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Elastic Resistive Force Theory: Development and Applications to Flexible Intruders[†]

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Dry granular intrusion and extraction occurs commonly for off-road vehicles, foundation work, and plant uprooting. Although many models exist to describe the response of the granular medium for such scenarios, reduced-order models, namely granular resistive force theory (RFT), offer a good balance between accuracy and computational cost. RFT is able to efficiently approximate a resistive force experienced by an intruder moving through a granular medium as a function of the intruder's velocity direction, geometry, and material parameters. Despite its accuracy for steady-state motions, due to the explicitly velocity-dependent nature of its formulation, RFT fails at modeling static conditions, the force build-up that occurs before flow, or intruders with stagnant points of motion. We propose elastic RFT to remedy this shortcoming. elastic RFT intrinsically models both granular elasticity and flow adjacent to intruders by splitting intruder motion into elastic and plastic parts, allowing force to build up elastically before flow and letting the plastic part of motion maintain RFT flow rules. We present the details of the underlying kinematic and constitutive assumptions for elastic RFT. We demonstrate its applications to sample plant uprooting problems through the use of a nonlinear inextensible beam solver to model the roots. We highlight potential applications areas for elastic RFT, including simulation results for soft wheel traction.

1 Introduction

Modeling the motion of bodies intruding through granular media is of interest in various applications including terra-mechanics^{1,2}, root-soil interactions^{3,4}, and impact penetration⁵. One approach to modeling the behavior of the granular medium is making use of discrete element methods (DEM) where each particle is modeled individually according to Newton's laws, offering extreme accuracy when grain details match those of the DEM⁶. However, for many granular intrusion or extraction problems, such as plant uprooting⁷ or track racing⁸ the region of interest can be quite large, easily containing over billions of particles^{9,10}. Granular resistive force theory (RFT), a reduced-order model to approximate the resistive forces experienced by an intruder moving through grains, has become an effective tool in recent years to model granular intrusion phenomena^{9,11–14}.

Initially developed by Gray and Hancock to model the motion of intruders through viscous fluids^{15,16}, resistive force theory

(RFT) has been adapted to granular media and shown to accurately model the motion of legged robots on granular terrain¹¹, terra-mechanics of rigid wheels⁹, impact intrusion¹⁷, and burrowing motions mimicking animals¹⁸. The accuracy of the predictions of resistive force theory in these works when compared to experimental and computational results even exceed those for resistive force theory in viscous fluids¹⁹. The simplicity and high accuracy of granular resistive force theory in modeling intrusion forces in granular media can be connected directly to the fact that the continuum-level response of a dry granular medium is well-captured by a frictional plasticity model^{20,21}.

At the core of RFT is a reduced-order formula to model the distribution of forces that arise along the surface of an arbitrary intruding body using material and kinematic parameters as inputs. Although RFT has been successful in modeling quasi-steady motion in granular media^{11,17,22}, the dependence of RFT's formulation on the velocity of the intruder to calculate/compute a resistive traction renders it unable to model resistive forces during static equilibria. For example, a buried object under self-weight that ought not move is unable to sit still using RFT because RFT requires there to be a velocity direction in order to have a resistive force response. Numerical RFT codes frequently exhibit this shortcoming in the form of 'kicks' that mobilize or remobilize buried intruders that should be static. This problem is very reminiscent of an analogous problem that happens in plasticity theory when a yield condition is not properly coupled to an elastic sub-

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[‡] Additional footnotes to the title and authors can be included *e.g.* 'Present address:' or 'These authors contributed equally to this work' as above using the symbols: ‡, §, and ¶. Please place the appropriate symbol next to the author's name and include a \footnotetext entry in the the correct place in the list.

yield response; in our case RFT is like a yield condition determining force levels necessary for intruder motion to commence and the lack of an elastic response prevents loaded static situations from being captured by the model.

In this work, we first provide an overview of RFT, focusing on 2-dimensional implementations and emphasizing its formulation. We highlight its current shortcomings, namely its inability model static solutions and its lack of pairing with deformable intruders. To address RFT's limitations, we present *elastic RFT*, which builds upon existing RFT formulations by introducing an elastic traction based on an elastic-plastic displacement split. We outline both theoretical and computational implementation details of elastic RFT. We demonstrate the ability of elastic RFT to handle static conditions through a simple computational beam drop example where RFT and elastic RFT results are compared side by side.

Through DEM and experiments for loading and unloading experienced for simple intruder shapes, we verify that introducing an elastic RFT stiffness is a realistic way to approximate the elastic response of granular media. We pair elastic RFT with root shapes whose deformation is modeled by nonlinear beam equations. We highlight the effect of accounting for intruder flexibility, which influences the resistive forces significantly. We demonstrate that eRFT is capable of modeling three regimes of elasto-plasticity, namely pure elastic response, small plasticity, and large plasticity through force controlled uprooting simulations. We mimic the same uprooting scenario with experiments, DEM paired with bonded particles, and eRFT paired with flexible intruders to validate that eRFT can match both real life and computational experiments.

Finally, we apply elastic RFT to a common granular intruder, deformable wheels, and show that it can match literature to show that lower wheel pressure provides better traction. With the work presented in this paper, along with the accompanying code, elastic RFT can be used to model a wide range of granular intrusion and extraction problems with great accuracy as demonstrated by the validations and reduced computational cost compared to continuum level or discrete element models.

2 RFT: a Brief Summary

For an arbitrary intruder moving below the surface of a granular medium, the total resistive force experienced by the intruder, $\mathbf{f}^{\text{total}}$ can be expressed as the integral of the traction $\mathbf{t}$ applied by the grains along the intruder's external surface area:

$$\mathbf{f}^{\text{total}} = \int_{\text{surf}} \mathbf{t} dA. \quad (1)$$

RFT assumes that the traction experienced on each submerged surface element of the intruder is independent of the rest and can be expressed as function of its face outward normal $\hat{\mathbf{n}}$, velocity direction $\hat{\mathbf{v}}$, gravity $\mathbf{g}$, and material properties describing the grains and their interaction with the intruder surface which we denote collectively as "mat"^{11,22}. Adopting the assumption that resistive forces grow linearly with depth due to pressure overburden, RFT presumes

$$\mathbf{t} = \alpha(\hat{\mathbf{n}}, \hat{\mathbf{v}}, \mathbf{g}; \text{mat})|z|, \quad (2)$$

where α is some vector-valued function. Although most imple-

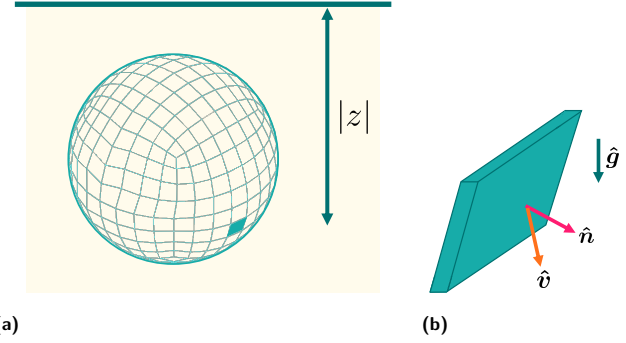


Fig. 1 Visual overview of rigid RFT assumptions: (a) an intruder moving through a granular medium with discretized surface elements (b) RFT function inputs surface normal $\hat{\mathbf{n}}$, velocity direction $\hat{\mathbf{v}}$, and gravity direction $\hat{\mathbf{g}}$ depicted on a single highlighted surface element

mentations of RFT are callous about the definition of the velocity direction since the granular medium in which the intruder travels is typically stationary, it should be noted that for RFT formulations to satisfy material frame indifference, the input $\hat{\mathbf{v}}$ must correspond to the direction of relative velocity between the intruder surface element and the background material²³. Since we use a stationary surrounding granular medium in this work, we will not continue to make this distinction moving forward.

To determine the vectorial α function, experimental (as in¹¹) or higher-order computational (as in²²) datasets are generated by moving a small rigid plate element with a set surface normal $\hat{\mathbf{n}}$ and velocity direction $\hat{\mathbf{v}}$ in the granular medium of interest. The test is repeated as these parameters are varied. Once enough data points are obtained, a fit can be performed. To have scalar input variables for these fit functions, the face normal $\hat{\mathbf{n}}$ and velocity direction $\hat{\mathbf{v}}$ can be related to the angles β and γ as shown in Fig 2 for the two-dimensional case. The variable β corresponds to the clockwise angle from the horizontal to the face of the plate element, while γ is the counterclockwise angle from the horizontal to the direction of velocity. With these scalar values defined, the components of the α function can be easily visualized as heat maps (see Fig 3) where each point on either map corresponds to one component of the α function. After performing the experiments to obtain data points describing α , trigonometric functions were used in¹¹ to calculate these fits while Agarwal et al.²² used trigonometric polynomial functions.

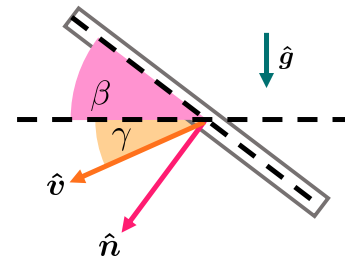


Fig. 2 Visual definitions of the tilt angle β and angle of attack γ angles corresponding to the orientation and velocity directions of a 2D surface plate element.

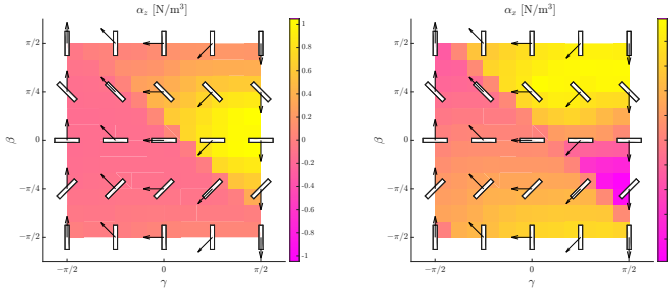


Fig. 3 Vertical ($\alpha_{\text{gen},z}$) and horizontal ($\alpha_{\text{gen},x}$) components of normalized traction per area per depth function α_{gen} generated using data from ²².

Upon investigating the values of the α function for various materials experimentally, it has been discovered that the overall shape of the heat maps for the components of the α functions do not vary much when the granular material is modified across different types of dry grains¹¹. Thus, once a fit for the α function is defined for one granular material and intruder surface type, the function only needs to be adjusted via a single scalar multiple ξ determined by material changes and gravity, provided the intruder has a fixed material-interface surface roughness²²,

$$\mathbf{t} = \alpha_{\text{gen}}(\hat{\mathbf{n}}, \hat{\mathbf{v}}, \hat{\mathbf{g}})\xi(\text{mat}, g)|z|. \quad (3)$$

This significantly reduces the effort required to apply RFT. The scaling material parameter function ξ depends directly on the critical density of the granular medium ρ_c , the magnitude of gravitational acceleration g , and a function of the internal friction of the grains μ_{int} ,

$$\xi = \rho_c g f(\mu_{\text{int}}). \quad (4)$$

If the intruder surface friction coefficient varies across tests, the tangential portion of the resistive traction calculated via the α functions must be rescaled according to the surface friction coefficient, following the process outlined in ²². With continuous definitions of traction per area per depth in terms of the plate orientation and velocity direction, any intruder can be prescribed a resistive force, provided that all plate elements have a well-defined velocity direction.

To define the α_{gen} functions used in this work, we have used the data from ²² obtained through computational experiments with a material point method (MPM) implementation of a non-dilatant plasticity model. The constitutive model in these continuum simulations had a rate-insensitive, Drucker-Prager yield criterion with incompressible plastic deformations, and no cohesion²⁴. The material had an internal friction μ_{int} of 0.4, surface friction μ_s of 0.15, and effective density ρ_c of 3000 kg/m³. The MPM data for traction measured was then normalized such that the value of $\alpha_{\text{gen},z}$ for a horizontal plate intruding down is set to 1. This α_{gen} , presented in figure 3, will be used for the rest of our work here because it was validated favorably against DEM tests²² and it is of high overall refinement across β and γ angles. In doing so, we make the approximation that the intruder surface friction interactions among our tests is similar enough to that of ²² to produce only a minor effect; the results in ²² show that the surface

friction coefficient has only a small contribution to the shape of the α functions. Because our tests will all be nominally 2D with in-plane symmetry, we only need to utilize the subset of the 3D MPM data that is under the same conditions, i.e. corresponding to ‘no twist angle’.

2.1 Using RFT for Extraction Problems

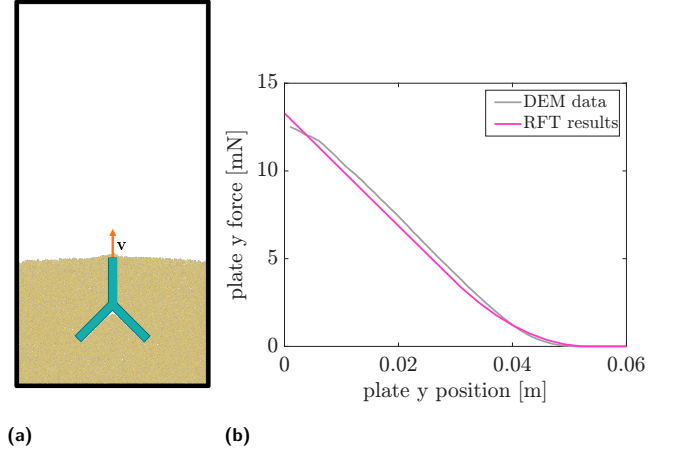


Fig. 4 (a) setup for DEM uprooting experiment of rigid inverse-Y shaped intruder. The intruder is extracted from a bed of Hookean grains with constant velocity applied at the top. (b) Force displacement results for Y-root extraction solutions using DEM and elastic RFT. The RFT functions α are calibrated with interaction parameter ξ derived from DEM experiment. DEM data is time averaged using a moving mean to reduce noise.

Since most previous applications of RFT were focused on granular intrusion problems, and our current aim is to explore uprooting, which is an extraction problem, we must ensure that RFT is similarly applicable and accurate for this type of problem. A DEM experiment of a rigid inverse-Y shape being extracted from a bed of Hookean grains was conducted. The grains used had diameters distributed randomly between 0.8-0.6 mm and density of 2600 kg/m³. The grains were modeled with spherical atom style, and a Hookean granular pair style with LAMMPS²⁵. The pair coefficients for the Hookean contact model in LAMMPS were specified as normal stiffness $k_n = 1\text{ kN/m}$, coefficient of restitution $e = 0.1$, tangential stiffness $k_t = 2/7 k_n$, and internal friction $\mu_s = 0.7$ with no tangential damping. The interaction between the grains and side walls, as well as the intruder were governed by the same Hookean contact model and parameters.

The inverse Y-shaped intruder geometry is defined by the parameters from figure 10. The branches have a thickness $t = 5$ mm, length $L = 3$ cm, and inter-branch angle $\theta = 45^\circ$. Since this is a 2D DEM simulation, the width perpendicular to the plane w is taken as the mean particle diameter, which is an approximation that yields realistic results while eliminating the need to run a computationally intensive 3D simulation.

From the force-displacement data obtained for this test, we calibrated the ξ material parameter for the media by fitting the shallow-regime quadratic portion of the smoothed force-

displacement corresponding to when the structure is exiting the surface. Using $\xi = 54.5 \text{ kN/m}^3$, the complete RFT response for the same motion was calculated using the third order function for RFT provided in²². The results of DEM and RFT agree throughout the entire displacement of the intruder despite using a single scalar value to calibrate RFT, affirming that RFT can be used to successfully model extraction problems as well as intrusion.

2.2 Shortcomings of Rigid RFT

The ability of RFT to capture the forces experienced by intruders in granular media under steady state kinematically controlled motions has been verified against both experimental results and continuum models^{11,17,22}. However, since the definition of the resistive force relies on the direction of velocity, in cases where a surface element has zero velocity the resistive force is not well defined. Moreover, there is no accounting for the continual build-up of resistive force on a buried intruder as applied loads increase; this resistive force only reaches the finite resistive force given by RFT when plastic granular flow is achieved. The absence of velocity can occur in various scenarios including the stagnant point at the bottom of a rotating wheel and any static equilibria. The lack of a well-defined traction in the absence of velocity for granular resistive force theory is akin to the lack of a well-defined stress tensor in the absence of plastic strains in ‘rigid plasticity’ models in which the material is presumed perfectly rigid when loads are under the yield strength. Motivated by this similarity, we rename RFT that has been developed so far *Rigid RFT*.

We now introduce elastic RFT, which defines a resistive traction from the granular medium even when the intruder is not moving. This transforms RFT from a rigid plastic model to an elastic-plastic one. Elastoplastic constitutive relations have been shown to accurately capture the continuum behavior of granular media and are used widely^{24,26–28}.

When comparing the results from rigid RFT to DEM simulations for reciprocal motion of an intruder, Zhang et al.²⁹ have found that while for rigid RFT the resistive force would switch directions and magnitudes following a square wave pattern, the DEM results provided resistive forces that gradually built up to the steady-state value of the resistive force. With elastic RFT we aim to mimic this gradual build-up of the resistive forces to their steady-state values.

Another unexplored avenue with rigid RFT is that thus far it has only been paired with either rigid intruders¹, or deformable intruders with prescribed kinematics^{11,18}. Many common intruders in grains such as reptiles, plant roots, soft robots have some elastic character and deform in response to the forces applied by the grains onto them. We anticipate the resulting change in shape of the intruder to in turn affect the resistive force calculations. We will hence also explore the effect of intruder flexibility for the case of root structure removals by modeling the roots with materials of different rigidities.

3 Elastic RFT: Theory

Elastic RFT improves upon Rigid RFT by accounting for an elastic traction experienced by the intruder before and during gran-

ular yielding. In the stiff elastic limit, elastic RFT is identical to rigid RFT, however starting from a stagnant position, elastic RFT allows for the resistive forces to develop continuously. The similarities and differences between elastic and rigid RFT can be explained visually by making use of simple one-dimensional analog models for the underlying mechanics.

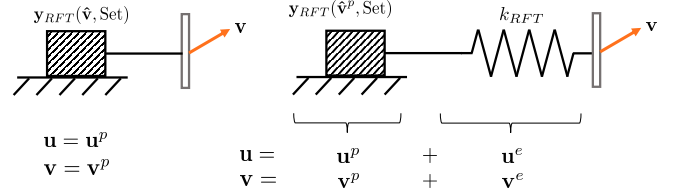


Fig. 5 Analog models for rigid RFT (left), elastic RFT (right). The slider represents the flow resistance of the grains while the spring captures their elastic response. The parameter Set consists of the face normal of the plate element $\hat{\mathbf{n}}$, gravity, and material parameters. Displacements and velocities are decomposed into elastic and plastic components on the corresponding representative elements. **LAL:split figure cleanly**

In the analog model for rigid RFT, a plate element which is our simple intruder of interest, is tied in series to a slider. The yield strength y_{RFT} of this slider is the RFT force calculated with the appropriate α function using the plate element’s velocity direction and Set parameters as input.

$$y_{RFT}(\hat{\mathbf{v}}, \text{Set}) = \alpha_{\text{gen}}(\hat{\mathbf{n}}, \hat{\mathbf{v}}, \hat{\mathbf{g}}) \xi(\text{mat}, |\mathbf{g}|) |z| \quad (5)$$

$$\text{Set} = \{\hat{\mathbf{n}}, \hat{\mathbf{g}}, \xi, |z|\} \quad (6)$$

As seen in this schematic, for any motion of the plate, the slider must also be moving. The slider’s response can be thought of as the plastic response of the surrounding grains. Since rigid RFT is derived for steady-state motion, it assumes that any motion of the plate results in a quasi-steady plastic flow of grains surrounding the intruder.

For elastic RFT, an elastic spring is added to the system in series. This allows for a gradual increase in the traction within the system as the plate moves, until the yield strength of the slider is reached. The slider is again governed by the forces that are described by the RFT functions, however in this case instead of the direction of the velocity of the plate, we define and use the plastic velocity direction as input for the RFT function,

$$y_{RFT}(\hat{\mathbf{v}}^p, \text{Set}) = \alpha_{\text{gen}}(\hat{\mathbf{n}}, \hat{\mathbf{v}}^p, \hat{\mathbf{g}}) \xi(\text{mat}, |\mathbf{g}|) |z|. \quad (7)$$

3.1 Kinematics

For rigid RFT, any displacement of the intruder is deemed plastic, corresponding to flow of the granular media around the plate element. For elastic RFT, we assume an additive decomposition of the displacement of the plate element in the form of:

$$\mathbf{u} = \mathbf{u}^e + \mathbf{u}^p. \quad (8)$$

The elastic displacement is recovered once the plate element is released. Returning to the analog models, this is the amount the spring stretches during the total displacement $\mathbf{u}$ experienced by the plate element. The plastic displacement $\mathbf{u}^p$ arises from plas-

tic granular strains around the intruder — motion that cannot be recovered upon unloading. The time rate of change of these quantities give the total, elastic, and plastic velocities:

$$\mathbf{v} = \mathbf{v}^e + \mathbf{v}^p. \quad (9)$$

3.2 Constitutive Response

3.2.1 Elastic Response

With rigid RFT, the traction experienced by the plate element always lies on the yield surface. To define the constitutive behavior when the traction lies within the elastic regime, we have outlined two approaches. These two approaches, which we denote as *codirectional nonlinear* elastic RFT and *non-codirectional linear* elastic RFT, differ in the manner in which the applied traction approaches the yield surface. In describing these methods, ‘codirectional’ refers to the presumption that elastic displacement is parallel to the plastic velocity direction, unlike the conventional use of this word in plasticity theory. Linear and nonlinear refer to the relationship between the elastic displacement and traction.

3.2.1.1 Non-codirectional linear elastic RFT In the non-codirectional linear approach, the traction in the system grows linearly with the elastic displacement as scaled by an “RFT stiffness”, which we denote as k_{RFT} .

$$\mathbf{t} = k_{RFT} \mathbf{u}^e = k_{RFT} (\mathbf{u} - \mathbf{u}^p) \leq \mathbf{y}_{RFT}(\hat{\mathbf{v}}^p, \text{Set}). \quad (10)$$

These approaches produce distinct ways for defining the traction in the elastic regime, however once this elastic traction reaches the yield surface, both collapse to the values on the yield locus defined by the RFT function. Since the force need not always lie on the yield locus, we can state that

$$\mathbf{t} = \mathbf{y}_{RFT}(\hat{\mathbf{v}}^p, \hat{\mathbf{n}}, \hat{\mathbf{g}}, |z|, \text{mat}) \quad \text{when} \quad |\mathbf{v}^p| > 0 \quad (11)$$

where the yield stress-per-depth $\mathbf{y}_{RFT}$ is identical to what was previously referred to as the RFT force function, α , multiplied by depth $|z|$.

3.2.1.2 Codirectional nonlinear elastic RFT For codirectional nonlinear elastic RFT, the elastic displacement and plastic velocities are assumed to be in the same direction. The traction grows with k_{RFT} and the magnitude of elastic displacement linearly, however it is not necessarily oriented along the elastic displacement direction. The direction of traction is determined by asserting that the plastic velocity direction matches that of the elastic displacement in the $\mathbf{y}_{RFT}$ function, i.e.,

$$\mathbf{t} = k_{RFT} |\mathbf{u}^e| \frac{\mathbf{y}_{RFT}(\hat{\mathbf{u}}^e, \text{Set})}{|\mathbf{y}_{RFT}(\hat{\mathbf{u}}^e, \text{Set})|} \leq \mathbf{y}_{RFT}(\hat{\mathbf{v}}^p, \text{Set}) \quad (12)$$

3.2.2 Plastic Response: Yield strength, flow

The sliders are governed by a flow strength $\mathbf{y}_{RFT}$ that correspond to the RFT forces. For a single choice of Set parameters, a yield locus can be created by probing the RFT function’s values at various points in traction space corresponding to different velocity directions. Figure 6 contains some examples of yield loci for different plate orientations.

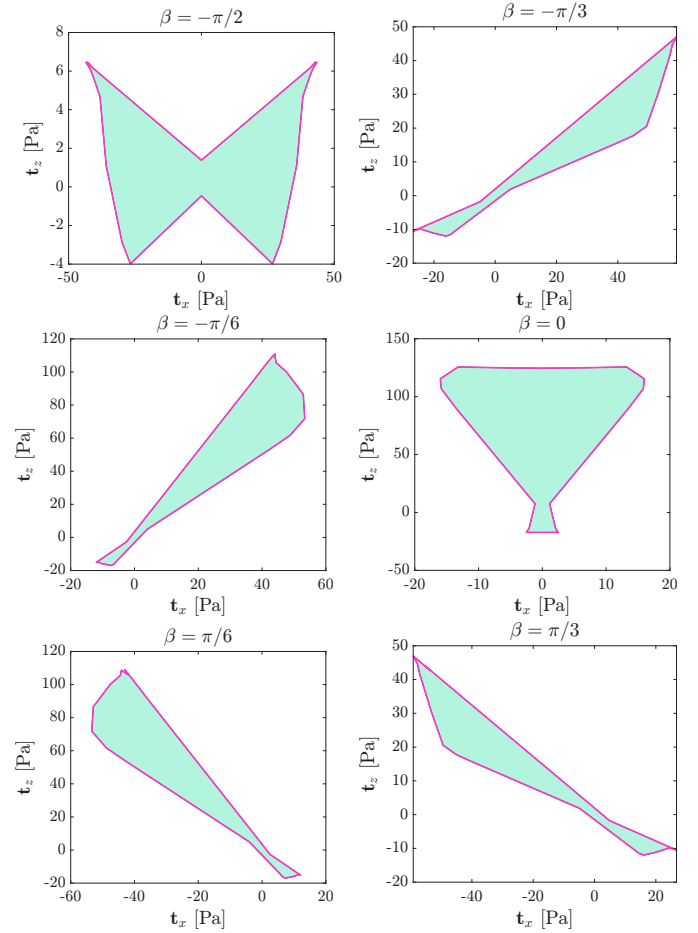


Fig. 6 Linearly interpolated yield traction loci for different plate orientations. The pink border indicates the yield tractions for different velocity directions. The light green interiors correspond to the elastic regions. The symmetry within certain yield loci and reflective symmetry across other loci match RFT’s symmetrical properties. The values of the yield tractions come from the MPM tests in²².

tractions correspond to the α_{gen} values from figure 3. In application, these values will be multiplied by a material scaling parameter ξ however the shape of the surfaces will remain the same. Here, we approximate the loci as polygons with vertices corresponding to experimentally evaluated values during flow. However, yield loci with finer resolution can be generated using RFT forces derived from α fit functions. However, using higher order polynomial fits²² often lead to non-simply-connected artifacts in the yield loci with self-intersecting points that can complicate the process of evaluating whether a point lies within the closed locus.

3.3 k_{RFT} : is it real?

To query whether k_{RFT} is a parameter that can be derived from the granular materials constitutive response via direct testing, we ran DEM tests. The grains and the grain-intruder interaction were modeled using the same Hookean contact model and parameters described in section 2.1. In these tests, a horizontally oriented flat plate was buried in a bed of grains. The plate moves in 3 stages: first it moves upward at a constant velocity toward the surface until some chosen depth is reached, then after a brief

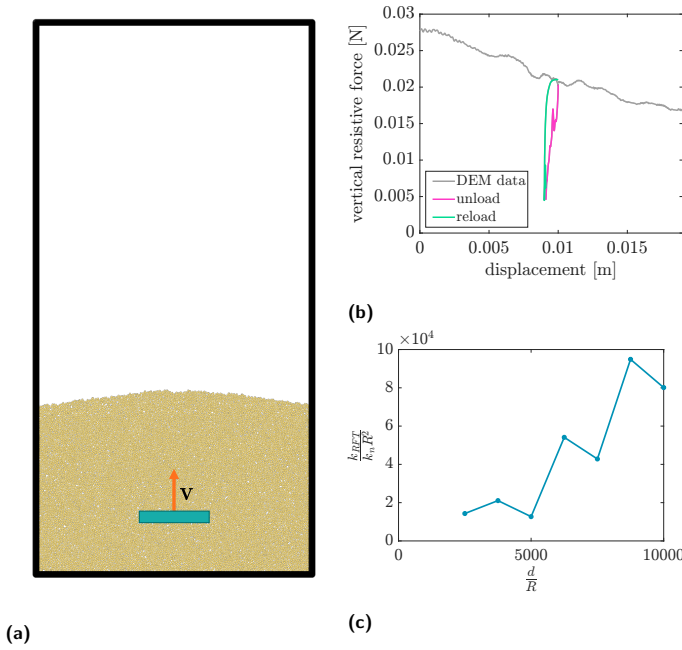


Fig. 7 (a) DEM setup for flat plate pull-release-pull experiments to test k_{RFT} . The rigid horizontal plate is moved through the Hookean grains with a velocity boundary condition. (b) Force displacement response for one of the load-unload-reload experiments. Although unloading and reloading slopes are slightly different, the overall force displacement behavior is reversible. (c) nondimensionalized depth (depth / nominal grain radius R_{grain}) versus k_{RFT} ($k_{RFT}/(k_{grain} R_{grain}^2)$) data from DEM experiments conducted at different depths

pause it reverses its motion for a much smaller total displacement to unload the force, lastly the plate moves up until it is removed entirely from the granular medium. This experiment was conducted at various reversal depths. The slope of the unload-reload curve was calculated and accepted as the RFT stiffness k_{RFT} . The force displacement data for all experiments show similar reversible unload-reload behavior. Although there is some small hysteresis, the overall force-displacement behavior in the unload-reload region remains reversible.

The results show that k_{RFT} is depth-dependent. However, for most depths the magnitude of k_{RFT} is quite large when compared to the traction observed at that same depth. Motivated by this difference in the order of magnitudes, we saw it sufficient to use a single large k_{RFT} value for our initial implementation of elastic RFT.

The existence of k_{RFT} is replicable in experimental settings as well. We conducted pull-release-pull experiments for both inverse-Y shapes and flat plates in one of our experimental setups, shown in figure 8c, where the intruder is placed in a 5-gallon bucket of dry Quikrete 1962 Medium Sand and then extracted using a fishing line connected to an INSTRON tensile testing machine. We observe for both of these shapes an elastic recovery in the load-displacement curves, as can be observed in figure 8.

4 Elastic RFT: Computation

To numerically simulate an intruder's motion under eRFT, we utilize a trial-return, explicit/implicit Euler solver inspired by similar

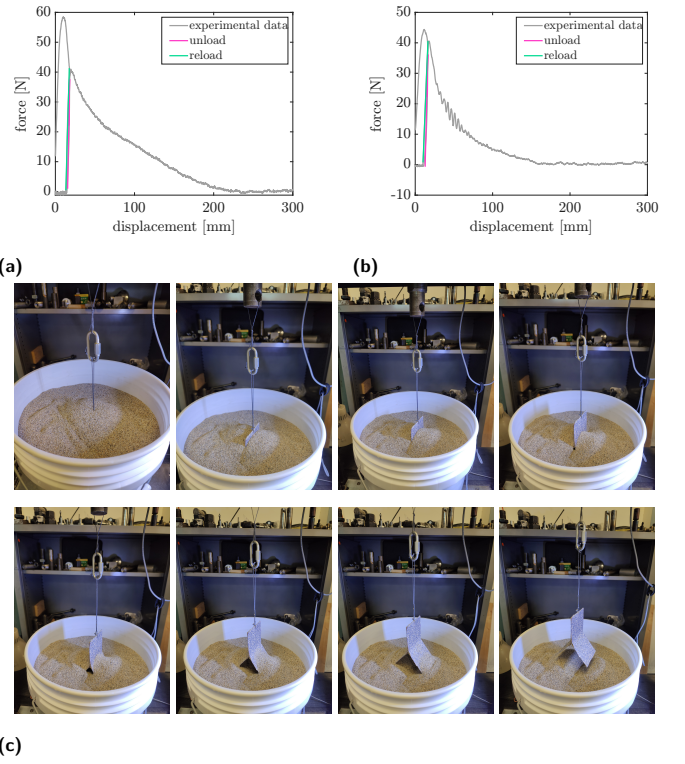


Fig. 8 (a) Experimental force displacement results for inverse-Y shape for a pull-release-pull test in an INSTRON machine. (b) Same results as (a) for a horizontal square intruder (c) sequence of inverse-Y shape testing

solving algorithms used in bulk elastoplastic deformation problems^{30,31}. This solver is implemented with a MATLAB script. With the traction, position, velocity, elastic velocity, plastic velocity and external loads known at time t with timestep dt , we first assume a purely elastic step and calculate a trial traction. We then check if the trial traction is inside the yield locus corresponding to the Set parameters of the plate element at timestep t utilizing the *inpoly* command available in MATLAB which evaluates whether the point defined by the traction vector is inside the closed polygon generated by the vertex points defining the $\mathbf{y}_{RFT}$ locus. If the trial traction vector is inside the locus, the elastic step is accepted and the solver advances to time $t + dt$. If however the trial traction is outside the yield locus, yield must be occurring in the timestep and we must correct the trial traction so it returns to the locus. This step is different for the two different elastic RFT methods. Regardless of the method for calculating the elastic RFT tractions, it can be paired with an intruder following the steps in Algorithm 1.

4.1 Nonlinear Codirectional Elastic RFT: Numerical Solution

For the computational implementation of nonlinear codirectional elastic RFT, summarized in Algorithm 2, we begin by assuming

Algorithm 1: Algorithm for elastic RFT on a node of (kinematically controlled) rigid intruder

given: material parameters, Δt , k_{RFT} , $\mathbf{y}_{RFT}(\mathbf{v}^p, \text{Set})$
known: $\mathbf{u}^n$, $\mathbf{u}^{p,n}$, $\mathbf{v}^n$, $\mathbf{v}^{p,n}$, $\mathbf{t}^n$
step 1: update kinematic boundary conditions
 output: $\mathbf{v}^{n+1}$
step 2: call eRFT routine to update traction boundary conditions
 input: $\mathbf{u}^n$, $\mathbf{u}^{p,n}$, $\mathbf{v}^{n+1}$
 output: $\mathbf{t}^{n+1}$, $\mathbf{u}^{n+1}$, $\mathbf{u}^{p,n+1}$, $\mathbf{v}^{p,n+1}$
step 3: repeat

an elastic trial step knowing $\mathbf{t}$, $\mathbf{u}$, $\mathbf{v}^{n+1}$, $\mathbf{u}^e$, $\mathbf{u}^p$ at timestep t .

$$\mathbf{v}^{e,n+1} = \mathbf{v}^{n+1} \quad (13)$$

$$\mathbf{u}^{e,n+1} = \mathbf{u}^e + \mathbf{v}^{n+1} \Delta t \quad (14)$$

$$\mathbf{t}^{trial} = \mathbf{t} + k_{RFT} |\mathbf{v}^{n+1} \Delta t| \frac{\mathbf{y}_{RFT}(\hat{\mathbf{u}}^e, \text{Set})}{|\mathbf{y}_{RFT}(\hat{\mathbf{u}}^e, \text{Set})|} \quad (15)$$

If the trial traction is within the elastic regime:

$$\mathbf{t}^{n+1} = \mathbf{t}^{trial} \quad (16)$$

$$\mathbf{v}^{p,n+1} = \mathbf{v}^p \quad (17)$$

$$\mathbf{u}^{p,n+1} = \mathbf{u}^p. \quad (18)$$

If the trial traction is outside the locus, we must solve for a plastic velocity that guarantees the ‘slider’ is on the yield locus with a traction that agrees with the traction experienced by the ‘spring’, i.e.

$$\begin{aligned} \mathbf{t}^{n+1} &= \mathbf{t}^n + k_{RFT} |\mathbf{v}^{e,n+1}| \frac{\mathbf{y}_{RFT}(\hat{\mathbf{u}}^e, \text{Set})}{|\mathbf{y}_{RFT}(\hat{\mathbf{u}}^e, \text{Set})|} \Delta t \\ &= \mathbf{y}_{RFT}(\hat{\mathbf{v}}^{p,n+1}, \text{Set}) \end{aligned} \quad (19)$$

$$\mathbf{v}^{n+1} = \mathbf{v}^{p,n+1} + \mathbf{v}^{e,n+1} \quad (20)$$

We can then update the rest of the variables for the new timestep.

$$\mathbf{u}^{e,n+1} = \mathbf{u}^e + \mathbf{v}^{e,n+1} \Delta t \quad (21)$$

$$\mathbf{u}^{p,n+1} = \mathbf{u}^p + \mathbf{v}^{p,n+1} \Delta t \quad (22)$$

Following these steps either method can be implemented. There are further details on the solution of plastic velocity in the appendix for both methods 7.1.

5 Elastic RFT: Application & Verification

5.1 Demonstration: Verifying claim of equilibrium

One of the key shortcomings of rigid RFT is its inability to achieve static equilibrium in load controlled scenarios. To highlight this shortcoming and demonstrate that elastic RFT is able to address it, we look at the test case of dropping a horizontal bar into a bed of grains under its self-weight. The experimental setup can be seen in figure 9a. A horizontal bar with initial downward velocity

Algorithm 2: process for numerical solution of codirectional nonlinear elastic RFT intrusion problem

Given: Δt , k_{RFT} , $\mathbf{y}_{RFT}(\hat{\mathbf{v}}^p, \text{Set})$
Input: $\mathbf{t}^n$, $\mathbf{u}^n$, $\mathbf{u}^{p,n}$, $\mathbf{v}^{n+1}$
Output: $\mathbf{t}^{n+1}$, $\mathbf{u}^{p,n+1}$
begin
 $\mathbf{u}^{e,n} = \mathbf{u}^n - \mathbf{u}^{p,n}$;
 $\mathbf{u}^{e,n+1} = \mathbf{u}^{e,n} + \mathbf{v}^{n+1} \Delta t$;
 $\mathbf{t}^{trial} = \mathbf{t}^n + k_{RFT} |\mathbf{v}^{n+1} \Delta t| \frac{\mathbf{y}_{RFT}(\hat{\mathbf{u}}^{e,n+1}, \text{Set})}{|\mathbf{y}_{RFT}(\hat{\mathbf{u}}^{e,n+1}, \text{Set})|}$;
 if $\mathbf{t}^{trial}$ *within yield locus* **then**
 $\mathbf{t}^{n+1} = \mathbf{t}^{trial}$;
 $\mathbf{u}^{p,n+1} = \mathbf{u}^{p,n}$;
 else
 Solve for $\mathbf{v}^{p,n+1}$ such that:
 $\mathbf{t}^n + k_{RFT} |\mathbf{v}^{e,n+1}| \frac{\mathbf{y}_{RFT}(\hat{\mathbf{u}}^{e,n+1}, \text{Set})}{|\mathbf{y}_{RFT}(\hat{\mathbf{u}}^{e,n+1}, \text{Set})|} \Delta t$
 $= \mathbf{y}_{RFT}(\hat{\mathbf{v}}^{p,n+1}, \text{Set})$,
 $\mathbf{v}^{n+1} = \mathbf{v}^{p,n+1} + \mathbf{v}^{e,n+1}$,
 $\mathbf{u}^{e,n+1} = \mathbf{u}^{e,n} + \mathbf{v}^{e,n+1} \Delta t$;
 $\mathbf{t}^{n+1} = \mathbf{y}_{RFT}(\hat{\mathbf{v}}^{p,n+1}, \text{Set})$;
 $\mathbf{u}^{p,n+1} = \mathbf{u}^{p,n} + \mathbf{v}^{p,n+1} \Delta t$;
 end
end

$\mathbf{v}_0$ and mass m is released at height h above the surface of a bed of grains under gravity g . An analytical solution for the final rest depth z_{final} the bar should reach can be calculated from a simple work-energy balance:

$$\frac{1}{2} m |\mathbf{v}_0|^2 = m g (z_0 - z_{\text{final}}) + \int_{z_{\text{surface}}}^{z_{\text{final}}} |\mathbf{f}_z| dz \quad (23)$$

where $\mathbf{f}_z$ is the z direction force the bar experiences due to RFT. The solutions to this load case are obtained with rigid and elastic RFT. For the results presented here, the beam has a square cross-section of 5 cm, length of 1 m and density of $\rho = 670 \text{ kg/m}^3$. The parameters used for rigid and elastic RFT are $\xi = 9.2 \text{ kN/m}^3$ and $k_{RFT} = 1 \text{ kN/m}^3$. The beam starts at height $y = 0 \text{ m}$, 0.1 m above the surface level with initial downward velocity of 0.1 m/s under gravity $g = 9.81 \text{ m/s}^2$. The results are compared to the analytical solution in figure 9b. Both rigid RFT and elastic RFT first hit the depth corresponding to the analytical solution. Once a balance between the resistive force and the bar’s weight is achieved, the bar is expected to have zero velocity in this static equilibrium. Without any velocity, rigid RFT is unable to provide a resistive force to balance the weight, so the bar begins to drift. After the initial dip of the bar below the analytical depth, the bar is pushed up and accelerates. In general, the resistive forces are much larger for intrusion than extraction, thus the bar faces less resistance moving upward, which results in the rigid RFT solution slowly lifting the bar up. The elastic RFT solution on the other hand, remains at the analytical solution, with small elastic oscillations; one can choose to damp these oscillations through the addition of a damping term, but we will not discuss that here. The ampli-

tude of the oscillation increases and the frequency decreases as we lower kRFT. This matches elastic oscillations expected from any dynamic system.

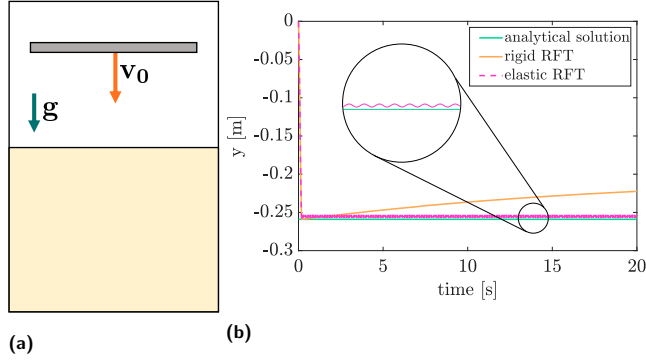


Fig. 9 (a) Setup for beam drop test: a horizontal beam with initial downward velocity is dropped into a bed of grains under gravity (b) time-position results of the beam drop test comparing elastic RFT solution, rigid RFT solution, and the analytical solution for final rest depth.

5.2 Demonstration: Uprooting

Previous applications of RFT have mostly focused on rigid or kinematically controlled intruders. However, many typical intruders such as rubber wheels, plant roots, crawlers are highly deformable. We have chosen to pair RFT with deformable intruders through the study of uprooting of plant root shapes. Studying uprooting can illuminate plant anchoring mechanisms. Plant roots improve terrain stability by anchoring the soil^{32,33} stabilizing river banks^{34,35}, and reducing the harm of landslides³⁶. The anchoring effect of root systems can be used in geoengineering applications to prevent erosion^{33,37}.

To model the deformation of the roots as they are being pulled out of granular soils, we choose beam elements due to the roots' slender nature. Since we anticipate large deformations of the roots as they move through the grains, we diverge from the commonplace linear Euler-Bernoulli beam theory by (i) making use of geometrically nonlinear beam theory to account for large bending displacements and (ii) asserting inextensibility in the roots, since bending is far easier than extension in slender roots. To satisfy the inextensibility kinematic constraint, we introduce the tension within the beams as a Lagrange multiplier to satisfy this constraint. Root structures mainly fail by either root breakage or root pullout, both of which are due to the internal loads in the roots which can be traced in this model³⁸. Root pullout is addressed by using resistive force theory, while the tension calculated inside the root structures can be compared to the plant's axial strength to consider breakage. One benefit of the inextensibility assumption is the removal of the severe numerical time step restriction that modeling stiff root stretching would impose. Physically, Wagner et al.'s³⁹ inspection of the elastic properties of a root in three orthogonal directions, longitudinal, radial, and tangential, demonstrates that the longitudinal stiffness of green wood is much larger than the other two, suggesting that an inextensibility assumption of roots along their length is realistic.

The details of nonlinear inextensible beam equations, and their

numerical application to root structures can be found in the appendix. The core governing equations are as follows. The tangential and normal direction equations of motion, respectively, are:

$$\frac{dT}{ds} + VK + f_t = \lambda \dot{\mathbf{v}} \cdot \hat{\mathbf{t}} \quad (24)$$

$$\frac{dV}{ds} + TK + f_n = \lambda \dot{\mathbf{v}} \cdot \hat{\mathbf{n}} \quad (25)$$

where s is the axial coordinate of the beam, T is the tension, and V is the shear force. K is the curvature of the beam, while f_t and f_n are the tangential and normal components of external applied forces, which are a combination of self-weight and RFT forces in our calculation. λ is the mass per length of the beam. The shear forces within the beams can be related to the moments as:

$$V = -\frac{dM}{ds} \quad (26)$$

The inextensibility of the beam is expressed by setting the scalar product of the tangent vector and change of the velocity along the beam to zero.

$$\hat{\mathbf{t}} \cdot \frac{d\mathbf{v}}{ds} = 0 \quad (27)$$

To satisfy inextensibility when solving the system computationally, we make use of the projection method inspired by its popular use for incompressible flows in fluid system simulations^{40,41}.

For multi-branch root structures, such as the inverse Y-shaped one presented in this paper, each section of the root structure is modeled as a separate beam. These beams are then connected at the center via two kinematical constraints. The first is to ensure that a joint at which two beams connect does not separate during uprooting. The second constraint fixes the angle between the beams at the joint. This is based on the assumption that junctions where multiple branches join are much more rigid than the rest of the root structure⁴².

With the nonlinear inextensible beam equations defined for the root structure we can pair it with elastic RFT to solve for the motion of a root shape in a granular medium. The steps for this process are outlined in Algorithm 3. Given the defining parameters of the problem such as material details, intruder geometry, boundary conditions and initial conditions, we first update the velocity and beam-related quantities explicitly by solving the momentum balance and constitutive equations, enforcing the boundary conditions. Then, with the new velocity, we call the elastic RFT routine to update the traction boundary conditions and the internal plastic displacement variable such that the traction is not outside the yield locus.

5.2.1 Force Controlled Uprooting: What is the threshold for uprooting?

Mimicking an uprooting scenario by loading our root structures vertically from the top, we investigate the threshold force required to uproot an inverse-Y shape root structure. Each branch has a length $L = 1\text{m}$, a square cross-section with a side width of $w = 5\text{cm}$, with the lower branches joining at an angle $\theta = 60^\circ$ following the definitions in figure 10. The branches had a density $\rho = 670\text{kg/m}^3$ used to calculate the linear density λ and Young's modulus of $E = 10\text{MPa}$. The damping coefficient used

Algorithm 3: Algorithm for elastic RFT on a nonlinear beam branch structure

given: material parameters, Δt , k_{RFT} , $y_{RFT}(\hat{\mathbf{v}}^p, \text{Set})$, λ , EI

known: $\mathbf{u}^n$, $\mathbf{u}^{p,n}$, $\mathbf{v}^n$, $\mathbf{v}^{p,n}$, $\mathbf{t}^n$, T^n , V^n , K^n at timestep t^n

step 1: call the momentum balance routine & constitutive routine. Explicitly update velocity and beam equations.

input: $\mathbf{t}^n$, T^n , V^n , K^n , $\mathbf{v}^n$, $\mathbf{u}^n$

output: $\mathbf{v}^{n+1}$, T^{n+1} , V^{n+1} , K^{n+1}

step 2: call eRFT routine to update traction boundary conditions

input: $\mathbf{u}^n$, $\mathbf{u}^{p,n}$, $\mathbf{v}^{n+1}$

output: $\mathbf{t}^{n+1}$, $\mathbf{u}^{n+1}$, $\mathbf{u}^{p,n+1}$

step 3: repeat

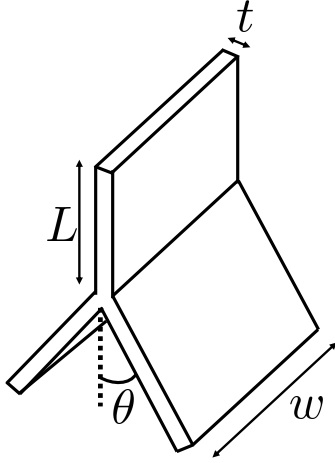


Fig. 10 Diagram for inverse Y root shapes with branch length L , thickness t , width w , and lower branch angle θ

is $\eta = 0.001$ Ns. No gravity is applied to be able to demonstrate the total displacement from initial state dropping to zero for the elastic case. For elastic RFT, the α functions are calibrated with $\xi = 158 \text{ kN/m}^3$ and $k_{RFT} = 1 \text{ MN/m}^3$ is used. This ξ value comes from experiments discussed in section 5.3. We tested the response of the root under 5N, 100N, and 4kN applied to the top of the root. All three roots were pulled for 4 seconds and then released for 4 seconds. Chronological screencaps can be seen in figure 11.

For the smallest applied load, the root structure barely deforms and elastic RFT is able to balance the applied load via only elastic tractions. This corresponds to none of the grains around the root flowing, but simply responding elastically. When we increase the load to 100N, we see the small plastic response. Around the vertical portion of the root red traction arrows corresponding to plastic flow of grains are present. This load is not enough to completely uproot the structure in 4 seconds however when the load is removed we can still see its effect via the residual tractions required to balance the new position of the root, similar to residual stresses in a partially plastically deformed metal after unloading due to lack of compatibility of strains in the plastic part. For loads large enough, such as 4kN for this root, the grains quickly yield resulting in red plastic traction arrows in the visualization. The root bends significantly under the large load, and most of the root is above the surface by the end of the 4 seconds of loading. These

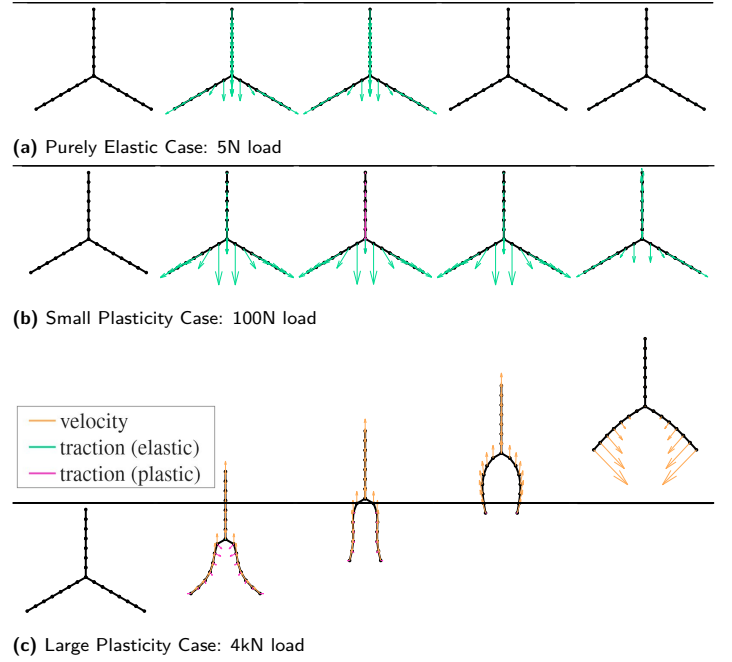


Fig. 11 Response of an identical root to three different applied load magnitudes at the top node. Force arrow magnitudes rescaled between cases. Load applied, held, and then released. The fourth image in every sequence corresponds to right after release of the loading at the top node.

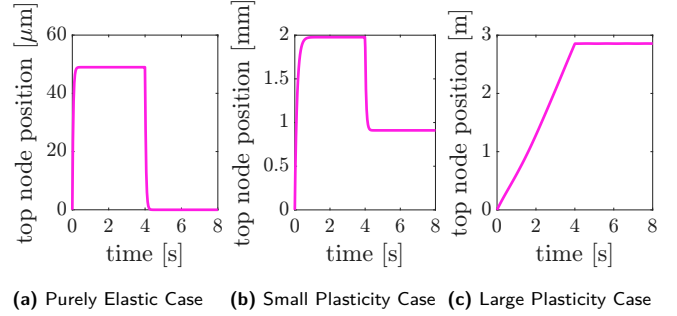


Fig. 12 time-position data of the top node for the three identical roots pulled at different load magnitudes.

results highlight elastic RFT's capability to model a wide range of elastoplastic response.

5.2.2 Effects of Intruder Flexibility

To the best of our knowledge, previous studies using RFT mainly focused on either rigid intruders or deforming intruders whose kinematics were assigned. In this section we query the effects of accounting for the flexibility of the intruder on the resistive forces calculated. The same velocity-controlled uprooting scenario with gravity applied was repeated for two Y-shaped root samples with identical geometries but different elastic moduli. The stiffness of the materials are 1 MPa and 0.1 GPa. Each branch has a length $L = 7$ cm, a thickness $t = 7$ mm and out of plane width $w = 9.5$ cm, and vertical branch connection angle $\theta = 60^\circ$ following figure 10. The mass per length parameter λ is 3.04 kg/m^2 . The damping parameter used is $\eta = 2$ nNs. The ξ value used to calibrate α_{gen} values was 158 kN/m^3 , derived from an experimental

setup explained in section 5.3. k_{RFT} is set to 100 MN/m^3 . The roots are subject to gravity $g = 9.81 \text{ m/s}^2$ and 15.6 mm/s velocity applied at the top node. In figure 14 the difference between the response of the two samples can be seen qualitatively. The soft sample bends much more in response to the resistive forces, while the stiff sample mostly maintains its initial shape.



Fig. 13 Response of identical roots with different material properties to constant velocity pull at the top node with gravity.

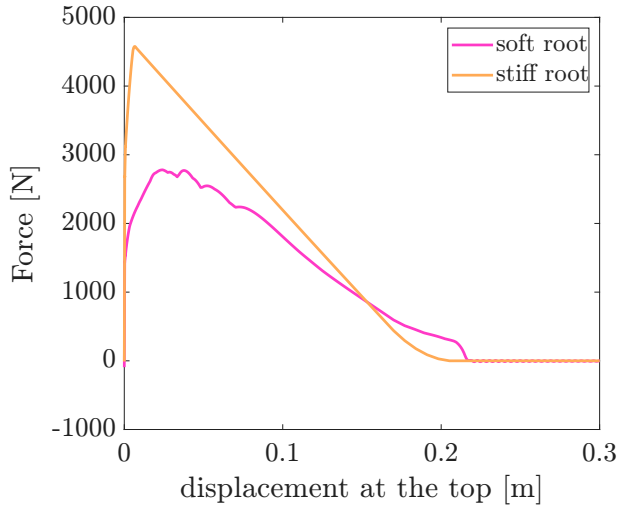


Fig. 14 force displacement results for uprooting of identical roots with different stiffnesses. The stiff root shape requires both a higher peak force and more work overall to extract.

In figure 14 the force required at the top node to apply a constant velocity uprooting is shown for the two samples. Overall, since the aluminum sample bends less, it maintains a larger area that is perpendicular to its direction of motion resulting in a larger peak resistive force. The rubber root structure requires a peak load that is over 30% lower than that required for the aluminum sample. Since the rubber sample bends significantly, its ends re-

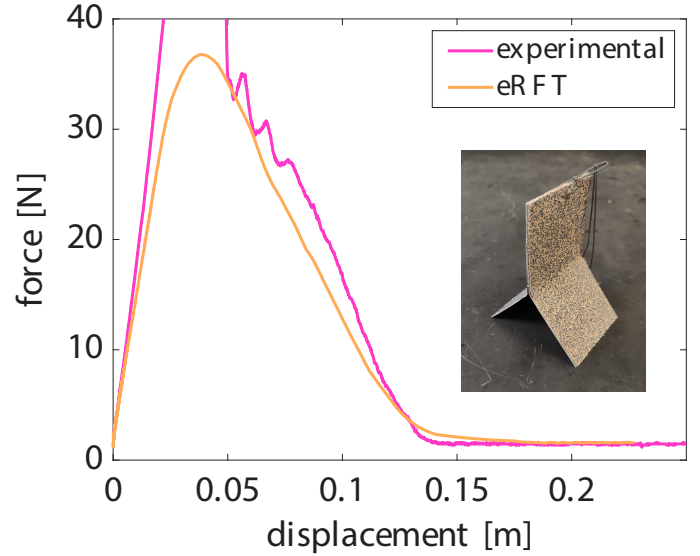


Fig. 15 Experimental and eRFT force-displacement results for the uprooting of an aluminum inverse-Y shape root **LALE: FIX FONT and find better picture of setup**

main in the sand for a larger total displacement. It can be seen in figure 14 that the aluminum sample is freed from the grains around 15 cm of displacement and no longer experiences any resistance from the grains while the rubber sample continues to experience traction from the grains until it travels 17 cm. These results highlight that both qualitatively and quantitatively, taking the flexibility of the intruder into account heavily influences the results one obtains from resistive force theory.

5.3 Experimental Validation

Although the main motivation to introduce k_{RFT} to RFT was to be able to handle static conditions when modeling intrusion, we also wanted to ensure elastic RFT can realistically model intrusion and extraction problems. In order to test this, a set of uprooting experiments were conducted to compare with elastic RFT. In the experiments, an inverse-Y root shape, shown in figure 10, made of aluminum sheets covered in coarse sandpaper to increase surface friction was buried in a large sandbox. The root shape has length $L = 4.5 \text{ cm}$, width $w = 12.5 \text{ cm}$, thickness $t = 3 \text{ mm}$. The branch angle is $\theta = 62^\circ$.

The inverse-Y root shape was placed at desired depth measured from the surface of the sandbox filled with dry Quikrete 1962 Medium Sand, a silica sand with a size range of 0.3-0.8 mm. The inverse-Y shape was then extracted from the sand using a pulley system connected to the sample via a fishing line. Force-time data was collected during the extraction. Using the constant velocity of 227 mm/s applied during the experiments through the pulley system, the force-time data was converted to force-displacement.

To calibrate elastic RFT parameters to the experiments, a set of single flat plate extraction experiments were conducted. The plate was made of the same aluminum material covered with sandpaper. The force-displacement results of the average of three flat plate extraction tests were used to calibrate the ξ parameter used in the elastic RFT formulation, resulting in $\xi = 158 \text{ kN/m}^3$.

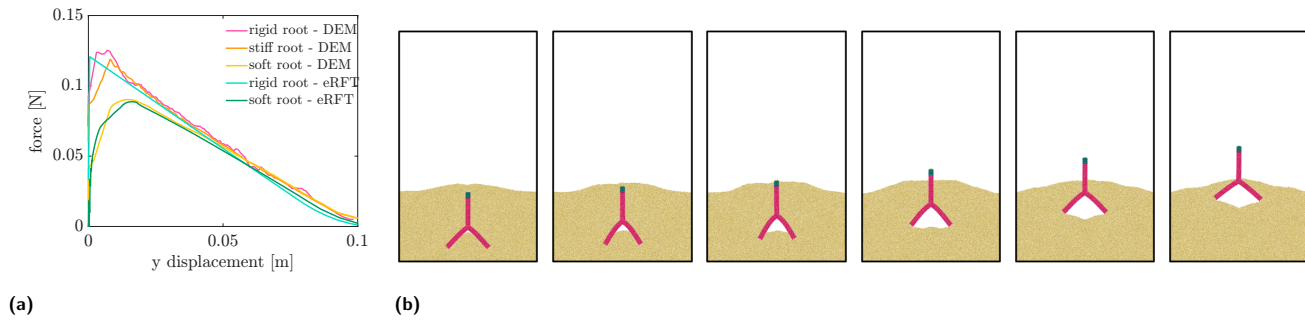


Fig. 16 (a) force displacement results obtained by DEM and elastic RFT for rigid, stiff, and soft root shapes. (b) sequence of uprooting of soft root made of bonded particles in DEM

For the Y-shape root, a density of $\rho = 2700\text{kg/m}^3$ is used to calculate the linear mass density λ . Since the aluminum sheet was covered in sandpaper which would reduce its modulus, a Young's modulus of $E = 0.1\text{GPa}$ is used. This also helps avoid having to use very small timesteps when simulating while still keeping the response of the material relatively stiff.

The results presented in figure 15 demonstrate good agreement between experimental results and elastic RFT's predictions. Elastic RFT cannot model the initial peak in the force response, since this is heavily preparation dependent. The RFT α are derived from forces measured during steady motion, thus the peak in force as the intruder initiates its motion is not something that RFT can capture. If instead the experiment were initialized by first pulling up on the structure until it reaches a flow state and then releasing it, this peak in force would no longer be present. This fact can be seen in the load-unload-reload results in section 3.3.

5.4 DEM Soft Intruder Validation

To validate our soft intruder – elastic RFT solver for the roots, we mimicked the same scenarios using DEM. An inverse Y root shape was generated using spherical particles which were bonded with linear elastic springs to their neighbors using the bonded particle method (BPM) package in LAMMPS⁴³. By tuning the spring constant k between the particles we were able to model intruders of different stiffnesses. We have derived global elastic moduli E for the bonded particle media based on the provided stiffness of the springs through a simple static analysis for hexagonally packed particles. This derived elastic modulus was validated in a separate tensile test using the same bonded particles in DEM. The grains used in these DEM experiments were identical to those in section 2.1. The elastic intruder defined with the bonded particles interacts with the grains with the same Hookean contact parameters.

Here we present 3 sets of force-displacement results from the DEM tests corresponding to a fully rigid Y root, a stiff elastic Y root ($k = 10\text{ kN/m}$) and a soft elastic Y root ($k = 1\text{ kN/m}$). The stiff and rigid roots have similar force-displacement responses. These Y roots have thickness $t = 1\text{cm}$, length ($L = 5\text{cm}$, branch angle $\theta = 60^\circ$). Since the DEM simulations were 2D, mean particle diameter was used as the out-of-plane width w . When we replicate these uprooting scenarios using elastic RFT paired to

deformable roots with elastic moduli determined by the different DEM k stiffnesses, we see from figure 16 that the results show remarkable agreement. The soft root experiences a lower force in DEM, and when replicated with the nonlinear elastic beams paired with elastic RFT, the results match even during the loading section.

5.5 Demonstration: Flexible Wheel Driving on Sand

In the past, RFT has been commonly used with wheels or other terrestrial locomoting bodies^{1,11}. In most of these applications, the wheel or robot leg has been modeled as rigid or kinematically controlled. Accounting for the elasticity of the intruder can influence the response significantly since, e.g., tire pressure can heavily influence traction response in loose terrain⁴⁴. To explore elastic RFT's ability to model terramechanics of soft wheels, we developed simulations of a 2D wheel driven with constant torque input. The wheel was modeled using the inextensible beam elements developed for the root shapes in the previous sections. Wheels with diameter of 0.5m, thickness of 5 cm, and width of 10 cm. The cylindrical wheels had rigid grousers placed at each node of width 0.039 m.

The wheels with different elastic moduli were tested with different torque inputs with elastic RFT using the α functions from experimental calibrations in section 5.3. At low applied torque, the wheels do not rotate and the tractions provided by elastic RFT are mostly elastic. As the torque is increased, the wheel begins to rotate and translate forward.

It is commonly known that in order to gain better traction for locomotion on sandy terrains, wheel pressure should be lower than what is recommended for asphalt or other terrain⁴⁵. As can be seen in our model results for 100 Nm applied torque in figure 17, our model confirms this; as the wheels stiffen, the work required to travel the same distance increases. Figure 18 shows successive snapshots of a soft and a stiff wheel driven with the same torque, indicating the soft wheel travels farther faster.

Conclusions

With this work, we introduced elastic RFT to improve upon rigid RFT by accounting for the elastic response of the granular media. By doing so, we have expanded RFT's domain of application to intrusion scenarios where static or zero-velocity conditions are expected. We have detailed the theory for codirectional nonlinear

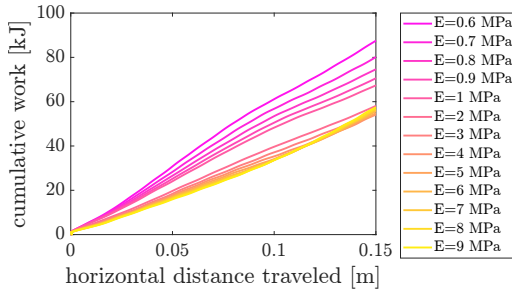


Fig. 17 work input with respect to horizontal travel for smooth elastic wheels of different stiffnesses supplied 100Nm constant torque input.

and non-codirectional linear formulations of elastic RFT. Steps for numerical implementation of both versions were outlined. Proof of elastic RFT's ability to address rigid RFT's shortcomings was demonstrated via beam drop stoppage and a study of intruder flexibility. With the introduction of elasticity to resistive force theory, its applicability has been expanded to load-controlled scenarios.

A nonlinear inextensible beam theory is discussed and numerically implemented to model root structures. Elastic RFT predictions compared favorably to both an experimental study of uprooting and a numerical DEM simulation of flexible uprooting.

The ability of the same eRFT – flexible intruder framework to accurately capture both root extraction and flexible wheel locomotion illustrates that this approach offers a numerically inexpensive solution to a variety of granular intrusion and extraction problems. The same framework can be applied to virtually any quasi-steady motion of a deformable intruder through dry granular media including modeling soft robots and studying animal locomotion.

The nonlinear inextensible beam model presented in this work is useful and accurate approach for modeling thin root structures. However, many intruders such as reptilian limbs and developed plant roots have more complex geometries along with constitutive behavior that requires more sophisticated models. In order to be able to apply eRFT to model these types of intruders, future efforts include using eRFT as a surface traction boundary condition with finite element software to increase its applicability.

Motivated by its success in modeling granular intrusions efficiently, granular RFT has been expanded in recent years for a 3D formulation²², dynamic formulation¹⁷, a cohesive powder formulation⁴⁶. Further research will undertake establishing a cohesive granular RFT to model uprooting in cohesive soils.

Author contributions

[stuff]

Conflicts of interest

The authors declare no conflict of interest.

Data availability

[stuff]

Acknowledgements

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Appendix Material

6 Nonlinear Inextensible Beams

6.1 Governing Equations for Nonlinear Beams

The tangent and normal direction vectors along the beam can be defined from the position of the points along the beam. These vectors combined then define the curvature as a function of the axial coordinate s :

$$K = \frac{d\hat{\mathbf{t}}}{ds} \cdot \hat{\mathbf{n}} \quad (28)$$

For the constitutive relationship between the moment and curvature, we use a linear elastic relationship:

$$M = EI K \quad (29)$$

The equations of motion governing a nonlinear beam with linear mass λ subject to bending moments M , tension T , shear forces V , and external loading f can be derived by examining the free body diagram in figure ??

Balancing linear and angular momentum at axial coordinate s yields the three governing equations of motion:

1. Tangential direction equation of motion:

$$\frac{dT}{ds} + V K + f_t = \lambda \dot{\mathbf{v}} \cdot \hat{\mathbf{t}} \quad (30)$$

2. Normal direction equation of motion:

$$\frac{dV}{ds} + T K + f_n = \lambda \dot{\mathbf{v}} \cdot \hat{\mathbf{n}} \quad (31)$$

3. Balance of moments, relating shear forces to moments:

$$V = -\frac{dM}{ds} \quad (32)$$

These 3 equations govern the motion of the beam elements. These are similar to linear Euler-Bernoulli beam governing equations. We can see that for small deformations (which is the regime in which linear beam theory applies) the curvature can be approximated to be zero dropping the nonlinear terms in both equations 30 and 31.

6.1.0.1 Curvature Rate Based Damping The inextensible beams are able to capture the overall elastic response of tree roots. However, in reality the plant material also displays some time-dependent behavior^{47,48}. To mimic this effect as well as to damp out the oscillations of the roots in our simulations, we introduce a curvature-rate-based damping into our system of equations. This approach helps mimic the resistance of a root to being bent and the damping increases the quicker and more intense the change in shape occurs. We do so by augmenting the moment response in the beam in response to the curvature change rate. We modify equation 29 as the following to account for the curvature rate based damping, where η is the damping coefficient.

$$M = EI K + \eta \dot{K} \quad (33)$$

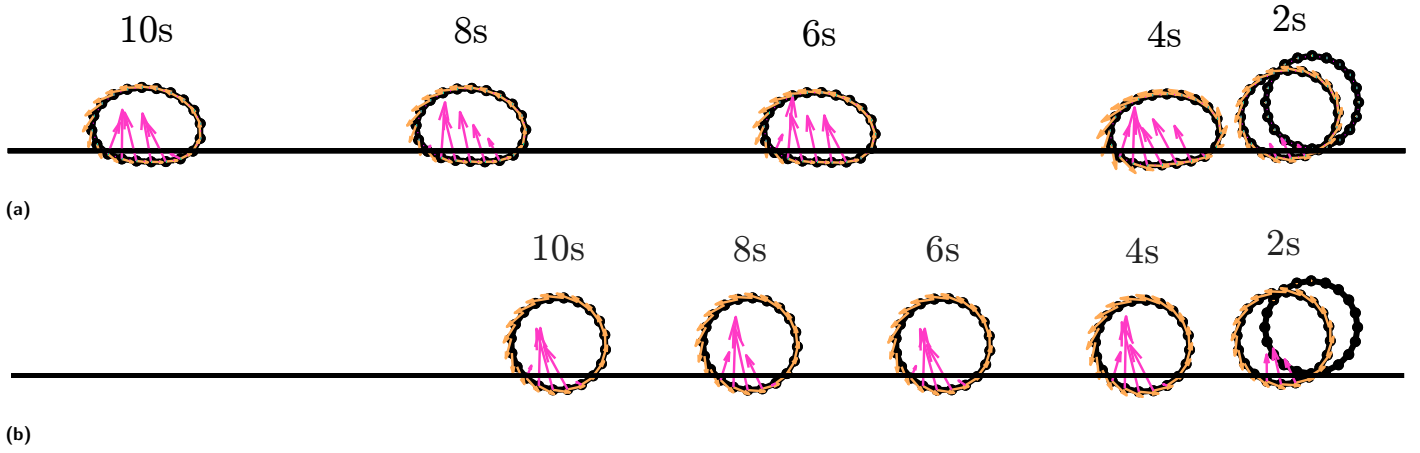


Fig. 18 Travel of soft (a, 1 MPa) and stiff (b, 10 MPa) wheels over time

Evaluating the time rate of change of curvature numerically requires us to store the past values of curvature. To avoid this and work with only the present values of variables we follow the steps outlined below. We can plug in the definition of the curvature along the beam as stated in equation 28.

$$\dot{K} = \frac{d}{dt} \left(\frac{d\hat{\mathbf{t}}}{ds} \cdot \hat{\mathbf{n}} \right) \quad (34)$$

When we apply the chain rule we obtain:

$$\left(\frac{d}{dt} \frac{d^2 \mathbf{x}}{ds^2} \right) \cdot \hat{\mathbf{n}} + \frac{d^2 \mathbf{x}}{ds^2} \cdot \frac{d\hat{\mathbf{n}}}{dt} \quad (35)$$

The first term scales with the rate of change of the curvature while the second term scales with the present curvature magnitude. If we rearrange the the order of the derivatives in the first term we get the second derivative of the normal velocity (equation 36). This can be evaluated in a single time step and is able to capture the resistance of the beam. This term is used in place of $\dot{K}$ in our beam code due to its computational ease. The second term can be accounted for by storing the face normal vectors between timesteps, however it has not been necessary to capture a realistic damping response thus far. This second term can be rearranged into the form in equation 36, which can be evaluated easily for all nodes by simply using the normal velocity values at the current timestep.

$$\frac{d^2 v_n}{ds^2} \quad (36)$$

6.2 Updating Beam Velocities While Satisfying Inextensibility: The Projection Method

[lale: reference this section in algorithm table from main body]
To solve for the velocities and positions of the nodes along the beam at a new timestep while also satisfying the inextensibility constraint, we make use of the projection method, a fractional step numerical method primarily developed and popularized by Chorin to reduce the difficulties of numerically solving the incompressible Navier-Stokes Equation for fluids⁴⁰.

We use the projection method to separate the calculations of ve-

locities and tensions for the beams representing the roots. At the start of a time-step, from the positions of the nodes we calculate the curvature 28. Once the curvature is known, the moments are defined following 29. Then the shear forces along the beam are defined from the derivative of the moments as 32. Then, knowing the old velocities, external line loads (due to resistive forces, gravity etc.), curvatures, and shear forces we can take a semi-explicit half step following equations 37, 38 to define mid-step star velocities, as is typical for the projection method.

$$v_t^* = (VK + f_t) \frac{dt}{\lambda} + v_t^{old} \quad (37)$$

$$v_n^* = \left(\frac{dV}{ds} + f_n \right) \frac{dt}{\lambda} + v_n^{old} \quad (38)$$

Using these star velocities, we solve the inextensibility constraint for the tensions. This is detailed in the following section. Once the tensions are known, the new velocities satisfying inextensibility are calculated using 39 & 40.

$$v_t^{new} = \left(\frac{dT}{ds} \right) \frac{dt}{\lambda} + v_t^* \quad (39)$$

$$v_n^{new} = TK \frac{dt}{\lambda} + v_n^* \quad (40)$$

Thus we are able to calculate the tensions and velocities in separate steps to satisfy the inextensibility constraint by making use of the projection method.

6.3 Calculations for Projection-Correction: Tensions Satisfying Inextensibility

In order to successfully implement the projection method, we need an expression of the inextensibility constraint and governing equations in terms of tension T . To do so we first take the derivative of equation 39 with respect to s and rearrange:

$$\frac{d^2 T}{ds^2} = \frac{\lambda}{dt} \left[\frac{d}{ds} (\mathbf{v}^{new} \cdot \mathbf{t} - \mathbf{v}^* \cdot \mathbf{t}) \right] \quad (41)$$

By applying the chain rule to the first term we obtain equation 42.

$$\frac{d^2T}{ds^2} = \frac{\lambda}{dt} \left[\frac{d\mathbf{v}^{new}}{ds} \cdot \mathbf{t} + \mathbf{v}^{new} \cdot \frac{d\mathbf{t}}{ds} - \frac{dv_t^*}{ds} \right] \quad (42)$$

Due to inextensibility [equation 27] the first term on the right hand side of equation 42 is zero. We can also write the second term as a product of components rather than a dot product. In equation 43 and the following, the subscripts t and n denote the components of the derivative of the tangent of the beam in the tangential and normal directions, respectively.

$$\frac{d^2T}{ds^2} = \frac{\lambda}{dt} \left[v_t^{new} \left(\frac{d\mathbf{t}}{ds} \right)_t + v_n^{new} \left(\frac{d\mathbf{t}}{ds} \right)_n - \frac{dv_t^*}{ds} \right] \quad (43)$$

Since we want to solve for the tension before we know the new tangential and normal velocities, we replace v_t^{new} and v_n^{new} with their expressions in equations 39 and 40 to substitute the star velocities and other quantities known before tension is calculated.

$$\frac{d^2T}{ds^2} = \frac{\lambda}{dt} \left[\left(\left(\frac{dT}{ds} \right) \frac{dt}{\lambda} + v_t^* \right) \left(\frac{d\mathbf{t}}{ds} \right)_t + \left(TK \frac{dt}{\lambda} + v_n^* \right) \left(\frac{d\mathbf{t}}{ds} \right)_n - \frac{dv_t^*}{ds} \right] \quad (44)$$

Rearranging we get a second order ordinary differential equation for T (equation 45) which can easily be numerically solved.

$$\frac{d^2T}{ds^2} - \frac{dT}{ds} \left(\frac{d\mathbf{t}}{ds} \right)_t - TK \left(\frac{d\mathbf{t}}{ds} \right)_n = \frac{\lambda}{dt} \left[v_t^* \left(\frac{d\mathbf{t}}{ds} \right)_t + v_n^* \left(\frac{d\mathbf{t}}{ds} \right)_n - \frac{dv_t^*}{ds} \right] \quad (45)$$

Combining equation 45 for T , equation 39 for v_t^{new} and equation 40 for v_n^{new} along with the boundary conditions, we can solve for the unknowns at all nodes.

6.4 Discrete Numerical Beams

So far, the derivation of the inextensible nonlinear beam theory has been a discussion based on a continuous beam with axial coordinate s . To numerically solve for the motion of these beams, we discretize them into nodes and elements.

The black squares denote the element centers while the orange circles denote the nodes. There are ghost elements at the free ends of the beams to help calculate certain derivatives at the nodes. The positions, velocities, curvatures, moments, and line forces are stored in and applied to the nodes. The tension T , shear force V are stored at the element centers. The ghost nodes have 0 shear force and typically have 0 tension (can be modified for force-controlled boundary conditions). Since the positions and velocities are stored at the nodes, the governing equations are solved for the nodes. Storing tensions and shear forces in the element centers allows for the derivative terms dT/ds and dV/ds to be centered at the nodes. For the tension and shear force terms that appear directly (paired with curvature K in the governing equations) we average the elemental values to the node centers. The existence of ghost elements makes the calculation of averages and centered finite difference derivatives more straightforward.

The tangents are first calculated for the elements from the positions of their 2 neighboring nodes. The normals are then defined for the elements via a 90 degree rotation of the tangent clockwise. Once the element tangent and normal vectors are defined,

they are averaged onto the nodes as nodal normal and tangential vectors.

Fig. 19 Inverted Y shaped branch made up of 3 combined beams. At the center orange node, kinematic constraint 1 applies. The angular constraint is achieved via blue massless rigid links

With all the equations governing the multi-branch structure, we assemble them into a linear system of equations which is solved at each time step as we iterate through time in our simulations. The linear system of equations is visualized in figure 20 where the unknowns are listed as $\mathbf{T}$ for tension, $\mathbf{f}_{top}$ for the force required at the top node to apply a velocity constraint, $\mathbf{f}_{rigid}$ for the forces in the rigid links satisfying the angular constraint between branches, and $\mathbf{f}_{conn}$ for connection forces at the center node between connecting branches ensuring they move together.

Fig. 20 Final form of the linear system of equations for a multi-branch root structure. Top node velocity controls are emitted for force controlled uprooting scenarios.

7 Other things in the appendix

7.1 Linear Noncodirectional Elastic RFT: Numerical Solution

Algorithm 4: process for numerical solution of non-codirectional linear elastic RFT intrusion problem

Given: $\Delta t, k_{RFT}, \mathbf{y}_{RFT}(\hat{\mathbf{v}}^p, \text{Set})$

Input: $\mathbf{t}^n, \mathbf{u}^n, \mathbf{v}^{n+1}, \mathbf{v}^{e,n}, \mathbf{v}^{p,n}$

Output: $\mathbf{t}^{n+1}, \mathbf{v}^{n+1}, \mathbf{v}^{p,n+1}$

begin

$\mathbf{v}^{e,n+1} = \mathbf{v}^{n+1};$

$\mathbf{t}^{trial} = \mathbf{t}^n + k_{RFT} \mathbf{v}^{n+1} \Delta t;$

if within yield locus then

$\mathbf{t}^{n+1} = \mathbf{t}^{trial};$

$\mathbf{v}^{p,n+1} = \mathbf{v}^{p,n};$

else

 outside locus, solve for $\mathbf{v}^{n+1} = \mathbf{v}^{p,n+1} + \mathbf{v}^{e,n+1};$

$\mathbf{t}^{n+1} = \mathbf{t}^n + k_{RFT} \mathbf{v}^{e,n+1} \Delta t = \mathbf{y}_{RFT}(\hat{\mathbf{v}}^{p,n+1}, \text{Set});$

end

end

7.2 Solving for the Plastic Velocity Direction for Linear Elastic Velocity

For linear noncodirectional elastic RFT, we can solve for the plastic velocity direction following the steps below. Assuming a fully elastic step, we update the resistive force as:

$$\mathbf{t}^{trial} = \mathbf{t}^{old} + k_{RFT} \mathbf{v}^{new} dt \quad (46)$$

If this trial force lies outside the yield locus, then it means that the grains are flowing and the trial traction must be returned back such that it lies on the yield locus at the new timestep. To perform this "return" step, we assume that the change in Set parameters, such as the face normal and depth, are negligible within the

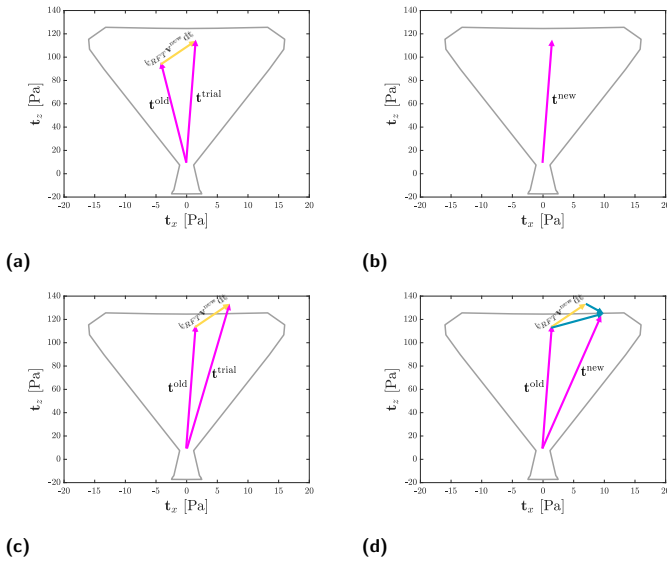


Fig. 21 Visual representation of the steps for the numerical implementation of the traction update through time and evaluation of the flow rule for linear noncodirectional elastic RFT

timestep.

$$\mathbf{t}^{new} = \mathbf{t}^{old} + k_{RFT}(\mathbf{v} - \mathbf{v}^p) dt = \mathbf{y}_{RFT}(\hat{\mathbf{v}}^p, \text{Set}) \quad (47)$$

Rearranging and decomposing equation 47, we obtain the following three equations.

$$\mathbf{t}^{trial} - \mathbf{y}_{RFT}(\hat{\mathbf{v}}^p, \text{Set}) = k_{RFT} \mathbf{v}^p dt \quad (48)$$

$$t_x^{trial} - \mathbf{y}_{RFT,x}(\hat{\mathbf{v}}^p, \text{Set}) = k_{RFT} |\mathbf{v}^p| \cos(\gamma^p - \pi) dt \quad (49)$$

$$t_y^{trial} - \mathbf{y}_{RFT,y}(\hat{\mathbf{v}}^p, \text{Set}) = k_{RFT} |\mathbf{v}^p| \sin(\gamma^p - \pi) dt \quad (50)$$

The $-\pi$ comes from the definition of γ in the RFT formulation. We can now divide the x-direction equation by the y-direction equation and rearrange to obtain the following:

$$(t_x^{trial} - \mathbf{y}_{RFT,x}(\mathbf{v}^p)) \sin(\gamma^p - \pi) = (t_y^{trial} - \mathbf{y}_{RFT,y}(\mathbf{v}^p)) \cos(\gamma^p - \pi) \quad (51)$$

We can then solve this by finding the intersections of these two curves which can be done using the `polyxpoly` command in MATLAB. Sometimes these curves have multiple intersections. In such cases the intersection that is closest to the trial traction is selected. If there are no intersections, the trial traction is accepted for that timestep which often results in a solution in the next iteration.

7.3 Solving for the Plastic Velocity Direction for Nonlinear Elastic Velocity and Proof of Codirectionality

For the nonlinear codirectional formulation of elastic RFT, the elastic traction is defined following equation 12. After the elastic trial step for this formulation, if the trial displacement leads to a traction outside the yield locus, it must be corrected by some correction factor $\mathbf{R}$ such that the new corrected elastic displacement at the end of the timestep yields the correct elastic traction which matches the yield traction due to the direction of plastic velocity

[equation 11].

$$\mathbf{u}^{e,new} = \mathbf{u}^{e,trial} - \mathbf{R} = \mathbf{u}^{e,trial} - \mathbf{u}^{e,old} + \mathbf{v}^{e,new} dt = \mathbf{u}^{e,trial} - \mathbf{v}^{p,new} dt \quad (52)$$

$$\mathbf{R} = \mathbf{v}^{p,new} dt \quad (53)$$

$$\mathbf{t}^{elastic}(\mathbf{u}^{e,new}) = \mathbf{y}_{RFT}(\hat{\mathbf{v}}^{p,new}) = \mathbf{y}_{RFT}\left(\frac{\hat{\mathbf{R}}}{dt}\right) \quad (54)$$

The elastic traction is defined following equation 12, such that its direction is already determined by the $\mathbf{y}_{RFT}$ function. Thus for the elastic and plastic traction to be equal to each other as in equation 54, the direction provided from the $\mathbf{y}_{RFT}$ function must be the same for both inputs $\hat{\mathbf{u}}^{e,new}$ and $\mathbf{v}^{p,new}$. For this to be true, the direction of both of these inputs must be the same since the yield traction function is one-to-one. The direction of the corrected elastic displacement must be along the direction of the updated plastic velocity. Both the trial and corrected elastic displacements are along the plastic velocity direction, earning this formulation of elastic RFT the "co-directional" title.

$$k_{RFT} |\mathbf{u}^{e,new}| \frac{\mathbf{y}_{RFT}(\hat{\mathbf{u}}^{e,new}, \text{Set})}{|\mathbf{y}_{RFT}(\hat{\mathbf{u}}^{e,new}, \text{Set})|} = \mathbf{y}_{RFT}(\hat{\mathbf{v}}^{p,new}, \text{Set}) \quad (55)$$

$$\mathbf{u}^{e,new} \parallel \mathbf{v}^{p,new} \quad (56)$$

$$\mathbf{u}^{e,trial} - \mathbf{R} \parallel \frac{\mathbf{R}}{dt} \quad (57)$$

$$\mathbf{u}^{e,trial} \parallel \mathbf{R} \quad (58)$$

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