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V. N. Gribov and I. Ya. Pomeranchuk[1] have argued that there is an

[1] V. N. Gribov and I. Ya. Pomeranchuk, in Proceedings of 1962 International Conference on High Energy Physics, p. 522.

essential singularity in the partial-wave scattering amplitude for the s channel which is proportional to the Mandelstam[2] double spectral function

[2] S. Mandelstam, Phys. Rev. 112, 1344 (1958).

$\rho(t,u)$. In this note we present an example of a nonlocal potential, suggested by the Mandelstam representation, which leads to this behavior at orbital angular momentum minus one.

Our proof is based on the radial momentum space Schroedinger equation for the elastic partial-wave amplitude at energy k^2:

$$\langle \kappa \ell \mid \vec{k} \rangle = \frac{\delta(\kappa - k)}{k^2} + \frac{1}{k^2 + \kappa^2 + i\epsilon} \int_0^\infty \langle \kappa \ell \mid V \mid \kappa' \ell \rangle d\kappa' \langle \kappa' \ell \mid \vec{k} \rangle . \tag{1}$$

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In particular, following H. A. Bethe and T. Kinoshita,[3] we consider the

[3] H. A. Bethe and T. Kinoshita, Phys. Rev. 128, 1418 (1962).

problem of finding Regge poles for large negative values of k^2. These are given by finding solutions of the homogeneous part of Eq. (1). For the case of a Yukawa potential it is well known that

$$\langle \kappa \ell \mid V \mid \kappa' \ell \rangle \sim Q_\ell \left[\frac{\kappa^2 + \kappa'^2 + \mu^2}{2\kappa\kappa'} \right] , \qquad (2)$$

where Q_ℓ is the Legendre function of the second kind. A different potential is obtained by requiring that in Born approximation the potential reproduces that part of the scattering amplitude which arises from the third double spectral function of the Mandelstam representation.[4]

[4] Compare, for example, G. F. Chew, S-Matrix Theory of Strong Interactions (W. A. Benjamin Co., New York, 1961), Ch. 7, p. 39, Eq. (7-21).

This leads in nonrelativistic approximation to

$$\langle \kappa \ell \mid V \mid \kappa' \ell \rangle \sim \frac{Q_\ell \left[\frac{\kappa^2 + \kappa'^2 + \mu^2}{2\kappa\kappa'} \right]}{\kappa^2 + \kappa'^2 + \mu^2} \qquad (3)$$

when $\rho(t,u)$ is assumed to be a delta function in the variables t and u at $\mu/2$ in the region where it is nonvanishing.

In the following paragraph we show that in the neighborhood of orbital angular momentum minus one the homogeneous equation for the Yukawa potential takes the approximate form

$$\langle \kappa \ell \mid \vec{k} \rangle = \frac{\lambda}{k^2 + \kappa^2} \int_0^\infty d\kappa' \langle \kappa' \ell \mid \vec{k} \rangle \frac{\kappa'}{2\kappa} , \tag{4}$$

whereas in the case of the nonlocal potential of Eq. (3) in the same vicinity one finds

$$\langle \kappa \ell \mid \vec{k} \rangle = \frac{\lambda}{k^2 + \kappa^2} \int_0^\infty d\kappa' \frac{\langle \kappa' \ell \mid \vec{k} \rangle}{\kappa^2 + \kappa'^2 + \mu^2} \frac{\kappa'}{2\kappa} . \tag{5}$$

In these equations

$$\lambda = -\pi \cot \pi \ell \tag{6}$$

is the eigenvalue that determines the appropriate relation between k and ℓ. In the first case the kernel of the integral equation is degenerate, so that there is only one eigensolution for large k in the vicinity of minus one. The second equation, which possesses a nondegenerate kernel, has an infinite number of eigensolutions and eigenvalues, λ_n, which imply the existence of an infinite number of Regge poles accumulating at orbital angular momentum minus one; thus, there is an essential singularity at this point.

The justification of the foregoing remarks is most easily seen by studying the solution of the homogeneous part of Eq. (1) in the case of a Yukawa potential. It is convenient to change the dependent variable to

$$\langle \kappa \ell \mid \phi \rangle = \kappa^{\ell} \langle \kappa \ell \mid \vec{k} \rangle \tag{7}$$

so that the integral equation becomes

$$\langle \kappa \ell \mid \phi \rangle = - \frac{1}{2(k^2 + \kappa^2)} \int_0^{\infty} \left(\frac{\kappa'}{\kappa} \right)^{\ell+1} Q_{\ell} \left[\frac{\kappa^2 + \kappa'^2 + \mu^2}{2\kappa\kappa'} \right] \langle \kappa \ell \mid \phi \rangle . \tag{8}$$

The solution of this equation in the neighborhood of interest is achieved by introducing

$$\langle \kappa \ell \mid \chi \rangle \equiv \int_0^{\kappa} d\kappa'' \langle \kappa'' \ell \mid \phi \rangle \tag{9}$$

and performing a partial integration. This leads to an equation for $\langle \kappa \ell \mid \chi \rangle$:

$$\langle \kappa \ell \mid \chi \rangle = - \frac{\langle \infty \ell \mid \chi \rangle \sqrt{\pi} \; \Gamma(\ell + 1)}{2k \, \Gamma(\ell + 3/2)} \tan^{-1} \frac{\kappa}{k} + \int_0^{\infty} \langle \kappa \ell \mid K \mid \kappa' \ell \rangle \, d\kappa' \langle \kappa' \ell \mid \chi \rangle , \tag{10}$$

where the value of $\langle \kappa \ell \mid \chi \rangle$ at infinity is a constant and $\langle \kappa \ell \mid K \mid \kappa' \ell \rangle$ is a kernel proportional to $\Gamma(\ell + 2)$ and with a complicated dependence on the variables κ and κ' . In addition it decreases with increasing k . Further, as will be shown in a forthcoming publication,

$$\langle \kappa \ell \mid K_2 \mid \kappa' \ell \rangle = \int \langle \kappa \ell \mid K \mid \kappa'' \ell \rangle \, d\kappa'' \langle \kappa'' \ell \mid K \mid \kappa' \ell \rangle$$

$$\left| \langle \kappa \ell \mid K_2 \mid \kappa' \ell \rangle \right| < \int \left| \langle \kappa \ell \mid K \mid \kappa'' \ell \rangle \right| d\kappa'' \left| \langle \kappa'' \ell \mid K \mid \kappa' \ell \rangle \right|$$

$$\ll \langle \kappa \ell \mid K \mid \kappa' \ell \rangle \ . \tag{11}$$

As a result, if Eq. (10) is considered as an inhomogeneous integral equation for $\langle \kappa \ell \mid X \rangle$, we find that the Neumann series for the solution is convergent and hence the solution is unique. In order that the solution be consistent, we obtain the eigenvalue condition

$$\frac{(\pi)^{3/2} \ \Gamma(\ell + 1)}{2k \ \Gamma(\ell + 3/2)} = -1 \ , \tag{12}$$

which leads to a single pole in the neighborhood of minus one. If one now asks what approximate equation would lead to this same result one is led immediately to Eq. (4).

The solution of Eq. (1) for the case of the nonlocal potential may be discussed by using an approach similar to the Yukawa case. The essential feature of the latter problem is that in the vicinity of minus one the Legendre function becomes approximately constant as a function of its argument. Making this same approximation in the present case, we find Eq. (5). This then leads to an infinity of solutions in the vicinity of minus one.