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A mass conservative well-balanced reconstruction at wet/dry interfaces for the Godunov-type shallow water model

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Summary

This paper presents a novel mass conservative, positivity preserving wetting and drying treatment for Godunov-type shallow water models with second-order bed elevation discretization. The novel method allows to compute water depths equal to machine accuracy without any restrictions on the time step or any threshold that defines whether the finite volume cell is considered to be wet or dry. The resulting scheme is second-order accurate in space and keeps the C-property condition at fully flooded area and also at the wet/dry interface. For the time integration, a second-order accurate Runge–Kutta method is used. The method is tested in two well-known computational benchmarks for which an analytical solution can be derived, a C-property benchmark and in an additional example where the experimental results are reproduced. Overall, the presented scheme shows very good agreement with the reference solutions. The method can also be used in the discontinuous Galerkin method. Copyright © 2016 John Wiley & Sons, Ltd.

1 Introduction

Godunov-type shallow water flow models are applied to a broad range of surface flows such as river hydraulics [1, 2](#), dam break simulations [3–5](#), urban flood modeling [6–8](#), and rainfall–runoff in natural catchments [9–11](#). In most of these cases, wetting and drying of cells occurs during the

simulation, for example, part of the computational domain may fall dry or the domain may start initially dry and get flooded during the simulation. The interface between a wet and a dry cell is usually referred to as the wet/dry front or wet/dry interface [12](#), [13](#). Especially wet/dry interfaces over complex topography are known to cause numerical errors such as spurious oscillations in the flow velocity and negative water depths [13-21](#).

In Godunov-type schemes, these numerical instabilities are caused by the reconstruction of the cell variables at the edges [18](#). In second-order Godunov-type schemes, the reconstruction is linear [22](#), [23](#), and a slope limiter is applied to ensure monotonicity [24](#), [25](#). The water depth is calculated with

$$h = d - B \quad (1)$$

where d is the elevation of the free surface. As B and d are reconstructed independently, unrealistic negative water depths may occur at wet/dry interfaces. The flow velocity u at cell edge, which is required for computing the flux, is computed as

$$u = q/h \quad (2)$$

where q denotes the unit discharge and h stands for the water depth. Thus, for the water depth approaching zero, the velocity becomes non-physically large and causes undesirable numerical instabilities.

A positivity preserving correction of the water depth has been proposed by Kurganov and Petrova [26](#). This correction is positivity and mass preserving and further modifies the reconstructed velocities to avoid non-physically large velocities caused by the numerical inaccuracy. However, the C-property, that is, quiescent steady state, is not satisfied [12](#), [13](#). Several other approaches for the wet/dry interface can be found in literature, such as the primitive variable reconstruction [27](#), which is not suitable for calculating a discontinuous solution [18](#), or the non-negative hydrostatic reconstruction method [14](#), which may fail at certain combinations of slope, water depth, and mesh cell size [28](#). Some of the schemes use a threshold [29](#), [30](#) to avoid the water depth to reach negative values. The water depth and velocity are considered to be zero when the water depth is smaller than this threshold. In [30](#), the water depth is set to zero in the cell that abuts the wet/dry interface. This step reduces the numerical flux, thus diminishes the probability of appearance of the negative water depth. However, finding a wet/dry interface and adjacent cells increases the computational demands, and accordingly to [31](#), the solver becomes very sensitive to the changes of this threshold when solving the bed friction problems parametrized by the Manning expression. Another approach is to limit the time step such that the cell does not dry out [15](#); however, this approach degenerates the time step and increases computational effort. Bollermann *et al.* [12](#) modify the correction of Kurganov and Petrova [26](#) to obtain a C-property satisfying scheme. The scheme restricts the time step as

in [15](#) to preserve positivity of water depth and adds water into the cell in special cases to satisfy the C-property. In the authors' opinion, this might result in loss of the mass conservation property of this scheme.

In this article, the correction at wet/dry interfaces by Kurganov and Petrova [26](#) is extended into a novel reconstruction scheme, which satisfies the C-property condition and is mass conservative and up to the machine accuracy, also known as the machine epsilon. The correction is implemented into a monotone upstream centered scheme for conservation laws, which uses the surface gradient method [32](#) to linearly reconstruct the edge values for the wet parts of the domain and switches to the novel correction method at wet/dry interfaces. The Harten–Lax–van Leer approximate Riemann solver is used to calculate the numerical flux. The novel scheme is verified in several benchmarks for the one-dimensional shallow water equations over complex topography.

This article is structured as follows: Firstly, the governing equations are presented; secondly, the numerical methods are discussed, and the novel correction method at wet/dry interface is presented; thirdly, computational examples are shown to verify the presented method; and finally, conclusions are given.

2 Governing Equations

The depth-averaged one-dimensional shallow water equations can be written in conservative form as

$$\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S} \quad (3)$$

where $\mathbf{W}$ is the vector of conservative variables described by the water depth h and flow velocity u as

$$\mathbf{W} = \begin{bmatrix} h \\ hu \end{bmatrix} \quad (4)$$

and $\mathbf{F}$ is the vector function of inviscid flux

$$\mathbf{F} = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix} \quad (5)$$

Here, g is the gravity acceleration, and $\mathbf{S}$ is the source term incorporating the bed slope source term $\mathbf{S}_b$ and the bottom friction source term $\mathbf{S}_f$, which both result from the vertical depth averaging. $\mathbf{S}_b$ can be written as

$$\mathbf{S}_b = \begin{bmatrix} 0 \\ -gh \frac{\partial B(x)}{\partial x} \end{bmatrix} \quad (6)$$

where $B(x)$ is the bottom topography. $B(x)$ might vary in time, however is considered time-independent in this work. $\mathbf{S}_f$ can be written as

$$\mathbf{S}_f = \begin{bmatrix} 0 \\ -\tau_f \end{bmatrix} \quad (7)$$

with τ_f standing for bed shear stress. This stress can be expressed via several well-known empirical friction laws [33](#), for example, Manning's law

$$\tau_f = C_f u |u| \quad (8)$$

Here,

$$C_f = (g n^2) / \sqrt[3]{h} \quad (9)$$

is the Manning's bed friction coefficient and n is Manning's bed roughness factor. Then, the source vector $\mathbf{S}$ is

$$\mathbf{S} = \mathbf{S}_b + \mathbf{S}_f \quad (10)$$

The source vector may further contain fluid viscosity effects, wind shear, turbulent viscosity, or Coriolis forces, depending on the case under consideration. These effects are neglected in this work.

3 Numerical Methods

3.1 Finite volume formulation

The mathematical model is solved by the finite volume method. The construction of the finite volume method is based on integral average in the finite cell. The system described in Equation [3](#) is integrated over the computational cell Ω_i

$$\int_{\Omega_i} \frac{\partial \mathbf{W}}{\partial t} d\Omega + \int_{\Omega_i} \frac{\partial \mathbf{F}}{\partial x} d\Omega = \int_{\Omega_i} \mathbf{S} d\Omega \quad (11)$$

The second integral is modified by the integration by parts

$$\int_{\Omega_i} \frac{\partial \mathbf{F}}{\partial x} d\Omega = \mathbf{F}_{i+\frac{1}{2}} - \mathbf{F}_{i-\frac{1}{2}} \quad (12)$$

where index $i \pm \frac{1}{2}$ indicates the corresponding edge of the computational cell over which the flux is calculated. With the assumption of constant boundary of the finite volume Ω_i in time, the

order of integration and derivation in the integral $\int_{\Omega_i} \frac{\partial \mathbf{W}}{\partial t} d\Omega$ can be changed, and the integral can be approximated by the integral average $\mathbf{W}_i$

$$\mathbf{W}_i = \frac{1}{\Delta x_i} \int_{\Omega_i} \mathbf{W} d\Omega \quad (13)$$

where Δx_i is the cell width. The same treatment can be applied to the bed slope source term, and Equation 11 results in

$$\frac{d\mathbf{W}_i}{dt} = -\frac{1}{\Delta x_i} \left(\mathbf{F}_{i+\frac{1}{2}} - \mathbf{F}_{i-\frac{1}{2}} \right) + \mathbf{S}_i \quad (14)$$

where the time derivative $\frac{d\mathbf{W}_i}{dt}$ is computed by the second-order Runge–Kutta method.

The flux $\mathbf{F}_{i+\frac{1}{2}}$ across the edge is calculated by the Harten–Lax–van Leer approximate Riemann solution 34

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{a_{i+\frac{1}{2}}^+ \mathbf{F}(\mathbf{W}_i^E) - a_{i+\frac{1}{2}}^- \mathbf{F}(\mathbf{W}_{i+1}^W)}{a_{i+\frac{1}{2}}^+ - a_{i+\frac{1}{2}}^-} + \frac{a_{i+\frac{1}{2}}^+ a_{i+\frac{1}{2}}^-}{a_{i+\frac{1}{2}}^+ - a_{i+\frac{1}{2}}^-} \left[\mathbf{W}_{i+1}^W - \mathbf{W}_i^E \right] \quad (15)$$

where $\mathbf{W}_i^W, \mathbf{W}_i^E$ are the values of the piecewise linear reconstruction of the vector of the

conservative variables, namely, $\mathbf{W}_i^W = \mathbf{W}_i \left(x_{i-\frac{1}{2}}, t \right)$ and $\mathbf{W}_i^E = \mathbf{W}_i \left(x_{i+\frac{1}{2}}, t \right)$. The description of this linear reconstruction is presented in the next section. The local propagation

speeds $a_{i+\frac{1}{2}}^\pm$ are computed as proposed in 35:

$$\begin{aligned} a_{i+\frac{1}{2}}^- &= \min \left\{ \lambda_1 \left(\mathbf{W}_{i+1}^W \right), \lambda_1 \left(\mathbf{W}_i^E \right), 0 \right\}, \\ a_{i+\frac{1}{2}}^+ &= \max \left\{ \lambda_2 \left(\mathbf{W}_{i+1}^W \right), \lambda_2 \left(\mathbf{W}_i^E \right), 0 \right\} \end{aligned} \quad (16)$$

Here, λ_1, λ_2 are the eigenvalues of the Jacobian $\frac{\partial \mathbf{F}}{\partial \mathbf{W}}$, that is,

$$\begin{aligned} \lambda_1 &= u - \sqrt{gh}, \\ \lambda_2 &= u + \sqrt{gh} \end{aligned} \quad (17)$$

The bed slope source term is approximated by the simple differencing formulae [26](#)

$$\mathbf{S}_{b_i} = \begin{bmatrix} 0 \\ -gh_i \frac{\Delta B_i}{\Delta x_i} \end{bmatrix} \quad (18)$$

where $\Delta B_i = B_{i+\frac{1}{2}} - B_{i-\frac{1}{2}}$ and $B_{i\pm\frac{1}{2}}$ are the values of the bed function at the corresponding edges.

When wetting and drying processes appear in the numerical simulation, the bed friction source term dominates the stability of the numerical scheme because of the water height in the denominator. In [36](#), the authors solve the bed friction source term by splitting point-implicit method. The time change of the vector of the conservative variables must be in balance with bed friction source term, that is,

$$\frac{d\mathbf{W}}{dt} = \mathbf{S}_f \quad (19)$$

The friction term does not affect the continuity equation. So, with the substitution $hu = q$, Equation [19](#) can be written as

$$\frac{q^{n+1} - q^n}{\Delta t} = -\tau_f^{n+1} \quad (20)$$

Here, the friction term τ_f can be approximated by the Taylor series

$$\tau_f^{n+1} = \tau_f^n + \left(\frac{\partial \tau_f}{\partial q} \right)^n \Delta q + \mathcal{O}(\Delta q^2) \quad (21)$$

where $\Delta q = q^{n+1} - q^n$. Ignoring the higher-order terms in Equation [21](#) and substitution into Equation [20](#) leads to

$$\frac{q^{n+1} - q^n}{\Delta t} = -\tau_f^n - \left(\frac{\partial \tau_f}{\partial q} \right)^n (q^{n+1} - q^n) \quad (22)$$

Then

$$q^{n+1} = q^n - \Delta t \left(\frac{\tau_f}{1 + \Delta t \frac{\partial \tau_f}{\partial q}} \right)^n = q^n - \Delta t \underbrace{\left(\frac{C_f u |u|}{1 + 2\Delta t C_f |q| h^{-2}} \right)^n}_{S_c^n} = q^n - \Delta t S_c^n \quad (23)$$

with C_f defined by [9](#). As mentioned earlier, the bed friction source term needs special treatment in the cases when the water depth approaches zero value during the wetting and drying processes. In these cases, the following limiter has to be used:

$$S_c^n = \frac{q^n}{\Delta t} \quad \text{if} \quad |\Delta t S_c^n| > |q^n| \quad (24)$$

If the absolute value of S_c^n is bigger than the absolute value of hu , then the flow is stopped. Without the limiter, the friction term would non-physically change the flow direction.

3.2 Piecewise linear reconstruction

The conservative variables and the bed function are approximated by a piecewise linear function. The bed function is approximated as shown in Figure 1. The approximation in Ω_i is defined as

$$\tilde{B}(x) = B_{i-\frac{1}{2}} + \frac{B_{i+\frac{1}{2}} - B_{i-\frac{1}{2}}}{\Delta x_i} \left(x - x_{i-\frac{1}{2}} \right), \quad \forall x : x_{i-\frac{1}{2}} \leq x \leq x_{i+\frac{1}{2}} \quad (25)$$

Figure 1

[Open in figure viewer](#) [PowerPoint](#)

Approximation of the bed function.

The tilde ($\sim$) implicates that the function is an approximation to the real bed. The point $B_{i+\frac{1}{2}}$ can be approximated as

$$B_{i+\frac{1}{2}} = \frac{\int_{x_i}^{x_{i+1}} B(x) dx}{\int_{x_i}^{x_{i+1}} dx} \quad (26)$$

In first-order accurate Godunov-type schemes, the values required to calculate the flux (cf. Equation 5) at the edges are set equal to the values at the middle of the cell. The accuracy of the scheme can be increased by the linear reconstruction of these variables, that is,

$$\phi(x) = \phi_i + \varphi_i (x - x_i) \quad (27)$$

where ϕ represents the conservative variable, that is, water depth h or discharge hu and x_i is the position of the cell center. φ_i is the gradient of the variable computed as 26

$$\varphi_i = L \left(\Theta \frac{h_i - h_{i-1}}{\Delta x_i}, \frac{h_{i+1} - h_{i-1}}{2\Delta x_i}, \Theta \frac{h_{i+1} - h_i}{\Delta x_i} \right) \quad (28)$$

where Θ can be used to control the numerical viscosity 26 and L stands for a limiter, ensuring a non-oscillatory nature of this reconstruction. One of the possible limiters is the so-called MINMOD limiter (cf., e.g., 24) defined as

$$L_{\text{minmod}}(a_1, a_2, a_3) = \begin{cases} \min(a_1, a_2, a_3) & \text{if } a_i > 0 \text{ for } i = 1, 2, 3 \\ \max(a_1, a_2, a_3) & \text{if } a_i < 0 \text{ for } i = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases} \quad (29)$$

In this work, the surface gradient method 32 is used to conserve the C-property in the fully flooded domain. This method consists of the transformation of the water depth $h(x)$ into the water level $d(x) = h(x) + B(x)$ before the reconstruction. The water level is computed as

$$d_i = h_i + \frac{B_{i-\frac{1}{2}} + B_{i+\frac{1}{2}}}{2} \quad (30)$$

After the reconstruction, the water depths at the cell edges are computed as

$$h_i^E = d_i + \varphi_i \frac{\Delta x_i}{2} - B_{i+\frac{1}{2}} \quad (31)$$

and

$$h_i^W = d_i - \varphi_i \frac{\Delta x_i}{2} - B_{i-\frac{1}{2}} \quad (32)$$

h_i^E stands for $h_i \left(x_{i+\frac{1}{2}} \right)$, and h_i^W stands for $h_i \left(x_{i-\frac{1}{2}} \right)$ (Figure 2).

Figure 2

[Open in figure viewer](#)[PowerPoint](#)

Linear reconstruction of the water depth with the negative water depth h_i^W .

3.3 Positivity preserving reconstruction

The linear reconstruction described in the previous section improves the accuracy of the numerical scheme and can produce negative water depths at the cell edge. An exemplary reconstruction resulting in negative water depth at the edge $i - \frac{1}{2}$ is shown in Figure 2. This problem can be addressed by the correction presented in 26. The negative water depth h_i^W at the edge $i - \frac{1}{2}$ is set to zero, and the water depth h_i^E is computed in such way that the water depth at cell center h_i remains unchanged:

$$\begin{aligned}
\underbrace{0}_{h_i^W} + B_{i-\frac{1}{2}} + h'_i \frac{\Delta x_i}{2} &= h_i + \frac{B_{i+\frac{1}{2}} + B_{i-\frac{1}{2}}}{2} \\
\Downarrow \\
h'_i &= \frac{2h_i + B_{i+\frac{1}{2}} - B_{i-\frac{1}{2}}}{\Delta x_i}
\end{aligned} \tag{33}$$

Here, h'_i is the slope of the water level. Then

$$h_i^E = h_i + \frac{B_{i+\frac{1}{2}} + B_{i-\frac{1}{2}}}{2} + h'_i \frac{\Delta x_i}{2} - B_{i+\frac{1}{2}} = 2h_i \tag{34}$$

The corrected reconstruction of the water depth is shown in Figure 3. The case $h_i^E < 0$ can be corrected in a similar way and the conditions of the correction can be summarized as

$$\begin{aligned}
\text{if } h_i^W < 0 &\begin{cases} h_i^W = 0 \\ h_i^E = 2h_i \end{cases}, \\
\text{if } h_i^E < 0 &\begin{cases} h_i^W = 2h_i \\ h_i^E = 0 \end{cases}
\end{aligned} \tag{35}$$

Figure 3

[Open in figure viewerPowerPoint](#)

Corrected linear reconstruction of the water depth.

Then, the non-negativity of the water depth is ensured. However, this water depth may become very small and cause problems with the computation of the velocity u , which is computed by the fraction $u = \frac{q}{h}$ and $u \rightarrow \infty$ for $h \rightarrow 0$. To avoid unfeasible velocities, Kurganov and Petrova [26](#) proposed the following formulae, which avoids the division by very small numbers:

$$u = \frac{\sqrt{2h(hu)}}{\sqrt{h^4 + \max(h^4, \epsilon_h)}} \quad (36)$$

The indices $i \pm \frac{1}{2}$ were omitted for the sake of simplicity, and ϵ_h is a small positive constant.

In [26](#), it is recommended to set this constant to $\epsilon_h = (\Delta x_i)^4$. But as our numerical simulations show, ϵ_h should not be larger than 10^{-7} . The discharge hu has to be recomputed as

$$hu = h \cdot u \quad (37)$$

otherwise, the scheme can produce negative water depth. The proof of the positivity preserving property of the scheme is given in [26](#).

3.4 Novel C-property preserving reconstruction

The aforementioned correction by Kurganov and Petrova [26](#) does not necessarily conserve the C-property at the wet/dry interface. For the initial conditions shown in Figure [4](#), the water depth correction leads to a non-physical discontinuity at the point $x_{i+\frac{1}{2}}$ as shown in Figure [5](#). Based on previous research of the authors [13](#), it is proposed to modify the bed source term instead of the water depth to preserve the C-property at wet/dry interfaces. The water depth h_i^W remains negative, and the flux across the edge $x_{i-\frac{1}{2}}$ is set to be zero. Hence, the total flux in the finite volume Ω_i is the flux across the edge $x_{i+\frac{1}{2}}$. Considering zero velocity and constant water level, this flux $\mathbf{F}_{i+\frac{1}{2}}$ is equal to

$$\mathbf{F}_{i+\frac{1}{2}} = \begin{bmatrix} 0 \\ \frac{1}{2\Delta x_i} g (h_i^E)^2 \end{bmatrix} \quad (38)$$

Figure 4
[Open in figure viewer](#) PowerPoint
 Initial condition of the 'lake at rest' with wet/dry interface.

Figure 5

[Open in figure viewer](#)[PowerPoint](#)

Water depth after the correction.

Bed slope source term [18](#) is equal to

$$\mathbf{S}_{b_i} = \begin{bmatrix} 0 \\ -g h_i \frac{\Delta B_i}{\Delta x_i} \end{bmatrix} \quad (39)$$

The component ΔB_i is chosen to be modified. The following equation is solved considering

$$h_i = \frac{h_i^W + h_i^E}{2}$$

$$\begin{bmatrix} 0 \\ -g h_i \frac{\Delta B_i}{\Delta x_i} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2\Delta x_i} g (h_i^E)^2 \end{bmatrix} \Rightarrow \Delta B_i = -\frac{(h_i^E)^2}{h_i^W + h_i^E} \quad (40)$$

The negative water depth h_i^W does not have to be changed, and the numerical flux $F_{i-\frac{1}{2}}$ is considered to be zero. Similarly, Equation 40 can be computed for the case when h_i^E is negative and conditions for ΔB_i can be summarized as

$$\Delta B_i = \begin{cases} -\frac{(h_i^E)^2}{h_i^W + h_i^E} & \text{when } h_i^W < 0, \\ \frac{(h_i^W)^2}{h_i^W + h_i^E} & \text{when } h_i^E < 0, \\ B_{i+\frac{1}{2}} - B_{i-\frac{1}{2}} & \text{in other cases} \end{cases} \quad (41)$$

With this modification, the bed slope source term and the flux term are well-balanced not only at the fully flooded area but also at the wet/dry interface. However, the water depth can ‘dry out’ to a negative value; therefore, this reconstruction does not ensure mass conservation.

3.5 Novel positivity and C-property preserving reconstruction

The aforementioned reconstruction methods differ only in bed slope source term computations and conservative variables, which are used in the computation of the numerical flux. Herein, a novel reconstruction method is proposed, which is essentially switching between both schemes, depending whether positivity or C-property needs to be preserved, and therefore combines the advantages of both methods while overcoming their limitations. In this section, a new predictor is introduced. This predictor governs the numerical approximation and chooses suitable approach for the computations at the wet/dry interface.

The predictor is based on the continuity equation that governs the mass flow. In the one-dimensional case, the continuity equation is given by

$$h_i^{n+1} = h_i^n - \frac{\Delta t}{\Delta x_i} \left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} \right) \quad (42)$$

where n is the time level and $F_{i\pm\frac{1}{2}}$ are the first components of the numerical fluxes. At the wet/dry interface, one of the numerical fluxes is zero. For the conditions illustrated in

Figures 2 and 3, the flux $F_{i-\frac{1}{2}}$ is zero and Equation 42 results in

$$h_i^{n+1} = h_i^n - \frac{\Delta t}{\Delta x_i} F_{i+\frac{1}{2}} \quad (43)$$

The water depth in the $n + 1^{\text{th}}$ time level must be non-negative

$$h_i^n - \frac{\Delta t}{\Delta x_i} F_{i+\frac{1}{2}} \geq 0 \quad (44)$$

and the condition in Equation 44 results in the final predictor, which defines whether the bed source term and conservative variables are computed by the C-property or positivity preserving reconstruction:

$$\mathbf{F}_{i+\frac{1}{2}}, \mathbf{S}_{b_i} = \begin{cases} \text{C-property approach if } h_i^n \Delta x_i \geq \Delta t F_{i+\frac{1}{2}}, \\ \text{positivity preserving approach in other cases} \end{cases} \quad (45)$$

The case, when h_i^E is negative, follows by symmetry. The scheme can be extended to two-dimensional problem. In 26, there is the two-dimensional positivity preserving scheme described. In two-dimensional, similar predictor to 45 and modification of the bed source similar to 41 can be used to obtain well-balanced positivity preserving scheme. The difference between one-dimensional and two-dimensional predictors is the appearance of the fluxes in the second dimension in condition 44.

The resulting reconstruction is positivity preserving and satisfies the C-property condition as demonstrated in the following computational examples.

4 Computational Examples

4.1 C-property and positivity preserving benchmark

This benchmark is the case where C-property and also positivity preservation are tested. The topography is given as

$$B(x) = \begin{cases} x - 1, & x \in \langle 1, 2 \rangle \\ 1, & x \in \langle 2.0, 2.5 \rangle \\ 0, & \text{otherwise} \end{cases} \quad (46)$$

The initial conditions of the water depth are

$$h(x) = \begin{cases} 0.8 - B(x), & x \in \langle 1, 1.8 \rangle \\ 0.6, & x \in \langle 2.5, 4 \rangle \\ 0, & \text{otherwise} \end{cases} \quad (47)$$

with zero velocity in the whole domain. Friction is neglected. The initial conditions are shown in Figure 6. The simulation after 0.8 s is shown in Figure 7. It can be seen that the scheme keeps the C-property at the wet/dry interface, which is set at the position $x = 2.5$ m. Eventually, due to the numerical viscosity, the flow becomes still, and initially flooded areas become dry as shown in

Figure 8. As shown in Figure 9, the total mass in the domain remains constant during the computations.

Figure 6

[Open in figure viewerPowerPoint](#)

Initial conditions of the test case.

Figure 7

[Open in figure viewerPowerPoint](#)

Simulation at time $t = 0.8$ s.

Figure 8

[Open in figure viewerPowerPoint](#)

Simulation at time $t = 1000$ s.

Figure 9

[Open in figure viewer](#)[PowerPoint](#)

Plot of the total mass in the computational domain.

4.2 Dam-break on dry bed

The dam-break problem has become *de facto* standard test in validation of shallow water equations solvers. The analytical solution for the water depth for the frictionless surge in a horizontal channel was proposed by Barré de Saint-Venant [37](#) as

$$\frac{x}{t\sqrt{gh_0}} = 2 - 3\sqrt{\frac{h}{h_0}} \quad \text{for} \quad -1 \leq \frac{x}{t\sqrt{gh_0}} \leq 2 \quad (48)$$

The initial height of the water depth is $h_0=5$ m, the water gate is set at distance $l_0=5$ m, and the velocity is zero. The length of the computational domain is 10.5 m, and simulation was stopped

at the time $t = \frac{5}{2\sqrt{gh_0}}$. In Figure 10, the comparison between the analytical and numerical solutions with 1000 of finite volume cells is shown.

Figure 10

[Open in figure viewer](#)[PowerPoint](#)

Comparison between the analytical and numerical solutions of the water depth at time

$$t = \frac{5}{2\sqrt{gh_0}}.$$

According to the numerical results, the model has a good agreement with the analytical solution.

4.3 Planar surface flow in a parabola

One of the classical benchmarks for testing bed slope and wetting and drying processes is the planar surface in a parabola without friction. The exact solution for the two-dimensional problem

was derived in [38](#). Here, the one-dimensional benchmark presented in [39](#) is implemented. The topography is a parabolic shape given by

$$B(x) = h_0 \left(\frac{1}{a^2} \left(x - \frac{L}{2} \right)^2 - 1 \right) \quad (49)$$

The analytical solution of the water depth is

$$h(x) = \begin{cases} -h_0 \left(\left(\frac{1}{a} \left(x - \frac{L}{2} \right) + \frac{C}{\sqrt{2gh_0}} \cos \left(\frac{\sqrt{2gh_0}}{a} t \right) \right)^2 - 1 \right) & \text{for } x_1(t) \leq x \leq x_2(t), \\ 0 & \text{otherwise} \end{cases} \quad (50)$$

with location of the wet/dry interfaces at time t being calculated as

$$\begin{aligned} x_1(t) &= -\frac{1}{2} \cos \left(\frac{\sqrt{2gh_0}}{a} t \right) - a + \frac{L}{2}, \\ x_2(t) &= -\frac{1}{2} \cos \left(\frac{\sqrt{2gh_0}}{a} t \right) + a + \frac{L}{2} \end{aligned} \quad (51)$$

and

$$C = \sqrt{\frac{2gh_0}{2a}} \quad (52)$$

Initial conditions of the water depth are given by [50](#) for $t = 0$ s, and the flow velocity is zero.

In the simulation, the following parameters were used: $a = 1$ m, $h_0 = 0.5$ m, and $L = 4$ m. The coefficient Θ from the limiter [28](#) was set to 0.5. The comparison between the original scheme proposed in [26](#) and novel scheme was performed at two positions of the water level. The time of

the first position is $t = 5T$, where $T = \frac{a\pi}{\sqrt{2gh_0}}$. At this position, the velocity of the flow is the smallest. The visual comparison can be seen in Figure [11](#). The comparison of the schemes for the case when the velocity takes the largest values can be at time $t = 5.5T$, and the results are plotted in Figure [12](#). Relations [35](#) are distorting the front face of the surge wave; thus, the proposed scheme gives better results than the original one because these relations are used only in the cases when the approach, described in Section [3.5](#), can cause the appearance of the negative water depth in the computational domain. Both schemes are losing the numerical viscosity with the highest number of the finite volumes as it can be seen in Figure [13](#). The L_2 errors of the schemes, computed at time $t = 5T$ for different number of finite volumes, can be seen in Table [1](#).

This error is computed as

$$L_2 \text{ error} = \sqrt{\frac{\sum_{i=1}^{n_I} (h_a(x_i) - h_n(x_i))^2 |\Omega_i|}{\sum_{i=1}^{n_I} |\Omega_i|}} \quad (53)$$

where n_i is the number of finite volumes, $h_a(x_i)$ means analytical, and $h_n(x_i)$ numerical solution of the water depth at the position x_i .

Table 1. L_2 error of the planar surface flow in a parabola simulation at time $t = 5T$.

Number of finite volume cells	Δx	ε_h	L_2 error of the original scheme	L_2 error of the novel scheme
200	2.000e-02	1.60e-07	3.42934e-02	2.74614e-02
300	1.333e-02	3.16e-08	2.35962e-02	1.86985e-02
400	1.000e-02	1.00e-08	1.79318e-02	1.41330e-02
500	8.000e-03	4.10e-09	1.44701e-02	1.13661e-02
600	6.667e-03	1.98e-09	1.21143e-02	9.49169e-03
700	5.714e-03	1.07e-09	1.04211e-02	8.15222e-03
800	5.000e-03	6.25e-10	9.14027e-03	7.13755e-03

Number of finite volume cells	Δx	ε_h	L_2 error of the original scheme	L_2 error of the novel scheme
900	4.444e-03	3.90e-10	8.14028e-03	6.35123e-03
1000	4.000e-03	2.56e-10	7.33701e-03	5.71963e-03
1100	3.636e-03	1.75e-10	6.67735e-03	5.20108e-03
1200	3.333e-03	1.23e-10	6.12803e-03	4.76891e-03
1300	3.077e-03	8.96e-11	5.66045e-03	4.40458e-03
1400	2.857e-03	6.66e-11	5.26049e-03	4.09107e-03
1500	2.667e-03	5.06e-11	4.91263e-03	3.82048e-03
1600	2.500e-03	3.91e-11	4.60853e-03	3.58240e-03

Number of finite volume cells	Δx	ε_h	L_2 error of the original scheme	L_2 error of the novel scheme
1700	2.353e-03	3.07e-11	4.33934e-03	3.37241e-03
1800	2.222e-03	2.44e-11	4.10054e-03	3.18582e-03
1900	2.105e-03	1.96e-11	3.88615e-03	3.02023e-03
2000	2.000e-03	1.60e-11	3.69349e-03	2.86945e-03

Figure 11
[Open in figure viewerPowerPoint](#)
 Solution at time $t = 5T$ simulated for 100 finite volumes and $\varepsilon_h = 10^{-7}$.

Figure 12

[Open in figure viewerPowerPoint](#)

Solution at time $t = 5.5T$ simulated for 100 finite volumes and $\varepsilon_n=10^{-7}$.

Figure 13

[Open in figure viewer](#)[PowerPoint](#)

Solution at time $t = 5T$ simulated for 1000 finite volumes and $\varepsilon_i = (\Delta x)^4$.

The order of accuracy of a scheme can be found by the slope of the $\log(L_2\text{error})$ dependent on $\log(n_i)$. The novel and the original schemes are compared with the scheme published in [29](#).

This scheme uses a threshold to cope with the wet/dry interface. In the computations, two different thresholds 10^{-8} and 10^{-13} are used. In [Figure 14](#), it can be seen that the threshold affects the order of the scheme convergence. Better accuracy is obtained with smaller threshold, but the phenomena of the non-physical velocities, described in [Section 3.3](#), are more likely to occur.

Thus, we had to implement a velocity limiter, and the velocity in a cell was set to zero if exceeded 3 m/s. The maximal theoretical velocity is 1.57 m/s [39](#). The maximal numerical velocity, computed by the novel scheme, is 1.76 m/s. The convergence of the reference scheme with threshold, original scheme proposed by Kurganov and novel scheme is shown in [Figure 14](#). The convergence of the L_2 errors are computed by the least square method, and the values are also shown in [Figure 14](#). Accordingly to the numerical results, our approach has the highest

accuracy and the highest rate of convergence. In the authors opinion, the lower convergence rate of the reference scheme can be caused by the fact that the bed function is approximated only by piecewise constant functions.

Figure 14

[Open in figure viewerPowerPoint](#)

Convergence of L_2 errors of the schemes and slope of the convergence.

5 Conclusions

A well-balanced shallow water equation solver with a novel reconstruction of variables at wet/dry interfaces was presented. The solver is positivity preserving for both wetting and drying processes and also satisfies the C-property condition at the wet/dry interface.

The scheme gives good results for both wetting and drying processes. The flows can be computed up to water depths approaching zero values with no need for any dry water depth threshold, which is used in many shallow water models to choose whether a computational cell is wet or dry and to avoid computations around zero value. The proposed method is mass conservative and calculates the water depth correctly up to the machine epsilon. The machine accuracy is provided by the library `float.h` as a variable `DBL_EPSILON`. The machine epsilon of the computer, where the scheme was tested, is $2.22\text{e-}16$. The numerical model also does not distort the time step during the drying processes. Several benchmarks were presented to demonstrate the novel method's capabilities. The method was able to approximate analytical and experimental reference solutions with very good accuracy. It is concluded that the presented scheme is suitable to solve practical applications featuring flow over complex topography, is applicable to problems with wetting and drying, and can be extended to two-dimensional problems. In the future work, results will be used within discontinuous Galerkin method for the shallow water equations.

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