Title
On the construction of polynomial minimal surfaces with Pythagorean normals

Authors
Farouki, Rida T
Knez, Marjeta
Vitrih, Vito
et al.

Publication Date
2022-12-01

DOI
10.1016/j.amc.2022.127439

Peer reviewed

On the construction of polynomial minimal surfaces with Pythagorean normals

Rida T. Farouki[a], Marjeta Knez[b,c], Vito Vitrih[d,e], Emil Žagar[b,c]

[a]*Mechanical and Aerospace Engineering, University of California, Davis, CA 95616, USA*
[b]*Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, Ljubljana, Slovenia*
[c]*Institute of Mathematics, Physics and Mechanics, Jadranska 19, Ljubljana, Slovenia*
[d]*Faculty of Mathematics, Natural Sciences and Information Technologies, University of Primorska, Glagoljaška 8, Koper, Slovenia*
[e]*Andrej Marušič Institute, University of Primorska, Muzejski trg 2, Koper, Slovenia*

Abstract

A novel approach to constructing polynomial minimal surfaces (surfaces of zero mean curvature) with isothermal parameterizations from Pythagorean triples of complex polynomials is presented, and it is shown that they are Pythagorean normal (PN) surfaces, i.e., their unit normal vectors have a rational dependence on the surface parameters. This construction generalizes a prior approach based on Pythagorean triples of real polynomials, and yields more shape freedoms for surfaces of a specified degree. Moreover, when one of the complex polynomials is just a constant, the minimal surfaces have the Pythagorean–hodograph (PH) preserving property — a planar PH curve in the parameter domain is mapped to a spatial PH curve on the surface. Cubic and quintic examples of these minimal PN surfaces are presented, including examples of solutions to the Plateau problem, with boundaries generated by planar PH curve segments in the parameter domain. The construction is also generalized to the case of minimal surfaces with non–isothermal parameterizations. Finally, an application to the problem of interpolating three given points in $\mathbb{R}^3$ as the corners of a triangular cubic minimal surface patch, such that the three patch sides have prescribed lengths, is addressed.

Keywords: Pythagorean–hodograph curves, Pythagorean–normal surfaces, minimal surfaces, Enneper–Weierstrass parameterization, Plateau's problem, quaternions.

1. Introduction

The study of *minimal surfaces* (i.e., surfaces of least area subject to given boundary conditions) has been a topic of great interest in the field of differential geometry for more than two centuries. The partial differential equation that characterizes the minimal surfaces was first derived by Joseph Louis Lagrange (1736–1813), and Jean Baptiste Meusnier (1754–1793) subsequently noted that the solutions to this equation must possess zero mean curvature — or, equivalently, their principal curvatures must be of equal magnitude but opposite sign at each point [7]. Thus, minimal surfaces

*Corresponding author

Email addresses: farouki@ucdavis.edu (Rida T. Farouki), marjetka.knez@fmf.uni-lj.si (Marjeta Knez), vito.vitrih@upr.si (Vito Vitrih), emil.zagar@fmf.uni-lj.si (Emil Žagar)

connect “local” and “global” geometry features in a remarkable manner. Joseph Antoine Ferdinand Plateau (1801–1883) observed [7] that, when a thin wire formed into a closed loop is immersed in a soap solution, the resulting soap film assumes the shape of least surface area (thus minimizing its surface tension energy). The problem of constructing minimal surfaces bounded by closed space curves (which is, in general, quite difficult) is now known as Plateau’s problem, and the existence of solutions for any boundary curve was first demonstrated [1] by Jesse Douglas in 1931.

The concepts and tools developed in the novel field of computer aided geometric design provide useful approaches to the construction of examples of minimal surfaces. The role of the Pythagorean hodograph (PH) curves and Pythagorean normal (PN) surfaces in such constructions is of particular interest. A PH curve has the distinctive property that its unit tangent has a rational dependence on the curve parameter [3], and a PN surface possesses a unit normal with a rational dependence [9, 12] on the surface parameters. Hao [4] demonstrated the construction of minimal surfaces from a given planar PH curve, the plane projection of a surface isoparametric curve being coincident with the chosen PH curve. Ueda [15] noted that an isothermal parameterization of the Enneper surface (a well–known minimal surface [11]) admits a PN representation, and isothermal parameterizations are closely related [6] to mappings that preserve the PH structure. Lávička and Vršek [10] identify a family of cubic polynomial PN surfaces, analogous to Tschirnhaus cubic (the unique [2] planar cubic PH curve), and identify instances that are minimal surfaces. Kozak, Krajnc, and Vitrih [8] introduced a quaternion formulation for constructing polynomial PN surfaces of general degree, and they formulate constraints on the quaternion coefficients that yield minimal surfaces.

The present study builds on prior investigations by developing a construction of polynomial minimal surfaces generated by three complex polynomials. When one of these polynomials is just a constant, this allows the construction of bounded minimal surface patches (solutions to Plateau’s problem) by using planar PH curves in the parameter domain to generate the patch boundary.

The plan for the remainder of the paper is as follows. Section 2 reviews some basic properties of minimal surfaces, and introduces the construction of polynomial minimal surfaces from triples of complex polynomials as a generalization of the approach in [4]. This methodology is elaborated in Section 3, with a focus on surfaces with isothermal parameterizations. It is shown that these are PN surfaces, and that planar PH curves in the parameter domain are mapped to spatial PH curves on the surface. The construction is illustrated by cubic and quintic isothermal cases, including bounded surface patches generated by specfiying a restricted parameter domain with a boundary defined by PH curve segments. In Section 4, the construction is generalized to the case of minimal surfaces having non–isothermal paramaterizations. As an example application to surface design problems, Section 5 presents a construction of a cubic triangular minimal surface patch with prescribed corner points and prescribed arc length for the three patch boundary curves. Finally, Section 6 summarizes the results of the present study, and identifies further possible directions of investigation.

2. Preliminaries

Let $\boldsymbol{r} : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a regular parameterization of a surface

$$\{\boldsymbol{r}(x,y) : \ \boldsymbol{r}_x(x,y) \times \boldsymbol{r}_y(x,y) \neq \boldsymbol{0}, \ (x,y) \in \Omega\} \, ,$$

where $\boldsymbol{r}_x = \frac{\partial \boldsymbol{r}}{\partial x}$ and $\boldsymbol{r}_y = \frac{\partial \boldsymbol{r}}{\partial y}$ denote the partial derivatives with respect to parameters x and y. We call $\boldsymbol{r}$ a parametric surface or, briefly, just a surface. The coefficients of the first and the second fundamental forms of $\boldsymbol{r}$ are defined by

$$E = \boldsymbol{r}_x \cdot \boldsymbol{r}_x, \quad F = \boldsymbol{r}_x \cdot \boldsymbol{r}_y, \quad G = \boldsymbol{r}_y \cdot \boldsymbol{r}_y, \tag{1a}$$

and

$$L = \frac{\boldsymbol{r}_x \times \boldsymbol{r}_y}{\|\boldsymbol{r}_x \times \boldsymbol{r}_y\|} \cdot \boldsymbol{r}_{xx}, \quad M = \frac{\boldsymbol{r}_x \times \boldsymbol{r}_y}{\|\boldsymbol{r}_x \times \boldsymbol{r}_y\|} \cdot \boldsymbol{r}_{xy}, \quad N = \frac{\boldsymbol{r}_x \times \boldsymbol{r}_y}{\|\boldsymbol{r}_x \times \boldsymbol{r}_y\|} \cdot \boldsymbol{r}_{yy}, \tag{1b}$$

respectively, where $\cdot$ and $\times$ denote the scalar and vector product in $\mathbb{R}^3$, while $\boldsymbol{r}_{xx}$, $\boldsymbol{r}_{xy}$ and $\boldsymbol{r}_{yy}$ are the second order partial derivatives of $\boldsymbol{r}$. If $E = G$ and $F = 0$, the parameterization of $\boldsymbol{r}$ is said to be *isothermal*. For such parameterizations the isoparametric curves $\boldsymbol{r}(\cdot, y_0)$ and $\boldsymbol{r}(x_0, \cdot)$ are orthogonal at the point $\boldsymbol{r}(x_0, y_0)$ and the partial derivatives at that point have equal magnitudes. The mean curvature H and Gaussian curvature K of the surface are computed [14] from the coefficients (1) as

$$H = \frac{EN - 2FM + GL}{2(EG - F^2)}, \quad K = \frac{LN - M^2}{EG - F^2}.$$

Note that H and K are invariant under regular reparameterizations of the surface.

Definition 1. *A parametric surface $\boldsymbol{r}$ is a minimal surface, if its mean curvature H is identically zero.*

It is well known [11] that a simply connected minimal surface $\boldsymbol{r}$ can be constructed by the Enneper–Weierstrass parameterization from three holomorphic functions Φ_1, Φ_2, Φ_3 that satisfy the condition

$$\Phi_1^2(\mathtt{z}) + \Phi_2^2(\mathtt{z}) + \Phi_3^2(\mathtt{z}) = 0. \tag{2}$$

More precisely, by computing the Weierstrass complex curve

$$\boldsymbol{\Psi}(\mathtt{z}) = \int_{\mathtt{z}_0}^{\mathtt{z}} (\Phi_1(\mathtt{z}), \Phi_2(\mathtt{z}), \Phi_3(\mathtt{z}))\, d\mathtt{z}, \tag{3}$$

its real part defines the minimal surfaces in isothermal parameters, $\boldsymbol{r}(x, y) = \mathrm{Re}(\boldsymbol{\Psi}(\mathtt{z}))$, where $\mathtt{z} = x + \mathtt{i}y$.

Remark 1. *Defining $\widetilde{\boldsymbol{r}}(x, y) = Im(\boldsymbol{\Psi}(\mathtt{z}))$ also gives a minimal surface, which is a conjugate of $\boldsymbol{r}$, meaning that*

$$\frac{\partial \boldsymbol{r}}{\partial x} = \frac{\partial \widetilde{\boldsymbol{r}}}{\partial y}, \quad \frac{\partial \boldsymbol{r}}{\partial y} = -\frac{\partial \widetilde{\boldsymbol{r}}}{\partial x}.$$

In [4] a special class of parametric polynomial minimal surfaces is derived, based on the connection of (2) with Pythagorean triples and planar Pythagorean-hodograph curves. The main idea relies on the following theorem and subsequent definition.

Theorem 1. *For three real polynomials $a(t)$, $b(t)$ and $c(t)$ the Pythagorean condition*

$$a^2(t) + b^2(t) = c^2(t)$$

is satisfied if and only if these polynomials can be expressed in terms of other real polynomials $u(t)$, $v(t)$ and $w(t)$ as

$$\begin{aligned} a(t) &= w(t)(u^2(t) - v^2(t)), \\ b(t) &= 2w(t)u(t)v(t), \\ c(t) &= w(t)(u^2(t) + v^2(t)). \end{aligned} \tag{4}$$

Definition 2. *A parametric polynomial curve $\boldsymbol{p} : [\alpha, \beta] \to \mathbb{R}^d$ is a Pythagorean–hodograph curve (PH curve) if the unit tangent $\boldsymbol{t} = \frac{\boldsymbol{p}'}{\|\boldsymbol{p}'\|}$ is rational. Namely, $\|\boldsymbol{p}'\| = \sigma$ for some polynomial function σ. We call $\boldsymbol{h} := \boldsymbol{p}'$ the hodograph of $\boldsymbol{p}$.*

Using Theorem 1 Hao constructs ([4, Theorem 4]) polynomial minimal surfaces in Enneper–Weierstrass parameterization from three real polynomials $u, v, w \in \mathbb{R}[t]$ by computing $a, b, c \in \mathbb{R}[t]$ as in (4) and choosing

$$\Phi_1(\mathtt{z}) = a(\mathtt{z}), \quad \Phi_2(\mathtt{z}) = b(\mathtt{z}), \quad \Phi_3(\mathtt{z}) = \mathtt{i}\, c(\mathtt{z}).$$

By the same theorem and Definition 2 the polynomials u, v, w — which we call the *preimage polynomials* — define a planar PH curve $\boldsymbol{p}(t) = \int_\alpha^t \boldsymbol{h}(u)du + \mathrm{const}$ by prescribing its hodograph as $\boldsymbol{h}(t) = (w(t)(u^2(t) - v^2(t), 2w(t)u(t)v(t))$, and vice–versa, i.e., for any planar PH curve there exist three preimage polynomials u, v, w that satisfy (4) for $(a, b) = \boldsymbol{p}'$. Therefore, any planar PH curve generates one polynomial minimal surface, and it is further shown in [4] that this generating curve lies on the minimal surface. Let us call the set of all minimal surfaces obtained from planar PH curves through Enneper–Weierstrass parameterization the *class* 1 minimal surfaces. However, we show in the next section that this class of polynomial minimal surfaces is just a particular subset of a more general class with twice as many free parameters.

3. Construction of polynomial minimal surfaces in isothermal parameterization

First, let us shortly recall some definitions from complex analysis [13]. A function $\mathrm{f} : U \subseteq \mathbb{C} \to \mathbb{C}$ is holomorphic on an open set $U \subseteq \mathbb{C}$ if it is complex differentiable at every point $\mathtt{z}_0 \in U$. Writing $\mathtt{z} = x + \mathtt{i}y$, we can express $\mathrm{f}(\mathtt{z}) = f_R(x, y) + \mathtt{i} f_I(x, y)$, where $f_R = \mathrm{Re}(\mathrm{f})$ and $f_I = \mathrm{Im}(\mathrm{f})$ denote the real and imaginary part of f. If f is holomorphic on U then the first-order partial derivatives of f_R and f_I with respect to x and y exist on U and satisfy the *Cauchy-Riemann* equations

$$\frac{\partial f_R}{\partial x} = \frac{\partial f_I}{\partial y}, \quad \frac{\partial f_R}{\partial y} = -\frac{\partial f_I}{\partial x}, \tag{5}$$

which imply that f_R and f_I are both harmonic functions.

Let us now choose two complex polynomials of degree n

$$\mathrm{u} : \mathbb{C} \to \mathbb{C}, \quad \mathrm{u}(\mathtt{z}) = \sum_{j=0}^{n} \mathrm{u}_j\, \mathtt{z}^j, \quad \mathrm{u}_j \in \mathbb{C},$$

$$\mathrm{v} : \mathbb{C} \to \mathbb{C}, \quad \mathrm{v}(\mathtt{z}) = \sum_{j=0}^{n} \mathrm{v}_j\, \mathtt{z}^j, \quad \mathrm{v}_j \in \mathbb{C},$$

and a complex polynomial

$$\mathrm{w} : \mathbb{C} \to \mathbb{C}, \quad \mathrm{w}(\mathtt{z}) = \sum_{j=0}^{k} \mathrm{w}_j\, \mathtt{z}^j, \quad \mathrm{w}_j \in \mathbb{C}$$

of degree k, and denote

$$u_R = \mathrm{Re}(\mathrm{u}), \quad u_I = \mathrm{Im}(\mathrm{u}), \quad v_R = \mathrm{Re}(\mathrm{v}), \quad v_I = \mathrm{Im}(\mathrm{v}), \quad w_R = \mathrm{Re}(\mathrm{w}), \quad w_I = \mathrm{Im}(\mathrm{w}). \tag{6}$$

These polynomials are clearly holomorphic on $\mathbb{C}$. If we choose

$$\begin{aligned} \Phi_1(\mathtt{z}) &= \mathrm{w}(\mathtt{z})(\mathrm{u}^2(\mathtt{z}) - \mathrm{v}^2(\mathtt{z})), \\ \Phi_2(\mathtt{z}) &= 2\,\mathrm{w}(\mathtt{z})\mathrm{u}(\mathtt{z})\mathrm{v}(\mathtt{z}), \\ \Phi_3(\mathtt{z}) &= \mathtt{i}\mathrm{w}(\mathtt{z})\left(\mathrm{u}^2(\mathtt{z}) + \mathrm{v}^2(\mathtt{z})\right), \end{aligned} \tag{7}$$

then Φ_j are also holomorphic, and the condition (2) is clearly satisfied. Thus, we obtain from the Weierstrass curve (3) the minimal surface $\boldsymbol{r} : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$\boldsymbol{r}(x, y) = \boldsymbol{r}_0 + \mathrm{Re}\left(\int_0^1 \boldsymbol{\Phi}\left(\xi(x + \mathtt{i}y)\right)(x + \mathtt{i}y)\, d\xi\right), \tag{8a}$$

where

$$\boldsymbol{\Phi}(\mathtt{z}) := (\Phi_1(\mathtt{z}), \Phi_2(\mathtt{z}), \Phi_3(\mathtt{z})) \tag{8b}$$

and $\boldsymbol{r}_0 \in \mathbb{R}^3$ is an arbitrary point. Denoting

$$\boldsymbol{g}_R = \mathrm{Re}(\boldsymbol{\Phi}) \quad \text{and} \quad \boldsymbol{g}_I = \mathrm{Im}(\boldsymbol{\Phi}),$$

we have two polynomial vector–valued functions of total degree $\leq 2n + k$ in the variables (x, y), and we obtain the following lemma.

Lemma 1. *For $\boldsymbol{r}$ defined by* (8)*, its partial derivatives with respect to x and y are equal to*

$$\frac{\partial \boldsymbol{r}}{\partial x} = \boldsymbol{g}_R, \quad \frac{\partial \boldsymbol{r}}{\partial y} = -\boldsymbol{g}_I.$$

Proof : First, we observe that

$$\mathrm{Re}(\boldsymbol{\Phi}\left(\xi(x + \mathtt{i}y)\right)(x + \mathtt{i}y)) = \boldsymbol{g}_R(\xi x, \xi y)\, x - \boldsymbol{g}_I(\xi x, \xi y)\, y.$$

From the Cauchy-Riemann equations it then follows that

$$\frac{\partial}{\partial x}\left(\boldsymbol{g}_R(\xi x, \xi y)\, x - \boldsymbol{g}_I(\xi x, \xi y)\, y\right) = \frac{\partial}{\partial x}\left(\boldsymbol{g}_R(\xi x, \xi y)\right) x + \boldsymbol{g}_R(\xi x, \xi y) + \frac{\partial}{\partial y}\left(\boldsymbol{g}_R(\xi x, \xi y)\right) y.$$

If we denote

$$\boldsymbol{g}_R(x, y) =: \sum_{0 \leq i+j \leq 2n+k} \boldsymbol{\alpha}_{i,j} x^i y^j,$$

then

$$\begin{aligned} \frac{\partial}{\partial x}\mathrm{Re}(\boldsymbol{\Phi}\left(\xi(x + \mathtt{i}y)\right)(x + \mathtt{i}y)) &= \frac{\partial}{\partial x}\left(\sum_{0 \leq i+j \leq 2n+k} \boldsymbol{\alpha}_{i,j} x^i y^j \xi^{i+j}\right) x + \boldsymbol{g}_R(\xi x, \xi y) \\ &+ \frac{\partial}{\partial y}\left(\sum_{0 \leq i+j \leq 2n+k} \boldsymbol{\alpha}_{i,j} x^i y^j \xi^{i+j}\right) y = \sum_{0 \leq i+j \leq 2n+k} \boldsymbol{\alpha}_{i,j}(i + j + 1) x^i y^j \xi^{i+j}. \end{aligned}$$

Therefore

$$\frac{\partial \boldsymbol{r}}{\partial x}(x,y) = \int_0^1 \left(\sum_{0\le i+j\le 2n+k} \boldsymbol{\alpha}_{i,j}(i+j+1)x^i y^j \xi^{i+j} \right) d\xi = \sum_{0\le i+j\le 2n+k} \boldsymbol{\alpha}_{i,j} x^i y^j = \boldsymbol{g}_R(x,y).$$

The second equality follows similarly. $\square$

Since $\frac{\partial \boldsymbol{g}_R}{\partial y} = -\frac{\partial \boldsymbol{g}_I}{\partial x}$, the parameterization of the surface (8) can be written as (see [8, Lemma 1])

$$\boldsymbol{r}(x,y) = \boldsymbol{r}_0 + \int_0^x \boldsymbol{g}_R(x,y)dx - \int_0^y \boldsymbol{g}_I(0,y)dy = \boldsymbol{r}_0 - \int_0^y \boldsymbol{g}_I(x,y)dy + \int_0^x \boldsymbol{g}_R(x,0)dx. \quad (9)$$

Its component functions are bivariate polynomials of total degree $\le 2n+k+1$.

In what follows we express $\boldsymbol{g}_R$ and $\boldsymbol{g}_I$ with quaternion polynomials and prove that the unit normal of $\boldsymbol{r}$ is rational. This condition characterizes the following class of parametric surfaces.

Definition 3. *A parametric surface $\boldsymbol{r} : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ is a Pythagorean normal (PN) surface if its normal vector field*

$$\boldsymbol{N} = \frac{\boldsymbol{r}_x \times \boldsymbol{r}_y}{\|\boldsymbol{r}_x \times \boldsymbol{r}_y\|}$$

is rational.

From (6) and (7) it is straightforward to compute that

$$\boldsymbol{g}_R = \begin{pmatrix} w_R\left(u_R^2 - u_I^2 - v_R^2 + v_I^2\right) - 2w_I\left(u_R u_I - v_R v_I\right) \\ 2w_R\left(u_R v_R - u_I v_I\right) - 2w_I\left(u_R v_I + u_I v_R\right) \\ -2w_R\left(u_R u_I + v_R v_I\right) - w_I\left(u_R^2 - u_I^2 + v_R^2 - v_I^2\right) \end{pmatrix} = w_R\,\mathcal{A}\,\boldsymbol{i}\,\bar{\mathcal{A}} - w_I\,\mathcal{A}\,\boldsymbol{k}\,\bar{\mathcal{A}} \quad \text{(10a)}$$

and

$$\boldsymbol{g}_I = \begin{pmatrix} 2w_R\left(u_R u_I - v_R v_I\right) + w_I\left(u_R^2 - u_I^2 - v_R^2 + v_I^2\right) \\ 2w_R\left(u_R v_I + u_I v_R\right) + 2w_I\left(u_R v_R - u_I v_I\right) \\ w_R\left(u_R^2 - u_I^2 + v_R^2 - v_I^2\right) - 2w_I\left(u_R u_I + v_R v_I\right) \end{pmatrix} = w_I\,\mathcal{A}\,\boldsymbol{i}\,\bar{\mathcal{A}} + w_R\,\mathcal{A}\,\boldsymbol{k}\,\bar{\mathcal{A}}, \quad \text{(10b)}$$

where

$$\mathcal{A} = u_R - v_I\,\boldsymbol{i} + u_I\,\boldsymbol{j} + v_R\,\boldsymbol{k}, \quad \text{(10c)}$$

and $\mathcal{A}\,\boldsymbol{i}\,\bar{\mathcal{A}}$, $\mathcal{A}\,\boldsymbol{k}\,\bar{\mathcal{A}}$ are identified with vectors in $\mathbb{R}^3$. This confirms the known fact that $\boldsymbol{r}$ has an isothermal parameterization:

$$\boldsymbol{r}_x \cdot \boldsymbol{r}_x = \boldsymbol{r}_y \cdot \boldsymbol{r}_y = \left(w_R^2 + w_I^2\right)\|\mathcal{A}\|^4, \quad \boldsymbol{r}_x \cdot \boldsymbol{r}_y = 0.$$

Moreover, we compute that

$$\boldsymbol{r}_x \times \boldsymbol{r}_y = \left(w_R^2 + w_I^2\right)\left(\mathcal{A}\,\boldsymbol{k}\,\bar{\mathcal{A}}\right) \times \left(\mathcal{A}\,\boldsymbol{i}\,\bar{\mathcal{A}}\right) = \left(w_R^2 + w_I^2\right)\|\mathcal{A}\|^2\,\mathcal{A}\,\boldsymbol{j}\,\bar{\mathcal{A}},$$

so the unit normal of the surface $\boldsymbol{r}$ is equal to

$$\boldsymbol{N} = \frac{\mathcal{A}\,\boldsymbol{j}\,\bar{\mathcal{A}}}{\|\mathcal{A}\|^2}, \quad (11)$$

which is a rational expression. This proves the next theorem.

Theorem 2. *For any complex polynomials* u, v *and* w, *the surface* $\boldsymbol{r}$ *defined by* (7) *and* (8) *is a PN surface.*

The next lemma reveals when $\boldsymbol{r}$ is a *PH preserving mapping*, i.e., it maps any planar PH curve in a parameter domain to a spatial PH curve on the surface.

Lemma 2. *Let a parametric surface* $\boldsymbol{r}$ *be given by* (7) *and* (8) *for some complex polynomials* u, v *and* w. *Suppose that* $\mathrm{w}(\mathtt{z}) = 1$ *and let* $\boldsymbol{q} : \mathbb{R} \to \mathbb{R}^2$ *be a planar polynomial PH curve. Then* $\boldsymbol{p} = \boldsymbol{r} \circ \boldsymbol{q} : \mathbb{R} \to \mathbb{R}^3$ *is a spatial polynomial PH curve.*

Proof : If we denote $\boldsymbol{q}(t) = (q_1(t), q_2(t))^T$, then

$$\boldsymbol{p}'(t) = \boldsymbol{r}_x(\boldsymbol{q}(t))q_1'(t) + \boldsymbol{r}_y(\boldsymbol{q}(t))q_2'(t).$$

Using the fact that $\boldsymbol{r}$ has an isothermal parameterization and the assumption that $\mathrm{w}(\mathtt{z}) = 1$, we compute

$$\begin{aligned}\|\boldsymbol{p}'(t)\| &= \sqrt{\|\boldsymbol{r}_x(\boldsymbol{q}(t))\|^2 q_1'(t)^2 + \|\boldsymbol{r}_y(\boldsymbol{q}(t))\|^2 q_2'(t)^2} \\ &= \sqrt{q_1'(t)^2 + q_2'(t)^2}\sqrt{w_R^2(\boldsymbol{q}(t)) + w_I^2(\boldsymbol{q}(t))}\,\|\mathcal{A}(\boldsymbol{q}(t))\|^2 = \sigma(t)\,\|\mathcal{A}(\boldsymbol{q}(t))\|^2\end{aligned}$$

where $\sigma(t) = \sqrt{q_1'(t)^2 + q_2'(t)^2}$ is a polynomial since $\boldsymbol{q}$ is a PH curve. This concludes the proof. $\square$

Remark 2. *The construction of minimal surfaces by Hao in [4] follows from our construction by choosing complex polynomials* u, v *and* w, *such that their coefficients are real, i.e.,* $\mathit{Im}(\mathrm{u}_j) = \mathit{Im}(\mathrm{v}_j) = 0$, $j = 0, 1, \ldots, n$, *and* $\mathit{Im}(\mathrm{w}_j) = 0$, $j = 0, 1, \ldots, k$.

Let us demonstrate the presented theory on the construction of cubic, quartic and quintic minimal surfaces.

Example 1 (Cubic case)**.** *To construct minimal surfaces in cubic parameterization we need to choose linear complex polynomials* u, v, *and constant polynomial* w. *Let them be chosen as*

$$\mathrm{u}(\mathtt{z}) = (1 + 2\mathtt{i}) + (3 - 2\mathtt{i})\,\mathtt{z}, \quad \mathrm{v}(\mathtt{z}) = (4 - \mathtt{i}) + (2 + 2\mathtt{i})\,\mathtt{z}, \quad \mathrm{w}(\mathtt{z}) = 1.$$

Holomorphic functions (7) *are then equal to*

$$\begin{aligned}\Phi_1(\mathtt{z}) &= \left(5x^2 + 40xy - 5y^2 - 6x + 4y - 18\right) + \mathtt{i}\left(-20x^2 + 10xy + 20y^2 - 4x - 6y + 12\right), \\ \Phi_2(\mathtt{z}) &= \left(20x^2 - 8xy - 20y^2 + 16x + 10y + 12\right) + \mathtt{i}\left(4x^2 + 40xy - 4y^2 - 10x + 16y + 14\right), \\ \Phi_3(\mathtt{z}) &= \left(4x^2 - 10xy - 4y^2 - 20x - 34y + 4\right) + \mathtt{i}\left(5x^2 + 8xy - 5y^2 + 34x - 20y + 12\right),\end{aligned}$$

and the surface parameterization is computed as

$$\begin{aligned}\boldsymbol{r}(x, y) = \Big(&\frac{5x^3}{3} + 20x^2y - 5xy^2 - \frac{20y^3}{3} - 3x^2 + 4xy + 3y^2 - 18x - 12y, \\ &\frac{20x^3}{3} - 4x^2y - 20xy^2 + \frac{4y^3}{3} + 8x^2 + 10xy - 8y^2 + 12x - 14y, \\ &\frac{4x^3}{3} - 5x^2y - 4xy^2 + \frac{5y^3}{3} - 10x^2 - 34xy + 10y^2 + 4x - 12y\Big).\end{aligned}$$

Quaternion polynomial (10c) *that corresponds to this surface simplifies to*

$$\mathcal{A}(x,y) = (3x+2y+1) + (-2x-2y+1)\boldsymbol{i} + (-2x+3y+2)\boldsymbol{j} + (2x-2y+4)\boldsymbol{k},$$

from where we compute that

$$\boldsymbol{N}(x,y) = \left(-\frac{2\left(2x^2+2y^2+20x+7y+2\right)}{21x^2+21y^2+10x-4y+22}, \frac{5x^2+5y^2-14x+36y-12}{21x^2+21y^2+10x-4y+22}, -\frac{2\left(10x^2+10y^2+3x-8y-9\right)}{21x^2+21y^2+10x-4y+22}\right),$$

which confirms that $\boldsymbol{r}$ *is a PN surface.*

Example 2 (Quartic case)**.** *Choosing linear complex polynomials* u*,* v *and* w *leads to quartic minimal surfaces. From*

$$\mathrm{u}(\mathtt{z}) = (1+2\mathtt{i}) + (3-2\mathtt{i})\,\mathtt{z}, \quad \mathrm{v}(\mathtt{z}) = (4-\mathtt{i}) + (2+2\mathtt{i})\,\mathtt{z}, \quad \mathrm{w}(\mathtt{z}) = (1-\mathtt{i}) + (-1+3\mathtt{i})\,\mathtt{z}$$

we obtain

$$\begin{aligned}
\Phi_1(\mathtt{z}) = &\left(55x^3 - 105x^2y - 165xy^2 + 35y^3 + 3x^2 + 78xy - 3y^2 - 28x + 64y - 6\right)\\
&+ \mathtt{i}\left(35x^3 + 165x^2y - 105xy^2 - 55y^3 - 39x^2 + 6xy + 39y^2 - 64x - 28y + 30\right),\\
\Phi_2(\mathtt{z}) = &\left(-32x^3 - 168x^2y + 96xy^2 + 56y^3 + 38x^2 - 84xy - 38y^2 - 48x + 4y + 26\right)\\
&+ \mathtt{i}\left(56x^3 - 96x^2y - 168xy^2 + 32y^3 + 42x^2 + 76xy - 42y^2 - 4x - 48y + 2\right),\\
\Phi_3(\mathtt{z}) = &\left(-19x^3 - 21x^2y + 57xy^2 + 7y^3 - 73x^2 + 186xy + 73y^2 - 26x - 54y + 16\right)\\
&+ \mathtt{i}\left(7x^3 - 57x^2y - 21xy^2 + 19y^3 - 93x^2 - 146xy + 93y^2 + 54x - 26y + 8\right),
\end{aligned}$$

that give the surface parameterization $\boldsymbol{r} = (r_1, r_2, r_3)$*,*

$$\begin{aligned}
r_1(x,y) = &\frac{55x^4}{4} - 35x^3y - \frac{165}{2}x^2y^2 + 35xy^3 + \frac{55y^4}{4} + x^3 + 39x^2y - 3xy^2 - 13y^3 +\\
&- 14x^2 + 64xy + 14y^2 - 6x - 30y,\\
r_2(x,y) = &- 8x^4 - 56x^3y + 48x^2y^2 + 56xy^3 - 8y^4 + \frac{38x^3}{3} - 42x^2y - 38xy^2 + 14y^3 +\\
&- 24x^2 + 4xy + 24y^2 + 26x - 2y,\\
r_3(x,y) = &- \frac{19x^4}{4} - 7x^3y + \frac{57x^2y^2}{2} + 7xy^3 - \frac{19y^4}{4} - \frac{73x^3}{3} + 93x^2y + 73xy^2 - 31y^3 +\\
&- 13x^2 - 54xy + 13y^2 + 16x - 8y.
\end{aligned}$$

The quaternion polynomial (10c) *that corresponds to this surface and the unit normal* $\boldsymbol{N}$ *are the same as in Example 1, which follows from the fact that polynomials* u *and* v *are the same as in the cubic case.*

Example 3 (Quintic case)**.** *Quintic minimal surfaces can be obtained by choosing quadratic complex polynomials* u*,* v*, and a constant polynomial* w*. From*

$$\mathrm{u}(\mathtt{z}) = (1+2\mathtt{i}) + (3-2\mathtt{i})\,\mathtt{z} + (4-3\mathtt{i})\,\mathtt{z}^2, \quad \mathrm{v}(\mathtt{z}) = (4-\mathtt{i}) + (2+2\mathtt{i})\,\mathtt{z} + (5+5\mathtt{i})\,\mathtt{z}^2, \quad \mathrm{w}(\mathtt{z}) = 1$$

we compute Φ_1, Φ_2 and Φ_3 of degree 4 by (7)*, and the surface parameterization* $\boldsymbol{r} = (r_1, r_2, r_3)$ *follows as*

$$\begin{aligned} r_1(x,y) =& \frac{7x^5}{5} + 74x^4y - 14x^3y^2 - 148x^2y^3 + 7xy^4 + \frac{74y^5}{5} + 3x^4 + 74x^3y - 18x^2y^2 - 74xy^3 \\ & + 3y^4 - \frac{25x^3}{3} + 40x^2y + 25xy^2 - \frac{40y^3}{3} - 3x^2 + 4xy + 3y^2 - 18x - 12y, \\ r_2(x,y) =& 14x^5 - 10x^4y - 140x^3y^2 + 20x^2y^3 + 70xy^4 - 2y^5 + \frac{39x^4}{2} - 14x^3y - 117x^2y^2 \\ & + 14xy^3 + \frac{39y^4}{2} + 12x^3 - 2x^2y - 36xy^2 + \frac{2y^3}{3} + 8x^2 + 10xy - 8y^2 + 12x - 14y, \\ r_3(x,y) =& -\frac{26x^5}{5} - 7x^4y + 52x^3y^2 + 14x^2y^3 - 26xy^4 - \frac{7y^5}{5} - \frac{3x^4}{2} - 12x^3y + 9x^2y^2 \\ & + 12xy^3 - \frac{3y^4}{2} - 12x^3 - 75x^2y + 36xy^2 + 25y^3 - 10x^2 - 34xy + 10y^2 + 4x - 12y. \end{aligned}$$

Furthermore, the quaternion polynomial (10c) *that corresponds to this surface equals*

$$\begin{aligned} \mathcal{A}(x,y) = & \left(4x^2 + 6xy - 4y^2 + 3x + 2y + 1\right) + \left(-5x^2 - 10xy + 5y^2 - 2x - 2y + 1\right) \boldsymbol{i} \\ & + \left(-3x^2 + 8xy + 3y^2 - 2x + 3y + 2\right) \boldsymbol{j} + \left(5x^2 - 10xy - 5y^2 + 2x - 2y + 4\right) \boldsymbol{k}, \end{aligned}$$

from where we can by (11) *directly compute the surface normal.*

To plot the resulting minimal surfaces we need to choose a bounded (planar) region $\mathcal{D}$ in the parameter domain over which we compute points on the minimal surface. However, to take advantage of the PH preserving property we can define the boundary of the planar region $\mathcal{D}$ by a closed PH spline curve. In the case when $\mathrm{w}(\mathrm{z})$ is constant, the image of such a planar PH spline obtained with the surface parameterization is the spatial PH spline. If we are able to find a simple polynomial or spline parameterization of $\mathcal{D}$ we can then easily plot a part of the minimal surface bounded by a spatial PH spline curve.

The simplest way to obtain such a region $\mathcal{D}$ is to choose $N \geq 3$ planar points $\boldsymbol{Q}_\ell$ and associated tangent directions $\boldsymbol{t}_\ell$, $\ell = 1, \ldots, N$. Then we compute N planar cubic PH curves $\boldsymbol{q}_\ell : [0,1] \to \mathbb{R}^2$, $\ell = 1, \ldots, N$, that geometrically interpolate two points $\boldsymbol{Q}_\ell$, $\boldsymbol{Q}_{\ell+1}$ and two tangent directions $\boldsymbol{t}_\ell$, $\boldsymbol{t}_{\ell+1}$, where $\boldsymbol{Q}_{N+1} := \boldsymbol{Q}_1$, $\boldsymbol{t}_{N+1} := \boldsymbol{t}_1$. The interpolant can be written in the Bézier form as

$$\boldsymbol{q}_\ell(t) = \sum_{j=0}^{3} \boldsymbol{b}_j^{(\ell)} B_j^3(t), \quad B_j^3(t) := \binom{3}{j} t^j (1-t)^{3-j},$$

with control points

$$\boldsymbol{b}_0^{(\ell)} = \boldsymbol{Q}_\ell, \quad \boldsymbol{b}_1^{(\ell)} = \boldsymbol{Q}_\ell + \lambda_{\ell,0} \boldsymbol{t}_\ell, \quad \boldsymbol{b}_2^{(\ell)} = \boldsymbol{Q}_{\ell+1} - \lambda_{\ell,1} \boldsymbol{t}_{\ell+1}, \quad \boldsymbol{b}_3^{(\ell)} = \boldsymbol{Q}_{\ell+1}$$

for some positive values $\lambda_{\ell,0}$, $\lambda_{\ell,1}$, that follow as a solution of a nonlinear system. This interpolation problem has been examined (considering also the shape preservation) e.g., in [5]. There it is shown that for the convex data, i.e., data with planar cross products $\boldsymbol{t}_\ell \times \Delta \boldsymbol{Q}_\ell$ and $\Delta \boldsymbol{Q}_\ell \times \boldsymbol{t}_{\ell+1}$

being of the same sign (where $\Delta \boldsymbol{Q}_\ell = \boldsymbol{Q}_{\ell+1} - \boldsymbol{Q}_\ell$), that additionally satisfy the angle conditions $\angle(\boldsymbol{t}_\ell, \Delta\boldsymbol{Q}_\ell) \leq \frac{\pi}{2}$, $\angle(\Delta\boldsymbol{Q}_\ell, \boldsymbol{t}_{\ell+1}) \leq \frac{\pi}{2}$, there exists a unique interpolating curve $\boldsymbol{q}_\ell$ such that its control points are also convex, i.e., $\boldsymbol{b}_j^{(\ell)} \times \boldsymbol{b}_{j+1}^{(\ell)}$ are of the same sign for $j = 0, 1, 2$.

We further use these control points and one additional point $\boldsymbol{Q}_{\mathrm{c}}$ from the convex hull of given points $\boldsymbol{Q}_\ell$ to define the geometry mappings $\boldsymbol{F}_\ell : \triangle_0 \to \mathbb{R}^2$, where

$$\triangle_0 = \{(\mu, \nu) : \mu \in [0,1],\, 0 \leq \nu \leq 1 - \mu\}$$

and

$$\boldsymbol{F}_\ell(\mu, \nu) = \sum_{(i_1,i_2,i_3)\in\mathbb{D}_3} \boldsymbol{f}^{(\ell)}_{i_1,i_2,i_3} B^3_{i_1,i_2,i_3}(\mu,\nu) \tag{12}$$

is a cubic bivariate polynomial written in the bivariate Bernstein basis

$$\left\{ B^3_{i_1,i_2,i_3}(\mu,\nu) := \frac{3!}{i_1! i_2! i_3!} \mu^{i_1} \nu^{i_2} (1-\mu-\nu)^{i_3}, \quad (i_1,i_2,i_3) \in \mathbb{D}_3 \right\},$$

where $\mathbb{D}_3 = \{(i_1,i_2,i_3) \in \mathbb{N}_0^3 : i_1 + i_2 + i_3 = 3\}$. The control points of $\boldsymbol{F}_\ell$ are chosen as

$$\begin{pmatrix} \boldsymbol{f}^{(\ell)}_{0,0,3} & & & \\ \boldsymbol{f}^{(\ell)}_{1,0,2} & \boldsymbol{f}^{(\ell)}_{0,1,2} & & \\ \boldsymbol{f}^{(\ell)}_{2,0,1} & \boldsymbol{f}^{(\ell)}_{1,1,1} & \boldsymbol{f}^{(\ell)}_{0,2,1} & \\ \boldsymbol{f}^{(\ell)}_{3,0,0} & \boldsymbol{f}^{(\ell)}_{2,1,0} & \boldsymbol{f}^{(\ell)}_{1,2,0} & \boldsymbol{f}^{(\ell)}_{0,3,0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{Q}_{\mathrm{c}} & & & \\ \frac{2}{3}\boldsymbol{Q}_{\mathrm{c}} + \frac{1}{3}\boldsymbol{Q}_\ell & \frac{2}{3}\boldsymbol{Q}_{\mathrm{c}} + \frac{1}{3}\boldsymbol{Q}_{\ell+1} & & \\ \frac{1}{3}\boldsymbol{Q}_{\mathrm{c}} + \frac{2}{3}\boldsymbol{Q}_\ell & \frac{1}{3}\left(\boldsymbol{Q}_{\mathrm{c}} + \boldsymbol{Q}_\ell + \boldsymbol{Q}_{\ell+1}\right) & \frac{1}{3}\boldsymbol{Q}_{\mathrm{c}} + \frac{2}{3}\boldsymbol{Q}_{\ell+1} & \\ \boldsymbol{Q}_\ell & \boldsymbol{b}_1^{(\ell)} & \boldsymbol{b}_2^{(\ell)} & \boldsymbol{Q}_{\ell+1} \end{pmatrix}$$

and the region $\mathcal{D}$ is defined as a union

$$\mathcal{D} = \bigcup_{\ell=1}^{N} \{\boldsymbol{F}_\ell(\mu,\nu) : (\mu,\nu) \in \triangle_0\}. \tag{13}$$

Clearly, in order to avoid overlapping planar patches, we need to choose $\boldsymbol{Q}_{\mathrm{c}}$ so that the triangles formed by $\boldsymbol{Q}_{\mathrm{c}}, \boldsymbol{Q}_\ell, \boldsymbol{Q}_{\ell+1}$ do not overlap. For convex data points, we can simplify choose $\boldsymbol{Q}_{\mathrm{c}} = \frac{1}{N}\sum_{\ell=1}^{N} \boldsymbol{Q}_\ell$.

In the case when $N = 3$ we can also parameterize the domain $\mathcal{D}$ by only one polynomial patch. Namely, as $\mathcal{D} = \{\boldsymbol{F}(\mu,\nu) : (\mu,\nu) \in \triangle_0\}$ for $\boldsymbol{F}$ of the form (12) with control points

$$\begin{pmatrix} \boldsymbol{Q}_3 & & & \\ \boldsymbol{b}_1^{(3)} & \boldsymbol{b}_2^{(2)} & & \\ \boldsymbol{b}_2^{(3)} & \boldsymbol{Q}_{\mathrm{c}} & \boldsymbol{b}_1^{(2)} & \\ \boldsymbol{Q}_1 & \boldsymbol{b}_1^{(1)} & \boldsymbol{b}_2^{(1)} & \boldsymbol{Q}_2 \end{pmatrix}.$$

As an example, let us take the interpolation data from the circle with radius ρ, centered at the origin. The interpolating planar cubic PH curves (blue curves), their control points (black) and control points of the geometry mappings for $N = 6$ and $\rho = 1/2$, together with the domain (13) (light blue) are shown in Figure 1 (top left). On the right there is a cubic minimal surface from Example 1 with the boundary defined by the spatial PH spline curve (blue), which is the image of the planar PH spline. On bottom left of Figure 1 one can see the graph of a quartic minimal surface from Example 2. Note that in this case the curve that bounds the surface area is not a PH spline, since $\mathbf{w}$ is non-constant. Quintic minimal surface from Example 3 is shown in Figure 1 (bottom right).

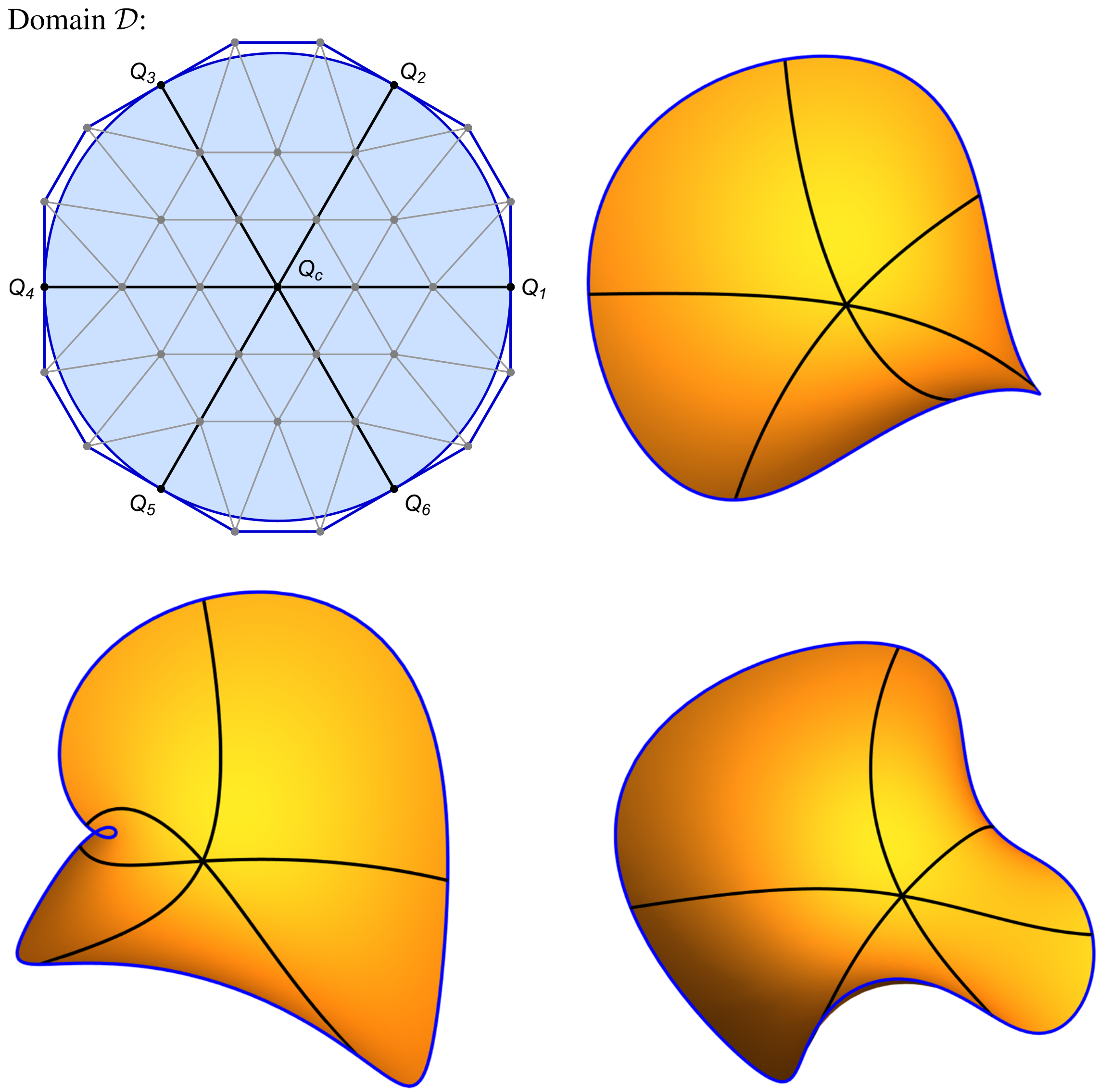

Figure 1: Domain $\mathcal{D}$ bounded by a planar cubic PH spline curve (top right) and graphs of a cubic minimal surface from Example 1 (top right), quartic minimal surface from Example 2 (bottom left) and quintic minimal surface from Example 3 (bottom right) over the domain $\mathcal{D}$.

Remark 3. *To be added: compare this construction with the construction of minimal PN surfaces from [8].*

4. Construction of polynomial minimal surfaces in non-isothermal parameterization

In this section we show how the presented construction of minimal surfaces can be generalized to minimal surfaces in non-isothermal parameterization.

We choose four real numbers $\alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathbb{R}$ and transform a complex variable $\mathtt{z} = x + \mathtt{i}y$

to $\mathtt{z} \mapsto \varphi(\mathtt{z}) = (\alpha_0 x + \beta_0 y) + \mathtt{i}\,(\alpha_1 x + \beta_1 y)$. Then we define two polynomials

$$\mathrm{u}(\mathtt{z}) = \sum_{j=0}^{n} \mathrm{u}_j\, \varphi(\mathtt{z})^j =: u_R(x,y) + \mathtt{i} u_I(x,y), \quad \mathrm{u}_j \in \mathbb{C},$$
$$\mathrm{v}(\mathtt{z}) = \sum_{j=0}^{n} \mathrm{v}_j\, \varphi(\mathtt{z})^j =: v_R(x,y) + \mathtt{i} v_I(x,y), \quad \mathrm{v}_j \in \mathbb{C},$$

and, for the sake of simplicity, we choose $\mathrm{w}(\mathtt{z}) = 1$. If $\varphi(\mathtt{z}) \neq \mathtt{z}$, the real bivariate polynomials u_R, u_I, v_R, v_I no longer satisfy the Cauchy-Riemann equations, but their partial derivatives are connected as

$$\begin{pmatrix} \beta_1 & -\alpha_1 \\ -\beta_0 & \alpha_0 \end{pmatrix} \begin{pmatrix} \frac{\partial u_R}{\partial x} \\ \frac{\partial u_R}{\partial y} \end{pmatrix} = \begin{pmatrix} \alpha_0 & \beta_0 \\ \alpha_1 & \beta_1 \end{pmatrix} \begin{pmatrix} \frac{\partial u_I}{\partial y} \\ -\frac{\partial u_I}{\partial x} \end{pmatrix} \tag{14}$$

for u_R, u_I, and similarly for v_R, v_I. Next, we define Φ_1, Φ_2 and Φ_3 as in (7). These functions still satisfy (2), but are no longer holomorphic, so we can not apply Enneper–Weierstrass parameterization. However, we can define a quaternion polynomial $\mathcal{A}$ as in (10c), and two other quaternions

$$\mathcal{U}_1 := \alpha_0\, \boldsymbol{i} - \alpha_1\, \boldsymbol{k} \quad \text{and} \quad \mathcal{U}_2 := \beta_0\, \boldsymbol{i} - \beta_1\, \boldsymbol{k}.$$

From the generalized Cauchy-Riemann equations (14) it follows that

$$\mathcal{A}_y\, \mathcal{U}_1 = \mathcal{A}_x\, \mathcal{U}_2.$$

Using this equality and choosing

$$\boldsymbol{g}_R = \mathcal{A}\mathcal{U}_1\bar{\mathcal{A}} \quad \text{and} \quad \boldsymbol{g}_I = \mathcal{A}\mathcal{U}_2\bar{\mathcal{A}}$$

we obtain

$$\begin{aligned} \frac{\partial \boldsymbol{g}_R}{\partial y} - \frac{\partial \boldsymbol{g}_I}{\partial x} &= \mathcal{A}_y\mathcal{U}_1\bar{\mathcal{A}} + \mathcal{A}\mathcal{U}_1\bar{\mathcal{A}}_y - \mathcal{A}_x\mathcal{U}_2\bar{\mathcal{A}} - \mathcal{A}\mathcal{U}_2\bar{\mathcal{A}}_x \\ &= (\mathcal{A}_y\, \mathcal{U}_1 - \mathcal{A}_x\, \mathcal{U}_2)\, \bar{\mathcal{A}} + \mathcal{A}\, \overline{(\mathcal{A}_y\, \mathcal{U}_1 - \mathcal{A}_x\, \mathcal{U}_2)} = \mathbf{0}. \end{aligned}$$

Therefore, prescribing partial derivatives of the surface as $\boldsymbol{r}_x = \boldsymbol{g}_R$ and $\boldsymbol{r}_y = \boldsymbol{g}_I$, the mixed second-order partial derivatives of $\boldsymbol{r}$ are equal, and the surface follows from (9). It is a PN surface, since its unit normal equals

$$\boldsymbol{N} = \frac{\boldsymbol{g}_R \times \boldsymbol{g}_I}{\|\boldsymbol{g}_R \times \boldsymbol{g}_I\|} = -\frac{\mathcal{A}\, \boldsymbol{j}\, \bar{\mathcal{A}}}{\|\mathcal{A}\|^2}.$$

By computing the coefficients of the first fundamental form, it is straightforward to see that $EN - 2FM + GL \equiv 0$, which shows that $\boldsymbol{r}$ is a minimal surface.

Example 4. *To be added.*

5. Application of cubic PN minimal surfaces

In this section we present an interpolation problem using cubic PN minimal surfaces . We limit the analysis to surfaces in isothermal parameterization with the *PH preserving* property, derived in Section 3. Suppose that $\boldsymbol{P}_\ell$, $\ell = 1,2,3$, are three given points in $\mathbb{R}^3$ and $\boldsymbol{Q}_\ell := (x_\ell, y_\ell)$, $\ell = 1,2,3$, three vertices in a parameter domain. Furthermore, let $\boldsymbol{q}_\ell = (q_{\ell,x}, q_{\ell,y})$, $\ell = 1,2,3$, be three planar parametric curves defined on $[0,1]$, such that $\boldsymbol{q}_\ell(0) = \boldsymbol{Q}_\ell$, $\boldsymbol{q}_\ell(1) = \boldsymbol{Q}_{\ell+1}$, $\ell = 1,2,3$, where $\boldsymbol{Q}_4 = \boldsymbol{Q}_1$, and let $\mathcal{D}$ denotes the region bounded by curves $\boldsymbol{q}_\ell$.

The task is to compute a cubic minimal surface $\boldsymbol{r} : \mathcal{D} \to \mathbb{R}^3$, given by (7) and (8) with

$$\mathrm{u}(\mathrm{z}) = \mathrm{u}_0 + \mathrm{u}_1\,\mathrm{z}, \quad \mathrm{v}(\mathrm{z}) = \mathrm{v}_0 + \mathrm{v}_1\,\mathrm{z}, \quad \text{and} \quad \mathrm{w}(\mathrm{z}) = 1, \tag{15}$$

that interpolates the given points $\boldsymbol{P}_\ell$ at parameters $\boldsymbol{Q}_{\boldsymbol{\ell}}$ and has a prescribed length of boundary curves,

$$\text{length}\,(\boldsymbol{r}|_{\partial\mathcal{D}}) = L, \tag{16}$$

for some chosen $L > \|\boldsymbol{P}_2 - \boldsymbol{P}_1\| + \|\boldsymbol{P}_3 - \boldsymbol{P}_2\| + \|\boldsymbol{P}_1 - \boldsymbol{P}_3\|$. The quaternion polynomial (10c) equals

$$\mathcal{A}(x,y) = \mathcal{A}_{0,0} + \mathcal{A}_{1,0}\,x + \mathcal{A}_{0,1}\,y,$$

where

$$\mathcal{A}_{\ell,0} = \mathrm{Re}(\mathrm{u}_\ell) - \mathrm{Im}(\mathrm{v}_\ell)\,\boldsymbol{i} + \mathrm{Im}(\mathrm{u}_\ell)\,\boldsymbol{j} + \mathrm{Re}(\mathrm{v}_\ell)\,\boldsymbol{k}, \quad \ell = 0,1, \quad \mathcal{A}_{0,1} = \mathcal{A}_{1,0}\,\boldsymbol{j}.$$

Note that from $\mathcal{A}_{0,0}$ and $\mathcal{A}_{1,0}$ one can uniquely determine the polynomials (15) and vise-versa. Let us define two operations

$$\begin{aligned}
\star_{\boldsymbol{i}} &: \mathbb{H} \times \mathbb{H} \to \mathbb{R}^3, \quad \mathcal{A} \star_{\boldsymbol{i}} \mathcal{B} := \frac{1}{2}\left(\mathcal{A}\,\boldsymbol{i}\,\bar{\mathcal{B}} + \bar{\mathcal{A}}\,\boldsymbol{i}\,\mathcal{B}\right), \\
\star_{\boldsymbol{k}} &: \mathbb{H} \times \mathbb{H} \to \mathbb{R}^3, \quad \mathcal{A} \star_{\boldsymbol{k}} \mathcal{B} := \frac{1}{2}\left(\mathcal{A}\,\boldsymbol{k}\,\bar{\mathcal{B}} + \bar{\mathcal{A}}\,\boldsymbol{k}\,\mathcal{B}\right),
\end{aligned}$$

and let

$$\mathcal{A}^{2\star_{\boldsymbol{i}}} := \mathcal{A} \star_{\boldsymbol{i}} \mathcal{A}, \quad \mathcal{A}^{2\star_{\boldsymbol{k}}} := \mathcal{A} \star_{\boldsymbol{k}} \mathcal{A}.$$

From (10a)–(10b) it follows that

$$\begin{aligned}
\boldsymbol{g}_R(x,y) = \mathcal{A}^{2\star_{\boldsymbol{i}}}(x,y) &= \mathcal{A}_{0,0}^{2\star_{\boldsymbol{i}}} + 2x\mathcal{A}_{0,0} \star_{\boldsymbol{i}} \mathcal{A}_{1,0} - 2y\mathcal{A}_{0,0} \star_{\boldsymbol{k}} \mathcal{A}_{1,0} + (x^2 - y^2)\mathcal{A}_{1,0}^{2\star_{\boldsymbol{i}}} - 2xy\mathcal{A}_{1,0}^{2\star_{\boldsymbol{k}}}, \\
\boldsymbol{g}_I(x,y) = \mathcal{A}^{2\star_{\boldsymbol{k}}}(x,y) &= \mathcal{A}_{0,0}^{2\star_{\boldsymbol{k}}} + 2x\mathcal{A}_{0,0} \star_{\boldsymbol{k}} \mathcal{A}_{1,0} + 2y\mathcal{A}_{0,0} \star_{\boldsymbol{i}} \mathcal{A}_{1,0} + (x^2 - y^2)\mathcal{A}_{1,0}^{2\star_{\boldsymbol{k}}} + 2xy\mathcal{A}_{1,0}^{2\star_{\boldsymbol{i}}},
\end{aligned}$$

and (9) gives

$$\begin{aligned}
\boldsymbol{r}(x,y) =& \boldsymbol{r}_0 + x\mathcal{A}_{0,0}^{2\star_{\boldsymbol{i}}} - y\mathcal{A}_{0,0}^{2\star_{\boldsymbol{k}}} + (x^2 - y^2)\mathcal{A}_{0,0} \star_{\boldsymbol{i}} \mathcal{A}_{1,0} - 2xy\mathcal{A}_{0,0} \star_{\boldsymbol{k}} \mathcal{A}_{1,0} \\
&+ \left(\frac{1}{3}x^3 - xy^2\right)\mathcal{A}_{1,0}^{2\star_{\boldsymbol{i}}} - x^2y\mathcal{A}_{1,0}^{2\star_{\boldsymbol{k}}} + \frac{1}{3}y^3\mathcal{A}_{1,0}^{2\star_{\boldsymbol{k}}},
\end{aligned}$$

from where we directly obtain the equations for point interpolation:

$$\boldsymbol{r}(x_\ell, y_\ell) = \boldsymbol{P}_\ell, \quad \ell = 1,2,3. \tag{17}$$

Following the proof of Lemma 2, the length interpolation equation can be expressed as

$$L = \sum_{\ell=1}^{3} \operatorname{length}(\boldsymbol{q}_\ell), \quad \operatorname{length}(\boldsymbol{q}_\ell) = \int_0^1 \|(\mathcal{A} \circ \boldsymbol{q}_\ell)(t)\|^2 \sigma_\ell(t)\, dt, \tag{18}$$

where $\sigma_\ell(t) := \|\boldsymbol{q}'_\ell(t)\|$. Moreover, from

$$\|\mathcal{A}(x,y)\|^2 = \|\mathcal{A}_{0,0}\|^2 + 2x \langle \mathcal{A}_{0,0}, \mathcal{A}_{1,0}\rangle + 2y \langle \mathcal{A}_{0,0}, \mathcal{A}_{1,0}\boldsymbol{j}\rangle + (x^2+y^2)\|\mathcal{A}_{1,0}\|^2 + 2xy \langle \mathcal{A}_{1,0}, \mathcal{A}_{1,0}\boldsymbol{j}\rangle,$$

we derive that (18) equals

$$L = C_0 \|\mathcal{A}_{0,0}\|^2 + C_1 \langle \mathcal{A}_{0,0}, \mathcal{A}_{1,0}\rangle + C_2 \langle \mathcal{A}_{0,0}, \mathcal{A}_{1,0}\,\boldsymbol{j}\rangle + C_3 \|\mathcal{A}_{1,0}\|^2 + C_4 \langle \mathcal{A}_{1,0}, \mathcal{A}_{1,0}\,\boldsymbol{j}\rangle, \tag{19}$$

where

$$C_0 = \sum_{\ell=1}^{3} \int_0^1 \sigma_\ell(t) dt, \quad C_1 = \sum_{\ell=1}^{3} \int_0^1 2q_{\ell,x}(t)\sigma_\ell(t) dt, \quad C_2 = \sum_{\ell=1}^{3} \int_0^1 2q_{\ell,y}(t)\sigma_\ell(t) dt,$$
$$C_3 = \sum_{\ell=1}^{3} \int_0^1 \left(q_{\ell,x}^2(t) + q_{\ell,y}^2(t)\right)\sigma_\ell(t) dt, \quad C_4 = \sum_{\ell=1}^{3} \int_0^1 2q_{\ell,x}(t) q_{\ell,y}(t)\sigma_\ell(t) dt$$

depend only on the chosen boundary curves $\boldsymbol{q}_\ell$. Now, (17) and (19) represent 10 nonlinear equations for 11 unknowns (point $\boldsymbol{r}_0$ and the components of $\mathcal{A}_{0,0}$ and $\mathcal{A}_{1,0}$), so the set of solutions is expected to be one-parametric. The system of equations is polynomial with the unknowns involved in a quadratic way, but the existence analysis is beyond the scope of this paper. However, let us apply the described construction to a few numerical examples.

Example 5. *The interpolation data are chosen as*

$$\boldsymbol{P}_1 = (0,0,1), \quad \boldsymbol{P}_2 = (1,2,3), \quad \boldsymbol{P}_3 = (-2,3,0), \quad L = 12.$$

Let us first assume that the vertices of the triangle T are equal to

$$\boldsymbol{Q}_1 = (0,0), \quad \boldsymbol{Q}_2 = (1,0), \quad \boldsymbol{Q}_3 = (0,1), \tag{20}$$

and that planar curves $\mathbf{q}_\ell$ are simply just lines:

$$\boldsymbol{q}_\ell(t) = (1-t)\boldsymbol{Q}_\ell + t\boldsymbol{Q}_{\ell+1}.$$

With this assumption the equations (17) *simplify to $\boldsymbol{r}_0 = \boldsymbol{P}_0$ and*

$$\begin{aligned} \mathcal{A}_{0,0}^{2\star_{\boldsymbol{i}}} + \mathcal{A}_{0,0} \star_{\boldsymbol{i}} \mathcal{A}_{1,0} + \frac{1}{3}\mathcal{A}_{1,0}^{2\star_{\boldsymbol{i}}} &= \boldsymbol{P}_2 - \boldsymbol{P}_1, \\ -\mathcal{A}_{0,0}^{2\star_{\boldsymbol{k}}} - \mathcal{A}_{0,0} \star_{\boldsymbol{i}} \mathcal{A}_{1,0} + \frac{1}{3}\mathcal{A}_{1,0}^{2\star_{\boldsymbol{k}}} &= \boldsymbol{P}_2 - \boldsymbol{P}_0, \end{aligned} \tag{21}$$

and the length interpolation equation equals (19) *with*

$$C_0 = 2 + \sqrt{2}, \quad C_1 = C_2 = 1 + \sqrt{2}, \quad C_3 = \frac{2}{3}(1+\sqrt{2}), \quad C_4 = \frac{\sqrt{2}}{6}.$$

To fix the one free parameter we choose $Re(\mathrm{u}_\ell) = 1$. *Using the program* `Mathematica` *and its function* `Solve[equations, unknowns, Reals]`, *that computes all real solutions of a polynomial system of equations, we get two such solutions:*

$$
\begin{aligned}
1.\,\text{solution}: \quad & \mathcal{A}_{0,0} = 1 + 1.11580\,\boldsymbol{i} + 0.112905\,\boldsymbol{j} + 0.916207\,\boldsymbol{k}, \\
& \mathcal{A}_{1,0} = 0.467454 - 0.994804\,\boldsymbol{i} - 0.874479\,\boldsymbol{j} - 0.00347057\,\boldsymbol{k}, \\
2.\,\text{solution}: \quad & \mathcal{A}_{0,0} = 1 + 1.54848\,\boldsymbol{i} + 0.688564\,\boldsymbol{j} + 0.552663\,\boldsymbol{k}, \\
& \mathcal{A}_{1,0} = 0.467454 - 0.411121\,\boldsymbol{i} - 0.662571\,\boldsymbol{j} - 1.07479 + 0.875019\,\boldsymbol{k}.
\end{aligned}
\tag{22}
$$

The interpolating cubic minimal surfaces are shown in Figure 2. To give a comparison between

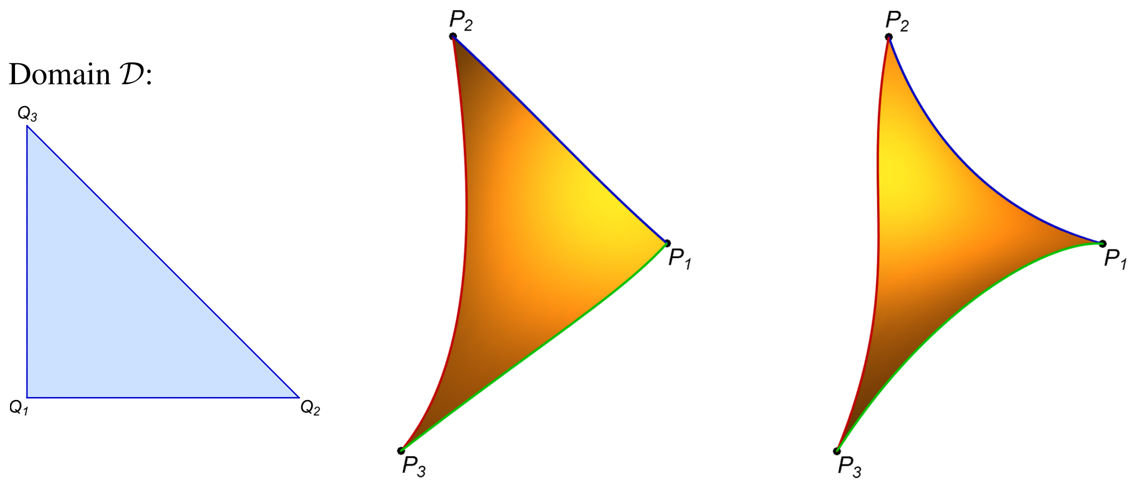

Figure 2: The domain triangle and the interpolating minimal surfaces from Example 5 computed from (22) with the length of boundary curves equal to $L = 12$.

them we compute the surface area E_{area} *and the* L^2 *norm of the Gauss curvature* E_{Gauss}. *For the first solution these values are equal to* $E_{\text{area}} = 5.43798$, $E_{\text{Gauss}} = 0.164709$ *while for the second solution we get* $E_{\text{area}} = 5.43887$, $E_{\text{Gauss}} = 0.211876$.

Example 6. *Suppose that the data points* $\boldsymbol{P}_\ell$ *and the domain points* $\boldsymbol{Q}_\ell$ *are the same as in Example 5. Now, we compute the planar curves* $\mathbf{q}_\ell$ *as cubic PH curves that interpolate the points* $\boldsymbol{Q}_\ell$ *and the tangent directions* $\mathbf{t}_\ell$, *chosen as vectors orthogonal to the bisectors of angles* $\angle(\Delta\boldsymbol{Q}_{\ell-1}, \Delta\boldsymbol{Q}_\ell)$. *Their Bézier control points are equal to*

$$
\begin{array}{llll}
\boldsymbol{b}_0^{(1)} = (0,0), & \boldsymbol{b}_1^{(1)} = (0.416426, -0.416426), & \boldsymbol{b}_2^{(1)} = (0.864422, -0.327314), & \boldsymbol{b}_3^{(1)} = (1,0), \\
\boldsymbol{b}_0^{(2)} = (1,0), & \boldsymbol{b}_1^{(2)} = (1.30656, 0.740108), & \boldsymbol{b}_2^{(2)} = (0.740108, 1.30656), & \boldsymbol{b}_3^{(2)} = (0,1), \\
\boldsymbol{b}_0^{(3)} = (0,1), & \boldsymbol{b}_1^{(3)} = (-0.327314, 0.864422), & \boldsymbol{b}_2^{(3)} = (-0.416426, 0.416426), & \boldsymbol{b}_3^{(3)} = (0,0),
\end{array}
$$

and they determine the geometry mapping $\boldsymbol{F}$ *as described in Section 3. The equations* (17) *remains the same as in* (21), *while the length interpolation equation equals* (19) *with*

$$C_0 = 4.30267, \quad C_1 = C_2 = 3.66142, \quad C_3 = 3.55736, \quad C_4 = 0.663949.$$

By choosing $L = 17$ and fixing $Re(\mathrm{u}_\ell) = 1.1$, we again obtain two real solutions that yield

$$\begin{aligned}
&1.\,\text{solution}: && \mathcal{A}_{0,0} = 1.1 + 0.509623\,\boldsymbol{i} - 0.615793\,\boldsymbol{j} + 1.46538\,\boldsymbol{k}, && E_{\text{area}} = 17.5802,\\
& && \mathcal{A}_{1,0} = 0.632456 - 0.503278\,\boldsymbol{i} + 0.188496\,\boldsymbol{j} - 1.22612\,\boldsymbol{k}, && E_{\text{Gauss}} = 0.154104,\\
&2.\,\text{solution}: && \mathcal{A}_{0,0} = 1.1 + 1.16752\,\boldsymbol{i} + 0.544340\,\boldsymbol{j} + 0.886994\,\boldsymbol{k}, && E_{\text{area}} = 18.0792,\\
& && \mathcal{A}_{1,0} = -1.35753 + 0.300035\,\boldsymbol{i} - 0.142571\,\boldsymbol{j} + 0.0683613\,\boldsymbol{k}, && E_{\text{Gauss}} = 0.132174.
\end{aligned}$$

The domain $\mathcal{D}$ and the interpolating cubic minimal surfaces are shown in Figure 3.

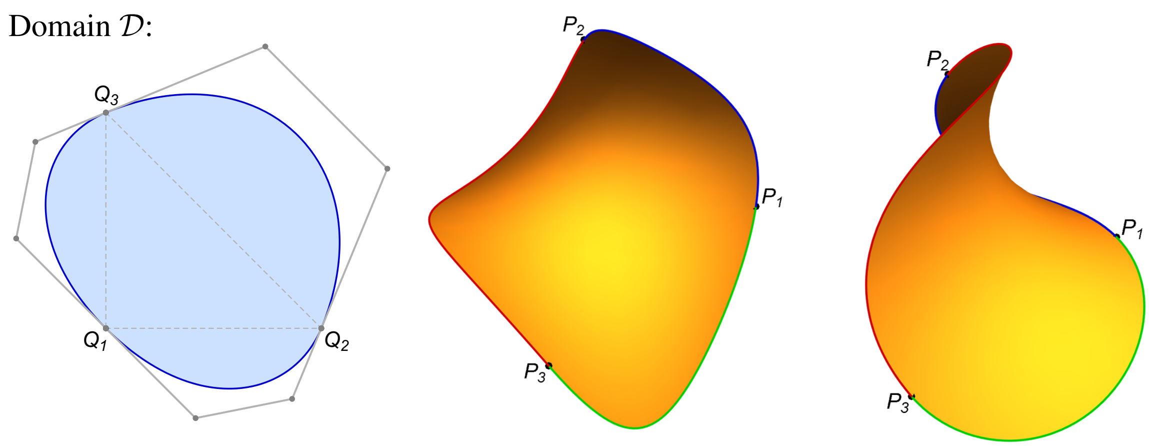

Figure 3: Domain $\mathcal{D}$ bounded by a cubic PH spline (left) and the two interpolating minimal surfaces defined over $\mathcal{D}$ with the length of boundary curves equal to $L = 17$.

Example 7. *As a final example we take the same data points $\boldsymbol{P}_\ell$ as in Example 5. For the domain vertices $\boldsymbol{Q}_\ell$ we choose the vertices of the triangle obtained by rotating and translating the spatial triangle formed by $\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3$ to the (x, y)–plane:*

$$\boldsymbol{Q}_1 = (0, 0), \quad \boldsymbol{Q}_2 = (3, 0), \quad \boldsymbol{Q}_3 = (2/3, \sqrt{122}/3).$$

Domain $\mathcal{D}$ is then chosen as the domain bounded by a cubic PH spline, computed as in Example 6 (see Figure 4). Among different solution interpolants we choose the one with the minimal value of E_{area}. The resulting minimal surfaces for different values of $L \in \{17, 18, 20, 25\}$ are shown in Figure 4.

6. Closure

A construction of polynomial minimal surfaces from a set of complex polynomials has been presented. The constructed surfaces have a number of favorable properties, including isothermal parameterizations, rational unit normal vectors, mapping of PH curves in the parameter domain to spatial PH curves on the surface, and a generalization to minimal surfaces with non–isothermal parameterizations. Among the practical implications of this novel construction, we note that the minimal surfaces possess rational offset (parallel) surfaces, and the ability to construct bounded

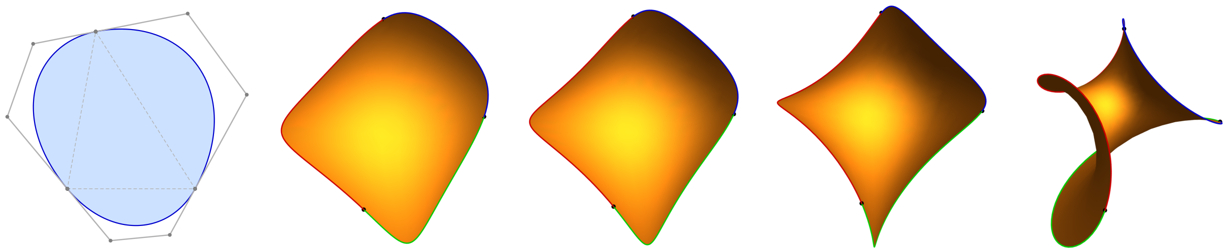

Figure 4: Domain $\mathcal{D}$ bounded by a cubic PH spline (left) and the interpolating minimal surfaces defined over $\mathcal{D}$ with the length of boundary curves equal to $L = 17, 18, 20, 25$ respectively.

minimal surface patches satisfying corner point and boundary length constraints. A number of low–degree examples have been included to illustrate the methodology.

The focus of this study has been on describing the construction scheme and basic properties of the resulting miminal surfaces. The application of the methodology to the design of polynomial minimal surface patches that satisfy various geometrical constraints — corner points and/or surface normal vectors, prescribed boundary curves, arc length or area constraints, etc. — deserves a separate detailed investigation, that we hope to pursue in a future study.

References

[1] J. Douglas (1931), Solution of the problem of Plateau, *Trans. Amer. Math. Soc.* **33**, 263–321.

[2] R. T. Farouki and T. Sakkalis (1990), Pythagorean hodographs, *IBM J. Res. Develop.* **34**, 736–752.

[3] R. T. Farouki (2008), *Pythagorean-Hodograph Curves: Algebra and Geometry Inseparable*, Geometry and Computing vol. 1, Springer, Berlin.

[4] Y–X. Hao (2020), A new method to construct polynomial minimal surfaces, *Comput. Appl. Math.* **39**, article 275.

[5] G. Jaklič, J. Kozak, M. Krajnc,V. Vitrih, E. Žagar (2010), On interpolation by planar cubic G^2 Pythagorean-hodograph spline curves, *Math. Comp.* **79**(269), 305–326.

[6] G. I. Kim and S. Lee (2008), Pythagorean–hodograph preserving mappings, *J. Comp. Appl. Math.* **216**, 217–226.

[7] M. Kline (1972), *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, New York.

[8] J. Kozak, M. Krajnc, V. Vitrih (2016), A quaternion approach to polynomial PN surfaces, *Comput. Aided Geom. Design* **47**, 172–188.

[9] M. Lávička and B. Bastl (2008), PN surfaces and their convolutions with rational surfaces, *Comput. Aided Geom. Design* **25**, 763–774.

[10] M. Lávička and J. Vršek (2012), On a special class of polynomial surfaces with Pythagorean normal vector fields, in: *Curves and Surfaces 2011* (J–D. Boissonnat et al., eds.), Lecture Notes in Computer Science 6920, pp. 431–444.

[11] R. Osserman (1986), *A Survey of Minimal Surfaces*, Dover Publications, New York.

[12] H. Pottmann (1995), Rational curves and surfaces with rational offsets, *Comput. Aided Geom. Design* **12**, 175–192.

[13] H. A. Priestley (1990), *Introduction to Complex Analysis*, Clarendon Press, Oxford.

[14] D. J. Struik (1961), *Lectures on Classical Differential Geometry* (2nd edition), Dover Publications, New York.

[15] K. Ueda (1998), Pythagorean–hodograph curves on isothermal surfaces, in: The Mathematics of Surfaces VIII (R. Cripps, ed.), Information Geometers, Winchester, pp. 339–353.