET's method in its entirety, stating "like the method that they counted the ones on the side with the corners and the ones on the top and bottom were counting…. which were the ones that were left out without counting the corners." As detailed in Argument 1 of NWR 1.1, Jose's statement was interpreted as adding the tiles on the left and right sides and the tiles remaining on the top and bottom, while excluding the corner tiles. Notably, this marked the first instance of stating a general method. This suggests that, unlike previous attempts to elicit such a statement, Not-Use-Tech effectively directed the class's attention to the essential features of the method and how they related to each other, ultimately supporting the articulation of Haleemah's and ET's method.

Rewatch-to-Review IO

In this section for Rewatch-to-Review, I present evidence to claim that the teacher's use of this IO was productive in supporting the emergence of stating a general method in the pool context and the game app context. I identified Rewatch-to-Review as a novel IO, specific to the technological environment used in this TE. I define Rewatch-to-Review as the teacher action of asking students to rewatch a previously viewed video clip for the purpose of reviewing prior material. I begin by presenting evidence that the teacher used Rewatch-to-Review, first in the pool context (Day 4) and again in the game app context (Day 7). Then, I make the case that Rewatch-to-Review supported the emergence of stating a general method in both the pool context and the game app context.

Evidence of a Rewatch-to-Review IO

Before presenting the evidence, I describe the relevant background for the use of the Rewatch-to-Review IO on Days 4 and 7. The teacher's actions on Days 4 and 7 stemmed from the sequence of events occurring the days before, on Days 3 and 6. Because the TE held sessions twice a week, this meant that in between Days 2 and 3 and Days 5 and 6, there was a four-day gap in between sessions. This meant that while the class demonstrated progress during Day 2, for example, they did not remember certain ideas by Day 3. Similarly, certain ideas discussed on Day 5 were not repeated or built upon on Day 6. This student forgetting hindered students' progress in stating a general method (see NWR 1.1 for details and evidence).

The teacher's use of video to revisit how Haleemah and ET formed their general methods, first in the pool context (Day 4) and later in the game app context (Day 7), was evidence of the Rewatch-to-Review IO. In the pool context, the teacher replayed a video on Day 4 after noticing that students had forgotten aspects of Haleemah's and ET's method on Day 3. This video was the first clip that students watched on Day 1, showing Haleemah's and ET's initial formulation of their method of finding the total number of tiles in a pool with ten tiles on a side. She introduced the video by saying, "I just wanted to refresh everyone's memory on Haleemah's and ET's method. So now you can work on this task." The teacher's emphasis on the students' "memory" suggests that she was using the video as a review rather than to generate new ideas, for instance, as with Discuss-the-Screen.

In the game app context, the teacher replayed a video on Day 7 in response to student difficulties on Day 6. The teacher showed the video right after introducing the naming convention of the "long method" and the "short method" (see Link-Screen-Board section below). This video, which she had first shown on Day 6, demonstrated Haleemah's and ET's initial formulation of the "long method" to find the total amount of money spent by three friends on game apps. She similarly introduced the video by saying, "So I'm going to just remind you guys of Haleemah's and ET's long method, and then I'm going to ask two group presenters to come up and explain what they saw in the video." Similarly to the use of the video in the pool context, the teacher framed the video to "remind" students, suggesting that the purpose of showing the video was to remind students of prior material. Thus, in both instances, the teacher's use of the video serves as evidence of Rewatch-to-Review.

Evidence of Rewatch-to-Review Supporting NWR 1.1

The use of Rewatch-to-Review in both the pool and game app contexts supported students in articulating their general methods. This is evident from the fact that, on the days Rewatch-to-Review was implemented (Days 4 and 7), students produced statements that met all three criteria of stating a general method. In the pool context, for example, Jose articulated Haleemah's and ET's general method shortly after the video review, during his work on the Comparing Visualizations Task (see section on Not-Use-Tech). Similarly, in the game app context, Gisela articulated the long method immediately following the Rewatch-to-Review clip. The close temporal connection between these articulations and the video reviews suggests that Rewatch-to-Review, by addressing students' difficulty in recalling previously discussed ideas, played a role in supporting the emergence of stating a general method.

Explain-the-Screen IO

In this section for Explain-the-Screen, I present evidence that the use of this IO was productive in supporting the emergence of stating a general method. Explain-the-Screen is characterized in Drijvers et al. (2010) as a whole-class explanation by the teacher, guided by what happens on-screen. These explanations may involve specific mathematical content, such as using a technological feature to explain why square roots are nonnegative, or a clarification of how tasks can be solved, for instance, by drawing students' attention to features of an applet before they start a task. Drijvers et al. (2010) specify that this explanation must move beyond explaining the technical aspects of the technology. This distinguishes Explain-the-Screen from the related IO, Technical-Demo. Furthermore, in this characterization, "screen" can refer not only to the physical projection of the technology onto a shared screen, but also to a digital task or any other aspect of the technological environment. I present evidence that the teacher implemented Explain-the-Screen, followed by evidence that the use of Explain-the-Screen supported student reasoning about the two general methods in the game app context, by clarifying the relationship between the task they are working on and the task in the video.

Evidence of an Explain-the-Screen IO

Before presenting evidence, I provide relevant background on the use of the Explain-the-Screen IO on Day 6. This orchestration was implemented in the game app context, which presented a unique instructional challenge. In this context, students watched videos of Haleemah and ET working on the Three Friends Task, while the class worked on a related but different task: the Two Friends Task. Additionally, the videos showed Haleemah and ET generalizing two methods. Meanwhile, students were expected not only to apply ideas from the video to their own work, but also to articulate the two general methods in the context of their situation with two friends, resulting in four methods to coordinate. This complexity compelled the teacher to play a more proactive role in helping students make sense of the video content.

Prior to employing Explain-the-Screen, the teacher prematurely asked students to write an algebraic equation that expressed one of the methods. This led to some students using symbols in problematic ways (see NWR 2.3 for details and evidence). The teacher's original plan was to show a video of Haleemah and ET expressing one of their methods with algebra and then use Discuss-the-Screen to facilitate a conversation about the meaning of their literal symbols. However, her initial attempt to spark discussion was met with one student expressing fatigue by saying he was "tired," followed by prolonged silence, and a request for clarification from Gisela ("Wait…can you ask the question again?").

To address the class's fatigue and potential confusion, the teacher adjusted her approach and clarified the four methods more explicitly, which was evidence of the Explain-the-Screen IO.

Teacher: Yeah…Okay, so there's Haleemah's and ET's situation, there's Sofia and Manuel, which you guys are doing. There's Esperanza's method and there's this method [gestures to the video screen]. So, there's a lot going on, right?

The exchange above illustrates how the teacher clarified two confusing aspects of the video. First, she distinguished the context in the video from the context the students were working on ("Haleemah's and ET's situation" and "Sofia's and Manuel's"). Second, she distinguished the two methods that students were meant to generalize. First, she mentioned "this method" while gesturing to the video screen, which referred Haleemah's and ET's way of finding the total amount of money spent by the friends. Second, she mentioned "Esperanza's method," which referred to another way of finding the total amount of money spent by the friends, which had been previously expressed by Esperanza. The teacher's aside clarified the relationship between the task they are working on and the task in the video, thus serving as evidence of Explain-the-Screen.

Evidence of Explain-the-Screen Supporting NWR 1.1

There is evidence to suggest that the use of Explain-the-Screen supported student reasoning about the two general methods in the game app context. For example, shortly after the use of Explain-the-Screen, Ahmed responded to the teacher probe, "Esperanza's method was what? Can you say it again?" by saying, "Esperanza's method was to add all the apps together and then multiply by the cost." Ahmed's statement satisfied the three criteria of stating a general method, as defined in NWR 1.1, indicating that the explanation of the screen helped clarify some confusion. The next day, Gisela's statement of the other general method, which involved adding together the total amount each friend spent one at a time, would appear in an argument, as detailed in Argument 1 for NWR 1.1. The timing of these articulations in close proximity and the teacher's clarification indicates that Explain-the-Screen, by distinguishing the class's task from the task from the video, played a role in supporting the emergence of stating a general method. Additionally, Explain-the-Screen laid the foundation for more intentional scaffolding the next day (Day 7) using Link-Screen-Board, which is explored in the next section.

Link-Screen-Board IO

I present evidence to claim that Link-Screen-Board was a productive IO for supporting the emergence of stating a general method. Link-Screen-Board is defined as the teacher action of establishing links between mathematics as it is represented in the technological environment and mathematical representations and techniques on the board (Drijvers et al., 2010). Link-Screen-Board moves beyond clarifying or explaining aspects of the technological environment, as is the case with Explain-the-Screen. While Link-Screen-Board may also serve the purpose of helping in explaining video content, this orchestration type explicitly involves linking video content to the board, for instance, by re-presenting mathematical representations from the video on the board. In this characterization, "board" can refer to any representation of conventional mathematics such as paper, book, or a whiteboard. In this section, I present evidence that the teacher used the Link-Screen-Board orchestration. Then, I provide evidence that the Link-Screen-Board supported students in interpreting a method demonstrated in the video, which was pivotal for stating a general method becoming normative.

Evidence of a Link-Screen-Board IO

As mentioned in the section for Explain-the-Screen, the game app context posed initial challenges in connecting video content to the students' task. In the pool context, students watched videos of Haleemah and ET tackling the same tasks as them, making it more straightforward for the teacher to establish a link between the on-screen mathematics and their work. The teacher asked students to watch and discuss the videos (see the Discuss-the-Screen section) and then apply the ideas to their tasks. In contrast, in the game app context, the video participants worked on the Three Friends Task while the students worked on the different but related Two Friends Task. Therefore, there were initial challenges in connecting video content to the students' work, as evidenced on Day 6 when the teacher did not provide sufficient support to distinguish between two methods demonstrated in the video (see the Skip-to-Screen section below for details and evidence).

The teacher's subsequent actions to address the confusion in the game app context (Day 7) are evidence of a Link-Screen-Board IO by naming the two methods demonstrated in the video and strategically organizing the board. She introduced distinct names for the two methods featured in the video—the "long method" and the "short method"—and noted these on the board, as depicted in Figure 37 in the case of the long method. Her statement, "So I'm going to be giving names to the methods we're going to be generalizing, just so that it doesn't get confusing" illustrated her intent to focus the viewer's attention on the two methods. Since both the long method and short methods constituted general methods, naming them helped connect the mathematical work in the video to the desired way of reasoning, stating a general method, as defined in NWR 1.1.

She additionally connected ideas from the video to the board via her organization of the board into two sections (see Figure 37). On the right side, she displayed recreated images that mirrored the video participants' drawing of the friends' names and the number of game apps. Namely, the drawing included a rectangle for each of the three friends, labeled with the friend's name, and a circle inside each rectangle for each game app the friend bought. In contrast, the left side depicted the students' Two Friends scenario. The teacher strategically designed the spatial arrangement on the board to provide a dedicated area for students to note ideas from the video before applying them to their own tasks. By strategically organizing the board, naming the video participants' methods, and re-using imagery from the video, the teacher established a link between the mathematics featured in the video and the board.

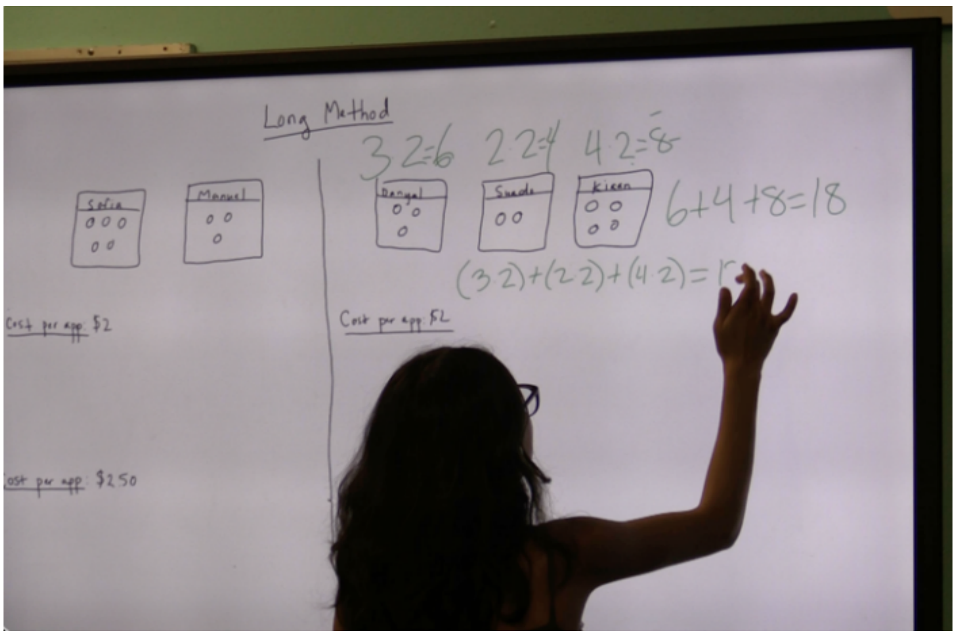


Figure 37: Link-Screen-Board IO

Evidence of Link-Screen-Board Supporting NWR 1.1

I now present evidence to show how the Link-Screen-Board IO supported NRW 1.1 (stating a general method). Specifically, using Link-Screen-Board facilitated the translation of mathematical ideas from the video onto the board and enabled students to articulate a general method using the previously established naming convention. For instance, after viewing the video where Haleemah and ET demonstrated the "long method" by writing the arithmetic equation "(3 • 2) + (2 • 2) + (4 • 2) = 18," Maricruz successfully recreated the arithmetic equation on the right side of the board, as shown in Figure 37. This replication, made possible by the board organization, provided a concrete example from which students could further interpret the long method, independent of the technological interface.

Subsequently, Gisela built upon this example. Using the "long method" terminology, she articulated its meaning by explaining, "the longer method…it means that you multiply the number of games each person has by the price, one at a time." This explanation by Gisela, detailed in Argument 2 of NWR 1.1, established a criterion for the DCA method, which signified that the idea had become normative in the classroom. In sum, the Link-Screen-Board orchestration, through both the strategic organization of the board and the use of naming convention, not only facilitated Maricruz's ability to connect ideas from the video to the board but also provided Gisela with the proper foundation to deepen her meaning of the method, which she articulated by leveraging the established terminology.

Discuss-Tech-Without-It IO

This section and the next (Work-and-Walk-By) are distinct from previous sections in this chapter because they present evidence of IOs supporting a different NWR, developing descriptive equivalence, an interpretation which treats two expressions as equivalent if they represent the same quantity (NWR 5.2). Specifically, in this section for Discuss-Tech-Without-It, I present evidence to claim that the IO was productive in supporting the emergence of developing descriptive equivalence. Discuss-Tech-Without-It refers to the teacher action of discussing the technology without the technology being physically present (Tabach, 2011). I begin by presenting evidence that the teacher used this IO on Day 9, followed by evidence that the use of Discuss-Tech-Without-It supported students' precision of language, which played an important role in the articulation of a descriptive equivalence interpretation on Day 10.

Evidence of a Discuss-Tech-Without-It IO

I begin by summarizing the relevant background on Day 8 prior to the use of Discuss-Tech-Without-It on Day 9. Discuss-Tech-Without-It was used in the game app context, where students were asked to interpret quantities in context. For instance, one instructional goal was for students to interpret an expression like 5 • 3 as the total amount a friend spent on game apps. A central challenge in supporting these interpretations was fostering students' use of precise mathematical language to clearly articulate quantitative referents. To address this, the teacher facilitated a discussion around a video clip that highlighted the importance of precision by specifying a quantity to the individual involved, Danyal. In the clip, Haleemah initially offered an imprecise interpretation of the quantity 15, referring to it as the "total amount spent." When asked if he agreed, ET responded by suggesting they "specify it to Danyal," prompting them to revise their interpretation to "the total amount spent by Danyal." Haleemah and ET's use of precise language became the focal point of the subsequent class discussion.

During the discussion, two students noted that, without specifying the friend who spent the money, the phrase "total amount spent" could be ambiguous, possibly referring to the total spent by all the friends ("all the money wasted on game apps") or by someone else ("it could be for another friend"). Building on this insight, the teacher invited students to apply the idea of precision to their own work. They were tasked with interpreting each quantity in the equation 5 • 7 + 3 • 7 = 56, which represented the "long method" applied to the cost per app of $7. In doing so, students practiced articulating each quantity with greater precision (see NWR 3.2 for details and evidence).

On Day 9, the teacher evidenced Discuss-Tech-Without-It by referencing an idea from the video without replaying the clip. The class tackled a similar task, which asked them to interpret each quantity in the equation 8 • 7 = 56. Instead of showing the clip again, the teacher harkened back to the video by saying, “Do you guys remember when ET asked Haleemah to specify it to Danyal? Okay, I want you guys to be precise and with your group, make sure you're being precise about what things mean.” By invoking ET’s suggestion to revise “total amount spent” to “total amount spent by Danyal,” the teacher reinforced the importance of using precise language when interpreting quantities. This exchange served as evidence of Discuss-Tech-Without-It because the teacher revisited an idea from the video without the direct use of the technology. In the next section, I argue that this IO supported students in articulating precise quantitative interpretations, which in turn contributed to the emergence of developing descriptive equivalence (NWR 5.2).

Evidence of Discuss-Tech-Without-It Supporting NWR 5.2

Immediately following the use of Discuss-Tech-Without-It on Day 9, the class used precise mathematical language to interpret the quantities in the equation 8 • 7 = 56. For instance, after completing the group task, Felicia defined 8 • 7 as the “total combined cost of Sofia’s and Manuel’s game apps” (see Figure 38). Felicia’s phrasing of 8 • 7 as the “total combined cost of Sofia’s and Manuel’s game apps” rather than “total game apps” suggested an awareness of the teacher’s intentional focus on the precision of language during Discuss-Tech-Without-It.

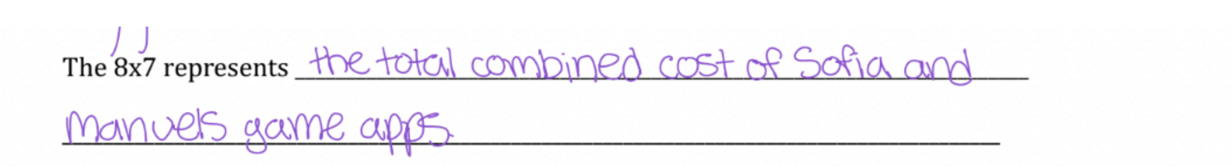


Figure 38: Discuss-Tech-Without-It IO

Developing students’ precision of language played a key role in eliciting the first instance of a descriptive equivalence interpretation of the equation $5c + 3c = 8c$ on Day 10, as detailed in Argument 1 of NWR 5.2. Specifically, a descriptive equivalence interpretation entails interpreting equivalence of $5c + 3c$ and $8c$ as the total amount Sofia spent ($5c$) together with the total amount Manuel spent ($3c$) was the same as the total combined amount the two friends spent together. Precise language is particularly important for a descriptive equivalence interpretation because it relies on the clear articulate quantitative referents. Fatima interpreted $5c + 3c = 8c$ as "Five *c* is Sofia's amount spent. Three *c* is Manuel's amount spent. Eight *c* is the total amount Sofia and Manuel spent for both friends." Her interpretation mirrored the precise interpretations offered the day before. For instance, her interpretation of $8c$ as "the total combined cost of Sofia's and Manuel's game apps" used precise and detailed language, much like the interpretation of 8 • 7 on Day 9 as "total combined cost of Sofia's and Manuel's game apps." Specifically, both interpretations specified that the total cost was associated with Sofia and Manuel.

The consistency between these interpretations suggests that Fatima's interpretation was a natural extension of the interpretations developed following Discuss-Tech-Without-It. The class's continued use of precise language following the use of Discuss-Tech-Without-It indicates that it enabled Fatima in offering a detailed articulation of a descriptive equivalence interpretation. Had the teacher not implemented Discuss-Tech-Without-It, Fatima may have interpreted an expression like $8c$ simply as the "total amount spent." Hence, Discuss-Tech-Without-It was important for the emergence developing descriptive equivalence interpretation on Day 10.

Work-and-Walk-By IO

In this section for Work-and-Walk-By, I present evidence to make the case that this IO was productive in facilitating NWR 5.2, developing descriptive equivalence. I follow Drijvers's (2012) characterization of Work-and-Walk-By to refer to the teacher action of walking among working students, monitoring their progress, and providing guidance as the need arises.

However, this IO is distinct from others presented in this Chapter because of its didactical configuration. Because the technology in this classroom was presented on one central screen, I adapt Drijvers's (2012) characterization with respect to the didactical configuration to refer to students working in groups of 3 to 4 on tasks related to the videos (as opposed to students working individually or in pairs with computers). Specifically, I present two examples with Group 1 and Group 4 as evidence that the teacher's guidance during small-group work supported developing descriptive equivalence in tentatively functioning as if shared among these groups by the end of Day 10.

Prior to the use of Work-and-Walk-By on Day 10, students had been working to make sense of the equation $5c + 3c = 8c$. The teacher's goal was to elicit a descriptive equivalence interpretation, which entailed interpreting equivalence of $5c + 3c$ and $8c$ as the total amount Sofia spent ($5c$) together with the total amount Manuel spent ($3c$) was the same as the total combined amount the two friends spent together. Such an interpretation relies on quantitative interpretations of $5c$, $3c$, and $8c$, which the teacher worked to refine before interpreting equivalence. When first presented with the equation $5c + 3c = 8c$, however, students justified the equivalence using a syntactic interpretation, relies on syntactic algebraic properties. For instance, one student claimed that $5c + 3c = 8c$ because the coefficients added together ($5 + 3 = 8$). The teacher then asked the class to work on the Equivalence Task in groups (see Figure 33), during which she worked to guide students toward a descriptive equivalence interpretation.

Evidence of a Work-and-Walk-By IO

The teacher's tailored scaffolding to each group demonstrated evidence of a Work-and-Walk-By IO. For instance, during the small-group discussion within Group 1, the teacher used a previously voiced interpretation of the video by a group member to guide their work. Earlier that day, the teacher asked students to interpret the idea of the quantitative product interpretation demonstrated in a video in their own words, where Jose interpreted 5*c* as representing "the amount of money she wasted on five apps." This interpretation was written on the board and accepted in the classroom (see Figure 39).

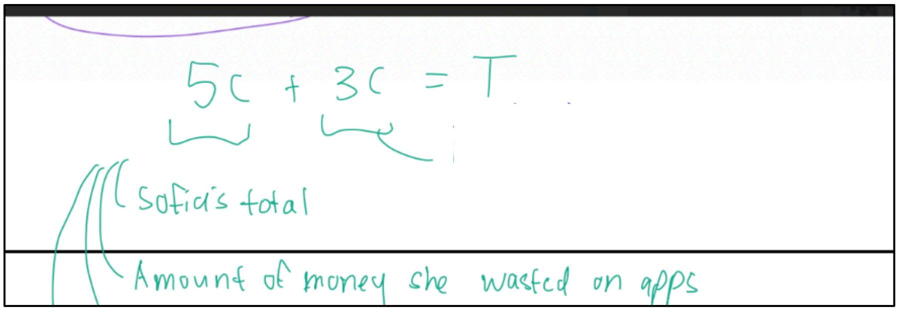


Figure 39: Jose's Interpretation of 5*c*

In the exchange below, the teacher uses Jose's language of "wasted" to guide the group in their transition from a syntactic interpretation of equivalence to descriptive equivalence.

Teacher: I like this idea of wasted. So why would this be true?
Jose: Wait, what would be true?
Felicia: Speaking of like terms…speaking of like terms, this would be the cost, right? This would automatically equal this. So like, in like terms cause since there's no number for *c* yet, you would just add them. When you add them, basically just 5 and 3. Like, the *c*'s not there.
Teacher: Okay, but what about using this idea of wasted?
Felicia: Is it because it combines the total amount they wasted together?

The exchange provides evidence that the teacher used Jose's language of "wasted" to guide students away from a syntactic interpretation and toward a descriptive equivalence interpretation, therefore evidencing the Work-and-Walk-By orchestration.

Evidence of Work-and-Walk-By Supporting NWR 5.2

The teacher's scaffolding appeared to support the group's articulation of a descriptive equivalence interpretation. Felicia initially interpreted $5c + 3c = 8c$ syntactically, by relying on the algebraic rule for combining "like terms." However, after the teacher asked the group to rely on a quantitative product interpretation, via her reference to Jose's language using "wasted," Felicia shifted to a descriptive interpretation. She said, "because it combines the total amount they wasted together," which seemed to refer to an interpretation of both $5c + 3c$ and $8c$. This suggests significant progress toward interpreting $5c + 3c = 8c$ as combining the total amount Sofia spent with the total amount Manuel spent to equal the total amount they spent together. Indeed, as detailed in Group 1's Argument in NWR 5.2, Felicia later elaborated on this idea to provide strong evidence of a descriptive equivalence interpretation (see corresponding section in Chapter 4).

Another instance of Work-and-Walk-By was evidenced when the teacher revoiced certain questions posed in the video to Group 4 to guide them toward the desired interpretation. When students got stuck, evidenced by commenting "What? I'm lost," the teacher responded by asking "What does $5c$ represent?", "What does $3c$ represent?", "What does $8c$ represent?" As a result, the students named the quantities associated with $5c$, $3c$, $8c$ and mentioned equality, providing evidence (albeit less strong evidence than for Group 1) of a descriptive equivalence interpretation (see subsection Group 2-4 Arguments of NWR 5.2 for details and evidence).

Sherpa-at-Work IO

In the following sections, I present evidence of three cases of IOs that I characterize as unproductive in supporting students' stating a general method (NWR 1.1). I direct the reader to the Discussion section at the end of this chapter for possible explanations for why these IOs were less effective. There is precedent for analyzing ineffective IOs. For instance, Moore-Russo et al. (2012) identified three qualities of ineffective orchestrations in a technological environment using digital games. Although their specific criteria did not align with my dataset, I adopt their analytical approach in identifying three unproductive orchestrations that were evidenced in my dataset and relevant to the emergence of stating a general method (NWR 1.1).

In the first section for Sherpa-at-Work, I present evidence to support the claim that this IO was unproductive in fostering students' articulation of a general method (NWR 1.1). Trouche's (2004) original characterization of Sherpa-at-Work involves asking a student to explain their work within a technological environment (in their case CAS calculators) in front of the class for the purpose of whole-class discussion. Trouche referred to the student as a "Sherpa" to evoke the figure who both guides and carries the load during Himalayan expeditions. This definition was not directly applicable to my dataset, as video does not constitute a technological environment in which students can actively work on tasks.

However, the teacher actions observed in the dataset align conceptually with Trouche's (2004) orchestration. Therefore, I adapt Trouche's (2004) Sherpa-at-Work to refer to the teacher action of asking a student to explain their work on a digital task at the board during a whole-class discussion, where the screen is displayed. By digital task, I mean a task that is either the same or related to the tasks demonstrated in the video (for an example of a non-digital task, see the section for Not-Use-Tech). When students explained their thinking related to digital tasks while a video screen was displayed, it functioned as a proxy for working within a technological environment. I note that other characterizations of Sherpa-at-Work in video-centered context may consider students' sharing of work on digital tasks, even without the video screen visible, as a sufficient proxy.

Moreover, this orchestration is distinct from others such as Discuss-the-Screen or Link-Screen-Board, which are primarily aimed at helping students make sense of the video content to apply ideas to their own tasks. In contrast, Sherpa-at-Work, as defined here, emphasizes the reverse direction: students explain their previously established ideas, which are intended to serve as a basis for making connections to the ideas presented in the video. In the sections that follow, I first present evidence that the teacher implemented Sherpa-at-Work, then present evidence that this IO was not productive in supporting students in articulating their general method in the pool context.

Evidence of a Sherpa-at-Work IO

Before presenting the evidence, I describe the background relevant to the use of Sherpa-at-Work on Day 3 in the pool context. As detailed in the Preliminary Related Reasoning subsection of Argument 1 for NWR 1.1 (in Chapter 4), during the review of students' work from Day 2, much of the prior meaning developed in the previous session was not repeated or spontaneously offered. In response, the teacher played a voiceover clip summarizing Haleemah's and ET's general method. As the narrator described the method, three different-sized pools were shown (see Figure 40), and segments were illuminated to highlight key elements of the method. Even though the clip explicitly stated Haleemah's and ET's method, asking the students whether they remembered the method did not elicit any similar statements by students. The teacher, attempting to provide additional support, made an on-the-spot decision to ask students to apply the method again, this time to a Size 12 pool in groups. Her goal was to provide additional practice while helping students make connections across different-sized pools, as was expressed in the voiceover clip.

The teacher's subsequent actions serve as evidence of Sherpa-at-Work. Before inviting groups to share their ideas with the class, she chose to keep the screen from the voiceover clip displayed, which showed three pools of different sizes (see Figure 40). The teacher called on two students, Diego and Felicia, to come to the board, where the screen remained visible, and explain the method in their own words using their work on the Size 12 task (see Figure 40). The teacher said, “Okay, so, I'm actually going to leave this thing up and I'm going to have Felicia and Diego come up and explain Haleemah and ET's method in their own words. You can pick whatever pool that's up here and just think about your ideas with the Size 12.” This exchange illustrates the teacher’s intentional action of keeping the screen displayed (“I’m actually going to leave this thing up”) while asking students to explain their work on the digital task (“think about your ideas with the Size 12”) to the whole class. Thus, her actions are evidence of Sherpa-at-Work.

Figure 40: Sherpa-at-Work IO

Evidence that Sherpa-at-Work was Unproductive in Supporting NWRs

The students’ responses immediately following Sherpa-at-Work serves as evidence that Sherpa-at-Work was not a productive IO in eliciting a statement of Haleemah’s and ET’s general method. Specifically, Diego and Felicia answered a different question than was asked. For instance, when explaining her ideas at the screen, Felicia explained the associative property of the addition in the method rather than offering a statement of the general method itself ("with the cumulative property…it could be switched"). Afterwards, Diego shared his ideas at the screen (see Figure 40), but he too did not articulate Haleemah's and ET's method. Instead, he described how he found the number of tiles in the border of the Size 12 pool using a different method entirely. These responses suggest not only the absence of a statement of the general method, but also a confusion of the question they were being asked by the teacher. Hence, this IO was not productive in facilitating an articulation of Haleemah's and ET's general method.

Skip-to-Screen IO

In this section for Skip-to-Screen, I provide evidence to argue that the teacher's implementation of the IO was unproductive in facilitating the emergence of stating a general method (NWR 1.1). Skip-to-Screen, like Rewatch-to-Review and Pause-and-Predict, was identified as a new orchestration type, which I define as the teacher action of responding to a students' idea by skipping to a video clip that she identified as relevant for further discussion. This IO assumes that the teacher has pre-existing knowledge of the content of the videos that would enable her to intentionally skip ahead.

Skip-to-Screen shares similarities with the previously identified Spot-and-Show, which refers to the teacher bringing up previous student work within the technological environment that they had identified as relevant for further discussion (Drijvers et al., 2010). Skip-to-Screen similarly identifies a relevant idea (spot) and uses the idea as a catalyst for discussion (show). However, there are two key different differences. First, rather than "spotting" previous student work, the teacher responds to a student idea and selects a clip that is related to that idea. Second, rather than "showing" the students' prior work, the teacher shows the clip that is related to the idea. These distinctions highlight the fact that in a video-centered environment, students cannot actively solve problems within a video as with applets or computer programs. Therefore, this IO was identified as a novel IO. In the following sections, I present evidence that the teacher employed Skip-to-Screen, followed by evidence that Skip-to-Screen was unproductive in supporting stating the short method in the game app context.

Evidence of a Skip-to-Screen IO

Before presenting the evidence, I provide relevant background for the use of Skip-to-Screen on Day 6 in the game app context. In this context, videos showed participants working on the Three Friends Task while the students worked on the different but related Two Friends Task (see Figure 11). The teacher intended for the class to follow a certain trajectory: (a) watch Haleemah and ET develop the “long method” to state a generalization for finding the total amount spent on game apps in the Three Friends Task, (b) apply the “long method” to the Two Friends Task, (c) watch Haleemah and ET develop the “short method” in the Three Friends Task, and (d) apply the “short method” to the Two Friends Task. In practice, however, the teacher struggled to respond to an unexpected student idea, resulting in a disruption of the trajectory.

First, the teacher showed the video of Haleemah and ET developing the long method in the Three Friends Task, as planned. Unexpectedly, however, when asked to explain the method, Esperanza commented that Haleemah and ET had made it “complicated” and proposed the alternative short method. The teacher’s initial response was to acknowledge that the short method would be revisited later in the TE, and she moved forward with asking the class to apply the long method to the Two Friends Task. Instead, several students adopted Esperanza’s method (i.e., the short method). The use of both methods indicated confusion about which method the students were meant to apply and ultimately generalize. To direct the class’s focus to just one method, the teacher asked the class whether they wanted to generalize Esperanza’s shorter method or the original method they had viewed in the video. The class agreed to use Esperanza's method.

The teacher's on-the-spot decision to skip to a later video showing the short method in response to Esperanza's idea was evidence of Skip-to-Screen.

Teacher: Okay, we're actually going to watch Haleemah and ET do Esperanza's method and then we'll talk about the connections between your ideas and their ideas. Okay. Okay. Okay, we're jumping a little bit.

The teacher's mention of "Esperanza's method" suggests that she was responding to Esperanza's idea expressed earlier in the TE. She also said, "we're jumping a little bit," indicating that she was skipping to another clip in response to Esperanza's idea. Lastly, she framed the video as a source of further discussion by saying, "and then we'll talk about the connections between your ideas and their ideas." In sum, the exchange provides evidence that she was responding to a students' idea by skipping to a video clip that she identified as relevant for further discussion, thereby satisfying the definition of Skip-to-Screen.

Evidence that Skip-to-Screen was Unproductive in Supporting NWRs

A sequence of student actions immediately following the use of Skip-to-Screen was evidence that this IO was unproductive in facilitating the articulation of the short method. First, after asking students to apply Esperanza's method (the short method) to a cost per app of $2.50 in the Two Friends Task, both methods were presented. For example, both equations (5 • 2.50) + (3 • 2.50) = 20 (representing the long method) and 8(2.50) = 20 (representing the short method) were written on the board. Additionally, when asked about the difference between the two methods, the students were unable to distinguish the two. When asked "Is this [(5 • 2.50) + (3 • 2.50) = 20] the same as Esperanza's method or different?", one student responded, "the same."

These responses indicated a confusion about which methods the students were meant to apply and generalize, suggesting that Skip-to-Screen was not ultimately productive in facilitating a statement of Esperanza's method (the short method). I direct the reader to the discussion section at the end of the chapter for an argument as to why this IO seemed to introduce confusion.

Technical-Demo IO

This section for Technical-Demo presents evidence to claim that the teacher's use of this IO was unproductive in supporting the emergence of any normative ways of reasoning. Technical-Demo is characterized by the teacher explaining the technical details for using the tool (Drijvers et al., 2010). The technical details of a video tool are simple because they are limited to pausing, fast-forwarding, and rewinding. The features of the TE further constrained students' use of these functionalities because the teacher controlled the technology on one central screen throughout the TE. The teacher intended to give students the ability to manipulate the video technology by giving them the option to signal that they wanted to pause or rewatch a video via placing color-coded sticky notes on their desk corresponding to each action. I first present evidence the teacher employed a Technical-Demo IO by explaining this intended use of the video. Then, I present evidence to claim that the Technical-Demo was unproductive in that in no instance after its implementation did students make use of the video as intended.

Evidence of a Technical-Demo IO

The teacher employed the Technical-Demo IO at the outset of the TE on Day 1. The teacher's explanations of the mechanism by which students could interact with the video are evidence of a Technical-Demo IO. She began by outlining how videos would be used during the TE sessions, noting that although she would primarily control the videos, students would also have opportunities to interact with the videos:

Teacher: In these sessions we're going to be working on problems and watching videos, and I will be controlling the videos up here. But I wanted you guys to also have some control over the videos.

She then distributed two stacks of sticky notes to each student, pink for "pause/stop" and blue for "rewatch," and explained their intended functions. By explaining how standard video actions (pause and rewatch) would operate in the TE, the teacher enacted a Technical-Demo IO.

She additionally demonstrated how students could use the sticky notes, specifically the pink sticky note representing the "pause" action, by showing a video that might warrant pausing the video. She began the demonstration by asking students to read the Pool Task (Figure 12), which asked them to find the number of tiles in a square pool with ten tiles on a side. She then framed the first video by saying, "We're going to watch a video and when you see something that's confusing or surprising to you, why don't you go ahead and put a red or pink sticky note on the desk to let me know you want to pause the video. If you don't find anything surprising or confusing, that's okay, too, okay? So, we're going to watch Haleemah and ET work on the same problem." The teacher not only explained how the sticky notes could be used but modeled their intended use in the context of watching a particular video, thus serving as evidence of a Technical-Demo IO.

Evidence that Technical-Demo was Unproductive in Supporting NWRs

Despite the teacher's intentional efforts to promote student interaction with the videos through the Technical-Demo IO, the orchestration proved unproductive in supporting the emergence of normative ways of reasoning. For example, after framing the video intended to prompt spontaneous use of the "pause" action, no students used the sticky notes. The clip featured Haleemah and ET incorrectly counting 40 tiles around the border of the pool, a moment the teacher anticipated would be sufficiently surprising or confusing to elicit a pause. When no one responded, she paused the video herself and asked whether students agreed with Haleemah and ET. Upon hearing disagreement, she followed up by asking, “I’m just curious, is there a reason no one put the sticky notes up?” but received no response. She reiterated the intended use: “In the future, if you want to stop or you see something that is different than what you thought, just let me know and you can put the sticky note.” Despite this clarification, students did not use the sticky notes for the remainder of the session, indicating that this IO was unproductive in facilitating their mathematical reasoning.

Discussion

The findings presented in this chapter address the researcher question: What IOs of the artifact of unscripted dialogic mathematics videos support the emergence of secondary students’ normative ways of reasoning? In particular, the study documents 11 distinct IOs enacted by one teacher, including eight that are adaptations of IOs from existing literature and three novel orchestrations that leverage the unique affordances of video. In this section, I reflect on three key contributions that these findings make to the literature: (a) conceptualizing video as an orchestrable technology, (b) examining dialogic mathematics videos as an instructional tool, and (c) advancing knowledge on supporting students’ quantitative reasoning.

Video as an Orchestrable Technology

Researchers working within the IO framework have emphasized that the management of technology in classrooms is non-trivial (Drijvers & Trouche, 2008). From this perspective, the use of video in mathematics instruction should move beyond simply showing clips and assigning tasks. By extending research on the orchestration of other technological tools (e.g., dynamic geometry software, graphing calculators), this study illustrates how video can also be productively orchestrated to support mathematical reasoning. Rather than discussing all 11 IOs in detail, I focus here on three that were particularly productive, each illustrating how the teacher moved beyond the superficial use of video to engage students in meaningful mathematical activity: Pause-and-Predict, Discuss-the-Screen, and Link-Screen-Board.

Pause-and-Predict

The first case of a highly productive IO in this study was Pause-and-Predict, in which the teacher strategically paused before key moments and asked students to predict what would happen next. Many of the IOs presented in existing literature focus on teachers' use of manipulable technologies, such as applets. As a result, they were not applicable to my TE, because the interactive features of video are limited to pausing, fast-forwarding, and rewinding. Moreover, a critical reader may argue that videos are not capable of being orchestrated in ways similar to other technologies because of the passive nature of watching video. However, the teacher's use of Pause-and-Predict illustrates how video afforded viewers a way to participate actively in learning mathematics. In this TE, the teacher leveraged the technological feature of pausing to engage students in making predictions, which not only disrupted passive viewing but also engaged students in important mathematical sense-making. This finding therefore challenges the assumption that videos are inherently passive and instead positions them as interactive tools that, when paired with pedagogical moves, can elicit student reasoning.

The effectiveness of Pause-and-Predict was evidenced by a noticeable shift in student participation. Rather than watching passively, students were actively engaged and offered a wide range of predictions. One reason this IO may have been so productive is that the teacher's strategic use of pauses created a "cliffhanger effect," which prompted an authentic need for students to share their thinking. However, it is important to note that Pause-and-Predict was productive given the didactical configuration of the video being displayed on a central screen and controlled by the teacher, which allowed her to select key moments to pause the video. On the other hand, the IO Technical-Demo, in which students were expected to pause the video themselves, was unproductive. It may be the case that the didactical configuration constrained other types of interactions with the video, suggesting that other configurations may afford learners with different opportunities to interact with the technology.

Discuss-the-Screen

Another case of a highly productive IO was Discuss-the-Screen, where the teacher facilitated discussion to support collective reasoning. While previous IO research has characterized this orchestration as an extension of general teaching practice, this study made the Discuss-the-Screen IO a central feature of its design. Specifically, students in this classroom often interpreted the same video in different ways, which is consistent with earlier findings on students' varied interpretations of video (Weinberg & Thomas, 2018; Foster et al., 2025). As a result, Discuss-the-Screen was not only productive, but essential for eliciting students' interpretations and building towards a shared understanding. This finding suggests that video-centered environments may especially benefit from this orchestration, as it provides a mechanism for establishing shared meaning after watching video.

In this study, the Discuss-the-Screen IO was implemented each time the class viewed a video, although some discussions were more productive than others. The orchestration was most productive when clips featured the everyday language of video participants. For instance, on Day 2, students were observed repeating and building on key ideas heard in the video during whole-class discussion. In contrast, a discussion centered on a clip with more formal narration, categorized in this chapter as a part of the Sherpa-at-Work IO, was unproductive. One possible explanation is that the formal register of the narration was misaligned with students' emergent reasoning, making it more difficult for them to connect with the content. These findings suggest while Discuss-the-Screen is well-suited to video-based learning environments, its effectiveness may depend on the intentional selection of clips that align with students' language and reasoning.

Link-Screen-Board

Finally, Link-Screen-Board was a productive IO, in which the teacher strategically linked the mathematics represented in the video to mathematics on the board. Previous research has shown that students often struggle to transfer mathematical understanding from digital environments to paper-and-pencil contexts (Billington, 2009; Bretscher, 2009; Kieran & Drijvers, 2006). However, instructional videos differ from other technologies in that they frequently feature paper-and-pencil techniques, raising the question of whether such a transfer would occur more naturally in a video context. In this study, the need for Link-Screen-Board emerged later in the TE, when students were asked to work on tasks *different* from those shown in the videos. This suggests that while the video format may afford more natural connections to traditional mathematical representations, explicit linking is still necessary, particularly when tasks diverge from the scenarios presented in the video.

The teacher's use of Link-Screen-Board in this study was characterized by a strategic organization of the board that created a designated space for interpreting the video task and a separate space for working on the assigned task. This IO was productive, as evidenced by a stark shift from confusion about an idea to the acceptance of the idea in the classroom. One reason this IO was particularly productive may be because the teacher created a visual structure that allowed students to offload cognitive work. In contrast, other efforts to connect student work to the video were apparently unproductive, as seen with the use of Skip-to-Screen where the teacher showed a relevant clip in response to a student idea. In this case, the teacher did not reduce students' confusion, but added to it, suggesting that the IO may have been more productive when used together with Link-Screen-Board. Thus, Link-Screen-Board may be especially valuable in moments of cognitive overload, providing the clarity and structure needed to support students in bridging video content with their own tasks.

Dialogic Mathematics Videos as an Instructional Tool

In addition to its contributions to the current literature on IO, this study extends the emerging literature on dialogic videos to consider their use in a classroom setting. Specifically, this study builds directly upon a prior interview study about how pairs of Grade 9 students made sense of the same set of dialogic mathematics videos used in my TE (Foster et al., in press). I now turn to discuss two significant insights into the use of dialogic videos for a pedagogical purpose.

Foster et al. (in press) found that while some student pairs evidenced changes in their thinking and expressed mathematical meanings consistent with those presented in the video, others evidenced a variety of meanings and an unreceptiveness to certain ideas. Based on these findings, I designed the TE to incorporate additional instructional supports such as facilitating whole-class discussions (aligned with the Discuss-the-Screen IO) and connecting the videos to students' own tasks (aligned with the Link-Screen-Board IO). Surprisingly, this study revealed that another category of orchestration, often described in the IO literature as "teacher-centered," was also productive in supporting students' reasoning. IOs characterized as teacher-centered are typically ones in which the teacher's ideas drive the classroom discourse. Specifically, two such orchestrations, Explain-the-Screen and Rewatch-to-Review, appeared to be productive. This finding is significant in that it challenges the common assumption that student-centered pedagogies are inherently more productive. One possible explanation is that unscripted dialogic videos, by design, are already deeply student-centered. As such, teacher-centered IOs may serve as valuable complements, helping to clarify, consolidate, or reframe ideas introduced in the video. In this way, teacher-centered orchestrations can enhance, rather than detract from, student learning in environments anchored by dialogic videos.

The dialogic videos used in this study are not only student-centered but also capture a learning trajectory of the video participants over time. However, it was initially unclear whether a classroom community would follow the trajectory featured in the videos because studies investigating students' engagement with dialogic videos over time have not had an explicit focus on comparing the learning trajectories of the students using the videos versus the video participants (Foster, 2023; Lobato et al., 2019; Lobato et al., 2023, Lobato & Walker, 2019). Entering the study, I hypothesized that both the teacher and the students would stay close to the trajectory modeled in the videos, but this was not always the case. For example, the teacher's use of Not-Use-Tech, an IO in which she intentionally decided to depart from the trajectory, was nevertheless productive in supporting students' reasoning of stating a general method. On the other hand, for other ideas, students did closely follow the trajectory depicted in the videos. While a detailed analysis of this alignment is beyond the scope of this thesis, these findings suggest that the video trajectory can serve as a helpful guide for instruction but need not be strictly followed to foster the desired mathematical reasoning.

Instructional Supports for Quantitative Reasoning

Previous research on algebraic reasoning has identified various instructional supports for helping students navigate specific challenges, such as encouraging non-sequential ordering when generalizing figural patterns (Kuchemann, 2010). In contrast, considerably less attention has been devoted to instructional strategies that promote quantitative reasoning. Specifically, there has been less focus on the precision of language required to distinguish between subtly different quantitative interpretations (e.g., “total amount spent” versus “total amount spent by Sofia”). This study addresses this gap by highlighting the productive role of fostering precision of language when developing quantitative meaning. In the Discuss-Tech-Without-It orchestration, the teacher emphasized distinctions in student language that were productive in supporting students’ descriptive equivalence interpretation of the equation $5c + 3c = 8c$. This finding suggests that motivating students to use precise language is a useful instructional support for fostering quantitative reasoning in algebra classrooms.

Recent studies on quantitative reasoning have also examined the role of teacher moves that help students engage in quantitative reasoning by attending to each number and symbol in context (White et al., 2023; White et al., 2024). In particular, White et al. (2024) investigated the effect of the teacher move known as *unfolding*, the act of breaking down compacted conceptual facets into their more elementary facets. Their study provided some evidence that unfolding moves, such as asking students what their numbers or points meant in context, were influential in supporting students’ development of quantitative reasoning. This study extends that work by illustrating how unfolding can be embedded within a classroom orchestration, like Work-and-Walk-By. In this case, the teacher asked students to interpret the meaning of their expressions within the game app context (e.g., “What does $5c$ represent?”, “What does $3c$ represent?”, “What does $8c$ represent?”) These findings provide additional evidence that unfolding moves, in conjunction with the use of dialogic videos, can support students in articulating descriptive equivalence interpretations of algebraic expressions. Taken together, these findings affirm the value of both fostering precision of language and unfolding as productive instructional supports for fostering quantitative reasoning in algebra.

Chapter 6: Conclusions

This dissertation pursued two research aims: (a) to explore how a classroom community develops meaning of algebra topics in context over time, and (b) to identify the instructional supports for the algebraic reasoning that functioned as if shared by the classroom community, specifically supports related to the use of dialogic mathematics videos. The first aim was grounded in a need to reframe algebra not as a set of procedures to memorize, but as a means of reasoning about quantities. The second aim responds to a need to better understand how to support this type of reasoning by using an innovative instructional approach, and specifically, one that leverages technology. At the heart of this study was the use of a novel digital tool: unscripted dialogic videos that depict authentic student problem-solving. On the surface, the design and investigation of this study may seem straightforward: create a learning environment in which students view the videos and engage in similar tasks and then document how their reasoning develops in parallel with the learners featured on screen. Yet, the findings revealed the nuanced complexities involved not only in students' emergence of meanings for algebraic expressions and equations but also in the teacher's facilitation of dialogic videos to support their emergence.

The study followed a classroom TE methodology in which 13 Grade 9 students worked together over the course of ten one-hour sessions, facilitated by the teacher-researcher (myself). To address the first research aim, I grounded the TE in a quantitative reasoning approach (Thompson, 1990), which emphasizes interpreting quantities in context. In contrast to much of the existing research that often examines algebraic topics in isolation (e.g., Blanton et al., 2017; Tondorf & Prediger, 2022), this study investigated the interrelatedness of four core algebraic ideas: (a) generalizing via figural patterns, (b) making meaning of literal symbols, (c) interpreting expressions as both processes and as products, and (d) developing interpretations of equivalence. Chapter 4 presents an analysis of the collective mathematical progress of the classroom, using the emergent perspective (Cobb & Yackel, 1996) and the Documenting Collective Activity (DCA) method (Rasmussen & Stephan, 2008). The DCA method treats the classroom community as the unit of analysis and conceptualizes learning as a process that emerges through collective argumentation. The resulting five-part learning trajectory documents how mathematics practices (i.e., clusters of related normative ways of reasoning) about these core algebraic ideas developed over time and ultimately, became accepted within the classroom community.

To address the second research aim, I integrated dialogic mathematics videos from Project MathTalk (www.mathtalk.org) as a central feature of the learning environment. The design efforts included intentional planning related to the arrangement of the classroom and the video technology, as well as specific teaching actions. In this study, the didactical configuration involved projecting the dialogic videos on a shared screen at the front of the classroom (see Figure 7), while students worked individually on paper worksheets on tasks related to the video content. This arrangement was intended to foster whole-class argumentation and aligned with the IO research tradition, where such configuations are commonly used. There were three categories of design decisions related to teaching actions: (a) routine teaching actions, informed by literature on algebraic and QR, dialogic videos, and classroom technology use, (b) episode-specific decisions, such as selecting which video to use, determining its length, and deciding when to pause, and (c) guidelines for task selection, including whether to use the same task featured in the video or introduce a related but distinct task.

Chapter 5 explores how I, as the teacher-researcher, leveraged this video tool to support students' reasoning about the unit's focal topics. To guide the analysis, I used the instrumental orchestration framework, which focuses on the teacher's role in intentionally and systematically using technology in a classroom setting (Trouche, 2004; Drijvers et al., 2010). This framework comprises three components, the physical arrangement of the classroom and the technological tools (didactical configuration), the teacher's planned approach for using the tool (exploitation mode), and teacher's in-the-moment teaching actions (didactical performance). Prior studies using the IO framework have identified a range of instrumental orchestrations based on these components. Using a combination of a priori codes from the literature and inductive coding, I identified 11 instrumental orchestration types, eight of which appeared to be productive in supporting students' algebraic reasoning and three of which seemed to be unproductive.

In the remainder of this chapter, I summarize the main findings of the study, discuss their significance and implications, discuss limitations of the study and future research directions, and finally reflect on my experience as the teacher-researcher.

Summary of the Findings

Findings from Chapter 4 resulted in a five-part learning trajectory: (a) stating a general method, (b) expressing a general method using literal symbols, (c) interpreting arithmetic expressions from quantitative process and quantitative product perspectives, (d) interpreting algebraic expressions from quantitative process and quantitative product perspectives, and (e) developing interpretations of descriptive equivalence. This trajectory was complex. While many of the conjectured ways of reasoning (see Chapter 2) were eventually accepted by the classroom community, the order in which ways of reasoning emerged and became accepted did not follow a linear sequence. Instead, the evolution of ideas overlapped, which both demonstrated the interrelatedness of ideas and suggested that students could draw upon emergent ideas before becoming accepted. Additionally, certain ideas were particularly challenging for students. Descriptive equivalence of algebraic expressions, for instance, defined as treating two expressions as equal if they represent the same quantity, showed only partial evidence of FAIS in the classroom by the final day of instruction.

Findings from Chapter 5 identified eight productive and three unproductive orchestrations of the dialogic video tool by the teacher. Productive orchestrations, such as Discuss-the-Screen, Pause-and-Predict, and Link-Screen-Board, provided evidence of use of the video as an interactive tool that supported students in interpreting, predicting, and sense-making. Unproductive ones, such as Technical-Demo and Sherpa-at-Work, provided evidence of instances where use of the video seemed to introduce confusion or impose unnecessary cognitive demands. Taken together, the study offers insights on the evolution of complex algebraic reasoning over time and how a teacher's use of dialogic videos can support students' reasoning.

Theoretical and Practical Contributions

This subsection is organized into the theoretical and practical contributions that this study makes to three bodies of literature: algebraic and quantitative reasoning, instrumental orchestration, and learning from dialogic videos.

Contributions to Algebraic and Quantitative Reasoning

A key theoretical contribution of this study is its focus on descriptive equivalence. Recently introduced by Cook and colleagues (2022) as a part of a broader framework for understanding equivalence across middle school, high school, and undergraduate mathematics, descriptive equivalence extends beyond the traditional dichotomy of operational and relational views of the equal sign that has dominated the literature. This construct was central to the present study because it aligns with Thompson’s (2011) notion of quantitative reasoning. However, despite its theoretical promise, empirical investigations of descriptive equivalence remain limited. This study is among the first to intentionally design for the emergence of descriptive equivalence through a sequence of interrelated mathematics practices.

One exception is a recent study by Tondorf and Prediger (2022), who documented the learning processes of fifth graders coming to understand descriptive equivalence of arithmetic expressions such as $3(10 + 5) = 3 \cdot 10 + 3 \cdot 5$. Building on this work, this study extends the construct to algebraic expressions. In doing so, it revealed four additional layers of reasoning, captured in Mathematics Practices 1-4, that appeared to be necessary precursors. However, findings also indicated that additional supports may be needed for interpreting equivalent algebraic expressions, as there was only partial evidence to claim that descriptive equivalence of algebraic expressions became normative in the classroom. In sum, the study contributes to theoretical understanding by identifying a trajectory towards descriptive equivalence in algebra as well as highlights the challenges of extending this interpretation from arithmetic to algebraic contexts.

A practical implication of this study is that precision of language can be a beneficial support for developing descriptive equivalence. While the importance of precise language in mathematics is widely acknowledged, it remains underexplored in the context of algebraic and quantitative reasoning. One notable exception is the work of Czocher et al. (2024), who identify “using precise and detailed language to refer to referents” (p. 13) as a scaffold for quantitative reasoning. This study adds empirical evidence to that claim, demonstrating that developing students’ precision of language, particularly evident in the Discuss-Tech-Without-It section, appear to strengthen students’ reasoning about descriptive equivalence. These findings suggest that precision of language may be a crucial and underutilized instructional support for promoting descriptive interpretations of equivalence.

Somewhat unexpectedly, students' precision of language developed gradually, contributing to the non-linear nature of the learning progression. For example, while students were able to express their general methods using literal symbols relatively early on, they did not consistently interpret arithmetic expressions as quantitative processes until later in the unit. Had precision of language been made an explicit instructional focus from the outset, the trajectory toward descriptive equivalence might have appeared more linear. These results suggest that educators using a quantitative-reasoning approach to foster interpretations of equivalence, or other algebraic topics, may benefit from deliberately foregrounding precision of language early in instruction.

Contributions to Instrumental Orchestration

This study contributes to the theory of instrumental orchestration by expanding the application of the framework to include the use of video, a tool that could be perceived as passive and non-interactive. Most orchestrations in the literature focus on the use of dynamic, manipulable technologies (such as graphing calculators or dynamic geometry software), where students engage directly with the tool. In contrast, video offers a narrower set of technical affordances, namely pausing, fast-forwarding, and rewinding. However, this study challenges the assumption that video lacks interactivity by providing a model of how deliberate pedagogical moves can reposition it as a tool for active learning. A key contribution was the identification of a novel orchestration, Pause-and-Predict, in which the teacher paused the video at key moments to prompt students to make predictions about what would come next. This move not only disrupted passive viewing but also engaged students in active meaning making. These findings suggest that when paired with orchestration, video can support interactive student engagement and should be considered a legitimate technology within the instrumental orchestration framework.

These findings also offer implications for teaching practice in how video technology can be orchestrated to support student reasoning. While some orchestration types identified in this study, such as Discuss-the-Screen and Link-Screen-Board, align with those found in research on other technologies (e.g., Drijvers et al. 2010), this study also identified three novel orchestration types specific to video: (a) Pause-and-Predict, (b) Rewatch-to-Review, and (c) Skip-to-Screen. Rewatch-to-Review emerged as a productive orchestration type in which the teacher prompted students to rewatch a previously viewed video for the purpose of reviewing prior material. In contrast, Skip-to-Screen was identified as unproductive in this context. This orchestration involved skipping ahead to a clip that the teacher identified as relevant for further discussion. While there was evidence that this orchestration introduced additional confusion in this study, with more intentional scaffolding, this move may be more effective in other classrooms. Together, these novel orchestration types offer concrete examples of how teachers can strategically use video in their classroom by leveraging its unique affordances (pausing, rewinding, and fast-forwarding).

While Pause-and-Predict may be a useful IO for any video-based tool, it highlights a unique affordance of dialogic videos: the visual display of students’ reasoning and strategies. In this TE, the teacher used Pause-and-Predict to make these strategies the focal point of classroom engagement. At key moments, the teacher paused the video and asked students to predict how the video participants might approach the task. Had a lecture-style video been used instead, student predictions would likely have centered on teacher explanations or worked examples. In contrast, dialogic videos allowed the teacher to engage the class in predicting and interpreting peers' thinking. In sum, beyond basic video functions, this orchestration demonstrates how teachers can leverage the visual display of student thinking in dialogic videos to support students' mathematical reasoning and engagement.

Contributions to Learning from Dialogic Videos

This study expands our understanding of vicarious learning (learning by observing others' reasoning; Chi et al., 2008) by extending the research setting to a classroom. Most prior work has examined individuals or pairs of students engaging with dialogic video with minimal instructor intervention (Chi et al., 2017; Foster, 2023; Muller et al., 2007). In contrast, this study investigated how a classroom community, guided by a teacher, reasoned over time while engaging with dialogic videos. There were at least two notable differences from previous research. First, students did not progress at the same pace as the video participants. For instance, even after the video participants articulated a general method, the class was still grasping pieces of the method, as evidenced in Mathematics Practice 1. Second, students often forgot ideas from the video in between sessions, unlike in prior studies where pairs appeared to remember and build upon ideas they gleaned from videos across consecutive sessions.

These findings highlight two realities of using dialogic video in classrooms. First, introducing more students into a shared learning environment brings greater variability in the pace at which students process ideas. Second, with less control of students' attention to the videos compared to one-on-one or paired settings, there is a natural tendency for ideas to be forgotten over time. In this context, teacher facilitation played a crucial role in addressing both challenges, by pausing for discussion (Discuss-the-Screen), and facilitating rewatching the video (Rewatch-to-Review).

Currently, there is no established use model for dialogic videos in mathematics instruction. Consequently, the findings of the study can be a starting point for teachers interested in integrating such tools into their practice. A key, and perhaps counterintuitive, finding is that certain teacher-centered orchestrations (IOs where teacher ideas drive the classroom discourse) were effective when paired with dialogic videos. Specifically, the orchestrations Explain-the-Screen, Rewatch-to-Review, and Link-Screen-Board, appeared to be productive in this context. These moves seemed to help clarify and reinforce video content, especially when students struggled to remember or make sense of what they had seen. This challenges a common assumption that student-centered pedagogies are inherently more productive. One explanation is that dialogic videos are inherently student-centered, allowing teacher-led orchestrations to function as supportive complements rather than as dominant modes of instruction. As such, this study offers an initial model for how teachers might integrate dialogic videos into their instruction in ways that include both student-centered and teacher-centered orchestrations.

Limitations

There are at least three limitations to this study that should be considered when interpreting its findings. First, the study followed a classroom TE design that took place in an after-school setting, which limits the generalizability of the results. While the long-term goal is to explore these research aims in typical classrooms, this intermediate step was important given how little is currently known about the use of dialogic videos in instruction. Typical classrooms often lack the inquiry-oriented norms that were foundational to this study, and even when such conditions are present, instruction may not align with the specific content focus of this study. Moreover, most classroom teachers are likely unfamiliar with dialogic videos, meaning that a classroom setting design may have required additional training. Taken together, these aspects of typical classrooms would have made it more difficult to maximize students' learning potential and to interpret how the instructional environment shaped the evolution of their reasoning.

Second, the original design of this teaching experiment was more ambitious than what proved feasible for this particular classroom community. While some of the complexity in the findings can be attributed to the mathematical content itself, other challenges emerged from instructional decisions and implementation. For instance, there were several areas where I miscalculated. I introduced literal symbols before students had formed a general method, and I waited until the latter half of the TE to make precision of language an explicit instructional goal. I also had to navigate how best to balance time across video viewing, whole-class discussion, small-group work on tasks, and building toward collective understandings. These tensions suggest a need for further iteration, likely through multiple classroom teaching experiments, before moving toward a classroom study.

Third, the DCA method, while useful for tracing evolution of reasoning, may have overlooked critical moments of reasoning in transition by focusing primarily on moments when ideas become normative. For instance, it is possible that students did evidence quantitative meanings earlier in the TE before ideas became normative, but in small groups rather than in the whole-class exchange. However, recent advancements by Dreyfus et al. (2023) and Rasmussen et al. (2024) have extended the method to include small-group data, which may address some of these analytical challenges.

Future Research

This dissertation explored the evolution of students' algebraic reasoning through orchestrated classroom interactions involving dialogic mathematics videos. However, it did not directly investigate the relationship between the video tool and students' reasoning. Looking ahead, I plan to make this a central analytic focus by investigating the ways in which the video tool shaped students' reasoning identified in the learning trajectory. Drawing on the framework of instrumental genesis and the construct of mediation, I aim to explore how features of the video shaped, supported, or constrained students' reasoning (Rabardel, 1995; White, 2008). Evidence of mediation may include students referencing ideas from the videos, attributing them to the participants, and applying them in their own work. Using thematic analysis, I will trace the patterns in which elements such as inscriptions, language, or strategies introduced in the videos mediate students' reasoning throughout the TE. This work will offer insight into which features of dialogic videos are meaningful for learning.

Another avenue of future research is a second iteration of the TE, in which several refinements would be made. First, more time would be devoted to interpretations of arithmetic expressions (Mathematics Practice 3) and articulating a general method (Mathematics Practice 1) before introducing literal symbols (Mathematics Practice 2). This adjustment could provide a stronger conceptual bridge from interpreting arithmetic expressions quantitatively to forming a general method to expressing that method with algebra. Second, descriptive equivalence of algebraic expressions would be scaffolded earlier and more explicitly. Rather than assuming that students would naturally transition from interpreting algebraic expressions quantitatively to a descriptive equivalence interpretation, I would provide tailored prompts such as asking students where they see the quantity on the left and right sides of the equal sign. Finally, precision of language would be emphasized from the outset. Given how crucial language became in helping students articulate quantities, future iterations would incorporate routine opportunities for students to articulate their mathematical descriptions with precision. Together, these adjustments would aim to provide a refined instructional sequence that could be the basis for instructional and curricular recommendations for supporting students' reasoning about these topics.

Reflections and Concluding Remarks

In this section, I reflect on several aspects of the TE that were not fully captured by the two research aims of this dissertation. I offer a set of reflections drawn from my experience as the teacher-researcher. I begin with reflections on algebra as a subject of instruction, then consider the broader role of video in the classroom and conclude with thoughts on the specific affordances and challenges of teaching with dialogic videos.

Learning Algebra with Meaning Is Messier Than Anticipated

I began this TE with an ambitious vision: to reframe algebra not as a series of step-by-step procedures, but as a space for constructing quantitative meaning. Inspired by scholars working in the tradition of quantitative reasoning, I sought to support students in shifting from symbolic manipulation to reasoning grounded in context. But the reality was more complex than I anticipated. Their progress did not develop in a predictable sequence (according to the trajectory expressed in the videos) but deepened gradually through repeated opportunities to voice and refine their ideas. For example, although I had anticipated that students would be ready to articulate a general method and express it algebraically by Day 2 of the TE, these did not become normative until the Day 8.

This experience brought into focus a central tension in algebra instruction. Teaching algebra through a quantitative reasoning approach, one that emphasizes meaning and context, can be generative, but also cognitively demanding and logistically difficult. In contrast, traditional approaches to algebra, which often strip away context, can feel more accessible but offer less opportunity for meaningful learning. This tension was highlighted in a classroom moment, when Esperanza expressed confusion after hearing a student's quantitative interpretation of an arithmetic equation. After a pause, Esperanza admitted that it "took [her] a minute to process it" and later reflected, "when people really look at [the equation] without like, seeing the game apps, they would just think that it doesn't symbolize anything, that it's just numbers." Her comment shows both the challenge and the value of a quantitative approach: it can be confusing at first, but it invites deeper sense-making. At the same time, it points to a major drawback of traditional instruction. As she put it, without context, equations "do not mean anything," they are "just numbers." This raises an important question: What should be the role and purpose of algebra in schools?

For over three decades, the role of algebra in school mathematics has been the subject of national debate (e.g., Kaput, 1995; Kilpatrick & Izsák, 2008; Eddy et al., 2015). Recent opinion pieces like "Is Algebra Necessary" (Hacker, 2012) and "Is It Time to Kill Calculus?" (Rockmore, 2020) have questioned its gate-keeping status and real-world relevance. More recently, Walkington et al. (2025) investigated how career scientists use algebra in practice, finding that the mathematical practices central to their work were very distanced from the symbolic manipulation emphasized in school algebra. Their findings suggest that algebra instruction should reflect the complexity of mathematics needed in the workplace. This TE sought to contribute to that conversation by arguing that a quantitative reasoning approach to algebra, though cognitively demanding, better reflects the complex thinking required in real-world settings, offering a promising direction for curriculum reform.

Yet, implementing this vision also revealed the instructional patience it demands. For instance, it wasn't until the third week of the unit that students began to use precise language like "total amount spent by Sofia." Although the idea had been introduced early on, it took explicit instructional support for this language to become normalized. This pacing highlights an essential point: developing quantitative meaning takes time. If we want students to engage in algebra meaningfully, we must design classroom environments that support the messy work of meaning making.

Are Mathematics Educators Using Video Technology All Wrong?

Video technology has been hailed as a transformative force in education, and accordingly, I sought to incorporate the regular use of videos in my teaching experiment. In mathematics education, three dominant models of video use have emerged: flipped classrooms, individualized video-based instruction (e.g., Khan Academy), and MOOCs (Massive Open Online Courses). Flipped classroom models typically involve students watching instructional videos at home and engaging in practice or dicscussion in class. Individualized video-based instruction, such as Khan Academy system, is designed for self-paced learning, including videos and mastery-based quizzes. MOOCs are large-scale online courses that deliver pre-recorded video lectures to massive audiences. Despite their differences, these models share a common feature: a focus on mathematical procedures and traditional expository instruction (Klinger & Walter, 2022). As a result, the use of video in these models often reinforces conventional modes of teaching rather than reimagining instruction in ways the support deeper learning (Chen et al., 2025).

This concern is especially relevant now as generative AI tools enter classrooms. Like video, AI promises personalization and scalability, but it too risks reinforcing traditional models of instruction. Many teacher-facing AI tools in mathematics education, such as Khanmigo (the AI chatbot developed by Khan Academy) are designed to make incremental improvements to existing approaches or enhance the efficiency of content delivery, rather than supporting more transformative pedagogies (Walkington, 2025; Meyer, 2024). As Muller (2025) observed, much of the rhetoric around AI in education echoes the hype around video in earlier decades. Both technologies have the potential to transform learning, but only if integrated with thoughtful pedagogy.

The dialogic mathematics video-based instructional approach presented in this study offers a compelling alternative for several reasons. First, the content featured in dialogic mathematics videos align with decades of research advocating for student-centered meaning making in mathematics. Second, designing for dialogic video use involved deliberate, episode-by-episode design decisions, including the intentional selection of clips, determining where to pause for discussion, and preparing follow-up questions that could extend or challenge students' thinking. It also required careful planning of the structure of the classroom: balancing time spent viewing videos, engaging in whole-class discussions, working in small groups on tasks, and building toward shared understanding. In this context, video fundamentally transformed how students engaged with mathematical ideas rather than reinforcing traditional modes of instruction.

Moreover, this approach reflects broader trends in educational technology, which are increasingly oriented toward tools that promote dialogue. For instance, the rise of educational podcasts and recent AI innovations like NotebookLM, which have the capabilities of turning content (e.g., PDFs, text files, YouTube videos) into podcast-like conversations, illustrate a growing appetite for tools that center conversation. These trends suggest that an instructional approach that leverages dialogic videos may offer one promising direction for the use of videos in mathematics instruction that is in tune with global shifts in emerging educational technologies.

Benefits and Constraints of Using Dialogic Videos

When I began this TE, I was new to teaching and apprehensive about the uncertainty of inquiry-oriented instruction. Dialogic videos became a powerful support, not just for students, but for me. In many ways, the videos acted as a co-teacher. While I introduced tasks and attempted to follow up on students' ideas, the dialogic videos, and particularly the voice of the facilitator off-screen, echoed and reinforced those same moves. During the Work-and-Walk-By orchestration, for example, I found myself restating prompts I had heard in the video. The presence of the dialogic video felt like having my own personal teaching assistant.

Another key affordance was the predictability the dialogic videos offered. One of the biggest challenges of inquiry-based instruction is the uncertainty it introduces because student ideas evolve in unexpected ways. I recall one moment in which a disagreement between students spiraled in an unanticipated direction. I wasn't sure how to resolve it without prematurely shutting down the conversation. Turning to the dialogic video offered a way back to the intended trajectory. In that moment, it felt like bowling with guardrails: having the freedom to explore, with enough guidance to stay on track.

Over time, I came to see dialogic videos as more than an instructional supplement, but as a tangible representation of broader reform goals. In my graduate training, I have encountered research calling for classrooms built around collaboration, argumentation, and equitable participation. These ideals were hard to imagine in practice, until I saw them depicted in video. Dialogic videos modeled the kinds of mathematical learning we hope to cultivate. While the learning in the dialogic videos didn't automatically replicate itself among viewers, it expanded my sense of what is possible in mathematics teaching and learning.

Still, there were moments when the dialogic videos constrained instruction. At times, the mathematical ideas in the video were misaligned with the needs of my classroom. In one instance, during the Not-Use-Tech orchestration, I found that a non-digital task better supported students in articulating a general method than rewatching or extending video-based tasks. This revealed that the dialogic videos, while powerful, are not responsive to the needs of specific classrooms. It falls to the teacher to decide when the video supports the moment and when it doesn't.

There were also moments when the videos introduced unnecessary cognitive load. Because dialogic videos reflect authentic student reasoning, including initially incomplete or incorrect ideas, they can be difficult to follow. While research suggests this aspect leads to deeper learning than traditional expository videos (e.g., Muller, 2008), it also means that teacher intervention is sometimes needed to scaffold understanding. In moments where the conceptual richness led to additional confusion, teacher-centered orchestrations like Explain-the-Screen or Rewatch-to-Review helped anchor students' understanding. In conclusion, I learned that despite these constraints of dialogic videos, they can serve as powerful instructional tools for collective meaning-making, when intentionally orchestrated.

References

Alqahtani, M. M., & Powell, A. B. (2017). Mediational activities in a dynamic geometry environment and teachers' specialized content knowledge. *The Journal of Mathematical Behavior*, *48*, 77-94.

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Roa Fuentes, S., Trigueros, M., & Weller, K. (2014). *APOS Theory: A framework for research and curriculum development in mathematics education*. Springer. https://doi.org/10.1007/978-1-4614-7966-6

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning*, *7*(3), 245–274. https://doi.org/10.1023/A:1022103903080

Ben-David Kolikant, Y., & Broza, O. (2011). The effect of using a video clip presenting a contextual story on low-achieving students' mathematical discourse. *Educational Studies in Mathematics*, *76*(1), 23–47. https://doi.org/10.1007/s10649-010-9262-5

Blanton, M., Brizuela, B. M., Gardiner, A. M., Sawrey, K., & Newman-Owens, A. (2017). A progression in first-grade children's thinking about variable and variable notation in functional relationships. *Educational Studies in Mathematics*, *95*(2), 181–202. https://doi.org/10.1007/s10649-016-9745-0

Blanton, M., Stephens, A., Knuth, E., Gardiner, A. M., Isler, I., & Kim, J.-S. (2015). The development of children's algebraic thinking: The impact of a comprehensive early algebra intervention in third grade. *Journal for Research in Mathematics Education*, *46*(1), 39–87. https://doi.org/10.5951/jresematheduc.46.1.0039

Boesdorfer, S., Lorsbach, A., & Morey, M. (2011). Using a vicarious learning event to create a conceptual change in preservice teachers' understandings of the seasons. *The Electronic Journal for Research in Science & Mathematics Education*, *15*(1). https://ejrsme.icrsme.com/article/view/7381

Booth, J. L., Barbieri, C., Eyer, F., & Paré-Blagoev, E. J. (2014). Persistent and pernicious errors in algebraic problem solving. *The Journal of Problem Solving*, *7*(1), 3.

Bourke, S., & Stacey, K. (1988). Assessing problem solving in mathematics: Some variables related to student performance. *The Australian Educational Researcher*, *15*(1), 73–83. https://doi.org/10.1007/BF03219402

Bowers, J., Cobb, P., & McClain, K. (1999). The evolution of mathematical practices: A case study. *Cognition and Instruction*, *17*(1), 25–66. https://doi.org/10.1207/s1532690xci1701_2

Bowers, J., Passentino, G., & Connors, C. (2012). What is the complement to a procedural video? *Journal of Computers in Mathematics and Science Teaching*, *31*(3), 213–248.

Bozkurt, G., & Ruthven, K. (2018). The activity structure of technology-based mathematics lessons: A case study of three teachers in English secondary schools. *Research in Mathematics Education*, *20*(3), 254–272. https://doi.org/10.1080/14794802.2018.1474798
Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics*, *23*(3), 247–285.

Byerley, C., & Thompson, P. W. (2017). Secondary mathematics teachers' meanings for measure, slope, and rate of change. *The Journal of Mathematical Behavior*, *48*, 168–193. https://doi.org/10.1016/j.jmathb.2017.09.003

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352–378.

Carlson, M. P., O'Bryan, A., & Rocha, A. (2022). Instructional conventions for conceptualizing, craphing, and symbolizing quantitative relationships. In G. Karagöz Akar, İ. Ö. Zembat, S. Arslan, & P. W. Thompson (Eds.), *Quantitative Reasoning in Mathematics and Science Education* (pp. 221–259). Springer International Publishing. https://doi.org/10.1007/978-3-031-14553-7_9

Charles, R. I. (2005). Big ideas and understandings as the foundation for elementary and middle school mathematics. *Journal of Mathematics Education, 7*(3), 9–24.

Chevallard, Y. (1999). Didactique? Is it a plaisanterie? You must be joking! A critical comment on terminology. *Instructional Science*, *27*(1/2), 5–7.

Chi, M. T. H., Kang, S., & Yaghmourian, D. L. (2017). Why students learn more from dialogue- than monologue-videos: Analyses of peer interactions. *Journal of the Learning Sciences*, *26*(1), 10–50. https://doi.org/10.1080/10508406.2016.1204546

Chi, M. T. H., Roy, M., & Hausmann, R. G. M. (2008). Observing tutorial dialogues collaboratively: Insights about human tutoring effectiveness from vicarious learning. *Cognitive Science*, *32*(2), 301–341. https://doi.org/10.1080/03640210701863396

Christou, K. P., Kyrvei, D. I., & Vamvakoussi, X. (2022). Interpreting literal symbols in algebra under the effects of the natural number bias. *Mathematical Thinking and Learning*, 1–14. https://doi.org/10.1080/10986065.2022.2128276

Christou, K. P., & Vosniadou, S. (2012). What kinds of numbers do students assign to literal symbols? Aspects of the transition from arithmetic to algebra. *Mathematical Thinking and Learning*, *14*(1), 1–27. https://doi.org/10.1080/10986065.2012.625074

Christou, K., & Vosniadou, S. (2009). Misinterpreting the use of literal symbols in algebra. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the*

International Group for the Psychology of Mathematics Education, Vol. 2 (pp. 329–336). Thessaloniki, Greece: PME.

Cobb, P. (1999). Individual and collective mathematical development: The case of statistical data analysis. *Mathematical Thinking and Learning*, *1*(1), 5–43.

Cobb, P. (2000). *Conducting teaching experiments in collaboration with teachers*. Lawrence Erlbaum Associates, Inc.

Cobb, P. (2003). Chapter 1: Investigating students' reasoning about linear measurement as a paradigm case of design research. *Journal for Research in Mathematics Education. Monograph*, *12*, 1–16.

Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, *32*(1), 9–13.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, *31*(3–4), 175–190. https://doi.org/10.1080/00461520.1996.9653265

Cole, R., Becker, N., Towns, M., Sweeney, G., Wawro, M., & Rasmussen, C. (2012). Adapting a methodology from mathematics education research to chemistry education research: Documenting collective activity. *International Journal of Science and Mathematics Education*, *10*(1), 193–211. https://doi.org/10.1007/s10763-011-9284-1

Common Core State Standards Initiative (CCSSI) (2010). *Common core state standards for mathematics*. http://www.corestandards.org/assets/CCSSi_Math%20Standards.pdf

Cook, J. P. (2012). *A guided reinvention of ring, integral domain, and field*. (Doctoral dissertation, University of Oklahoma).

Cook, J. P., Reed, Z., & Lockwood, E. (2022). An initial framework for analyzing students' reasoning with equivalence across mathematical domains. *The Journal of Mathematical Behavior*, *66*, 100935. https://doi.org/10.1016/j.jmathb.2022.100935

Cooper, K. M., Ding, L., Stephens, M. D., Chi, M. T. H., & Brownell, S. E. (2018). A course-embedded comparison of instructor-generated videos of either an instructor alone or an instructor and a student. *CBE—Life Sciences Education*, *17*(2), ar31. https://doi.org/10.1187/cbe.17-12-0288

Closson, T. (2024, May 22). The Algebra Problem: How Middle School Math Became a National Flashpoint (Published 2024). *The New York Times.* https://www.nytimes.com/2024/05/22/nyregion/middle-school-math-algebra.html

Craig, S., Chi, M., & Vanlehn, K. (2009). Improving classroom learning by collaboratively observing human tutoring videos while problem solving. *Journal of Educational Psychology*, *101*, 779–789. https://doi.org/10.1037/a0016601

Craig, S. D., Gholson, B., Brittingham, J. K., Williams, J., & Shubeck, K. T. (2012). Promoting vicarious learning of physics using deep questions with explanations. *Computers & Education, 58*, 1042–1048.

Craig, S. D., Gholson, B., Ventura, M., Graesser, A. C., & Tutoring Research Group. (2000). Overhearing dialogues and monologues in virtual tutoring sessions: Effects on questioning and vicarious learning. *International Journal of Artificial Intelligence in Education, 11*, 242–253.

Craig, S. D., Sullins, J., Witherspoon, A., & Gholson, B. (2006). The deep-level-reasoning-question effect: The role of dialogue and deep-level-reasoning questions during vicarious learning. *Cognition and Instruction*, *24*(4), 565–591.

Czocher, J., Roan, E., & Kularajan, S. S. (2024). Scaffolding Moves for Constructing Cognitively Meaningful Models of a Predator-Prey Scenario. *PRIMUS*, *34*(6), 638-652.

Dimitrakopoulou, S. A., & Christou, K. P. (2018). Treatment of variables in greek mathematics textbooks and the different ways that students interpret them. *Journal for Research in Mathematics Education (Erevna Gia Ti Didaktiki Ton Mathimatikon), 11*, 31–52. https://doi.org/10.12681/enedim.18938

Ding, L., Cooper, K., Stephens, M., Chi, M., & Brownell, S. (2021). Learning from error episodes in dialogue-videos: The influence of prior knowledge. *Australasian Journal of Educational Technology*, *37*(4), 20–32.

Doorman, M., Drijvers, P., Gravemeijer, K., Boon, P., & Reed, H. (2012). Tool use and the development of the function concept: From repeated calculations to functional thinking. *International Journal of Science and Mathematics Education*, *10*, 1243–1267.

Drijvers, P. (2012). Teachers transforming resources into orchestrations. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), *From text to "lived" resources: Mathematics curriculum materials and teacher development* (pp. 265–281). Springer Netherlands. https://doi.org/10.1007/978-94-007-1966-8_14

Drijvers, P. (2015). Digital technology in mathematics education: Why it works (or doesn't). In *Selected regular lectures from the 12th international congress on mathematical education* (pp. 135–151). Springer International Publishing.

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: Instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics*, *75*(2), 213–234. https://doi.org/10.1007/s10649-010-9254-5

Drijvers, P., Grauwin, S., & Trouche, L. (2020). When bibliometrics met mathematics education research: the case of instrumental orchestration. *ZDM Mathematics Education*. https ://doi. org/10.1007/s11858-020-01169-3

Drijvers, P., & Trouche, L. (2008). From artifacts to instruments. *Research on Technology and the Teaching and Learning of Mathematics*, *2*, 363–391.

Driscoll, M. (1999). *Fostering Algebraic Thinking: A Guide for Teachers, Grades 6-10*. Heinemann, 361 Hanover Street, Portsmouth, NH 03801-3912.

Driscoll, D. M., Craig, S. D., Gholson, B., Ventura, M., Hu, X., & Graesser, A. C. (2003). Vicarious learning: Effects of overhearing dialog and monologue-like discourse in a virtual tutoring session. *Journal of Educational Computing Research*, *29*(4), 431–450. https://doi.org/10.2190/Q8CM-FH7L-6HJU-DT9W

Drouhard, J. P., & Teppo, A. R. (2004). Symbols and language. *The future of the teaching and learning of algebra the 12th ICMI study*, 225–264.

Dubinsky, E. (1991). Constructive aspects of reflective abstraction in advanced mathematics. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 160–202). Springer. https://doi.org/10.1007/978-1-4612-3178-3_9

Durkin, K., Rittle-Johnson, B., Star, J. R., & Loehr, A. (2023). Comparing and discussing multiple strategies: An approach to improving algebra instruction. *The Journal of Experimental Education*, *91*(1), 1–19.

Eddy, C. M., Quebec Fuentes, S., Ward, E. K., Parker, Y. A., Cooper, S., Jasper, W. A., ... & Wilkerson, T. L. (2015). Unifying the algebra for all movement. *Journal of Advanced Academics*, *26*(1), 59-92.

Eckert, A., & Nilsson, P. (2022). The emergence of the “FlexTech” orchestration of inferential reasoning on pattern generalization. *Digital Experiences in Mathematics Education*, *8*(1), 1–26. https://doi.org/10.1007/s40751-021-00098-4

Ellis, A. B. (2007). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction*, *25*(4), 439–478.

Ellis, A. B. (2011). Generalizing-promoting actions: How classroom collaborations can support students' mathematical generalizations. *Journal for research in mathematics education*, *42*(4), 308-345.

Ellis, A. B., Ozgur, Z., Kulow, T., Dogan, M. F., & Amidon, J. (2016). An exponential growth learning trajectory: Students’ emerging understanding of exponential growth through covariation. *Mathematical thinking and learning*, *18*(3), 151-181.

Ellis, A., Ely, R., Singleton, B., & Tasova, H. (2020). Scaling-continuous variation: Supporting students' algebraic reasoning. *Educational Studies in Mathematics*, *104*(1), 87–103. https://doi.org/10.1007/s10649-020-09951-6

Ellis, A. B., & Özgür, Z. (2024). Trends, insights, and developments in research on the teaching and learning of algebra. *ZDM–Mathematics Education*, *56*(2), 199-210.

Ellis, A. B., Özgür, Z., Kulow, T., Dogan, M. F., & Amidon, J. (2016). An exponential growth learning trajectory: Students' emerging understanding of exponential growth through covariation. *Mathematical Thinking and Learning*, *18*(3), 151–181.

Finkelstein, N., Fong, A., Tiffany-Morales, J., Shields, P., & Huang, M. (2012). College Bound in Middle School & High School? How Math Course Sequences Matter. *Center for the Future of Teaching and Learning at WestEd*.

Falkner, K. P., Levi, L., & Carpenter, T. P. (1999). Children's understanding of equality: A foundation for algebra. *Teaching Children Mathematics*, *6*(4), 232–236. JSTOR. http://www.jstor.org/stable/41197398

Foster, M., White, I., & Lobato, J. (in press, 2025). How do students make sense of dialogic mathematics videos? *Journal of Computers in Mathematics and Science Teaching*.

Fox Tree, J. E. (1999). Listening in on monologues and dialogues. *Discourse Processes*, *27*(1), 35–53. https://doi.org/10.1080/01638539909545049

Gates, B. (2021, December 7). More students flunk this high school course than any other. *Gates Notes*. https://www.gatesnotes.com/helping-students-succeed-in-algebra

Gates, B. (2023, October 3). What does popcorn have to do with math? *Gates Notes*. https://www.gatesnotes.com/modernizing-math-education

Gholson, B., & Craig, S. D. (2006). Promoting constructive activities that support vicarious learning during computer-based instruction. *Educational Psychology Review*, *18*(2), 119–139. https://doi.org/10.1007/s10648-006-9006-3

Gholson, B., Witherspoon, A., Morgan, B., Brittingham, J. K., Coles, R., Graesser, A. C., ... & Craig, S. D. (2009). Exploring the deep-level reasoning questions effect during vicarious learning among eighth to eleventh graders in the domains of computer literacy and Newtonian physics. *Instructional Science*, *37*, 487–493.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education*, *25*(2), 116–140. https://doi.org/10.5951/jresematheduc.25.2.0116

Gravemeijer, K. (2020). A socio-constructivist elaboration of realistic mathematics education. *National reflections on the Netherlands didactics of mathematics: Teaching and learning in the context of realistic mathematics education*, 217–233.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, *1*(2), 155–177.

Gruver, J. (2018). A trajectory for developing conceptual understanding of logarithmic relationships. *The Journal of Mathematical Behavior*, *50*, 1–22. https://doi.org/10.1016/j.jmathb.2017.12.003
Harel, G., & Soto, O. (2017). Structural reasoning. *International Journal of Research in Undergraduate Mathematics Education*, *3*(1), 225-242.

Haspekian, M. (2005). An "instrumental approach" to study the integration of a computer tool into mathematics teaching: the case of spreadsheets. *International Journal of Computers for Mathematical Learning*, *10*, 109–141.

Haspekian, M., Kieran, C., Drijvers, P., Bråting, K., & Tabach, M. (2024). Algebra Education and Digital Resources: A Long-Distance Relationship?. In *Handbook of digital resources in mathematics education* (pp. 439-470). Cham: Springer International Publishing.

Hallagan, J. E. (2006). The case of Bruce: A teacher's model of his students' algebraic thinking about equivalent expressions. *Mathematics Education Research Journal*, *18*(1), 103–123.

Hohensee, C. (2017). Preparing elementary prospective teachers to teach early algebra. *Journal of Mathematics Teacher Education*, *20*(3), 231–257. https://doi.org/10.1007/s10857-015-9324-9

Jankvist, U. T., & Misfeldt, M. (2015). CAS-induced difficulties in learning mathematics?. *For the Learning of Mathematics*, *35*(1), 15–20.

Johnson, H. L. (2015a). Secondary students' quantification of ratio and rate: A framework for reasoning about change in covarying quantities. *Mathematical Thinking and Learning*, *17*(1), 64–90. https://doi.org/10.1080/10986065.2015.981946

Johnson, H. L. (2015b). Together yet separate: Students' associating amounts of change in quantities involved in rate of change. *Educational Studies in Mathematics*, *89*(1), 89–110. https://doi.org/10.1007/s10649-014-9590-y

Kaput, J. J. (1995, October 21–24). *A Research Base Supporting Long Term Algebra Reform?*[Paper presentation]. Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Columbus, OH. https://eric.ed.gov/?id=ED389539

Kaput, J. J. (2008). 1 What Is Algebra? What Is Algebraic Reasoning? In *Algebra in the Early Grades*. Routledge.

Khan, S. (2012, September 6). *Combining like terms introduction | Introduction to algebra | Algebra I | Khan Academy*. [Video] YouTube. https://www.youtube.com/watch?v=CLWpkv6ccpA

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational studies in Mathematics*, *12*, 317–326.

Kieran, C. (1992). The learning and teaching of school algebra. *Handbook of Research on Mathematics Teaching and Learning*, 390–419.

Kieran, C. (2018a). Seeking, using, and expressing structure in numbers and numerical operations: A fundamental path to developing early algebraic thinking. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds: The global evolution of an emerging field of research and practice* (pp. 79–105). Springer International Publishing. https://doi.org/10.1007/978-3-319-68351-5_4

Kieran, C. (Ed.). (2018b). *Teaching and learning algebraic thinking with 5- to 12-year-olds*. Springer International Publishing. https://doi.org/10.1007/978-3-319-68351-5

Kieran, C. (2022). The multi-dimensionality of early algebraic thinking: Background, overarching dimensions, and new directions. *ZDM – Mathematics Education*, *54*(6), 1131–1150. https://doi.org/10.1007/s11858-022-01435-6

Kieran, C., & Drijvers, P. (2006). The co-emergence of machine techniques, paper-and-pencil techniques, and theoretical reflection: A study of CAS use in secondary school algebra. *International Journal of Computers for Mathematical Learning*, *11*, 205–263.

Kieran, C., & Sfard, A. (1999). Seeing through symbols: The case of equivalent expressions. *Focus on Learning Problems in Mathematics*, *21*.

Kilhamn, C., Bråting, K., Helenius, O., & Mason, J. (2022). Variables in early algebra: Exploring didactic potentials in programming activities. *ZDM–Mathematics Education, 54*(6), 1273–1288.

Kilpatrick and Iszak (2008). A history of algebra in the school curriculum. In C. Greenes and R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics* (pp. 3–18). Reston, VA: National Council of Teachers of Mathematics.

Klinger, M., & Walter, D. (2022). How users review frequently used apps and videos containing mathematics. *International Journal for Technology in Mathematics Education*, *29*(1), 25–36.

Knuth, E. J., Alibali, M. W., McNeil, N. M., Weinberg, A., & Stephens, A. C. (2005). Middle school students' understanding of core algebraic concepts: Equivalence & Variable. *Zentralblatt für Didaktik der Mathematik*, *37*(1), 68–76.

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education*, *37*(4), 297–312. https://doi.org/10.2307/30034852

Kuchemann, D. (1978). Children's understanding of numerical variables. *Mathematics in School*, *7*(4), 23–26.

Kuchemann, D. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics* (Vols. 11–16, pp. 102–119). John Murray.

Kuchemann, D. (2010). Using patterns generically to see structure. *Pedagogies: An International Journal*, *5*(3), 233–250.

Lannin, J. K., Barker, D. D., & Townsend, B. E. (2006). Recursive and explicit rules: How can we build student algebraic understanding? *The Journal of Mathematical Behavior*, *25*(4), 299–317. https://doi.org/10.1016/j.jmathb.2006.11.004

Lannin, J. K. (2005). Generalization and justification: The challenge of introducing algebraic reasoning through patterning activities. *Mathematical Thinking and Learning*, *7*(3), 231–258.

Larnell, G. V. (2016). More than just skill: Examining mathematics identities, racialized narratives, and remediation among black undergraduates. *Journal for Research in Mathematics Education*, *47*(3), 233-269.

Liang, B., & Moore, K. C. (2021). Figurative and operative partitioning activity: Students' meanings for amounts of change in covarying quantities. *Mathematical Thinking and Learning*, *23*(4), 291–317. https://doi.org/10.1080/10986065.2020.1789930

Litke, E. (2020). The nature and quality of algebra instruction: Using a content-focused observation tool as a lens for understanding and improving instructional practice. *Cognition and Instruction*, *38*(1), 57–86. https://doi.org/10.1080/07370008.2019.1616740

Lobato, J., & Siebert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *The Journal of Mathematical Behavior*, *21*(1), 87–116.

Lobato, J., Walters, C. D., Walker, C., & Voigt, M. (2019). How do learners approach dialogic, on-line mathematics videos? *Digital Experiences in Mathematics Education*, *5*(1), 1–35. https://doi.org/10.1007/s40751-018-0043-6

Lobato, J., & Walker, C. (2019). How viewers orient toward student dialogue in online math videos. *Journal of Computers in Mathematics and Science Teaching*, *38*(2), 177–200.

Lobato, J., & Walters, C. D. (2017). A taxonomy of approaches to learning trajectories and progressions. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 74–101). Reston, VA: National Council of Teachers of Mathematics.

Lobato, J., Gruver, J., & Foster, M. (2023). Students' development of mathematical meanings while participating vicariously in conversations between other students in instructional videos. *The Journal of Mathematical Behavior*, *71*, 101068.

Lucariello, J., Tine, M. T., & Ganley, C. M. (2014). A formative assessment of students' algebraic variable misconceptions. *The Journal of Mathematical Behavior*, *33*, 30–41.

MacGregor, M., & Stacey, K. (1993). Cognitive models underlying students' formulation of simple linear equations. *Journal for research in mathematics education*, *24*(3), 217-232.

MacGregor, M., & Stacey, K. (1997). Students' understanding of algebraic notation: 11–15. *Educational studies in mathematics*, *33*(1), 1-19.

Maracci, M., Pocalana, G., & Carlino, G. (2023). The role of artifacts and resources in the transition between mathematical experiences within and between digital and non-digital contexts. *Digital Experiences in Mathematics Education*, *9*(1), 56–88. https://doi.org/10.1007/s40751-022-00122-1

Mariotti, M. A. (2000). Introduction to proof: The mediation of a dynamic software environment. *Educational Studies in Mathematics*, *44*(1), 25–53. https://doi.org/10.1023/A:1012733122556

Markworth, K. A. (2010). *Growing and growing: Promoting functional thinking with geometric growing patterns* [Ph.D., The University of North Carolina at Chapel Hill]. Retrieved December 5, 2022, from https://www.proquest.com/docview/750620744/abstract/74B0636EDEFA47ABPQ/1

Mason, J., Graham, A., Pimm, D., & Gowar, N. (1985). *Routes to / Roots of Algebra*. University Press.

Mason, J. (1996). Expressing Generality and Roots of Algebra. In N. Bernarz, C. Kieran, & L. Lee (Eds.), *Approaches to Algebra: Perspectives for Research and Teaching* (pp. 65–86). Springer Netherlands. https://doi.org/10.1007/978-94-009-1732-3_5

Maxwell, J. A. (2005). Conceptual framework: What do you think is going on. *Qualitative Research Design: An Interactive Approach*, *41*, 33–63.

McKendree, J., Stenning, K., Mayes, T., Lee, J., & Cox, R. (1998). Why observing a dialogue may benefit learning. *Journal of Computer Assisted Learning*, *14*(2), 110–119. https://doi.org/10.1046/j.1365-2729.1998.1420110.x

McNeil, N. M., & Alibali, M. W. (2004). You'll see what you mean: Students encode equations based on their knowledge of arithmetic. *Cognitive Science*, *28*(3), 451–466. https://doi.org/10.1207/s15516709cog2803_7

McNeil, N. M., & Alibali, M. W. (2005). Why won't you change your mind? Knowledge of operational patterns hinders learning and performance on equations. *Child development*, *76*(4), 883–899.

McNeil, N. M., Grandau, L., Knuth, E. J., Alibali, M. W., Stephens, A. C., Hattikudur, S., & Krill, D. E. (2006). Middle-school students' understanding of the equal sign: The books they read can't help. *Cognition and instruction*, *24*(3), 367–385.

Meyer, D. (2024, April). Khanmigo WANTS to love kids but doesn't know how. *Mathworlds*. https://danmeyer.substack.com/p/khanmigo-wants-to-love-kids-but-doesnt

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis: An expanded sourcebook*. SAGE.

Molina, M., Rodríguez-Domingo, S., Cañadas, M. C., & Castro, E. (2017). Secondary school students' errors in the translation of algebraic statements. *International Journal of Science and Mathematics Education*, *15*, 1137-1156.

Moore, K. C., & Carlson, M. P. (2012). Students' images of problem contexts when solving applied problems. *The Journal of Mathematical Behavior*, *31*(1), 48–59. https://doi.org/10.1016/j.jmathb.2011.09.001

Moore, K. C., Liang, B., Stevens, I. E., Tasova, H. I., & Paoletti, T. (2022). Abstracted Quantitative Structures: Using Quantitative Reasoning to Define Concept Construction. In G. Karagöz Akar, İ. Ö. Zembat, S. Arslan, & P. W. Thompson (Eds.), *Quantitative reasoning in mathematics and science education* (pp. 35–69). Springer International Publishing. https://doi.org/10.1007/978-3-031-14553-7_3

Moore-Russo, D., Diletti, J., Strzelec, J., Reeb, C., Schillace, J., Martin, A., Arabeyyat, T., Prabucki, K., & Scanlon, S. (2015). A study of how angry birds has been used in mathematics education. *Digital Experiences in Mathematics Education*, *1*(2), 107–132. https://doi.org/10.1007/s40751-015-0008-y

Moses, R., & Cobb, C. (2001). *Radical equations*. Boston: Beacon Press.

Muldner, K., Lam, R., & Chi, M. T. H. (2014). Comparing learning from observing and from human tutoring. *Journal of Educational Psychology*, *106*(1), 69–85. https://doi.org/10.1037/a0034448

Muller, D. A. (2025, April 8). Veritasium: What Everyone Gets Wrong About AI and Learning – Derek Muller Explains [Video]. YouTube.

Muller, D. A., Bewes, J., Sharma, M. D., & Reimann, P. (2008). Saying the wrong thing: Improving learning with multimedia by including misconceptions. *Journal of Computer Assisted Learning*, *24*(2), 144-155.

Muller, D. A., Sharma, M. D., Eklund, J., & Reimann, P. (2007). Conceptual change through vicarious learning in an authentic physics setting. *Instructional Science*, *35*, 519-533.

National Council of Teachers of Mathematics. (2000). Principles and standards for school mathematics. Reston, VA: Author.

National Council of Teachers of Mathematics (2020). Standards for the preparation of secondary mathematics teachers. Reston, VA.

National Mathematics Advisory Panel (2008). Foundations for success: *The final report of the National Mathematics Advisory Panel*. Washington, DC: US Department of Education.

Oehrtman, M., Carlson, M., & Thompson, P. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. Rasmussen (Eds.), *Making the Connection: Research and Teaching in Undergraduate Mathematics Education* (pp. 27– 42). Mathematical Association of America. https://doi.org/10.5948/UPO9780883859759.004

Ottmar, E., Lee, J. E., Vanacore, K., Pradhan, S., Decker-Woodrow, L., & Mason, C. A. (2023). Data from the Efficacy Study of From Here to There! A Dynamic Technology for Improving Algebraic Understanding. *Grantee Submission*, *11*(5), 1-15.

Pantziara, M., & Philippou, G. (2012). Levels of students' "conception" of fractions. *Educational Studies in Mathematics*, *79*(1), 61–83.
https://doi.org/10.1007/s10649-011-9338-x

Paoletti, T., Gantt, A. L., & Corven, J. (2023). A local instructional theory for middle school students' emergent reasoning. *Journal for Research in Mathematics Education, 54*(3).
https://doi.org/10.5951/jresematheduc-2021-0066

Paoletti, T., & Moore, K. C. (2018). A covariational understanding of function: Putting a horse before the cart. *For the Learning of Mathematics*, *38*(3), 37–43.
https://www.jstor.org/stable/26548510

Paoletti, T., Stevens, I. E., Acharya, S., Margolis, C., Olshefke-Clark, A., & Gantt, A. L. (2024). Exploring and promoting a student's covariational reasoning and developing graphing meanings. *The Journal of Mathematical Behavior*, *74*, 101156.

Paoletti, T., & Vishnubhotla, M. (2023). Constructing covariational relationships and distinguishing nonlinear and linear relationships. In G. Karagöz Akar, İ. Ö. Zembat, S. Arslan, & P. W. Thompson (Eds.), *Quantitative Reasoning in Mathematics and Science Education* (pp. 133–167). Springer International Publishing.
https://doi.org/10.1007/978-3-031-14553-7_6

Philipp, R. A. (1992). The many uses of algebraic variables. *The Mathematics Teacher*, *85*(7), 557–561. https://www.jstor.org/stable/27967771

Powell, S. R. (2012). Equations and the equal sign in elementary mathematics textbooks. *The Elementary school journal, 112*(4), 627-648.

Prediger, S. (2010). How to develop mathematics-for-teaching and for understanding: The case of meanings of the equal sign. *Journal of Mathematics Teacher Education*, *13*(1), 73–93. https://doi.org/10.1007/s10857-009-9119-y

Rabardel, P. (2002). *People and Technology. A Cognitive Approach to Contemporary Instruments*. Paris: Universitë Paris 8.

Rabardel, P., & Samurçay, R. (2001, March). From Artifact to Instrument-Mediated Learning. In *Symposium on New Challenges to Research on Learning* (pp. 21–23).

Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: A semiotic analysis. *Educational Studies in Mathematics*, *42*, 237–268. https://doi.org/10.1007/s11858-007-0061-0

Ralston, N., & Li, M. (2022). Student conceptions of the equal sign: Knowledge trajectories across the elementary grades. *The Elementary School Journal*, *122*(3), 411–432. https://doi.org/10.1086/717999

Rasmussen, C., & Stephan, M. (2008). A methodology for documenting collective activity. In A. E. Kelly, R. A. Lesh, & J. Y. Baek (Eds.), *Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching* (pp. 195–215). New York: Routledge.

Rasmussen, C., Wawro, M., & Zandieh, M. (2015). Examining individual and collective level mathematical progress. *Educational Studies in Mathematics*, *88*, 259–281.

Rasmussen, C., Wawro, M., & Zandieh, M. (2024). An integrated methodological approach for documenting individual and collective mathematical progress: Reinventing the Euler Method algorithmic tool. *Education Sciences*, *14*(3), 335.

Rezat, S., & Sträßer, R. (2012). From the didactical triangle to the socio-didactical tetrahedron: artifacts as fundamental constituents of the didactical situation. *ZDM*, *44*, 641–651.

Rich, K. M. (2023). Teacher Orchestrations of Transitions Within and Beyond Fractions Virtual Manipulatives. *Digital Experiences in Mathematics Education*, *9*(1), 163–208. https://doi.org/10.1007/s40751-022-00111-4

Rickel, J., Szatrowski, A., & Rosemond, C. (2023). Efficacy Analysis of Zearn Math: Findings from Implementation in Louisiana. *Online Submission*.

Rivera, F. D. (2006). Research, reflection, practice: Changing the face of arithmetic: Teaching children algebra. *Teaching Children Mathematics*, *12*(6), 306–311.

Rivera, F. D., & Becker, J. (2005). Teacher to teacher: figural and numerical modes of generalizing in algebra. *Mathematics Teaching in the Middle School*, *11*(4), 198–204.

Rivera, F. D., & Becker, J. R. (2007). Abduction–induction (generalization) processes of elementary majors on figural patterns in algebra. *The Journal of Mathematical Behavior*, *26*(2), 140–155.

Rivera, F. D., & Becker, J. R. (2008). Middle school children's cognitive perceptions of constructive and deconstructive generalizations involving linear figural patterns. *ZDM*, *40*(1), 65–82. https://doi.org/10.1007/s11858-007-0062-z

Rivera, F. D., & Becker, J. R. (2011). Formation of Pattern Generalization Involving Linear Figural Patterns Among Middle School Students: Results of a Three-Year Study. In J. Cai & E. Knuth (Eds.), *Early Algebraization: A Global Dialogue from Multiple Perspectives* (pp. 323–366). Springer. https://doi.org/10.1007/978-3-642-17735-4_18

Roorda, G., Vos, P., Drijvers, P., & Goedhart, M. (2016). Solving rate of change tasks with a graphing calculator: A case study on instrumental genesis. *Digital Experiences in Mathematics Education*, *2*, 228-252.

Savuran, R., & Akkoç, H. (2023). Examining pre-service mathematics teachers' use of technology from a sociomathematical norm perspective. *International Journal of Mathematical Education in Science and Technology*, *54*(1), 74–98. https://doi.org/10.1080/0020739X.2021.1966529

Seethaler, S., Burgasser, A. J., Bussey, T. J., Eggers, J., Lo, S. M., Rabin, J. M., Stevens, L., & Weizman, H. (2020). A Research-Based Checklist for Development and Critique of STEM Instructional Videos. *Journal of College Science Teaching*, *50*(1), 21–27. https://www.jstor.org/stable/27119226

Shoecraft, P. (1989). "Equals" Means "Is the Same As." *The Arithmetic Teacher, 36*(8), 36-40.

Solares, A., & Kieran, C. (2013). Articulating syntactic and numeric perspectives on equivalence: the case of rational expressions. *Educational Studies in Mathematics*, *84*, 115-148.

Soneira, C. (2022). The use of representations when solving algebra word problems and the sources of solution errors. *International Journal of Science and Mathematics Education*, *20*(5), 1037-1056.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*(1), 1–36. https://doi.org/10.1007/BF00302715

Sfard, A. (1992). Operational origin of mathematical objects and the quandary of reification – the case of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspectsof* *epistemology and pedagogy* (pp. 59–84). Washington, DC: Mathematical Association of America.

Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification: The case of algebra. *Educational Studies in Mathematics*, *26*(2/3), 191–228. https://www.jstor.org/stable/3482784

Sfard, A., & Thompson, P. (2022). *Problems of Reification: Representations and Mathematical Objects*.

Sharples, M., & Anastopoulou, S. (2012). Designing orchestration for inquiry learning. In *Orchestrating inquiry learning* (pp. 69–85). Routledge.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*(2), 114–145. https://doi.org/10.5951/jresematheduc.26.2.0114

Smith, J., & Thompson, P. (2007). Quantitative reasoning and the development of algebraic reasoning. In J. Kaput & D. Carraher (Eds.), *Algebra in the Early Grades* (pp. 95–132). New York, NY: Lawrence Erlbaum Associates.

Song, Y. (2021). A review of how class orchestration with technology has been conducted for pedagogical practices. *Educational Technology Research and Development*, *69*(3), 1477–1503. https://doi.org/10.1007/s11423-021-10001-y

Stacey, K. (1989). Finding and using patterns in linear generalising problems. *Educational Studies in Mathematics*, *20*(2), 147–164. https://doi.org/10.1007/BF00579460

Stacey, K., & MacGregor, M. (1997). Ideas about symbolism that students bring to algebra. *The Mathematics Teacher*, *90*(2), 110–113. https://doi.org/10.5951/MT.90.2.0110

Steffe, L. P. (1983). Children's algorithms as schemes. *Educational Studies in Mathematics*, *14*(2), 109–125.

Steffe, L., & Thompson, P. (2000). *Teaching experiment methodology: Underlying principles and essential elements* (pp. 267–307).

Stein, M. K., Kaufman, J. H., Sherman, M., & Hillen, A. F. (2011). Algebra: A challenge at the crossroads of policy and practice. *Review of Educational Research*, *81*(4), 453–492.

Stephan, M., & Rasmussen, C. (2002). Classroom mathematical practices in differential equations. *The Journal of Mathematical Behavior*, *21*(4), 459–490. https://doi.org/10.1016/S0732-3123(02)00145-1

Sztajn, P., & Wilson, P. H. (2019). Learning trajectories and professional development. *Learning trajectories for teachers: Designing effective professional development for math instruction*, 1–14. Teachers College Press.

Tabach, M. (2011). A mathematics teacher's practice in a technological environment: A case study analysis using two complementary theories. *Technology, Knowledge and Learning*, *16*(3), 247–265. https://doi.org/10.1007/s10758-011-9186-x

Tabach, M., Hershkowitz, R., & Dreyfus, T. (2013). Learning beginning algebra in a computer-intensive environment. *ZDM*, *45*(3), 377–391. https://doi.org/10.1007/s11858-012-0458-2
Tall, D., Thomas, M., Davis, G., Gray, E., & Simpson, A. (1999). What is the object of the encapsulation of a process? *The Journal of Mathematical Behavior*, *18*(2), 223–241. https://doi.org/10.1016/S0732-3123(99)00029-2

Thompson P. W. (1988). Quantitative concepts as a foundation for algebraic reasoning: sufficiency, necessity, and cognitive obstacles. In M. Behr, C. Lacampagne & M. Wheeler (Eds.), *Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education Vol. 1*, 163–170.

Thompson, P. W. (1990). *A theoretical model of quantity-based reasoning in arithmetic and algebra.* Unpublished manuscript. *Center for Research in Mathematics & Science Education: San Diego State University*. https://doi.org/10.13140/RG.2.2.23908.45447

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 3*, 165–208.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181–234). Albany, NY: SUNY Press.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield S. Chamberlain, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education* (Vol. 1, pp. 33–57). University of Wyoming. http://bit.ly/15II27f

Thompson, P. W. (2013). In the Absence of Meaning…. In *Vital Directions for Mathematics Education Research* (pp. 57–93). https://doi.org/10.1007/978-1-4614-6977-3_4

Thompson, P. W., & Sfard, A. (1994). Problems of reification: Representations and mathematical objects. In *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education—North America, plenary sessions* (Vol. 1, pp. 1–32).

Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In K. C. Moore, L. P. Steffe & L. L. Hatfield (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing*. WISDOMe Monographs (Vol. 4, pp. 1–24). Laramie, WY: University of Wyoming.

Thompson, P. W., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. *Research companion to the principles and standards for school mathematics*, 95–113.

Thompson, A. G., & Thompson, P. W. (1995). A cognitive perspective on the mathematical preparation of teachers: The case of algebra. In C. Lacampagne, W. Blair, & J. Kaput (Eds.), *The Algebra Initiative Colloquium* (Vol. 1, pp. 95–116). Washington D.C.: U.S. Department of Education and the Government Printing Office.

Tondorf, A., & Prediger, S. (2022). Connecting characterizations of equivalence of expressions: Design research in Grade 5 by bridging graphical and symbolic representations. *Educational Studies in Mathematics*, *111*(3), 399–422. https://doi.org/10.1007/s10649-022-10158-0

Trigueros, M., & Jacobs, S. (2008). On developing a rich conception of variable. *Making the connection: Research and Teaching in Undergraduate Mathematics Education*, *73*, 3–13.

Trouche, L. (2003). From artifact to instrument: Mathematics teaching mediated by symbolic calculators. *Interacting with Computers*, *15*(6), 783–800. https://doi.org/10.1016/j.intcom.2003.09.004

Trouche, L. (2004). Managing the Complexity of Human/Machine Interactions in Computerized Learning Environments: Guiding Students' Command Process through Instrumental Orchestrations. *International Journal of Computers for Mathematical Learning*, *9*(3), 281–307. https://doi.org/10.1007/s10758-004-3468-5

Trouche, L. (2005). Instrumental Genesis, Individual and Social Aspects. In D. Guin, K. Ruthven, & L. Trouche (Eds.), *The Didactical Challenge of Symbolic Calculators: Turning a Computational Device into a Mathematical Instrument* (pp. 197–230). Springer US. https://doi.org/10.1007/0-387-23435-7_9

Trouche, L., & Drijvers, P. (2014). Webbing and orchestration. Two interrelated views on digital tools in mathematics education. *Teaching Mathematics and Its Applications: International Journal of the IMA*, *33*(3), 193–209. https://doi.org/10.1093/teamat/hru014

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. *The ideas of algebra, K-12*, *8*, 19.

Van Hoof, J., Vamvakoussi, X., Van Dooren, W., & Verschaffel, L. (2017). Chapter 5—The transition from natural to rational number knowledge. In D. C. Geary, D. B. Berch, R. J. Ochsendorf, & K. M. Koepke (Eds.), *Acquisition of Complex Arithmetic Skills and Higher-Order Mathematics Concepts* (pp. 101–123). Academic Press. https://doi.org/10.1016/B978-0-12-805086-6.00005-9

Ventura, A. C., Brizuela, B. M., Blanton, M., Sawrey, K., Gardiner, A. M., & Newman-Owens, A. (2021). A learning trajectory in Kindergarten and first grade students' thinking of variable and use of variable notation to represent indeterminate quantities. *The Journal of Mathematical Behavior*, *62*, 100866.

Verillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of though in relation to instrumented activity. *European Journal of Psychology of Education*, *10*(1), 77–101. https://www.jstor.org/stable/23420087

Walkington, C., Bernacki, M., Leyva, E., & Istas, B. (2025). When Am I (N) ever Going to Use This? How Algebraic Functions Are Used in STEM-Related Careers. *Journal for Research in Mathematics Education*, *56*(2), 63-89.

Walkowiak, T. A. (2014). Elementary and middle school students' analyses of pictorial growth patterns. *The Journal of Mathematical Behavior*, *33*, 56–71. https://doi.org/10.1016/j.jmathb.2013.09.004

Weinberg, A., Corey, D. L., Tallman, M., Jones, S. R., & Martin, J. (2022). Observing intellectual need and its relationship with undergraduate students' learning of calculus. *International Journal of Research in Undergraduate Mathematics Education*. https://doi.org/10.1007/s40753-022-00192-x

White, I., Foster, M., & Lobato, J. (2023). Making Sense of Algebraic Expressions in Context. *Mathematics Teacher: Learning and Teaching PK-12*, *116*(8), 604–612.

Wilkie, K. J., & Clarke, D. M. (2016). Developing students' functional thinking in algebra through different visualisations of a growing pattern's structure. *Mathematics Education Research Journal*, *28*, 223–243.

Wilkie, K. J. (2022). Generalization of quadratic figural patterns: Shifts in student noticing. *The Journal of Mathematical Behavior*, *65*, 100917. https://doi.org/10.1016/j.jmathb.2021.100917

Wong, M., & Evans, D. (2007). Students' conceptual understanding of equivalent fractions. In *Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia* (pp. 824–833). Adelaide, SA: MERGA.

Yao, X., & Manouchehri, A. (2019). Middle school students' generalizations about properties of geometric transformations in a dynamic geometry environment. *The Journal of Mathematical Behavior*, *55*, 100703. https://doi.org/10.1016/j.jmathb.2019.04.002

Yilmaz, K. (2013). Comparison of quantitative and qualitative research traditions: Epistemological, theoretical, and methodological differences. *European Journal of Education*, *48*(2), 311–325.

Zazkis, R., & Liljedahl, P. (2002). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational studies in mathematics*, *49*, 379-402.

Zwetzschler, L., & Prediger, S. (2013). Conceptual challenges for understanding the equivalence of expressions–a case study. In *Proceedings of the Eighth Congress of European Research in Mathematics Education (CERME 8)* (pp. 558–567).

Appendix A: Screening Instrument

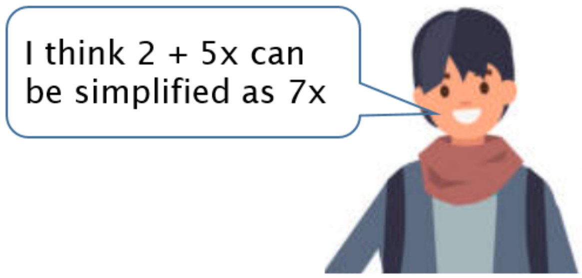
Write your answers and explanations below.

If you cannot answer a question, write "not sure" or "don't know."

1. Do you agree or disagree with Angelo?

Answer: ___________________

Explain why:



Angelo

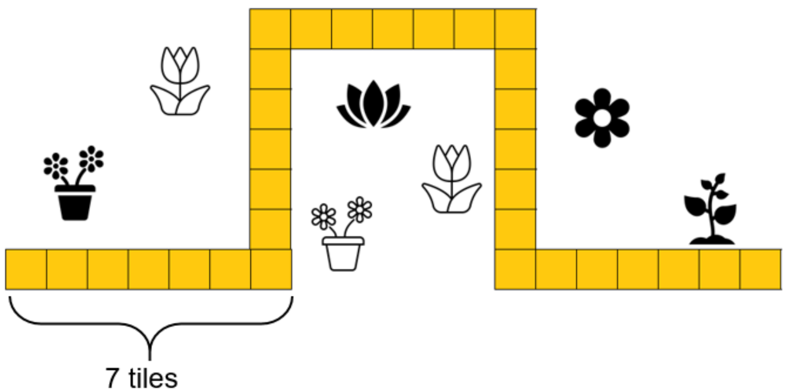
2. Describe a real world situation that can be represented by $4c$.

3. Write down all the numbers you can think of that each **letter** shown below (*a*, *b*, *g* or *x*) can stand for:

a	***-b***

$4g = 3$	$3x = -6$

4. The picture below shows a garden with a pathway made from yellow tiles. One segment of the pathway has 7 tiles.

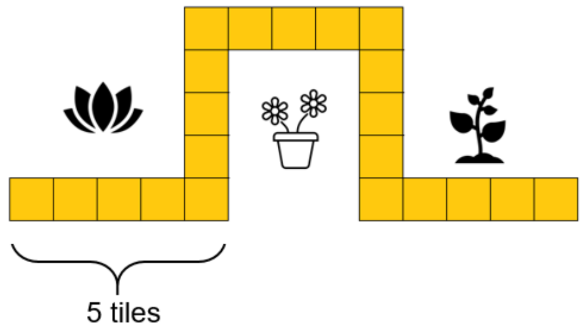


A. What is the total number of tiles in the whole pathway?

Answer: ______________________

Describe how you found the answer. You can label the picture.

B. Apply your method to find the total number of tiles in a pathway with 5 tiles in one segment.



Answer: ____________________

Write an arithmetic equation that shows your method for finding the total number of tiles in the pathway, when there are 5 tiles in one segment:

C. Apply your method to find the total number of tiles in a pathway with 20 tiles in one segment. Show your work.

D. **Write an algebra equation** that generalizes your method for finding the total number of tiles in the pathway (***y***), when there is an unknown number of tiles (***x***) in one segment.

5. A car is traveling at speed of 60 miles per hour for *n* number of hours. Two friends, Jorge and Kiyana, think that the situation can be represented by 60*n*. But they see the meaning of 60*n* differently:

Jorge: I think 60*n* means speed times time

Kiyana: I think 60*n* shows how far the car has traveled

Who is correct – Jorge, Kiyana, or both of them?

Answer: ___________________________________

Explain why: