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Authors
Ichiba, Tomoyuki
Pang, Guodong
Taqqu, Murad S

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Semimartingale properties of a generalized fractional Brownian motion and its mixtures with applications in finance

TOMOYUKI ICHIBA, GUODONG PANG, AND MURAD S. TAQQU

ABSTRACT. We study the semimartingale properties for the generalized fractional Brownian motion (GFBM) introduced by Pang and Taqqu (2019). We discuss the applications of the GFBM and its mixtures in financial models, including stock price models, arbitrage and rough volatility. The GFBM is self-similar and has non-stationary increments, whose Hurst parameter $H \in (0,1)$ is determined by two parameters. We identify the region of these two parameter values in which the GFBM is a semimartingale. We also establish the p-variation results of the GFBM, which are used to provide an alternative proof of the non-semimartingale property when $H < 1/2$. We then study the semimartingale properties of the mixed process of an independent Brownian motion and a GFBM when the Hurst parameter $H \in (1/2, 1)$, and derive the associated equivalent Brownian measure.

1. INTRODUCTION

Semimartingale and non-semimartingale properties of the standard fractional Brownian motion (FBM) B^H and its mixtures are well understood. These properties are important in modeling stock price [23, 34], constructing arbitrage strategies and hedging policies [33, 41, 37, 13], and modeling rough volatility [20, 7, 44]. The standard FBM B^H captures short/long-range dependence, and possesses the self-similar and stationary increment properties, as well as regular path properties. It may arise as the limit process of scaled random walks with long-range dependence or an integrated shot noise process [32].

A generalized fractional Brownian motion (GFBM) X, introduced by Pang and Taqqu [31], is a self-similar Gaussian process, but does not have the stationary increments property. See the definition of the process in (2.1). The Hurst parameter $H \in (0,1)$ is determined by two-parameters (α, γ) in the range shown in Figure 1. The GFBM X is derived as the limit of integrated power-law shot noise processes in [31] (see a brief review in Section 6.1). We have studied in [21] some important path properties of the GFBM X, including the Hölder continuity property, the differentiability and non-differentiability properties, and functional and local law of the iterated logarithm, see Section 2 for a summary of its fundamental properties.

Key words and phrases. Generalized fractional Brownian motion, semimartingale, p-variations, mixture of Brownian motion and GFBM, stock price model, arbitrage, rough volatility.

In this paper we focus on the semimartingale properties of the GFBM and its mixtures. We identify the regions of the two parameter values (α, γ) in which the GFBM X is a semimartingale (see the regions (I) and (II) in Figure 1 and Proposition 3.1). To do so, we first establish the necessary and sufficient condition for the square integrability of the derivative of the kernel of this Gaussian process. We use this to distinguish the parameter regions for the semimartingale property. We can then apply the characterization of the spectral representation of Gaussian semimartingales by Basse [5]. Another approach for establishing the semimartingale property is to study the so-called p-variations as was done for FBM B^H in [36, 12] and for bifractional Brownian motion $B^{H,K}$ in [35]. We also establish the p-variation ($p < 1/H, p = 1/H, p > 1/H$) results for the GFBM X (see Proposition 4.1) and use them to conclude the non-semimartingale property of the GFBM X when $H < 1/2$ (see Proposition 4.2). It is worth noting that the standard FBM B^H is a semimartingale if and only if $H = 1/2$ while the GFBM X can be a semimartingale for $H \in (1/2, 1)$ in certain regions of parameters of (α, γ), and in the very special case when the GFBM X becomes a BM (see Remark 3.1 and Figure 1 for more discussions, noting that there is a quadrilateral shape of (α, γ) that result in $H \in (1/2, 1)$, and there is a line segment of (α, γ) over $[0, 1]$ resulting $H = 1/2$ but only $\gamma = 0$ gives a BM).

We then study the semimartingale properties of the mixed GFBM process (sum of an independent BM and GFBM). It is shown in [11, 10, 8] that the mixed FBM B^H process is a semimartingale with respect to its own filtration if and only if $H \in \{1/2\} \cup (3/4, 1]$. We show in Proposition 5.1 that the mixed GFBM process is a semimartingale in the region of the two parameter values (α, γ) that is equivalent to $H \in (1/2, 1)$ (see region (III) in Figure 2), as well as in the very special case when GFBM is also a BM. It is also worth noting the wide range of values of the Hurst index H for which the mixed GFBM process is a semimartingale. We use the characterization of the equivalence of Gaussian measures in Shepp [40], with solutions to the associated Wiener–Hopf integral equations. For that purpose, we establish that $H \in (1/2, 1)$ is the necessary and sufficient condition for the second partial derivative function fo the covariance function of the GFBM X to be square integrable (see Lemma 5.1). We also conjecture that the mixed process is not a semimartingale when $H \in (0, 1/2)$ (see further discussions in Remark 5.1).

We then use the GFBM and its mixtures to financial models. The first application is stock price models. The price models we introduce in Section 6.2 generalize those using shot noise process and FBMs in the literature [23, 2, 38, 43]. When the mixed BM and GFBM processes are semimartingales, we derive the Radon–Nikodym derivative for the equivalent martingale measure (Proposition 6.1), which can then be used for the price dynamics of various options. In our framework, the larger ranges of the Hurst index thus provides greater flexibility in modeling and for further theoretical analysis through the use of the Itô's formula and the properties of semimartingales.

The second application is on the arbitrage in asset pricing. With FBM B^H, arbitrage in fractional Bachelier and Black-Scholes models has been well studied in [33, 41, 37, 13]. Rogers [33] pointed out that it is possible to construct a process similar to the FBM to model long-range dependence of returns while avoiding arbitrage. In particular, Rogers [33] proposed a process $\mathcal{X}(t)$:

$$\mathcal{X}(t) = \int_{-\infty}^{t} \varphi(t-s)\mathrm{d}B(s) - \int_{-\infty}^{0} \varphi(-s)\mathrm{d}B(s)\,,$$

and discussed conditions on the function $\varphi(\cdot)$ under which the process $\mathcal{X}$ remains as a semimartingale while exhibiting long–range dependence as FBM. One may extend the quest with a construction by considering an integrand of type $\tilde{\varphi}(t,s)$ such that it depends on both (t,s), instead of the difference $t-s$ as in $\varphi(t-s)$. The GFBM X in (2.1) provides an example of such a process with these desirable properties. We identify self-financing arbitrage strategies in the Bachelier and Black-Scholes models with the GFBM X for the region of parameters that results in $H \in (1/2, 1)$ in which the process X is not a semimartingale. We take the approach in Shirayaev [41]. See more discussions in Section 6.3.

The third application is on rough volatility models. There have been many activities in the study of rough volatility using FBM since the seminal research was initiated by Gatheral et al. [20], see the complete literature on [44]. We propose five models for the volatility process using

- the GFBM,
- the mixed GFBM process,
- the generalized fractional Ornstein-Ulenbeck (fOU) processes driven by the GFBM,
- the generalized fOU by the mixed GFBM process,
- Rough Bergomi model driven by the GFBM.

These models may be more advantageous since the GFBM and its mixture can be a semimartingale for a wide range of Hurst parameter values while possessing the long range dependence and roughness properties.

The paper is organized as follows. In Section 2, we give the precise definition of the GFBM and summarize some of its properties. In Sections 3 and 4, we study the semimartingale properties of the GFBM X. In Section 5, the semimartingale property of the mixed BM and GFBM process is investigated. We present the applications in financial models in Section 6.

2. A GENERALIZED FRACTIONAL BROWNIAN MOTION

A generalized fractional Brownian motion (GFBM) $X := \{X(t) : t \in \mathbb{R}_+\}$ is defined via the following (time-domain) integral representation:

$$\{X(t)\}_{t\in\mathbb{R}} \stackrel{\mathrm{d}}{=} \left\{ c\int_{\mathbb{R}} \left((t-u)_+^{\alpha} - (-u)_+^{\alpha}\right) |u|^{-\gamma/2} B(\mathrm{d}u) \right\}_{t\in\mathbb{R}}, \tag{2.1}$$

where

$$i = 1, 2, \quad \gamma \in [0, 1), \quad \alpha \in \Big(-\frac{1}{2} + \frac{\gamma}{2}, \, \frac{1}{2} + \frac{\gamma}{2} \Big) \,, \tag{2.2}$$

$c = c(\alpha, \gamma) \in \mathbb{R}_+$ is the normalization constant (as given in Lemma 2.1 of [21]), and $B(\mathrm{d}u)$ is a Gaussian random measure on $\mathbb{R}$ with the Lebesgue control measure $\mathrm{d}u$. Let

$$H := \alpha - \frac{\gamma}{2} + \frac{1}{2} \in (0, 1) \,.$$

The GFBM X is a H-self-similar process. It can be also written as

$$\{X(t)\}_{t \in \mathbb{R}} \stackrel{\mathrm{d}}{=} \bigg\{ c \int_{\mathbb{R}} \Big((t-u)_+^{H - \frac{1}{2} + \frac{\gamma}{2}} - (-u)_+^{H - \frac{1}{2} + \frac{\gamma}{2}} \Big) |u|^{-\gamma/2} B(\mathrm{d}u) \bigg\}_{t \in \mathbb{R}} \,. \tag{2.3}$$

It is clear that when $\gamma = 0$, this becomes the standard FBM B^H with Hurst parameter $H = \alpha + 1/2 \in (0, 1)$ (equivalently, $\alpha \in (-1/2, 1/2)$):

$$\{B^H(t)\}_{t \in \mathbb{R}} \stackrel{\mathrm{d}}{=} \bigg\{ c \int_{\mathbb{R}} \Big((t-u)_+^{H - \frac{1}{2}} - (-u)_+^{H - \frac{1}{2}} \Big) B(\mathrm{d}u) \bigg\}_{t \in \mathbb{R}} \,. \tag{2.4}$$

One may regard $\gamma \in (0, 1)$ as a scale/shift parameter. It appears that the component $|u|^{-\gamma/2}$ renders the paths rougher. On the one hand, it is somewhat surprising that the GFBM X has the Hölder continuity property with the same parameter $H - \epsilon$ for $\epsilon > 0$ as the FBM B^H. On the other hand, the parameter γ does affect the differentiability of the paths (see (vi) below).

The GFBM process X has the following properties:

(i) $X(0) = 0$ and $\mathbb{E}[X(t)] = 0$ for all $t \ge 0$;
(ii) X is a Gaussian process and $\mathbb{E}[X(t)^2] = t^H$ for $t \ge 0$;
(iii) X has continuous sample paths almost surely;
(iv) X is self-similar with Hurst parameter $H \in (0, 1)$;
(v) the paths of X are Hölder continuous with parameter $H - \epsilon$ for $\epsilon > 0$;
(vi) the paths of X is non-differentiable if $\alpha \in (0, 1/2]$ and differentiable if $\alpha > 1/2$ almost surely (see non-differentiable region (II) $\alpha \in (0, 1/2]$ and differentiable region (I) $\alpha > 1/2$ in Figure 1).

These properties are established in [31, 21]. See Proposition 5.1 [31] for properties (iii) and (iv), and Theorems 3.1 and 4.1 in [21] for (v) and (vi).

Recall that the FBM B^H has stationary increments. Namely, the second moment of its increment:

$$\mathbb{E}\big[(B^H(s) - B^H(t))^2\big] = c^2 |t - s|^{2H} \,,$$

and the covariance function

$$\mathbb{E}\big[B^H(s) B^H(t)\big] = \frac{1}{2} c^2 \big(t^{2H} + s^{2H} - |t - s|^{2H} \big) \,. \tag{2.5}$$

When $\gamma \in (0,1)$, in comparison with the FBM B^H, the process X loses the stationary increment property. In particular, the second moment of its increment is

$$\begin{aligned}\Phi(s,t) &:= \mathbb{E}\big[(X(t)-X(s))^2\big] \\ &= c^2 \int_{\mathbb{R}} \Big((t-u)_+^{\alpha} - (s-u)_+^{\alpha}\Big)^2 |u|^{-\gamma} \mathrm{d}u \\ &= c^2 \int_s^t (t-u)^{2\alpha} u^{-\gamma} \mathrm{d}u + c^2 \int_0^s ((t-u)^{\alpha} - (s-u)^{\alpha})^2 u^{-\gamma} \mathrm{d}u \\ &\quad + \int_0^{\infty} ((t+u)^{\alpha} - (s+u)^{\alpha})^2 u^{-\gamma} \mathrm{d}u\,, \end{aligned} \tag{2.6}$$

and the covariance function is

$$\begin{aligned}\Psi(s,t) &= \mathrm{Cov}(X(t),X(s)) = \mathbb{E}[X(s)X(t)] \\ &= c^2 \int_{\mathbb{R}} \Big(\big((t-u)_+^{\alpha} - (-u)_+^{\alpha}\big) \big((s-u)_+^{\alpha} - (-u)_+^{\alpha}\big) \Big) |u|^{-\gamma} \mathrm{d}u \\ &= c^2 \int_0^s (t-u)^{\alpha} (s-u)^{\alpha} u^{-\gamma} \mathrm{d}u \\ &\quad + c^2 \int_0^{\infty} ((t+u)^{\alpha} - u^{\alpha})((s+u)^{\alpha} - u^{\alpha}) u^{-\gamma} \mathrm{d}u\,, \end{aligned} \tag{2.7}$$

for $0 \le s \le t$.

In this paper we focus on the semimartingale properties associated with the process X and its mixtures. For FBM B^H, it is shown in [33] that B^H is not a semimartingale for $H \in (0,1/2) \cup (1/2,1)$. When $H = 1/2$, B^H is a Brownian motion, and is a martingale (thus, also a semimartingale). In [11], it is shown that the mixture of independent BM and FBM is a semimartingale for $H \in (3/4,1)$. This important finding is proved using a filtering approach in [10]. These results have significant implications in financial applications, in particular, arbitrage theory and pricing, see, e.g., [33, 13]. The proofs for these results rely heavily upon the stationary increments property of the FBM. The lack of stationary increments of the GFBM X requires new approaches to establish similar results. On the other hand, the GFBM X has two parameters α and γ, which provides more flexibility in the modeling. We will identify regions of (α,γ) in the domain $\{(\alpha,\gamma) \in \mathbb{R}^2 : \gamma \in (0,1),\ \alpha \in (-1/2+\gamma/2, 1/2+\gamma/2)\}$, in which the process X and its mixture with an independent BM are semimartingales.

We also remark that generalized FBMs are stated in a more general form with the additional terms involving $(t-u)_-^{\alpha} - (-u)_-^{\alpha}$ in the integrands in [31, Sections 5.1 and 5.2]. In this paper we focus on the representations of X in (2.1) since the other forms with additional terms can be treated similarly.

3. When is the GFBM a semimartingale?

Any centered Gaussian process X with right-continuous sample paths has a spectral representation in distribution [25], that is,

$$X(t) \stackrel{\mathrm{d}}{=} \int_{-\infty}^{t} K_t(s)\mathrm{d}N(s), \quad t \geq 0\,,$$

where N is an independently scattered centered Gaussian random measure and $(t, s) \to K_t(s)$ is a square-integrable deterministic function. Basse [5, Theorem 4.6] characterizes the spectral representation of Gaussian semimartingales, identifying the family of kernels $K_t(s)$ for which the representation $\big\{ \int_{-\infty}^{t} K_t(s)\mathrm{d}N(s) : t \geq 0 \big\}$ is a semimartingale with respect to the natural filtration $\{\mathcal{F}_t^N : t \geq 0\}$. Specifically for one-dimensional case $N(\cdot) \equiv B(\cdot)$ being the Brownian Gaussian random measure, it says that $\{X(t) : t \geq 0\}$ is an $\{\mathcal{F}_t^N : t \geq 0\}$ semimartingale if and only if for $t \geq 0$, the kernel can be represented as

$$K_t(s) \;=\; g(s) + \int_0^t \Psi_r(s)\mu(\mathrm{d}r)\,, \tag{3.1}$$

where $g : \mathbb{R}_+ \to \mathbb{R}$ is locally square integrable with respect to the Lebesgue measure, $\mu(\cdot)$ is a Radon measure on $\mathbb{R}_+$ and a measurable mapping $\Psi_r(s) : (r, s) \to \mathbb{R}$ is square integrable with respect to the Lebesgue measure, $\int_{-\infty}^{\infty} |\Psi_r(s)|^2\mathrm{d}s = 1$, and $\Psi_r(s) = 0$, $r < s$.

In the case of FBM B^H, we have the kernel function

$$K_t(s) = (t-s)_+^{H-1/2} - (-s)_+^{H-1/2}\,,$$

and $N(\cdot) = B(\cdot)$. Since it is a $\{\mathcal{F}_t^B : t \geq 0\}$ semimartingale if and only if $H = 1/2$, i.e., a Brownian motion, applying [5, Theorem 4.6], we have $g \equiv 0$, and $\Psi_r(s) \equiv 1$.

For the GFBM X, we have the kernel function

$$K_t(s) = \left[((t-s)_+)^{\alpha} - ((-s)_+)^{\alpha}\right] |s|^{-\gamma/2}\,, \tag{3.2}$$

and $N(\cdot) = B(\cdot)$.

Define the function $\Psi_t(\cdot)$ by

$$\Psi_t(s) \;:=\; C_t^{-1}\left[\alpha(t-s)^{\alpha-1}s^{-\gamma/2} \cdot \mathbf{1}_{\{0<s<t\}} + \alpha(t-s)^{\alpha-1}(-s)^{-\gamma/2} \cdot \mathbf{1}_{\{s<0\}}\right],$$

where C_t is a time-dependent, normalizing constant defined by

$$C_t := \alpha\, t^H \left(\mathrm{Beta}(1-\gamma, 2\alpha-1) + \mathrm{Beta}(1-\gamma, 1-2\alpha+\gamma)\right)^{1/2}, \quad t > 0\,.$$

Lemma 3.1. *The function $\Psi_t(\cdot)$ is square integrable with respect to the Lebesgue measure if and only if $1/2 < \alpha < (1+\gamma)/2$ and $\gamma \in (0,1)$ (region (I) in Figure 1). In this case, $\int_{-\infty}^{\infty} |\Psi_t(s)|^2\mathrm{d}s = 1$.*

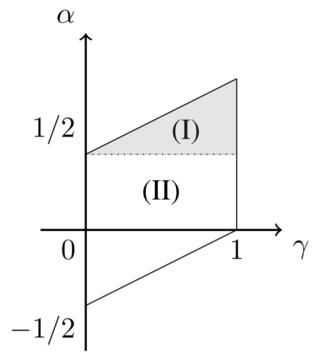

FIGURE 1. The parameter sets (γ, α) for semimartingale region (I) and for non-semimartingale region (II).

Proof. If $1/2 < \alpha < (1+\gamma)/2$ and $\gamma \in (0,1)$, it follows from the definition that

$$\alpha^{-2} C_r^2 \int_{-\infty}^{\infty} |\Psi_r(s)|^2 \mathrm{d}s = \int_0^r (r-s)^{2(\alpha-1)} s^{-\gamma} \mathrm{d}s + \int_{-\infty}^0 (r-s)^{2(\alpha-1)} (-s)^{-\gamma} \mathrm{d}s \,,$$

where the first and the second terms are rewritten by the change of variables as

$$\begin{aligned} \int_0^r (r-s)^{2(\alpha-1)} s^{-\gamma} \mathrm{d}s \,&=\, r^{2H} \int_0^1 (1-u)^{2(\alpha-1)} u^{-\gamma} \mathrm{d}u \,=\, r^{2H} \mathrm{Beta}(1-\gamma, 2\alpha-1) \,, \\ \int_{-\infty}^0 (r-s)^{2(\alpha-1)} (-s)^{-\gamma} \mathrm{d}s \,&=\, r^{2H} \int_0^{\infty} (1+u)^{2(\alpha-1)} u^{-\gamma} \mathrm{d}u \,=\, r^{2H} \mathrm{Beta}(1-\gamma, 1+\gamma-2\alpha) \,. \end{aligned} \tag{3.3}$$

Thus, in this case, $\int_{-\infty}^{\infty} |\Psi_t(s)|^2 \mathrm{d}s = 1$. On the other hand, if the conditions on the parameters α and γ are not satisfied, then these terms are not integrable. Thus we conclude the proof. □

Proposition 3.1. *The following properties hold:*

(i) *If $\gamma \in (0,1)$ and $1/2 < \alpha < (1+\gamma)/2$ (region (I) in Figure 1), then $\{X(t), t \ge 0\}$ in (2.1) is a semimartingale with respect to $\mathcal{F}^B(\cdot)$.*

(ii) *If $\gamma = 0$ and $\alpha \in (-1/2, 0) \cup (0, 1/2)$, $\{X(t), t \ge 0\}$ in (2.1) is reduced to a fractional Brownian motion and is not semimartingale. If $\gamma \in (0,1)$ and $\alpha \le 1/2$ (region (II) in Figure 1), then $\{X(t), t \ge 0\}$ in (2.1) is not a semimartingale with respect to $\mathcal{F}^B(\cdot)$.*

(iii) *Particularly, if $\gamma = 0$ and $\alpha = 0$, that is, $H = 1/2$, it is a Brownian motion and thus a semimartingale. However, for $\gamma = 2\alpha \in (0,1)$ with $H = 1/2$, it is not a semimartingale with respect to $\mathcal{F}^B(\cdot)$.*

Remark 3.1. In the range (region (I) of Figure 1) of parameters (α, γ): $\gamma \in (0,1)$ and $1/2 < \alpha < (1+\gamma)/2$, the Hurst parameter $H = \alpha - \frac{\gamma}{2} + \frac{1}{2} \in \left(\frac{1}{2}, 1\right)$. Note that when α is close to $1/2$, and γ is close to 1, the Hurst parameter H is also close to $1/2$, which differs from the standard FBM case with $\gamma = 0$ and $\alpha = 0$ resulting in $H = 1/2$. The range of values of Hurst parameter H possessing the semimartingale property is expanded from a single value $1/2$ for the standard FBM, to the half interval $[1/2, 1)$ for the process $X(t)$. (Note that $H = 1/2$ here only corresponds to the singular point $\alpha = 0, \gamma = 0$. The GFBM X can have $H = 1/2$ on the line segment $\alpha = \gamma/2$, which is a BM if and only if $\alpha = 0$). This provides more flexibility for stochastic integrals with respect to FBMs.

Remark 3.2 (Differentiability)**.** It is shown in [21] that the regions (I) and (II) of Figure 1 correspond to the regions of almost sure differentiable and non-differentiable paths, respectively, that is, in the region (I) the sample path of GFBM is differentiable, while in region (II) the sample path of GFBM is not differentiable. This is not just a coincidence but it turns out that when $\alpha > 1/2$, it is a semimartingale and its (local) martingale part in the semimartingale decomposition is zero, and its finite variation part is the integral of a Gaussian process. When $\alpha = \gamma = 0$, it is a Brownian motion with non-differentiable sample path.

Proof. (i) Let us consider the case $1/2 < \alpha < (1+\gamma)/2$ and $\gamma \in (0,1)$. Thanks to the integrability of (3.3) in this parameter set, the square integrability of $\Psi_t(\cdot)$ is assured, by Lemma 3.1, and hence, by Theorem 4.6 of Basse (2009), we have the representation

$$X(t) \,=\, \int_0^t \Big(\int_{-\infty}^t \Psi_r(s) \mathrm{d}B(s) \Big) \mu(\mathrm{d}r) \,, \quad t \ge 0 \,,$$

where $\mu(\mathrm{d}r) = \mathbf{1}_{\mathcal{C}_r} \, \mathrm{d}r$ is a Radon measure, $\mathcal{C}_r = (-\infty, r]$ and $\Psi_t(\cdot)$ is defined by

$$\Psi_t(s) \,:=\, \alpha (t-s)^{\alpha-1} s^{-\gamma/2} \cdot \mathbf{1}_{\{0<s<t\}} + \alpha(t-s)^{\alpha-1}(-s)^{-\gamma/2} \cdot \mathbf{1}_{\{s<0\}} \tag{3.4}$$

for $s < t$, and $\Psi_t(s) := 0$ for $s > t$. Thus it is a process of finite variation, in particular, it is a semimartingale.

(ii) When $\gamma = 0$, $X(\cdot)$ in (2.1) is a FBM with Hurst index $\alpha + 1/2$, and it is not a semimartingale if $\alpha \in (-1/2, 0) \cup (0, 1/2)$. Thus, let us consider the case $\gamma \in (0,1)$, $\alpha \le 1/2$ and show the claim by contradiction.

Suppose that $X(\cdot)$ in (2.1) is a semimartingale with respect to $\mathcal{F}^B(\cdot)$. We know $\mathbb{E}[X^2(t)] \,=\, t^{2\alpha-\gamma+1}$ for $t \ge 0$. Then by Theorem 4.6 of [5] with $N. \,=\, B(\cdot)$, $g(s) \,:=\, 0$, $C_t \,=\, (-\infty, t]$, there is a canonical decomposition

$$X(t) \,=\, \int_{-\infty}^t K_t(s) \mathrm{d}B(s) \,=\, \int_{-\infty}^t [((t-s)_+)^\alpha - ((-s)_+)^\alpha] \, |u|^{-\gamma/2} \mathrm{d}B(s)$$

for $t \ge 0$, where the integrand $K_t(s)$ has the form:

$$K_t(s) \,:=\, g(s) + \int_0^t \Psi_r(s) \mu(\mathrm{d}r) \,, \quad 0 \le s \le t \,. \tag{3.5}$$

Here $g(\cdot)$ is square integrable with respect to the Lebesgue measure, $\mu(\cdot)$ is a Radon measure on $\mathbb{R}_+$, and $\Psi_t(s)$ is a measurable mapping satisfying

$$\int_{-\infty}^\infty |\Psi_t(s)|^2 \mathrm{d}s \,=\, 1 \,, \quad \text{ and } \quad \Psi_t(s) \equiv 0 \;\; s(> t) \,. \tag{3.6}$$

Taking derivatives with respect to t in (3.5), we have

$$\frac{\mathrm{d}K_t(s)}{\mathrm{d}t} = \Psi_t(s) \cdot \frac{\mu(\mathrm{d}t)}{\mathrm{d}t},$$

while the integrand $K_t(s)$ in (3.2) has the derivative with respect to t:

$$\frac{\mathrm{d}K_t(s)}{\mathrm{d}t} = \alpha(t-s)^{\alpha-1}s^{-\gamma/2} \cdot \mathbf{1}_{\{0<s<t\}} + \alpha(t-s)^{\alpha-1}(-s)^{-\gamma/2} \cdot \mathbf{1}_{\{s<0\}}.$$

Thus by comparing these two expressions and by setting $\mu(\mathrm{d}r) = \mathbf{1}_{\mathcal{C}_r} \cdot \mathrm{d}r$ with $\mathcal{C}_r = (-\infty, r]$ as the Lebesgue measure, we identify $\Psi_t(s)$ as

$$\Psi_t(s) = \alpha(t-s)^{\alpha-1}s^{-\gamma/2} \cdot \mathbf{1}_{\{0<s<t\}} + \alpha(t-s)^{\alpha-1}(-s)^{-\gamma/2} \cdot \mathbf{1}_{\{s<0\}}$$

for $s < t$, and $\Psi_t(s) = 0$ for $s > t$. However, as in Lemma 3.1, if $\alpha \le 1/2$ and $\gamma \in (0,1)$, $\Psi_t(\cdot)$ is not square integrable for every $t > 0$. This yields a contradiction to (3.6). Thus, we claim that $X(\cdot)$ in (2.1) is not semimartingale with respect to $\mathcal{F}^B(\cdot)$, if $\gamma \in (0,1)$ and $\alpha \le 1/2$.

(iii) The standard Brownian motion case $H = 1/2$ is indeed a semimartingale. The second statement on the parameter sets $\gamma = 2\alpha \in (0,1)$ with $H = 1/2$ is proved as a special case of (ii). $\square$

4. VARIATIONS AND NON-SEMIMARTINGALE PROPERTY OF THE GFBM X WHEN $H < 1/2$

For the standard FBM B^H, it is shown in [36, Proposition 3.14] that B^H has a $1/H$-variation, that is,

$$\mathrm{ucp} - \lim_{\varepsilon \downarrow 0} \int_0^t \frac{1}{\varepsilon}|B^H(s+\varepsilon) - B^H(s)|^{1/H}\mathrm{d}s = \varrho_H t,$$

where $\varrho_H = \mathbb{E}[|Z|^{1/H}]$ for $Z \sim N(0,1)$. Here, the limit is in the sense of convergence in probability uniformly on every compact interval (ucp). It can be also shown that the classical variation

$$\sum_{i=0}^{n-1} |B^H_{t_{i+1}} - B^H_{t_i}|^{1/H} \xrightarrow[n\to\infty]{L^1} \varrho_H t,$$

where $0 = t_0 < \cdots < t_n = t$ is a partition of $[0,t]$, see Proposition 3.14 in [36] and Remark 1 in [35]. Then by Propositions 1.9 and 1.11 of [12] one can conclude that B^H is not a semimartingale with respect to $\mathcal{F}^B(\cdot)$, if $H < 1/2$. Note that the results in [12] involve the more general case of weak semimartingales.

This approach of evaluating the variations is also used in [35] to show that the bifractional Brownian motion $B^{H,K}$ with parameters (H,K), $H \in (0,1)$, $K \in (0,1]$ is not a semimartingale, if $HK \neq 1/2$, see Propositions 1–3 and Remark 1 there. Recall that the bifractional Brownian motion $B^{H,K}$ is a centered Gaussian process with $B^{H,K}(0) = 0$ and covariance function

$$\mathbb{E}[B^{H,K}(s)\, B^{H,K}(t)] = \frac{1}{2^K}((t^{2H} + s^{2H})^K - |t-s|^{2HK}), \quad s,t \ge 0.$$

The bifractional Brownian motion $B^{H,K}$ is a FBM with Hurst index $H \in (0,1)$ if $K = 1$.

We use this approach to establish the following properties of the GFBM X in (2.1).

Proposition 4.1. *Let*

$$\rho_{\alpha,\gamma} := (c^2 \,\mathrm{Beta}(1+2\alpha, 1-\gamma))^{1/(2H)} \sqrt{2^{1/H}/\pi}\,\Gamma((1+(1/H))/2)\,.$$

Then we have the convergence in L^1 *:*

$$\frac{1}{\varepsilon}\int_0^t |X(s+\varepsilon)-X(s)|^{1/H}\mathrm{d}s \xrightarrow[\varepsilon\downarrow 0]{L^1} \rho_{\alpha,\gamma}\, t\,, \quad t>0\,, \tag{4.1}$$

and similarly, for every partition $\pi : 0 = t_0 < \cdots < t_n = t$ *of* $[0,t]$ *with size* $|\pi| := \max_{1\le i\le n}|t_i - t_{i-1}|$*, we have*

$$\sum_{i=0}^{n-1}|X(t_{i+1})-X(t_i)|^{1/H} \xrightarrow[n\to\infty]{L^1} \rho_{\alpha,\gamma} t\,. \tag{4.2}$$

It is immediate that for $p < 1/H$, we have

$$\sum_{i=0}^{n-1}|X(t_{i+1})-X(t_i)|^{p} \xrightarrow[n\to\infty]{L^1} +\infty\,, \tag{4.3}$$

and for $p > 1/H$, we have

$$\sum_{i=0}^{n-1}|X(t_{i+1})-X(t_i)|^{p} \xrightarrow[n\to\infty]{L^1} 0\,. \tag{4.4}$$

As a consequence, by Propositions 1.9 and 1.11 of [12] we obtain the following property.

Proposition 4.2. *The process* X *is not a semimartingale with respect to* $\mathcal{F}^X(\cdot)$ *if* $H < 1/2$*, that is,* $\gamma \in (0,1)$ *and* $\alpha \in (-1/2+\gamma/2, 0)$*. Since* $\mathcal{F}^X \subset \mathcal{F}^B$*, it is not a semimartingale with respect to* $\mathcal{F}^B(\cdot)$ *if* $H < 1/2$ *(which is part of Proposition 3.1 (ii)).*

Proposition 4.1 is proved below. It is immediate to establish the following lemma.

Lemma 4.1. *For every* $0 < u < v$*, the limits of the covariance of increments at* u, v *are given by*

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon^{2H}}\,\mathbb{E}\big[(X(u+\varepsilon)-X(u))^2\big] = c^2\,\mathrm{Beta}(1+2\alpha,1-\gamma) > 0\,,\\
&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon^{2H}}\,\mathbb{E}\big[(X(u+\varepsilon)-X(u))(X(v+\varepsilon)-X(v))\big] = 0\,.
\end{aligned} \tag{4.5}$$

Proof. First, we consider the case $u = v$. In this case, since the expectation in (4.5) becomes the second moment $\Phi(u+\varepsilon, u)$ of the increment in (2.6), by a straightforward calculation (see, e.g., [21]) for $s < t$,

$$\begin{aligned}
\mathbb{E}[|X(t)-X(s)|^2] = c^2(t-s)^{2H}\Bigg(&\mathrm{Beta}(1+2\alpha,1-\gamma) + \int_{s/(t-s)}^{\infty}[(1+v)^\alpha - v^\alpha]^2\Big(v-\frac{s}{t-s}\Big)^{-\gamma}\mathrm{d}v\\
&+\Big[\int_0^1 (1-w)^{-\gamma}((1+xw)^\alpha-(xw)^\alpha)^2 x^{1-\gamma}\mathrm{d}w\Big]\Big|_{\{x=s/(t-s)\}}\Bigg)\,,
\end{aligned}$$

and hence, substituting $t = u+\varepsilon$ and $s = u$ and dividing by ε^{2H} on both sides, we obtain the first claim in (4.5). Indeed, since $\int_0^\infty[(1+v)^\alpha - v^\alpha]^2 v^{-\gamma}\mathrm{d}v < \infty$ under (2.2) with $\gamma\in(0,1)$, we have

$$\int_{u/\varepsilon}^{\infty}[(1+v)^\alpha - v^\alpha]^2\Big(v-\frac{u}{\varepsilon}\Big)^{-\gamma}\mathrm{d}v = \int_{u/\varepsilon}^{\infty}\Big[\Big(1+v-\frac{u}{\varepsilon}\Big)^\alpha - \Big(v-\frac{u}{\varepsilon}\Big)^\alpha\Big]^2 v^{-\gamma}\mathrm{d}v \xrightarrow[\varepsilon\downarrow 0]{} 0\,, \quad u>0\,,$$

for the second term, and for the third term, we apply the dominated convergence theorem to obtain

$$\int_0^1 (1-w)^{-\gamma}((1+xw)^\alpha - (xw)^\alpha)^2 x^{1-\gamma} \mathrm{d}w \xrightarrow[x\to\infty]{} 0\,,$$

because for every $x \ge 1$, by the monotonicity of $x \mapsto ((1+xw)^\alpha - (xw)^\alpha)x^{1-\gamma}$,

$$\int_0^1 (1-w)^{-\gamma}((1+xw)^\alpha - (xw)^\alpha)^2 x^{1-\gamma} \mathrm{d}w \;\le\; \int_0^1 (1-w)^{-\gamma}((1+w)^\alpha - w^\alpha)^2 \mathrm{d}w < \infty$$

in the case of $\alpha > 0$, and similarly for every $x \ge 1$,

$$\int_0^1 (1-w)^{-\gamma}((1+xw)^\alpha - (xw)^\alpha)^2 x^{1-\gamma} \mathrm{d}w \;\le\; \int_0^1 (1-w)^{-\gamma} w^{2\alpha} \mathrm{d}w \;=\; \mathrm{Beta}(1-\gamma, 1+2\alpha) < \infty$$

in the case of $\alpha < 0$. See the proof of Theorem 3.1 in [21] for the derivation of these upper bounds.

Next, we consider the case $u < v$ by evaluating the sum of four terms

$$\frac{1}{c^2}\mathbb{E}\big[(X(u+\varepsilon) - X(u))(X(v+\varepsilon) - X(v))\big] \;=\; C_{1,\varepsilon} + C_{2,\varepsilon} + C_{3,\varepsilon} + C_{4,\varepsilon}\,, \tag{4.6}$$

where

$$\begin{aligned}
C_{1,\varepsilon} \;&:=\; \int_0^u ((u+\varepsilon-w)^\alpha - (u-w)^\alpha)((v+\varepsilon-w)^\alpha - (v-w)^\alpha)|w|^{-\gamma}\mathrm{d}w\,,\\
C_{2,\varepsilon} \;&:=\; \int_u^{u+\varepsilon} (u+\varepsilon-w)^\alpha((v+\varepsilon-w)^\alpha - (v-w)^\alpha)|w|^{-\gamma}\mathrm{d}w\,,\\
C_{3,\varepsilon} \;&:=\; \int_0^1 ((u+\varepsilon+w)^\alpha - (u+w)^\alpha)((v+\varepsilon+w)^\alpha - (v+w)^\alpha)|w|^{-\gamma}\mathrm{d}w\,,\\
C_{4,\varepsilon} \;&:=\; \int_1^\infty ((u+\varepsilon+w)^\alpha - (u+w)^\alpha)((v+\varepsilon+w)^\alpha - (v+w)^\alpha)|w|^{-\gamma}\mathrm{d}w\,.
\end{aligned}$$

For the first term $C_{1,\varepsilon}$, we consider the case $\alpha > 0$ first. Note that a simple application of the Hölder continuity, i.e., $x^\alpha - y^\alpha \le |x-y|^\alpha$, $x, y > 0$ implies $C_{1,\varepsilon} \le \int_0^u \varepsilon^{2\alpha}|w|^{-\gamma}\mathrm{d}w \;=\; u^{1-\gamma}\varepsilon^{2\alpha}$ but then we may not use this inequality to show $\lim_{\varepsilon\downarrow 0} C_{1,\varepsilon}/\varepsilon^{2H} \;=\; 0$, because $2\alpha \le 2H \;=\; 2\alpha - \gamma + 1$. We shall use the estimates of the difference $x^\alpha - y^\alpha$ for $0 < y < x$ outside the neighborhood of the origin. If $\alpha > 0$, using the inequality

$$(v+\varepsilon-w)^\alpha - (v-w)^\alpha \;=\; (v-w)^\alpha\Big(\Big(1 + \frac{\varepsilon}{v-w}\Big)^\alpha - 1\Big) \;\le\; \alpha(v-u)^{\alpha-1}\varepsilon \tag{4.7}$$

for $0 < w < u < v$, we have

$$C_{1,\varepsilon} \;\le\; \alpha(v-u)^{\alpha-1}\varepsilon \cdot \int_0^u ((u+\varepsilon-w)^\alpha - (u-w)^\alpha)w^{-\gamma}\mathrm{d}w\,, \tag{4.8}$$

where the integral on the right hand is evaluated by

$$\begin{aligned}
&\int_0^u ((u+\varepsilon-w)^\alpha - (u-w)^\alpha)w^{-\gamma}\mathrm{d}w\\
&= \int_0^{u+\varepsilon} (u+\varepsilon-w)^\alpha w^{-\gamma}\mathrm{d}w - \int_0^u (u-w)^\alpha w^{-\gamma}\mathrm{d}w - \int_u^{u+\varepsilon} (u+\varepsilon-w)^\alpha w^{-\gamma}\mathrm{d}w\\
&= \big((u+\varepsilon)^{\alpha-\gamma+1} - u^{\alpha-\gamma+1}\big)\,\mathrm{Beta}(1+\alpha, 1-\gamma) - \int_u^{u+\varepsilon} (u+\varepsilon-w)^\alpha w^{-\gamma}\mathrm{d}w
\end{aligned}$$

$$\leq (\alpha-\gamma+1)u^{\alpha-\gamma}\text{Beta}(1+\alpha,1-\gamma)\varepsilon - \int_u^{u+\varepsilon}(u+\varepsilon-w)^{\alpha}w^{-\gamma}\mathrm{d}w\,, \tag{4.9}$$

and

$$0 \leq \int_u^{u+\varepsilon}(u+\varepsilon-w)^{\alpha}w^{-\gamma}\mathrm{d}w \leq u^{-\gamma}\int_u^{u+\varepsilon}(u+\varepsilon-w)^{\alpha}\mathrm{d}w \;=\; \frac{u^{-\gamma}}{1+\alpha}\cdot\varepsilon^{\alpha+1}\,.$$

Here, $\alpha-\gamma+1>0$. Thus, if $\alpha>0$, combining these inequalities together with (4.8), we obtain

$$\begin{aligned}
\frac{1}{\varepsilon^{2H}}C_{1,\varepsilon} \;&\leq\; \alpha(v-u)^{\alpha-1}\varepsilon^{1-2H}\Big((\alpha-\gamma+1)u^{\alpha-\gamma}\text{Beta}(1+\alpha,1-\gamma)\varepsilon + \frac{u^{-\gamma}}{1+\alpha}\cdot\varepsilon^{\alpha+1}\Big)\\
&=\; \alpha(v-u)^{\alpha-1}(\alpha-\gamma+1)u^{\alpha-\gamma}\text{Beta}(1+\alpha,1-\gamma)\varepsilon^{2(1-H)}\\
&\quad + \alpha(v-u)^{\alpha-1}\frac{u^{-\gamma}}{1+\alpha}\cdot\varepsilon^{2(1-H)+\alpha}\\
&\xrightarrow[\varepsilon\downarrow 0]{}\; 0\,,
\end{aligned}$$

because $0<H<1$ and $2(1-H)+\alpha \;=\; 1-\alpha+\gamma>0$.

Still for the first term $C_{1,\varepsilon}$, we consider the case $\alpha<0$. Let $\widetilde{\alpha} \;:=\; -\alpha>0$. Then using (4.7) with the power $\widetilde{\alpha}>0$, instead of the power α, we have

$$\big|(v+\varepsilon-w)^{\alpha}-(v-w)^{\alpha}\big| \;=\; \left|\frac{(v+\varepsilon-w)^{\widetilde{\alpha}}-(v-w)^{\widetilde{\alpha}}}{(v+\varepsilon-w)^{\widetilde{\alpha}}\,(v-w)^{\widetilde{\alpha}}}\right| \leq \left|\frac{\widetilde{\alpha}(v-u)^{\widetilde{\alpha}-1}}{(v+\varepsilon-w)^{\widetilde{\alpha}}\,(v-w)^{\widetilde{\alpha}}}\right|\varepsilon$$

for $0<w<u$. Using this inequality, we obtain

$$|C_{1,\varepsilon}| \leq \left|\frac{\widetilde{\alpha}(v-u)^{\widetilde{\alpha}-1}}{(v+\varepsilon-u)^{\widetilde{\alpha}}\,(v-u)^{\widetilde{\alpha}}}\right|\varepsilon\cdot\int_0^u|(u+\varepsilon-w)^{\alpha}-(u-w)^{\alpha}|w^{-\gamma}\mathrm{d}w\,,$$

where the integral on the right hand side is now evaluated by

$$\begin{aligned}
&\int_0^u|(u+\varepsilon-w)^{\alpha}-(u-w)^{\alpha}|w^{-\gamma}\mathrm{d}w\\
&= -\int_0^{u+\varepsilon}(u+\varepsilon-w)^{\alpha}w^{-\gamma}\mathrm{d}w + \int_0^u(u-w)^{\alpha}w^{-\gamma}\mathrm{d}w + \int_u^{u+\varepsilon}(u+\varepsilon-w)^{\alpha}w^{-\gamma}\mathrm{d}w\\
&\leq\; \Big|(u+\varepsilon)^{\alpha-\gamma+1}-u^{\alpha-\gamma+1}\Big|\,\text{Beta}(1+\alpha,1-\gamma) + \int_u^{u+\varepsilon}(u+\varepsilon-w)^{\alpha}w^{-\gamma}\mathrm{d}w\,.
\end{aligned}$$

Thus, from here, we may follow the same steps after the inequality (4.9) and we obtain $\lim_{\varepsilon\downarrow 0}\,|C_{1,\varepsilon}|\,/\,\varepsilon^{2H} \;=\; 0$ under the case $\alpha<0$ as well.

With similar reasoning, we obtain $\lim_{\varepsilon\downarrow 0}|C_{i,\varepsilon}|\,/\,\varepsilon^{2H} \;=\; 0$ for $i \;=\; 2,3,4$. Therefore, combining these estimate with (4.6) we conclude the second claim in (4.5). □

We are now ready to prove Proposition 4.1.

Proof of Proposition 4.1. Recall that if $\tilde{Z}$ is a normal random variable with mean 0 and variance $\sigma^2>0$, then $\mathbb{E}[|\tilde{Z}|^p] \;=\; \sigma^p\sqrt{2^p/\pi}\,\Gamma((1+p)/2)$ for $p\geq 1$ by the Gamma function formula. Thus, combining this fact with the consequence

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon^{2H}}\text{Var}(X(u+\varepsilon)-X(u)) = c^2\text{Beta}(1+2\alpha,1-\gamma)\,,$$

from (4.5), we obtain

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}\big[|X(u+\varepsilon)-X(u)|^{1/H}\big] = \big(c^2\mathrm{Beta}(1+2\alpha,1-\gamma)\big)^{1/(2H)}\cdot\sqrt{\frac{2^{1/H}}{\pi}}\Gamma\Big(\frac{1+(1/H)}{2}\Big) = \rho_{\alpha,\gamma}\,,$$

as the convergence of the moments of normal random variables. Thus it suffices to show

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\bigg[\Big(\frac{1}{\varepsilon}\int_0^t|X(s+\varepsilon)-X(s)|^{1/H}\mathrm{d}s\Big)^2\bigg] = (\rho_{\alpha,\gamma}t)^2\,,\quad t>0\,. \tag{4.10}$$

To show (4.10), we follow the proofs of Proposition 1 of [35] and Proposition 3.14 of [36], and use our Lemma 4.1. By the Fubini theorem, we rewrite

$$\mathbb{E}\bigg[\Big(\frac{1}{\varepsilon}\int_0^t|X(s+\varepsilon)-X(s)|^{1/H}\mathrm{d}s\Big)^2\bigg] = 2\int_{[0,t]^2}\mathbf{m}_\varepsilon\cdot\mathbf{1}_{\{u<v\}}\mathrm{d}u\mathrm{d}v\,, \tag{4.11}$$

where we define

$$\mathbf{m}_\varepsilon := \frac{1}{\varepsilon^2}\mathbb{E}\Big[|(X(u+\varepsilon)-X(u))(X(v+\varepsilon)-X(v))|^{1/H}\Big]\,,\quad \varepsilon>0\,. \tag{4.12}$$

The joint distribution of of the increments $X(u+\varepsilon)-X(u)$ and $X(v+\varepsilon)-X(v)$ is normal with zero mean and variances defined by

$$\sigma_{1,\varepsilon}^2 := \mathbb{E}\big[(X(u+\varepsilon)-X(u))^2\big] = \Phi(u+\varepsilon,u)\,,\quad \sigma_{2,\varepsilon}^2 := \mathbb{E}\big[(X(v+\varepsilon)-X(v))^2\big] = \Phi(v+\varepsilon,v)$$

and the covariance defined by

$$\vartheta_\varepsilon := \mathbb{E}\big[(X(u+\varepsilon)-X(u))(X(v+\varepsilon)-X(v))\big]\,.$$

By the the conditional distribution of $X(v+\varepsilon)-X(v)$, given $X(u+\varepsilon)-X(u)$, we compute $\mathbf{m}_\varepsilon$ in (4.12) as

$$\begin{aligned}\mathbf{m}_\varepsilon &= \frac{1}{\varepsilon^2}\mathbb{E}\Bigg[\big|\,\sigma_{1,\varepsilon}\cdot Z_1\big|^{1/H}\cdot\bigg|\frac{\vartheta_\varepsilon}{\sigma_{1,\varepsilon}}\cdot Z_1+\sqrt{1-\frac{{\vartheta_\varepsilon}^2}{\sigma_{1,\varepsilon}^2\,\sigma_{2,\varepsilon}^2}}\,\sigma_{2,\varepsilon}\,Z_2\bigg|^{1/H}\Bigg]\\ &= \mathbb{E}\Bigg[\big|\,Z_1\,\big|^{1/H}\cdot\bigg|\frac{\vartheta_\varepsilon}{\varepsilon^{2H}}\cdot Z_1+\sqrt{1-\frac{{\vartheta_\varepsilon}^2}{\sigma_{1,\varepsilon}^2\,\sigma_{2,\varepsilon}^2}}\,\frac{\sigma_{1,\varepsilon}\sigma_{2,\varepsilon}}{\varepsilon^{2H}}\,Z_2\bigg|^{1/H}\Bigg]\,.\end{aligned}$$

Here, Z_i, $i=1,2$ are i.i.d. standard normal random variables with mean 0 and variance 1. Then it follows from Lemma 4.1 that

$$\lim_{\varepsilon\downarrow 0}\sigma_{i,\varepsilon}^2\,/\,\varepsilon^{2H} = c^2\mathrm{Beta}(1+2\alpha,1-\gamma)\quad\text{for}\quad i=1,2\,,$$

and

$$\lim_{\varepsilon\downarrow 0}\frac{\vartheta_\varepsilon}{\varepsilon^{2H}} = 0\,,\quad \lim_{\varepsilon\downarrow 0}\frac{\vartheta_\varepsilon^2}{\sigma_{1,\varepsilon}^2\sigma_{2,\varepsilon}^2} = 0\,,\quad \lim_{\varepsilon\downarrow 0}\frac{\sigma_{1,\varepsilon}\sigma_{2,\varepsilon}}{\varepsilon^{2H}} = c^2\mathrm{Beta}(1+2\alpha,1-\gamma)\,.$$

Thus substituting them into (4.11), we obtain (4.10) and conclude (4.1). The proof of (4.2) is similar. $\square$

5. MIXED BM AND GFBM

In this section we consider the semimartingale properties of the following process

$$Y(t) = \widetilde{B}(t) + X(t), \quad t \geq 0, \tag{5.1}$$

where $\widetilde{B}(t)$ is a standard Brownian motion and $X(t)$ is the GFBM defined in (2.1), independent of $\widetilde{B}(t)$. Let us call Y the *mixed GFBM*.

In the case of FBM B^H, we denote

$$Y^H(t) = \widetilde{B}(t) + B^H(t), \quad H \in (0,1), \quad t \geq 0.$$

It is shown in [11, Theorem 1.7] and [10, Theorem 2.7] (see also [8]) that $Y^H(t)$ is a semimartingale with respect to its own filtration if and only if $H \in \{\frac{1}{2}\} \cup (\frac{3}{4}, 1]$. In [11], the concept of weak semimartingale and a theorem on Gaussian processes in [42] is used. On the other hand, in [10], the filtering approach is used. In particular, the mixed FBM Y^H is innovated by a martingale in its natural filtration for all $H \in (0,1]$. Then the equivalence property with respect to the Wiener measure is established for $H \in (3/4, 1]$ and the equivalence property with respect to the Wiener measure is established for $H \in (1/4)$. The associated Randon-Nikodym density formulas are then derived in these ranges of the parameter H.

Let $\mu^{Y,H}$ be the probability measure induced by Y^H on the space of its paths in $\mathbb{C}(\mathbb{R}_+;\mathbb{R})$, and μ^B be the Wiener measure. For $H > 1/2$, the covariance function of $B^H(t)$ in (2.5) is written as

$$\Psi^H(t,s) = \mathbb{E}\big[B^H(t)B^H(s)\big] = \int_0^t \int_0^s K^H(u,v)\mathrm{d}u\mathrm{d}v, \tag{5.2}$$

where

$$K^H(t,s) = \frac{\partial^2}{\partial t \partial s} E\big[B^H(s)B^H(t)\big] = c_H |t-s|^{2H-2}. \tag{5.3}$$

with $c_H := c^2 H(2H-1)$. If $H > 3/4$, $K^H(\cdot,\cdot) \in L^2([0,T]^2)$, and $\mu^{Y,H} \sim \mu^B$ (equivalence) by the general criterion in Shepp [40], and in addition, Shepp's Randon-Nikodym derivative can be written in the form

$$\frac{d\mu^{Y,H}}{d\mu^B}(Y^H) = \exp\left(- \int_0^T \varphi_t(Y^H)\mathrm{d}Y^H(t) - \frac{1}{2}\int_0^T \varphi_t^2(Y^H)\mathrm{d}Y^H(t) \right),$$

where $L^H \in L^2([0,T]^2)$ is the unique solution of the Wiener-Hopf integral equation

$$L^H(s,t) + c_H \int_0^t L^H(r,t)|r-s|^{2H-2}\mathrm{d}r = -c_H|s-t|^{2H}, \quad 0 \leq s \leq t \leq T,$$

and

$$\varphi_t(Y^H) := \int_0^t L^H(s,t)\mathrm{d}Y_s^H, \quad 0 \leq t \leq T. \tag{5.4}$$

The second partial derivative $K(u,v)$ of the covariance function Ψ in (2.7) is given by

$$K(u,v) = \frac{\partial^2 \Psi}{\partial u \partial v}(u,v) = c^2(f_1(u \wedge v, u \vee v) + f_2(u \wedge v, u \vee v)), \tag{5.5}$$

where

$$f_1(u,v) := \int_0^u (v-\theta)^{\alpha-1}(u-\theta)^{\alpha-1}\theta^{-\gamma}\mathrm{d}\theta\,,$$
$$f_2(u,v) := \int_0^\infty (v+\theta)^{\alpha-1}(u+\theta)^{\alpha-1}\theta^{-\gamma}\mathrm{d}\theta\,.$$

Lemma 5.1. *The function* $K(\cdot,\cdot)$ *in* (5.5) *is square integrable with respect to the Lebesgue measure in* $(0,T)\times(0,T)$ *for every* $T>0$, *if and only if* $\gamma/2<\alpha<1/2+\gamma/2$ *and* $0<\gamma<1$ *(equivalently,* $H\in(1/2,1)$, *region (III) in Figure 2).*

Proof. (i) Suppose that $\gamma/2<\alpha<1/2+\gamma/2$ and $0<\gamma<1$. Using the inequality $(x+y)^2\le 2(x^2+y^2)$ for $x,y>0$ and the symmetry of integral region, we obtain

$$\int_0^T\int_0^T\left[\frac{\partial^2\Psi}{\partial u\partial v}(u,v)\right]^2\mathrm{d}u\mathrm{d}v \le 4c^4\int_0^T\left(\int_0^v (f_1(u,v))^2+(f_2(u,v))^2\mathrm{d}u\right)\mathrm{d}v\,. \tag{5.6}$$

Here, the first term $f_1(u,v)$ is bounded by

$$f_1(u,v) \le \int_0^u v^{\alpha-1}(u-\theta)^{\alpha-1}\theta^{-\gamma}\mathrm{d}\theta = \mathrm{Beta}(\alpha,1-\gamma)\,u^{\alpha-\gamma}v^{\alpha-1},$$

for $u<v$. Since $0<\alpha<1$ and $(uw)^{\alpha-1}(vw)^{\alpha-1}\ge(1+uw)^{\alpha-1}(1+vw)^{\alpha-1}$ for $u,v>0$, we have a bound for the second term $f_2(u,v)$,

$$\begin{aligned}
f_2(u,v) &= \int_0^\infty (uv)^{\alpha-\gamma+1}(1+uw)^{\alpha-1}(1+vw)^{\alpha-1}w^{-\gamma}\mathrm{d}w\\
&\le (uv)^{2\alpha-\gamma}\int_1^\infty w^{2\alpha-2-\gamma}\mathrm{d}w + (uv)^{\alpha-\gamma+1}\int_0^1 (1+uw)^{\alpha-1}(1+vw)^{\alpha-1}w^{-\gamma}\mathrm{d}w\\
&\le \frac{(uv)^{2H-1}}{2-2H}+\frac{(uv)^{\alpha-\gamma+1}}{1-\gamma}\,.
\end{aligned}$$

Substituting these upper bounds of both terms and using again the inequality $(x+y)^2\le 2(x^2+y^2)$, $x,y>0$ for the second term, we obtain the estimates

$$\begin{aligned}
\int_0^T\int_0^v (f_1(u,v))^2\mathrm{d}u\mathrm{d}v &\le [\mathrm{Beta}(\alpha,1-\gamma)]^2\int_0^T\left(\int_0^v u^{2\alpha-2\gamma}v^{2\alpha-2}\mathrm{d}u\right)\mathrm{d}v\\
&= \frac{[\mathrm{Beta}(\alpha,1-\gamma)]^2\,T^{4\alpha-2\gamma}}{2(2\alpha-\gamma)(2\alpha-2\gamma+1)}<\infty\,,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^T\int_0^v (f_2(u,v))^2\mathrm{d}u\mathrm{d}v &\le \int_0^T\left[\int_0^v\left(\frac{(uv)^{2H-1}}{2-2H}+\frac{(uv)^{\alpha-\gamma+1}}{1-\gamma}\right)^2\mathrm{d}u\right]\mathrm{d}v\\
&\le 2\int_0^T\int_0^v\left(\frac{(uv)^{4H-2}}{(2-2H)^2}+\frac{(uv)^{2\alpha-2\gamma+2}}{(1-\gamma)^2}\right)\mathrm{d}u\mathrm{d}v\\
&= \frac{T^{8H+1}}{(2-2H)^2(4H-1)(4H+1)}+\frac{T^{4H-2\gamma+4}}{(1-\gamma)^2(4H-2\gamma+3)(2H-\gamma+2)}\\
&<\infty\,.
\end{aligned}$$

The right hand sides are finite when $2\alpha>\gamma$ and $0<\gamma<1$.

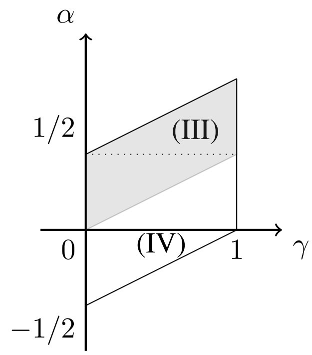

FIGURE 2. The parameter sets (γ, α) for the square integrable region (III) and for the non-square integrable region (IV) in Lemma 5.1.

Therefore, combining these estimates with (5.6), we conclude the second derivative $K(u,v)$ is square integrable in $(0,T)\times(0,T)$.

(ii) Suppose that $-1/2+\gamma/2<\alpha<\gamma/2$ and $0<\gamma<1$. Since $(x+y)^2\ge x^2$ for $x,y>0$, we shall show

$$\int_0^T\int_0^T [K(u,v)]^2 \mathrm{d}u\mathrm{d}v \,\ge\, 2\int_0^T\int_0^v [f_1(u,v)]^2\mathrm{d}u\mathrm{d}v \,=\, \infty\,.$$

To do so, by the change-of-variable and by Jensen's inequality, we observe that if $u\le v$,

$$\begin{aligned} f_1(u,v) &= u^{\alpha-\gamma}\int_0^1 (v-uw)^{\alpha-1}w^{-\gamma}(1-w)^{\alpha-1}\mathrm{d}w \\ &= u^{\alpha-\gamma}\mathrm{Beta}(\alpha,1-\gamma)\int_0^1 (v-uw)^{\alpha-1}\frac{w^{-\gamma}(1-w)^{\alpha-1}}{\mathrm{Beta}(\alpha,1-\gamma)}\mathrm{d}w \\ &\ge \mathrm{Beta}(\alpha,1-\gamma)\,u^{\alpha-\gamma}\left(v-u\cdot\frac{\alpha}{\alpha+1-\gamma}\right)^{\alpha-1}, \end{aligned}$$

because $w\mapsto(v-uw)^{\alpha-1}$, $0<w<1$ is a convex function and the expectation of Beta distribution with parameters $(\alpha,1-\gamma)$ is $\alpha/(\alpha+1-\gamma)$. Thus, we have a lower bound for $\int_0^v[f_1(u,v)]^2\mathrm{d}u$, that is,

$$\begin{aligned} \int_0^v [f_1(u,v)]^2\mathrm{d}u \,&\ge \int_0^v [\mathrm{Beta}(\alpha,1-\gamma)]^2\,u^{2(\alpha-\gamma)}\left(v-u\cdot\frac{\alpha}{\alpha+1-\gamma}\right)^{2(\alpha-1)}\mathrm{d}u \\ &= [\mathrm{Beta}(\alpha,1-\gamma)]^2 v^{4\alpha-2\gamma-1}\int_0^1\left(1-\theta\cdot\frac{\alpha}{\alpha+1-\gamma}\right)^{2\alpha-2}\theta^{2\alpha-2\gamma}\mathrm{d}\theta \\ &= [\mathrm{Beta}(\alpha,1-\gamma)]^2\cdot\left(\frac{1-\gamma}{\alpha+1-\gamma}\right)^{2\alpha-2}\frac{v^{4\alpha-2\gamma-1}}{2\alpha-2\gamma+1} \end{aligned}$$

for $0<v<T$. However, if $2\alpha<\gamma$, this lower bound is not integrable over $(0,T)$, and hence, $K(\cdot,\cdot)$ is not square integrable over $(0,T)\times(0,T)$. $\square$

Suppose that $\gamma/2 < \alpha < 1/2 + \gamma/2$ and $0 < \gamma < 1$. Let $L(s,t) \in L^2([0,T]^2)$ be the unique solution to the Wiener–Hopf integral equation

$$L(s,t) + \int_0^t L(r,t)K(r,s)\mathrm{d}r \,=\, -K(s,t)\,, \quad 0 \le s \le t \le T\,, \tag{5.7}$$

and define

$$\varphi_t(Y) \,:=\, \int_0^t L(s,t)\mathrm{d}Y(s)\,, \quad 0 \le t \le T\,. \tag{5.8}$$

Also, let $\ell(s,t) \in L^2([0,T]^2)$ be the unique solution to the Volterra equation

$$\ell(s,t) + \int_s^t \ell(r,t)L(s,r)\mathrm{d}r \,=\, L(s,t)\,, \quad 0 \le s \le t \le T\,. \tag{5.9}$$

As a direct consequence of [40] and Lemma 5.1, we obtain the following absolute continuity of $Y(\cdot)$ in (5.1) with respect to the Brownian motion $\widetilde{B}(\cdot)$.

Proposition 5.1. *Suppose that $\gamma/2 < \alpha < 1/2 + \gamma/2$ and $0 < \gamma < 1$, i.e., region (III) in Figure 2. The probability measure μ^Y induced by Y in (5.1) is absolutely continuous with respect to the Wiener measure $\mu^{\widetilde{B}}$ over $[0,T]$ with the Radon–Nikodym density*

$$\frac{\mathrm{d}\mu^Y}{\mathrm{d}\mu^{\widetilde{B}}}(Y) \,=\, \exp\bigg(- \int_0^T \varphi_t(Y)\mathrm{d}Y(t) - \frac{1}{2}\int_0^T [\varphi_t(Y)]^2\mathrm{d}t\bigg)\,.$$

By the Girsanov theorem

$$\overline{W}_t \,:=\, Y(t) + \int_0^t \varphi_s(Y)\mathrm{d}s \,=\, Y(t) + \int_0^t\int_0^s L(r,s)\mathrm{d}Y(r)\mathrm{d}s\,, \quad 0 \le t \le T \tag{5.10}$$

is a Brownian motion with respect to its own filtration. Moreover, $Y(t)$ can be written as

$$Y(t) \,=\, \overline{W}(t) - \int_0^t\int_0^s \ell(r,s)\mathrm{d}\overline{W}_r\mathrm{d}s\,, \quad 0 \le t \le T\,. \tag{5.11}$$

Particularly, the filtration $\mathcal{F}^Y(\cdot)$ generated by Y and the filtration $\mathcal{F}^{\widetilde{B}}(\cdot)$ satisfy the identities $\mathcal{F}^Y(t) = \mathcal{F}^{\widetilde{B}}(t)$ for $0 \le t \le T$. Therefore, $Y(t)$ is a semimartingale for the pair (α,γ) values in this region.

Remark 5.1. We conjecture that the mixture process Y is not a semimartingale with respect to its own filtration in the parameter region $\gamma/2 - 1/2 < \alpha \le \gamma/2$ and $\gamma \in (0,1)$ (region (IV) including the boundary line segment $\alpha = \gamma/2$, $\gamma \in (0,1)$ in Figure 2). The boundary point $\gamma = \alpha = 0$ represents the standard Brownian motion B^H with $H = 1/2$ (written as $B^{1/2}$ without confusion) and $Y = \widetilde{B} + B^{1/2}$ becomes a semimartingale. For standard FBM B^H, in Cai et al. [10], representations of the FBM with the Riemann–Liouville fractional integrals and derivatives are used to prove the innovation representations in Theorem 2.4 for $H < 1/2$, and equivalence of the measures for $\widetilde{B} + B^H$ and B^H for $H < 1/4$. However, for the GFBM X, it still remains open to establish the Riemann–Liouville fractional integrals and derivatives. Therefore, we leave it as future work to prove the non-semimartingale property of the mixture process Y for the parameter pair (α,γ) in region (IV) of Figure 2.

5.1. **Generalized Riemann–Liouville FBM and its Mixture.** In this section, we discuss the semimartingale properties of the generalized Riemann–Liouville (R-L) FBM and its mixtures. The generalized R-L FBM is introduced in Remark 5.1 in [31], and further studied in Section 2.2 in [21]. It is defined by

$$X(t) = c\int_0^t (t-u)^{\alpha} u^{-\gamma/2} B(\mathrm{d}u)\,, \quad t \ge 0\,, \tag{5.12}$$

where $B(\mathrm{d}u)$ is a Gaussian random measure on $\mathbb{R}$ with the Lebesgue control measure $\mathrm{d}u$ and $c \in \mathbb{R}$, $\gamma \in [0,1)$ and $\alpha \in (\gamma/2 - 1/2, \gamma/2 + 1/2)$. It is a continuous self-similar Gaussian process with Hurst parameter $H = \alpha - \gamma/2 + 1/2 \in (0,1)$. When $\gamma = 0$, it reduces to the standard R-L FBM

$$B^H(t) = c\int_0^t (t-u)^{\alpha} B(\mathrm{d}u), \quad t \ge 0\,.$$

It is clear that the semimartingale properties in Proposition 3.1 hold for the process X in (5.12). In particular, by letting the natural kernel $K_t(s) := (t-s)^{\alpha} s^{-\gamma/2}$, we have the spectral representation $X(t) \overset{\mathrm{d}}{=} \int_0^t K_t(s) N(\mathrm{d}s)$, for a Gaussian measure $N(\cdot)$. Define

$$\Psi_t(s) = C_t^{-1} \alpha (t-s)^{\alpha-1} s^{-\gamma/2}\,,$$

for a time-dependent normalization constant C_t. As shown in the proof of Lemma 3.1, the function $\Psi_1(\cdot)$ is square integrable with respect to the Lebesgue measure if and only if $1/2 < \alpha < 1/2 + \gamma/2$ and $\gamma \in (0,1)$, and thus by Basse's characterization of the spectral representation of Gaussian semimartingales ([5, Theorem 4.6]), we can conclude the the semimartingale property in part (ii) of Proposition 3.1. The non-semimartingale property in part (i) of Proposition 3.1 also follows from a similar argument as in the proof of the proposition. In addition, the variation properties in Proposition 4.1 also hold for the process X in (5.12), from which we can also conclude the non-semimartingale property as in the proof of Proposition 4.2.

We next discuss the mixed process $Y = \widetilde{B} + X$ with X in (5.12) and an independent BM $\widetilde{B}$. Let $L(s,t) \in L^2([0,T]^2)$ be the unique solution to the Wiener-Hopf integral equation (5.7) with

$$K(u,v) = c^2 \int_0^u (v-\theta)^{\alpha-1}(u-\theta)^{\alpha-1}\theta^{-\gamma}\mathrm{d}\theta\,,$$

and let $\ell(s,t) \in L^2([0,T]^2)$ be the unique solution to the Volterra equation (5.9). Then the properties in Proposition 5.1 hold, in particular, $Y = \widetilde{B} + X$ with X in (5.12) is a semimartingale with respect to the filtration generated by itself in the parameter range $\gamma/2 < \alpha < 1/2 + \gamma/2$ and $0 < \gamma < 1$.

6. Applications in Financial Models

6.1. **Shot noise process and integrated shot noise.** In modeling of financial markets, Brownian motion and compound Poisson processes or more generally, Lévy processes are widely utilized to capture effects of various noises in the financial markets. Recently, Pang and Taqqu [31] studied a non-stationary, power-law shot noise process $\mathcal{Z}^* = \{\mathcal{Z}^*(y) : y \in \mathbb{R}\}$ on the whole real line defined by

$$\mathcal{Z}^*(y) := \sum_{j=-\infty}^{\infty} g^*(y-\tau_j)R_j\,, \quad y \in \mathbb{R}\,, \tag{6.1}$$

where $\{\tau_j : j \in \mathbb{Z}\}$ is a sequence of Poisson arrival times of shots with rate λ on the whole real line, and each R_j is the noise associated with shot j at time τ_j for $j \in \mathbb{Z}$. It is a generalization of the compound Poisson process on the positive half line. The variables $\{R_j, j \in \mathbb{Z}\}$ are conditionally independent, given $\{\tau_j\}$, and the marginal distribution of R_j depends on the shot arrival time τ_j, that is, $\mathbb{P}(R_j \le r \,|\, \tau_j = u) =: F_u(r)$, $r \in \mathbb{R}$, $u \in \mathbb{R}$ for every $j \in \mathbb{Z}$. Assume that

(i) (the power-law property) the function g^* satisfies $g^*(y) = y^{-(1-\alpha)}L^*(y)$ for $y \ge 0$ and $g^*(y) = 0$ for $y < 0$ and $\alpha \in (0, 1/2)$, where L^* is a positive slowly varying function at $+\infty$, and

(ii) (the moment conditions) the common conditional distribution F_t of the noises $R_\cdot$, given $\tau_\cdot = t$, satisfies the zero mean $K_1(t) := \int_{\mathbb{R}} r \mathrm{d}F_t(r) = 0$ for every $u \in \mathbb{R}$ and finite variance $K_2(t) = \int_{\mathbb{R}} r^2 \mathrm{d}F_t(r) = t^{-\gamma}\tilde{L}_+(t)$ for $t > 0$ and $K_2(t) = |t|^{-\gamma}\tilde{L}_-(t)$ for $t < 0$, where $\gamma \in (0,1)$, and $\tilde{L}_\pm$ are some positive slowly varying functions.

The shot noise process $\mathcal{Z}^*$ in (6.1) is a generalization of the compound Poisson process, because if $g^*(y) := \mathbf{1}_{\{y>0\}}$, $y \in \mathbb{R}$, then $\mathcal{Z}^*$ is a compound process. The integrated shot noise process $\mathcal{Z} = \{\mathcal{Z}(t) : t \in \mathbb{R}_+\}$ is defined by

$$\mathcal{Z}(t) := \int_0^t \mathcal{Z}^*(y)\mathrm{d}y \;=\; \sum_{j=-\infty}^{\infty} (g(t-\tau_j) - g(-\tau_j))R_j\,, \quad t \ge 0\,, \tag{6.2}$$

where the shot shape function $g(\cdot)$ is differentiable with its derivative g^*, i.e., $g(t) := \int_0^t g^*(y)\mathrm{d}y$ for $t \ge 0$.

6.2. **Stock price models with the (mixed) GFBM.** Stock pricing models with a shot-noice component have been developed to study credit and insurance risks [2, 38, 43]. In particular, the stock price $P(t)$ is modeled as

$$P(t) := P(0)\exp\left(\Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma\widetilde{B}(t) + \sigma\int_0^t \sum_{\tau_i \le s} \mathfrak{f}(s-\tau_i, R_i)\mathrm{d}s\right), \quad t \ge 0\,, \tag{6.3}$$

where $\{(\tau_i, R_i) : i \in \mathbb{N}\}$ is a marked point process, independent of the Brownian motion $\widetilde{B}$, with arrival times τ_i and marks (noises) $U_i \in \mathbb{R}^d$, and the function $\mathfrak{f} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ is the deterministic shot shape function. $\mu \in \mathbb{R}$ and $\sigma > 0$ are some real constants. Equivalent martingale measures for this price process are studied in [38, 39]. In [38], it is also discussed when the shot-noise component is Markovian or a semimartingale. This is usually when the function $\mathfrak{f}$ takes a particular form (exponential function for the Markovian property). In these studies, the noises $\{R_i\}$ are assumed to be i.i.d. with finite variance.

Since it is usually more difficult to work with the shot noise process directly, one may use the diffusion approximations. For example, Klüppelberg and Kühn [23] showed that under regular variation conditions, a Poisson shot noise process can be approximated by an FBM (under proper scaling and validated by a functional central limit theorem), and then used the limiting FBM as a stock pricing model.

Here, as a pre-limit, we consider the usual random walk noise and the shock noises on the stock price. We evaluate the effects of these noises, when the frequency of arrival of shot noises is very high with appropriate scaling. Given a scaling parameter $\varepsilon > 0$, we model a pre-limit of price process by

$$\begin{aligned} P_\varepsilon(t) &:= P(0)\exp\bigg(\Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma\varepsilon^{1/2}\sum_{j=1}^{\lfloor t/\varepsilon\rfloor}\xi_j + \sigma\int_0^t \frac{1}{\varepsilon^{1-H}}\mathcal{Z}^*\Big(\frac{u}{\varepsilon}\Big)\mathrm{d}u\bigg) \\ &= P(0)\exp\bigg(\Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma\varepsilon^{1/2}\sum_{j=1}^{\lfloor t/\varepsilon\rfloor}\xi_j + \sigma\varepsilon^{H}\int_0^{t/\varepsilon}\mathcal{Z}^*(u)\mathrm{d}u\bigg) \\ &= P(0)\exp\bigg(\Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma\varepsilon^{1/2}\sum_{j=1}^{\lfloor t/\varepsilon\rfloor}\xi_j + \sigma\varepsilon^{H}\mathcal{Z}\Big(\frac{t}{\varepsilon}\Big)\bigg), \quad t \ge 0, \end{aligned} \tag{6.4}$$

where $\{\xi_j, j \in \mathbb{N}\}$ are i.i.d. random variables with zero mean and unit variance, independent of the shot noise $\mathcal{Z}^*$ in (6.1), and $\mathcal{Z}$ is the integrated shot noise process in (6.2). Here, μ and $\sigma > 0$ are some real constants. (One may also choose a model without the random walk component, in which case the model in (6.5) will have only the process X instead of the mixed GFBM.)

Under certain regularity conditions and with a proper scaling, Pang and Taqqu [31] have shown that the scaled process $\widehat{\mathcal{Z}}^\varepsilon(t) := \varepsilon^H\mathcal{Z}(t/\varepsilon)$ converge weakly to the GFBM X, as $\varepsilon \to 0$. The random walk term $\varepsilon^{1/2}\sum_{j=1}^{\lfloor t/\varepsilon\rfloor}\xi_j$, $t \ge 0$ converges weakly to the standard BM, independent of X.

Suppose that the parameters (α, γ) are in the semimartingale region: $\gamma/2 < \alpha < 1/2 + \gamma/2$ and $0 < \gamma < 1$ (i.e., $H \in (1/2, 1)$), as in the assumption of Proposition 5.1. As a scaling limit of (6.4), we propose a stock price model using the mixed GFBM as follows:

$$\begin{aligned} P(t) &= P(0)\exp\Big(\Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma(\widetilde{B}(t) + X(t))\Big) \\ &= P(0)\exp\Big(\Big(\mu - \frac{\sigma^2}{2}\Big)t + \sigma Y(t)\Big), \quad t \ge 0, \end{aligned} \tag{6.5}$$

where $Y(\cdot) = \widetilde{B} + X$ is the mixed GFBM in (5.1). (For a recent account of weak convergence in financial models, we refer to Kreps [24].) Under the above parameter range, Y is a semimartingale. The price dynamics is determined as the unique strong solution of the linear stochastic differential equation

$$\mathrm{d}P(t) = P(t)(\mu\mathrm{d}t + \sigma\mathrm{d}Y(t)); \quad t \ge 0, \tag{6.6}$$

driven by the semimartingale Y, where μ is a drift and σ is volatility of stock price under a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$.

As an example with this stock price model (6.5), we consider an investor who trades this stock with price (6.5) and money market account with an instantaneous interest rate $r(>0)$. Recall that in the case of standard FBM B^H, as shown in [33, 11, 10], the mixed process $Y^H = \widetilde{B} + B^H$ is a semimartingale if and only if $H = 1/2$ (the Brownian case) and $H \in (3/4, 1)$. Of course, with a BM, i.e., $H = 1/2$, the standard results of stock pricing and equivalence of martingale measure can be applied. On the other hand, with $H \in (3/4, 1)$, we also obtain the Radon–Nikodym derivative in (6.7) where Y is replaced by Y^H, and the function $\varphi_t(Y)$ is replaced by $\varphi_t(Y^H)$ in (5.4). Also recall that for the GFBM X with $H = 1/2$, the parameters (α, γ) lies on the line segment $\alpha = \gamma/2$. It is only a semimartingale when $\gamma = 0$, which becomes the special case of Brownian pricing model; otherwise, there does not exist an equivalent martingale measure.

Proposition 6.1. *Assume $\gamma/2 < \alpha < 1/2 + \gamma/2$ and $0 < \gamma < 1$ (i.e., $H \in (1/2, 1)$). Under the stock price process dynamics (6.6), the discounted stock price process $e^{-rt}P(t)$, $0 \le t \le T$ is a martingale under the new measure $\mathbb{Q}$ defined by*

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_T} := \exp\bigg(- \int_0^T (\theta - \varphi_t(Y))\mathrm{d}Y(t) - \frac{1}{2}\int_0^T (\theta^2 - |\varphi_t(Y)|^2)\mathrm{d}t \bigg), \tag{6.7}$$

where $\theta := (\mu - r)/\sigma$ is the market price of risk and $\varphi_\cdot(Y)$ is defined in (5.8).

Proof. It follows from Proposition 5.1 and the Girsanov theorem that the process $\overline{W}(\cdot) = Y(\cdot) + \int_0^\cdot \varphi_s(Y)\mathrm{d}s$ in (5.10) is a Brownian motion for $0 \le t \le T$ under $\mathbb{P}$.

By the simple application of the product rule to (6.6), we have the discounted stock price process

$$\begin{aligned} e^{-rt}P(t) &= P(0) + \int_0^t \sigma e^{-rs}P(s)\mathrm{d}\Big(Y(s) + \frac{\mu - r}{\sigma}s\Big) \\ &= P(0) + \int_0^t \sigma e^{-rs}P(s)\mathrm{d}\Big(\overline{W}(s) - \int_0^s \varphi_u(Y)\mathrm{d}u + \theta s\Big), \quad t \ge 0, \end{aligned}$$

with $\theta := (\mu - r)/\sigma$. By another application of the Girsanov theorem, $Y(t) + \theta t$, $0 \le t \le T$ is a Brownian motion under the measure $\mathbb{Q}$. In particular, the discounted price process $e^{-rt}P(t)$, $0 \le t \le T$ is a martingale under $\mathbb{Q}$. □

Consequently, the time-t price of European option on this stock with payoff function $\mathfrak{g}$ and with maturity T is given by

$$\mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}\,\mathfrak{g}(P(T))\,|\,\mathcal{F}_t] = \mathbb{E}^{\mathbb{P}}\bigg[e^{-r(T-t)}\,\mathfrak{g}(P(T)) \cdot \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_T}\,\bigg|\mathcal{F}_t\bigg],$$

where the conditional expectations $\mathbb{E}^{\mathbb{Q}}$ and $\mathbb{E}^{\mathbb{P}}$ are calculated under $\mathbb{Q}$ and $\mathbb{P}$, respectively, given the filtration $\{\mathcal{F}_t\}_{t \ge 0}$. Thus, under the stock price model with the mixed GFBM Y and with the parameter sets $\{\gamma/2 < \alpha < 1/2 + \gamma/2, 0 < \gamma < 1\}$ or $\{\gamma = 0, 1/4 < \alpha < 1/2\}$, the pricing problems of various options (such as American and Asian options) are solved in the same way as in the standard Black-Scholes model.

6.3. **Arbitrage.** In the theory of asset pricing, the "First Fundamental Asset Pricing Theorem" requires the existence of an equivalent martingale measure for no arbitrage, and works in the framework of semimartingales for pricing models. For stock price models with FBM B^H, since B^H is a semimartingale if and only if $H = 1/2$ [26, 33], arbitrage strategies have been discussed in both fractional Bachelier and Black-Scholes models [33, 41, 37, 13]. In particular, Rogers [33] constructed arbitrage for the fractional Bachelier model: a market with a money account $\xi_t = 1$ with zero interest rate and a risky stock (no dividends or transaction costs) with price $\tilde{P}(t) = \tilde{P}(0) + \nu t + \sigma B^H(t)$, for $H \in (0, 1/2) \cup (1/2, 1)$ for $t \geq 0$.

As mentioned in the introduction, the GFBM X in (2.1) is an example for the desirable process Rogers [33] has quested. In particular, as shown in Proposition 3.1, the process X in (2.1) is a semimartingale with respect to $\mathcal{F}^B(\cdot)$ for $\alpha \in (1/2, 1/2 + \gamma/2)$ and $\gamma \in (0,1)$, which exhibits long-range dependence since $H \in (1/2, 1)$ in this parameter range. In the meantime, for $\alpha \in (\gamma/2, 1/2)$ and $\gamma \in (0,1)$, the process X also has $H \in (1/2, 1)$, but it is not a semimartingale.

Therefore, for $\alpha \in (1/2, 1/2+\gamma/2)$ and $\gamma \in (0,1)$ (resulting in $H \in (1/2,1)$), an equivalent martingale measure can be constructed and there is no arbitrage opportunity. For $\alpha \in (\gamma/2, 1/2)$ and $\gamma \in (0,1)$ (also resulting in $H \in (1/2,1)$), we construct the following arbitrage strategy using an approach similar that of Shiryaev [41] since the p-variation of the process X is finite for $p > 1/H$ for $H \in (1/2,1)$ and well defined, so that pathwise Riemann–Stieltjes stochastic integral with respect to X is well defined [41, 37]. (See further discussions on stochastic integrals with respect to the GFBM X in Section 6.5.) Indeed, replacing the FBM B^H in [41] by the GFBM X in (2.1), we find that the self-financing, admissible strategy $\boldsymbol{\pi} := (\boldsymbol{\beta}, \boldsymbol{\gamma})$ in the Bachelier model $\widetilde{P}(t) = \widetilde{P}_0 + X(t)$, $t \geq 0$ with a risk free bond of zero interest rate yields the portfolio value

$$\mathcal{V}_t^{\pi} = \boldsymbol{\beta}_t + \boldsymbol{\gamma}_t \widetilde{P}(t) = \mathcal{V}_0^{\pi} + \int_0^t \boldsymbol{\gamma}_u \mathrm{d}\widetilde{P}(u), \quad t \geq 0, \tag{6.8}$$

where the stochastic integral is understood as the pathwise Riemann–Stieltjes integral with respect to X, and then with $\widetilde{P}_0 := 1$, $\boldsymbol{\gamma}_t := 2X(t)$, $\boldsymbol{\beta}_t := -X(t)^2 - 2X(t)$, $t \geq 0$, the resulting portfolio value satisfies $\mathbb{P}(\mathcal{V}_0^{\pi} = 0, \mathcal{V}_t^{\pi} > 0) = 1$ for every $t > 0$, because $\mathcal{V}_t^{\pi} = \boldsymbol{\beta}_t + \boldsymbol{\gamma}_t \widetilde{P}(t) = X(t)^2$, $t \geq 0$.

Similarly, in the fractional Black-Scholes model, which is a market with a money market account $\xi_t = e^{rt}$ and a risky stock (no dividends or transaction costs) $\widetilde{P}(t) = \widetilde{P}_0 \exp\big(rt + B^H(t)\big)$, Shiryaev [41] constructed a self-finance arbitrage strategy for $H \in (1/2, 1)$. For the GFBM X, with $\widetilde{P}(t) = \widetilde{P}_0 \exp\big(rt + X(t)\big)$, we can analogously choose the portfolio $\boldsymbol{\beta}_t = 1 - e^{2X(t)}$ and $\boldsymbol{\gamma}_t = 2(e^{X(t)} - 1)$ for a self-financing arbitrage opportunity with the portfolio value process $\mathcal{V}_t^\pi$, $t \geq 0$, defined by

$$\mathcal{V}_t^\pi := \boldsymbol{\beta}_t e^{rt} + \boldsymbol{\gamma}_t \widetilde{P}(t) = e^{rt}(e^{X(t)} - 1)^2 = \int_0^t \boldsymbol{\beta}_s r \mathrm{d}s + \boldsymbol{\gamma}_s \mathrm{d}\widetilde{P}(s) , \quad t \geq 0 . \tag{6.9}$$

However, for the mixed process Y in (5.1), by Proposition 5.1, for $\gamma/2 < \alpha < 1/2 + \gamma/2$ and $0 < \gamma < 1$ (equivalently, $H \in (1/2, 1)$ for the GFBM X), it is a semimartingale, and thus, there is no arbitrage opportunity in the associated fractional Bachelier and Black-Scholes models.

On the other hand, for $\alpha \in (\gamma/2 - 1/2, \gamma/2)$ and $\gamma \in (0, 1)$ (resulting in $H \in (0, 1/2)$), we cannot use the approach in Shiryaev [41] to construct arbitrage strategies. It is potential to construct arbitrage strategies by extending the approaches in [13], which we leave as future work.

6.4. **Rough Fractional Stochastic Volatility.** In the seminal paper Gatheral et al. [20] conducted empirical study on stochastic volatility and discovered that the log-volatility behaves like a FBM with Hurst exponent $H < 1/2$ (mostly between 0.08 and 0.2), and thus proposed a "rough" volatility model. For the recent developments in the empirical studies of the Hurst parameter of financial data and the microstructure of leverage effects, see also [1, 30, 18] and papers listed on the webpage [44]. In particular, for a given asset with log-price taking the form

$$\frac{\mathrm{d}P(t)}{P(t)} = \mu(t)\mathrm{d}t + \sigma(t)\mathrm{d}W(t) , \quad t \geq 0 ,$$

where $\mu(t)$ is a drift term and $W(t)$ is a one-dimensional BM, the stochastic volatility $\sigma(t)$ is modeled as

$$\sigma(t) = \sigma(0) \exp(Z(t)) , \quad t \geq 0 ,$$

where the process $Z(t)$ is a stationary fractional Ornsten-Uhlenbeck (fOU) process, given by the stationary solution of the SDE:

$$dZ(t) = -a(Z(t) - m)\mathrm{d}t + \nu \mathrm{d}B^H(t)$$

with $m \in \mathbb{R}$ and a and ν being positive constant parameters. Here the fOU process Z has an explicit solution given by

$$Z(t) = Z(0)e^{-at} + m(1 - e^{-at}) + \nu \int_{-\infty}^t e^{-a(t-s)} \mathrm{d}B^H(s) ,$$

where the stochastic integral with respect to the FBM B^H is a pathwise Reimann-Stieltjes integral (see [14]). This model has received great attention in the community of stochastic volatility (see, e.g., [7] and the complete relevant literature on [44]).

In addition to the estimation of the Hurst parameter being $H < 1/2$, there are several important findings in the empirical study of [20]. First, the distribution of the increments of the log-volatility is approximately normal. Second, although the log-volatility estimations are smooth over most intervals, there is observable non-smoothness for some stock prices and indices and for certain time windows. It is also argued that the reason why the above model of stochastic volatility is used, instead of $\sigma(t) = \sigma \exp(\nu B^H(t))$, is because the above model is stationary. Although mathematical tractability is desirable with the stationary model, it is evident that non-stationarity (in terms of increments) is prominent in financial data (see also [6, 27]).

Therefore, we propose the following candidates (i)-(iv) to model the volatility process $\sigma(t)$.

(i) GFBM:

$$\ln(\sigma(t)/\sigma(0)) = X(t)\,, \tag{6.10}$$

where $X(t)$ is the GFBM given in (2.1). Note that the log-volatility $\ln\sigma(t)$ in (6.10) is a semimartingale with respect to the filtration generated by the BM $\mathcal{F}^B(\cdot)$ in the parameter range of (α,γ) with $\alpha \in (1/2, 1/2+\gamma/2)$ and $\gamma \in (0,1)$ (resulting in $H \in (1/2,1)$), but it is not a semimartingale in the range with $\alpha \in (1/2-\gamma/2, 1/2) \cup \{\gamma/2\}$ and $\gamma \in (0,1)$ (resulting in $H \in (0,1)$), by Propositions 3.1 and 4.2.

We remark that although in the conventional stochastic volatility models, the volatility σ_t is often modeled as a continuous Brownian semi-martingale, for example, the Heston model, the Hull and White model and the SABR model, one may also model the log-volatility process as a semimartingale in certain scenarios.

(ii) Mixed GFBM:

$$\ln(\sigma(t)/\sigma(0)) = Y(t) = \widetilde{B}(t) + X(t), \tag{6.11}$$

where $Y(t)$ is the mixture process in (5.1), of a standard BM $\widetilde{B}(t)$ and the process $X(t)$ in (2.1). Here the log-volatility $\ln\sigma(t)$ in (6.11) is a semimartingale with respect to the filtration of its own in the parameter range of (α,γ) with $\alpha \in (\gamma/2, \gamma/2+1/2)$ and $\gamma \in (0,1)$ such that $H > 1/2$, by Proposition 5.1. When $\alpha \in (\gamma/2-1/2, \gamma/2)$ and $\gamma \in (0,1)$ (resulting $H < 1/2$), we conjecture that the log-volatility is not a semimartingale with respect to its own filtration.

(iii) The generalized fractional Ornstein-Ulenbeck (fOU) processes driven by the GFBM:

$$\ln(\sigma(t)/\sigma(0)) = Z(t), \tag{6.12}$$

where $Z(t)$ is the solution to the SDE driven by the GFBM $X(t)$ in (2.1):

$$\mathrm{d}Z(t) = -a(Z(t)-m)\mathrm{d}t + \nu \mathrm{d}X(t) \tag{6.13}$$

with $m \in \mathbb{R}$ and a and ν being positive constant parameters. Using pathwise Reimann-Stieltjes integral, we can also write the solution as

$$Z(t) = Z(0)e^{-at} + m(1-e^{-at}) + \nu \int_{-\infty}^{t} e^{-a(t-s)}\mathrm{d}X(s). \tag{6.14}$$

We refer to this as the generalized fOU process.

(iv) The generalized fOU by the mixed GFBM process: In (6.12), instead of having Z in (6.13), the process Z is the solution to the SDE driven by the mixed GFBM process Y in (5.1):

$$\mathrm{d}Z(t) = -a(Z(t)-m)\mathrm{d}t + \nu \mathrm{d}Y(t).$$

Again, using pathwise Reimann-Stieltjes integral, we can also write the solution as

$$Z(t) = Z(0)e^{-at} + m(1 - e^{-at}) + \nu \int_{-\infty}^{t} e^{-a(t-s)} \mathrm{d}Y(s). \tag{6.15}$$

It is clear that in models (iii) and (iv), the log-volatility process $\sigma(t)$ is not a semimartingale.

6.4.1. *Rough Bergomi model.* In [7], the rough Bergomi (rBergomi) model was introduced as a non-Markovian generalization of Bergomi model with FBM B^H. Specifically, the stock price process $P(u)$ and the instantaneous volatility $v(u),\ u \geq t$ under the physical measure are defined by

$$\begin{aligned} \frac{\mathrm{d}P(u)}{P(u)} &= \mu(u)\mathrm{d}u + \sqrt{v(u)}\mathrm{d}\widetilde{B}(u), \\ v(u) &= \xi(t) \exp\left(\eta\sqrt{2H} \int_t^u (u-s)^{H-1/2} \mathrm{d}B(s) - \frac{1}{2}\eta^2 (u-t)^{2H} \right), \quad u \geq t, \end{aligned} \tag{6.16}$$

where μ is an expected log return process, η is a constant, $\xi(t) = \mathbb{E}[v(u) \,|\, \mathcal{F}(t)],\ u \geq t$ is the forward variance curve, $H \in (0, 1/2)$, and $\widetilde{B} = \rho B + \sqrt{1-\rho^2} B^{\perp}$ for two independent standard Brownian motions $B, B^{\perp}$ with correlation coefficient $\rho \in (-1, 1)$. Here, the filtration $\mathcal{F}(t)$, $t \geq 0$ is generated by the price process $P(t)$, $t \geq 0$, and the process $\int_t^u (u-s)^{H-1/2} \mathrm{d}B(s)$ is the so-called Riemann-Liouville FBM or Volterra fractional Brownian motion. In [7], it was discussed as a first approximation that $P(u)$ becomes a true martingale by the deterministic change of measure under a fixed time horizon $t \leq u \leq T$ in the rough Bergomi model under the equivalent martingale measure. For the details of the martingale property of the rough Bergomi model, see [19]. The interested readers are recommended to refer to the remarkable variance swap curve estimation in [7]. Recently, the rough Bergomi model is studied in the limiting case $H \to 0$ in [16].

Using the generalized Riemann-Liouville FBM $X(t)$ in (5.12), we may modify the above model with replacement of v in (6.16) by

$$\begin{aligned} v(u) &= \xi(t) \exp\left(\eta(X(u) - X(t)) - \frac{1}{2}\eta^2 \mathbb{E}[|X(u) - X(t)|^2] \right) \\ &= \xi(t) \exp\left(\eta c \int_t^u (u-s)^{\alpha} s^{-\gamma/2} B(\mathrm{d}s) - \frac{1}{2}\eta^2 c^2 \int_t^u (u-s)^{2\alpha} s^{-\gamma} \mathrm{d}s \right), \quad u \geq t, \end{aligned} \tag{6.17}$$

where $\alpha \in ((1-\gamma)/2, \gamma/2)$, $\gamma \in (0,1)$ and $H = \alpha - \gamma/2 + 1/2 \in (0, 1/2)$.

We remark that in Remark 2.1 [7], the authors stated that the Riemann-Liouville FBM is an example of a Brownian semistationary (BSS) process [4], which is of the form $X(t) = \int_{-\infty}^t g(t-s)\sigma(s)\mathrm{d}B(s)$ for some deterministic function g and an adapted intermittency process $\sigma(s)$. However, our generalized Riemann-Liouville FBM (5.12) does not belong to this class (BSS) of processes, since the process $u^{-\gamma/2}$ in the definition of the $X(t)$ in (5.12) violates all the "(semi)stationarity" conditions imposed upon $\sigma(t)$ (see, e.g., [3]). With the generalization stemming from the additional parameter set (α, γ), the estimation problem of variance swap curve with (6.17) needs additional care, and it is an ongoing research project.

We remark that the GFBM process can be also potentially used to generalize the fractional Cox-Ingersoll-Ross process in [22] and the rough Heston models in [15]. These will be interesting future work.

6.5. **Comments on stochastic integrals with respect to the GFBM.** Observe that we have used stochastic integrals with respect with the GFBM X in (6.8) and (6.9) in portfolio optimization and in (6.14) and (6.15) in rough volatility models. In (6.8) and (6.9), the stochastic integral is of the type $\int f(X)\mathrm{d}X$, while in (6.14) and (6.15), the integrand is a deterministic function $f(t-s)$.

Stochastic integrals with respect to FBM B^H have been extensively studied in the literature (see for example [32, Chapter 7], [28, Chapter 3] and [9, 29]). For $H > 1/2$, the stochastic integral $\int f(B^H)\mathrm{d}B^H$ can be defined pathwise using Young integral due to the regularity of the sample paths of B^H, see, e.g., [28, Chapter 3.1]. Thanks to the variation property of the GFBM X in Proposition 4.1 when $H > 1/2$, the integral $\int f(X)\mathrm{d}X$ for the GFBM X can also be well defined pathwsie using Young integral. On the other hand, for the standard FBM B^H with $H < 1/2$, the stochastic integral is studied using rough path theory [17] and/or Malliavin calculus [9, 29]. This study for the GFBM X with $H < 1/2$ is out of the scope of this paper, and will be investigated in the future.

When the integrand is a deterministic function, in particular, of the form $\int_0^t f(s)\mathrm{d}B^H(s)$ or $\int_0^t f(t-s)\mathrm{d}B^H(s)$, the integral can be defined as a pathwise Riemann-Stieltjes integral, see Proposition A.1 in [14]. By a slight modification of the proof of that proposition, using self-similarity and the Hölder continuity properties of the GFBM X, the same conclusions in Proposition A.1 of [14] also hold by replacing B^H with the GFBM X. Therefore, the generalized fOU process Z in (6.14) is well defined, and so is the process in (6.15) driven by the mixed GFBM process Y.

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DEPARTMENT OF STATISTICS & APPLIED PROBABILITY, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CA 93106
Email address: ichiba@pstat.ucsb.edu

THE HAROLD AND INGE MARCUS DEPARTMENT OF INDUSTRIAL AND MANUFACTURING ENGINEERING, COLLEGE OF ENGINEERING, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802
Email address: gup3@psu.edu

DEPARTMENT OF MATHEMATICS AND STATISTICS, BOSTON UNIVERSITY, BOSTON, MA 02215
Email address: murad@math.bu.edu