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Complexity of Outer Automorphisms of Free Groups and Higher Rank Lattices

A dissertation submitted in partial satisfaction
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Doctor of Philosophy
in
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by

Paige Kathleen Hillen

Committee in charge:

Professor Darren D Long, Chair
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4. A hyperbolic free-by-cyclic group determined by its finite quotients 2024
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Preprint, available at `arXiv:2410.17817` [math.GR].
To appear in Glasgow Mathematical Journal

3. Latent symmetry of graphs and stretch factors in $\mathrm{Out}(F_r)$ 2024
Preprint, available at `arXiv:2409.19446` [math.GR].
To appear in Groups, Geometry, and Dynamics

2. Low complexity among principal fully irreducible elements of $\mathrm{Out}(F_3)$ 2024
(with Naomi Andrew, Robert Alonzo Lyman, and Catherine Pfaff)
Preprint, available at `arXiv:2405.0368` [math.GR].
To appear in Algebraic and Geometric Topology

1. Non-uniform lattices of large systole containing a fixed 3-manifold group. 2024
Preprint, available at `arXiv:2403.14081` [math.GT].
To appear in Algebraic and Geometric Topology

Abstract

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This thesis explores the relationship between symmetry, geometry, and complexity through two projects.

The first project concerns irreducible outer automorphisms of free groups, each of which is topologically represented by an irreducible train track map $f : \Gamma \to \Gamma$ for some graph Γ. Moreover, f can always be expressed as a composition of "folds" and a graph isomorphism. We establish a lower bound on the stretch factor of an irreducible outer automorphism in terms of the number of folds of f and the number of edges in Γ. In the case where f is a bijection on the vertices of Γ, we show that a precise notion of the latent symmetry of Γ provides a lower bound on the number of folds required. Using this notion, we classify all possible irreducible single fold train track maps.

The second project concerns arithmetic lattices. Let $d \geq 2$ be a square free integer and $\mathbb{Q}(\sqrt{d})$ a totally real quadratic field over $\mathbb{Q}$. We construct an arithmetic lattice $\mathcal{L}$ in $SL(8, \mathbb{R})$ with entries in the ring of integers of $\mathbb{Q}(\sqrt{d})$, along with a sequence of lattices Λ_n commensurable to $\mathcal{L}$, such that the systole of the locally symmetric finite-volume manifold $\Lambda_n \diagdown SL(8, \mathbb{R}) \diagup SO(8)$ tends to infinity as $n \to \infty$. Nevertheless, every Λ_n contains the same hyperbolic 3-manifold group Π, a finite-index subgroup of the arithmetic hyperbolic 3-manifold vol3. Notably, such an example does not exist in rank one, making this a phenomenon unique to higher rank lattices.

Contents

Overview

Two projects form the foundation of this thesis, one concerned with stretch factors of outer automorphisms of free groups and the other with properties of higher rank lattices. The two projects are united by their common motivating question:

how do geometry and symmetry affect complexity?

Project 1: Out(F_r)

In this project, we explore how geometry and symmetry influence the dynamical complexity of outer automorphisms of free groups. We measure the dynamical complexity of $\varphi \in \mathrm{Out}(F_r)$ by its *stretch factor*, which records the asymptotic growth rate of words in F_r under repeated applications of φ. In many ways, $\mathrm{Out}(F_r)$ plays a role for graphs analogous to that of the mapping class group for surfaces: a homotopy equivalence from a finite graph to itself induces an outer automorphism on its fundamental group, just as a homeomorphism of a surface S induces an element of the mapping class group of S. In the mapping class group setting, much work has focused on understanding the minimal stretch factor among *pseudo-Anosov* mapping classes for a fixed surface. In the same vein, our results aim to determine the minimal stretch factor among *irreducible* outer automorphisms of F_r for each fixed rank r. We determine this minimum precisely for

$r = 3$, develop a framework for investigating $r \geq 4$, and explore some consequences of these results.

Given an irreducible $\varphi \in \text{Out}(F_r)$, there is some rank r graph Γ and a particularly nice choice of homotopy equivalence $f : \Gamma \to \Gamma$ which induces φ. Such a nice choice of representative $f : \Gamma \to \Gamma$ is called an *irreducible train track map*. These nice representatives carry dynamical information about φ, and in particular give a way to compute the stretch factor of φ. We can write an irreducible train track map as a composition of really simple graph maps called *folds*, followed by a graph isomorphism. Each fold is a quotient map which identifies two whole or partial edges. More folds in the fold decomposition of a train track map f corresponds to a more dynamically complicated outer automorphism φ. We make this idea precise by establishing a lower bound on the stretch factor in terms of the number of fold in the fold decomposition of f and the number of edges in the graph Γ.

With the lower bound in hand and the goal of determining minimal stretch factors in mind, we then turn to a systematic study of which graphs Γ can possibly admit irreducible train track maps with fold decompositions consisting of a very small number of folds. We develop a notion of the latent symmetry of Γ and make precise the idea that graphs with less latent symmetry require more folds to build an irreducible train track map. This allows us to determine exactly which graphs can admit irreducible train track maps with a single fold in their fold decomposition. Finally, we combine the classification of single fold irreducible train track maps and the lower bound result to determine the minimal stretch factor among irreducible elements of $\text{Out}(F_3)$, and show that the element attaining this minimum is unique up to $\text{Out}(F_3)$ conjugation. The work in this chapter also appears in [Hil24a]

Project 2: Lattices

We investigate the relationship between the geometric and topological complexity of higher rank lattices in $\mathrm{SL}(n, \mathbb{R})$. Higher rank lattices are finite-volume quotients of symmetric spaces with flat subspaces of dimension ≥ 2. These lattices serve as a natural counterpart to hyperbolic manifolds; while they share some features, the presence of flat subspaces introduces unique behaviors and properties.

Here, we measure geometric complexity via the *systole*: the minimal length of a non-contractible closed loop. Topological complexity is assessed through geometrically meaningful subgroups of the fundamental group, such as the *systolic genus*: the minimal genus of a surface subgroup. In hyperbolic manifolds, the systolic genus is bounded below in terms of the systole. Consequently, a sequence of hyperbolic manifolds with systoles tending to infinity must also exhibit diverging topological complexity, as measured by the systolic genus.

However, this relationship does not extend to higher rank lattices. We construct a sequence of higher rank lattices with systoles diverging to infinity, yet each contains a fixed hyperbolic 3-manifold group. Since this 3-manifold group has surface subgroups, the systolic genus of these lattices remains bounded. Thus, in higher rank, topological complexity is not tied to the behavior of the systole, unlike in the hyperbolic setting. The work in this chapter also appears in [Hil24b]

Chapter 1

Outer Automorphisms of Free Groups

1.1 Introduction

1.1.1 From Aut(F_r) to Out(F_r)

From our perspective, the primary objects in the study of Out(F_r) are connected finite graphs, allowing multiple edges and self-loops (see Definition 1.2.2). Given such a graph Γ, consider a homotopy equivalence $f : \Gamma \to \Gamma$. Fixing a base point $p \in \Gamma$, we obtain an induced isomorphism on the fundamental group:

$$f_* : \pi_1(\Gamma, p) \to \pi_1(\Gamma, f(p)).$$

By shrinking a maximal tree in Γ down to a point via a deformation retract, we see that any finite connected graph is homotopy equivalent to a wedge of r circles for some $r \in \mathbb{Z}_{\geq 0}$. Thus, we have

$$\pi_1(\Gamma, p) \cong F_r,$$

where F_r denotes the free group of rank r. Fix an identification of $\pi_1(\Gamma, p)$ with $F_r = \langle x_1, \ldots, x_r \rangle$. Although $\pi_1(\Gamma, f(p))$ is isomorphic to $\pi_1(\Gamma, p)$, they are not literally the same group. Thus, even after identifying $\pi_1(\Gamma, p)$ with F_r, the map f_* is not an element of $\mathrm{Aut}(F_r)$.

To remedy this, we choose a path α in Γ from p to $f(p)$ and define

$$h_\alpha : \pi_1(\Gamma, f(p)) \to \pi_1(\Gamma, p)$$

by sending a loop ℓ based at $f(p)$ to the loop $\overline{\alpha}\ell\alpha$ based at p, where $\overline{\alpha}$ is α traversed backwards. Now,

$$h_\alpha \circ f_* : \pi_1(\Gamma, p) \to \pi_1(\Gamma, p)$$

is a well-defined element of $\mathrm{Aut}(F_r)$. However, different choices of α yield different automorphisms. Suppose β is another path from p to $f(p)$, and let $g \in \pi_1(\Gamma, p)$ be the element represented by the loop $\alpha\overline{\beta}$. Let $\gamma_g \in \mathrm{Aut}(F_r)$ denote conjugation by g. We observe that

$$h_\beta \circ f_* = \gamma_g \circ (h_\alpha \circ f_*).$$

Since $h_\beta \circ f_*$ and $h_\alpha \circ f_*$ differ by the inner automorphism γ_g, they represent the same element in

$$\mathrm{Out}(F_r) := \mathrm{Aut}(F_r)/\mathrm{Inn}(F_r).$$

Thus, we refer to f_* as an element of $\mathrm{Out}(F_r)$ without specifying a path from p to $f(p)$ and we study $\mathrm{Out}(F_r)$ via homotopy equivalences of graphs, analogous to how the mapping class group of a finite-type surface S is studied through homeomorphisms of S.

1.1.2 Stretch Factors

We are interested in the dynamics of elements in $\mathrm{Out}(F_r)$, particularly those that are dynamically rich, in the sense that they are irreducible (Definition 1.2.1) yet exhibit low dynamical complexity. We measure the dynamical complexity of $\varphi \in \mathrm{Out}(F_r)$ via its *stretch factor*, defined as

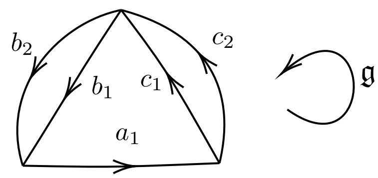
$$\lambda(\varphi) := \sup_{w \in F_r} \lim_{n\to\infty} ||\varphi^n(w)||^{1/n},$$

where $||w||$ denotes the cyclically reduced word length of $w \in F_r$ with respect to a fixed basis $x_1, \ldots, x_r$ of F_r. Since $||w||$ is constant within a conjugacy class, the value $||\varphi^n(w)||$ is well defined without specifying a particular representative of the outer class φ.

The limit $\lim_{n\to\infty} ||\varphi^n(w)||^{1/n}$ only detects exponential growth, so the stretch factor measures the maximal asymptotic exponential growth rate of words under repeated applications of φ. There are elements $\varphi \in \mathrm{Out}(F_r)$ and words $w \in F_r$ such that $||\varphi^n(w)||$ grows polynomially as $n \to \infty$. However, when φ is irreducible, every non-periodic conjugacy class grows exponentially with the same exponential growth rate.

Irreducible elements of $\mathrm{Out}(F_r)$ have an *irreducible train track representative*, (Definitions 1.2.11 and 1.2.13), that is a self homotopy equivalence of a graph of rank r, which induces φ on the fundamental group and has certain desirable properties under iteration [BH92]. The stretch factor of φ appears as the leading eigenvalue of the *transition matrix* (Definition 1.2.9) of such a train track representative, and hence is a *weak Perron number*, that is, a real positive algebraic integer which is larger than or equal to its algebraic conjugates in modulus.

Example 1.1.1 *Consider the self graph map $\mathfrak{g}$ as pictured below.*



$\mathfrak{g}:\ b_1 \mapsto c_1 \mapsto a_1 \mapsto b_2 \mapsto c_2 \mapsto \overline{b_1}\ \overline{c_1}$

This is an irreducible train track map representing the fully irreducible outer automorphism φ equal to the outer class of the automorphism $x_1 \mapsto x_2 \mapsto x_3 \mapsto x_3x_1^{-1}$ of $F_3 = \langle x_1, x_2, x_3\rangle$. The outer automorphism φ has stretch factor equal to the largest root of the polynomial $x^5 - x - 1$. In fact, φ attains the minimal stretch factor among fully irreducible elements of Out(F_3) [AHLP24b].

Conversely, Thurston showed every weak Perron number is the stretch factor of some outer automorphism [Thu14], [DDH$^+$24]. In Thurston's proof, he explicitly constructs an irreducible train track map with stretch factor equal to a given weak Perron number. The maps he constructs are all on $(1, N)-$bipartite graphs with 7 edges between the single vertex and each vertex in the N vertex set. There is no control on N, and hence no control on the rank of the corresponding free group. It remains an interesting question which weak Perron numbers can occur as stretch factors in a fixed rank. In particular, we are concerned with finding the minimal such stretch factor among irreducible elements:

$$\underline{\lambda}_r := \min\{\lambda(\varphi) : \ \varphi \in \mathrm{Out}(F_r) \text{ is irreducible}\}.$$

Since there are reducible outer automorphisms with exponential growth, and there are irreducible outer automorphisms which are not *fully irreducible* (Definition 1.2.1), one could also study two related minima:

(i) $\underline{\lambda}_r^{\mathrm{all}} := \min\{\lambda(\varphi) : \ \varphi \in \mathrm{Out}(F_r) \text{ and } \lambda(\varphi) > 1\}$, and

(ii) $\underline{\lambda}_r^{\mathrm{fully}} := \min\{\lambda(\varphi) : \ \varphi \in \mathrm{Out}(F_r) \text{ is fully irreducible}\}$.

We immediately see that

$$1 < \underline{\lambda}_r^{\text{all}} \leq \underline{\lambda}_r \leq \underline{\lambda}_r^{\text{fully}}.$$

One can also ask: when is there a strict inequality among these minima and when is the minimizing outer automorphism unique up to $\text{Out}(F_r)$ conjugation?

Progress has been made toward these questions:

(i) [AKR15] gives an upper and lower bound for this minimum in terms of the rank r,

$$3^{\frac{1}{3r-3}} \leq \underline{\lambda}_r \leq 2^{\frac{1}{r}},$$

(ii) [AHLP24b] precisely determines $\underline{\lambda}_3^{\text{fully}}$,

$$\underline{\lambda}_3^{\text{fully}} = \text{the largest root of } x^5 - x - 1,$$

(iii) in Corollary 1.8.1 (and in [Hil24a]), we show that

$$\underline{\lambda}_3 = \underline{\lambda}_3^{\text{fully}},$$

and the outer automorphism attaining this minimum is unique up to $\text{Out}(F_3)$ conjugation.

Since the stretch factor is the same among elements in the same $\text{Out}(F_r)$ conjugacy class, one could also study the number of conjugacy classes with a bounded stretch factor in a fixed rank r. In this direction, [KP24] explore the behavior of the number of fully irreducible conjugacy classes $[\varphi]$ in $\text{Out}(F_r)$ with $\log \lambda(\varphi) \leq L$ as $L \to \infty$.

Further, it would be interesting to determine the asymptotic behavior of these minima as the rank $r \to \infty$. By example, we know $\lim_{r\to\infty} \underline{\lambda}_r = 1$ and similarly for $\underline{\lambda}_r^{\text{all}}$ and $\underline{\lambda}_r^{\text{fully}}$.

However the limit of $\underline{\lambda}_r^r$ as $r \to \infty$ remains interesting. The bounds from [AKR15] imply

$$3^{\frac{1}{3}} \ \leq \ \lim_{r\to\infty} \underline{\lambda}_r^r \ \leq \ 2,$$

presuming the limit exists.

1.1.3 Surface Analogy

The mapping class group of a finite type surface S, denoted $\mathcal{MCG}(S)$, is the set of isotopy classes of homeomorphisms on S.[1] In the analogy between $\mathrm{Out}(F_r)$ and $\mathcal{MCG}(S)$, irreducible elements of $\mathrm{Out}(F_r)$ correspond to pseudo-Anosov elements of $\mathcal{MCG}(S)$[2]. In the mapping class group setting, every stretch factor of a pseudo-Anosov is a *bi-Perron* algebraic unit [T$^+$88], but it is still unknown exactly which such units can occur. In 2021, Pankau and Liechti used Thurston's construction of pseudo-Anosov homeomorphisms to show every bi-Perron unit λ whose algebraic conjugates lie in $S^1 \cup \mathbb{R}$ has a power which is a stretch factor of a pseudo-Anosov homeomorphism on a closed orientable surface of genus coarsely determined by the algebraic degree of λ [LP21]. However, there is no control on how large of a power one needs to take.

In 1991, Penner showed bounds on the minimal stretch factor in terms of the genus g for closed surfaces [Pen91]:

$$(A)^{\frac{1}{g}} \leq \min\{\lambda : \text{ pseudo-Anosov } f : S_g \to S_g \text{ has stretch factor } \lambda\} \leq (B)^{\frac{1}{g}}$$

for explicit constants A and B. Since then, minimal stretch factors among pseudo-Anosov elements on a fixed surface of finite type have been widely studied. Until very recently, the

[1]We allow for orientation reversing homeomorphisms in $\mathcal{MCG}(S)$.

[2]In different contexts, different types of outer automorphisms may be the appropriate analogous objects to pseduo-Anosov elements of $\mathcal{MCG}(S)$.

minimum was only known for a few isolated examples of surfaces. Consider an orientable surface of finite type with genus g and p punctures. Previously the minimum stretch factor among all pseudo-Anosov mapping classes was precisely known only for small values of (g, p), for example $(1, 0)$ and $(1, 1)$, $(2, 0)$ [CH08], and $(0, p)$ for $p = 3, 4, 5, 6, 7, 8$ ([SKL02], [HS07], [LT11a]). Recent breakthrough work of [TZ24] also computes the minimum stretch factor for $(0, p)$ for large p. In particular, they find

$$\lim_{p\to\infty}(\min\{\lambda : \text{ pseudo-Anosov } f : S_{0,p} \to S_{0,p} \text{ has stretch factor } \lambda\})^p = (2 + \sqrt{3})^2.$$

Among orientable pseudo-Anosov mapping classes, the minimum is known for 0 punctures and genus $g = 2, 3, 4, 5, 8$ ([LT11b], [Hir10]). For genus g surfaces with $p > 0$ punctures, $\pi_1(S_{g,p})$ is isomorphic to the free group of rank $r = 2g + p - 1$, and hence elements of the mapping class group correspond to outer automorphisms of F_r. Such outer automorphisms are called *geometric*. In a certain sense, outer automorphisms are generically not geometric, meaning they cannot be realized as a homeomorphism on a surface [Riv08]. Thus, while the study of $\mathrm{Out}(F_r)$ is often inspired by results in $\mathcal{MCG}(S)$, we also see new and unique behavior in $\mathrm{Out}(F_r)$.

1.1.4 Teichmüller Space and Outer Space

The Teichmüller space of a finite type surface S, denoted $\mathcal{T}(S)$, is the space of marked Riemann surfaces homeomorphic to S. The mapping class group of S acts properly discontinuously on $\mathcal{T}(S)$ by changing the marking. Pseudo-Anosov elements are exactly the mapping classes which act loxodromically on $\mathcal{T}(S)$ [Ber78]. Moreover, the translation length of a pseudo-Anosov $\varphi \in \mathcal{MCG}(S)$ along the invariant axis is exactly $\log \lambda(\varphi)$. The quotient of $\mathcal{T}(S)$ by the action of $\mathcal{MCG}(S)$ yields the moduli space of Riemann surfaces homeomorphic to S. Hence $\log \underline{\lambda}_S$ is the minimal length of a non-contractible loop in the moduli space of S and, more generally, stretch factors in $\mathcal{MCG}(S)$ give information about the geometry of the moduli space of S.

Culler-Vogtmann Outer Space, denoted CV_r, is similar to Teichmüller space, in the sense that it is a marked parameter space with a natural $\mathrm{Out}(F_r)$ action on the marking. Developed in the early 1980's as a tool for studying $\mathrm{Out}(F_r)$ [CV86], CV_r is the space of marked rank r graphs Γ such that each vertex $v \in \mathcal{V}\Gamma$ has valence ≥ 3, along with a choice of positive length $\ell(e)$ on each edge $e \in \mathcal{E}\Gamma$ such that $\sum \ell(e) = 1$. Fixing a rank r graph Γ with n edges and allowing the edge lengths to vary yields an $n-1$ dimensional open simplex in CV_r. There is a natural action of $\mathrm{Out}(F_r)$ on CV_r by changing the marking. Irreducible elements of $\mathrm{Out}(F_r)$ act loxodromically on CV_r, however, some reducible elements also act loxodromically [Bes11]. In order for $\varphi \in \mathrm{Out}(F_r)$ to act loxodromically, there must be some word $w \in F_r$ such that $||\varphi^n(w)||$ is growing exponentially as $n \to \infty$, and hence we must have $\lambda(\varphi) > 1$. However, this is not a sufficient condition, so we may consider another family of minima:

$$\underline{\lambda}_r^{\mathrm{lox}} := \min\{\lambda(\varphi) : \ \varphi \in \mathrm{Out}(F_r) \text{ acts loxodromically on } CV_r\}$$

and note that we now have the following inequalities:

$$1 < \ \underline{\lambda}_r^{\mathrm{all}} \ \leq \ \underline{\lambda}_r^{\mathrm{lox}} \ \leq \ \underline{\lambda}_r \ \leq \ \underline{\lambda}_r^{\mathrm{fully}}.$$

There is a natural non-symmetric metric on CV_r, and a given loxodromic element φ has minimal translation distance equal to $\log \lambda(\varphi)$ [3]. Similar to the surface case, quotienting CV_r by the $\mathrm{Out}(F_r)$ action yields the moduli space of metric rank r graphs. Hence when the moduli space is endowed with the induced non-symmetric metric from

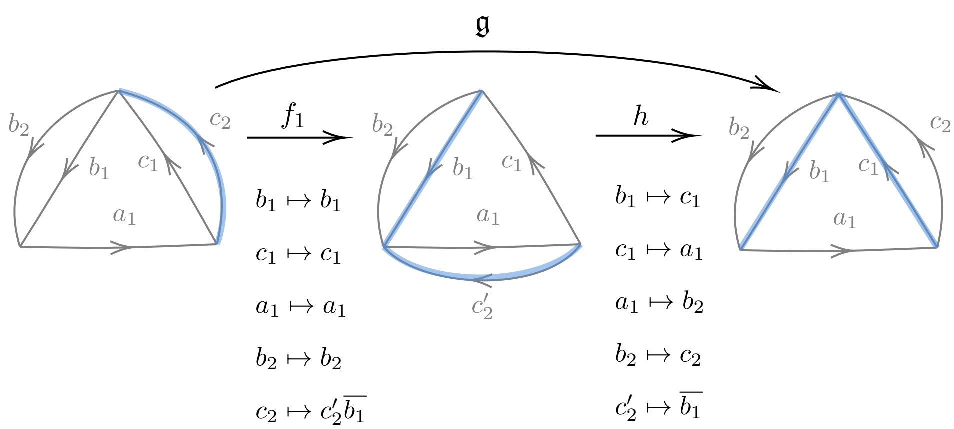
[3] In contrast to the surface case, there is generally not a unique invariant axis.

CV_r, $\log \underline{\lambda}_r^{\mathrm{lox}}$ is the minimal length of a non-contractible directed loop in this moduli space.

1.1.5 Summary of Results

Given a surjective *graph map* f, (see Definition 1.2.4), there is a non-unique decomposition of f as a sequence of graphs maps called *folds* (see Definition 1.2.19) and a graph isomorphism [Sta83].

Example 1.1.2 *The train track map $\mathfrak{g}$ in Example 1.1.1 can be decomposed as a single fold f_1 followed by a graph isomorphism h:*



Intuitively, one might expect that more folds in the fold decomposition of a train track map corresponds to a more complicated induced outer automorphism. This is captured in the following result.

Theorem A. *Suppose $f : \Gamma \to \Gamma$ is an irreducible homotopy equivalence self graph map with fold decomposition consisting of m total folds. Let $n = |\mathcal{E}\Gamma|$, where $\mathcal{E}\Gamma$ is the edge set of Γ. Then*

$$(m+1)^{\frac{1}{n}} \leq \lambda_f$$

where λ_f is the largest eigenvalue of the transition matrix of f.

Remark 1.1.3 *When f is an irreducible train track representative of $\varphi \in Out(F_r)$, we have $\lambda_f = \lambda(\varphi)$. Hence, given a specific stretch factor λ in some rank r, the above theorem gives a finite list of pairs (number of edges, number of folds) which could possibly correspond to an irreducible train track map with stretch factor less than λ.*

Remark 1.1.3 suggests a computational strategy for finding minimal stretch factors in $\mathrm{Out}(F_r)$. Knowing which rank r graphs can possibly support an irreducible train track map with at most m folds would reduce the computation involved in this procedure. As we require $f : \Gamma \to \Gamma$ is *irreducible* on the edges of Γ, and folds help ensure irreducibility, there is a delicate balance between reducing the number of folds and maintaining mixing amongst the edges of Γ under applications of f. With this in mind, and taking inspiration from the language of stacks and mixing edges introduced in [AKR15], we define a graph invariant called the *stack score* (see Definition 1.6.1), denoted $\mathfrak{S}(\Gamma)$. The stack score is a natural number which measures the latent symmetry of Γ. Informally, a lower stack score indicates a higher degree of latent symmetry. More latent symmetry facilitates greater mixing in the graph isomorphism that follows the folds, thereby reducing the required number of folds needed to attain an irreducible graph map.

Theorem B. *Any irreducible expanding homotopy equivalence self graph map $f : \Gamma \to \Gamma$ which is periodic on the vertex set of Γ must have at least $\mathfrak{S}(\Gamma)$ folds.*

It appears the condition that f is periodic on the vertex set (equivalently, f is a bijection on the vertex set) is not too restricive. For example, f having a Stallings fold decomposition consisting of only proper full folds (and a graph isomorphism) is enough to guarantee periodicity of the vertex set. However, if f has complete and partial folds, it may or may not be periodic on the vertices.

The stretch factor of $\varphi \in \mathrm{Out}(F_r)$ represented by an irreducible train track map $f : \Gamma \to \Gamma$ is the leading eigenvalue of the integral $|\mathcal{E}\Gamma| \times |\mathcal{E}\Gamma|$ transition matrix of f

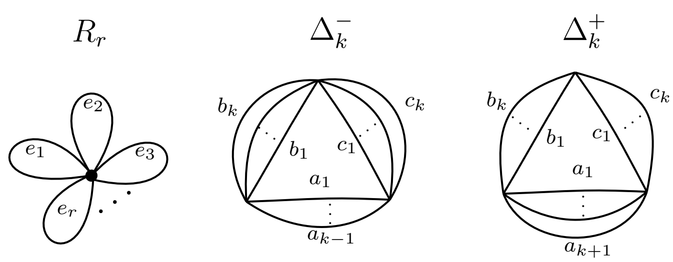
[BH92]. Hence the algebraic degree of the stretch factor is bounded from above by the number of edges of Γ. The following corollary, directly implied by Theorems A and B, is another example of a property of Γ affecting the set of possible stretch factors of train track maps on Γ.

Corollary 1.6.3 *Let $f : \Gamma \to \Gamma$ be an irreducible expanding homotopy equivalence self graph map which is periodic on the vertex set of* Γ*. Let $n = |\mathcal{E}\Gamma|$. Then*

$$(\mathfrak{S}(\Gamma) + 1)^{\frac{1}{n}} \leq \lambda_f$$

where $\mathfrak{S}(\Gamma)$ is the stack score of Γ *and λ_f is the leading eigenvalue of the transition matrix of f.*

Leveraging the restriction that a single fold irreducible self graph map must be periodic on the vertices and take place on a graph with stack score equal to 1, we obtain the following result, concerning the three families of graphs pictured below.



Theorem C. *Suppose* Γ *is a connected rank r graph and $f : \Gamma \to \Gamma$ is a single fold irreducible homotopy equivalence self graph map. Then* Γ *is isomorphic to R_r, Δ_k^+, or Δ_k^- for some $k \geq 2$. In particular:*

(i) if $r \equiv 0$ mod 3, then $\Gamma \cong G \in \{R_r, \Delta_k^-\}$,

(ii) if $r \equiv 1 \mod 3$*, then* $\Gamma \cong R_r$*, and*

(iii) if $r \equiv 2 \mod 3$*, then* $\Gamma \cong G \in \{R_r, \Delta_k^+\}$*,*

for appropriate values of k*.*

Examples 6.2 and 6.3 in [AKR15] are single fold irreducible train track maps on R_r and $\{\Delta_k^-, \Delta_k^+\}$, respectively. Algom-Kfir and Rafi conjecture these maps on Δ_k^+ and Δ_k^- attain the minimal stretch factor in their rank. For fully irreducible elements of $\mathrm{Out}(F_3)$, [AHLP24b] shows this is indeed the case for Δ_2^-, see Example 1.2.18. As a consequence of Theorems A and C, the $\mathrm{Out}(F_r)$ conjugacy class determined by $\mathfrak{g}$ on Δ_2^- is in fact the unique minimizing conjugacy class among irreducible elements in $\mathrm{Out}(F_3)$, see Corollary 1.8.1.

1.2 Background

Let $r \in \mathbb{Z}_{\geq 2}$ and F_r be the free group of rank r. We are interested in the *outer automorphisms of* F_r,

$$\mathrm{Out}(F_r) := \mathrm{Aut}(F_r)/\mathrm{Inn}(F_r).$$

In 1974, Thurston classified elements of $\mathcal{MCG}(S)$ as either reducible, finite-order, or pseudo-Anosov [T^+88]. Upon announcing his work, it was realized Nielsen made a similar discovery from a different perspective, and this classification is now known as the Nielsen–Thurston classification. Using the technology of train track maps on graphs, Bestvina and Handel developed an analogous classification of elements in $\mathrm{Out}(F_r)$ [BH92].

Definition 1.2.1 *(Reducible, Irreducible, Fully Irreducible) An element* $\varphi \in Out(F_r)$ *is called* ***reducible*** *if there are free factors* $A, B_1, \ldots, B_k$ *for* $k > 0$*, such that* $F_r =$

*$A * B_1 * \cdots * B_k$ and φ transitively permutes the conjugacy classes of the B_i. Otherwise, φ is **irreducible**. We say φ is **fully irreducible** if every power of φ is irreducible.*

Definition 1.2.2 *(Graph, Directed Graph) A **graph** Γ is a 1-dimensional CW complex whose 0-simplices are vertices, denoted $\mathcal{V}G$, and whose 1-simplices are edges, denoted $\mathcal{E}\Gamma$. Note that we allow for multiple edges between vertices, as well as self loops. We will always assume our graphs have finitely many edges and vertices.*

*When there is a choice of orientation on each edge, Γ is a **directed graph** and we let $\mathcal{E}\Gamma$ denote the set of positively oriented edges, $\mathcal{E}^-\Gamma$ the negatively oriented edges, and $\mathcal{E}^\pm\Gamma$ the union of both. We let $\overline{e}$ denote the edge e with reversed orientation. We have initial and terminal maps*

$$\iota, \tau : \mathcal{E}^\pm\Gamma \to \mathcal{V}\Gamma$$

given by $\iota(e) =$ initial vertex of e and $\tau(e) =$ terminal vertex of e.

Definition 1.2.3 *(Edge Path) An **edge path** in Γ is a nonempty concatenation of oriented edges $e_1 \dots e_k$ such that $\tau(e_i) = \iota(e_{i+1})$ for all $1 \leq i \leq k-1$. If $u = e_1 \dots e_k$ is an edge path, then*

(i) $\iota(u) := \iota(e_1)$,

(ii) $\tau(u) := \tau(e_k)$, *and*

(iii) $\overline{u} := \overline{e_k} \dots \overline{e_1}$.

Let $\mathcal{EP}\Gamma$ denote the set of edge paths in Γ. Note that we can interpret $\mathcal{E}^\pm\Gamma$ as a subset of $\mathcal{EP}\Gamma$ by identifying an oriented edge e with the edge path equal to e.

Definition 1.2.4 *(Graph Map) Given graphs Γ_1 and Γ_2, a **graph map** $f : \Gamma_1 \to \Gamma_2$ consists of maps* *(i)* $f_V : \mathcal{V}\Gamma_1 \to \mathcal{V}\Gamma_2$, *and*

(ii) $f_E : \mathcal{E}^{\pm}\Gamma_1 \to \mathcal{EP}\Gamma_2$ *such that* $f_V(\iota(e)) = \iota(f_E(e))$ *and* $f_E(\bar{e}) = \overline{f_E(e)}$ *for every* $e \in \mathcal{E}^{\pm}\Gamma$.

Notation 1.2.5 *Given an edge path* u *in a graph* Γ*, we use* $|u|$ *to denote the number of edges in* u*. We say* u *traverses* $e \in \mathcal{E}\Gamma$ *if* e *or* $\bar{e}$ *appears as an edge in* u*. Note that if a sequence* $e\bar{e}$ *appears in an edge path* u*, both* e *and* $\bar{e}$ *contribute to the number of edges in* u*. In other words, we do not tighten the path* u *before counting the number of edges. Thus* $|f(u)| \geq |u|$ *for any graph map* f *and edge path* u*.*

Definition 1.2.6 *(Graph Isomorphism, Graph Automorphism) A graph map* $f : \Gamma_1 \to \Gamma_2$ *is a* ***graph isomorphism*** *if*

(i) f_V *is a bijection, and*

(ii) f_E *is injective with image equal to* $\mathcal{E}^{\pm}\Gamma_2$.

A graph isomorphism $f : \Gamma \to \Gamma$ *is a* ***graph automorphism***.

Notation 1.2.7 *Given a graph map* $f : \Gamma_1 \to \Gamma_2$*, we often drop the subscripts on the corresponding maps on the vertices and edges, and just write* $f(e)$ *for* $f_E(e)$ *and* $f(v)$ *for* $f_V(v)$ *when it is clear that* e *is an edge and* v *is a vertex.*

Notation 1.2.8 *When* Γ_1 *has no isolated vertices, a graph map* $f : \Gamma_1 \to \Gamma_2$ *is entirely determined by* f_E *restricted to the set of positively oriented edges of* Γ_1*. We will often define a graph map by just giving its image on every positively oriented edge.*

In order to define graph maps on Γ, we always assume our graphs have an orientation on each edge. However, since edge paths can traverse edges backwards, these orientations do not carry meaningful information about the nature of the graph itself (with the exception of stack graphs, see Definition 1.4.1).

If $f : \Gamma \to \Gamma$ is a homotopy equivalence on a connected rank r graph Γ, after a choice of identification of $\pi_1(\Gamma)$ with F_r, we say that $f : \Gamma \to \Gamma$ *topologically represents* the outer automorphism φ induced by f on $\pi_1(\Gamma)$. Different choices of identification of $\pi_1(\Gamma)$ with F_r give $\mathrm{Out}(F_r)$-conjugate outer automorphisms.

Definition 1.2.9 *(Transition Matrix) Given a self graph map $f : \Gamma \to \Gamma$, and an order on the set of edges $\{e_1, \dots, e_n\}$, the **transition matrix** of f, denoted $T(f)$, is the $|\mathcal{E}\Gamma| \times |\mathcal{E}\Gamma|$ matrix (a_{ij}) where a_{ij} is the number of times $f(e_i)$ traverses e_j in either direction.*

Definition 1.2.10 *(Irreducible, Primitive) Let M be an $n \times n$ matrix.*

*(i) M is **irreducible** if for each $1 \leq i, j \leq n$, there is a power k such that the ij-th entry of M^k is positive. When M is non-negative, this is equivalent to requiring that M has no non-trivial proper invariant coordinate subspaces. The coordinate subspaces are those which are spanned by a subset of the standard basis elements in $\mathbb{R}^n$.*

*(ii) M is **primitive** if it is non-negative and there is a power k such that all entries of M^k are positive.*

Definition 1.2.11 *(Irreducible Graph Map) We call a self graph map $f : \Gamma \to \Gamma$ **irreducible** if $T(f)$ is an irreducible matrix and the valence of every vertex in Γ is at least 3.*

Definition 1.2.12 *(Expanding Graph Map) A self graph map $f : \Gamma \to \Gamma$ is **expanding** if $|f^n(e)| \to \infty$ as $n \to \infty$ for every edge $e \in \mathcal{E}\Gamma$. When f is an irreducible homotopy equivalence, this is equivalent to requiring the largest eigenvalue of $T(f)$ is strictly greater than 1 in modulus (see Lemma 1.2.21).*

Definition 1.2.13 *(Train Track Map) A self graph map $f : \Gamma \to \Gamma$ is a* ***train track map*** *if it is a homotopy equivalence and for all powers $n \in \mathbb{N}$, f^n is locally injective on the interior of every edge e.*

We will sometimes refer to an irreducible train track map as an i.t.t. map and an irreducible homotopy equivalence graph map as an i.h.e. map. Our proofs do not use the locally injective property of train track maps, and hence our results are stated for i.h.e. maps.

The following theorem reduces the question of stretch factors of irreducible outer automorphisms to a question about leading eigenvalues of their i.t.t. representatives.

Theorem 1.2.14 ([BH92]) *Every irreducible outer automorphism $\varphi \in Out(F_r)$ is represented by an irreducible train track map $f : \Gamma \to \Gamma$ on a connected rank r graph Γ. The leading eigenvalue of $T(f)$, denoted λ_f, is real, positive, and equal to the stretch factor of φ. Moreover, there is a length function ℓ on the edges of Γ such that f is uniformly λ_f-expanding on (Γ, ℓ). That is, $\ell(f(e)) = \lambda_f \ell(e)$ for every $e \in \mathcal{E}\Gamma$. Further, φ is a finite-order homeomorphism if and only if $\lambda_f = 1$.*

However, it should be noted that while every irreducible outer automorphism has an i.t.t. representative, a given i.t.t. map could induce an outer automorphism which is reducible.

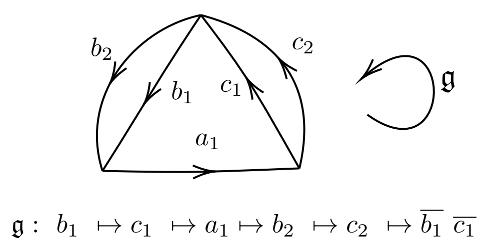
In [AKR15], Algom-Kfir and Rafi define *mixing edges* and *stacks* of graph maps. We recall their definitions here.

Definition 1.2.15 *[AKR15] (Mixing Edge) Given a graph map $f : \Gamma_1 \to \Gamma_2$, an edge e is called a* ***mixing edge*** *if $f(e)$ is an edge path consisting of more than one edge.*

Definition 1.2.16 *(Surplus Edge) Given a graph map $f : \Gamma_1 \to \Gamma_2$, an edge e is called a* ***surplus edge*** *if e is non-mixing and $f(e) \in \{f(u), \overline{f(u)}\}$ for some edge $u \in \mathcal{E}\Gamma_1$ with $u \notin \{e, \overline{e}\}$.*

Definition 1.2.17 *[AKR15] (Stack) Given a self graph map $f : \Gamma \to \Gamma$, let $\sim$ be an equivalence relation on the unoriented edges of Γ generated by $e \sim f(e)$ if e is non-mixing and non-surplus. An equivalence class of edges is called a* ***stack*** [4]. *The stacks of f partition $\mathcal{E}\Gamma$.*

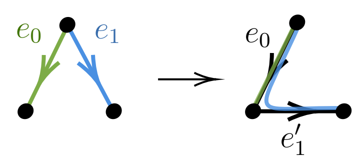
Example 1.2.18 *Let $\mathfrak{g} : \Delta_2^- \to \Delta_2^-$ be as in Example 1.1.1. Observe that $\mathfrak{g}$ has a single stack equal to $\mathcal{E}\Delta_2^-$ and a single mixing edge, c_2.*



This is an expanding i.t.t. map representing the fully irreducible outer automorphism $\varphi : x_1 \mapsto x_2 \mapsto x_3 \mapsto x_3x_1^{-1}$, which has minimal stretch factor among fully irreducible elements of Out(F_3) [AHLP24b].

Definition 1.2.19 *(Folds) Given a directed graph Γ and two edges $e_0, e_1 \in \mathcal{E}^{\pm}\Gamma$ such that $\iota(e_0) = \iota(e_1)$, there are three procedures, called* ***folds****, to form a new graph Γ' and a surjective graph map $f : \Gamma \to \Gamma'$. We describe these three types of folds first in terms of a procedure. Then, we give the equivalent definition of these folds in terms of a quotient graph and a quotient map. The latter definition is more standard, but the former definition determines our convention for labels on Γ'.*

(i) (Proper Full Fold) Let Γ' be the graph with $\mathcal{V}\Gamma' = \mathcal{V}\Gamma$ and $\mathcal{E}\,\Gamma' = (\mathcal{E}\Gamma - \{e_1\}) \cup \{e_1'\}$, where e_1' has $\iota(e_1') := \tau(e_0)$ and $\tau(e_1') := \tau(e_1)$. Let $f : \Gamma \to \Gamma'$ be given by:

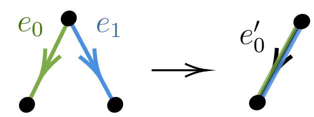


$$f(e) = \begin{cases} e_0e_1' & \text{if } e = e_1 \\ e & \text{otherwise} \end{cases}$$

[4] This definition of stack differs slightly from that in [AKR15], as we allow $e \sim f(e)$ even if $f(e)$ appears in the image of a mixing edge.

f is called the *proper full fold of* e_1 *over* e_0. Equivalently, subdivide $e_1 \in \mathcal{E}\Gamma$: let v' be a new vertex in the middle of e_1 and relabel e_1 as two edges e_1'' and e_1', oriented so that e_1 is now equal to the edge path $e_1''e_1'$. Now, let $\Gamma' = \Gamma/e_1'' \sim e_0$, and let $f : \Gamma \to \Gamma'$ be the quotient map.

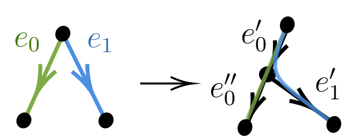
(ii) (Complete Fold) Let Γ' be the graph resulting from identifying the vertices $\iota(e_0)$ and $\iota(e_1)$ and identifying the edges e_0 and e_1 into a new edge labelled e_0'. Let $f : \Gamma \to \Gamma'$ be given by:



$$f(e) = \begin{cases} e_0' & \text{if } e \in \{e_0, e_1\} \\ e & \text{otherwise} \end{cases}$$

f is called the *complete fold of* e_1 *and* e_0. Equivalently, let $\Gamma' = \Gamma/e_1 \sim e_0$, and let $f : \Gamma \to \Gamma'$ be the quotient map. If f is a fold in a fold decomposition of a homotopy equivalence, then $\tau(e_0) \neq \tau(e_1)$.

(iii) (Partial Fold) Let Γ' be the graph with $\mathcal{V}\Gamma' = \mathcal{V}\Gamma \cup \{v'\}$ and $\mathcal{E}\Gamma' = (\mathcal{E}\Gamma - \{e_0, e_1\}) \cup \{e_0', e_0'', e_1'\}$, where e_0' joins $\iota(e_0)$ to v', e_0'' joins v' to $\tau(e_0)$, and e_1' joins v' to $\tau(e_1)$. Let $f : \Gamma \to \Gamma'$ be given by:



$$f(e) = \begin{cases} e_0'e_0'' & \text{if } e = e_0 \\ e_0'e_1' & \text{if } e = e_1 \\ e & \text{otherwise} \end{cases}$$

f is called the *partial fold of* e_1 *over* e_0. Equivalently, subdivide $e_0 \in \mathcal{E}\Gamma$: let v' be a new vertex in the middle of e_0 and relabel e_0 as two edges e_0' and e_0'', oriented so that e_0 is now equal to the edge path $e_0'e_0''$. Subdivide $e_1 \in \mathcal{E}\Gamma$: let v'' be a new vertex in the middle of e_1 and relabel e_1 as two edges e_1'' and e_1', oriented so that e_1' is now equal to the edge path $e_1''e_1'$. Now, let $\Gamma' = \Gamma/e_1'' \sim e_0'$, and let $f : \Gamma \to \Gamma'$ be the quotient map.

Theorem 1.2.20 ([Sta83]) *Every surjective homotopy equivalence graph map $f : \Gamma \to \Gamma'$ can be decomposed as $f = h \circ f_m \circ \cdots \circ f_2 \circ f_1$ where $\Gamma_1 = \Gamma$, each $f_i : \Gamma_i \to \Gamma_{i+1}$ is a fold, and $h : \Gamma_{m+1} \to \Gamma'$ is an graph isomorphism.*

In particular, i.h.e. maps are surjective, and thus have such a fold decomposition. For instance, in Example 1.1.2, $\mathfrak{g}$ is decomposed as a single proper full fold of c_2 over $\overline{b_1}$ and a graph isomorphism.

We collect some known observations in the following lemma.

Lemma 1.2.21 *Suppose $f : \Gamma \to \Gamma$ is an i.h.e. graph map with fold decomposition consisting of m folds and a graph isomorphism $h : \Gamma' \to \Gamma$. Let λ_f denote the greatest eigenvalue of $T(f)$ in modulus. Then there is a choice of positive length ℓ on each edge in Γ such that for every $e \in \mathcal{E}\Gamma$, we have $\ell(f(e)) = \lambda_f \ell(e)$ where $\ell(u) := \sum_{i=1}^{k} \ell(b_i)$ when $u = b_1 b_2 \dots b_k$ is an edge path. Moreover, the following are equivalent:*

(i) $m = 0$,

(ii) *there is a power $n \in \mathbb{N}$ such that f^n is the identity on Γ,*

(iii) $\lambda_f = 1$,

(iv) *f is not expanding.*

Proof: Suppose $T(f)$ is the transition matrix of f with respect to an edge ordering $\{e_1, \dots, e_n\} = \mathcal{E}\Gamma$. Since $T(f)$ is irreducible, the Perron–Frobenius Theorem guarantees there is a left eigenvector $\vec{v}$ with positive entries such that $\vec{v}\,T(f) = \lambda_f \vec{v}$. Use the entries of $\vec{v} = [v_1, \ldots, v_n]$ to assign the length v_i to the corresponding edge e_i. Letting $\{a_i^1, \ldots, a_i^n\}$ denote the entries of the $i-$th column of $T(f)$, we have

$$\begin{aligned}\ell(f(e_i)) &= \sum_{j=1}^{n} a_i^j \ell(e_j)\\ &= \sum_{j=1}^{n} a_i^j v_j\\ &= \lambda_f v_i.\end{aligned}$$

Hence $\ell(f(e)) = \lambda_f \ell(e)$ for each $e \in \mathcal{E}\Gamma$.

(i) $\Rightarrow$ (ii): Suppose $m = 0$. Then f is a graph isomorphism and hence a bijection on the set of oriented edges of Γ. Thus there is a power n such that f^n is equal to the identity.

(ii) $\Rightarrow$ (iii): If f^n is the identity, then $(\lambda_f)^n = 1$, so $|\lambda_f| = 1$. The Perron–Frobenius theorem guarantees λ_f is real, positive and greater than or equal to 1. Thus $\lambda_f = 1$.

(iii) $\Rightarrow$ (iv): Now suppose $\lambda_f = 1$. Thus $\ell(f^n(e)) = \ell(e)$ for each $e \in \mathcal{E}\Gamma$ and power $n \in \mathbb{N}$. Since the length of each edge is positive, $|f^n(e)|$ is bounded from above for all $n \in \mathbb{N}$. Hence f is not expanding.

(iv) $\Rightarrow$ (i): Proceeding by contrapositive, suppose $m > 0$. If the fold decomposition consisted of only complete folds, then $|\mathcal{V}\Gamma'| < |\mathcal{V}\Gamma|$, contradicting that $h : \Gamma' \to \Gamma$ is a graph isomorphism. Thus there is at least one fold which is a proper full fold or a partial fold, and hence some edge $b \in \mathcal{E}\Gamma$ with $|f(b)| > 1$. Let $e \in \mathcal{E}\Gamma$ be any edge. Since f is irreducible, there is a power k such that $f^k(e)$ traverses b, and a power p such that $f^p(b)$ traverses b. Hence $f^{np}(f^k(e))$ traverses b for each $n \in \mathbb{N}$. Since $|f(b)| > 1$, we have $|f^{np+k+1}(e)| > |f^{np+k}(e)|$ for each $n \in \mathbb{N}$. Since $|f(u)| \geq |u|$ for any edge path u,

$$\{|f^n(e)|\}_{n=1}^{\infty}$$

is a non-decreasing sequence of integers which strictly increases for each $n \equiv k+1 \mod p$. Therefore $|f^n(e)| \to \infty$ and hence f is expanding.

■

1.3 Folds and Mixing

The following lemmas relating folds, mixing edges, and stacks will provide key facts for our lower bound and symmetry results.

Lemma 1.3.1 *Suppose $f : \Gamma \to \Gamma$ is an expanding i.h.e. map. Then each stack of f has the form $\mathcal{K} = \{e, f(e), f^2(e), \ldots, f^s(e)\}$ with only $f^s(e)$ either mixing or surplus.*

Proof: Let $\mathcal{K}$ be a stack of f and suppose $e \in \mathcal{K}$. If $f^t(e)$ is non-mixing and non-surplus for all $0 \le t \le k$, then

$$\{e, f(e), \ldots, f^k(e), f^{k+1}(e)\} \subseteq \mathcal{K}.$$

By definition of stack, these edges are distinct as unoriented edges, except possibly $f^{k+1}(e) \in \{e, \overline{e}\}$. Suppose $f^{k+1}(e) \in \{e, \overline{e}\}$. Then for any $b \in \{e, f(e), \ldots, f^k(e)\}$, we have $f^n(b)$ or $f^n(\overline{b})$ is an edge in this same set. By irreducibility of $T(f)$, we must have

$$\{e, f(e), \ldots, f^k(e)\} = \mathcal{E}\Gamma.$$

Thus $T(f)$ is a permutation matrix, so $\lambda_f = 1$. By Lemma 1.2.21, this contradicts that f is expanding. Thus $f^{k+1}(e) \notin \{e, \overline{e}\}$.

Since $\mathcal{E}\Gamma$ is finite, eventually there is a first power s such that $f^s(e)$ is either mixing or surplus. Suppose $\mathcal{K} - \{e, f(e), \dots, f^s(e)\} \neq \emptyset$. Then there must be an edge e' such that $f(e') = e$. Thus

$$\{e', f(e'), f^2(e'), \dots, f^{s+1}(e')\} \subseteq \mathcal{K}.$$

Once again, if $\mathcal{K} - \{e', f(e'), \dots, f^{s+1}(e')\} \neq \emptyset$, there is a e'' such that $f(e'') = e'$, so

$$\{e'', f(e''), f^2(e''), \dots, f^{s+2}(e'')\} \subseteq \mathcal{K}.$$

Since $\mathcal{E}\Gamma$ is finite, this process eventually terminates, so $\mathcal{K}$ has the desired format.

■

Definition 1.3.2 *(Root Edge, Final Edge) Given a stack $\mathcal{K} = \{e, f(e), f^2(e), \dots, f^s(e)\}$, we call e the root edge of $\mathcal{K}$ and $f^s(e)$ the final edge of $\mathcal{K}$.*

Lemma 1.3.3 *Suppose $f : \Gamma \to \Gamma$ is an expanding i.h.e. map with fold decomposition consisting of m total folds and p total stacks. Then*

$$m \leq \sum_{e \in \mathcal{E}\Gamma} \Big(|f(e)| - 1 \Big). \tag{1.1}$$

Moreover, if f is periodic on the vertices of Γ, then $p \leq m$.

Proof: Write $f = h \circ f_m \circ \cdots \circ f_2 \circ f_1$ where $\Gamma_1 = \Gamma$, each $f_i : \Gamma_i \to \Gamma_{i+1}$ is a fold and $h : \Gamma_{m+1} \to \Gamma$ is a graph isomorphism. To keep track of the number of edges in the image as each fold f_i is applied, let $T_0 = 0$ and

$$T_i = \sum_{e \in \mathcal{E}\Gamma} \Big(|(f_i \circ \cdots \circ f_1)(e)| - 1 \Big).$$

Claim:

(i) If f_i is a proper full fold, then $T_i \geq 1 + T_{i-1}$ and $|\mathcal{V}\Gamma_{i+1}| = |\mathcal{V}\Gamma_i|$.

(ii) If f_i is a complete fold, then $T_i = T_{i-1}$ and $|\mathcal{V}\Gamma_{i+1}| = |\mathcal{V}\Gamma_i| - 1$.

(iii) If f_i is a partial fold, then $T_i \geq 2 + T_{i-1}$ and $|\mathcal{V}\Gamma_{i+1}| = |\mathcal{V}\Gamma_i| + 1$.

Assuming the claim for now, we have

$$T_m \geq (\text{number of proper full folds}) + 2(\text{number of partial folds})$$

and

$$|V\Gamma_{m+1}| = |V\Gamma| + (\text{number of partial folds}) - (\text{number of complete folds}).$$

Since $h : \Gamma_{m+1} \to \Gamma$ is a graph isomorphism, $|\mathcal{V}\Gamma_{m+1}| = |\mathcal{V}\Gamma|$, so the number of complete folds must be equal to the number of partial folds. Further, for any edge path u, we have $|h(u)| = |u|$, again since h is a graph isomorphism. Therefore

$$\begin{aligned}
\sum_{e \in \mathcal{E}\Gamma} \Big(|f(e)| - 1\Big) &= T_m \\
&\geq (\text{number of proper full folds}) + 2(\text{number of partial folds}) \\
&= (\text{number of proper full folds}) + (\text{number of partial folds}) \\
&\qquad + (\text{number of complete folds}) \\
&= m.
\end{aligned}$$

This completes the proof of Equation 1.1. We now prove the claim and subsequently argue that $p \leq m$ when f is periodic on $\mathcal{V}\Gamma$.

Proof of Claim (i): Suppose $f_i : \Gamma_i \to \Gamma_{i+1}$ is a proper full fold of e_1 over e_0. By definition, $|\mathcal{V}\Gamma_{i+1}| = |\mathcal{V}\Gamma_i|$ and

$$f_i(e) = \begin{cases} e_0' e_1 & e = e_1 \\ e & \text{otherwise} \end{cases}$$

Let $u \in \mathcal{E}\Gamma$. If $(f_{i-1} \circ \cdots \circ f_1)(u)$ traverses e_1 a total of k times, then $|(f_i \circ \cdots \circ f_1)(u)| = |(f_{i-1} \circ \cdots \circ f_1)(u)| + k$. Since each f_j is surjective, there must be at least one u with $k > 0$. Hence $T_i \geq T_{i-1} + 1$. ◇

Proof of Claim (ii): Suppose $f_i : \Gamma_i \to \Gamma_{i+1}$ is a complete fold of e_1 and e_0. Since f is a homotopy equivalence, $\tau(e_0) \neq \tau(e_1)$. Thus $|\mathcal{V}\Gamma_{i+1}| = |\mathcal{V}\Gamma_i| - 1$. By definition,

$$f_i(e) = \begin{cases} e_0' & e \in \{e_0, e_1\} \\ e & \text{otherwise} \end{cases}$$

For all $u \in \mathcal{E}\Gamma$, we have $|(f_i \circ \cdots \circ f_1)(u)| = |(f_{i-1} \circ \cdots \circ f_1)(u)|$, so $T_i = T_{i+1}$. ◇

Proof of Claim (iii): Suppose $f_i : \Gamma_i \to \Gamma_{i+1}$ is a partial fold of e_1 over e_0. By definition, $|\mathcal{V}\Gamma_{i+1}| = |\mathcal{V}\Gamma_i| + 1$ and

$$f_i(e) = \begin{cases} e_0' e_0'' & e = e_0 \\ e_0' e_1' & e = e_1 \\ e & \text{otherwise} \end{cases}$$

Let $u \in \mathcal{E}\Gamma$. If $(f_{i-1} \circ \cdots \circ f_1)(u)$ traverses e_0 and e_1 a total of k times, then $|(f_i \circ \cdots \circ f_1)(u)| = |(f_{i-1} \circ \cdots \circ f_1)(u)| + k$. Since each f_j is surjective, there must be at least one u with $(f_{i-1} \circ \cdots \circ f_1)(u)$ traversing e_0 at least once, and at least one u with $(f_{i-1} \circ \cdots \circ f_1)(u)$ traversing e_1 at least once. Hence $T_i \geq T_{i-1} + 2$. ◇

Now suppose f is periodic on the vertices of Γ. Suppose distinct edges $e_1, e_2 \in \mathcal{E}\Gamma$ are surplus and $f(e_1) = f(e_2)$. Since f is a bijection on the vertices, we must have

$\iota(e_1) = \iota(e_2)$ and $\tau(e_1) = \tau(e_2)$. Hence $e_1\overline{e_2}$ is a closed loop Γ which is not null-homotopic. However, $f(e_1\overline{e_2}) = f(e_1)\overline{f(e_1)}$ is null-homotopic, contradicting that f is a homotopy equivalence. Therefore there are no surplus edges, and hence by Lemma 1.3.1, the final edge in each stack is mixing. Let $\alpha_1, \ldots, \alpha_p$ denote these final mixing edges. We will make an assignment of each α_k to a fold f_{i_k} in the following way:

Recursively label $f_i(\alpha_k)$ as $\alpha_k \in \mathcal{E}\Gamma_{i+1}$ whenever $|f_i(\alpha_k)| = 1$. This agrees with the labelling determined in Definition 1.2.19. If α_k nor $\overline{\alpha_k}$ is never properly folded over an edge, nor involved in a partial fold, then $|f(\alpha_k)| = 1$ contradicting that α_k is mixing. Thus, possibly replacing α_k with $\overline{\alpha_k}$, there must exist a first fold f_{i_k} and some $e_0 \in \mathcal{E}\Gamma_{i_k}$ such that

(i) f_{i_k} is a proper full fold of α_k over e_0 and $f_{i_k}(\alpha_k) = \alpha_k' e_0$, or

(ii) f_{i_k} is a partial fold of α_k over e_0 and $f_{i_k}(\alpha_k) = \alpha_k' e_0'$, or

(iii) f_{i_k} is a partial fold of e_0 over α_k and $f_{i_k}(\alpha_k) = e_0'' e_0$.

To each proper full fold, either one or zero mixing edges are assigned. To each partial fold, either two, one, or zero mixing edges are assigned. As argued above, the number of partial folds is equal to the number of complete folds. Since all p mixing edges are assigned to some proper full fold or partial fold, there are at least p folds.

■

1.4 Stack Graphs

To prove Theorem A, we introduce a tool called the stack graph, which captures the interactions between the stacks in a graph map. Alternatively, combining Lemma 1.3.3

with Lemma 1.5.1 ([HS07]) yields a proof of Theorem A for i.h.e. maps with primitive transition matrices, which avoids the need for stack graphs.

For the duration of this section, let $f : \Gamma \to \Gamma$ be an irreducible expanding self graph map with stacks $\mathcal{K}_1, \dots, \mathcal{K}_p$. For each $1 \leq i \leq p$, let n_i be the number of edges in stack $\mathcal{K}_i$ and α_i the final edge in stack $\mathcal{K}_i$. Let n be the total number of edges in Γ and note that $n = \sum_{i=1}^{p} n_i$.

Definition 1.4.1 *(Stack Graph, Weight ω) The **stack graph of f**, denoted $\mathcal{SG}(f)$, is a directed graph with vertex set $\mathcal{V}(\mathcal{SG}(f)) = \{\mathcal{K}_1, \dots, \mathcal{K}_p\}$ and directed edges:*

$$\mathcal{E}^+\mathcal{SG}(f) = \{[\mathcal{K}_i, \mathcal{K}_j] \mid f(\alpha_i) \text{ contains an edge in } \mathcal{K}_j\}.$$

We assign a weight ω to the vertices of $\mathcal{SG}(f)$:

$$\omega(\mathcal{K}_i) := |f(\alpha_i)| - 1.$$

Note that $\omega(\mathcal{K}_i) = 0$ if and only if the final edge α_i is surplus, instead of mixing.

Observation 1.4.2 *Any non-final edge e is non-mixing, and hence has $|f(e)| = 1$. When f is an expanding i.h.e. map, by Lemma 1.3.3 we have*

$$\begin{aligned}\sum_{j=1}^{p} \omega(\mathcal{K}_j) &= \sum_{j=1}^{p} \Big(|f(\alpha_j)| - 1\Big) \\ &= \sum_{e \in \mathcal{E}\Gamma} \Big(|f(e)| - 1\Big) \geq m.\end{aligned}$$

where m is the number of folds in the fold decomposition of f.

Definition 1.4.3 *(Length s, Directed ball of size d) We assign a length s to the edges of $\mathcal{SG}(f)$:*

$$s([\mathcal{K}_i, \mathcal{K}_j]) := \min\{s \mid f^s(\alpha_i) \text{ traverses } \alpha_j\}$$

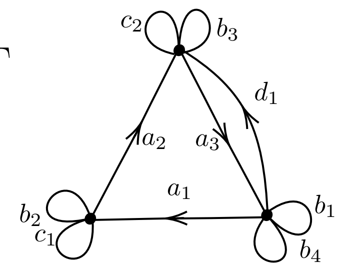
*Observe that by the definition of $\mathcal{E}^+\mathcal{SG}(f)$, we have $s([\mathcal{K}_i, \mathcal{K}_j]) \leq n_j$. For any number d and $\mathcal{K}_i \in \mathcal{V}(\mathcal{SG}(f))$, let the **directed ball of size** d **at** $\mathcal{K}_i$, be*

$$B_d(\mathcal{K}_i) := \{\mathcal{K}_j \in \mathcal{V}(\mathcal{SG}(f)) \mid \text{there is a directed edge path } P \text{ in } \mathcal{SG}(f) \text{ from } \mathcal{K}_i \text{ to } \mathcal{K}_j \text{ with } s(P) \leq d\}.$$

where $P = E_1 \dots E_k$ must only traverse edges with positive orientation and $s(P) := \sum_{i=1}^{k} s(E_i)$

Example 1.4.4 *Consider the irreducible expanding self graph map $\mathfrak{f} : \Gamma \to \Gamma$, written in stack format to the right.*

Γ



Below and to the right is the stack graph of $\mathfrak{f}$, $\mathcal{SG}(\mathfrak{f})$. The length of edges are labeled, and the weight of each vertex in $\mathcal{SG}(\mathfrak{f})$ is listed. For example, a_3 is the final edge in stack a and

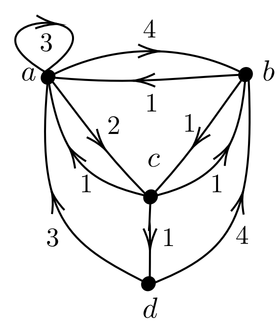
$$\mathfrak{f} : \begin{cases} a_1 \mapsto a_2 \mapsto a_3 \mapsto b_1 a_1 c_1 \\ b_1 \mapsto b_2 \mapsto b_3 \mapsto b_4 \mapsto c_1 a_2 c_2 a_3 a_1 \\ c_1 \mapsto c_2 \mapsto d_1 b_3 a_3 b_4 b_1 \\ d_1 \mapsto \overline{a_1} b_1 \end{cases}$$

$$\mathfrak{f}^3(a_3) = b_3 a_3 d_1 b_3 a_3 b_4 b_1$$

contains the final edge in stacks a, b, and d. There are directed paths of length 3 in $\mathcal{SG}(\mathfrak{f})$ from a to a, b, and d. In contrast, there is no directed path of length 3 from a to c.

$\mathcal{SG}(\mathfrak{f})$

$\omega(a) = 2$
$\omega(b) = 4$
$\omega(c) = 4$
$\omega(d) = 1$



Lemma 1.4.5 *If there is a directed path P in $\mathcal{SG}(f)$ from $\mathcal{K}_i$ to $\mathcal{K}_j$ with $s(P) = d$, then $f^d(\alpha_i)$ traverses α_j.*

Proof: Suppose a directed path P with $s(P) = d$ has vertices $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_k$ and let $s_i = s([\mathcal{K}_i, \mathcal{K}_{i+1}])$. Hence $d = \sum_{i=1}^{k} s_i$. By definition of s, $f^{s_i}(\alpha_i)$ traverses α_{i+1}. Hence $f^d(\alpha_1) = f^{s_k} \circ \cdots \circ f^{s_1}(\alpha_1)$ traverses α_k. ■

Lemma 1.4.6 *$\mathcal{SG}(f)$ is strongly connected and for any $\mathcal{K}_i \in \mathcal{V}(\mathcal{SG}(f))$, we have*

$$\mathcal{V}(\mathcal{SG}(f)) \subseteq B_{n-n_i}(\mathcal{K}_i).$$

Proof: Let $\mathcal{K}_i, \mathcal{K}_j \in \mathcal{V}(\mathcal{SG}(f))$. Since f is irreducible, there is a power s such that $f^s(\alpha_i)$ traverses α_j.

- Let b_s be either α_j or $\overline{\alpha_j}$, whichever appears in $f^s(\alpha_i)$.
- Let b_{s-1} be a single edge in $f^{s-1}(\alpha_i)$ such that b_s appears in $f(b_{s-1})$.
- For $2 \leq t \leq s$, let b_{s-t} be a single edge in $f^{s-t}(\alpha_i)$ such that b_{s-t+1} appears in $f(b_{s-t})$.

Hence $b_0 = \alpha_i$, and $f(b_t)$ contains b_{t+1} for all $0 \leq t \leq s-1$. Whenever b_t is a non-final edge, $f(b_t) = b_{t+1}$, so both are in the same stack. Whenever b_t is a final edge and $f(b_t)$ traverses b_{t+1}, there is an edge in $\mathcal{SG}(f)$ from the the stack containing b_t to the stack containing b_{t+1}. Following the sequence of stacks containing the edges $\{b_t\}_{t=0}^{s}$ gives a directed path in $\mathcal{SG}(f)$ from $\mathcal{K}_i$ to $\mathcal{K}_j$. Thus $\mathcal{SG}(f)$ is strongly connected.

Let $\mathcal{K}_i, \mathcal{K}_j \in \mathcal{V}(\mathcal{SG}(f))$. If $\mathcal{K}_j = \mathcal{K}_i$, it is immediate that $\mathcal{K}_j \in B_{n-n_i}(\mathcal{K}_i)$. Suppose $\mathcal{K}_j \neq \mathcal{K}_i$. Since $\mathcal{SG}(f)$ is strongly connected, there is a path P in $\mathcal{SG}(f)$ from $\mathcal{K}_i$ to $\mathcal{K}_j$. Choose P so that every vertex in P appears only once. Since each vertex in P appears only once, we have at most one edge with terminal vertex $\mathcal{K}$ for each $\mathcal{K} \in \mathcal{V}(\mathcal{SG}(f))$. Moreover, since P starts at $\mathcal{K}_i$ and ends at $\mathcal{K}_j \neq \mathcal{K}_i$, no edge in P has terminal vertex $\mathcal{K}_i$. Observe that for any edge $E \in \mathcal{E}^+\mathcal{SG}(f)$, $s(E) \leq n_t$ where $\mathcal{K}_t$ is the terminal vertex of E. Thus

$$s(P) = \sum_{E \in P} s(E) \leq \sum_{t \neq i} n_t = n - n_i$$

Therefore $\mathcal{K}_j \in B_{n-n_i}(\mathcal{K}_i)$. Since j is arbitrary, $\mathcal{V}(\mathcal{SG}(f)) \subseteq B_{n-n_i}(\mathcal{K}_i)$. ■

Lemma 1.4.7 *For any* $d \in \mathbb{Z}_{\geq 0}$,

$$|f^{d+1}(\alpha_i)| \geq \Big(\sum_{\mathcal{K}_j \in B_d(\mathcal{K}_i)} \omega(\mathcal{K}_j) \Big) + 1.$$

Proof: We prove this by induction on d. When $d = 0$, $B_0(\mathcal{K}_i) = \{\mathcal{K}_i\}$, so

$$\begin{aligned} |f(\alpha_i)| &= (|f(\alpha_i)| - 1) + 1 \\ &= \Big(\sum_{\mathcal{K}_j \in B_0(S)} \omega(\mathcal{K}_j) \Big) + 1. \end{aligned}$$

Now let $d \geq 1$ and suppose the inequality holds for $d - 1$. Let $B_d(\mathcal{K}_i) - B_{d-1}(\mathcal{K}_i) = \{\mathcal{K}_{t_1}, \ldots, \mathcal{K}_{t_k}\}$. Then for each t_q, there is a directed path from $\mathcal{K}_i$ to $\mathcal{K}_{t_q}$ with length exactly d, so by Lemma 1.4.5, $f^d(\alpha_i)$ traverses α_{t_q}.

Let $\delta = |f^d(\alpha_i)|$ and let $\alpha_{t_1}, \ldots, \alpha_{t_k}, b_{k+1}, \ldots, b_\delta$ denote the edges appearing in $f^d(\alpha_i)$, with multiplicity. Thus

$$\begin{aligned} |f^{d+1}(\alpha_i)| &= |f(\alpha_{t_1})| + \cdots + |f(\alpha_{t_k})| + |f(b_{k+1})| + \cdots + |f(b_\delta)| \\ &\geq (\omega(\mathcal{K}_{t_1}) + 1) + \cdots + (\omega(\mathcal{K}_{t_k}) + 1) + (\delta - k) \\ &= \Big(\sum_{q=1}^{k} \omega(\mathcal{K}_{t_q}) \Big) + \delta \\ &= \Big(\sum_{t=1}^{k} \omega(\mathcal{K}_{t_q}) \Big) + |f^d(\alpha_i)|. \end{aligned}$$

By our induction hypothesis,

$$|f^d(\alpha_i)| \geq \Big(\sum_{\mathcal{K}_t \in B_{d-1}(\mathcal{K}_i)} \omega(\mathcal{K}_t) \Big) + 1.$$

Moreover, we have

$$B_d(\mathcal{K}_i) = B_{d-1}(\mathcal{K}_i) \ \cup \ \Big\{ \ \mathcal{K}_{t_q} \in B_d(\mathcal{K}_i) \mid q \in \{1, \dots k\} \ \Big\},$$

and hence

$$\begin{aligned} |f^{d+1}(\alpha_i)| &\geq \Big(\sum_{t=1}^{k} \omega(\mathcal{K}_{t_q}) \Big) + |f^d(\alpha_i)| \\ &\geq \Big(\sum_{q=1}^{k} \omega(\mathcal{K}_{t_q}) \Big) + \Big(\sum_{\mathcal{K}_t \in B_{d-1}(\mathcal{K}_i)} \omega(\mathcal{K}_t) \Big) + 1 \\ &= \Big(\sum_{\mathcal{K}_t \in B_d(\mathcal{K}_i)} \omega(\mathcal{K}_t) \Big) + 1. \end{aligned}$$

This completes the proof of the lemma. ■

1.5 Lower Bound Proof

Theorem A. *Suppose $f : \Gamma \to \Gamma$ is an irreducible homotopy equivalence self graph map with fold decomposition consisting of m total folds. Let $n = |\mathcal{E}\Gamma|$. Then*

$$(m+1)^{\frac{1}{n}} \leq \lambda_f$$

where λ_f is the largest eigenvalue of the transition matrix of f.

Proof: If f is not expanding, then by Lemma 1.2.21 we have $m = 0$ and $\lambda_f = 1$, so the inequality holds. We now assume f is expanding.

Let $\lambda = \lambda_f$ and let ℓ be the metric on Γ from Lemma 1.2.21, so that f is uniformly $\lambda-$expanding on (Γ, ℓ). Let $e \in \mathcal{E}\Gamma$ be an edge with the shortest length $\ell(e)$. Uniformly scale ℓ so that $\ell(e) = 1$. We claim that e must be the root edge in some stack of f. Otherwise, $e = f(a)$ for some edge a. Since f is uniformly $\lambda-$expanding, $\ell(e) = \lambda\ell(a)$. Since $\lambda > 1$, $\ell(e) > \ell(a)$, contradicting that e is the shortest edge.

Without loss of generality, suppose e is the root edge in stack $\mathcal{K}_1$. Let n_1 be the number of edges in $\mathcal{K}_1$, so $f^{n_1-1}(e)$ is the final edge of $\mathcal{K}_1$.

By Lemma 1.4.6, $\mathcal{V}(\mathcal{SG}(f)) \subseteq B_{n-n_1}(\mathcal{K}_1)$. Thus by Lemma 1.4.7 with $d = n - n_1$,

$$|f^n(e)| = |f^{(n-n_1)+1}(f^{n_1-1}(e))| \geq \left(\sum_{j=1}^{p} \omega(\mathcal{K}_j)\right) + 1,$$

where p is the number of stacks in f. By observation 1.4.2,

$$\left(\sum_{j=1}^{p} \omega(\mathcal{K}_j)\right) \geq m$$

Since every edge has length greater than or equal to $\ell(e) = 1$,

$$\begin{aligned} \lambda^n = \ell(f^n(e)) &\geq |f^n(e)| \\ &\geq \left(\sum_{j=1}^{p} \omega(\mathcal{K}_j)\right) + 1 \geq m + 1 \end{aligned}$$

Therefore, $(m+1)^{\frac{1}{n}} \leq \lambda_f$. ■

Using the following lemma, (Lemma 3.1 in [HS07]), we provide an alternative proof of Theorem A for irreducible homotopy equivalence self graph maps with primitive transition matrices. In particular, if f is an i.t.t. representative of a fully irreducible outer automorphism, then $T(f)$ is primitive (Lemma 2.4(2) in [Kap14]).

Lemma 1.5.1 *[HS07] Suppose M is a non-negative integral primitive $n \times n$ matrix with*

$\lambda > 1$ *its largest eigenvalue. Then*

$$\lambda^n \geq |M| - n + 1$$

where $|M|$ *denotes the sum of the entries of* M*.*

Alternative Proof of Theorem A for i.h.e. maps with primitive transition matrix: Suppose f is an irreducible homotopy equivalence self graph map with $T(f)$ primitive. Since $|T(f)| = \sum_{e\in\mathcal{E}(\Gamma)} |f(e)|$, and $T(f)$ is non-negative and integral, by Lemma 1.5.1 and Lemma 1.3.3,

$$\begin{aligned}\lambda^n &\geq \Big(\sum_{e\in\mathcal{E}\Gamma} (|f(e)|)\Big) - n + 1\\ &= \Big(\sum_{e\in\mathcal{E}\Gamma} (|f(e)| - 1)\Big) + 1\\ &\geq m + 1.\end{aligned}$$

■

1.6 Latent Symmetry

In order for a graph to admit an i.h.e. map with very few folds in its fold decomposition, the graph isomorphism following the folds needs to sufficiently mix the edges. The stack score is designed to measure how much mixing the graph isomorphism can possibly admit, with a smaller stack score indicating more mixing is possible in the graph isomorphism.

Definition 1.6.1 *(Stack Score) A graph* G *a supergraph of* Γ *if* Γ *is a subgraph of* G*. Given a supergraph* G *of* Γ *with* $\mathcal{V}G = \mathcal{V}\Gamma$*, and* $\psi \in Aut(G)$*, we define an equivalence relation* $\sim_\psi$ *on* $\mathcal{E}\Gamma$ *generated by* $a \sim_\psi \psi(a)$ *whenever* $\psi(a) \in \mathcal{E}\Gamma$*. The* ***stack score*** *of a graph* Γ *is*

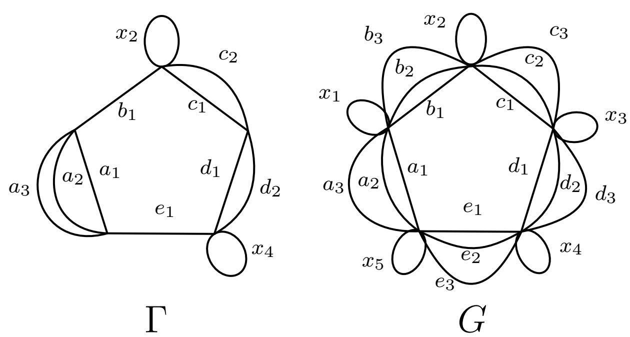
$$\mathfrak{S}(\Gamma) := \min\{\textit{number of } \sim_\psi \textit{ equivalence classes} \mid G \textit{ is a supergraph of } \Gamma, \mathcal{V}G = \mathcal{V}\Gamma, \textit{ and } \psi \in Aut(G)\}$$

Similarly, let $\mathfrak{O}(\Gamma)$ be the minimum number of ψ edge orbits over all pairs (G, ψ), where G is a supergraph of Γ with $\mathcal{V}G = \mathcal{V}\Gamma$ and $\psi \in \text{Aut}(G)$. Then $\mathfrak{O}(\Gamma)$ is a similar graph invariant to $\mathfrak{S}(\Gamma)$. While $\mathfrak{O}(\Gamma)$ is slightly easier to conceptualize and compute, we have

$$\mathfrak{O}(\Gamma) \leq \mathfrak{S}(\Gamma)$$

and there are cases when the inequality is strict. Below, Example 1.6.2 gives a graph Γ with $\mathfrak{O}(\Gamma) = 2$ and $\mathfrak{S}(\Gamma) = 3$.

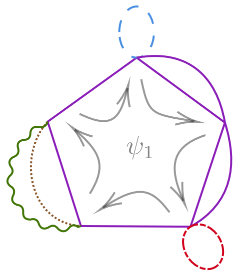
Example 1.6.2 *Consider the graph Γ along with a supergraph G as pictured below.*



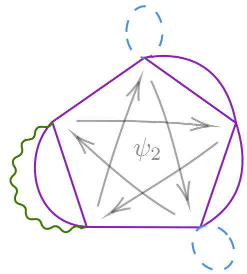
Let $\psi_1 \in Aut(G)$ rotate vertices in G clockwise by one, and map the edges of G as follows:

$$\psi_1 : \begin{cases} x_1 \mapsto x_2 \mapsto x_3 \mapsto x_4 \mapsto x_5 \mapsto x_1 \\ c_i \mapsto d_i \mapsto e_i \mapsto a_i \mapsto b_i \mapsto c_{i+1} \end{cases}$$

for $1 \leq i \leq 3$, with the exception that $b_3 \mapsto c_1$. Then $[c_1]_{\sim_{\psi_1}} = \{c_1, d_1, e_1, a_1, b_1, c_2, d_2\}$ and a_2, a_3, x_2, x_4 are each their own equivalence class. Below, $(\Gamma, \sim_{\psi_1})$ shows Γ with edges colored and dashed to distinguish the $\sim_{\psi_1}$ equivalence classes.



$(\Gamma,\ \sim_{\psi_1})$



$(\Gamma,\ \sim_{\psi_2})$

Let $\psi_2 \in Aut(G)$ rotate vertices in G clockwise by two and map the edges of g as follows:

$$\psi_2 : \begin{cases} x_1 \mapsto x_3 \mapsto x_5 \mapsto x_2 \mapsto x_4 \mapsto x_1 \\ d_i \mapsto a_i \mapsto c_i \mapsto e_i \mapsto b_i \mapsto d_{i+1} \end{cases}$$

for $1 \leq i \leq 3$, with the exception that $b_3 \mapsto d_1$. Then $[d_1]_{\sim_{\psi_2}} = \{d_1, a_1, c_1, e_1, b_1, d_2, a_2, c_2\}$, $[x_2]_{\sim_{\psi_2}} = \{x_2, x_4\}$, and $[a_3]_{\sim_{\psi_2}} = \{a_3\}$. Above, $(\Gamma, \sim_{\psi_2})$ shows Γ with edges colored and dashed to distinguish the $\sim_{\psi_2}$ equivalence classes. For this graph Γ, $\sim_{\psi_2}$ gives the minimal number of equivalence classes, so $\mathfrak{S}(\Gamma) = 3$.

Theorem B. *Any irreducible expanding homotopy equivalence self graph map $f : \Gamma \to \Gamma$ which is periodic on the vertex set of Γ must have at least $\mathfrak{S}(\Gamma)$ folds.*

Proof: Suppose f has p stacks of sizes $n_1, n_2, \ldots, n_p$ and root edges $e_1, \ldots, e_p$. Then f is given by:

$$f : \begin{cases} e_1 \mapsto f(e_1) \mapsto \cdots \mapsto f^{n_1-1}(e_1) \mapsto f^{n_1}(e_1) \\ e_2 \mapsto f(e_2) \mapsto \cdots \mapsto f^{n_2-1}(e_2) \mapsto f^{n_2}(e_2) \\ \vdots \\ e_p \mapsto f(e_p) \mapsto \cdots \mapsto f^{n_p-1}(e_p) \mapsto f^{n_p}(e_p) \end{cases}$$

For each $i \in \{1, \ldots, p\}$, let $v_i := \iota(e_i)$ and $w_i := \tau(e_i)$. Since f is periodic on $\mathcal{V}\Gamma$, there is some power k_i of f such that $f^{k_i}(v_i) = v_i$ and some power t_i such that $f^{t_i}(w_i) = w_i$. Let q_i be a multiple of $k_i t_i$ such that $n_i \leq q_i$. Build a supergraph G of Γ by adding edges

b_i^j for $n_i \le j \le q_i - 1$ joining $f^j(v_i)$ to $f^j(w_i)$. Define a graph map $\psi : G \to G$ by letting $\psi_V = f_V$ and

$$\psi : \begin{cases} e_1 \mapsto f(e_1) \mapsto \cdots \mapsto f^{n_1-1}(e_1) \mapsto b_1^{n_1} \mapsto \cdots \mapsto b_1^{q_1-1} \mapsto e_1 \\ e_2 \mapsto f(e_2) \mapsto \cdots \mapsto f^{n_2-1}(e_2) \mapsto b_2^{n_2} \mapsto \cdots \mapsto b_2^{q_2-1} \mapsto e_2 \\ \vdots \\ e_p \mapsto f(e_p) \mapsto \cdots \mapsto f^{n_p-1}(e_p) \mapsto b_p^{n_p} \mapsto \cdots \mapsto b_p^{q_p-1} \mapsto e_p \end{cases}$$

on the edges of g. We claim ψ is an automorphism of G. By definition, ψ_E is a bijection from $\mathcal{E}G$ to itself. We also have $\psi_V = f_V$ is a bijection by our hypothesis on f. It remains to show that ψ is a graph map. For any edge b_i^j, with $n_i \le j \le q_i - 2$ and $1 \le i \le p$, we have

$$\psi(\iota(b_i^j)) = \psi(f^j(v_i)) = f^{j+1}(v_i) = \iota(b_i^{j+1}) = \iota(\psi(b_i^j)).$$

For any edge $b_i^{q_i-1}$ with $1 \le i \le p$, we have

$$\begin{aligned} \psi(\iota(b_i^{q_i-1})) = \psi(f^{q_i-1}(v_i)) = f^{q_i}(v_i) = v_i = \iota(e_i) \\ = \iota(\psi(b_i^{q_i-1})). \end{aligned}$$

For the edges $f^{n_i-1}(e_i)$ with $1 \le i \le p$, we have

$$\begin{aligned} \psi(\iota(f^{n_i-1}(e_i))) = \psi(f^{n_i-1}(v_i)) = f^{n_i}(v_i) = \iota(b_i^1) \\ = \iota(\psi(f^{n_i-1}(e_i))). \end{aligned}$$

Thus $\psi(\iota(e)) = \iota(\psi(e))$ for every edge $e \in \mathcal{E}G$. Similarly, $\psi(\tau(e)) = \tau(\psi(e))$, so indeed ψ is a graph map.

Observe that $\sim_\psi$ partitions $\mathcal{E}\Gamma$ into exactly p equivalence classes. Hence $\mathfrak{S}(\Gamma) \le p$.

Since f is periodic on the vertices of Γ, by Lemma 1.3.3, p is less than or equal to the number of folds in the fold decomposition of f. Hence f has at least $\mathfrak{S}(\Gamma)$ many folds.

■

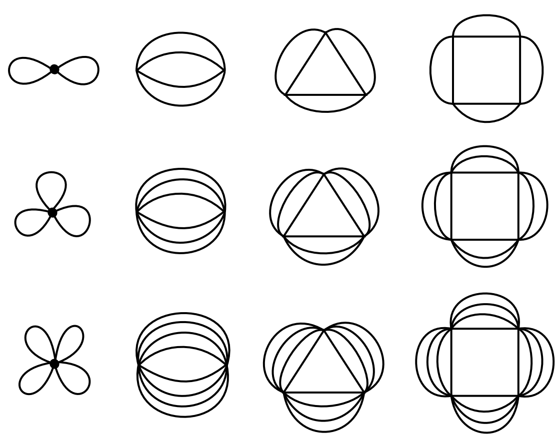
Theorems A and B immediately give the following corollary.

Corollary 1.6.3 *Let $f : \Gamma \to \Gamma$ be an irreducible expanding homotopy equivalence self graph map which is periodic on the vertex set of Γ. Let $n = |\mathcal{E}\Gamma|$. Then*

$$(\mathfrak{S}(\Gamma) + 1)^{\frac{1}{n}} \leq \lambda_f,$$

where $\mathfrak{S}(\Gamma)$ is the stack score of Γ and λ_f is the leading eigenvalue of the transition matrix of f.

1.7 Single Fold Maps



Definition 1.7.1 *(Polygonal Graph) Let $P_{s,k}$ be a graph with vertex set $\mathcal{V}P_{s,k} = \{v_0, \ldots, v_{s-1}\}$ and edges*

$$\mathcal{E}P_{s,k} = \{e_i^j : \ 1 \leq j \leq k, \ 0 \leq i \leq s-1, \},$$

*where an edge e_i^j joins v_i to v_{i+1}, with vertex subscripts taken modulo s. We call $P_{s,k}$ the **s-gonal graph of depth k**.*

*A **side** of $P_{s,k}$ is $\mathfrak{s}_i := \{e_i^j \mid 1 \leq j \leq k\} \subseteq \mathcal{E}P_{s,k}$. The sides of $P_{s,k}$ partition $\mathcal{E}P_{s,k}$.*

Observe that each polygonal graph has an edge transitive automorphism. Hence $\mathfrak{S}(P_{s,k}) = 1$ for any $s, k \in \mathbb{N}$. The following lemma provides a converse to this statement in the special case that a graph G has an automorphism which is both edge and vertex transitive.

Lemma 1.7.2 *If G is a connected graph and there exists a $\psi \in Aut(G)$ such that the cyclic subgroup of $Aut(G)$ generated by ψ, denoted $\langle\psi\rangle$, acts transitively on both $\mathcal{V}G$ and $\mathcal{E}G$, then G is isomorphic to some polygonal graph $P_{s,k}$.*

Proof: Let $\mathcal{V}G = \{v_0, \ldots, v_{s-1}\}$. Since $\langle\psi\rangle$ is transitive on $\mathcal{V}G$, we can assume the vertices are labeled so that $\psi(v_i) = v_{i+1}$, with subscripts taken modulo s. Suppose e is an edge joining v_0 to v_j. Thus for any power m, $\psi^m(e)$ is an edge joining v_m to v_{j+m}. Since $\langle\psi\rangle$ is transitive on $\mathcal{E}G$,

$$\{\psi^m(e) | m \in \mathbb{Z}\} = \mathcal{E}G.$$

Hence each $a \in \mathcal{E}G$ joins v_i to v_{j+i} for some i. In other words, there is an edge between v_{i_1} and v_{i_2} if and only if $|i_1 - i_2| = j$.

Suppose there are precisely k distinct edges in G joining v_0 to v_j. Since ψ is an automorphism, there must be exactly k edges joining $\psi^m(v_0) = v_m$ to $\psi^m(v_j) = v_{m+j}$ for each power m. To summarize, given any two vertices v_{i_1} and v_{i_2}, there are exactly k edges joining v_{i_1} to v_{i_2} if $|i_1 - i_2| = j$, and zero edges joining v_{i_1} to v_{i_2} otherwise. Since G is connected, G is isomorphic to $P_{s,k}$. ■

The following lemma classifies the structure of connected subgraphs of polygonal graphs with stack score equal to 1. In particular, the number of edges in each side of the polygonal graph which are also in the subgraph can vary by at most 1.

Lemma 1.7.3 *Suppose* Γ *is a connected subgraph of* $P_{s,k}$ *for* $s \geq 3$ *and there exists an edge transitive automorphism* $\psi \in Aut(P_{s,k})$ *and an edge* $e \in \mathcal{E}\Gamma$ *such that*

$$\{e, \psi(e), \ldots, \psi^{n-1}(e)\} = \mathcal{E}\Gamma. \tag{1.2}$$

Let $\mathfrak{s}_0, \ldots, \mathfrak{s}_{s-1}$ *denote the sides of* $P_{s,k}$ *and write* $n = sm + t$ *for* $m \in \{1, \ldots, k\}$ *and* $t \in \{0, \ldots, m-1\}$*. Then*

(i) there are precisely t *sides such that* $|\mathfrak{s}_i \cap \mathcal{E}\Gamma| = m + 1$*, and*

(ii) the remaining $s - t$ *sides have* $|\mathfrak{s}_i \cap \mathcal{E}\Gamma| = m$*.*

Proof: By the definition of a graph automorphism, $\psi(\iota(a)) = \iota(\psi(a))$ for every $a \in \mathcal{E}^{\pm}P_{s,k}$. Thus ψ descends to a bijection on the sides of $P_{s,k}$. Relabel the sides of $P_{s,k}$ so that $e \in \mathfrak{s}_0$ and ψ on the sides is given by

$$\psi : \mathfrak{s}_0 \mapsto \mathfrak{s}_1 \mapsto \cdots \mapsto \mathfrak{s}_{s-1} \mapsto \mathfrak{s}_0.$$

Hence by (1.2),

$$\mathfrak{s}_i \cap \mathcal{E}\Gamma = \{\psi^j(e) \mid j \in \{0, 1, \ldots, n-1\} \text{ and } j \equiv i \bmod s\}.$$

Therefore,

$$|\mathfrak{s}_i \cap \mathcal{E}\Gamma| = \begin{cases} m+1 & \text{if } 1 \leq i \leq t-1 \\ m & \text{it } t \leq i \leq s-1. \end{cases}$$

This completes the proof of the lemma. ■

Definition 1.7.4 *(Almost 3-gonal graphs) For any* $k \in \mathbb{N}$*, we define two graphs called the almost 3-gonal graphs of depth* k*.*

(i) *Let* Δ_k^- *be* $P_{3,k}$ *with edge* e_0^k *removed. Note that the choice of removed edge does not change the isomorphism class of* Δ_k^-*. We have*

$$Rank(\Delta_k^-) = 3k - 3.$$

(ii) *Let* Δ_k^+ *be* $P_{3,k+1}$ *with edges* e_0^{k+1} *and* e_1^{k+1} *removed. The choice of removed edges from two distinct sides of* $P_{3,k+1}$ *does not change the isomorphism class of* Δ_k^+*. We have*

$$Rank(\Delta_k^+) = 3k - 1.$$

Definition 1.7.5 *(Rose) For any* $r \in \mathbb{N}$*, the rose with* r *petals is* $R_r = P_{1,r}$*. We have* $Rank(R_r) = r$*.*

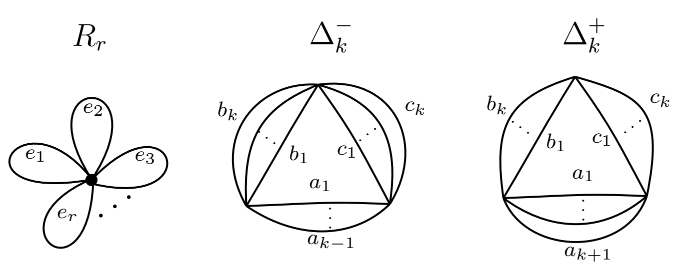
Theorem C. *Suppose* Γ *is a connected rank* r *graph and* $f : \Gamma \to \Gamma$ *is a single fold irreducible homotopy equivalence self graph map. Then* Γ *is isomorphic to* R_r, Δ_k^+*, or* Δ_k^- *for some* $k \geq 2$*. In particular:*

(i) *if* $r \equiv 0$ *mod* 3*, then* $\Gamma \cong G \in \{R_r, \Delta_k^-\}$*,*

(ii) *if* $r \equiv 1$ *mod* 3*, then* $\Gamma \cong R_r$*, and*

(iii) *if* $r \equiv 2$ *mod* 3*, then* $\Gamma \cong G \in \{R_r, \Delta_k^+\}$*,*

for appropriate values of k*.*



To prove this theorem, we first we argue that Γ satisfies the hypotheses of Lemma 1.7.3. Next, we show that Γ must be a subgraph $P_{1,k}$ or $P_{3,k}$. Finally, we determine which subgraphs of $P_{3,k}$ are admissible.

Proof: We can write $f = h \circ f_1$, where $f_1 : \Gamma \to \Gamma'$ is a fold and $h : \Gamma' \to \Gamma$ is a graph isomorphism. Since Γ' must be isomorphic to Γ, the fold f_1 must be a proper full fold, as complete and partial folds change the number of vertices of Γ'. Hence f must be periodic on the vertex set. Moreover, since f has a fold, f is expanding. Thus by Theorem B, $\mathfrak{S}(\Gamma) = 1$.

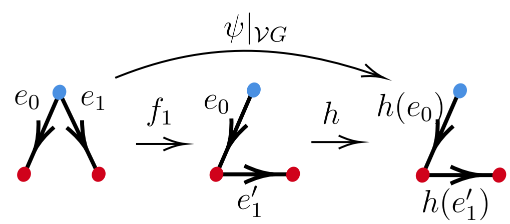
By the definition of a stack score, there exists a supergraph G of Γ and an automorphism $\psi \in \operatorname{Aut}(G)$ such that $\sim_\psi$ partitions the edges of Γ into a single set. By the proof of Theorem B, we can assume ψ can be written:

$$\psi : e \mapsto f(e) \mapsto f^2(e) \mapsto \cdots \mapsto f^{n-1}(e) \mapsto b_1 \mapsto \cdots \mapsto b_j \mapsto e \tag{1.3}$$

where $\{e, f(e), \ldots, f^{n-1}(e)\} = \mathcal{E}\Gamma$ and $\{b_1, \ldots, b_j\} = \mathcal{E}G - \mathcal{E}\Gamma$. Hence $\psi_V = f_V$, and $\langle\psi\rangle$ acts transitively on $\mathcal{E}G$.

Claim: $\langle\psi\rangle$ also acts transitively on $\mathcal{V}G$.

Proof of Claim: Suppose $\langle\psi\rangle$ does not act transitively on $\mathcal{V}G$. By Theorem 2.1 in [LS16] G is bipartite and the action of $\langle\psi\rangle$ on $\mathcal{V}G$ has two orbits, X and Y, which form the partition of $\mathcal{V}G$. Suppose f_1 is a proper full fold of e_1 over e_0. Assume $\iota(e_1) = \iota(e_0) \in X$ and $\tau(e_1), \tau(e_0) \in Y$.



Since f_1 is the identity on $\mathcal{V}\Gamma$, $\psi_V = f_V$, and the sets X and Y are invariant under ψ, we have

$$\iota(h(e'_1)), \tau(h(e'_1) \in Y.$$

However, X and Y form the bipartition of $\mathcal{V}G$, so this is a contradiction. Hence $\langle\psi\rangle$ acts transitively on $\mathcal{V}G$. ◇

By Lemma 1.7.2, G is an s-gonal graph of depth k, for some $s, k \in \mathbb{N}$. Hence by (1.3), Γ satisfies the hypotheses of Lemma 1.7.3.

We now argue that in fact G is either a 1-gonal graph (and hence isomorphic to a rose R_k) or a 3-gonal graph.

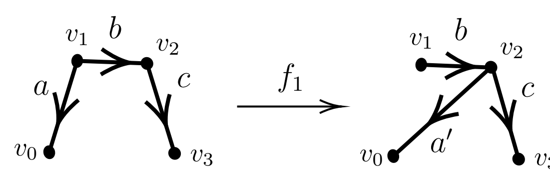
Claim: If $s \geq 4$ then Γ' cannot be isomorphic to Γ.

Proof of Claim: Suppose $s \geq 4$. A single proper full fold between edges in Γ in the same side yields a graph Γ' with a self loop, and hence Γ' is not isomorphic to Γ. Otherwise, the single fold $f_1 : \Gamma \to \Gamma'$ must be between edges in adjacent sides. Without loss of generality, suppose f_1 is the proper full fold of an edge a from v_1 to v_0 over an edge b from v_1 to v_2. By definition of proper full fold, Γ' has an edge a' from v_2 to v_0. Since the valence of every vertex in Γ is at least 3, Lemma 1.7.3 guarantees that for each side $\mathfrak{s}_i$ of G, we have

$$|\mathfrak{s}_i \cap \mathcal{E}\Gamma| \geq 1.$$

Therefore, there must be an edge $c \in \mathcal{E}\Gamma$ from v_2 to v_3.

Observe that in Γ', the vertex v_2 is adjacent to vertices v_0, v_1, and v_3. However, every vertex in a subgraph of an s-gonal graph is adjacent to at most two vertices.



Hence Γ' cannot be isomorphic to Γ. ◇

Since $h : \Gamma' \to \Gamma$ is a graph isomorphism, Γ' must be isomorphic to Γ. Hence $1 \leq s \leq 3$.

If $s = 1$, then $\Gamma \cong R_k$. Any subgraph of R_k is another rose R_j for some $j \leq k$. Since the rank of R_k is equal to k, we can build a rose with any rank.

If $s = 2$, then G is a (1,1)-bipartite graph. As a connected non-empty subgraph of G, the graph Γ is also a (1,1)-bipartite graph. Any single proper full fold in Γ yields an edge e_1' with $\iota(e_1') = \tau(e_1')$. Hence Γ' is not bipartite, and thus not isomorphic to Γ, a contradiction. Hence $s \in \{1, 3\}$.

Suppose $s = 3$. Then $G \cong P_{3,k}$. By Lemma 1.7.3, up to relabeling of the sides $\mathfrak{s}_i$, we have the following three cases:

(i) If $n = 3m$ for some $m \in \mathbb{N}$, then

$$(|(\mathfrak{s}_0 \cap \mathcal{E}\Gamma)|, |(\mathfrak{s}_1 \cap \mathcal{E}\Gamma)|, |(\mathfrak{s}_2 \cap \mathcal{E}\Gamma)|) = (m, m, m).$$

Hence $\Gamma \cong P_{3,m}$.

(ii) If $n = 3m + 1$ for some $m \in \mathbb{N}$, then

$$(|(\mathfrak{s}_0 \cap \mathcal{E}\Gamma)|, |(\mathfrak{s}_1 \cap \mathcal{E}\Gamma)|, |(\mathfrak{s}_2 \cap \mathcal{E}\Gamma)|) = (m + 1, m, m).$$

Hence $\Gamma \cong \Delta_m^+$.

(iii) If $n = 3m + 2$ for some $m \in \mathbb{N}$, then

$$(|(\mathfrak{s}_0 \cap \mathcal{E}\Gamma)|, |(\mathfrak{s}_1 \cap \mathcal{E}\Gamma)|, |(\mathfrak{s}_2 \cap \mathcal{E}\Gamma)|) = (m + 1, m + 1, m).$$

Hence $\Gamma \cong \Delta_{m+1}^-$.

Observe that when $s = 3$, we have $|\mathcal{V}\Gamma| = 3$. Hence by the Euler characteristic formula, the rank r of Γ is computed as

$$\begin{aligned} r &= |\mathcal{E}\Gamma| - |\mathcal{V}\Gamma| + 1 \\ &= n - 2. \end{aligned}$$

Thus, the above cases correspond to $r \equiv 1, 2, 0 \bmod 3$ respectively.

Now we need only rule out the possibility that Γ is isomorphic to $P_{3,m}$. In this case, any single proper full fold yields a graph with a self loop or a $3-$gonal graph with side depths $(m, m-1, m+1)$. Hence Γ' is not isomorphic to $P_{3,m}$, a contradiction. ■

1.8 Further Observations and Questions

1.8.1 Unique Minimizer in Out(F_3)

We have the following application of Theorems A and C.

Corollary 1.8.1 *The element $\varphi \in Out(F_3)$ given by $\varphi : x \mapsto y \mapsto z \mapsto zx^{-1}$ defines the unique $Out(F_3)-$conjugacy class of infinite order irreducible elements realizing the minimal stretch factor $\lambda \approx 1.167$, the largest real root of $x^5 - x - 1$.*

Proof: The element φ is Example 1.2.18. It is shown in [AHLP24b] that φ has stretch factor $\lambda(\varphi) \approx 1.167$, the largest real root of $x^5 - x - 1$. Suppose $\phi \in \mathrm{Out}(F_3)$ is an infinite order irreducible element with $\lambda(\phi) \leq \lambda(\varphi)$. Let $f : \Gamma \to \Gamma$ be an irreducible train track representative of ϕ on a connected rank 3 graph Γ. Since ϕ is infinite order, $\lambda_f > 1$ by Theorem 1.2.14. Thus by Lemma 1.2.21, f must have at least one fold in its fold decomposition. Since

$$\lambda_f \leq \lambda(\varphi) < 2^{\frac{1}{4}} < 3^{\frac{1}{6}},$$

by Theorem A, f must have exactly one fold in its fold decomposition and Γ must have at least 5 edges. As the vertices of Γ have valence at least 3 and Γ has rank 3, an Euler characteristic argument shows Γ can have no more than 6 edges. Hence by Theorem C, $\Gamma \cong \Delta_2^-$.

Suppose $f = h \circ f_1$ is a fold decomposition, so $f_1 : \Gamma \to \Gamma'$ is a proper full fold and $h : \Gamma' \to \Gamma$ is a graph isomorphism. Up to relabeling the edges, the only proper full fold on Δ_2^- which yields an isomorphic graph is the proper full fold of c_2 over $\overline{b_1}$. Without loss of generality, suppose $\Gamma = \Delta_2^-$, give Γ the labels in Example 1.2.18, and assume f_1 is the proper full fold of c_2 over $\overline{b_1}$. By continuity, we must have $h(c_1) \in \{a_1, \overline{a_1}\}$.

Suppose $h(c_1) = \overline{a_1}$. If $h(a_1) = \overline{c_1}$, then $f(c_1) = \overline{a_1}$ and $f(a_1) = \overline{c_1}$, so f is reducible. This leaves two ways h could map the remaining edges:

(i) $h : a_1 \mapsto \overline{c_2}$, $c_2' \mapsto c_1$, $b_1 \mapsto \overline{b_1}$, and $b_2 \mapsto \overline{b_2}$.

In this case $f(b_1) = \overline{b_1}$, so f is reducible.

(ii) $h : a_1 \mapsto \overline{c_2}$. $c_2' \mapsto c_1$. $b_1 \mapsto \overline{b_2}$. $b_2 \mapsto \overline{b_1}$.

In this case, $f(b_1) = \overline{b_2}$ and $f(b_2) = \overline{b_1}$, so again f is reducible.

Thus $h(c_1) \neq \overline{a_1}$, so we must have $h(c_1) = a_1$. Then h maps the remaining edges in one of the following four ways:

(i) $h : b_1 \mapsto c_1,\ b_2 \mapsto c_2,\ a_1 \mapsto b_2$, and $c_2' \mapsto \overline{b_1}$.

In this case, f is equal to $\mathfrak{g}$ in Example 1.2.18 and hence ϕ is $\mathrm{Out}(F_3)-$conjugate to φ.

(ii) $h : b_1 \mapsto c_2,\ b_2 \mapsto c_1,\ a_1 \mapsto b_2$, and $c_2' \mapsto \overline{b_1}$.

In this case, we have $f(a_1) = b_2$, $f(b_2) = c_1$ and $f(c_1) = a_1$, so f is reducible.

(iii) $h : b_1 \mapsto c_1,\ b_2 \mapsto c_2,\ a_1 \mapsto b_1,\ c_2' \mapsto \overline{b_2}$.

In this case, we have $f(a_1) = b_1$, $f(b_1) = c_1$, and $f(c_1) = a_1$, so f is reducible..

(iv) $h : b_1 \mapsto c_2,\ \ b_2 \mapsto c_1,\ \ a_1 \mapsto b_1,\ \ c_2' \mapsto \overline{b_2}$.

In this case, we have $f : b_2 \mapsto c_1 \mapsto a_1 \mapsto b_1 \mapsto c_2 \mapsto \overline{b_2}\overline{c_2}$. Then λ_f is equal to the largest root of $x^5 - x^4 - 1$, which is larger than $\lambda(\varphi)$.

Therefore, if ϕ is an infinite order irreducible element of $\mathrm{Out}(F_3)$ with $\lambda(\phi) \leq \lambda(\varphi)$, then ϕ is $\mathrm{Out}(F_3)-$conjugate to φ, and hence has $\lambda(\phi) = \lambda(\varphi)$. ■

As a consequence of this Corollary and deep work of Hughes–Kudlinska [HK23] on profinite invariants for free-by-cyclic groups, we have the following result, elaborated on in [AHLP24a].

Theorem 1.8.2 *[AHLP24a] The group $G = \langle x, y, z, t \mid txt^{-1} = y, tyt^{-1} = z, tzt^{-1} = zx^{-1}\rangle$ is profinitely rigid amongst free-by-cyclic groups.*

1.8.2 Single Fold Irreducible Train Track Map on a Disconnected Graph

The hypothesis that Γ is connected in Theorem C is in fact necessary.

Example 1.8.3 *Let Γ be the graph consisting of the union of two disjoint copies of Δ_2^-. For the first copy of Δ_2^-, use the same labels for edges as in Example 1.2.18, and use a_1', b_1', b_2', c_1', and c_2' as edge labels for the second copy of Δ_2^-. Now define $f : \Gamma \to \Gamma$ by*

$$f : \begin{cases} b_1 \mapsto b_1' \mapsto c_1 \\ c_1 \mapsto c_1' \mapsto a_1 \\ a_1 \mapsto a_1' \mapsto b_2 \\ b_2 \mapsto b_2' \mapsto c_2 \\ c_2 \mapsto c_2' \mapsto \overline{b_1}\overline{c_1} \end{cases}$$

Then f is a single fold irreducible train track map and the leading eigenvalue of $T(f)$ is $\lambda^{\frac{1}{2}}$ for λ equal to the largest root of $x^5 - x - 1$.

Taking n copies of Δ_2^-, this example can be generalized to build a single fold irreducible train track map with leading eigenvalue $\lambda^{\frac{1}{n}}$. When Γ is disconnected, homotopy equivalences on Γ don't correspond to outer automorphisms of F_r. However, these examples are still relevant to the theory of minimal stretch factors in $\mathrm{Out}(F_r)$. In particular, when we allow for $\varphi \in \mathrm{Out}(F_r)$ to be reducible, we are no longer guaranteed an irreducible train track representative of φ, but we do have a *relative train track representative* $f : \Gamma \to \Gamma$ (see [BH92] for a definition of relative train track). It is possible that n disconnected copies of Δ_2^- form an $f-$invariant subgraph of Γ, and hence correspond to a stratum with stretch factor $\lambda^{\frac{1}{n}}$. In this project, we have been interested in the minimal stretch factor among irreducible elements of $\mathrm{Out}(F_r)$. However, as mentioned in the introduction, one could alternatively investigate

$$\underline{\lambda}_r^{\text{all}} := \min\{\lambda(\varphi) : \ \varphi \in \mathrm{Out}(F_r) \text{ such that } \lambda(\varphi) > 1\},$$

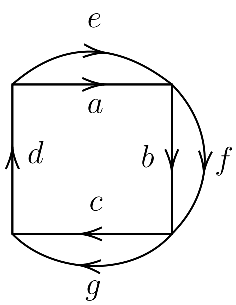
in which case, these reducible outer automorphisms with stretch factor $\lambda^{\frac{1}{n}}$ could be important to consider.

1.8.3 Candidate for Minimal Rank 4 Stretch Factor

By Theorem C, the only single fold i.t.t. maps on connected rank 4 graphs are on R_4. Among the single folds on R_4, the map sending $e_1 \mapsto e_2 \mapsto e_3 \mapsto e_4 \mapsto e_1e_2$ has the smallest stretch factor, which is the largest root of $x^4 - x - 1$, approximately 1.221. However, this is not minimal among irreducible elements of $\mathrm{Out}(F_4)$.

Example 1.8.4 *Consider the following single stack, 2 fold irreducible train track map γ*

on a subgraph of the 4-gonal graph of depth 2:



$$\gamma: \ a \mapsto b \ \mapsto c \ \mapsto d \ \mapsto e \ \mapsto f \ \mapsto g \mapsto \bar{c}\bar{b}\bar{a}$$

This represents the irreducible outer automorphism, $\varphi : w \mapsto x \mapsto y \mapsto z \mapsto zw^{-1}$, which has stretch factor λ_γ equal to the largest root of $x^7 - x^2 - x - 1$, approximately $\lambda_\gamma \approx 1.203$.

By the proof of Theorem A in [AHLP24b], every irreducible $\varphi \in \mathrm{Out}(F_4)$ has an i.t.t. representative on a graph with at most $3(4) - 4 = 8$ edges. Since

$$\lambda_\gamma < 3^{\frac{1}{5}} < 4^{\frac{1}{7}} < 5^{\frac{1}{8}},$$

Theorem A implies any irreducible $\varphi \in \mathrm{Out}(F_4)$ with stretch factor less than λ_γ must have an i.t.t. representative which is either 2 folds on a graph with 6, 7 or 8 edges or 3 folds on a graph with 8 edges.

In forthcoming work, we will use the technology of curve complexes for a variant of stack graphs, developed in [McM14], to rule out the possibility of 2 folds on 6 edges and 3 folds on 8 edges with stretch factor smaller than λ_γ. Further, we will classify the 2 fold maps on rank 4 graphs with 7 and 8 edges and ultimately show that the map $\varphi \in \mathrm{Out}(F_4)$ attains the minimal stretch factor among irreducible elements of $\mathrm{Out}(F_4)$.

Chapter 2

Higher Rank Lattices

2.1 Introduction

The relationship between the the *systolic genus*, the minimal genus surface subgroup, and the *systole*, the minimal length of a non-contractible closed geodesic, is notably different in higher rank. In 2012, Belolipetsky [Bel13] (see also [BD20]) showed that in the hyperbolic setting, the systolic genus is bounded from below in terms of the systole. In contrast, Long and Reid [LR19] found a family of sequences of lattices in $SL(3,\mathbb{R})$ (commensurable to an arbitrary non-uniform arithmetic lattice not commensurable to $SL(3,\mathbb{Z})$) with systole going to infinity, yet each contains a genus 3 surface subgroup. They found a similar result in the uniform case. Hence in higher rank, the systolic genus is not linked to the systole in the same way as in the hyperbolic setting.

Our result continues in this line of research. We expand Long and Reid's result to the existence of a fixed 3-manifold group in a sequence of commensurable lattices with arbitrarily large systole. More specifically, we find an infinite family of non-uniform arithmetic lattices in $SL(8,\mathbb{R})$ each with a sequence of commensurable lattices whose systole $\to \infty$, however every lattice in the sequence contains the same hyperbolic 3-

manifold group. Our non-uniform arithmetic lattices are indexed by square free numbers $d \geq 2$: for each such d, we consider the integral special unitary group

$$SU(I_8; \mathcal{O}_d, \tau) := \{A \in SL(8, \mathcal{O}_d) : \tau(A)^\top A = I_8\} < SL(8, \mathbb{R})$$

where I_8 = the identity matrix in $SL(8, \mathbb{R})$, $\mathcal{O}_d$ is the ring of integers of $\mathbb{Q}(\sqrt{d})$ and $\tau \in \mathrm{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q})$ is the non-trivial involution sending $\sqrt{d}$ to $-\sqrt{d}$.

Theorem D. *Fix square free $d \in \mathbb{Z}_{\geq 2}$ and let*

$$\mathcal{L} := SU(I_8; \mathcal{O}_d, \tau).$$

There exists a sequence of non-uniform arithmetic lattices $\Lambda_n < SL(8, \mathbb{R})$ commensurable to $\mathcal{L}$ such that

$$sys(\Lambda_n) \to \infty$$

as $n \to \infty$, yet every Λ_n contains a fixed hyperbolic 3-manifold group Π, a finite index subgroup of vol3.

In this setting, rank and determinant up to τ-Hermitian square classify SU-equivalent τ-Hermitian forms [Lan35]. In turn, equivalent τ-Hermitian forms yield commensurable integral special unitary groups. Hence $\mathrm{SU}(I_8; \mathcal{O}_d, \tau)$ is commensurable to $\mathrm{SU}(J; \mathcal{O}_d, \tau)$ for any non-degenerate τ-Hermitian form J over $\mathbb{Q}(\sqrt{d})^8$ such that $\det(J)$ is a τ-Hermitian square.

Critical to the proof is an 8 dimensional version of the family of discrete and faithful representations ρ_t of the hyperbolic 3-manifold vol3 found by Cooper, Long, and Thistlethwaite [CLT06] (see also [CLT07]). We were unable to choose values of t for which the entries of ρ_t lie in a ring of integers. However we are able to choose such specialized values of t for a certain conjugate of $\rho_t \oplus \rho_t$. From this family, we obtain a sequence of representations of vol3 into lattices $SU(J; \mathcal{O}_d, \tau)$ for non-degenerate forms J. To ensure the systole is going to infinity, we then consider principal congruence subgroups of each $SU(J; \mathcal{O}_d, \tau)$ of level p for an increasing sequence of primes p. We can no longer guarantee vol3 is contained in a principal congruence subgroup, but by carefully choosing the primes p, we can guarantee the fundamental group of a certain fixed 320-sheeted cover of vol3 is still contained in each principal congruence subgroup.

It is worth remarking that our finite index subgroup of vol3, denoted Π, contains surface subgroups. Therefore our result provides an alternative proof to Long and Reid's [LR19] result that systolic genus can be bounded in a sequence of commensurable higher rank lattices whose systole is diverging to infinity.

2.2 Background

Consider the following measures of the geometry and topology of a space:

Definition 2.2.1 *The **systole** of a Riemannian manifold M, denoted sys(M), is the minimal length of a non-contractible closed geodesic in M. We will interchangeably refer to the systole of M as the systole of $\pi_1(M)$.*

Definition 2.2.2 *Let S_g denote the closed surface with genus $g \geq 2$. The minimal g such that $\pi_1(S_g)$ injects into $\pi_1(M)$ is called the **systolic genus of** M, denoted sysg(M) (See [Bel13]).*

For hyperbolic manifolds, the behavior of the systole puts some restrictions on the behavior of the systolic genus. This is clear for hyperbolic surfaces, since Besicovitch's inequality

$$\mathrm{sys}(S_g)^2 \leq 2\mathrm{area}(S_g)$$

combined with Gauss-Bonnet theorem (for a hyperbolic metric on S_g),

$$\mathrm{area}(S_g) \leq 4\pi(g-1)$$

together show that a sequence of closed hyperbolic surfaces with systole $\to \infty$ requires the genera of the surfaces also tends toward infinity. In fact, Belolipetsky showed this generalizes to higher dimensions in the following sense (see Theorem 5.1 in [Bel13]):

Theorem 2.2.3 *Let $M_n = \mathbb{H}^m/\Gamma_n$ be a sequence of closed hyperbolic m-manifolds such that sys$(\Gamma_n) \to \infty$ as $n \to \infty$. Then the systolic genus sysg$(M_n) \to \infty$ as well.*

Intuitively, if the manifolds are getting complicated enough for the systole to grow arbitrarily large, their topology must be getting complicated as well. However, once we leave the hyperbolic setting, this is no longer true. In particular, the presence of flats seems to allow enough space for systoles to grow, without the systolic genus increasing.

Definition 2.2.4 *A discrete subgroup $\Gamma < SL(m, \mathbb{R})$ is a **lattice** if the quotient orbifold*

$$M_\Gamma := \Gamma \backslash SL(m, \mathbb{R})/SO(m)$$

*has finite volume. Note that M_Γ is a manifold if and only if Γ is torsion free. A lattice Γ is **uniform** (or cocompact) if M_Γ is compact. Otherwise, Γ is **non-uniform**.*

In the theory of lattices, passing to a finite index subgroup usually results in only minor differences. Since we often like to ignore these minor differences, we usually care about lattices up to commensurability.

Definition 2.2.5 *Lattices $\Gamma_1, \Gamma_2 < SL(m, \mathbb{R})$ are* ***commensurable*** *if for some $g \in SL(m, \mathbb{R})$*

$$[\Gamma_1 : \Gamma_1 \cap g\Gamma_2 g^{-1}] < \infty$$

Equivalently, their corresponding manifolds M_{Γ_1} and M_{Γ_2} have a common finite-sheeted cover (i.e. $M_{\Gamma_1 \cap g\Gamma_2 g^{-1}}$).

We review a construction of a family of non-uniform arithmetic lattices in $SL(m, \mathbb{R})$. See [Mor15] Chapter 6.8 for more details.

Construction 1 *Fix a square-free $d \in \mathbb{Z}_{>1}$ and $m \geq 3$. Then*

- *$F = \mathbb{Q}(\sqrt{d})$ is a totally real algebraic number field,*
- *$\tau \in Gal(F/\mathbb{Q})$ is the non-trivial involution sending $\sqrt{d} \mapsto -\sqrt{d}$, and*
- $\mathcal{O}_d = \begin{cases} \mathbb{Z}[\sqrt{d}] & d \equiv 2, 3 \ mod\ 4 \\ \mathbb{Z}[\frac{1+\sqrt{d}}{2}] & d \equiv 1 \ mod\ 4 \end{cases}$ *is the ring of integers of F.*

Let $J \in M_{m\times m}(F)$ be sesqui-symmetric matrix with respect to τ (i.e. $J^\top = \tau(J)$). We view J as a τ-Hermitian form on the F-vector space F^m. The associated special unitary group is

$$SU(J; F, \tau) = \{M \in SL(m, F) : M^* J M = J\}$$

where $M^* := \tau(M)^\top$. The integer points of this group form the associated integral special unitary group

$$SU(J; \mathcal{O}_d, \tau) = \{M \in SL(m, \mathcal{O}_d) : M^* J M = J\}.$$

Definition 2.2.6 *Two τ-Hermitian forms J and J' on F^m are* ***SU-equivalent*** *if $J' = P^*JP$ for some change of basis matrix $P \in GL(m, F)$.*

Observe that

$$\mathrm{SU}(P^*JP; F, \tau) = P^{-1}(\mathrm{SU}(J; F, \tau))P.$$

The corresponding integral groups may not be conjugate, but $P \in GL(m, F)$ is in the commensurator of both integral groups, so they are commensurable lattices. Our interest in commensurability classes of arithmetic lattices with entries in $\mathcal{O}_d$ leads to the question: when are two τ-Hermitian forms over F^m equivalent?

Definition 2.2.7 *Let $F = \mathbb{Q}(\sqrt{d})$ and $\tau \in Gal(F/\mathbb{Q})$ be the involution sending $\sqrt{d} \mapsto -\sqrt{d}$. A $\tau-$**Hermitian square** is an element $g \in F$ such that $g = \tau(h)h$ for some $h \in F$.*

By Landherr [Lan35], (see [Lew82] section 3) an equivalence class of τ-Hermitian forms on F^m for $m \geq 3$ is uniquely determined by

- the rank of the form, and
- the discriminant of the form up to $\tau-$Hermitian square.[1]

Proposition 2.2.8 *Suppose J is a full rank τ-Hermitian form on F^m. Then the group $SU(J; \mathcal{O}_d, \tau)$ as constructed above is a non-uniform arithmetic lattice. Moreover, $SU(J; \mathcal{O}_d, \tau)$ is commensurable to $SU(J'; \mathcal{O}_d, \tau)$ for any full rank $\tau-$Hermitian form J' such that $|det(J) - det(J')|$ is a $\tau-$Hermitian square.*

Proof: That $SU(J; \mathcal{O}_d, \tau)$ is an arithmetic lattice follows from Proposition (6.8.14) in [Mor15]. The same proposition tells us our lattice is non-uniform if and only if there

[1] Since $(\sqrt{d})\tau(\sqrt{d}) = -d$, there is no signature in this setting.

exists a nonzero $x \in F^n$ such that $x^* J x = 0$.

By the classification above, $SU(J; \mathcal{O}_d, \tau)$ is commensurable to

$$SU(\text{diag}(1, -1, -\det(J), 1 \ldots, 1); \mathcal{O}_d, \tau).$$

For $x = [1, 1, 0, \ldots, 0]$, clearly

$$x^* \text{diag}(1, -1, -\det(J), 1, \ldots, 1) x = 0.$$

Thus $SU(J; \mathcal{O}_d, \tau)$ is non-uniform. ■

2.3 Systolic Growth

Next, we find a way to control the systole of certain lattices commensurable to those built by construction 1. There is a 1-1 correspondence between closed geodesics in M_Γ and $\Gamma-$conjugacy classes of semi-simple elements in Γ. The length of the geodesic corresponding to the conjugacy class of a semi simple $\gamma \in \Gamma$ is proportional to the translation length of γ on the geodesic it leaves invariant in $SL(m, \mathbb{R})/SO(m)$. Let $l(\gamma)$ denote this length. Hence

$$\text{sys}(\Gamma) = \inf\{l(\gamma) : \text{ semi-simple } \gamma \in \Gamma\}.$$

The translation lengths are then bounded from below in terms of the trace ([LLM17], theorem 3.1):

Theorem 2.3.1 ***(Trace-Length Bounds)*** *Let $\gamma \in SL(m, \mathbb{R})$ be semi-simple with $|tr(\gamma)| \geq 1$. Then*

$$l(\gamma) \geq \sqrt{2}\ arccosh\Big(max\Big\{1, \frac{|tr(\gamma)|}{m}\Big\}\Big).$$

Since $\lim_{x\to\infty} \operatorname{arccosh}(x) = \infty$, we can control the lower bound for the systole of M_Γ by controlling the lower bound for the traces of semisimple elements in Γ. Our tool for controlling the lower bound of the trace is principal congruence subgroups.

Definition 2.3.2 *Let p be a rational prime and $\Gamma < SL(m, \mathbb{R})$ an arithmetic lattice with entries in a ring of integers $\mathcal{O}$. Then*

$$\Gamma^{(p)} = Ker(\pi_p : \Gamma \to SL(m, \mathcal{O}/(p)))$$

is the ***principal congruence subgroup of*** Γ ***of level*** p*, where π_p is projection modulo (p).*

$\Gamma^{(p)}$ is a normal subgroup of finite index in Γ. We will need the following proposition, whose proof uses ideas from the proof of Theorem 5.1 and Corollary 5.2 in [LLM17].

Proposition 2.3.3 *Fix square free $d \in \mathbb{Z}_{\geq 2}$. Let $F = \mathbb{Q}(\sqrt{d})$, $\mathcal{O}_d$ the ring of integers of F, and τ the non-trivial Galois automorphism of F over $\mathbb{Q}$. Suppose $\{J_n\}_{n\in\mathbb{N}}$ is a sequence of $\tau-$Hermitian forms over F^m. For each n, let*

$$\Gamma_n = SU(J_n; \mathcal{O}_d, \tau)$$

If $\{p_n\}_{n\in\mathbb{N}}$ is a sequence of rational primes diverging to ∞, then

$$sys(\Gamma_n^{(p_n)}) \to \infty$$

where $\Gamma_n^{(p_n)}$ denotes the principal congruence subgroup of Γ_n of level p_n.

Proof: Let $k \in \mathbb{R}$. Since $\lim_{x\to\infty} \operatorname{arccosh}(x) = \infty$, there exists $M \geq 2$ such that

$$\frac{2\sqrt{2}}{m} \operatorname{arccosh}(M-1) \geq k.$$

Since $p_n \to \infty$, there exists $N \in \mathbb{N}$ such that $p_n > mM$ for all $n \geq N$. Fix $n \geq N$ and suppose $\gamma \in \Gamma_n^{(p_n)}$ is semi-simple. Then for each power q of γ, γ^q is a semi simple element of $\Gamma_n^{(p_n)}$. Hence $\mathrm{tr}(\gamma^q) \equiv m \mod p_n$, so

$$\mathrm{tr}(\gamma^q) = p_n \alpha_q + m$$

for some $\alpha_q \in \mathcal{O}_d$. By the argument in the second paragraph[2] of the proof of Theorem 5.1 in [LLM17], there exists an integer q, $|q| \leq \frac{m}{2}$ such that $\mathrm{tr}(\gamma^q) \neq m$. Set $\alpha := \alpha_q$. Since $\mathrm{tr}(\gamma^q) \neq m$, $\alpha \neq 0$. Hence

$$|\mathrm{tr}(\gamma^q)| > m(M|\alpha| - 1) \qquad \text{and}$$
$$|\tau(\mathrm{tr}(\gamma^q))| > m(M|\tau(\alpha)| - 1)$$

For any $\alpha \in \mathcal{O}_d$, $\max(|\alpha|, |\tau(\alpha)|) \geq 1$. Thus

$$\max\left\{\frac{|\mathrm{tr}(\gamma^q)|}{m}, \frac{|\tau(\mathrm{tr}(\gamma^q))|}{m}\right\} > M - 1.$$

By definition of the special unitary group, $\gamma^* = J_n \gamma^{-1} J_n^{-1}$. Hence

$$\tau(\mathrm{tr}(\gamma^q)) = \mathrm{tr}((\gamma^q)^*) = \mathrm{tr}(\gamma^{-q}).$$

Since $l(\gamma^q) = l(\gamma^{-q})$, Theorem 3 implies

$$l(\gamma^q) \geq \sqrt{2}\,\mathrm{arccosh}\left(\max\left\{1, \frac{|\mathrm{tr}(\gamma^q)|}{m}, \frac{|\tau(\mathrm{tr}(\gamma^q))|}{m}\right\}\right).$$

[2]The argument in the aforementioned paragraph does not use that Γ is derived from a central simple algebra, only that γ is a semi-simple element and Newton's identities to obtain a formula for the characteristic polynomial of γ in terms of the trace of powers of γ.

Since arccosh is increasing on $[1, \infty)$ and $M \geq 2$,

$$l(\gamma^q) \geq \sqrt{2}\,\text{arccosh}\Big(M - 1\Big).$$

Since $l(\gamma^q) = |q| l(\gamma)$ and $|q| \leq \frac{m}{2}$,

$$l(\gamma) \geq \frac{2\sqrt{2}}{m}\,\text{arccosh}\Big(M - 1\Big) \geq k.$$

Hence

$$\text{sys}(\Gamma_n^{(p_n)}) \geq k.$$

Since k is arbitrary, this completes the proof. ■

2.4 Result

Theorem D. *Fix square free $d \in \mathbb{Z}_{\geq 2}$ and let*

$$\mathcal{L} := SU(I_8; \mathcal{O}_d, \tau).$$

There exists a sequence of non-uniform arithmetic lattices $\Lambda_n < SL(8, \mathbb{R})$ commensurable to $\mathcal{L}$ such that

$$sys(\Lambda_n) \to \infty$$

as $n \to \infty$, yet every Λ_n contains a fixed hyperbolic 3-manifold group Π, a finite index subgroup of vol3.

Remark 2.4.1 *The 3-manifold group Π contains surface groups, so each Λ_n contains a fixed surface group. In particular, the systolic genus of these lattices is bounded from above for all n.*

The fundamental group of vol3, which we will also refer to as vol3, has presentation

$$\text{vol3} = \langle a, b \mid aabbABAbb;\ aBaBabaaab \rangle$$

where $A = a^{-1}$ and $B = b^{-1}$. The hyperbolic representation of vol3 into $SO(3,1)$ admits discrete and faithful deformations in $SL(4,\mathbb{R})$. An explicit one-parameter family of these deformations was found by Cooper, Long, and Thistlethwaite [CLT06]. The manifold vol3 covers an orbifold, denoted $\text{vol3}/\langle u \rangle$, which has a simpler representation

$$\rho_t : \text{vol3}/\langle u \rangle \to SL(4, \mathbb{Q}(t, \sqrt{t^2-1}, \sqrt{t^2+2}))$$

for $t \geq 1$. The representation in [CLT06] uses parameter v instead of t, with the substitution $v = 2t$. Two elements, denoted here u and c, generate $\text{vol3}/\langle u \rangle$, and the image of these elements under ρ_t are listed in the appendix, as well as in the accompanying mathematica file [Hil24c]. To recover the manifold group, one can use the relations $a = u^2c$ and $b = (aua)^{-1}u$. In practice, we work with the orbifold group when interacting with the explicit matrix representation.

The hyperbolic representation is at $t = 1$ and for real values $t \geq 1$ the representation is the holonomy of a real projective structure on vol3, and is thus discrete and faithful. It is not clear how to specialize t to ensure the entries of the image all lie in a ring of integers over some field. As luck would have it, we found 16 specific elements $\{1, g_1, \ldots, g_{15}\} \in \text{vol3}/\langle u \rangle$ such that

- $\mathcal{B}_v = \{\rho_v(1), \rho_v(g_1), \ldots, \rho_v(g_{15})\}$ is a basis for the vector space $M_{4\times 4}(\mathbb{R})$.
- The left regular representation of $\text{vol3}/\langle u \rangle$ with respect to the basis $\mathcal{B}_v$ yields a representation

$$\eta_t : \text{vol3}/\langle u \rangle \to SL(16, \mathbb{Q}(t, \sqrt{t^2-1}))$$

- Both $\eta_t(u)$ and $\eta_t(c)$ have entries in $\mathbb{Z}[t, \sqrt{t^2-1}]$

In an attempt to find an integral representation of smaller dimension, we study invariant subspaces of this 16 dimensional representation. By considering eigenspaces of elements in the centralizer of $\eta_t(\mathrm{vol3}/\langle u \rangle)$, we were able to find an 8 dimensional invariant subspace, whose corresponding representation has entries in $\mathbb{Z}[t, \sqrt{t^2-1}]$. Let

$$\omega_t : \mathrm{vol3}/\langle u \rangle \to SL(8, \mathbb{Z}[t, \sqrt{t^2-1}])$$

denote this representation. The matrices $\omega_t(u)$ and $\omega_t(c)$ are listed in the appendix. This representation is conjugate to $\rho_t \oplus \rho_t$, and hence faithful. Computations confirming ω_t is conjugate to $\rho_t \oplus \rho_t$, the 16 dimensional representation η_t, as well as the the explicit 8 dimensional subspace mentioned above can all be found in the accompanying mathematica file [Hil24c].

2.5 Proof

For the remainder of this paper, fix square free $d \in \mathbb{Z}_{\geq 2}$ and let

$$\mathcal{L} := SU(I_8; \mathcal{O}_d, \tau).$$

Interpreting t as transcendental, let $\tau \in \mathrm{Gal}(\mathbb{Q}(t, \sqrt{t^2-1})/\mathbb{Q}(t))$ be the involution sending $\sqrt{t^2-1} \mapsto -\sqrt{t^2-1}$. For a $\tau-$Hermitian form $J_t \in (\mathbb{Q}(t, \sqrt{t^2-1}))^m$, let

$$SU(J_t; \mathbb{Q}(t, \sqrt{t^2-1}), \tau) := \{M \in SL(m, \mathbb{Q}(t))) \mid M^* J_t M = J_t\}$$

for $M^* := \tau(M)^\top$.

We start the proof of **Theorem D** by finding a sequence of arithmetic lattices commensurable to $\mathcal{L}$ which contain vol3.

Lemma 2.5.1 *There exists a family of Hermitian forms $J_t \in SL(8, \mathbb{Q}(t))$ such that*

- *For $t \geq 1$, J_t is full rank with $det(J_t)$ equal to a square in $\mathbb{Q}(t)$, and*
- *$\omega_t(vol3) < SU(J_t; \mathbb{Q}(t, \sqrt{t^2-1}), \tau)$.*

Moreover, there exists a sequence $t_n \to \infty$ such that

$$\omega_{t_n}(vol3) < SU(J_{t_n}; \mathcal{O}_d, \tau)$$

for all $n \in \mathbb{N}$. By Proposition 2.2.8, the $SU(J_{t_n}; \mathcal{O}_d, \tau)$ are commensurable to $\mathcal{L}$ for $n >> 0$.

Proof: The first part of the lemma follows from a computation: We solve for $J = J_t \in GL(8, \mathbb{R})$ such that $\omega_t(u)^* J \omega_t(u) = J$ and $\omega_t(c)^* J \omega_t(c) = J$. By replacing J with $J + J^*$, we can ensure J is sesqui-symmetric. There are 4 free variables in the solution for J, and by making a choice of numerical value for each free variable[3] we obtain a τ-Hermitian form J which is full rank (for all but finitely many choices of t) . Indeed,

$$\det(J_t) = \frac{16(3-4t^2)^4}{(1-4t^2)^2}$$

which is a square in $\mathbb{Q}(t)$. Moreover, for $t \geq 1$, $\det(J_t)$ is nonzero, so J_t is full rank. The matrix J_t can be found in the accompanying mathematica file [Hil24c].

To guarantee ω_t(vol3) lies in an *integral* special unitary group, is it necessary for the entries of $\omega_t(a)$ and $\omega_t(b)$ to lie in $\mathcal{O}_d$. It is sufficient to choose $t \in \mathbb{N}$ so that $\sqrt{t^2-1} \in \mathcal{O}_d$.

[3]Even leaving all four free variables in J as unknowns, the determinant of J is a square in $\mathbb{Q}(t)$. Thus choosing values in $\mathbb{Q}(t)$ for the free variables does not change the resulting commensurability class of the lattice.

Equivalently, we need to find infinitely many integral solutions (t, y) to Pell's equation:

$$t^2 - dy^2 = 1.$$

It is well known that for any positive non-square $d \in \mathbb{Z}$, Pell's equation has a fundamental solution $(t_1, y_1) \in \mathbb{N}^2$ and the other solutions are exactly the integers (t_n, y_n) such that

$$u^n = t_n + y_n\sqrt{d}$$

for $u = t_1 + y_1\sqrt{d}$. Therefore, for this sequence $\{t_n\}_{n=1}^{\infty}$,

$$\omega_{t_n}(vol3) < SU(J_{t_n}; \mathcal{O}_d, \tau)$$

for all $n \in \mathbb{N}$. We can write $t_n = \frac{1}{2}(u^n + u^{-n})$, so the sequence $t_n \to \infty$ as $n \to \infty$. ■

Let $\Gamma_n := SU(J_{t_n}; \mathcal{O}_d, \tau)$ for the sequence $\{t_n\}_{n\in\mathbb{N}}$ from Lemma 2.5.1. Our next goal is to find finite index subgroups of Γ_n whose systole goes to infinity as $n \to \infty$. This is accomplished by considering the principal congruence subgroups described in Section 3. We will let

$$\Lambda_n := \Gamma_n^{(p_n)}$$

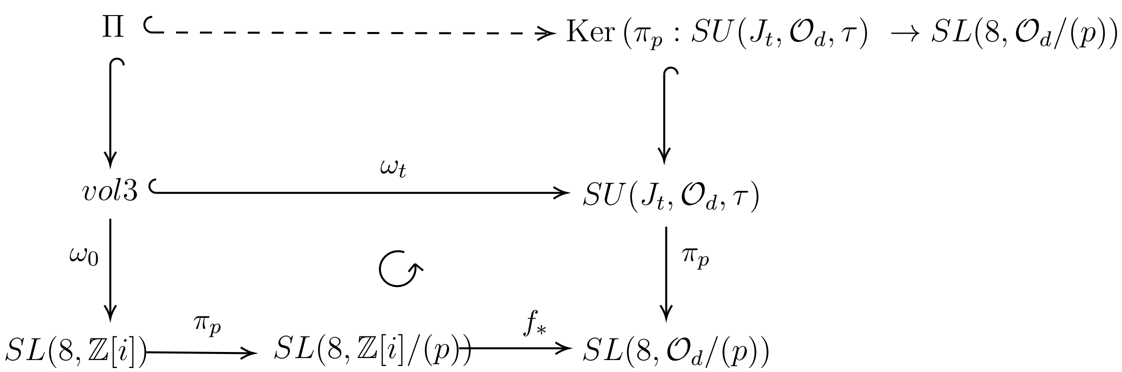
for carefully chosen primes $p_n \to \infty$. By Proposition 2.3.3, $\text{sys}(\Lambda_n) \to \infty$ as $n \to \infty$.

We will choose primes p_n so that $\omega_{t_n}(\Pi) < \Lambda_n$ for $\Pi < \text{vol3}$ a finite index subgroup. More specifically, set

$$\Pi = \text{Ker}(\omega_0 : \text{vol3} \to SL(8, \mathbb{Z}[i])).$$

At $t = 0$, one can check that $|\omega_0(\text{vol3})| = 320$ computationally, so indeed Π is finite index in vol3.

Consider the following diagram:



The primes we choose will divide t. Since our choice of t solves Pell's equation, we have $(pk)^2 - 1 = dy^2$ for integers k and d. Thus

$$f : \mathbb{Z}[i]/(p) \to \mathcal{O}_d/(p)$$

induced by sending $\overline{1} \mapsto \overline{1}$ and $\overline{i} \mapsto \overline{y\sqrt{d}}$ is a ring homomorphism. This induces a group homomorphism

$$f_* : SL(8, \mathbb{Z}[i]/(p)) \to SL(8, \mathcal{O}_d/(p)).$$

Our purpose in constructing this diagram is in the observation that, as long as the diagram commutes, we can guarantee $\omega_t(\Pi) < \mathrm{Ker}(\pi_p)$. In fact, by inspecting the representation (see Appendix), p dividing t is sufficient to guarantee the diagram commutes.

Lemma 2.5.2 *Let $\{t_n\}$ be as in Lemma 1. Possibly passing to a subsequence of t_n, there* *exists a sequence of primes $p_n \to \infty$ with each p_n dividing t_n. Hence*

$$\omega_{t_n}(\Pi) < \Gamma_n^{(p_n)}$$

for all $n \in \mathbb{N}$.

Proof: For each n, we would like a prime p_n dividing t_n, but not dividing t_m for $m < n$. The latter condition will ensure a subsequence of p_n is strictly increasing. Some results from number theory will help us accomplish this. We start with the following definition from [BHV01]:

Definition 2.5.3 *A pair of algebraic integers (α, β) is called a* ***Lucas pair*** *if $\alpha + \beta$ and $\alpha\beta$ are non-zero coprime rational integers and $\frac{\alpha}{\beta}$ is not a root of unity.*

By [Car13], (or see Theorem A in [BHV01]) if (α, β) is a Lucas pair, then for $n >> 0$ the n-th term of the sequence

$$S_n := \alpha^n + \beta^n$$

has a primitive prime divisor, i.e. a prime p_n such that p_n divides S_n but not S_m for $m < n$.

Claim: Let $u = t_1 + y_1\sqrt{d}$ be as in the proof of Lemma 1. Then $(u, 1/u)$ is a Lucas pair.

Observe that

$$2t_n = u^n + \left(\frac{1}{u}\right)^n$$

Therefore, pending the claim above, the sequence $2t_n$ has a corresponding sequence of primitive prime divisors, p_n for $n >> 0$. Passing to a subsequence, we may assume the p_n are strictly increasing, each prime $p_n \neq 2$. Hence each p_n divides t_n and $p_n \to \infty$.

Hence the diagram above commutes and therefore

$$\omega_{t_n}(\Pi) < \Gamma_n^{(p_n)}.$$

Proof of Claim: Since $u \in \mathbb{Z}[\sqrt{d}] \subseteq \mathcal{O}_d$, we know u is an algebraic integer. Since $t_1^2 - dy_1^2 = 1$,

$$\frac{1}{u} = t_1 - y_1\sqrt{d}$$

Hence $\frac{1}{u}$ is also an algebraic integer. Further, $u + \frac{1}{u} = 2t_1$ and $u(\frac{1}{u}) = 1$, so $u + \frac{1}{u}$ and $u(\frac{1}{u})$ are non-zero coprime rational integers. Moreover $\frac{u}{1/u} = u^2$ is not a root of unity. Hence $(u, \frac{1}{u})$ is a Lucas pair. ■

Now we put these pieces together to prove the main theorem:

Theorem D. *Fix square free $d \in \mathbb{Z}_{\geq 2}$ and let*

$$\mathcal{L} := SU(I_8; \mathcal{O}_d, \tau).$$

There exists a sequence of non-uniform arithmetic lattices $\Lambda_n < SL(8, \mathbb{R})$ commensurable to $\mathcal{L}$ such that

$$sys(\Lambda_n) \to \infty$$

as $n \to \infty$, yet every Λ_n contains a fixed hyperbolic 3-manifold group Π, a finite index subgroup of vol3.

Proof: For each $n \in \mathbb{N}$, let $\Gamma_n := SU(J_{t_n}; \mathcal{O}_d, \tau)$ for forms J_{t_n} from Lemma 1 and let

$$\Lambda_n := \Gamma_n^{(p_n)}$$

be the principal congruence subgroup of Γ_n of level p_n for primes p_n from Lemma 2. By Lemma 1, each Γ_n is a non-uniform lattice commensurable to $\mathcal{L}$. Since Λ_n is a finite index subgroup of Γ_n, each Λ_n is also a non-uniform lattice commensurable to $\mathcal{L}$. By Lemma 2, $\omega_{t_n}(\Pi) < \Lambda_n$ and by Proposition 2.3.3, $\mathrm{sys}(\Lambda_n) \to \infty$ as $n \to \infty$. ■

2.6 Examples

Example 2.6.1 *We describe how to find the first few terms of the sequence (t_n, p_n) in the case that $d = 3$. Since $3 \not\equiv 1 (mod\ 4)$, we have $\mathcal{O}_3 = \mathbb{Z}[\sqrt{3}]$.*

To guarantee the entries of $\omega_t(u)$ and $\omega_t(c)$ lie in $\mathcal{O}_3$, we solve for t in Pell's equation:

$$t^2 - 3y^2 = 1.$$

The fundamental solution is $(t_1, y_1) = (2, 1)$. Let $u = 2 + \sqrt{3}$. Powers of u allow us to find all the other solutions. To ensure the diagram commutes and consequently that $\omega_t(\Pi) < \Gamma_n^{(p)}$, we need to find corresponding primes p_n such that $t_n = 0$ mod p_n.

n	u^n	t_n	p_n
1	$2 + \sqrt{3}$	2	2
2	$7 + 4\sqrt{3}$	7	7
3	$26 + 15\sqrt{3}$	26	13
4	$97 + 56\sqrt{3}$	97	97
5	$362 + 209\sqrt{5}$	362	181

Observe each prime does not divide any of the previous values of t_n.

Example 2.6.2 *Now let* $d = 5$*. Since* $5 \equiv 1 \pmod 4$*,* $\mathcal{O}_5 = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$*. We first solve Pell's equation:*

$$t^2 - 5y^2 = 1.$$

The fundamental solution is $(t_1, y_1) = (9, 4)$*. Following the same process as above, we find the first 5 terms of the sequences* t_n *and* p_n*.*

n	u^n	t_n	p_n
1	$9 + 4\sqrt{5}$	9	3
2	$161 + 72\sqrt{5}$	161	7
3	$2889 + 1292\sqrt{5}$	2889	107
4	$51841 + 23184\sqrt{5}$	51841	1103
5	$930249 + 416020\sqrt{5}$	930249	2521

2.7 Appendix

2.7.1 The original 4 dimensional representation of vol3

The hyperbolic 3-manifold vol3 has presentation

$$\text{vol3} = \langle a, b \mid aabbABAbb;\ aBaBabaaab \rangle$$

where $A = a^{-1}$ and $B = b^{-1}$. The orbifold group vol3 $/\langle u \rangle$ which contains vol3 as a subgroup is generated by u and c, of order 4 and 2 respectively. To recover vol3, we have $a = u^2c$ and $b = (aua)^{-1}u$. A conjugate of the image of u and c under the original 4D representation from section 2.4 of [CLT07] with the substitution $t = \frac{v}{2}$ are:

$$\rho_t(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{(t^2-1)}{(2+t^2)}} & 1 \\ 0 & 0 & -\frac{(1+2t^2)}{(2+t^2)} & -\sqrt{\frac{(t^2-1)}{(2+t^2)}} \end{pmatrix}$$

$$\rho_t(c) = \begin{pmatrix} \frac{1}{2}(t+\sqrt{2+t^2}) & 0 & \frac{1}{2}(1-t^2-t\sqrt{2+t^2}) & 0 \\ 0 & \frac{1}{2}(t-\sqrt{2+t^2}) & 0 & \frac{1}{2}(-1+t^2-t\sqrt{2+t^2}) \\ 1 & 0 & \frac{1}{2}(-t-\sqrt{2+t^2}) & 0 \\ 0 & -1 & 0 & \frac{1}{2}(-t+\sqrt{2+t^2}) \end{pmatrix}$$

The exact representation from [CLT07] and the matrix which conjugates it to ρ_t listed above can be found in the accompanying mathematica file [Hil24c].

Let $\tau \in \mathrm{Gal}(\mathbb{Q}(t,\sqrt{t^2-1})/\mathbb{Q}(t))$ sending $\sqrt{t^2-1}$ to $-\sqrt{t^2-1}$. There is one $\tau-$Hermitian form (up to scalar multiples) which both generators $\rho_t(u)$ and $\rho_t(c)$ preserve:

$$M_t = \begin{pmatrix} -\frac{(2\sqrt{2+t^2})}{(2t+t^3-\sqrt{2+t^2}+t^2\sqrt{2+t^2})} & 0 & 0 & 0 \\ 0 & \frac{2(2+t^2)}{((1+2t^2)(1-t^2+t\sqrt{2+t^2}))} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{(2+t^2)}{(1+2t^2)} \end{pmatrix}$$

Note that for $t = 1$, we have $\sqrt{t^2-1} = 0$, so the involution τ is trivial. Since M_1 is a diagonal matrix with signature $(3,1)$, it is clear that ρ_1 lies in $SO(3,1)$.

2.7.2 The explicit 8 dimensional representation of vol3:

Our 8D representation which is integral for values of t solving Pell's equation is generated by:

$$\omega_t(u) = \begin{pmatrix} 1 & 0 & 0 & 2t & -2t & 2t & 0 & 0 \\ 0 & 1 & 0 & t-\sqrt{t^2-1} & -t+\sqrt{t^2-1} & t-\sqrt{t^2-1} & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -t+\sqrt{t^2-1} & -t-\sqrt{t^2-1} \\ 0 & 0 & 1 & -1 & 0 & 1 & -t+\sqrt{t^2-1} & -t-\sqrt{t^2-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\omega_t(c) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}$$

An explicit computation found in [Hil24c] shows this representation is conjugate to $\rho_t \oplus \rho_t$. One can also find the τ- Hermitian form J_t which both $\omega_t(u)$ and $\omega_t(c)$ preserve in the mathematica file in [Hil24c].

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