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Tamely Ramified Automorphic Function Theory of the Rational Function Field

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by

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Abstract

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This thesis studies the fundamental automorphic function theory associated to a marked genus zero curve over a finite field. Following insights from topological field theory, one expects this theory is deeply related to the unramified automorphic representation theory of general function fields. There are two main contributions. First I present an explicit description of the action of Hecke operators for the groups PGL_2 and SL_3. Second, I give a conjecture, along with some evidence, that characterizes the action of Hecke operators on Eisenstein series for any group.

To Ammu and Abbu

Contents

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Chapter 1

Introduction

1.1 What is in this thesis?

This thesis is about the decomposition under Hecke operators of the automorphic function theory of a rational function field. The research contributions focus on automorphic functions that are unramified everywhere except for tame ramification at finitely many places. Geometrically, these are functions on the moduli of $\underline{G}$-bundles on $\mathbb{P}^1$ with Iwahori level structure at finitely many points S.

$$\text{Automorphic Theory} := \text{Fun}(\text{Bun}_{\underline{G}}(\mathbb{P}^1, S))$$

The main contribution of this thesis is a conjecture, along with some evidence, characterizing the action of Hecke correspondences on the subspace of Eisenstein series (functions induced from the maximal torus). I give a precise statement of this conjecture in Section 1.2 and the proof of the partial results are presented in Chapter 4.

Chapters 2 and 3 contain an exposition of the Hecke algebra and its action by correspondences on moduli spaces of parabolic bundles. I intend for the exposition to be accessible given familiarity with the theory of reductive groups schemes (e.g. Borel, Milne, Humphreys, Springer). Chapters 4 explains the main results about the tamely ramified automorphic function theory of a rational function field. In Section 1.3 I explain how this problem is motivated by the Langlands program and especially connections to topological field theory.

1.2 Main Result

The central object of this thesis is a vector space of functions, C_{Aut}, which I call the automorphic function theory. This automorphic theory is associated to a reductive group scheme $\underline{G}$ and a curve $X = \mathbb{P}^1$, both defined over the finite field $\mathbb{F}_q$. The curve also carries some markings $S \subset X(\mathbb{F}_q)$ which is why I call the theory "tamely ramified" (the main results are about the case $S = \{0, 1, \infty\}$). C_{Aut} has several pairwise commuting actions of $\mathcal{H}_{\text{aff}}$, the affine

Hecke algeba, labelled by markings $s \in S$. [1] I start by constructing $\mathcal{H}_{\text{aff}}$. Later I will explain how it controls local modifications of "parabolic" vector bundles.

Affine Hecke Algebra - following Iwahori-Matsumoto

Fix a reductive group scheme $\underline{G}$ over $\mathbb{F}_q$ and let $\underline{B}$ be a Borel subgroup. Let $G = \underline{G}(\mathbb{F}_q)$ and $B = \underline{B}(\mathbb{F}_q)$. Now, define $G((t)) := \text{Map}(\text{Spec}(\mathbb{F}_q((t))), \underline{G})$ as the loop group and $G[[t]] := \text{Map}(\text{Spec}(\mathbb{F}_q[[t]]), \underline{G})$ as the subgroup of maps that extend over the origin. If $\underline{G}$ is a matrix group, then concretely $G((t))$ is consists of matrices with $\mathbb{F}_q((t))$ entries and $G[[t]]$ consists of matrices with $\mathbb{F}_q[[t]]$ entries (whose inverse also has $\mathbb{F}_q[[t]]$ entries). Let $I \subset G[[t]]$ be the inverse image of B under the evaluation map $G[[t]] \xrightarrow{t=0} G$.

Definition 1.2.1 (Affine Hecke Algebra)**.** The affine Hecke algebra $\mathcal{H}_{\text{aff}}$ is the algebra of finitely supported functions on $I\backslash G((t))/I$ where the product is given by convolution. For $u, v \in \mathcal{H}_{\text{aff}}$,

$$u \cdot v(g) = \int_{G((t))} u(h)v(h^{-1}g)\mathrm{d}h$$

where $\mathrm{d}h$ is an invariant measure on $G((t))$ that assigns I unit measure. Most of the Hecke algebras considered in this thesis are affine Hecke algebras, so I will usually just write $\mathcal{H}$.

The points of $I\backslash G((t))/I$ are in bijection with the affine Weyl group of $\underline{G}$.

$$I\backslash G((t))/I \leftrightarrow W^{\text{aff}} \cong \Lambda \rtimes W$$

$\Lambda := \text{Hom}(\mathbb{G}_m, \underline{T})$ is the lattice of coweights of $\underline{G}$ and W is the finite Weylgroup. W^{aff} is the group of affine linear automorphisms of Λ. It decomposes into translations by lattice elements and W, which is the stabilizer subgroup of 0. It will be useful to have a running example.

Example 1.2.1 ($\underline{G} = \text{GL}_n$)**.** $W \cong S_n$ is the symmetric group and $\Lambda \cong \mathbb{Z}^n$ by identifying

$$(t \mapsto \text{diag}(t^{a_1}, ..., t^{a_n})) \mapsto (a_1, ..., a_n).$$

In particular, the cosets $I\backslash G((t))/I$ are represented by permutation matrices with monomial entries.

For $w \in W^{\text{aff}}$, let $T_w \in \mathcal{H}$ denote the corresponding function that takes value one on the coset represented by w and vanishes elsewhere. Iwahori and Matsumoto explicitly computed the product in $\mathcal{H}$ using the T_w basis. Let $\Lambda_R \subset \Lambda$ be the sublattice generated by coroots and define $W^{\text{aff},R} \subset W^{\text{aff}}$ as

[1] The automorphic theory also has actions by the spherical Hecke algebra away from markings but in the setting I consider these are recovered from the affine Hecke action via central degeneration to marked points. See [8]

$$W^{\text{aff},R} \cong \Lambda_R \rtimes W.$$

The subgroup $W^{\text{aff},R}$ is generated by the set S_{aff} of affine simple reflections. Moreover, W^{aff} decomposes as

$$W^{\text{aff}} \cong W^{\text{aff},R} \rtimes \Lambda/\Lambda_R$$

In particular, Λ/Λ_R embeds as a subgroup in W^{aff}.

Example 1.2.2 ($\underline{G} = \mathrm{GL}_n$)**.** $\Lambda_R \subset \Lambda \cong \mathbb{Z}^n$ is the subset of tuples whose coordinates sum to zero. The set of affine simple reflections consists of the $n-1$ simple reflections and one affine simple reflection. In coordinates, the simple reflection s_i, for $i = 1, 2, ..., n-1$, exchanges the ith and $i+1$th coordinate. The affine simple reflection, s_0, is given by

$$s_0(a_1, ..., a_n) = (a_n - 1, a_2, ..., a_{n-1}, a_1 + 1)$$

The affine simple reflections generate a subgroup, $\Lambda_R \rtimes S_n$, consisting of those automorphisms that do not change the sum of the coordinates. The complementary subgroup $\Lambda/\Lambda_R \cong \mathbb{Z}$ is generated by τ_0,

$$\tau_0(a_1, ..., a_n) = (a_2, ..., a_n, a_1 + 1)$$

The relations in $\mathcal{H}$ are as follows. Let the length function, $\ell : W^{\text{aff},R} \to \mathbb{Z}$, be defined so that the length of $w \in W^{\text{aff},R}$ is the length of its minimal expression in the generators S_{aff}. Extend this to a length function, $\ell : W^{\text{aff}} \to \mathbb{Z}$ that is invariant under right multiplication by Λ/Λ_R. For $w \in W^{\text{aff}}, s \in S_{\text{aff}}$, and $\tau \in \Lambda/\Lambda_R$,

$$T_\tau T_w = T_{\tau w}$$
$$T_s T_w = \begin{cases} T_{sw} & \ell(sw) > \ell(w) \\ qT_{sw} + (q-1)T_w & \ell(sw) < \ell(w) \end{cases}$$

Observe that setting $q = 1$ recovers the group algebra of W^{aff}.

Automorphic Functions and Hecke Action

I'll now define the vector space C_{Aut}, associated to a reductive group scheme $\underline{G}$ and a curve $X = \mathbb{P}^1$ with markings $S \subset X(\mathbb{F}_q)$. I will also explain how $\mathcal{H}^{\otimes S}$ acts on C_{Aut}.

Let $\underline{\mathrm{Bun}}_{\underline{G}}(X, S)$ denote the moduli stack of $\underline{G}$ bundles on X with Borel reductions near S. For example, if $\underline{G} = \mathrm{GL}_n$, this classifies pairs $(\mathcal{E}, \{F^s\}_{s \in S})$ where $\mathcal{E}$ is a rank n vector bundle on X and $F^s \subset \mathcal{E}|_s$ is a complete flag in the fiber above s. For general $\underline{G}$ I sometimes refer to the data classified by this moduli stack as a parabolic $\underline{G}$ bundle (hopefully it is implicit from context what the extra parabolic data is). Let $\mathrm{Bun}_{\underline{G}}(X,S)$ denote the set of rational points of $\underline{\mathrm{Bun}}_{\underline{G}}(X,S)$. This set comes with a natural measure [2] , μ, given by

$$\mu((\mathcal{E},\{F^s\})) = |\mathrm{Aut}(\mathcal{E},\{F^s\})|^{-1}$$

$\mathrm{Aut}(\mathcal{E},\{F^s\})$ means the subgroup of the automorphism group of the vector bundle $\mathcal{E}$ such that the induced automorphism of $\mathcal{E}|_s$ fixes the flags F^s (and in general fixes the extra parabolic data).

Definition 1.2.2 (Tamely Ramified Automorphic Function Theory)**.** The tamely ramified at S automorphic function theory of the X is the vector space of finitely supported [3] functions on $\mathrm{Bun}_{\underline{G}}(X,S)$

$$C_{\mathrm{Aut}} := C_c(\mathrm{Bun}_{\underline{G}}(X,S))$$

$\mathcal{H}$ has commuting actions, indexed by S, on C_{Aut}. First, I will formally define the action $\mathcal{H}\otimes C_{\mathrm{Aut}} \to C_{\mathrm{Aut}}$ at $s \in S$. Then I will illustrate explicitly the action of the generators of $\mathcal{H}$ when $\underline{G} = \mathrm{GL}_n$. Let $\mathrm{Spec}(\mathbb{F}_q[[t]]) \to X$ be a formal disk centered at s. Consider the following diagram that organizes local modifications of parabolic $\underline{G}$ bundles at s.

$$\begin{array}{ccccc} \mathrm{Bun}_{\underline{G}}(X,S) & \xleftarrow[\pi_1]{} & \mathrm{HeckeMod}^s & \xrightarrow{\pi_2} & \mathrm{Bun}_{\underline{G}}(X,S) \\ & & \downarrow{\scriptstyle \mathrm{res}} & & \\ & & I\backslash G((t))/I & & \end{array}$$

$\mathrm{HeckeMod}^s$ classifies the data of a triple $((\mathcal{E}_1,\{F_1^s\}),(\mathcal{E}_2,\{F_2^s\}),T)$ consisting of two parabolic $\underline{G}$-bundles on $\mathbb{P}^1$ and an isomorphism of their restrictions away from s. res is the restriction of the parabolic bundles along the map $\mathrm{Spec}(\mathbb{F}_q[[t]]) \to \mathbb{P}^1$. The key point is that $I\backslash G((t))/I$ parametrizes the data of a pair of parabolic bundles on a formal disk that are identified away from the origin.

Definition 1.2.3 (Hecke action on automorphic functions)**.** The action $\mathcal{H}\otimes C_{\mathrm{Aut}} \to C_{\mathrm{Aut}}$ is given by

$$A \otimes f \mapsto \pi_{2!}(\mathrm{res}^* A \otimes {\pi_1}^* f)$$

Pullback of functions means the obvious thing. To define the pushforward, one needs to take into account that the source and target sets have natural measures coming from the automorphism group of objects. Using this measure one identifies functions with measures and computes the pushforward of measures instead.

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[2] As defined, μ is a function, but the set of isomorphism classes of parabolic bundles is countable, so in fact it is measure that assigns positive measure to each point.

[3] The point of considering finitely supported functions is so it is easier to state the main conjecture. In Chapter 5, which is about spectral theory of Hecke operators, I will consider the space of square integrable functions.

Now I describe the Hecke action at $s \in S$ explicitly when $\underline{G} = \mathrm{GL}_n$.

Hecke Modification of Parabolic Vector bundles ($\underline{G} = \mathrm{GL}_n$)

I'll make a slight notation switch for this subsection only, where I refer to the point of modification as p instead of s. Instead $s \in S$ will denote an arbitrary marking.

Recall that in this case the affine Hecke algebra is generated by three types of elements:

1. simple reflections, s_i, for $i = 1, ..., n-1$
2. affine simple reflection s_0
3. generator, τ_0, of Λ/Λ_R

First I will describe $T_{s_i} : C_{\mathrm{Aut}} \to C_{\mathrm{Aut}}$ for s_i a simple reflection. Let $(\mathcal{E}, \{F^s\}) \in \mathrm{Bun}_{\underline{G}}(X, S)$ be a parabolic vector bundle. $\mathcal{E}$ is a rank n vector bundle and $\mathcal{F}^s$ is a complete flag in the fiber $\mathcal{E}|_s$. Let $F^s = \{V_i^s\}_{i=1}^n$, where $V_1^s \subset V_2^s \subset ... \subset V_{n-1}^s \subset \mathcal{E}|_s$ and $\dim V_i^s = i$.

Definition 1.2.4 (Elementary Modification of Parabolic Vector Bundles)**.** Two parabolic vector bundles $(\mathcal{E}_1, \{F_1^s\})$ and $(\mathcal{E}_2, \{F_2^s\})$ are related by an elementary modification of type i at $p \in S$, denoted $(\mathcal{E}_1, \{F_1^s\}) \sim_{i,p} (\mathcal{E}_2, \{F_2^s\})$ if:

1. $\mathcal{E}_1 = \mathcal{E}_2$
2. $F_1^s = F_2^s$ for $s \neq p$
3. $(V_j^p)_1 = (V_j^p)_2$ for $j \neq i$
4. $(V_i^p)_1 \neq (V_i^p)_2$

The action of T_{s_i} at $p \in S$ is

$$T_{s_i}((\mathcal{E}_1, \{F_1^s\})) = \sum_{(\mathcal{E}_1, \{F_1^s\}) \sim_{i,p} (\mathcal{E}_2, \{F_2^s\})} \frac{|\mathrm{Aut}(\mathcal{E}_2, \{F_2^s\})|}{|\mathrm{Aut}(\mathcal{E}_1, \{F_1^s\}) \cap \mathrm{Aut}(\mathcal{E}_2, \{F_2^s\})|} (\mathcal{E}_2, \{F_2^s\})$$

This is a slight abuse of notation because I am denoting elements of C_{Aut} by finite formal sums of elements of $\mathrm{Bun}_{\underline{G}}(X, S)$. The quotient in the sum records the relative measure and comes from computing pushforward of functions. It takes into account the situation that two different elementary modifications may produce isomorphic parabolic bundles.

Next, I'll describe the action, $T_{s_0} : C_{\mathrm{Aut}} \to C_{\mathrm{Aut}}$, of the affine simple reflection at some point $p \in S$.

The data of a flag $F^s \subset \mathcal{E}|_s$ is equivalent to a chain of subbundles $\mathcal{E}_1^s \subset \mathcal{E}_2^s \subset ... \subset \mathcal{E}_{n-1}^s \subset \mathcal{E}_n^s = \mathcal{E}$, such that each successive quotient $\mathcal{E}_{i+1}^s / \mathcal{E}_i^s$ is supported at s and whose fiber at s is a one dimensional. In particular, the local sections of $\mathcal{E}_i^s \subset \mathcal{E}$ are the sections of $\mathcal{E}$ whose value at s is contained in V_i.

Definition 1.2.5 (Affine Elementary Modification of Parabolic Vector Bundles)**.** Parabolic vector bundles $(\mathcal{E}_1, \{F_1^s\})$ and $(\mathcal{E}_2, \{F_2^s\})$ are related by an elementary modification of type 0 at $p \in S$, denoted $(\mathcal{E}_1, \{F_1^s\}) \sim_{0,p} (\mathcal{E}_2, \{F_2^s\})$ if:

1. $(\mathcal{E}_i^s)_1 = (\mathcal{E}_i^s)_2$ for $i = 1, 2, ..., n-1$ and all $s \in S$
2. $(\mathcal{E}_{n-1}^p)_1 \subset \mathcal{E}_2 \subset (\mathcal{E}_1^p)_1 \otimes \mathcal{O}(p)$
3. $\mathcal{E}_1 \ncong \mathcal{E}_2$

In particular, $\mathcal{E}_2$ only differs from $\mathcal{E}_1$ at p. The following quotient is supported s and has a two dimensional fiber at s.

$$((\mathcal{E}_1^p)_1 \otimes \mathcal{O}(p))/(\mathcal{E}_{n-1}^p)_1$$

The choice of $\mathcal{E}_2$ is parameterized by the projectivization of that fiber (minus one point corresponding to $\mathcal{E}_1 \cong \mathcal{E}_2$).

Using the previous notation conventions, the action of T_{s_0} at $p \in S$ is

$$T_{s_i}((\mathcal{E}_1, \{F_1^s\})) = \sum_{(\mathcal{E}_1,\{F_1^s\}) \sim_{0,p} (\mathcal{E}_2,\{F_2^s\})} \frac{|\operatorname{Aut}(\mathcal{E}_2, \{F_2^s\})|}{|\operatorname{Aut}(\mathcal{E}_1, \{F_1^s\}) \cap \operatorname{Aut}(\mathcal{E}_2, \{F_2^s\})|} (\mathcal{E}_2, \{F_2^s\})$$

Finally, I'll describe the action, $T_{\tau_0} : C_{\text{Aut}} \to C_{\text{Aut}}$, of τ_0 at a point $p \in S$. This action comes from a coarse symmetry of $\underline{\text{Bun}}_{\underline{G}}(X, S)$, called the Atkin-Lehner symmetry.

Definition 1.2.6 (Atkin-Lehner Modificaton of Parabolic Vector Bundles)**.** $(\mathcal{E}_2, \{F_2^s\})$ is the Atkin-Lehner modification of $(\mathcal{E}_1, \{F_1^s\})$ at p if the following is true:

1. $(\mathcal{E}_i^s)_1 = (\mathcal{E}_i^s)_2$ for $i = 1, 2, ..., n-1$ and $s \neq p$
2. $\mathcal{E}_2 = (\mathcal{E}_1^p)_1 \otimes \mathcal{O}(p)$
3. $(\mathcal{E}_i^p)_2 = (\mathcal{E}_{i+1}^p)_1$ for $i = 1, 2, ..., n-1$

T_{τ_0} sends a parabolic bundle (thought of as a function in C_{Aut}) to its Atkin-Lehner modification at p. Note that in W^{aff} conjugation by τ sends s_i to s_{i+1}, and the same is true in the affine Hecke algebra. Therefore, it is possible to express all simple reflection Hecke operators T_{s_i} in terms of T_{s_0} and T_{τ_0}.

This explicit presentation of the Hecke action can easily be adapated to $\underline{G} = \text{PGL}_n$. A PGL_n bundle is a rank n vector bundle up to tensoring with a line bundle.

$$\underline{\text{Vect}}_n(\mathbb{P}^1)/\underline{\text{Pic}}(\mathbb{P}^1) \cong \underline{\text{Bun}}_{\text{PGL}_n}(\mathbb{P}^1)$$

In particular, the Hecke algebra is still generated by affine simple reflections, T_{s_i} for $i = 0, 1, ..., n-1$ and τ_0. However, $T_{\tau_0}^n = 1$ because it corresponds to tensoring with $\mathcal{O}(p)$.

Eisenstein Submodule

The main conjecture is about the Hecke action on the Eisenstein submodule of C_{Aut}. This is the subspace of functions induced from the maximal torus. In Chapter 4, I will give a detailed explanation of what this means and why it is significant. For now, the following definition is sufficient.

First, define the following special isomorphism class of parabolic bundles. The G orbits of the diagonal action $G \circlearrowright \mathcal{B}^S$ parametrize parabolic level structures on a trivial vector bundle, $\mathcal{E}$. This is because the automorphism group of the bundle, $\text{Aut}(\mathcal{E}) \cong G$, acts diagonally on the fibers of marked points $\prod_{s \in S} \mathcal{E}|_s$ (after canonically identifying them with each other). There is a point of $\text{Bun}_{\underline{G}}(X, S)$ where the bundle is trivial and the flags in the fibers are all identified. Let $\text{Eis}_0 \in C_{\text{Aut}}$ be the function that takes value one on this point at vanishes elsewhere.

Definition 1.2.7 (Eisenstein Submodule)**.** The Eisenstein submodule $C_{\text{Eis}} \subset C_{\text{Aut}}$ is the closure under $\mathcal{H}^{\otimes S}$ of the space spanned by Eis_0.

The main result is about the structure of C_{Eis} as a $\mathcal{H}^{\otimes S}$ module. It is easier to state using a different presentation of $\mathcal{H}$.

Affine Hecke Algebra - Bernstein's presentation

The affine Hecke algebra is generated by two important subalgebras.

1. (Maximal Commutative Subalgebra) The assignment $\lambda \mapsto T_\lambda$, for $\lambda \in \Lambda$ antidominant extends to a homomorphism $\mathbb{C}[\Lambda] \to \mathcal{H}$. Let J_λ denote the image of λ under this homomorphism; that is, $J_\lambda = T_{\lambda_1} T_{\lambda}^{-1}$, where $\lambda = \lambda_1 - \lambda_2$ with λ_1, λ_2 antidominant. I will refer to this subalgebra as the algebra of translation operators.

2. (Finite Hecke Algebra) The operators T_w for $w \in W$ generate a finite dimensional subalgebra, $\mathcal{H}_{\text{fin}}$ that I will sometimes call the algebra of reflection operators. It is a deformation of the group algebra of W.

It is not hard to see that as vector spaces, $\mathcal{H} \cong \mathbb{C}[\Lambda] \otimes \mathcal{H}_{\text{fin}}$. Lusztig (based on unpublished work of Bernstein) computed the following relation between the two subalgebras [19]:

$$J_\lambda T_{s_\alpha} = q^{-\check{\alpha}(\lambda)} T_{s_\alpha} J_{s_\alpha(\lambda)} + (q-1) \frac{J_\lambda - q^{-\check{\alpha}(\lambda)} J_{s_\alpha(\lambda)}}{1 - q J_\alpha}$$

s_α denotes the simple reflection about the plane normal to α. Observe that evaluating $q = 1$ recovers the standard relation in the affine Weyl group.

It is worth noting that the geometric meaning of Bernstein's presentation is clarified by Kazhdan and Lusztig's coherent realization of the algebra in terms of the Langlands dual group.

$$\mathcal{H} \cong K^{\underline{G}^L \times \mathbb{G}_m}(\mathrm{St}^L)$$

St^L is the Steinberg variety of the Langlands dual group. The parameter q in $\mathcal{H}$ appears on the RHS as the equivariant parameter for $\mathbb{G}_m$. Up to normalization by a power of q, the isomorphism sends J_λ to the class of $\triangle_*(\mathcal{O}_{\tilde{\mathcal{N}}^L}(\lambda))$, where $\triangle : \tilde{\mathcal{N}}^L \to \mathrm{St}^L$ is the diagonal map. In other words, $\mathbb{C}[\Lambda]$ is naturally identified with the group algebra of equivariant line bundles on the flag variety of the Langlands dual group.

Main Conjecture

Assume the following mild technical restriction on $\underline{G}$ that I will not explain until later chapters. Let $\Lambda^\vee := \mathrm{Hom}(\underline{T}, \mathbb{G}_m)$ denote the lattice of weights of $\underline{G}$. Assume that for all roots $\check{\alpha} \in \Lambda^\vee$, the map $\Lambda \to \mathbb{Z}$ given by $\lambda \mapsto \langle \check{\alpha}, \lambda \rangle$ is surjective. For example, PGL_2 and SL_3 satisfy this condition, but SL_2 does not. The adjoint form of a group will always satisfy this condition.

For A in $\mathcal{H}$ and $s \in S$, let A^s the corresponding local Hecke operator at s.

Conjecture 1.2.1. C_{Eis} is the affine Hecke tri-module generated by a distinguished function Eis_0 and following relations

1. (Translation Relation) For any $\lambda \in \Lambda$

$$J_\lambda^0 \mathrm{Eis}_0 = J_\lambda^1 \mathrm{Eis}_0 = J_\lambda^\infty \mathrm{Eis}_0$$

2. (Reflection Relation) For any simple reflection, $s_\alpha \in W$

$$(1 + T_{s_\alpha}^0)(1 + T_{s_\alpha}^1)\mathrm{Eis}_0 = (1 + T_{s_\alpha}^0)(1 + T_{s_\alpha}^\infty)\mathrm{Eis}_0 = (1 + T_{s_\alpha}^1)(1 + T_{s_\alpha}^\infty)\mathrm{Eis}_0$$

There is a natural generalization of this, Conjecture 4.6.1, to arbitrary tame ramification $S \subset \mathbb{P}^1(\mathbb{F}_q)$. When S consists of one or two points, the conjecture follows from the Radon transform which identifies C_{Eis} with the regular bimodule for the Hecke algebra. I will prove Conjecture 1.2.1 when $G = \mathrm{PGL}(2)$ or $\mathrm{SL}(3)$ (Theorems 4.2.1 and 4.5.1) as well as in the following generic sense. Let $\widetilde{C}$ be the quotient of $\mathcal{H}^{\otimes S}$ by the left ideal generated by the Translation and Reflection relations.

Theorem 1.2.1. There is a surjective map $\widetilde{C} \to C_{\mathrm{Eis}}$ of affine Hecke tri-modules, given by $1 \mapsto \mathrm{Eis}_0$ such that rationalizing the action of translation operators at 0 yields an isomorphism

$$\mathrm{Frac}(\mathbb{C}[\Lambda]^0) \otimes_{\mathbb{C}[\Lambda]^0} \widetilde{C} \xrightarrow{\cong} \mathrm{Frac}(\mathbb{C}[\Lambda]^0) \otimes_{\mathbb{C}[\Lambda]^0} C_{\mathrm{Eis}}.$$

The proof of this conjecture for PGL_2 and SL_3 relies on some very explicit computations about the geometry of level structures. Observe that the fiber of $\mathrm{Bun}_{\underline{G}}(X, S) \to \mathrm{Bun}_{\underline{G}}(X)$ is the following:

$$\begin{array}{ccc} \mathrm{Aut}(\mathcal{E})\backslash\mathcal{B}^S & \longrightarrow & \mathrm{Bun}_{\underline{G}}(X,S) \\ \downarrow & & \downarrow \\ \{\mathcal{E}\} & \hookrightarrow & \mathrm{Bun}_{\underline{G}}(X) \end{array}$$

$\mathcal{B} = G/B$ denotes the set of rational points of flag variety over $\mathbb{F}_q$. The fiber is the quotient of the triple flag variety. The $\mathcal{H}_{\mathrm{fin}}$ action at s only changes the level structure at s and doesn't change the underlying bundle. More precisely, the following is a decomposition into $\mathcal{H}_{\mathrm{fin}}^{\otimes S}$ submodules:

$$C_{\mathrm{Aut}} \cong \oplus_{\mathcal{E}} \mathrm{Fun}(\mathrm{Aut}(\mathcal{E})\backslash\mathcal{B}^S)$$

Moreover, the translation relation of Conjecture 1.2.1 is simply a realization of the non-trivial but classically known fact that Eisenstein series are compatible with Hecke modification. For any $s \in S$ and $\lambda, \mu \in \Lambda$,

$$J^s_\lambda \mathrm{Eis}_\mu = \mathrm{Eis}_{\lambda+\mu}$$

Therefore, the proofs for PGL_2 and SL_3 are completed by analyzing the finite Hecke action on the Eisenstein objects Eis_λ and showing that they are reproduced by the reflection relations of Conjecture 1.2.1. However, the calculations turns out to be quite involved and rely on computer algebra software to verify some computations. I don't expect this method will work to prove the conjecture in general. In Chapter 4 I give some hints at how it might be proved in general. In particular, the conjecture is reduced to showing that the formal quotient module $\tilde{C}$ is free over $\mathbb{C}[\Lambda]^0$, the algebra of translation operators at zero. I imagine the proof may involve identifying the reflection relation as a version of the functional equation satisfied by Eisenstein series.

1.3 Historical Motivations

The rest of this chapter presents the motivation for this thesis. The starting point is the fundamental automorphic representation.

The Fundamental Automorphic Representation

Let $\underline{G}$ be a reductive group scheme over a global field F and let $\mathbb{A}$ be the adele ring of F. A central problem in representation theory is to describe the representation

$$C^\infty(\underline{G}(\mathbb{A})) \circlearrowright L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A}))$$

in terms of the Langlands dual group $\underline{G}^L$ and the Galois group $\mathrm{Gal}(\overline{F}/F)$. This is a vast generalization of the familiar situation from Fourier theory where given a full rank lattice $\Gamma \subset V$ in a real vector space and $\Gamma^* \subset V^*$ its dual lattice,

$$L^2(\Gamma\backslash V) \cong \ell^2(\Gamma^*).$$

The isomorphism decomposes the LHS into characters, $f \mapsto \hat{f}(\lambda^*)$, of the natural $C^\infty(V)$ action.

Given a compact open subgroup $K \subset \underline{G}(\mathbb{A})$, consider the subspace of the automorphic representation consisting of K invariant functions. By varying the stabilizer subgroup K, one probes the smooth representation theory of $\underline{G}(\mathbb{A})$. For each place v of F, let $K_v \subset \underline{G}(F_v)$ denote a maximal compact subgroup (for v non-Archimedean we can take $K_v = \underline{G}(\mathcal{O}_v)$) and let $K_0 = \prod_v K_v$. The topology on $\underline{G}(\mathbb{A})$ has a basis of translates of compact open subgroups $K' \subset K_0$ of finite index. In particular, one loses nothing by considering functions invariant under some group $K' = \prod_v K'_v$ where $K'_v \subset K_v$ has finite index and $K'_v = K_v$ for all but finitely many places. Moreover, the group action on the invariant subspace survives [4] as

$$L^2(\underline{G}(F)\backslash\underline{G}(\mathbb{A})/K') \circlearrowleft C^\infty(K'\backslash\underline{G}(\mathbb{A})/K') \cong \bigotimes_v{}' C^\infty(K'_v\backslash\underline{G}(F_v)/K'_v)$$

The algebra of operators on the RHS is called the (global) Hecke algebra [5]. Remarkably, it contains a commutative [6] subalgebra over which it is finite called the (global) *spherical* Hecke algebra.

$$\text{Spherical Hecke Algebra} = C^\infty(K_0\backslash\underline{G}(\mathbb{A})/K_0) = \otimes'_v C^\infty(K_v\backslash\underline{G}(F_v)/K_v)$$

In this thesis I study the Hecke action on the automorphic function theory in the case where $F = \mathbb{F}_q(t)$ is the rational function field and the ramification subgroups K'_v are not much smaller than K_v.

Geometric Formulation

Weil observed that if $F = \mathbb{F}_q(X)$ is the function field of a curve $X/\mathbb{F}_q$, the unramified automorphic theory (functions stabilized by K_0) is a linearization of the moduli space of $\underline{G}$ bundles on X.

$$\underline{G}(F)\backslash\underline{G}(\mathbb{A})/K \cong \mathrm{Bun}_{\underline{G}}(X)$$

Precisely, this is an isomorphism of the LHS with the set of rational points of the moduli stack on the RHS that identifies the (descended) Haar measure with the groupoid counting

[4] One matches equivariance when convolving functions the same way one would match dimensions when multiplying matrices. Viewing $H_1\backslash G/H_2$ as $BH_1 \times_{BG} BH_2$, it is evident that both are instances of the same principle.

[5] The decoration $\otimes'$ indicates a restricted product, analogous to how the adele ring is defined as a restricted product of its local fields.

[6] I suspect there is in fact a *central* subalgebra isomorphic to spherical Hecke. This is true in the cases considered in this thesis because of Gaitsgory's central sheaves [8]. Perhaps the argument can be modified to work for arbitrary subgroups K'_v?

measure [7]. For the ramified automorphic theory, one needs to identify the fiber of the projection with some geometric level structure at the ramified places:

$$\begin{array}{ccc} \mathrm{Aut}(\mathcal{E})\backslash \prod_v K_v/K'_v & \longrightarrow & \underline{G}(F)\backslash\underline{G}(\mathbb{A})/K' \\ \downarrow & & \downarrow \\ \{\mathcal{E}\} & \hookrightarrow & \underline{G}(F)\backslash\underline{G}(\mathbb{A})/K \end{array}$$

I will focuses on the case of *tame ramification*. Let $S \subset X(\mathbb{F}_q)$ be a subset of the rational points of X. For each $v \in S$, picking a uniformizer $t \in F_v$ identifies $\underline{G}(F_v)$ with the loop group $G((t))$ and $\underline{G}(\mathcal{O}_v)$ with the arc group $G[[t]]$ [8]. The Iwahori subgroup $I \subset G[[t]]$ consists of matrices that are sent to B by evaluation $G[[t]] \xrightarrow{t=0} G$. The tamely ramified at S automorphic function theory consists of functions that are invariant under $K_S = \prod_v K'_v$, where $K'_v = I$ for $v \in S$ and $K'_v = K_v$ otherwise. The associated level structure is a B-reduction near S. More classically this is the data of a flag in the fiber $\mathcal{E}|_v$ of the bundle $\mathcal{E}$ for each $v \in S$.

$$\underline{G}(F)\backslash\underline{G}(\mathbb{A})/K_S \cong \mathrm{Bun}_{\underline{G}}(X,S)$$

The Hecke action on the automorphic theory in the unramified and tamely ramified cases can be understood geometrically as an action by correspondences. The local Hecke action at an unramified place v is given by functions on

$$G[[t]]\backslash G((t))/G[[t]] \cong W\backslash W^{\mathrm{aff}}/W \cong \Lambda_+$$

[9] These are isomorphisms of sets only. This algebra of operators is called the (local) spherical Hecke algebra. The Hecke operator indexed by $\lambda \in \Lambda_+$ acts through a correspondence

$$\mathrm{Bun}_{\underline{G}}(X,S) \longleftarrow \mathrm{Corr} \longrightarrow \mathrm{Bun}_{\underline{G}}(X,S)$$

The correspondence $\mathrm{Corr} \subset \mathrm{Bun}_{\underline{G}}(X,S) \times \mathrm{Bun}_{\underline{G}}(X,S)$ consists of pairs of bundles (with appropriate level structure) that are identified away from v and have relative position λ at v. For example, in the case where $\underline{G} = \mathrm{GL}_n$, this classifies pairs $(\mathcal{E},\mathcal{E}')$ of rank n vector bundles along with a short exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \delta_{\lambda,v} \to 0,$$

where $\delta_{\lambda,v}$ is a supported at v with structure determined by λ.

---

[7] This is also called the motivic measure.

[8] Assume $\underline{G}$ is the base change of a group scheme over $\mathbb{F}_q$, which we also denote $\underline{G}$. Then, $G := \underline{G}(\mathbb{F}_q)$ and $B := \underline{G}(\mathbb{F}_q)$.

[9] W and W^{aff} denote the finite and affine Weyl group, and $\Lambda_+ \subset \Lambda := \mathrm{Hom}(\mathbb{G}_m, \underline{T})$ is the set of dominant coweights.

There is a similar interpretation of the local Hecke action at tamely ramified places where the correspondences are labelled by

$$I\backslash G((t))/I \cong W^{\mathrm{aff}}.$$

This algebra of operators is called the affine Hecke algebra.

Automorphic Gluing

This thesis focuses on the tamely ramified automorphic theory of a rational function field $F = \mathbb{F}_q(t)$. One motivation for this setting comes from topological field theory. In this section I'll explain why one expects that the unramified automorphic theory of a general curve X can be understood in terms of $\mathbb{P}^1$ with up to three points of tame ramification. I'll be imprecise about the field over which X is defined (it could be $\mathbb{F}_q$ or $\mathbb{C}$) and about exactly which linearization of $\mathrm{Bun}_{\underline{G}}(X)$ the automorphic theory refers to (it could be a category of sheaves or a function theory). This is because many of the the ideas are not yet well formulated and are illustrated only to provide motivation. For further discussion, I suggest [20] and [4].

Kapustin and Witten explained that the assignment

$$X \mapsto [\text{Unramified Automorphic Theory of } X]$$

should be thought of as the partition function of a supersymmetric gauge theory associated to a $\underline{G}$. Furthermore, level structure can be incorporated into the theory via defects. Following their proposal, one expects that the automorphic theory admits a description that is insensitive to varying X. The strategy for producing a "topological" description then consists of two central facets.

Step 1 (Automorphic Gluing)**.** Establish a Verlinde formula description for automorphic theory of X whose basic building blocks are the automorphic theory of genus zero curves, tamely ramified at up to three marked points.

This is called a Verlinde formula because it is analogous to the formula for dimensions of conformal blocks in conformal field theory, where factorization rules allow degenerating X to a nodal graph of $\mathbb{P}^1$s and therefore a reduction to genus zero curves with up to three marks. The automorphic gluing conjecture is precisely formulated by Nadler and Yun in [20] for Betti Geometric Langlands [10]. Furthermore, it is expected that an anaolgous conjecture can be formulated when X is a curve over $\mathbb{F}_q$ and then related to the function theory through Grothendieck's sheaf-function dictionary.

Step 2 (Fusion Product of Hecke Representations)**.** Explicitly describe the automorphic theory for genus zero curves, tamely ramified at marked points, S, as affine Hecke modules, for $|S| \leq 3$.

[10] This is a version of the automorphic theory formulated by Ben-Zvi Nadler where one studies the category of nilpotent constructible sheaves on $\underline{Bun_{\underline{G}}}(X)$.

The cases when $|S| = 0, 1, 2$ are well understood by the Radon transform to be the spherical, vector, and regular Hecke representations, respectively. The function theory for $|S| = 3$ produces a Hecke tri-module, which can be understood as a fusion product of Hecke representations. This is analogous to the situation in conformal field theory where conformal blocks for a genus zero curve with three marked points are fusion coefficients for the loop group representations that label the markings.

Eisenstein Series

Following Harish-Chandra's philosophy, the spectral decomposition of the automorphic function theory is organized by parabolic induction and restriction maps called Eisenstein Series and Constant Term.

In the unramified function field case induction and restriction have a straightforward geometric meaning. Let $\underline{P} \subset \underline{G}$ be a parabolic subgroup and $\underline{L} := \underline{P}/[\underline{P}, \underline{P}]$ be the associated Levi subgroup.

$$\mathrm{Bun}_{\underline{L}}(X) \xleftarrow{q} \mathrm{Bun}_{\underline{P}}(X) \xrightarrow{p} \mathrm{Bun}_{\underline{G}}(X)$$

Eisenstein series and constant term are the adjoint pair $\mathrm{Eis}_{\underline{P}} := p_! q^*$ and $\mathrm{CT}_{\underline{P}} := q_! p^*$.

$$\mathrm{Eis}_{\underline{P}} : \mathrm{Fun}(\mathrm{Bun}_{\underline{L}}(X)) \longleftrightarrow \mathrm{Fun}(\mathrm{Bun}_{\underline{G}}(X)) : \mathrm{CT}_{\underline{P}}$$

The subspace $\mathrm{Cusp}(X; \underline{G}) \subset \mathrm{Fun}(\mathrm{Bun}_{\underline{G}(X)})$ of *cusp forms* is the intersection of the kernels of all constant term maps.

$$\mathrm{Cusp}(X; \underline{G}) := \bigcap_{\underline{P} \neq \underline{G}} \ker(\mathrm{CT}_{\underline{P}})$$

The orthogonal decomposition by cusp forms of (conjugacy classes of) Levi subgroups:

$$\mathrm{Fun}(\mathrm{Bun}_{\underline{G}}(X)) = \mathrm{Cusp}(X; \underline{G}) \oplus \bigoplus_{\underline{B} \subset \underline{P} \neq \underline{G}} \mathrm{Eis}_{\underline{P}} \, \mathrm{Cusp}(X; \underline{L})$$

It is expected that the component corresponding to $\underline{L}$ further decomposes by local systems on X for the Langlands dual group $\underline{L}^L$ (and some additional data when $\underline{L} \neq \underline{G}$). The celebrated work of L. Lafforgue, based on Drinfeld's shtukas, establishes this for the case that $\underline{G} = \mathrm{GL}_n$ and $\underline{L} = \underline{G}$ [15]. V. Lafforgue extends this to arbitrary groups $\underline{G}$ [16]. Their work applies in the setting of arbitrary ramification and proves the hardest part of Langlands correspondence for global function fields.

Langlands computed the inner product of functions induced from $\underline{L} \neq \underline{G}$ in terms of root data [17]. In principal, combined with Lafforgues' work on cusp forms, this characterizes the spectral decomposition of the component corresponding to $\underline{L}$, but the description is very complicated. Recently, Kazhdan and Okounkov gave a simple description, in the unramified case, of the spectrum of Eisenstein series induced from $\underline{T}$ that correspondend to the trivial $\underline{T}^L$ local system on X [14, 13]. Their decomposition is by nilpoltent conjugacy classes for the dual group $\underline{G}^L$. The research presented in this thesis is about Eisenstein series induced from $\underline{T}$, in the tamely ramified case, for $X = \mathbb{P}^1$ (although I expect the results to hold for Eisenstein series corresponding to the trivial $\underline{T}^L$ local system for arbitrary X).

Chapter 2

Corresondence Algebras

To motivate the exposition in this chapter, I start by explaining how to think of the group algebra of a finite group as a correspondence algebra.

Let G be a finite group and k a field. $k[G]$ denotes the group algebra, consisting of functions $f : G \to k$ with product given by

$$f \cdot g(x) = \sum_{y \in G} f(y) g(y^{-1}x)$$

Let $k^G[G \times G]$ denote the vector space of functions $u : G \times G \to k$ that are invariant with respect to the action $G \times (G \times G) \to G \times G$ given by $x \cdot (x_1, x_2) = (xx_1, xx_2)$. The following diagram organizes a product on $k^G[G \times G]$.

$$\begin{array}{ccccc} G \times G & \xleftarrow[\pi_{1,2}]{} & G \times G \times G & \xrightarrow{\pi_{1,3}} & G \times G \\ & & \big\downarrow{\scriptstyle \pi_{2,3}} & & \\ & & G \times G & & \end{array}$$

Explicitly the product of $T, U \in k^G[G \times G]$ is given by the formula:

$$T \cdot U(x_1, x_2) = \pi_{1,3!}(\pi_{1,2}^*(u) \otimes \pi_{2,3}^*(v))(g_1, g_2) = \sum_{y \in G} u(x_1, y) v(y, x_2) \tag{2.1}$$

There is an isomorphism of vector spaces $\phi : k[G] \to k^G[G \times G]$ given by $\phi(f)(x_1, x_2) = f(x_1^{-1}x_2)$ with inverse $\phi^{-1}(T)(x) = u(1, x)$.

Claim 2.0.1. For $f, g \in k[G]$, $\phi(f \cdot g) = \phi(f) \cdot \phi(g)$

Proof.

$$\phi(f \cdot g)(x_1, x_2) = (f \cdot g)(x_1^{-1}x_2) = \sum_{y \in G} f(y) g(y^{-1}x_1^{-1}x_2) = \sum_{y \in G} f(x_1^{-1}y) g(y^{-1}x_2)$$

$$= \sum_{y \in G} \phi(f)(x_1, y) \phi(g)(y, x_2) = (\phi(f) \cdot \phi(g))(x_1, x_2)$$

□

I will explain what I mean by correspondence algebra. In the previous discussion, if we identify function $G \times G \to k$ with matrices whose rows columns are indexed by G, then Equation 2.1 is the formula for matrix multiplication. Matrix algebras are simple examples of correspondence algebras. In general by a correspondence I mean a cycle $Y \subset X \times X$. Composition of correspondence is organized by diagrams analogous to Equation 2. For $Y, Z \subset X \times X$ satisfying natural assumptions, at the level of spaces, $Y \cdot Z = \pi_{1,3}(\pi_{1,2}^{-1}(Y) \cap \pi_{2,3}^{-1}(Z))$. Alternatively, one can consider linearizations of this formula, for example in cohomology, K-theory, or, as considered in this thesis, function theory. In all these cases, the formula for convolution of cycles is given, as before, by the formula

$$Y \cdot Z = \pi_{1,3!}(\pi_{1,2}^*(Y) \otimes \pi_{1,3}^*(Z)).$$

Note that the composition of cycles isn't always a cycle, so it is necessary to work in the broader setting of a linear theory.

2.1 Groupoids

In this section I will define the space of functions on a (nice) groupoid. If the automorphism groups of objects are finite, the space of complex valued functions is naturally an inner product space.

Definition of Groupoid

Definition 2.1.1 (Groupoid)**.** A groupoid is a category all of whose morphisms are invertible. A morphism of groupoids is a functor between the underlying categories. Denote the category of groupoids by **Grpd**.

There is a useful notion of equivalence for groupoids that is different from isomorphism.

Definition 2.1.2 (Equivalence of Groupoids)**.** An equivalence of groupoids is a morphism $X \to Y$ of groupoids that is an equivalence of the underlying categories.

Example 2.1.1. Let **FinSet** denote the groupoid of finite sets (all morphisms are isomorphisms) and let **SkFinSet** be the full sub category whose objects are sets $[n] = \{1, 2, ..., n\}$. **SkFinSet** → **FinSet** is an equivalence of groupoids.

The need for equivalence of groupoids hints that the category **Grpd** is not exactly the correct object to consider. I will explain more in Section 2.1.

Example: Sets

There is a fully faithful functor **Set** → **Grpd** that sends a set, S, to the groupoid with objects $s \in S$ and all of whose morphisms are identity.

Example: Group Actions

Given a group action on a set $G \times X \to X$, the quotient $G\backslash X$ is a groupoid with objects $x \in X$ and morphisms $x \xrightarrow{(g,x)} gx$. The automorphism group of an object, $\mathrm{Aut}(x)$, is its stabilizer subgroup $\mathrm{Stab}(x) \subset G$.

Consider two group actions $G \times X \to X$ and $H \times Y \to Y$ related by a group homomorphism $\phi : G \to H$ and an equivariant map $f : X \to Y$. Then, there is morphism of groupoids $G\backslash X \to H\backslash Y$ that sends objects $x \mapsto f(x)$ and morphisms $(g,x) \mapsto (\phi(g), f(x))$.

The previous two examples explain that there is a functor from the category of group actions to **Grpd**.

Example 2.1.2. If $G \times Y \to Y$ is a free group action, there is an equivalence of groupoids between the quotient groupoid $G\backslash Y$, and the orbit set $\{G\text{-orbits in } Y\}$ viewed as a groupoid with no nontrivial morphisms. It is not an isomorphism, as the objects are different.

Example 2.1.3. Suppose $G \times X \to X$ is a transitive group action. Let $x_0 \in X$ be an arbitrary element and $\mathrm{Stab}(x_0) \subset G$ its stabilizer. There is an equivalence of groupoids between the quotient groupoid $G\backslash X$, and the one object groupoid $\mathrm{Stab}(x_0)\backslash\{x_0\}$. Again, it is not an isomorphism. This is a groupoid level enhancement of the fact that X is identified with the set of left cosets of $\mathrm{Stab}(x_0)$ in G.

A slightly more conceptually useful version of the previous two observations is the following:

Example 2.1.4. Suppose $G \times X \to X$ and $G \times Y \to Y$ are group actions such that G acts transitively on X. Let $x_0 \in X$ be an arbitrary element and $\mathrm{Stab}(x_0) \subset G$ its stabilizer. There is an equivalence of groupoids $\mathrm{Stab}(x_0)\backslash Y \to G\backslash(X \times Y)$

Proof. The forward map sends objects $y \mapsto (x_0, y)$ and morphisms

$$(s : y \to sy) \mapsto (s : (x_0, y) \to (x_0, sy)),$$

for all $s \in \mathrm{Stab}(x_0)$ and $y \in Y$. A suitable reverse map can be constructed after making some choices.

- Pick a section of the following map sets $G \to G/\mathrm{Stab}(x_0)$. This means picking a representative $g_i \in G$ for every left coset $i \in G/\mathrm{Stab}(x_0)$.

Because G acts transitively on X, this data provides provides for any $x \in X$, a canonical group element, g_i, such that $x = g_i x_0$. At the level of objects, the reverse map sends $(x,y) \mapsto (x_0, g_i^{-1} y)$. If $g : (x,y) \to (x', y')$ is a morphism, let g_i, g_j be the canonical representatives such that $x = g_i x_0$ and $x' = g_j x_0$. Then, the reverse map sends morphisms

$$(g : (x,y) \to (x', y')) \to (g_j^{-1} g g_i : g_i^{-1} y \to g_j^{-1} y').$$

Verify that $g_j^{-1} g g_i \in \mathrm{Stab}(x_0)$. I will leave at as an exercise to check that these maps define an equivalence of groupoids. □

Cartesian Diagrams of Groupoids (Why one needs 2-categories)

Now, I'll brief make some remarks about Cartesian diagrams in **Grpd** that will be useful for later geometric constructions.

Given groupoids $f : X \to Z$ and $g : Y \to Z$, one can check that their 1-categorical product, $X \times^1_Z Y$, is the following groupoid:

1. The objects are pairs of objects (x, y) where $x \in X$ and $y \in Y$, such that $f(x) = g(y)$.
2. The set of morphisms from (x, y) to (x', y') is the set of pairs (α, β) such that $\alpha : x \to x'$ is a morphism in X and $\beta : y \to y'$ is a morphism in Y with the additional property that $f(\alpha) = g(\beta)$. In other words,

$$\mathrm{Hom}((x,y),(x',y')) = \mathrm{Hom}(x,x') \times_{\mathrm{Hom}(z,z')} \mathrm{Hom}(y,y'),$$

   where $z = f(x) = g(y)$ and $z' = f(x') = g(y')$.

The following example illustrates a critical failure of 1-categorical products that is fixed by using more enriched objects.

Example 2.1.5 (Double Quotients and Classifying Spaces)**.** For subgroups $H_1, H_2 \subset G$, the following is a Cartesian diagram in groupoids:

$$\begin{array}{ccc} \mathrm{pt}/(H_1 \cap H_2) & \longrightarrow & \mathrm{pt}/H_1 \\ \downarrow & & \downarrow \\ \mathrm{pt}/H_2 & \longrightarrow & \mathrm{pt}/G \end{array}$$

However, for classifying spaces, one really wants $BH_1 \times_{BG} BH_2$ to be the double coset space $H_1 \backslash G / H_2$ (where H_2 acts through its inverse, so it remains a left action).

In the remainder of this thesis, a Cartesian diagram, pullback square, or fiber product, will all refer to a product in the following 2-categorical sense.

Definition 2.1.3 (2-Category of Groupoids)**.** The 2-category of groupoids, $\mathbf{Grpd}_2$, is the full subcategory of the 2-category of categories. The underlying 1-category is **Grpd** and the 2-morphisms are given by natural transformations.

The preceding definition isn't really necessary. It is only to motivate the following.

Definition 2.1.4 (Fiber Product of Groupoids)**.** Given groupoids $f : X \to Z$ and $g : Y \to Z$, their product $X \times_Z Y$ is the following groupoid:

1. The objects are triples (x, y, ϕ), where $x \in X$ and $y \in Y$ are objects and $\phi : f(x) \cong g(y)$ is an isomorphism in z.
2. The set of morphisms from (x, y, ϕ) to (x, y, ϕ') is the set of pairs (α, β) of morphisms $\alpha : x \to x'$ and $\beta : y \to y'$ such that $\phi' \cdot f(\alpha) = g(\beta) \cdot \phi$.

Functions on Groupoids

Counting Measure on Groupoids

Definition 2.1.5 (Functions on Groupoids)**.** Suppose X is a groupoid such that the isomorphism classes of objects of X form a set. For a field k, define $\mathrm{Fun}(X,k)$ be the set of k-valued functions on X. $\mathrm{Fun}(X,k)$ is a k-algebra under pointwise multiplication (which I denote $\otimes$). For a subset, S, of the set of isomorphism classes of objects of X, 1_S will denote the function that takes value one on $x \in S$ and zero otherwise. $\underline{1}$ is the constant function taking value one everywhere. The point of the decorated symbol is the distinguish from 1, which may represent a function that is a unit of a convolution algebra.

Definition 2.1.6 (Inner Product of Functions)**.** Given two functions $f, g \in \mathrm{Fun}(X,\mathbb{C})$, their inner product is:

$$\langle f, g\rangle = \sum_{x\in X} \overline{f(x)} g(x) \frac{1}{|\mathrm{Aut}(x)|}$$

The sum is over isomorphism classes of objects. The quotient is sometimes called the counting or motivic measure on the set of isomorphism classes.

Example 2.1.6 (Groupoid of Finite Sets)**.** The function theory $\mathrm{Fun}(\mathbf{FinSet},k)$ is the vector space of functions on $\mathbb{N}$. The total measure of the set of isomorphism classes of objects is

$$\langle \underline{1}, \underline{1}\rangle = \sum_{n} \frac{1}{|\mathrm{Aut}([n])|} = \exp(1)$$

The previous example is of no relevance to the situations considered in this thesis, but it illustrates that even very "large" groupoids have concrete and interesting function theories.

Operations with Functions

I will describe some constructions in **Grpd**. This subsection can be skipped and referenced later.

Definition 2.1.7 (Integration and Restriction)**.** Let $p : X \to Y$ be a map of groupoids.

1. The pullback map $p^* : \mathrm{Fun}(Y,k) \to \mathrm{Fun}(X,k)$ is defined by $(p^*f)(x) = f(p(x))$. Check that pullback commutes with pointwise multiplication of functions and sends $\underline{1}$ to $\underline{1}$. In particular, it is a k-algebra homomorphism.

2. Assume k is characteristic zero for simplicity. The pushforward $p_! : \mathrm{Fun}(X,k) \to \mathrm{Fun}(Y,k)$ is defined when for every object $y \in Y$, there are finitely many isomorphism classes x such that $p(x) \cong y$. In that case,

$$(p_! f)(y) = \sum_{x \xrightarrow{f} y} f(x) \frac{|\operatorname{Aut}(y)|}{|\operatorname{Aut}(x)|}$$

The sum is over objects $x \in X$, considered up to isomorphism, such that $f(x) \cong y$.

Proposition 2.1.1. If $p : X \to Y$ is an equivalence of groupoids, then p^* and $p_!$ are inverses. In particular $\operatorname{Fun}(X, k)$ and $\operatorname{Fun}(Y, k)$ are isomorphic k-algebras. When $k = \mathbb{C}$, the isomorphism respects the inner product.

In general, pushforward is defined in such a way that it is adjoint to pullback.

Proposition 2.1.2. Let $p : X \to Y$ be a map of groupoids, $f \in \operatorname{Fun}(X, \mathbb{C})$, and $g \in \operatorname{Fun}(Y, \mathbb{C})$.

$$\langle f, p^* g \rangle = \langle p_! f, g \rangle$$

A more general version of this principle is the following proposition.

Proposition 2.1.3 (Base Change)**.** Given a Cartesian diagram of groupoids as below, $q^* s_! = p_! r^*$ (assuming both pushforwards make sense).

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle q} \\ Z & \xrightarrow{s} & W \end{array}$$

Proof. For $f \in \operatorname{Fun}(Z, k)$ and objects $y \in Y$,

$$q^* s_!(y) = \sum_{z \xrightarrow{s} q(y)} f(z) \frac{|\operatorname{Aut}(q(y))|}{|\operatorname{Aut}(z)|} \tag{2.2}$$

The sum is over objects $z \in Z$, considered up to isomoprhism, such that $q(y) \cong s(z)$.

On the other hand,

$$p_! r^*(y) = \sum_{x \xrightarrow{p} y} f(r(x)) \frac{|\operatorname{Aut}(y)|}{|\operatorname{Aut}(x)|} \tag{2.3}$$

This time the sum is over objects $x \in X$, considered up to isomorphism such that $p(x) \cong y$. By the previous comments on fiber products of groupoids, this is the same as the data of a pair of objects $y' \in Y$ and $z \in Z$ along with an isomorphism $\phi : q(y') \cong s(z)$ such that $y' \cong y$. Up to isomorphism in X, one can assume that $y' = y$, and rewrite Equation 2.3 as follows.

$$p_! r^*(y) = \sum_{z \xrightarrow{s} q(y)} \sum_{\phi : q(y) \cong s(z)} f(z) \frac{|\operatorname{Aut}(y)|}{|\operatorname{Aut}(y, z, \phi)|} \tag{2.4}$$

The outer sum is over objects $z \in Z$, considered up to isomorphism, such that $q(y) \cong s(z)$. The inner sum is over isomorphisms $\phi : q(y) \cong s(z)$, but is worth spelling out the notion of equivalence that these are considered up to.

- $\phi \sim \phi'$ if $\phi' = \beta\phi\alpha^{-1}$, for $\alpha \in \mathrm{Aut}(y)$ and $\beta \in \mathrm{Aut}(z)$.

The automorphism group of $\mathrm{Aut}(y, z, \phi)$ consists of exactly those pairs $(\alpha, \beta) \in \mathrm{Aut}(y) \times \mathrm{Aut}(z)$ such that $\beta\phi\alpha^{-1} = \phi$. By orbit stabilizer with respect to the action of $\mathrm{Aut}(y)\times\mathrm{Aut}(z)$ on $\mathrm{Iso}(q(y), s(z))$, Equation 2.4 can be rewritten as follows.

$$p_! r^*(y) = \sum_{z \xrightarrow{s} q(y)} f(z) \frac{|\mathrm{Aut}(y)||\mathrm{Iso}(q(y), s(z))|}{|\mathrm{Aut}(y)||\mathrm{Aut}(z)|} \tag{2.5}$$

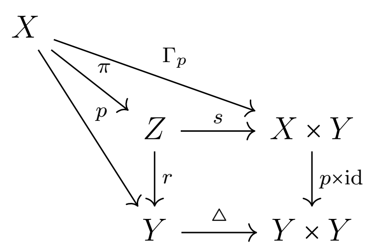
Finally, the equality of the RHS of Equations 2.2 and 2.5 follows after observing that $\mathrm{Iso}(q(y), s(z))$ is a torsor for $\mathrm{Aut}(q(y))$.

□

Remark 2.1.1. A special case of the base change formula that will be useful is that given $p : X \to Y$, $f \in \mathrm{Fun}(X, k)$, and $g \in \mathrm{Fun}(Y, k)$,

$$p_! f \otimes g = p_!(f \otimes p^* g)$$

Proof. Consider the following diagram, where the square is Cartesian.



The map $\Gamma_p : X \to X \times Y$ is the graph of p, given by $x \mapsto (x, p(x))$. The maps r and s are from the construction of the Cartesian product. Let $f \boxtimes g \in \mathrm{Fun}(X \times Y, k)$ be the function given by $f \boxtimes g(x, y) = f(x)g(y)$. Then,

$$p_! f \otimes g = \triangle^*(p_! \times \mathrm{id})(f \boxtimes g) = r_! s^*(f \boxtimes g).$$

On the other hand,

$$p_!(f \otimes p^* g) = p_! \triangle_p^* (f \boxtimes g) = r_! \pi_! \pi^* s^*(f \boxtimes g).$$

Therefore, it suffices to show that $\pi_! \pi^*$ is the identity. I argue that π is in fact an equivalence of groupoids. Z is described by the following data:

1. An object of Z is a triple $((x, y_1), y_2, \phi)$ of objects $(x, y_1) \in X \times Y$ and $y_2 \in Y$ along with an isomorphism $\phi : (p(x), y_1) \cong (y_2, y_2)$. Unwrapping this a bit further this is a tuple $(x, y_1, y_2, \phi_1, \phi_2)$ of objects $x \in X$ and $y_1, y_2 \in Y$ along with isomorphisms $\phi_1 : p(x) \cong y_2$ and $\phi_2 : y_1 \cong y_2$.

2. A morphism $(x, y_1, y_2, \phi_1, \phi_2) \to (x', y_1', y_2', \phi_1', \phi_2')$ consists of a triple (α, β, γ) of morphisms $\alpha : x \to x'$, $\beta : y_1 \to y_1'$, and $\gamma : y_2 \to y_2'$ such that $\phi_1' \cdot p(\alpha) = \gamma \cdot \phi_1$ and $\phi_2' \cdot \beta = \gamma \cdot \phi_2$.

Check that π is fully faithful and essentially surjective. For objects $x \in X$, $\pi(x)$ is the object $(x, p(x), p(x), \mathrm{id}, \mathrm{id}) \in Z$. For any other object $(x, y_1, y_2, \phi_1, \phi_2) \in Z$, the morphism $(\mathrm{id}, \phi_2^{-1}\phi_1, \phi_1)$ is from $\pi(x)$ to $(x, y_1, y_2, \phi_1, \phi_2)$. This shows π is essentially surjective. To see that it is fully faithful, observe that a morphism from $\pi(x)$ to $\pi(x')$ is given by a triple (α, β, γ), where $\alpha \in \mathrm{Hom}(x, x')$ and $\beta = \gamma = p(\alpha)$; in other words $\mathrm{Hom}(\pi(x), \pi(x')) = \mathrm{Hom}(x, x')$.

□

Example 2.1.7. A group homomorphism $H \hookrightarrow G$ induces a map $p : H\backslash \mathrm{pt} \to G\backslash \mathrm{pt}$. Identifying, $\mathrm{Fun}(H\backslash \mathrm{pt}, k) \cong k \cong \mathrm{Fun}(G\backslash \mathrm{pt}, k)$, $p^* = 1 \in \mathrm{End}(k)$ and $p_! = [G : H] \in \mathrm{End}(k)$.

Example 2.1.8 (Characters of a finite group)**.** If G is a finite group, then the space of functions on the adjoint quotient groupoid $\mathrm{Fun}(G\backslash G, \mathbb{C})$ is the space of class functions. Moreover, if $H \to G$ is a morphism of finite groups, and $p : H\backslash H \to G\backslash G$ is the corresponding map between their adjoint quotients, then characters of representations under induction and restriction are given by $p_!$ and p^*, respectively. This is not hard to prove; the key is to factor $H \to G$ as a surjection followed by an injection. Induction along a surjection is coinvariants of the kernel.

2.2 Finite Hecke Algebra

In this section let $G = \underline{G}(\mathbb{F}_q)$ and $B = \underline{B}(\mathbb{F}_q)$, for $\underline{G}$ a reductive algebraic group over $\mathbb{F}_q$ and $\underline{B} \subset \underline{G}$ a Borel subgroup. The finite Hecke algebra is usually presented as the algebra of B bi-invariant functions on G. I will take a slightly alternative approach and present it as a correspondence algebra. This is a well-known rephrasing of the standard approach that is appropriate for the present geometric context. I have not found it written explicitly in the literature.

Let $\mathcal{B} = \underline{\mathcal{B}}(\mathbb{F}_q) \cong G/B$. (see Appendix on group theory to review what this is). Bi-invariant functions on G are identified with functions on the groupoid $B\backslash G/B \cong B\backslash\mathcal{B}$. Furthermore, by Example 2.1.4 there is an equivalence of groupoids

$$B\backslash\mathcal{B} \to G\backslash(\mathcal{B} \times \mathcal{B}),$$

where G acts diagonally on the double flag variety. At the level of objects it is given by $B' \mapsto (B, B')$.

Define the finite Hecke algebra, $\mathcal{H}_{\text{fin}}$ as the algebra of equivariant correspondences in $\mathcal{B} \times \mathcal{B}$.

Definition 2.2.1 (Finite Hecke Algebra)**.** The following diagram defines an associative product on $\text{Fun}(G\backslash(\mathcal{B} \times \mathcal{B}))$.

$$\begin{array}{ccccc} G\backslash(\mathcal{B}\times\mathcal{B}) & \xleftarrow{\pi_{1,2}} & G\backslash(\mathcal{B}\times\mathcal{B}\times\mathcal{B}) & \xrightarrow{\pi_{1,3}} & G\backslash(\mathcal{B}\times\mathcal{B}) \\ & & \downarrow{\scriptstyle \pi_{2,3}} & & \\ & & G\backslash(\mathcal{B}\times\mathcal{B}) & & \end{array}$$

$$u \cdot v(B_1, B_2) = \pi_{1,3!}(\pi_{1,2}^*(u) \otimes \pi_{2,3}^*(v))(B_1, B_2)$$

Call this algebra $\mathcal{H}_{\text{fin}}$. It is associative because it is isomorphic to a subalgebra of the matrix algebra $\text{End}(\text{Fun}(\mathcal{B}))$.

Claim 2.2.1. $\mathcal{H}_{\text{fin}} \to \text{End}(\text{Fun}(\mathcal{B}))$ is a homomorphism of algebras.

Proof. For $f, g \in \text{Fun}(G\backslash(\mathcal{B} \times \mathcal{B}))$

$$f \cdot g(B_1.B_2) = \sum_{(x,y,z)\in\pi_{1,3}^{-1}(B_1,B_2)} f(x,y)g(y,z)\frac{|\text{Aut}\,(x,z)|}{|\text{Aut}(x,y,z)|}$$

The sum is over isomorphism classes of objects $(x, y, z) \in G\backslash(\mathcal{B} \times \mathcal{B} \times \mathcal{B})$ such that $(x, z) \cong (B_1, B_2)$ in $G\backslash(\mathcal{B} \times \mathcal{B})$. Therefore, in order to compute the sum it is sufficient to restrict to triples (B_1, y, B_2), but considered up to isomorphism. In particular, one may compute the following sum instead:

$$f \cdot g(B_1, B_2) = \sum_{y\in\text{Aut}(B_1,B_2)\backslash\mathcal{B}} f(B_1, y)g(y, B_2)\frac{|\text{Aut}\,(B_1, B_2)|}{|\text{Aut}(B_1, y, B_2)|}$$

The sum is now over isomorphism classes in the groupoid $\text{Aut}(B_1, B_2)\backslash\mathcal{B}$. By orbit-stabilizer, the size of the isomorphism class of y in this groupoid is exactly the quotient in the sum above. Therefore,

$$f \cdot g(B_1, B_2) = \sum_{y\in\mathcal{B}} f(B_1, y)g(y, B_2).$$

□

Remark 2.2.1. I hope the proof of the claim clarifies why the relative measure must be included when integrating functions. Another perspective is that associativity of the product follows from an argument relying on Proposition 2.1.3.

Before presenting the general structure theorem on the finite Hecke algebra, it is useful to understand two examples.

Examples: GL_2 and GL_3

$\underline{G} = \mathrm{GL}_2$

Identify $\mathcal{B} \cong \mathbb{P}^1(\mathbb{F}_q)$ and think of objects of $\mathcal{B}$ as lines $\ell \subset \mathbb{F}_q^2$. There are two isomorphism classes of objects $x = (\ell_1, \ell_2)$ of $G\backslash(\mathcal{B} \times \mathcal{B})$:

1. $\ell_1 = \ell_2$. Then, $\mathrm{Aut}(x) = \mathrm{Stab}_G(\ell_1) \cong B$.
2. $\ell_1 \neq \ell_2$. Then, $\mathrm{Aut}(x) = \mathrm{Stab}_G(\ell_1) \cap \mathrm{Stab}_G(\ell_2) \cong T$.

Let $1_{\ell_1=\ell_2}$ and $1_{\ell_1\neq\ell_2}$ denote the characteristic function of the isomorphism classes. Under the injection $\mathcal{H}_{\mathrm{fin}} \to \mathrm{End}(\mathrm{Fun}(\mathcal{B}))$, $1_{\ell_1=\ell_2}$ is sent to the identity operator. Furthermore, $1_{\ell=\ell_2} + 1_{\ell_1\neq\ell_2}$ is sent to the matrix all of whose entries are one. In particular,

$$(1_{\ell=\ell_2} + 1_{\ell_1\neq\ell_2})^2 = |\mathcal{B}|(1_{\ell=\ell_2} + 1_{\ell_1\neq\ell_2}).$$

Therefore,

$$\mathcal{H}_{\mathrm{fin}} \cong \mathbb{C}[T]/(T^2 - (q-1)T - q)$$

The identification is by $1_{\ell=\ell_2} \mapsto 1$ and $1_{\ell\neq\ell_2} \mapsto T$.

$\underline{G} = \mathrm{GL}_3$

Consider the natural identification $\underline{\mathcal{B}} \subset \mathrm{Gr}(1,3) \times \mathrm{Gr}(2,3)$ and think of objects of $\mathcal{B}$ as pairs (ℓ, p) consisting of a line and a plane $\ell \subset p \subset \mathbb{F}_q^3$. Before listing isomorphism classes in $G\backslash(\mathcal{B} \times \mathcal{B})$, I will name some subgroups that occur as automorphism groups of special configurations (namely the torus fixed points).

$$B = \mathrm{Stab}_G(\langle e_1\rangle, \langle e_1, e_2\rangle) = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

$$H_1 = B \cap \mathrm{Stab}_G(\langle e_2\rangle, \langle e_1, e_2\rangle) = \mathrm{Stab}_G(\langle e_1\rangle, \langle e_2\rangle, \langle e_1, e_2\rangle) = \begin{bmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

$$H_2 = B \cap \mathrm{Stab}_G(\langle e_1\rangle, \langle e_1, e_3\rangle) = \mathrm{Stab}_G(\langle e_1\rangle, \langle e_1, e_2\rangle, \langle e_1, e_3\rangle) = \begin{bmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$$

$$H_{21} = B \cap \mathrm{Stab}_G(\langle e_2\rangle, \langle e_2, e_3\rangle) = \mathrm{Stab}_G(\langle e_1\rangle, \langle e_2\rangle, \langle e_2, e_3\rangle) = \begin{bmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

$$H_{12} = B \cap \mathrm{Stab}_G(\langle e_3\rangle, \langle e_1, e_3\rangle) = \mathrm{Stab}_G(\langle e_1\rangle, \langle e_3\rangle, \langle e_1, e_2\rangle) = \begin{bmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$$

$$T = B \cap \mathrm{Stab}_G(\langle e_3\rangle, \langle e_2, e_3\rangle) = \mathrm{Stab}_G(\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle) = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$$

There are six isomorphism classes of object $x = ((\ell_1, p_2), (\ell_2, p_2))$ of $G\backslash(\mathcal{B} \times \mathcal{B})$:

1. $\ell_1 = \ell_2$ and $p_1 = p_2$. Then, $\mathrm{Aut}(x) = \mathrm{Stab}_G(\ell_1, p_1) \cong B$.
2. $\ell_1 \neq \ell_2$ and $p_1 = p_2$. Then, $\mathrm{Aut}(x) = \mathrm{Stab}_G(\ell_1, p_1, \ell_2) \cong H_1$.
3. $\ell_1 = \ell_2$ and $p_1 \neq p_2$. Then, $\mathrm{Aut}(x) = \mathrm{Stab}_G(\ell_1, p_1, p_2) \cong H_2$.
4. $\ell_2 \subset p_1$ and $\ell_1 \not\subset p_2$. Then, $\mathrm{Aut}(x) = \mathrm{Stab}_G(\ell_1, \ell_2, p_2) \cong H_{21}$.
5. $\ell_2 \not\subset p_1$ and $\ell_1 \subset p_2$. Then, $\mathrm{Aut}(x) = \mathrm{Stab}_G(\ell_1, p_1, \ell_2) \cong H_{12}$.
6. $\ell_2 \not\subset p_1$ and $\ell_1 \not\subset p_2$. Then, $\mathrm{Aut}(x) = \mathrm{Stab}_G(\ell_1, p_1, \ell_2, p_2) \cong T$.

For the same reason as before, the characteristic function of the first listed isomorphism class is the identity of $\mathcal{H}_{\mathrm{fin}}$. Label the characteristic functions of the other five classes, in order T_1, T_2, T_{21}, T_{12}, and T_3. The meaning of this labelling will become clear.

Claim 2.2.2. $T_1^2 - (q-1)T_1 - q = 0$ and $T_2^2 - (q-1)T_2 - q = 0$.

Proof. I'll prove the first claim and leave the second as an exercise.

$$T_1 \cdot T_1((\ell_1, p_1), (\ell_2, p_2)) = \sum_{*} \frac{|\,\mathrm{Aut}((\ell_1, p_1), (\ell_2, p_2)|}{|\,\mathrm{Aut}((\ell_1, p_1), (\ell, p), (\ell_2, p_2))|}$$

The sum is over isomorphism classes of objects $((\ell_1, p_1), (\ell, p), (\ell_2, p_2)) \in G\backslash(\mathcal{B} \times \mathcal{B} \times \mathcal{B})$ satisfying the following conditions, which I have abbreviated as $(*)$:

1. $\ell_1 \neq \ell$ and $p_1 = p$
2. $\ell \neq \ell_2$ and $p = p_2$

I claim that there are there two isomorphism classes of objects satisfying these conditions. There is an element of G sending the common plane $p_1 = p = p_2$ to $\langle e_1, e_2 \rangle$. If ℓ_1, ℓ, ℓ_2 are pairwise distinct, there is a further automorphism of $\langle e_1, e_2 \rangle$ the images of ℓ_1, ℓ, ℓ_2 to $\langle e_1 \rangle$, $\langle e_1 + e_2 \rangle$, $\langle e_2 \rangle$, respectively (this is because $\mathrm{Aut}(\mathbb{P}^1)$ acts transitively on triples of distinct lines). Then, the isomorphism class of $((\ell_1, p_1), (\ell_2, p_2))$ is the second of the listed classes and

$$\frac{|\mathrm{Aut}((\ell_1, p_1), (\ell_2, p_2)|}{|\mathrm{Aut}((\ell_1, p_1), (\ell, p), (\ell_2, p_2))|} = \frac{|\mathrm{Stab}_{\mathrm{SL}_2}(\langle e_1 \rangle, \langle e_2 \rangle)|}{|\mathrm{Stab}_{\mathrm{SL}_2}(\langle e_1 \rangle, \langle e_1 + e_2 \rangle, \langle e_2 \rangle)|}$$

$$= \left|\mathrm{Orbit}_{\mathrm{Stab}_{\mathrm{SL}_2}(\langle e_1 \rangle, \langle e_2 \rangle)}(\langle e_1 + e_2 \rangle)\right| = \left|\mathbb{P}^1(\mathbb{F}_q) \smallsetminus \{\langle e_1 \rangle, \langle e_2 \rangle\}\right| = q - 1.$$

On the other hand, if $\ell_1 = \ell_2$, then there is an isomorphism sending $\ell_1 = \ell_2$ to $\langle e_1 \rangle$ and ℓ to $\langle e_2 \rangle$.T he isomorphism class of $((\ell_1, p_1), (\ell_2, p_2))$ is the first of the listed classes and

$$\frac{|\mathrm{Aut}((\ell_1, p_1), (\ell_2, p_2)|}{|\mathrm{Aut}((\ell_1, p_1), (\ell, p), (\ell_2, p_2))|} = \frac{|\mathrm{Stab}_{\mathrm{SL}_2}(\langle e_1 \rangle)|}{|\mathrm{Stab}_{\mathrm{SL}_2}(\langle e_1 \rangle, \langle e_2 \rangle)|}$$

$$= \left|\mathrm{Orbit}_{\mathrm{Stab}_{\mathrm{SL}_2}(\langle e_1 \rangle)}(\langle e_2 \rangle)\right| = \left|\mathbb{P}^1(\mathbb{F}_q) \smallsetminus \{\langle e_1 \rangle\}\right| = q.$$

□

It is no coincidence that T_1 and T_2 satisfy the same relation as T in the previous example. This will become clear after considering general groups.

Claim 2.2.3. $T_2 \cdot T_1 = T_{21}$ and $T_1 \cdot T_2 = T_{12}$.

Proof. I'll only prove the first claim and leave the second as an exercise. The argument is essentially the same.

$$T_2 \cdot T_1((\ell_1, p_1), (\ell_2, p_2)) = \sum_{**} \frac{|\mathrm{Aut}((\ell_1, p_1), (\ell_2, p_2)|}{|\mathrm{Aut}((\ell_1, p_1), (\ell, p), (\ell_2, p_2))|}$$

The sum is over isomorphism classes of objects $((\ell_1, p_1), (\ell, p), (\ell_2, p_2)) \in G\backslash(\mathcal{B} \times \mathcal{B} \times \mathcal{B})$ satisfying the following conditions, which I have abbreviated as $(**)$:

1. $\ell_1 \neq \ell$ and $p_1 = p$

2. $\ell = \ell_2$ and $p \neq p_2$

Given any such triple, pick a splitting $p_2 \cong \ell_2 \oplus \ell_3$. Because $p \neq p_2$ and $p \cong \ell \oplus \ell_1 = \ell_2 \oplus \ell_1$, it follows that $\ell_1 \neq \ell_3$; in particular, there is a group element sending ℓ_1, ℓ_2, ℓ_3 to $\langle e_1 \rangle, \langle e_2 \rangle, \langle e_3 \rangle$. It follows that the isomorphism class of $((\ell_1, p_1), (\ell_2, p_2))$ is the fourth of the listed classes. Moreover, the data (ℓ, p) and is determined by (ℓ_1, p_1) and (ℓ_2, p_2), so the quotient in the sum, which records the relative stabilizer, is one. It follows that

$$T_2 \cdot T_1 = T_{21}.$$

□

Claim 2.2.4. $T_1 \cdot T_{21} = T_3$ and $T_2 \cdot T_{12} = T_3$.

Proof. Again I'll prove the first claim and leave the second as an exercise.

$$T_1 \cdot T_{21}((\ell_1, p_1), (\ell_2, p_2)) = \sum_{***} \frac{|\operatorname{Aut}((\ell_1, p_1), (\ell_2, p_2)|}{|\operatorname{Aut}((\ell_1, p_1), (\ell, p), (\ell_2, p_2))|}$$

The sum is over isomorphism classes of objects $((\ell_1, p_1), (\ell, p), (\ell_2, p_2)) \in G \backslash (\mathcal{B} \times \mathcal{B} \times \mathcal{B})$ satisfying the following conditions, which I have abbreviated as $(***)$:

1. $\ell \subset p_1$ and $\ell_1 \notin p$

2. $\ell \neq \ell_2$ and $p = p_2$

Given any such triple, I will argue that $\ell_2 \notin p_1$ and $\ell_1 \notin p_2$ (equivalently, that the isomorphism class of $((\ell_1, p_1), (\ell_2, p_2))$ is the fourth of the listed classes).

ℓ_2 is contained in p_2 and therefore also in p. $p \neq p_1$, so their intersection contains a unique line, which must be ℓ; therefore, because ℓ_2 is not equal to ℓ it must be that $\ell_2 \notin p_1$.

ℓ_1 is not contained in p and therefore not in p_2.

Finally, the data (ℓ, p) and is determined by (ℓ_1, p_1) and (ℓ_2, p_2) because $p = p_2$ and $\ell = p \cap p_1$. Therefore, the quotient in the sum, which records the relative stabilizer, is one. It follows that

$$T_1 \cdot T_{21} = T_3.$$

□

Therefore, in this case, $\mathcal{H}_{\text{fin}}$ is generated by T_1 and T_2 with the relations

1. $T_i^2 - (q-1)T_i - q = 0$ for $i = 1, 2$

2. $T_1 T_2 T_1 = T_2 T_1 T_2$

In particular, the $q = 1$ degeneration produces the Weyl group S_3.

General Groups

First, I'll remark that $\mathcal{H}_{\text{fin}}$ only depends on the Dynkin diagram. If $\underline{G} \to \underline{G}_0$ is a homomorphism with central kernel, $\underline{Z}$, then the groups have the same flag variety and $G\backslash\mathcal{B} \cong (G_0\backslash\mathcal{B}) \times (Z\backslash\text{pt})$. The convolution diagram in Definition 2.2.1 for $\underline{G}$ and $\underline{G}_0$ are the same up to equivalence of groupoids.

Now let $\underline{G}$ be an arbitrary reductive group scheme over $\mathbb{F}_q$. As before pick a Borel subgroup $\underline{B} \subset \underline{G}$. Now, also pick a splitting of the universal Cartan $\underline{T} := \underline{B}/[\underline{B},\underline{B}]$ into the Borel $\underline{T} \hookrightarrow \underline{B}$. The Weyl group is the quotient $W := N_{\underline{G}}(\underline{T})/Z_{\underline{G}}(\underline{T})$. Recall the Bruhat decomposition

$$G = \bigsqcup_{w \in W} B\dot{w}B$$

The formula indicates a choice of lift $\dot{w} \in N_G(T)$ for each Weyl group element, which may not be canonical. However, the double coset $B\dot{w}B$ doesn't depend on the choice of lift because $Z_G(T) \subset B$. By the Bruhat decomposition, isomorphism classes in $B\backslash\mathcal{B}$, and therefore in $G\backslash(\mathcal{B}\times\mathcal{B})$, are in bijection with w. Let $T_w \in \mathcal{H}_{\text{fin}}$ denote the characteristic function of the isomorphism class of $(B, \dot{w}^{-1}B)$ [1].

Theorem 2.2.1 (Finite Hecke Algebra Structure Theorem)**.** The product in $\mathcal{H}_{\text{fin}}$ is given as follows. For $s, w \in W$ with s a simple reflection,

$$T_sT_w = \begin{cases} T_{sw} & \ell(sw) > \ell(w) \\ qT_{sw} + (q-1)T_w & \ell(sw) < \ell(w) \end{cases}$$

The result follows from two claims.

Claim 2.2.5. For $w_1, w_2 \in W$ with $\ell(w_2w_1) = \ell(w_2) + \ell(w_1)$,

$$T_{w_2}T_{w_1} = T_{w_2w_1}.$$

Claim 2.2.6. For simple reflection $s \in W$,

$$T_s^2 - (q-1)T_s - q = 0.$$

In order to prove Claim 2.2.5, I'll use the following key fact from group theory.

Proposition 2.2.1. The $\underline{B}$-orbit of $\underline{\mathcal{B}}$ corresponding to w is an affine space of dimension $\ell(w)$.

By orbit-stabilizer, this shows that the automorphism group of the isomorphism class corresponding to w in $B\backslash\mathcal{B}$ (and therefore also in $G\backslash(\mathcal{B}\times\mathcal{B})$) has order $|B|/q^{\ell(w)}$.

[1] There is an inverse because humans read the convolution diagram 2.2.1 in the opposite direction that they compose matrices.

Proof of Claim 2.2.5. For objects $(B_1, B_2) \in G\backslash(\mathcal{B}\times\mathcal{B})$,

$$T_{w_2}\cdot T_{w_1}(B_1,B_2) = \sum_{*} \frac{|\operatorname{Aut}(x,z)|}{|\operatorname{Aut}(x,y,z)|}.$$

The sum is over isomorphic classes of triples (x,y,z) such that $(x,y)\cong(B,\dot{w}_1^{-1}B)$ and $(y,z)\cong(B,\dot{w}_2^{-1}B)$, where $\dot{w}_1,\dot{w}_2\in N_G(T)$ of the Weyl group element. The product $\dot{w}_2\dot{w}_1\in N_G(T)$ is also a lift of $w_2w_1\in W$; in particular, $(B,\dot{w}_1^{-1}B,\dot{w}_1^{-1}\dot{w}_2^{-1}B)$ is one such isomorphism classes, where $(x,z)\cong(B,(\dot{w}_2\dot{w}_1)^{-1}B)$. Because all the terms in the sum are nonnegative integers, it follows that

$$T_{w_2}\cdot T_{w_1} = T_{w_2w_1} + A,$$

where A is a nonnegative integral combination of the characteristic functions T_w. In order to prove the claim it suffices to show that

$$\langle T_{w_2}\cdot T_{w_1},\underline{1}\rangle = \langle T_{w_2w_1},\underline{1}\rangle$$

By Proposition 2.2.1,

$$\langle T_{w_2w_1},\underline{1}\rangle = \frac{1}{|\operatorname{Aut}(B,(w_2w_1)^{-1}B)|} = q^{\ell(w_2w_1)}/|B|$$

To compute the other inner product, consider the following diagram.

$$\begin{array}{ccccc} & & G\backslash(\mathcal{B}\times\mathcal{B}\times\mathcal{B}) & \xrightarrow{\pi_{1,3}} & G\backslash(\mathcal{B}\times\mathcal{B}) \\ & & \downarrow{\scriptstyle p} & & \\ G\backslash(\mathcal{B}\times\mathcal{B}) & \xleftarrow[\pi_L]{} & (G\backslash(\mathcal{B}\times\mathcal{B}))\times(G\backslash(\mathcal{B}\times\mathcal{B})) & \xrightarrow{\pi_R} & G\backslash(\mathcal{B}\times\mathcal{B}) \end{array}$$

This is essentially a reorganization of diagram 2.2.1. The map p is induced by $\pi_{1,2}$ and $\pi_{2,3}$. Using Proposition 2.1.2 with respect to $\pi_{1,3}$,

$$\langle T_{w_2}\cdot T_{w_1},\underline{1}\rangle = \langle \pi_{1,2}^*T_{w_1}\otimes\pi_{2,3}^*T_{w_2},\pi_{1,3}^*\underline{1}\rangle = \langle p^*\pi_L^*T_{w_1}\otimes p^*\pi_R^*T_{w_2},\underline{1}\rangle = \langle p^*\pi_L^*T_{w_1}\otimes p^*\pi_R^*T_{w_2},p^*\underline{1}\rangle$$

Then, using Proposition 2.1.2 and Remark 2.1.1 both with respect to the map p,

$$\langle p^*\pi_L^*T_{w_1}\otimes p^*\pi_R^*T_{w_2},p^*\underline{1}\rangle = \langle p_!(p^*\pi_L^*T_{w_1}\otimes p^*\pi_R^*T_{w_2}),\underline{1}\rangle = \langle p_!p^*\pi_L^*T_{w_1}\otimes\pi_R^*T_{w_2},\underline{1}\rangle$$

The operator $p_!p^*$ essentially measures the fibers of p.

Lemma 2.2.1. For $f\in\operatorname{Fun}((G\backslash(\mathcal{B}\times\mathcal{B}))\times(G\backslash(\mathcal{B}\times\mathcal{B})))$, $p_!p^*f = |B|f$.

Proof. For objects $x \in (G\backslash(\mathcal{B}\times\mathcal{B}))\times(G\backslash(\mathcal{B}\times\mathcal{B}))$,

$$p_! p^* f(x) = f(x)|\operatorname{Aut}(x)| \sum_{y \xrightarrow{p} x} \frac{1}{|\operatorname{Aut}(y)|}$$

The sum is over isomorphism classes of objects, y, such that $p(y) \cong x$. The sum can be reinterpreted as a count of objects.

$$\sum_{y \xrightarrow{p} x} \frac{1}{|\operatorname{Aut}(y)|} = \frac{1}{|G|}\big|\{y : p(y) \cong x\}\big|$$

The count on the RHS is of objects, rather than isomorphism classes. Let $x = (x_1, x_2)$ for objects $x_1, x_2 \in G\backslash(\mathcal{B}\times\mathcal{B})$. Count objects $y = (B_1, B_2, B_3)$, where $(B_1, B_2) \cong x_1$ and $(B_2, B_3) \cong x_2$. Without loss of generality assume $B_2 = B$ and multiply the count by $|\mathcal{B}|$. The choices of B_1 and B_3 are then $B/\operatorname{Aut}(x_1)$ and $B/\operatorname{Aut}(x_2)$, respectively. Therefore,

$$\big|\{y : p(y) \cong x\}\big| = |\mathcal{B}|\frac{|B|^2}{|\operatorname{Aut}(x_1)||\operatorname{Aut}(x_2)|}$$

The claim follows after observing that $\operatorname{Aut}(x) \cong \operatorname{Aut}(x_1)\times\operatorname{Aut}(x_2)$. □

Remark 2.2.2. Another way to prove the Lemma is to consider the following map of groupoids.

$$(B\backslash G)\times_B (G/B) \to (B\backslash G/B)\times(B\backslash G/B)$$

This map is the same as p up to replacing the source and target by equivalent (non-isomorphic) groupoids. It is more readily apparent that $p_! p^* f = |B| f$ in this setting.

Using the lemma,

$$\begin{aligned}\langle p_! p^* \pi_L^* T_{w_1} \otimes \pi_R^* T_{w_2}, \underline{1}\rangle &= |B|\langle \pi_L^* T_{w_1} \otimes \pi_R^* T_{w_2}, \underline{1}\rangle \\ &= |B|\frac{1}{|\operatorname{Aut}(B, w_1^{-1}B)|}\frac{1}{|\operatorname{Aut}(B, w_2^{-1}B)|} = \frac{q^{\ell(w_1)+\ell(w_2)}}{|B|}\end{aligned}$$

I've used that isomorphism classes in the product groupoid $(G\backslash(\mathcal{B}\times\mathcal{B}))\times(G\backslash(\mathcal{B}\times\mathcal{B}))$ are pairs of isomorphism classes of $G\backslash(\mathcal{B}\times\mathcal{B})$, and that automorphism groups are identified accordingly.

Finally, the claim follows because $\ell(w_2 w_1) = \ell(w_2) + \ell(w_1)$. □

The proof of Claim 2.2.6 is by reduction to the GL_2 (technically, SL_2) example.

Proof of Claim 2.2.6. Let $s = s_\alpha$ for some simple root α and let $\underline{P_\alpha} \supset \underline{B}$ be the corresponding almost minimal parabolic subgroup. There is a corresponding partial flag variety $\underline{\mathcal{P}}_\alpha \cong \underline{G}/\underline{P}_\alpha$ along with a forgetful map $\underline{\mathcal{B}} \to \underline{\mathcal{P}}_\alpha$. Consider the map

$$p_\alpha : G\backslash(\mathcal{B}\times\mathcal{B}) \to G\backslash(\mathcal{B}\times\mathcal{P}_\alpha).$$

At the level of objects, the map is identity in the first factor and forgetting in the second factor. The key technical point to proving the claim is that for $f \in \mathcal{H}_{\text{fin}}$,

$$(1+T_{s_\alpha})f = p_\alpha^* p_{\alpha!} f$$

I will prove this assertion by interpreting both sides as the convolution of f against a specific kernel in $G\backslash(\mathcal{B}\times\mathcal{B}\times\mathcal{B})$.

Let Y denote the full subcategory of $G\backslash(\mathcal{B}\times\mathcal{B})$ consisting of objects (B_1, B_2) whose reductions to $\mathcal{P}_\alpha$ coincide. Let X denote the full subcategory of $G\backslash(\mathcal{B}\times\mathcal{B}\times\mathcal{B})$ consisting of objects (B_1, B_2, B_3) such that the reductions of B_2 and B_3 to $\mathcal{P}_\alpha$ coincide. In particular, there is the following diagram where both squares are Cartesian.

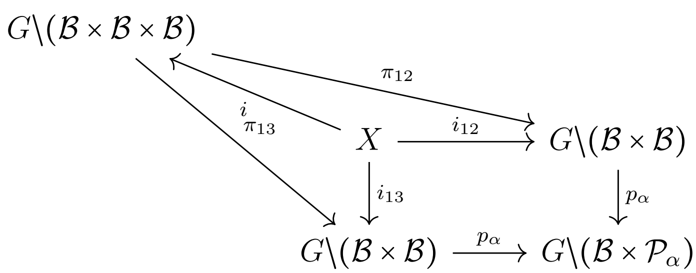
$$\begin{array}{ccc}
X & \xrightarrow{\;i\;} & G\backslash(\mathcal{B}\times\mathcal{B}\times\mathcal{B}) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi_{23}} \\
Y & \xrightarrow{\;j\;} & G\backslash(\mathcal{B}\times\mathcal{B}) \\
\downarrow & & \downarrow \\
G\backslash\mathcal{P}_\alpha & \xrightarrow{\;\triangle\;} & G\backslash(P_\alpha\times P_\alpha)
\end{array}$$

$1+T_{s_\alpha}$ is the characteristic function of the locus Y in $G\backslash(\mathcal{B}\times\mathcal{B})$. In particular,

$$(1+T_{s\alpha})f = \pi_{13!}(\pi_{12}^* f \otimes \pi_{23}^* j_! \underline{1}_Y) = \pi_{13!}(\pi_{12}^* f \otimes i_! \underline{1}_X)$$

In particular, $(1+T_{s_\alpha})$ is convolution against the kernel $i_!\underline{1}_X$.

Now consider the following diagram, where X is as before.



The square is Cartesian, so

$$p_\alpha^* p_{\alpha!} f = i_{13!} i_{12}^* f = \pi_{13!} i_! i^* \pi_{12}^* f = \pi_{13!} i_!(i^* \pi_{12}^* f \otimes \underline{1}_X) = \pi_{13!}(\pi_{12}^* f \otimes i_! \underline{1}_X)$$

This identifies $p_\alpha^* p_{\alpha!}$ with convolution against the kernel $i_!\underline{1}_X$, as well.

It remains to show that the following is true.

$$p_\alpha^* p_{\alpha!} p_\alpha^* p_{\alpha!} f = (q+1) p_\alpha^* p_{\alpha!}$$

However, similar to as in Lemma 2.2.1, $p_{\alpha!} p_\alpha^*$ measures fibers of p_α, which have total size (counted with respect to the groupoid measure) $|\mathcal{B}|/|\mathcal{P}_\alpha| = q+1$.

□

Chapter 3

Vector Bundles on Curves

In preparation for studying the action of Hecke correspondences on parabolic bundles, I will explain some basic results on vector bundles over a curve, $X/\mathbb{F}_q$ and their local modifications. I won't prove everything because for the purposes of this thesis it is more important to understand a few examples. Everything is well documented in the literature and I'll briefly point out how many of the arguments of Chapter 2 can be modified for the results presented here.

3.1 Loop Group Preliminaries

Let $\underline{G}$ be a split reductive group scheme, $\underline{B} \subset \underline{G}$ a Borel subgroup, and $\underline{T} := \underline{B}/[\underline{B}, \underline{B}]$ the universal torus (doesn't depend on the choice of $\underline{B}$). Define $G((t)) := \mathrm{Map}(\mathrm{Spec}(k((t))), \underline{G})$ as the loop group and $G[[t]] := \mathrm{Map}(\mathrm{Spec}(k[[t]]), \underline{G})$ as the subgroup of maps that extend over the origin. Let $\Lambda := \mathrm{Hom}(\mathbb{G}_m, \underline{T})$ be the lattice of coweights and let $\Lambda_R \subset \Lambda$ be the sublattice generated by coroots. There is a short exact sequence of Abelian groups.

$$0 \to \Lambda_R \to \Lambda \to \pi_1(\underline{G}) \to 0$$

The affine Weyl group, W^{aff} is the group of affine linear automorphisms of Λ. [1] Naturally, W^{aff} factors into translations and the stabilizer of $0 \in \Lambda$.

$$W^{aff} \cong \Lambda \rtimes W$$

There is also a factorization

$$W^{aff} \cong W^{aff,R} \rtimes \Lambda/\Lambda_R$$

where $W^{aff,R} \cong \Lambda_R \rtimes W \subset W^{aff}$ is the subgroup that sends the coroot lattice to itself. Evidently, the difference between these lattices is controlled by the center of $\underline{G}$. $W^{aff,R}$ is

[1] In most literature this is called the extended affine Weyl group, but I do not like that name.

important because it is generated by affine simple reflections. Given a root α, there is a reflection $s_\alpha : \Lambda \to \Lambda$ given by

$$s_\alpha(\lambda) = \lambda - \langle \alpha, \lambda \rangle \check{\alpha}$$

There is an affine reflection given by

$$s_0(\lambda) = s_{\alpha_0}(\lambda) - \check{\alpha}_0,$$

where α_0 is the positive long root. $W^{aff,R}$ is generated by simple reflections s_α for positive simple roots α and s_0.

Example 3.1.1 ($\underline{G} = \mathrm{SL}_2$)**.** $\Lambda \cong \mathbb{Z}$ is generated by $\check{\alpha} := t \mapsto \mathrm{diag}(t, t^{-1})$. This is also a coroot because it pairs to two with the simple root $\mathrm{diag}(a, b) \mapsto ab^{-1}$ The simple reflection corresponding to the unique simple root is $\lambda \mapsto -\lambda$. The affine simple reflection is $\lambda \mapsto -\lambda - \check{\alpha}$. This is reflection about $-1/2$ times the generator of Λ. There is no difference between Λ and Λ_R.

Example 3.1.2 ($\underline{G} = \mathrm{PGL}_2$)**.** $\Lambda \cong \mathbb{Z}$ generated by $\frac{1}{2}\check{\alpha} := t \mapsto \mathrm{diag}(t, 1)$. $\check{\alpha}$ is gain a coroot because it pairs to two with the simple root $\mathrm{diag}(a, b) \mapsto ab^{-1}$ The simple reflection corresponding to the unique simple root is $\lambda \mapsto -\lambda$. The affine simple reflection is $\lambda \mapsto -\lambda - \check{\alpha}$. This is reflection about -1 times the generator. $\Lambda_R \subset \Lambda$ is index two.

Example 3.1.3 ($\underline{G} = \mathrm{GL}_n$)**.** $\Lambda \cong \mathbb{Z}^n$ by

$$(t \mapsto \mathrm{diag}(t^{a_1}, .., t^{a_n})) \mapsto (a_1, .., a_n)$$

The coroot $\check{\alpha}_i$ for $1 \leq i \leq n-1$ is given by the tuple who's ith coordinate is 1 and whose $i+1$th coordinate is -1. The corresponding simple reflection exchanges the ith and $i+1$th coordinates. $\Lambda_R \subset \Lambda$ is the sublattice of tuples whose coordinates sum to zero. The affine simple reflection is given by

$$s_0(a_1, ..., a_n) = (a_n - 1, ..., a_1 + 1)$$

Let $I \subset G((t))$ denote the Iwahori subgroup which is the inverse image of B under evaluation at $t = 0$, $\mathrm{ev} : G[[t]] \to G$. The Iwahori subgroup essentially functions as a minimal parabolic subgroup of the loop group.

Analogous to the finite situation, W^{aff}, admits a presentation as $N_{G((t))}(T[[t]])/T[[t]]$. In particular, group elements $w \in W^{\mathrm{aff}}$ admit non-canonical $\dot{w} \in N_{G((t))}(T[[t]])$. There is a Bruhat decomposition

$$G((t)) = \bigsqcup_{w \in W^{\mathrm{aff}}} I\dot{w}I$$

The double coset $I\dot{w}I$ is independent of the chosen lift because $T[[t]] \subset I$.

There is also a parabolic analogue of the Bruhat decomposition that will be useful in the affine case.

$$G(((t)) = \bigsqcup_{\lambda \in W\backslash W^{\text{aff}}/W} G[[t]]\dot{\lambda}G[[t]]$$

Note that $W^{\text{aff}}/W \cong \Lambda$ because W^{aff} acts transitively on Λ and the stabilizer of 0 is W. Therefore, $W\backslash W^{\text{aff}}/W \cong W\backslash\Lambda$ is identified with the set of dominant coweights.

3.2 Affine Grassmanian and Affine Flags

I'll introduce important geometric objects built from the loop group that control local modifications of curves.

Affine Grassmanian

Define the affine Grassmanian, Gr, as the set of cosets $G((t))/G[[t]]$. There is an ind-scheme, $\underline{\mathrm{Gr}}$, over $\mathbb{F}_q$ whose set of rational points is Gr. It is probably a more appropriate object to call the affine Grassmanian, but it is not necessary for this thesis so I won't use it. However, the following moduli interpretation is stil useful:

$$\mathrm{Gr} \cong \{(\text{trivial})\ \underline{G}\text{-bundle on } \mathrm{Spec}(\mathbb{F}_q[[t]]) \text{ with a trivialization on } \mathrm{Spec}(\mathbb{F}_q((t)))\}$$

The left action $G((t)) \times \mathrm{Gr} \to \mathrm{Gr}$ modifies the trivialization away from the origin. Because all bundles on $\mathrm{F}_q[[t]]$ are trivial, the $G((t))$ action is transitive and the element corresponding to a trivialization that extends over the origin is stablized by $G[[t]]$.[2]

Example 3.2.1 ($\underline{G} = \mathrm{GL}_n$)**.** There is a $G((t))$ equivariant identification.

$$\mathrm{Gr} \cong \{\Lambda \subset \mathbb{F}_q((t))^n : \Lambda \text{ is a free rank } n\ \mathbb{F}_q[[t]] \text{ module}\}$$

It is given by sending the coset of $1 \in G((t))$ to the module $\mathbb{F}_q[[t]]^n \subset \mathbb{F}_q((t))^n$, which has stabilizer $G[[t]]$. Such submodules Λ are sometimes called lattices, in analogy to the situation over an Archimedean local field where lattices in $\mathbb{R}^n$ full rank $\mathbb{Z}$-submodules.

Example 3.2.2 ($\underline{G} = \mathrm{PGL}_n$)**.** In this case, there is a $G((t))$ equivariant identification

$$\mathrm{Gr} \cong \mathrm{Gr}_{\mathrm{GL}_n}/(\Lambda \sim t\Lambda)$$

[2] This was very hard for me to understand the first time. Another thing one can do is fix a trivial bundle over the formal disk. Then the choice of $G((t))$ is a section away from the origin, which gives a trivalization away from the origin. However, the quotient by $G[[t]]$ indicates that the original trivialization of a bundle on the formal disk was a choice.

Example 3.2.3 ($\underline{G} = \mathrm{SL}_n$)**.** In this case, there is a $G((t))$ equivariant identification

$$\mathrm{Gr} \cong \{\Lambda \in \mathrm{Gr}_{\mathrm{GL}_n} : \mathrm{val}_t(\det\Lambda) = 0\}$$

$\det(\Lambda)$ is the determinant of the matrix formed by generators of Λ over $\mathbb{F}_q[[t]]$. It is not well defined as an element of $\mathbb{F}_q((t))$, as the generating matrix is only defined up to $\mathrm{GL}_n(\mathbb{F}_q[[t]])$. However, the order of vanishing at $t = 0$ is.

The Bruhat decomposition from the previous section implies that the $G[[t]]$ orbits of Gr in bijection with Λ_+, the set of dominant weights [3]. Moreover, there is a "spherical" length function $\ell_{\mathrm{sph}} : \Lambda_+ \to \mathbb{Z}$ given by the minimal length of the corresponding double coset in $W\backslash W^{\mathrm{aff}}/W$.

Example 3.2.4 ($\underline{G} = \mathrm{GL}_n$)**.** Using the notation of Example 3.1.3, dominant coweights are identified with tuples $\lambda = (a_1, ..., a_n)$ such that $a_1 \geq a_2 \geq ... \geq a_n$. The orbit of λ is characterized as follows. The lattice $\Lambda_0 := \mathbb{F}_q[[t]]^n$ is invariant fixed by $G[[t]]$. Therefore, for any integer k, the following quantity is fixed under $G[[t]]$.

$$d_k = \dim_{\mathbb{F}_q}(\Lambda/(\Lambda \cap t^k\Lambda_0)) + \dim_{\mathbb{F}_q}(t^k\Lambda_0/(\Lambda \cap t^k\Lambda_0))$$

Note that if one replaces minus in the expression on the right with a plus, then the quantity only depends on $\mathrm{val}_t(\det\Lambda)$.

Plotting (k, d_k) one obtains a parabola carrying some information in its bulk. For $k << 0$, $d_{k+1} - d_k = -n$ and for $k >> 0$, $d_{k+1} - d_k = n$. $\Lambda \in G[[t]]\lambda$ if and only if

$$d_{k+1} - d_k \geq -n + 2i \iff k \geq a_i.$$

Example 3.2.5 (Regular Tree ($\underline{G} = \mathrm{PGL}_2$))**.** This example fascinated me when I first learned it. It is deeply related to the spectral theory of regular graphs and explains why the local Langlands correspondence says something about graphs.

Following Example 3.1.2, the dominant coweights are identified with the set of nonnegative integers. The $G[[t]]$ orbit in Gr corresponding to k has size 1 if $k = 0$ and size $(q+1)q^{k-1}$ otherwise [4]. Moreover, there is an underlying combinatorial structure. Given $\Lambda_1, \Lambda_2 \in \mathrm{Gr}$, define the relation $\Lambda_1 \sim \Lambda_2$ to mean that there are lattice representatives such that

$$t\Lambda_1 \subsetneq \Lambda_2 \subsetneq \Lambda_1$$

Check that the relation is symmetric. There is a graph (in the combinatorial sense) whose vertices are elements of Gr and whose edges are the relation. The neighborhood of Λ is identified, after picking a lattice representative, with $\mathbb{P}(\Lambda/t\Lambda)$ which has cardinality $q + 1$. This graph associated to Gr is isomorphic to the connected $(q+1)$-regular tree. If one roots

[3] In fact, as in the finite case, there is a distinguished $T[[t]]$? fixed representative in each orbit.

[4] In general $G[[t]]$ orbits $\underline{\mathrm{Gr}}$ are affine space bundles over partial flag varieties. The $(q+1)$ factor comes from the cardinality of the flag variety.

it at $\Lambda_0 := \mathbb{F}_q[[t]]^2$, then the $G[[t]]$ orbit labelled by k consists of vertices that are distance k from the root. [5]

Affine Flag Variety

Define the affine flag variety, Fl, as the set of cosets $G((t))/I$. There is an ind-scheme, $\underline{\mathrm{Fl}}$ whose rational points are Fl but I wont use it. The moduli interpretation is that an object of Fl is the data of an object of Gr along with a flag in the fiber above the origin. In particular, the fiber of $\mathrm{Fl} \to \mathrm{Gr}$ is naturally $\mathcal{B}$.

Example 3.2.6 ($\underline{G} = \mathrm{GL}_n$)**.** There is a $G((t))$ equivariant identification.

$$\mathrm{Fl} \cong \{(\Lambda_1, ..., \Lambda_n) : \ \Lambda_i \in \mathrm{Gr}, \ \Lambda_1 t \subsetneq \Lambda_n \subsetneq \Lambda_1 \subsetneq \Lambda_2 ... \subsetneq \Lambda_n\}$$

Such a chain of lattices is also specified by a single lattice $\Lambda := \Lambda_n$ along with a complete flag in $\Lambda/t\Lambda$.

There are similar lattice interpretations for the affine flag varieties of SL_n and PGL_n.

The Bruhat decomposition implies that the I orbits of Fl are in bijection with W^{aff}. Analogous to Proposition 2.2.1 for the finite case, the cardinality of the I-orbit labelled by w is $q^{\ell(w)}$. The geometric version is as follows.

Proposition 3.2.1. Moreover the I-orbit of $\underline{\mathrm{Fl}}$ corresponding to $w \in W^{\mathrm{aff}}$ is an affine space of dimension $\ell(w)$.

3.3 Affine Hecke Algebra

In this section I'll present the affine Hecke algebra $\mathcal{H}$ of $\underline{G}$. As stated in Chapter 1, the easiest definition is as the space of functions on the groupoid $I\backslash G((t))/I$.

$$\mathcal{H} = \mathrm{Fun}(I\backslash G((t))/I, \mathbb{C})$$

The product is as in Definition 1.2.1. As in the finite case there is an equivalence of groupoids

$$I\backslash G((t))/I \to G((t))\backslash(\mathrm{Fl} \times \mathrm{Fl})$$

The product is then given by the following diagram.

[5] An interesting feature of this combinatorial interpretation is that the Bott-Samelson resolution of the closure of the orbit corresponding to k is the space that parametrizes length k walks starting from Λ_0. Over the complex numbers, one way to construct Gr is to start with $\sqcup_{k\geq 0}(\mathbb{CP}^1)^k$, which parametrizes all walks with possible steps $\mathbb{P}^1$, and quotient by the relation that identifies any walks $wxyw' \sim ww'$ whenever x and y are antipodal steps and w, w' are arbitrary subwalks. Compare to the fact that there is a unique non-backtracking walk from the root of a tree to any other node. (I learned this construction of $\mathrm{Gr}_{\mathrm{PGL}_2(\mathbb{C})}$ from David Nadler.)

$$\begin{array}{ccccc} G((t))\backslash(\mathrm{Fl}\times\mathrm{Fl}) & \xleftarrow{\pi_{1,2}} & G((t))\backslash(\mathrm{Fl}\times\mathrm{Fl}\times\mathrm{Fl}) & \xrightarrow{\pi_{1,3}} & G((t))\backslash(\mathrm{Fl}\times\mathrm{Fl}) \\ & & \downarrow{\scriptstyle \pi_{2,3}} & & \\ & & G((t))\backslash(\mathrm{Fl}\times\mathrm{Fl}) & & \end{array}$$

For $u, v \in \mathcal{H}$,

$$u \cdot v = \pi_{1,3!}(\pi_{1,2}^*(u) \otimes \pi_{2,3}^*(v))$$

One has to take care interpreting the pushforward in this situation. The automorphism groups of objects are infinite, but they are always conjugate to finite index subgroups of I [6]. In particular, if $\pi_{1,3}(x) = y$, then $\mathrm{Aut}(x) \subset \mathrm{Aut}(y)$ is finite index so the relative measure is defined.

I will now recall Iwahori-Matusomoto's relations for $\mathcal{H}$. I won't state the proof because the proof of Theorem 2.2.1 can be adapted to this case taking into account the comment about relative measures. Let $T_w \in \mathcal{H}$ denote the characteristic function of the isomorphism class of $(I, \dot{w}^{-1}I)$.

Theorem 3.3.1 (Affine Hecke Algebra Structure Theorem)**.** The product in $\mathcal{H}$ is given as follows. For $w \in W^{\mathrm{aff}}, s \in S_{\mathrm{aff}}$, and $\tau \in \Lambda/\Lambda_R$,

$$T_\tau T_w = T_{\tau w}$$

$$T_s T_w = \begin{cases} T_{sw} & \ell(sw) > \ell(w) \\ qT_{sw} + (q-1)T_w & \ell(sw) < \ell(w) \end{cases}$$

It will be useful to work with Bernstein's presentation of $\mathcal{H}$, which I now recall.

Definition 3.3.1 (Translation Operator)**.** For $\lambda \in \Lambda$, define the *translation operator* $J_\lambda = (T_{-\lambda_1})^{-1}T_{-\lambda_2}$, where $\lambda = \lambda_1 - \lambda_2$, with $\lambda_i \in \Lambda_+$.

Remark 3.3.1. The definition does not depend on the choice λ_1, λ_2.

Theorem 3.3.2 (Bernstein's Relations)**.** The operators T_w $w \in W$ form a basis for the subalgebra $\mathcal{H}^{fin}$, called the finite Hecke algebra. The relations are as follows:

- $T_{w_1}T_{w_2} = T_{w_1w_2}$ if $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$
- $T_{s_\alpha}^2 = (q-1)T_{s_\alpha} + q$ if $s_\alpha \in W$ is simple

[6] Intuitively, this is because any finite collection of lattices in $\mathrm{Gr}_{\mathrm{GL}_n}$ contain each other up to finite translations. In other words, they are commensurable in the sense of group theory.

The operators T_{s_α} and J_λ for $s_\alpha \in W$ simple and $\lambda \in \Lambda$ satisfy

$$J_\lambda T_{s_\alpha} = q^{-\check{\alpha}(\lambda)} T_{s_\alpha} J_{s_\alpha(\lambda)} + (q-1)\frac{J_\lambda - q^{-\check{\alpha}(\lambda)} J_{s_\alpha(\lambda)}}{1 - qJ_\alpha}$$

Proof. See Proposition 3.6 of [19] for the original proof by Lusztig based on unpublished work of Bernstein. [10], [9], and [22] were also helpful references for me.

□

It follows that $\lambda \mapsto J_\lambda$ is an injective homomorphism $\mathbb{C}[\Lambda] \to H^{aff}$. Its image is maximal commutative subalgebra.

The geometric basis elements, $\{T_w\}_{w \in W^{aff}}$, are partially ordered by the length function on W^{aff}. Both the sets $\{J_\lambda T_w\}_{w \in W, \lambda \in \Lambda}$ and $\{T_w J_\lambda\}_{w \in W, \lambda \in \Lambda}$ are upper triangular with respect to the geometric basis, by Theorem 3.3.2. In particular, they are bases.

3.4 Hecke Operators on Moduli Space of Bundles

Recall the automorphic setup from Chapter 1. Let $X/\mathbb{F}_q$ be a curve and $S \subset X(\mathbb{F}_q)$ a finite set of markings. $\underline{\mathrm{Bun}_{\underline{G}}}(X,S)$ is the moduli stack of $\underline{G}$ bundles on X with Borel reductions near S and $\mathrm{Bun}_{\underline{G}}(X,S)$ its groupoid of rational points. Let C_{Aut} be the vector space of complex valued functions on $\mathrm{Bun}_{\underline{G}}(X,S)$.

For $s \in S$ the Hecke operators $\mathcal{H}^s$ are constructed as follows. I will construction a left action of $\mathcal{H}$ on C_{Aut}. Picking a uniformizer in the completed local ring at s defines a map $\mathrm{Spec}(\mathbb{F}_q[[t]]) \to \mathbb{P}^1$ sending the closed point to s. consider the following diagram

$$\begin{array}{ccccc} \mathrm{Bun}_G(\mathbb{P}^1,S) & \xleftarrow[\pi_1]{} & \mathrm{HeckeMod}^s & \xrightarrow{\pi_2} & \mathrm{Bun}_G(\mathbb{P}^1,S) \\ & & \big\downarrow{\scriptstyle \mathrm{res}} & & \\ & & G((t))\backslash(\mathrm{Fl}\times\mathrm{Fl}) & & \end{array}$$

$\mathrm{HeckeMod}^s$ classifies data the data of a triple $(\mathcal{E}_1, \mathcal{E}_2, T)$ where $\mathcal{E}_1, \mathcal{E}_2$ are parabolic G-bundles on $\mathbb{P}^1$ and T is an isomorphism of their restrictions away from s [7]. res is the restriction of the bundles along the map $\mathrm{Spec}(\mathbb{F}_q[[t]]) \to \mathbb{P}^1$. For $T \in \mathcal{H}$, the Hecke operator $T^s : C_{\mathrm{Aut}} \to C_{\mathrm{Aut}}$ is defined as

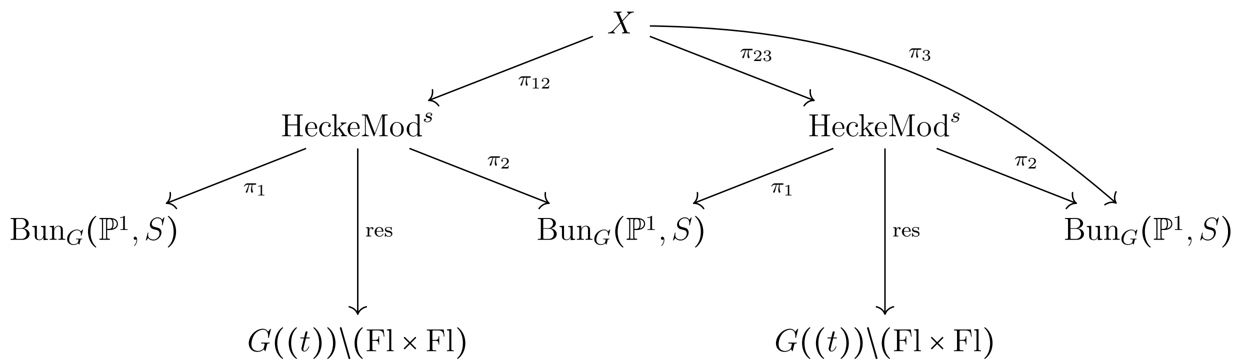
$$T^s f = \pi_{2!}(\mathrm{res}^* T \otimes \pi_1{}^* f)$$

The operator A^s is independent of the choice of uniformizer. It is worth noting that there are no difficulties interpreting the pushforward $\pi_{2!}$ because automorphism groups are finite.

I'll explain why this action is associative. It is essentially the same reason matrix multiplication is associative.

Consider the following diagram:

[7] Another way to think of this is as the fiber product of $\mathrm{Bun}_G(\mathbb{P}^1, S)$ with itself over restriction away from s.



X is the groupoid classifying triples $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ along with isomorphisms amongst their restrictions away from s. The square is Cartesian. First, I show that for $T_1, T_2 \in \mathcal{H}$ and $f \in C_{\mathrm{Aut}}$,

$$T_2(T_1 f) = \pi_{3!}(\pi_{23}^* \mathrm{res}^* T_2 \otimes \pi_{12}^* \mathrm{res}^* T_1 \otimes \pi_{12}^* \pi_1^* f).$$

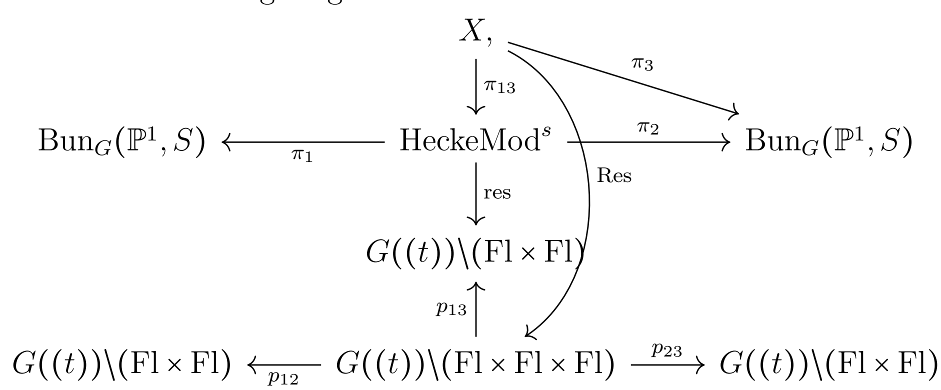
Starting from the definition of the action

$$T_2(T_1 f) = \pi_{2!}(\mathrm{res}^* T_2 \otimes \pi_1^*(\pi_{2!}(\mathrm{res}^* T_1 \otimes \pi_1{}^* f))) = \pi_{2!}(\mathrm{res}^* T_2 \otimes \pi_{23!}\pi_{12}^*(\mathrm{res}^* T_1 \otimes \pi_1{}^* f))$$

$$= \pi_{2!}\pi_{23!}(\pi_{23}{}^* \mathrm{res}^* T_2 \otimes \pi_{12}^*(\mathrm{res}^* T_1 \otimes \pi_1{}^* f)) = \pi_{3!}(\pi_{23}^* \mathrm{res}^* T_2 \otimes \pi_{12}^* \mathrm{res}^* T_1 \otimes \pi_{12}^* \pi_1^* f)$$

In the last step I've used that pullback and tensor product commute.

Now, consider the following diagram.



The square (it doesn't look like a square but it is) is Cartesian. I'll show that

$$(T_2 T_1) f = \pi_{3!}(\mathrm{Res}^* p_{23}^* T_2 \otimes \mathrm{Res}^* p_{12}^* T_1 \otimes \pi_{13}^* \pi_1^* f)$$

Starting from definitions,

$$(T_2 T_1) f = \pi_{2!}(\mathrm{res}^*(p_{13!}(p_{23}^* T_2 \otimes p_{12}^* T_1)) \otimes \pi_1{}^* f) = \pi_{2!}(\pi_{13!}\mathrm{Res}^*(p_{23}^* T_2 \otimes p_{12}^* T_1) \otimes \pi_1{}^* f)$$

$$= \pi_{2!}\pi_{13!}(\mathrm{Res}^*(p_{23}^* T_2 \otimes p_{12}^* T_1) \otimes \pi_{13}{}^*\pi_1{}^* f) = \pi_{3!}(\mathrm{Res}^* p_{23}^* T_2 \otimes \mathrm{Res}^* p_{12}^* T_1 \otimes \pi_{13}^* \pi_1^* f)$$

In the last step I have again used the tensor product commutes with pullback.

In order to show associativity, one needs to check the following.

$$\pi_{3!}(\pi_{23}^* \mathrm{res}^* T_2 \otimes \pi_{12}^* \mathrm{res}^* T_1 \otimes \pi_{12}^* \pi_1^* f) = \pi_{3!}(\mathrm{Res}^* p_{23}^* T_2 \otimes \mathrm{Res}^* p_{12}^* T_1 \otimes \pi_{13}^* \pi_1^* f)$$

Verify the requisite commutative diagrams: $\mathrm{res}\pi_{23} = p_{23}\mathrm{Res}, \mathrm{res}\pi_{12} = p_{12}\mathrm{res}, \pi_1\pi_{12} = \pi_1\pi_{13}$.

I will now remark that Hecke operators act by geometric correspondences. For $w \in W^{aff}$, the following is true.

$$\mathrm{Bun}_G(\mathbb{P}^1, S) \xleftarrow[\pi_1]{} \mathrm{Corr}_w^s \xrightarrow{\pi_2} \mathrm{Bun}_G(\mathbb{P}^1, S)$$

$$T_w^s f = \pi_{2!}\pi_1{}^* f$$

Corr_w^s is the subspace of $\mathrm{HeckeMod}^s$ where $\mathcal{E}_1$ and $\mathcal{E}_2$ restricted to a formal disk around s are in relative position w.

Definition 3.4.1 (Simultaneous Modification at Marked Points)**.** For $T \in \mathcal{H}$ and $R \subset S$, T^R will denote the product of the Hecke operators T^s for $s \in R$.

$$T^R := \prod_{s \in R} T^s$$

This is a product of commuting operators so the order of the product doesn't matter.

Reflection Operators

Definition 3.4.2 (Reflection Operator)**.** For simple reflections $s_\alpha \in W$ and $s \in S$, define the *reflection operator* $\mathrm{Avg}_{s_\alpha}^s := 1 + T_{s_\alpha}^s$.

$\mathrm{Avg}_{s_\alpha}^s$ has the following interpretation. Let P_{s_α} denote the almost minimal parabolic corresponding to the simple coroot α.

Let $\mathrm{Bun}_G(\mathbb{P}^1, S, s, s_\alpha)$ be the moduli stack of G-bundles on $\mathbb{P}^1$ with Borel reduction at $S \setminus \{s\}$ and P_{s_α} reduction at s. For example, for $G = GL_n$, it classifies pairs $(\mathcal{E}, \{F_p\}_{p \in S})$,

- $\mathcal{E}$ is a rank n vector bundle on $\mathbb{P}^1$
- For $p \neq s$, F_p is a full flag in the fiber $\mathcal{E}|_p$
- F_s is an almost full flag in the fiber $\mathcal{E}|_s$, consisting of a space of each dimension except the one corresponding to s_α.

There is a map $\pi : \mathrm{Bun}_G(\mathbb{P}^1, S) \to \mathrm{Bun}_G(\mathbb{P}^1, S, s, s_\alpha)$. For $F \in C_{Aut}$,

$$\mathrm{Avg}^{s}_{s_\alpha} \cdot F = \pi^* \pi_! F$$

The proof is virtually identical to the proof of main technical point of Claim 2.2.6. All the relevant moduli spaces need to be replaced by global analogues.

Chapter 4

Hecke Action on Eisenstein Series

This chapter contains the proofs of the main results, Conjecture 1.2.1 and Theorem 1.2.1, of the thesis. I will explain what are Eisenstein series and the relations satisfied by Hecke operators on Eisenstein series in the tamely ramified automorphic function theory of $\mathbb{P}^1/\mathbb{F}_q$. I will also explain that these relations are complete for three (simple) markings when $\underline{G} = \mathrm{SL}_2$ or $\underline{G} = \mathrm{PGL}_3$, and in a generic sense for arbitrary $\underline{G}$.

4.1 Pseudo-Eisenstein Series

Given a compactly supported function $f : \Lambda \to \mathbb{C}$, the pseudo-Eisenstein series Eis_f is defined by the following induction diagram.

$$\Lambda \otimes \mathrm{Pic}(\mathbb{P}^1) \cong \mathrm{Bun}_{\underline{T}}(\mathbb{P}^1) \xleftarrow{p} \mathrm{Bun}_{\underline{B}}(\mathbb{P}^1) \xrightarrow{q} \mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, S)$$

$$\mathrm{Eis}_f = q_! p^* f$$

p is the map associating the induced $\underline{T}$-bundle to a $\underline{B}$-bundle. q is the map that associates the induced $\underline{G}$-bundle and remembers the $\underline{B}$ structure along S. Define $\mathrm{Bun}_{\underline{B}}^{\lambda}(\mathbb{P}^1)$ as the preimage of the component $\lambda \in \mathrm{Bun}_{\underline{T}}(\mathbb{P}^1)$ and $q_\lambda : \mathrm{Bun}_{\underline{B}}^{\lambda}(\mathbb{P}^1) \to \mathrm{Bun}_{\underline{G}}(X, S)$ the restriction of q. then

$$\mathrm{Eis}_\lambda := \mathrm{Eis}_{\underline{1}_\lambda} = q_\lambda ! \underline{1}$$

Pseudo-Eisenstein series form a subpace of C_{Aut}. It is closed under spherical Hecke operators at $p \notin S$ but not under affine Hecke operators.

Definition 4.1.1 (Eisenstein Module)**.** The Eisenstein module, C_{Eis}, is the subspace of C_{Aut} generated by the action of all affine Hecke operators on all pseudo-Eisenstein series.

The space of Eisenstein series is also closed under spherical Hecke operators.

Compatibility of Eisenstein series and Translation

I describe the standard compatibility of Eisenstein induction with Hecke operators.

Theorem 4.1.1. For compactly supported $f : \Lambda \to \mathbb{C}$ and $\mu \in \Lambda$,

$$J^s_\mu \cdot \mathrm{Eis}_f = \mathrm{Eis}_{\mu \cdot f}$$

where $\mu \cdot f(\lambda) = f(\lambda - \mu)$.

Proof. It suffices to show $J^s_\mu \mathrm{Eis}_\lambda = \mathrm{Eis}_{\lambda+\mu}$ for $\mu, \lambda \in \Lambda$ with μ anti-dominant. In this case $J_\mu = T_\mu$. I show that there is a diagram, where the left square is Cartesian and the upper left horizontal arrow is a homemorphism:

$$\begin{array}{ccccc} \mathrm{Bun}^\lambda_{\underline{B}}(X) & \xleftarrow[t_1]{} & \Gamma & \xrightarrow{\cong} & \mathrm{Bun}^{\lambda+\mu}_{\underline{B}}(X) \\ \downarrow{\scriptstyle q_\lambda} & & \downarrow{\scriptstyle t_2} & & \downarrow{\scriptstyle q_{\lambda+\mu}} \\ \mathrm{Bun}_{\underline{G}}(X,S) & \xleftarrow[\pi_1]{} & \mathrm{Corr}^s_\mu & \xrightarrow{\pi_2} & \mathrm{Bun}_{\underline{G}}(X,S) \end{array}$$

Assuming such a diagram exists,

$$\mathrm{J}^s_\mu \cdot \mathrm{Eis}_\mu = \pi_{2!}\pi_1^* q_{\lambda!}\underline{1} = \pi_{2!}t_{2!}t_1^*\underline{1} = q_{\lambda+\mu!}\underline{1} = \mathrm{Eis}_{\lambda+\mu}$$

The existence of such a diagram is shown in Lemma 2.4.4 of [21].

□

4.2 Example: G=PGL(2)

Fix $G = \mathrm{PGL}(2)$. Identify $\Lambda \cong \mathbb{Z}$ by $(t \mapsto \mathrm{diag}(t^k, 1)) \mapsto k$. First, I'll describe the geometry of $\mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, S)$ and compute the finite Hecke action on C_{Aut} in the geometric basis of points of the moduli space. Then, I will calculate the structure of C_{Aut} as a $\mathcal{H}_{\mathrm{fin}}$ trimodule. In this case $C_{\mathrm{Aut}} = C_{\mathrm{Eis}}$. Finally, I will prove Theorem 4.2.1 characterizing C_{Aut} as a $\mathcal{H}$ trimodule and confirm Conjecture 1.2.1 for PGL(2).

Finite Hecke Action

There is a unique simple reflection. $\mathcal{H}_{\mathrm{fin}}$ is generated by the operator $\mathrm{Avg} = 1 + T_{s_\alpha}$, which satisfies the quadratic relation $\mathrm{Avg} \cdot \mathrm{Avg} = (q+1)\mathrm{Avg}$. Now, compute the action of $\mathcal{H}_{\mathrm{fin}}$ at $0 \in S$. The formulas for the action at other points are completely analogous.

Organize the calculation according to the following maps, given by forgetting parabolic structure.

$$\mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, S) \xrightarrow{\pi^0} \mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, \{1, \infty\}) \to \mathrm{Bun}_{\underline{G}}(\mathbb{P}^1)$$

Recall that $\mathrm{Avg}^0 = (\pi^0)^*\pi^0{}_!$. I list the rational points of $\mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, S)$ and record the fibers of the map π^0, in order to compute the operator Avg^0. Organize the information by fibers of the projection of $\mathrm{Bun}_{\underline{G}}(\mathbb{P}^1)$.

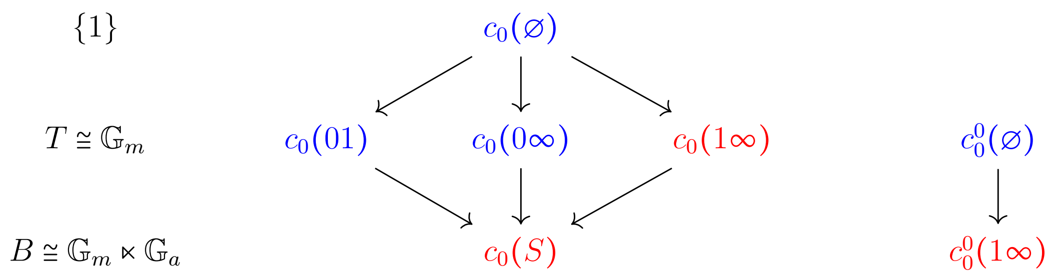
There is a short exact sequence, $1 \to \mathbb{G}_m \to \mathrm{GL}_2 \to G \to 1$, so by the vanishing of the Brauer group of a curve,

$$\mathrm{Vect}_2(\mathbb{P}^1)/\mathrm{Pic}(\mathbb{P}^1) \cong \mathrm{Bun}_{\underline{G}}(\mathbb{P}^1)$$

An of object of $\mathrm{Bun}_{\underline{G}}(\mathbb{P}^1)$ is represented by a rank 2 vector bundle, $\mathcal{E}$, up to tensoring with a line bundle. An object of $\mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, R)$, for $R \subset S$ is represented by a rank 2 vector bundle, $\mathcal{E}$, up to tensoring with line bundle, and a line ℓ_s in the fiber $\mathcal{E}|_s$, for $s \in R$.

$\mathcal{E} \cong \mathcal{O} \oplus \mathcal{O}$

The first column records the automorphism group of the object. The next two columns record the poset of points of $\mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, S)$ and $\mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, \{1, \infty\})$, respectively. $x \to y$ means y lies in the closure of x. The fibers of π^0 are indicated by color.



Identify the fibers $\mathcal{E}|_s$ for $s \in S$. For $R \subset S$, $c_0(R)$ denotes the locus where ℓ_s coincide for $s \in R$. Similarly, for $R \subset \{1, \infty\}$ $c_0^0(R)$ denotes the locus where ℓ_s coincide for $s \in R$. Avg^0 acts as follows.

$$\mathrm{Avg}^0 \underline{1}_{c_0(S)} = (\pi^0)^*\pi^0_! 1_{c_0(S)} = (\pi^0)^* \underline{1}_{c_0^0(S)} \frac{|\mathrm{Aut}(c_0^0(1\infty))|}{|\mathrm{Aut}(c_0(S))|} = \underline{1}_{c_0(S)} + \underline{1}_{c_0(1\infty)}$$

$$\mathrm{Avg}^0 \underline{1}_{c_0(1\infty)} = (\pi^0)^*\pi^0_! 1_{c_0(1\infty)} = (\pi^0)^* \underline{1}_{c_0^0(S)} \frac{|\mathrm{Aut}(c_0^0(1\infty))|}{|\mathrm{Aut}(c_0(1\infty))|} = q\underline{1}_{c_0(S)} + q\underline{1}_{c_0(1\infty)}$$

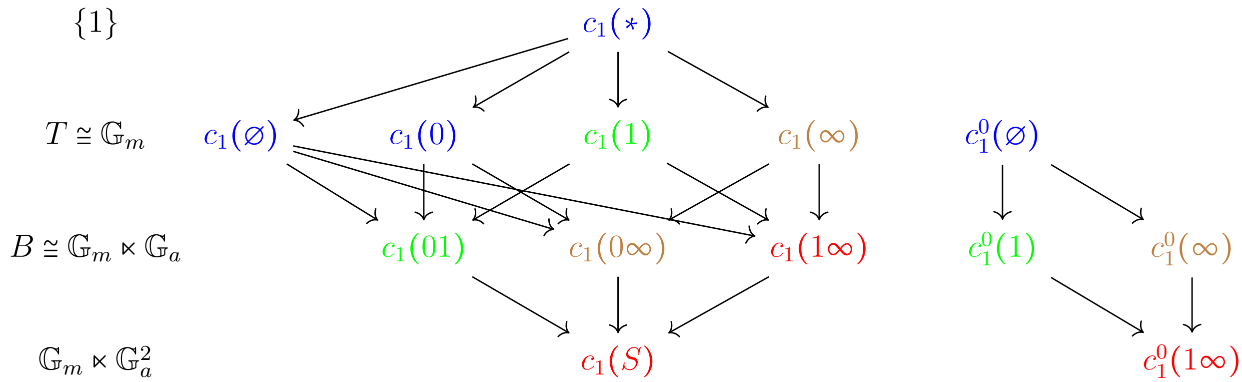
$$\mathrm{Avg}^0 \underline{1}_{c_0(01)} = (\pi^0)^*\pi^0_! 1_{c_0(01)} = (\pi^0)^* \underline{1}_{c_0^0(\varnothing)} \frac{|\mathrm{Aut}(c_0^0(\varnothing))|}{|\mathrm{Aut}(c_0(01))|} = \underline{1}_{c_0(01)} + \underline{1}_{c_0(0\infty)} + \underline{1}_{c_0(\varnothing)}$$

$$\mathrm{Avg}^0 \underline{1}_{c_0(0\infty)} = (\pi^0)^*\pi^0_! 1_{c_0(0\infty)} = (\pi^0)^* \underline{1}_{c_0^0(\varnothing)} \frac{|\mathrm{Aut}(c_0^0(\varnothing))|}{|\mathrm{Aut}(c_0(0\infty))|} = \underline{1}_{c_0(01)} + \underline{1}_{c_0(0\infty)} + \underline{1}_{c_0(\varnothing)}$$

$$\mathrm{Avg}^0 \underline{1}_{c_0(\varnothing)} = (\pi^0)^*\pi^0_! 1_{c_0(\varnothing)} = (\pi^0)^* \underline{1}_{c_0^0(\varnothing)} \frac{|\mathrm{Aut}(c_0^0(\varnothing))|}{|\mathrm{Aut}(c_0(\varnothing))|} = (q-1)\underline{1}_{c_0(01)} + (q-1)\underline{1}_{c_0(0\infty)}$$
$$+ (q-1)\underline{1}_{c_0(\varnothing)}$$

$\mathcal{E} \cong \mathcal{O}(1) \oplus \mathcal{O}$

I use the same conventions as before.

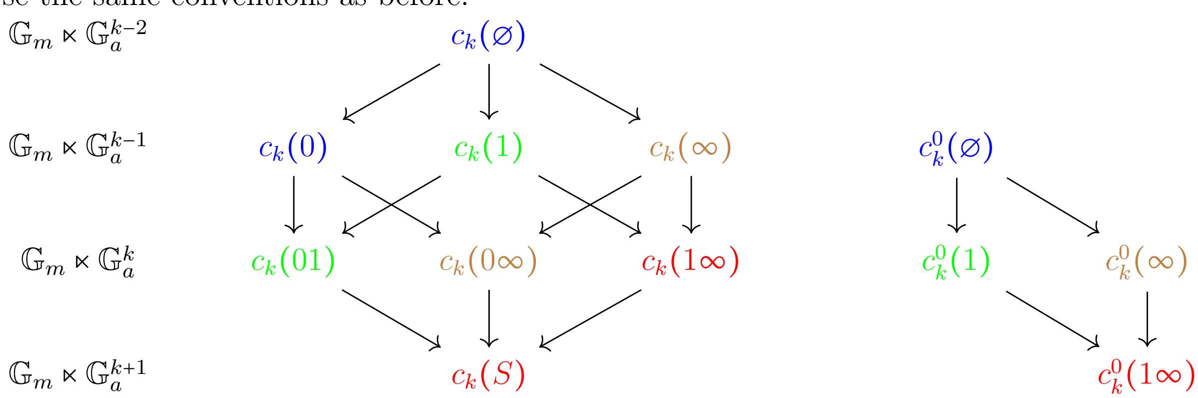


For $R \subset S$, $c_1(R)$ denotes the locus where there is a sub-bundle $\mathcal{O} \subset \mathcal{E}$, such that ℓ_s is contained in $\mathcal{O}(1)$ for $s \in R$ and ℓ_s is contained in the $\mathcal{O}$ sub-bundle for $s \notin R$. $c_1(*)$ is the generic configuration where no line ℓ_s lies in $\mathcal{O}(1)$ and there is no sub-bundle $\mathcal{O} \subset \mathcal{E}$ whose image contains the lines all the lines ℓ_s. For $R \subset \{1, \infty\}$, $c_1^0(R)$ is the locus where the line ℓ_s is contained in $\mathcal{O}(1)$ if and only if $s \in R$. The action of Avg^0 is as follows. I omit some of the intermediate computations.

$$\mathrm{Avg}^0 \underline{1}_{c_1(S)} = q^{-1}\mathrm{Avg}^0 \underline{1}_{c_1(1\infty)} = \underline{1}_{c_1(S)} + \underline{1}_{c_1(1\infty)}$$
$$\mathrm{Avg}^0 \underline{1}_{c_1(01)} = q^{-1}\mathrm{Avg}^0 \underline{1}_{c_1(1)} = \underline{1}_{c_1(01)} + \underline{1}_{c_1(1)}$$
$$\mathrm{Avg}^0 \underline{1}_{c_1(0\infty)} = q^{-1}\mathrm{Avg}^0 \underline{1}_{c_1(\infty)} = \underline{1}_{c_1(0\infty)} + \underline{1}_{c_1(\infty)}$$
$$\mathrm{Avg}^0 \underline{1}_{c_1(\varnothing)} = \mathrm{Avg}^0 \underline{1}_{c_1(0)} = (q-1)^{-1}\mathrm{Avg}^0 \underline{1}_{c_1(*)} = \underline{1}_{c_1(\varnothing)} + \underline{1}_{c_1(0)} + \underline{1}_{c_1(*)}$$

$\mathcal{E} \cong \mathcal{O}(k) \oplus \mathcal{O}$, $k \geq 2$

Use the same conventions as before.



For $R \subset S$, $c_k(R)$ denotes the locus where ℓ_s is contained in $\mathcal{O}(k)$ if and only if $s \in R$. For $R \subset \{1, \infty\}$, $c_k^0(R)$ is the locus where the line ℓ_s is contained in $\mathcal{O}(1)$ if and only if $s \in R$. The action of Avg^0 is given by the following formula. For $R \subset \{1, \infty\}$,

$$\mathrm{Avg}^0 \underline{1}_{c_k(R\cup\{0\})} = q^{-1}\mathrm{Avg}^0 \underline{1}_{c_k(R)} = \underline{1}_{c_k(R\cup\{0\})} + \underline{1}_{c_k(R)}$$

Finite Hecke Trimodule Structure

Definition 4.2.1. For $k \in \Lambda_+$, let $C^k_{\mathrm{Aut}} \subset C_{\mathrm{Aut}}$ denote the subspace of functions that take nonzero values only on points lying over the bundle type $k \in \mathrm{Bun}_{\underline{G}}(\mathbb{P}^1)$.

C^k_{Aut} is closed under finite Hecke operators at any $s \in S$. The space of automorphic functions admits the following decomposition into $\mathcal{H}_{\mathrm{fin}}$ trimodules.

$$C_{\mathrm{Aut}} = \oplus_{k\geq 0} C^k_{\mathrm{Aut}}$$

The calculations of the previous section imply the following theorem.

Proposition 4.2.1. C_{Aut} is the $(\mathcal{H}_{\mathrm{fin}})^{\otimes S}$ module generated by $\{\underline{1}_{c_k(S)}\}_{k\geq 0} \cup \{\underline{1}_{c_1(\varnothing)}\}$ with the relations

$$\mathrm{Avg}^{\{0,1\}} \underline{1}_{c_0(S)} = \mathrm{Avg}^{\{0,\infty\}} \underline{1}_{c_0(S)} = \mathrm{Avg}^{\{1,\infty\}} \underline{1}_{c_0(S)} \tag{4.1}$$

$$\mathrm{Avg}^s \underline{1}_{c_1(\varnothing)} = \mathrm{Avg}^s T^{S\smallsetminus\{s\}}_{s_\alpha} \underline{1}_{c_1(S)}, \text{ for } s \in S \tag{4.2}$$

Proof. For $k \geq 2$ and $R \subset S$,

$$T^R_{s_\alpha} \underline{1}_{c_k(S)} = \underline{1}_{c_k(S\smallsetminus R)}.$$

In particular, C^k_{Aut} is freely generated by $c_k(S)$.

C^0_{Aut} is generated by $\underline{1}_{c_0(S)}$. Check that $\mathrm{Avg}^{\{0,1\}}$ is the constant function on the locus where the bundle is trivial. Relation 4.1 follows. It is easy to see that there are no other relations.

I will check that C^1_{Aut} is generated by $c_1(*)$ and $c_1(S)$ generate C^1_{Aut} with a relations given by Equation 4.2. First check that equation 4.2 is true using the calculations from the previous section. To see that relations are sufficient, observe that $\underline{1}_{c_1(S)}$ generates a free rank one $(\mathrm{H}^{fin})^{\otimes S}$ submodule of C^1_{Aut} consisting of functions, f, satisfying $f(c_1(*)) = f(c_1(\varnothing))$. This submodule has codimension one in C^1_{Aut}. □

Hecke Trimodule Structure

I state and prove Theorem 4.2.1, confirming Conjecture 1.2.1 in this case. First identify the Eisenstein functions.

Eisenstein Objects

Proposition 4.2.2 (Eisenstein Objects)**.** $\mathrm{Eis}_k = \underline{1}_{c_k(S)}$ for $k \geq 0$ and $\mathrm{Eis}_{-1} = \underline{1}_{c_1(\varnothing)}$.

Proof. Recall the induction diagram

$$\mathrm{Bun}_{\underline{T}}(\mathbb{P}^1) \leftarrow \mathrm{Bun}_{\underline{B}}(\mathbb{P}^1) \rightarrow \mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, S)$$

Objects of $\mathrm{Bun}_{\underline{B}}(\mathbb{P}^1)$ are represented by pairs $(\mathcal{L}, \mathcal{E})$, $\mathcal{E}$ a rank 2 vector bundle, and $\mathcal{L} \subset \mathcal{E}$ a rank 1 sub-bundle, up to tensoring with a line bundle. The fiber above $k \in \Lambda$, $\mathrm{Bun}^k_B(\mathbb{P}^1)$, is the locus of pairs $(\mathcal{L}, \mathcal{E})$, where $\mathcal{L} \cong \mathcal{O}(k)$ and $\mathcal{E}/\mathcal{L} \cong \mathcal{O}$. Eis_k is the pushforward of the constant function on $\mathrm{Bun}^k_B(\mathbb{P}^1)$.

Fix $k \geq -1$. I show that the image of $\mathrm{Bun}^k_B(\mathbb{P}^1)$ in $\mathrm{Bun}_{\underline{G}}(\mathbb{P}, S)$ is a single point. Suppose one has a short exact sequence of vector bundles bundles,

$$\mathcal{O}(k) \rightarrow \mathcal{E} \rightarrow \mathcal{O},$$

$\mathrm{Ext}(\mathcal{O}(-k), \mathcal{O}) = 0$, so the short exact sequence splits. Therefore, $\mathrm{Bun}^k_B(\mathbb{P}^1, S)$ has a single point. If $k \geq 0$ the image of that point in $\mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, S)$ is $c_k(S)$, and if $k = -1$, the image is $c_1(\varnothing)$. There are three cases.

1. For $k > 0$, comparing stabilisers, one finds $\mathrm{Aut}((\mathcal{L}, \mathcal{E})) \cong \mathbb{G}_m \ltimes \mathbb{G}_a^{k+1} \cong \mathrm{Aut}(c_k(S))$. It follows that $\mathrm{Eis}_k = \underline{1}_{c_k(S)}$.
2. For $k = 0$, $\mathrm{Aut}((\mathcal{L}, \mathcal{E})) \cong B \cong \mathrm{Aut}(c_0(S))$. It follows that $\mathrm{Eis}_0 = \underline{1}_{c_0(S)}$.
3. Finally, for $k = -1$, $\mathrm{Aut}((\mathcal{L}, \mathcal{E})) \cong T \cong \mathrm{Aut}(c_1(\varnothing))$. $\mathrm{Eis}_{-1} = \underline{1}_{c_1(\varnothing)}$.

It is not necessary for the following calculations but one can calculate Eis_k for $k \leq -2$. For example

$$\mathrm{Eis}_{-2} = \underline{1}_{c_0(\varnothing)} + \underline{1}_{c_2(\varnothing)}.$$

In general, Eis_k for $k \leq -2$ is nonzero only on points of moduli space where the bundle is $\mathcal{E} \cong \mathcal{O}(r) \oplus \mathcal{O}$ with $0 \leq r \leq -k$ the same parity as k.

□

Main Theorem

Theorem 4.2.1. C_{Aut} is the $\mathcal{H}^{\otimes S}$ module generated by Eis_0 with the relations

$$J^0_k \mathrm{Eis}_0 = J^1_k \mathrm{Eis}_0 = J^\infty_k \mathrm{Eis}_0 \text{ for } k \in \Lambda \tag{4.3}$$

$$\mathrm{Avg}^{\{0,1\}} \mathrm{Eis}_0 = \mathrm{Avg}^{\{0,\infty\}} \mathrm{Eis}_0 = \mathrm{Avg}^{\{1,\infty\}} \mathrm{Eis}_0 \tag{4.4}$$

Proof. By Proposition 4.2.1, C_{Aut} is generated by Eisenstein functions under Hecke operators. By Theorem 4.1.1 all Eisenstein functions are generated by Eis_0 under translation Hecke operators. Therefore, C_{Aut} is generated by Eis_0.

I check that the stated relations hold. Equation 4.3 is a consequence Theorem 4.1.1 on compatibility of translation Hecke operators with Eisenstein induction. By Proposition 4.2.2, $\mathrm{Eis}_0 = \underline{1}_{c_0(S)}$, so Equation 4.4 follows from Equation 4.1 of Proposition 4.2.1.

I show that there are no other relations. Let $\widetilde{C}$ denote the quotient of $\mathcal{H}^{\otimes S}$ by the left ideal generated by the relations stated in Equations 4.3 and 4.4. There is a surjection of $\mathcal{H}^{\otimes S}$ modules

$$\widetilde{C} \to C_{\mathrm{Aut}}$$

I will show that this an injective map of $(\mathcal{H}_{\mathrm{fin}})^{\otimes S}$ modules. Let $\widetilde{C}_+ \subset \widetilde{C}$ be the $(\mathcal{H}_{\mathrm{fin}})^{\otimes S}$ submodule generated by $\{J^0_k\}_{k \geq -1}$. By Proposition 4.2.1 and Proposition 4.2.2, it suffices to show that the following are true in $\widetilde{C}$:

$$\mathrm{Avg}^s J^0_{-1} = \mathrm{Avg}^s T^{S \setminus \{s\}}_{s_\alpha} J_1, \text{ for } s \in S \tag{4.5}$$

$$J_k \in \widetilde{C}_+ \text{ for } k \leq -2 \tag{4.6}$$

I have omitted the superscript, $s \in S$, on the operators J_k because of the defining relations of $\widetilde{C}$. The formulas follow from Proposition 4.4.1.

□

4.3 Proof of Theorem 1.2.1

For this section, assume G is such that $\rho \in \Lambda$.

Definition 4.3.1. The *algebraic* Eisenstein module, $\widetilde{C}$, is the quotient $\mathcal{H}^S$ module which is the quotient of $\mathcal{H}^S$ by the left ideal generated by the relations

$$J^0_\lambda = J^1_\lambda = J^\infty_\lambda \text{ for } \lambda \in \Lambda$$

$$\mathrm{Avg}^{\{0,1\}}_{s_\alpha} = \mathrm{Avg}^{\{0,\infty\}}_{s_\alpha} = \mathrm{Avg}^{\{1,\infty\}}_{s_\alpha} \text{ for simple } s_\alpha \in W$$

Theorem 4.3.1. There is a surjective map of $\mathcal{H}^{\otimes S}$ modules, $\widetilde{C} \to C_{\mathrm{Eis}}$ given by $1 \mapsto \mathrm{Eis}_0$.

Proof. This is equivalent to checking the Translation and Reflection relations on C_{Eis}. By Theorem 4.1.1 $J^s_\lambda \mathrm{Eis}_0 = \mathrm{Eis}_\lambda$ for any $s \in S$ and $\lambda \in \Lambda$. In particular, $J^s_\lambda \mathrm{Eis}_0$ is independent of s.

Fix a simple coroot α. Let P_{s_α} the almost minimal parabolic associated with s_α. For $R \subset S$, let $\mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, S, R, s_\alpha)$ denote the moduli of stack of $\underline{G}$-bundles on $\mathbb{P}^1$ with Borel reductions at $S \smallsetminus R$ and P_{s_α} reduction at R. There is a map $\pi : \mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, S) \to \mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, \{0,1\}, s_\alpha)$ forgetting parabolic structure at 0 and 1. Consider the following diagram, where the square is Cartesian

$$\begin{array}{ccccc}
\mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, S) & & & & \\
\downarrow{\scriptstyle \pi_0} & & & & \\
\mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, S, \{0\}, s_\alpha) & \xleftarrow[\pi_0]{} & \mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, S), & & \\
\downarrow{\scriptstyle \pi_{01}} & & \downarrow{\scriptstyle \pi_1} & & \\
\mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, S, \{0,1\}, s_\alpha) & \xleftarrow[\pi_{10}]{} & \mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, S, \{1\}, s_\alpha) & \xleftarrow[\pi_1]{} & \mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, S)
\end{array}$$

For $F \in C_{\mathrm{Aut}}$,

$$\mathrm{Avg}_{s_\alpha}^{\{0,1\}} F = \mathrm{Avg}_{s_\alpha}^{1} \mathrm{Avg}_{s_\alpha}^{0} F = \pi_1^* \pi_{1!} \pi_0^* \pi_0 ! F = \pi_1^* \pi_{10}^* \pi_{01!} \pi_0 ! F = \pi^* \pi_! F$$

There is a point $\mathrm{pt}/B \to \mathrm{Bun}(\mathbb{P}^1, S)$ classifying trivial bundles with the same Borel reduction at all points of S. There is also a point $\mathrm{pt}/B \to \mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, \{0,1\}, s_\alpha)$ classifying trivial bundles with the same Parabolic structure at all points of S (and the unique, up to automorphism, of the further reduction of the structure group to B at ∞). The following diagram commutes

$$\begin{array}{ccc}
\mathrm{pt}/B & \longrightarrow & \mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, S) \\
& \searrow & \downarrow{\scriptstyle \pi} \\
& & \mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, \{0,1\}, s_\alpha)
\end{array}$$

Therefore,

$$\pi_! \mathrm{Eis}_0 = \pi_! \underline{1}_{\mathrm{pt}/B} = \underline{1}_{\mathrm{pt}/B}$$

The fiber above pt/B of π is the locus where the bundle is trivial and the Borel reductions at the points of S have the same P_{s_α} reduction. $\mathrm{Avg}_{s_\alpha}^{\{0,1\}} \mathrm{Eis}_0 = \pi^* \underline{1}_{\mathrm{pt}/B}$ is the constant function on this locus. By symmetry, see that $\mathrm{Avg}_{s_\alpha}^{S \smallsetminus \{s\}} \mathrm{Eis}_0$ is independent of s.

□

$\widetilde{C}$ and C_{Eis} are $\mathcal{H}^{\otimes}$ modules. By restriction of scalars through $\mathbb{C}[\Lambda] \to \mathcal{H}^0$, these become modules over algebra of translation operators at 0. I make some observations about these modules.

Proposition 4.3.1. C_{Eis} is finitely generated over $\mathbb{C}[\Lambda]$.

Proof. I show that C_{Eis} is generated by the $|W|^3$ elements $T^0_{w_0} T^1_{w_1} T^\infty_{w_\infty} \mathrm{Eis}_0$ for $w_0, w_1, w_\infty \in W$. Recall that $\{T_w J_\lambda\}_{w \in W, \lambda \in \Lambda}$ is a basis for $\mathcal{H}$. Therefore, the Eisenstein module is spanned by functions elements

$$T^0_{w_0} J^0_{\lambda_0} T^1_{w_1} J^1_{\lambda_1} T^\infty_{w_\infty} J^\infty_{\lambda_\infty} \mathrm{Eis}_0,$$

$w_0, w_1, w_\infty \in \Lambda$ and $\lambda_0, \lambda_1, \lambda_\infty \in \Lambda$. By the translation relation

$$T^0_{w_0}J^0_{\lambda_0}T^1_{w_1}J^1_{\lambda_1}T^\infty_{w_\infty}J^\infty_{\lambda_\infty} = T^0_{w_0}T^1_{w_1}T^\infty_{w_\infty}J^0_\lambda \mathrm{Eis}_0,$$

where $\lambda = \lambda_0 + \lambda_1 + \lambda_\infty$. because $\{J_\lambda T_w\}_{w\in W,\lambda\in\Lambda}$ is another basis for $\mathcal{H}$, C_{Eis} is spanned by functions

$$J^0_\lambda T^0_{w_0}T^1_{w_1}T^\infty_{w_\infty}\mathrm{Eis}_0.$$

In particular, C_{Eis} is generated over $\mathbb{C}[\Lambda]$ by functions $T^0_{w_0}T^1_{w_1}T^\infty_{w_\infty}\mathrm{Eis}_0$.

□

Proposition 4.3.2. C_{Eis} contains a free $\mathbb{C}[\Lambda]$ submodule of rank $|W|^2$.

Proof. I claim that $|W|^2$ elements $T^1_{w_1}T^\infty_{w_\infty}\mathrm{Eis}_0$ are independent over $\mathbb{C}[\Lambda]$. Suppose there is some nontrivial finite linear combination

$$\sum_i c_i J^0_{\lambda_i}T^1_{w_{1,i}}T^\infty_{w_{\infty},i}\mathrm{Eis}_0 = 0$$

Pick a weight λ such that $\mu + \lambda_i - \rho \in \Lambda_+$. J_λ is invertible, so

$$\sum_i c_i J^0_{\lambda_i}T^1_{w_{1,i}}T^\infty_{w_{\infty},i}\mathrm{Eis}_0 = 0 \iff \sum_i c_i J^0_{\mu+\lambda_i}T^1_{w_{1,i}}T^\infty_{w_{\infty},i}\mathrm{Eis}_0 = 0 \iff \sum_i c_i T^1_{w_{1,i}}T^\infty_{w_{\infty},i}\mathrm{Eis}_{\mu+\lambda_i}$$

In particular, it suffices to show that the functions $\{T^1_{w_{1,i}}T^\infty_{w_{\infty},i}\mathrm{Eis}_\lambda\}$, for $w_1, w_\infty \in W$ and $\lambda - \rho \in \Lambda_+$ are linearly independent.

Identify isomorphism classes of $\underline{G}$-bundles on $\mathbb{P}^1$ with $W\backslash W^{aff}/W \cong \Lambda_+$. If $\mathcal{E}_\lambda$ is a $\underline{G}$ bundle corresponding to $\lambda \in \Lambda_+$ such that $\lambda - \rho \in \Lambda_+$, then there is a $\underline{B}$-bundle $\mathcal{E}_{B,\lambda}$, stable under $\mathrm{Aut}(\mathcal{E}_\lambda)$. In particular, for $s \in S$ there is a flag $F_s \subset \mathcal{E}_\lambda|_s$ that is stable under $\mathrm{Aut}(\mathcal{E}_\lambda)$. The function $\{T^1_{w_{1,i}}T^\infty_{w_{\infty},i}\mathrm{Eis}_\lambda\}$ is supported only on points of the locus classifying parabolic bundles $(\mathcal{E}, \{F'_s\}_{s\in S})$ where $\mathcal{E} \cong \mathcal{E}_\lambda$ and the parabolic structure at $s \in \{1, \infty\}$ is in relative position w_s to F'_s.

□

Remark 4.3.1. If $\lambda - 2\rho \in \Lambda_+$, then the isomorphism class of an object in $\mathrm{Bun}_G(\mathbb{P}^1, S)$ with underlying bundle $\mathcal{E}_\lambda$ is determined by the relative positions, for $s \in S$, of the parabolic structure F'_s to F_s. I haven't proved this observation as it isn't needed for any of the results of this thesis.

Conjecture 4.3.1. $\widetilde{C}$ is free of rank $|W|^2$ over $\mathbb{C}[\Lambda]$.

Example 4.3.1. Conjecture 4.3.1 is true for $G = \mathrm{PGL}(2)$. C_{Eis} is generated over $\mathbb{C}[\Lambda]$ by the following functions:

$$\mathrm{Eis}_0, T^1\mathrm{Eis}_0, T^\infty\mathrm{Eis}_0, T^{\{1,\infty\}}\mathrm{Eis}_0, T^0\mathrm{Eis}_0$$

The first four generators are independent over $\mathbb{C}[\Lambda]$. One can check that

$$(q^2 J_2^0 - 1)(T^1 T^\infty - qT_0) = (q-1)(T^1 - q)(T^\infty - q) \tag{4.7}$$

Therefore, $\{\mathrm{Eis}_0, T^1\mathrm{Eis}_0, T^\infty\mathrm{Eis}_0, (T^{\{1,\infty\}} - qT^0)\mathrm{Eis}_0\}$ is a basis over $\mathbb{C}[\Lambda]$.

Conjecture 4.3.1 together with Propositions 4.3.1 and 4.3.2 imply that $\widetilde{C} \to C_{\mathrm{Eis}}$ is an isomorphism. I show that 4.3.1 is generically true over $\mathbb{C}[\Lambda]$.

Proposition 4.3.3. $\mathrm{Frac}(\mathbb{C}[\Lambda]) \otimes_{\mathbb{C}[\Lambda]} \widetilde{C}$ has dimension $|W|^2$ over $\mathcal{K}^0$.

Let us postpone the proof of Proposition 4.3.3 briefly. It will be easier to filter the vector space $\widetilde{C}$ and work with the associated graded vector space.

$\mathcal{H}$ is filtered by length $\ell : W \to \mathbb{Z}_{\geq 0}$. For $i \in \mathbb{Z}_{\geq 0}$ The ith filtered component $F^i(\mathcal{H}) \subset \mathcal{H}$ is spanned by $T_w J_\lambda$ for $w \in W$ with $\ell(w) \leq i$ and $\lambda \in \Lambda$. $F^0(\mathcal{H}) \cong \mathbb{C}[\Lambda]$ is the subalgebra of translation operators. Note that $F^i(\mathcal{H})$ is also spanned by $J_\lambda T_w$ for $w \in W$ with $\ell(w) \leq i$ and $\lambda \in \Lambda$. In general,

$$F^i(\mathcal{H}) \cdot F^j(\mathcal{H}) \subset F^{i+j}(\mathcal{H}) \ \forall \ i,j \in \mathbb{Z}_{\geq 0}$$

Filter $\widetilde{C}$ so that the following are true:

1. The $\widetilde{C}$ is a filtered module for the filtered algebra $\mathcal{H}^0$ of Hecke operators at 0. That is,

$$F^i(\mathcal{H}^0) \cdot F^j(\widetilde{C}) \subset F^{i+j}\widetilde{C}, \ \forall \ i,j \in \mathbb{Z}_{\geq 0}$$

2. The filtration on $\widetilde{C}$ is preserved by Hecke operators at $S \setminus \{0\}$. That is, for $s \in S \setminus \{0\}$,

$$F^i(\mathcal{H}^s) \cdot F^j(\widetilde{C}) \subset F^j\widetilde{C}, \ \forall \ i,j \in \mathbb{Z}_{\geq 0}$$

Note that a consequence of the first requirement is that the filtered components of $\widetilde{C}$ are modules for the algebra translation operators at 0, $\mathbb{C}[\Lambda]$.

Definition 4.3.2 (Filtration of $\widetilde{C}$)**.** The ith filtered component $F^i(\widetilde{C}) \subset \widetilde{C}$ is spanned by $A^0 T^1_{w_1} T^\infty_{w_\infty}$, for $w_1, w_\infty \in W$ and $A \in F^i(\mathcal{H})$. Alternatively, it is spanned by $T^0_{w_0} T^1_{w_1} T^\infty_{w_\infty} J_\lambda$ for $\lambda \in \Lambda$ and $w_0, w_1, w_\infty \in W$ with $\ell(w_0) \leq i$.

The first requirement on the filtration of $\widetilde{C}$ is automatically satisfied by construction. The second condition is also satisfied because of the translation relation. Now I'll prove Proposition 4.3.3

Proof of Proposition 4.3.3. After rationalization, there is the isomorphism

$$\mathrm{Frac}(\mathbb{C}[\Lambda]) \otimes_{\mathbb{C}[\Lambda]} \widetilde{C} \cong \bigoplus_i \mathrm{Frac}(\mathbb{C}[\Lambda]) \otimes_{\mathbb{C}[\Lambda]} \mathrm{Gr}^i(\widetilde{C})$$

$\mathrm{Gr}^i(\widetilde{C})$ is generated as a $\mathbb{C}[\Lambda] \otimes \mathcal{H}^{\otimes\{1,\infty\}}$ module by T^0_w, for $w \in W$ with $\ell(w) = i$. Therefore, one needs only to show that for $w \in W$ of length $\ell(w) = i$, there is $A \in \mathbb{C}[\Lambda]$ such that

$$A^0 \cdot T^0_w \in F^{i-1}(\widetilde{C}).$$

Pick a simple reflection s_α such that $\ell(ws_\alpha) = \ell - 1$. Pick $\lambda \in \Lambda$ such that $\langle \check{\alpha}, \lambda \rangle = 1$. Start with the equation from Proposition 4.4.2,

$$\begin{aligned} & T^0_{s_\alpha}(J_\lambda - J_{s_\alpha(\lambda)}) \in F^0(\widetilde{C}) \\ \Longrightarrow\ & J^1_{-s_\alpha(\lambda)} T^0_{s_\alpha}(J_\lambda - J_{s_\alpha(\lambda)}) \in F^0(\widetilde{C}) \\ \Longrightarrow\ & T^0_{s_\alpha}(J_\alpha - 1) \in F^0(\widetilde{C}) \\ \Longrightarrow\ & T^0_w(J_\alpha - 1) \in F^0(\widetilde{C}) \in F^{i-1}(\widetilde{C}) \end{aligned}$$

Observe that for some integer n, $T_w J_\alpha - q^n J_{w\cdot\alpha} T_w \in \mathbb{C}[\Lambda]$, so

$$(q^n J^0_{w\cdot\alpha} - 1) T^0_w \in F^0(\widetilde{C}) \in F^{i-1}(\widetilde{C})$$

□

Remark 4.3.2. $w \cdot \alpha$ is always a negative coroot. n is given by the explicity formula $n = \langle \check{\rho}, w \cdot \alpha \rangle - 1$. For Proposition 4.3.3 one only needed to invert the polynomial

$$\prod_{\alpha \in R_+} \left(q^{\langle \rho, \alpha \rangle + 1} J_\alpha - 1 \right)$$

This is not the standard discriminant polynomial. In particular, $q^{\langle \rho, \alpha \rangle + 1} J_\alpha - 1$ is not homogeneous with respect to the natural q-twisted $\mathbb{G}_m$ action on $\mathbb{C}[\Lambda]$.

Theorem 1.2.1 follows from Propositions 4.3.1, 4.3.2, and 4.3.3.

4.4 Some Formulas for Algebraic Eisenstein Module

In this section I prove some formulas that hold in the module $\widetilde{C}$ formally generated over $\mathcal{H}^{\otimes S}$ by one generator subject to the translation and reflection relations. I have postponed these calculations to this section as they don't fit the flow of the arguments where they are used. It is helpful to first understand the $G = \mathrm{PGL}(2)$ example.

Proposition 4.4.1 (Functional Equation for Algebraic Eisenstein Module)**.** Let $\widetilde{C}$ be the quotient of $\mathcal{H}^{\otimes S}$ by the left ideal generated by relations:

$$J^0_\lambda = J^1_\lambda = J^\infty_\lambda \text{ for } \lambda \in \Lambda$$

$$\mathrm{Avg}^{\{0,1\}}_{s_\alpha} = \mathrm{Avg}^{\{0,\infty\}}_{s_\alpha} = \mathrm{Avg}^{\{1,\infty\}}_{s_\alpha} \text{ for simple } s_\alpha \in W$$

Assume that the map $\check{\alpha} : \Lambda \to \mathbb{Z}$ given by $\lambda \mapsto \langle \check{\alpha}, \lambda \rangle$ is surjective. Then, for any simple reflection s_α,

$$\mathrm{Avg}^s_{s_\alpha} J_\lambda = \mathrm{Avg}^s_{s_\alpha} T^{S \setminus \{s\}}_{s_\alpha} J_{s_\alpha(\lambda)} \text{ if } \langle \check{\alpha}, \lambda \rangle = -1$$

$$J_\lambda \in \mathrm{Span}_{(\mathcal{H}_{\mathrm{fin}})^{\otimes S}} \{J_\mu\}_{\mu \in R(\lambda,\alpha)} \text{ if } \langle \check{\alpha}, \lambda \rangle \le -2$$

where $R(\lambda, \alpha) \subset \Lambda$ consists of coweights μ, such that $\mu - \lambda$ is an integral multiple of α and $-1 \le \langle \check{\alpha}, \mu \rangle \le -\langle \check{\alpha}, \lambda \rangle$.

Proof. Fix the simple coroot α and let $T := T_{s_\alpha}$, $\mathrm{Avg} := \mathrm{Avg}_{s_\alpha}$.Fix λ so that $\langle \check{\alpha}, \lambda \rangle = 1$. Start with the reflection relation

$$T^1 \mathrm{Avg}^0 = T^\infty \mathrm{Avg}^0$$

$$T^1 J^1_\lambda T^1 \mathrm{Avg}^0 = T^1 J^1_\lambda T^\infty \mathrm{Avg}^0$$

Observe that $T J_\lambda T = J_{\lambda - \alpha}$.

$$\mathrm{Avg}^0 J_{\lambda-\alpha} = \mathrm{Avg}^0 T^{1\infty} J_\lambda$$

This proves the first part of the proposition. Continuing with the previous equality

$$T^0 J^0_\lambda \mathrm{Avg}^0 J_{\lambda-\alpha} = T^0 J^0_\lambda \mathrm{Avg}^0 T^{1\infty} J_\lambda$$

$$J_{2\lambda-2\alpha} + T^0 J_{2\lambda-\alpha} = T^S J_{2\lambda} + T^{1\infty} J_{2\lambda-\alpha}$$

$$J_{2\lambda-2\alpha} = T^S J_{2\lambda} + (T^{1\infty} - T^0) J_{2\lambda-\alpha}$$

Now, let $\lambda' \in \Lambda$ be such that $\langle \check{\alpha}, \lambda' \rangle \le -2$. Define $\mu := \lambda' - 2\lambda$ and $n := -\langle \check{\alpha}, \mu \rangle \in \mathbb{Z}_{\ge 0}$.

$$J_{\lambda'} = J^0_\mu T^S J_{2\lambda} + J^0_\mu (T^{1\infty} - T^0) J_{2\lambda-\alpha} \in \mathrm{Span}_{(\mathcal{H}_{\mathrm{fin}})^{\otimes S}} \{J_{\lambda'+k\alpha}\}^n_{k=1}$$

The second part of the proposition follows by induction on n.

□

Proposition 4.4.2. Let $\widetilde{C}$ be as in Proposition 4.4.1. If $\lambda \in \Lambda$ such that $\langle \check{\alpha}, \lambda \rangle = 1$, then the following is true in $\widetilde{C}$:

$$T^0_{s_\alpha}(J_\lambda - J_{s_\alpha(\lambda)}) = -T^{\{1,\infty\}}_{s_\alpha}(J_\lambda - q^{-1} J_{s_\alpha(\lambda)}) - (1 + T^1_{s_\alpha} + T^\infty_{s_\alpha}) q^{-1/2} J_{s_\alpha(\lambda)}$$

Proof. For ease of notation, let $T := T_{s_\alpha}$. Introduce the operator $D \in \mathcal{H}$,

$$D := q^{1/2} J_\lambda - q^{-1/2} J_{s_\alpha(\lambda)}$$

Observe that

$$DT = -TD + \frac{2(q-1)D}{1-qJ_\alpha} = -TD - 2(q-1)q^{-1/2} J_{s_\alpha(\lambda)} \tag{4.8}$$

Start with the reflection relation.

$$(1+T^1)(T^0 - T^\infty) = 0$$

$$\implies (D^1 - D^0)(1+T^1)(T^0 - T^\infty) = 0$$

Using Equation 4.8 to move all D operators to the right and simplifying obtain the following. In light of the translation relation, the superscript is omitted from all translation that appear as the rightmost term of an expression.

$$(T^0 + T^{\{1,\infty\}})D = (1 + T^1 + T^\infty - T^0)\frac{(q-1)D}{1-qJ_\alpha}$$

$$\implies T^0\left(D + \frac{(q-1)D}{1-qJ_\alpha}\right) = -T^1T^\infty D + (1+T^1+T^\infty)\frac{(q-1)D}{1-qJ_\alpha}$$

$$\implies q^{1/2}T^0(J_\lambda - J_{s_\alpha(\lambda)}) = -T^1T^\infty D + (1+T^1+T^\infty)\frac{(q-1)D}{1-qJ_\alpha}$$

□

4.5 Example: G=SL(3)

This section has been included to provide some evidence that the conjecture is true and give some intuition for the general structure of C_{Eis}. Fix $G = \text{SL}(3)$ for this section. I will prove Theorem 4.5.1 verifying Conjecture 1.2.1 in this case.

Theorem 4.5.1. Conjecture 1.2.1 is true when $G = \text{SL}(3)$.

Our approach is to study the Eisenstein module as a finite Hecke trimodule. It is not expected that this approach will generalize to arbitrary G.

Identify the coweight lattice

$$\Lambda \cong \{(k_1, k_2, k_3) \in \mathbb{Z}^3 : \ k_1 + k_2 + k_3 = 0\}$$

by $(t \mapsto \text{diag}(t^{k_1}, t^{k_2}, t^{k_3})) \mapsto (k_1, k_2, k_3)$. There are two simple coroots, $\alpha_1 = (1,-1,0)$ and $\alpha_2 = (0,1,-1)$. $\rho = (1,0,-1)$. The Weyl group is identified $W \cong S_3$ with it's standard action on $\mathbb{Z}^3$. s_{α_i} is identified with the standard generator $s_i \in S_3$. Reflection normal to the long root is identified with $s_3 \in S_3$, $s_3 = s_1 s_2 s_1 = s_2 s_1 s_2$. To simplify notation, define T_i and $\mathrm{Avg}_i \in \mathcal{H}$, for $i = 1, 2$ as $T_i = T_{s_{\alpha_i}}$ and $\mathrm{Avg}_i = \mathrm{Avg}_{s_{\alpha_i}}$.

Let $\widetilde{C}$ be the algbraic Eisenstein module as in Definition 4.3.1. By Theorem 4.3.1 there is a surjective map of $\mathcal{H}^{\otimes S}$ modules $\widetilde{C} \to C_{\mathrm{Eis}}$. By Proposition 4.4.1, $\widetilde{C}$ is generated over $(\mathcal{H}_{\mathrm{fin}})^{\otimes S}$ by J_λ for $\lambda \in \Lambda$ such that $\lambda + \rho$ is dominant. Further, the following relations hold amongst the generators (see Figure 4.1):

1. $\lambda = 0$ (Principal Orbit)

$$\mathrm{Avg}_i^{\{0,1\}} J_0 = \mathrm{Avg}_i^{\{0,\infty\}} J_0 = \mathrm{Avg}_i^{\{1,\infty\}} J_0 \text{ for } i \in \{1,2\} \tag{4.9}$$

2. $\lambda \in W \cdot \rho$

$$\mathrm{Avg}_1^s J_{\alpha_2} = \mathrm{Avg}_1^s T_1^{S\smallsetminus\{s\}} J_\rho \text{ for } s \in S \tag{4.10}$$

$$\mathrm{Avg}_2^s J_{\alpha_1} = \mathrm{Avg}_2^s T_2^{S\smallsetminus\{s\}} J_\rho \text{ for } s \in S \tag{4.11}$$

$$\mathrm{Avg}_i^s J_{-\rho} \in \mathrm{Span}_{(\mathcal{H}_{\mathrm{fin}})^{\otimes S}}\{J_0, J_{\alpha_1}, J_{\alpha_2}, J_\rho\} \text{ for } i \in \{1,2\},\ s \in S \tag{4.12}$$

3. $\langle \check{\alpha}_i, \lambda \rangle = 0,\ \lambda \neq 0$ (walls of dominant cone)

$$\mathrm{Avg}_i^{\{0,1\}} J_\lambda = \mathrm{Avg}_i^{\{0,\infty\}} J_\lambda = \mathrm{Avg}_i^{\{1,\infty\}} J_\lambda \tag{4.13}$$

4. $\langle \check{\alpha}_i, \lambda \rangle = -1,\ \lambda \neq -\rho$ (walls of $-\rho$ shifted dominant cone)

$$\mathrm{Avg}_i^s J_\lambda = \mathrm{Avg}_i^s T_i^{S\smallsetminus\{s\}} J_{s_i \cdot \lambda} \text{ for } s \in S \tag{4.14}$$

One wants to show that $\widetilde{C} \to C_{\mathrm{Eis}}$ given by $1 \mapsto \mathrm{Eis}_0$ is an isomorphism. Study the map with respect to the $(\mathcal{H}_{\mathrm{fin}})^{\otimes S}$ action. For $\lambda \in \Lambda_+$, define the the $(\mathcal{H}_{\mathrm{fin}})^{\otimes S}$ submodules $\widetilde{C}^\lambda \subset \widetilde{C}$ and $C^\lambda_{Eis} \subset C_{\mathrm{Eis}}$ as follows.

$$\widetilde{C}^\lambda := \mathrm{Span}_{(\mathcal{H}_{\mathrm{fin}})^{\otimes S}}\{J_{w\cdot\lambda} : w \in W,\ w \cdot \lambda \in -\rho + \Lambda_+\}$$

$$C^\lambda_{\mathrm{Eis}} := \mathrm{Span}_{(\mathcal{H}_{\mathrm{fin}})^{\otimes S}}\{\mathrm{Eis}_{w\cdot\lambda} : w \in W, w \cdot\ \lambda \in -\rho + \Lambda_+\}$$

It suffices to show that

$$C_{\mathrm{Eis}} \cong \oplus_{\lambda\in\Lambda_+} C^\lambda_{\mathrm{Eis}} \tag{4.15}$$

$$\dim_{\mathbb{C}}(C^\lambda_{\mathrm{Eis}}) \geq \dim_{\mathbb{C}}(\widetilde{C}^\lambda) \text{ for } \lambda \in \Lambda_+ \tag{4.16}$$

These are established by Propositions 4.5.1 and 4.5.2.

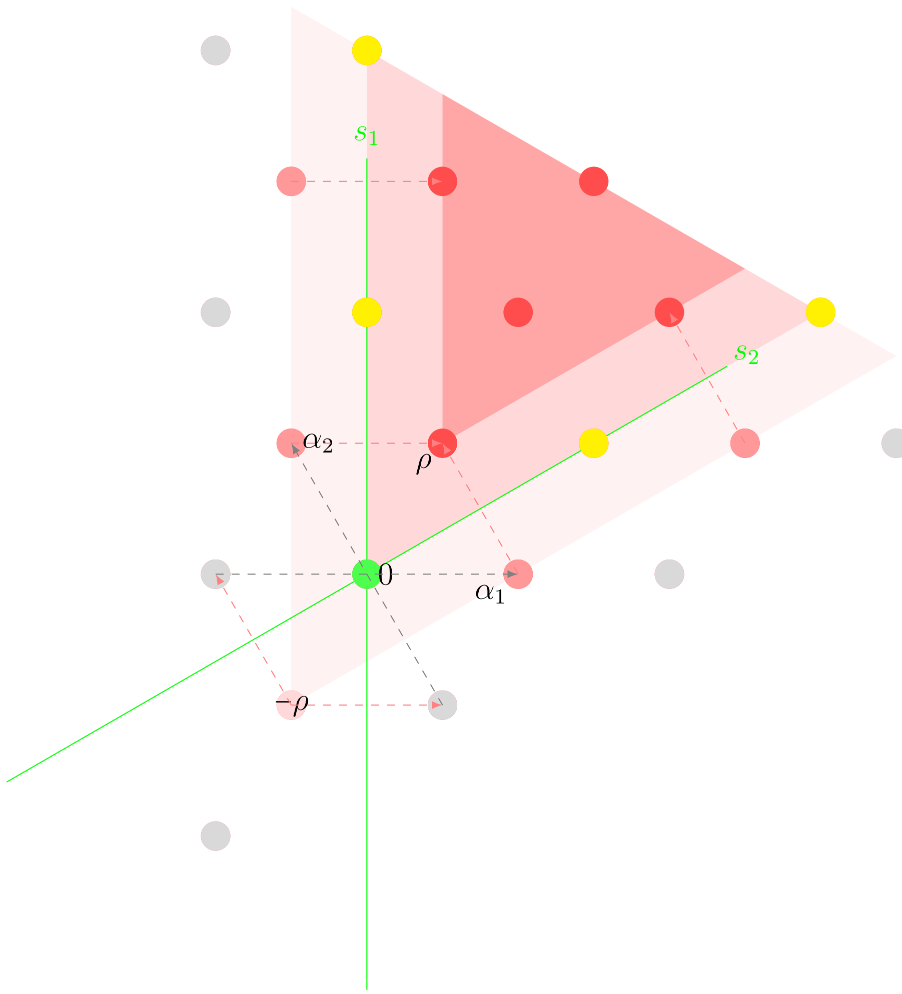


Figure 4.1: Lattice of coweights of SL(3); depicts the structure of $\widetilde{C}$ as a $(\mathcal{H}_{\text{fin}})^{\otimes S}$ module. The module is generated by the shifted dominant cone $-\rho + \Lambda_+$. The generator 0 satisfies the reflection relation. Generators (colored yellow) along a wall satisfy the reflection relations for only one simple root. Dashed red arrow indicated generators are related as Eis_{-1} and Eis_1 (see PGL(2) example).

Proposition 4.5.1. Equation 4.15 is true and

$$\dim_{\mathbb{C}}(C^{\lambda}_{\mathrm{Eis}}) = \begin{cases} 69 & \lambda = 0 \\ 6^3 + 3^3 + 3^3 + 1^3 & \lambda = \rho \\ 3^3 \cdot 5 & \langle \check{\alpha}_i, \lambda \rangle = 0,\ \lambda \neq 0 \\ 3^3 \cdot (2^3 + 1) & \langle \check{\alpha}_i, \lambda \rangle = 1,\ \lambda \neq \rho \\ 6^3 & \lambda \in 2\rho + \Lambda_+ \end{cases}$$

Proposition 4.5.2.

$$\dim_{\mathbb{C}}(\widetilde{C}^{\lambda}) \leq \begin{cases} 69 & \lambda = 0 \\ 6^3 + 3^3 + 3^3 + 1^3 & \lambda = \rho \\ 3^3 \cdot 5 & \langle \check{\alpha}_i, \lambda \rangle = 0,\ \lambda \neq 0 \\ 3^3 \cdot (2^3 + 1) & \langle \check{\alpha}_i, \lambda \rangle = 1,\ \lambda \neq \rho \\ 6^3 & \lambda \in 2\rho + \Lambda_+ \end{cases}$$

Proof of Proposition 4.5.1

To prove Proposition 4.5.1, I'll first describe the geometry of the fibers $\mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, S) \to \mathrm{Bun}_{\underline{G}}(\mathbb{P}^1)$ and identify the Eisenstein objects Eis_λ for $\lambda \in -\rho + \Lambda_+$. For $\lambda \in \Lambda_+$, let $\mathrm{Bun}^{\lambda}_{\underline{G}}(\mathbb{P}^1, S)$ denote the fiber above $\lambda \in \Lambda_+$. Except for $\mathrm{Eis}_{-\rho}$, all these Eisenstein objects are nonzero only on a single point, which lies in $\mathrm{Bun}^{\tilde{\lambda}}_{\underline{G}}(\mathbb{P}^1, S)$, where $\tilde{\lambda} \in \Lambda_+$ is in the W orbit of λ. Additionally, for $\lambda \in \Lambda_+ \smallsetminus \{0, \rho\}$, $C^{\lambda}_{\mathrm{Eis}}$ is equal to the space of all automorphic functions taking nonzero values only on points of $\mathrm{Bun}^{\lambda}_{\underline{G}}(\mathbb{P}^1, S)$.

Objects of $\mathrm{Bun}_{\underline{G}}(\mathbb{P}^1)$ are represented by rank 3 vector bundles $\mathcal{E}$, whose determinant bundle is trivial. Objects of $\mathrm{Bun}_{\underline{G}}(\mathbb{P}^1, S)$ are represented by $\mathcal{E} \in \mathrm{Bun}_{\underline{G}}(\mathbb{P}^1)$ with flags $F_s = (\ell_s, p_s)$, $\ell_s \subset p_s \subset \mathcal{E}|_s$.

$\mathcal{E} \cong \mathcal{O}(0)$

$\mathrm{Bun}^{0}_{\underline{G}}(\mathbb{P}^1, S)$ is identified with the orbits of the triple flag variety $G \backslash \mathcal{B}^S$. The generic configuration is when the flags are pairwise transverse and the following two conditions are satisfied:

- The lines ℓ_s are not coplanar.
- The planes p_s are not concurrent.

For $(p, q) \in S \times S$ with $p \neq q$, there is a map $\pi_{p,q} : G \backslash \mathcal{B}^S \to G \backslash (\mathcal{B} \times \mathcal{B})$. Identify the points of $G \backslash (\mathcal{B} \times \mathcal{B}) \cong B \backslash G / B$ with W by relative position of flags. Explicitly, for $w \in W$, (F_1, F_2), is in relative position w,

- $w = 1$ if $\ell_1 = \ell_2$ and $p_1 = p_2$
- $w = s_1$ if $\ell_1 \neq \ell_2$ and $p_1 = p_2$
- $w = s_2$ if $\ell_1 = \ell_2$ and $p_1 \neq p_2$
- $w = s_2 s_1$ if $\ell_2 \in p_1$, $\ell_1 \notin p_2$
- $w = s_1 s_2$ if $\ell_2 \notin p_1$, $\ell_1 \in p_2$
- $w = s_3$ if $\ell_2 \notin p_1$, $\ell_1 \notin p_2$

If (F_0, F_1) are in relative position w and (F_1, F_∞) are in relative position w', then the possible relative positions of (F_0, F_∞) are exactly those $w'' \in W$ such that $T_{w''}$ has a nonzero coefficient in $T_{w'}T_w \in \mathcal{H}_{\text{fin}}$. In particular, if $\ell(w) + \ell(w') = \ell(w'w)$, then the relative position of (F_0, F_∞) must be $w'w$. The other cases are

- $w = w' = s_i$. $T_{w'}T_w = (q-1)T_{s_i} + q$.
- $w = s_i$, $w' = s_j s_i$. $T_{w'}T_w = (q-1)T_{s_j s_i} + qT_{s_j}$.
- $w = s_i$, $w' = s_3$. $T_{w'}T_w = (q-1)T_{s_3} + qT_{s_i s_j}$.
- $w = s_i s_j$, $w' = s_i$. $T_{w'}T_w = (q-1)T_{s_i s_j} + qT_{s_j}$.
- $w = s_i s_j$, $w' = s_j s_i$. $T_{w'}T_w = (q-1)T_{s_3} + q(q-1)T_{s_j} + q^2$
- $w = w' = s_i s_j$. $T_{w'}T_w = (q-1)T_{s_3} + qT_{s_j s_i}$
- $w = s_i s_j$, $w' = s_3$. $T_{w'}T_w = (q-1)^2 T_{s_3} + q(q-1)T_{s_j s_i} + q(q-1)T_{s_i s_j} + q^2 T_{s_i}$.
- $w = s_3$, $w' = s_i$. $T_{w'}T_w = (q-1)T_{s_3} + qT_{s_j s_i}$.
- $w = s_3$, $w' = s_j s_i$. $T_{w'}T_w = (q-1)^2 T_{s_3} + q(q-1)T_{s_j s_i} + q(q-1)T_{s_i s_j} + q^2 T_{s_i}$.
- $w = s_3$, $w' = s_3$. $T_{w'}T_w = (q-1)(q^2-q+1)T_{s_3} + q(q-1)^2 T_{s_i s_j} + q(q-1)^2 T_{s_j s_i} + q^2(q-1)T_{s_i} + q^2(q-1)T_{s_j} + q^3$

s_i is one simple reflection, s_j is the other. Let $\pi : G\backslash(\mathcal{B}^S) \to (B\backslash G/B)^3$ be given by $(\pi_{0,1}, \pi_{1,\infty}, \pi_{0,\infty})$. Let $c_0(w, w', w'')$ be the preimage of (w, w', w''). From the above calculation, $c_0(w, w', w'')$ is nonempty exactly for the following triples:

1. (w, w', w''), with $w'' = w'w$. There are exactly 36 such triples.

2. (w, w', w'') is one of the following 33 triples:

$$
\begin{aligned}
&(s_1,s_1,s_1),(s_2,s_2,s_2),(s_1,s_2s_1,s_2s_1),(s_2,s_1s_2,s_1s_2),(s_1,s_3,s_3),(s_2,s_3,s_3),\\
&(s_1s_2,s_1,s_1s_2),(s_2s_1,s_2,s_2s_1),(s_1s_2,s_2s_1,s_3),(s_1s_2,s_2s_1,s_1),(s_2s_1,s_1s_2,s_3),\\
&(s_2s_1,s_1s_2,s_2),(s_1s_2,s_1s_2,s_3),(s_2s_1,s_2s_1,s_3),(s_1s_2,s_3,s_3),(s_1s_2,s_3,s_1s_2),\\
&(s_1s_2,s_3,s_2s_1),(s_2s_1,s_3,s_3),(s_2s_1,s_3,s_2s_1),(s_2s_1,s_3,s_1s_2),(s_3,s_1,s_3),(s_3,s_2,s_3),\\
&(s_3,s_2s_1,s_3),(s_3,s_2s_1,s_2s_1),(s_3,s_2s_1,s_1s_2),(s_3,s_1s_2,s_3),(s_3,s_1s_2,s_1s_2),(s_3,s_1s_2,s_2s_1)\\
&,(s_3,s_3,s_1),(s_3,s_3,s_2),(s_3,s_3,s_1s_2),(s_3,s_3,s_2s_1),(s_3,s_3,s_3)
\end{aligned}
$$

One can check that each of these loci $c_0(w, w', w'')$ has exactly one isomorphism class of objects, except for the locus $c_0(s_3, s_3, s_3)$, classifying triples of pairwise transverse flags. This locus is as follows:

$$
\begin{array}{ccccc}
 & & c_0(s_3,s_3,s_3;\varnothing) & & \\
 & \swarrow & & \searrow & \\
c_0(s_3,s_3,s_3;\{s_1\}) & & & & c_0(s_3,s_3,s_3;\{s_2\}) \\
 & \searrow & & \swarrow & \\
 & & c_0(s_3,s_3,s_3;\{s_1,s_2\}) & &
\end{array}
$$

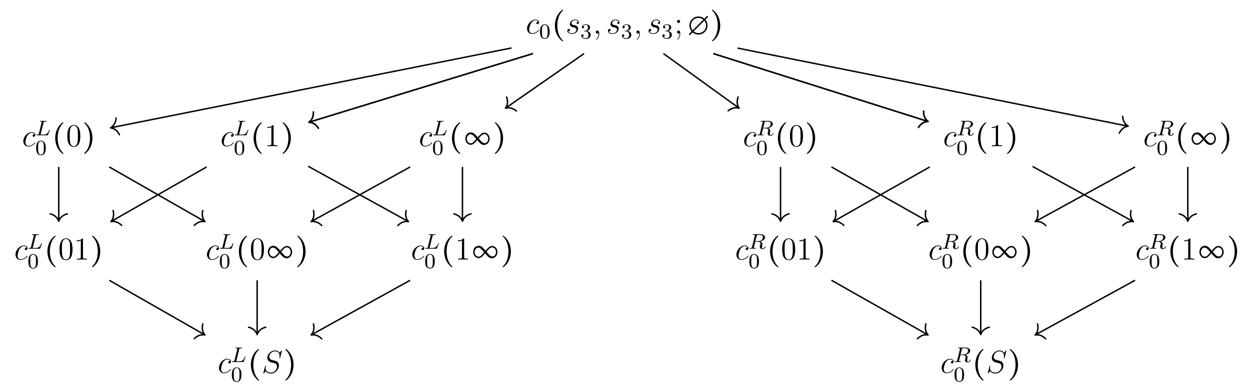
$c_0(s_3, s_3, s_3; \{s_1, s_2\})$ is the configuration where ℓ_s are coplanar and p_s are concurrent. On the other hand, $c_0(s_3, s_3, s_3; \{s_1\})$ is the configuration where ℓ_s are coplanar and p_s are non concurrent. $c_0(s_3, s_3, s_3; \{s_2\})$ is the configuration where ℓ_s are not coplanar and p_s are concurrent. $c_0(s_3, s_3, s_3; \varnothing)$ is the generic configuration. The loci of flags in generic configuration forms a space isomorphic to the complement of three points in $\mathbb{P}^1$; the remaining subloci of $c_0(s_3, s_3, s_3)$ have a unique point.

Definition 4.5.1. $C^0_{loc} \subset C_{\text{Aut}}$ is the subspace of functions supported on points where the bundle is trivial and constant along the generic locus $c_0(s_3, s_3, s_3; \varnothing)$.

By enumerating points I have shown that $\dim_{\mathbb{C}}(C^0_{loc}) = 72$. C^0_{Eis} is generated over $(\mathcal{H}_{\text{fin}})^{\otimes S}$ by $\text{Eis}_0 = \underline{1}_{c_0(1,1,1)}$. Furthermore, all functions on C^0_{Eis} are constant along the generic locus $c_0(s_3, s_3, s_3; \varnothing)$. Therefore, $C^0_{\text{Eis}} \subset C^0_{loc}$.

Lemma 4.5.1. C^0_{Eis} is codimension three in C^0_{loc}.

Proof. I describe the equations describing C^0_{Eis} in C^0_{loc}. Let $c_0(*)$ denote the locus where ℓ_s are not coplanar and p_s are not concurrent. The following are subloci of $c_0(*)$ organized so that $x \to y$ means y is contained in the closure of x.



For $R \subset S$, $c_0^L(R) \subset c_0(*)$ is the sublocus where $\ell_s \in p_{L(s)}$ if and only if $s \in R$, where $L(s)$ denotes the predecessor of s in the cyclic ordering $0 \to 1 \to \infty \to 0$. $c_0(R) \subset c_0(*)$ is the sublocus where $\ell_s \in P_{R(s)}$ if and only if $s \in R$, where $R(s)$ denotes the successor of s in the same cyclic ordering. In the previous notation

1. $c_0^L(S) = c_0(s_2s_1, s_2s_1, s_1s_2)$
2. $c^L(01) = c_0(s_2s_1, s_3, s_1s_2), c^L(0\infty) = c_0(s_3, s_2s_1, s_1s_2), c_0^L(1\infty) = c_0(s_2s_1, s_2s_1, s_3)$
3. $c^L(0) = c_0(s_3, s_3, s_1s_2), c^L(1) = c_0(s_2s_1, s_3, s_3), c_0^L(\infty) = c_0(s_3, s_2s_1, s_3)$
4. $c_0^R(S) = c_0(s_1s_2, s_1s_2, s_2s_1)$
5. $c^R(01) = c_0(s_1s_2, s_1s_2, s_3), c^R(0\infty) = c_0(s_1s_2, s_3, s_2s_1), c_0^R(1\infty) = c_0(s_3, s_1s_2, s_2s_1)$
6. $c^R(0) = c_0(s_1s_2, s_3, s_3), c^R(1) = c_0(s_3, s_1s_2, s_3), c_0^R(\infty) = c_0(s_3, s_3, s_2s_1)$

$C^0_{\text{Eis}} \subset C^0_{loc}$ is the subspace of functions, f, such that

$$\begin{aligned}
&f(c_0(s_3, s_3, s_3; \varnothing)) - f(c_0(s_3, s_3, s_3; \{s_1\})) - f(c_0(s_3, s_3, s_3; \{s_2\})) \\
&+ f(c_0(s_3, s_3, s_3; \{s_1, s_2\})) = 0 \\
&f(c_0(s_3, s_3, s_3; \varnothing)) + \sum_{R \subset S; R \neq \varnothing} (-1)^{|R|} f(c_0^L(R)) = 0 \\
&f(c_0(s_3, s_3, s_3; \varnothing)) + \sum_{R \subset S; R \neq \varnothing} (-1)^{|R|} f(c_0^R(R)) = 0
\end{aligned}$$

$f(c_0(s_3, s_3, s_3; \varnothing))$ is the common value of f on any point of the generic locus $c_0(s_3, s_3, s_3; \varnothing)$.
□

Interlude on Bundles with a Positive Splitting

The following is a special case. The general principle will be elaborated upon in a future document. Suppose that $\mathcal{E} \cong \mathcal{O}(\lambda)$ admits a *positive* splitting $\mathcal{E} \cong \mathcal{E}_1 \oplus \mathcal{E}_2$, which means that $\operatorname{Hom}(\mathcal{E}_1, \mathcal{E}_2) = 0$. For example, if $\mathcal{E}_1 \cong \mathcal{O}(m) \oplus \mathcal{O}(n)$ and $\mathcal{E}_2 \cong \mathcal{O}(k)$ then the positivity condition is $m, n \geq k+1$. Let $P \supset B$ be the parabolic subgroup corresponding to the splitting.

If $\mathcal{E}_1$ is rank two, then $P = P_{s_1}$ and if $\mathcal{E}_1$ is rank one, then $P = P_{s_2}$. The subbundle $\mathcal{E}_1$ is stable under $\mathrm{Aut}(\mathcal{E})$, so there is a subspace $\mathcal{E}_s^{\mathrm{stab}} \subset \mathcal{E}|_s$ given by restriction of $\mathcal{E}_1$. Let $\mathrm{fib}_s : \mathrm{Bun}_{\underline{G}}^{\lambda}(\mathbb{P}^1, S) \to P\backslash G/B$ be given by relative position of $(\mathcal{E}_s^{\mathrm{stab}}, F_s)$. For example, if $\mathcal{E}_1$ is rank two, the relative position, w is given by:

- $w = 1$ if $p_s = \mathcal{E}_s^{\mathrm{stab}}$
- $w = s_2$ if $\ell_s \subset \mathcal{E}_s^{\mathrm{stab}}$ but $p_s \neq \mathcal{E}|_s$
- $w = s_1 s_2$ if $\ell \notin \mathcal{E}_s^{\mathrm{stab}}$

There is also a map $\mathrm{Bun}_{\underline{G}}^{\lambda}(\mathbb{P}^1) \to \mathrm{Bun}_L^{\lambda_1}(\mathbb{P}^1)$, given by $\mathcal{E} \mapsto \mathcal{E}_1 \oplus \mathcal{E}/\mathcal{E}_1$, where $L \subset P$ is the Levi subgroup and $\mathcal{E}_1 \cong \mathcal{O}(\lambda_1)$. At the level of rational points, this can be lifted to included parabolic structure:

$$\mathrm{split} : \mathrm{Bun}_{\underline{G}}^{\lambda}(\mathbb{P}^1, S) \to \mathrm{Bun}_L^{\lambda_1}(\mathbb{P}^1, S)$$

For example, if $\mathcal{E}_1$ is rank two, the parabolic structure for $\mathcal{E}_1$ at s is given by $p_s \cap \mathcal{E}_s^{\mathrm{stab}}$ if p_s is transverse to $\mathcal{E}_s^{\mathrm{stab}}$ and otherwise by ℓ_s. The splitting map is not continuous on the underlying moduli spaces.

Suppose further that the splitting $\mathcal{E} \cong \mathcal{E}_1 \oplus \mathcal{E}_2$ is *very positive*, which means $\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2 \otimes \omega_{\mathbb{P}^1}(S)) = 0$. For example, if $\mathcal{E}_1 \cong \mathcal{O}(m) \oplus \mathcal{O}(n)$ and $\mathcal{E}_2 \cong \mathcal{O}(k)$ then the condition is $m, n \geq k + 2$. Calculating the action of $\mathrm{Aut}(\mathcal{E})$ on $\prod_{s \in S} \mathcal{E}|_s$ shows that the product of the splitting map and $\mathrm{fib} := \prod_{s \in S} \mathrm{fib}_s$ is a bijection on points.

$$(P\backslash G/B)^S \leftarrow \mathrm{Bun}_{\underline{G}}^{\lambda}(\mathbb{P}^1, S) \to \mathrm{Bun}_L^{\lambda_1}(\mathbb{P}^1, S)$$

$\mathcal{E} \cong \mathcal{O}(\rho)$

The $\underline{B}$-bundle $\mathcal{O}(1) \subset \mathcal{O}(1) \oplus \mathcal{O} \subset \mathcal{E}$ is stable under $\mathrm{Aut}(\mathcal{E})$. For $s \in S$, there is a flag $F_s^{\mathrm{stab}} = (\ell_s^{\mathrm{stab}}, p_s^{\mathrm{stab}}) \subset \mathcal{E}|_s$, given by restriction of the stable $\underline{B}$-bundle, also invariant under $\mathrm{Aut}(\mathcal{E})$. There is a map $\mathrm{fib}_s : \mathrm{Bun}_{\underline{G}}^{\rho}(\mathbb{P}^1, S) \to B\backslash G/B$ given by the relative position $(F_s^{\mathrm{stab}}, F_s)$ of the flag F_s in the fiber at s to the stable $\underline{B}$-bundle. Let $c_\rho(w_0, w_1, w_\infty)$ denote the locus where the relative position of F_s to the stable flag is $w_s \in W$.

There are two splitting maps, for $i = 1, 2$:

$$\mathrm{split}_i : \mathrm{Bun}_{\underline{G}}^{\rho}(\mathbb{P}^1, S) \to \mathrm{Bun}_{\mathrm{PGL}(2)}^{1}(\mathbb{P}^1, S)$$

The parabolic structure at $s \in S$ for split_1 is given by the distinguished line $\ell_s^{\mathrm{dist},1} \subset p_s^{\mathrm{stab}}$ defined as $\ell_s^{\mathrm{dist},1} = p_s \cap p_s^{\mathrm{stab}}$ if F_s is transverse to p_s^{stab} and ℓ_s otherwise. The parabolic structure for split_2 is given by the distinguished plane $\ell_s^{\mathrm{dist},2} \subset \mathcal{E}|_s / \ell_s^{\mathrm{stab}}$ given by $(\ell_s \oplus \ell_s^{\mathrm{stab}})/\ell_s^{\mathrm{stab}}$ if F_s is transverse to ℓ_s^{stab} and $p_s/\ell_s^{\mathrm{stab}}$, otherwise.

Explicitly, the points of $c_\rho(w_0, w_1, w_\infty)$ are as follows.

1. If for each $i = 1, 2$, there is at least one $s \in S$ such that $\ell(w_s s_1) > \ell(w_s)$, then the locus consists of a single point.

2. $\ell(w_s s_1) < \ell(w_s)$ for all $s \in S$, but there is at least one $s' \in S$ such that $\ell(w_{s'} s_2) > \ell(w_{s'})$. This locus consists of two points. The generic configuration, $c_\rho(w_0, w_1, w_\infty; \varnothing)$ is where the distinguished lines $\ell_s^{\text{dist},1}$ are not contained in the image of a map $\mathcal{O} \to \mathcal{O}(1) \oplus \mathcal{O}$. The degenerate locus, $c_\rho(w_0, w_1, w_\infty; \{s_1\})$ is where there is such a map.

3. $\ell(w_s s_2) < \ell(w_s)$ for all $s \in S$, but there is at least one $s' \in S$ such that $\ell(w_{s'} s_1) > \ell(w_{s'})$. The generic configuration, $c_\rho(w_0, w_1, w_\infty; \varnothing)$ is where the distinguished lines $\ell_s^{\text{dist},2}$ are not contained in the image of a map $\mathcal{O}(-1) \to \mathcal{E}/\mathcal{O}(1)$. The degenerate locus, $c_\rho(w_0, w_1, w_\infty; \{s_2\})$ is where there is such a map.

4. $w_0 = w_1 = w_\infty = s_3$. This locus has four points

$$\begin{array}{ccc} & c_\rho(s_3, s_3, s_3; \varnothing) & \\ \swarrow & & \searrow \\ c_\rho(s_3, s_3, s_3; \{s_1\}) & & c_\rho(s_3, s_3, s_3; \{s_2\}) \\ \searrow & & \swarrow \\ & c_\rho(s_3, s_3, s_3; \{s_1, s_2\}) & \end{array}$$

For $\delta \subset \{s_1, s_2\}$ $c_\rho(s_3, s_3, s_3; \delta)$ is the locus where the distinguished lines $\ell_s^{\text{dist},1} = p_s \cap p_s^{\text{stab}}$ are contained in the image of a map $\mathcal{O} \to \mathcal{O}(1) \oplus \mathcal{O}$ if and only if $s_1 \in \delta$ and the distinguished lines $\ell_s^{\text{dist},2} = p_s / \ell_s^{\text{stab}}$ is contained in the image of a map $\mathcal{O}(-1) \to \mathcal{E}/\mathcal{O}(1)$ if and only if $s_2 \in \delta$.

The Eisenstein objects in C_{Eis}^{ρ} are

$$\begin{aligned} \text{Eis}_\rho &= \underline{1}_{c_\rho(1,1,1)} \\ \text{Eis}_{s_1 \cdot \rho} &= \underline{1}_{c_\rho(s_1, s_1, s_1; \{s_1\})} \\ \text{Eis}_{s_2 \cdot \rho} &= \underline{1}_{c_\rho(s_2, s_2, s_2; \{s_2\})} \\ \text{Eis}_{-\rho} &= \underline{1}_{c_\rho(s_3, s_3, s_3; \{s_1, s_2\})} + \underline{1}_{c_0(s_3, s_3, s_3; \varnothing)} \end{aligned}$$

The finite Hecke module generated by Eis_ρ consists of all functions on the points $\text{Bun}_G^\rho(\mathbb{P}^1, S)$ constant along the loci $c_\rho(w_0, w_1, w_\infty)$. $\text{Eis}_{s_i \cdot \rho}$ generates, under finite Hecke modification, the constant function on points $c_\rho(w_0, w_1, w_\infty; \{s_i\})$ for $(w_0, w_1, w_\infty) \neq (s_3, s_3, s_3)$ as well as the function

$$\underline{1}_{c_\rho(s_3, s_3, s_3; \{s_1, s_2\})} + \underline{1}_{c_\rho(s_3, s_3, s_3; \{s_i\})}.$$

Therefore, C_{Eis}^{ρ} consists of functions, f, vanishing away from the points of the loci $\text{Bun}_{\underline{G}}^\rho(\mathbb{P}^1, S)$ and $c_0(s_3, s_3, s_3; \varnothing)$ that are constant along $c_0(s_3, s_3, s_3; \varnothing)$ and satisfy

$$f(c_0(s_3,s_3,s_3;\varnothing)) = f(c_\rho(s_3,s_3,s_3;\{s_1,s_2\}).$$

It follows that

$$\dim_{\mathbb{C}}(C^{\rho}_{\text{Eis}}) = \left|\text{Bun}^{\rho}_{\underline{G}}(\mathbb{P}^1,S)\right| = 6^3+3^3+3^3+1^3.$$

$\mathcal{E}\cong\mathcal{O}(\lambda)$, $\langle\check{\alpha}_i,\lambda\rangle = 0$, $\lambda\neq 0$

Without loss of generality, assume $\langle\check{\alpha}_1,\lambda\rangle = 0$. Then $\mathcal{E}\cong\mathcal{O}(k)\oplus\mathcal{O}(k)\oplus\mathcal{O}(-2k)$, for some $k\geq 1$. There is a bijection of points

$$\text{Bun}^{\lambda}_{\underline{G}}(\mathbb{P}^1,S)\leftrightarrow (P_{s_1}\backslash G/B)^S\times\text{Bun}^0_{\text{PGL}(2)}(\mathbb{P}^1,S)$$

Eis_λ is the constant function on the point corresponding to $c_0(S)\times(1,1,1)$. By the PGL(2) calculation, for every point, pt, of $\text{Bun}^0_{\text{PGL}(2)}(\mathbb{P}^1,S)$, C^{λ}_{Eis} contains the constant function on the point corresponding to $\text{pt}\times(1,1,1)$. Furthermore, for $w_0,w_1,w_\infty\in\{1,s_2,s_1s_2\}\cong P_{s_1}\backslash G/B$,

$$T^0_{w_0}T^1_{w_1}T^\infty_{w_\infty}\underline{1}_{\text{pt}\times(1,1,1)} = \underline{1}_{\text{pt}\times(w_0,w_1,w_\infty)}.$$

Therefore, C^{λ}_{Eis} consists of all functions taking nonzero value only on the points of $\text{Bun}^{\lambda}_{\underline{G}}(\mathbb{P}^1,S)$, so

$$\dim_{\mathbb{C}}(C^{\lambda}_{\text{Eis}}) = \left|(P_{s_1}\backslash G/B)^S\times\text{Bun}^0_{\text{PGL}(2)}(\mathbb{P}^1,S)\right| = 3^3\cdot 5$$

$\mathcal{E}\cong\mathcal{O}(\lambda)$, $\langle\check{\alpha}_i,\lambda\rangle = 1$, $\lambda\neq \rho$

Without loss of generality, assume $\langle\check{\alpha}_1,\lambda\rangle = 1$. Then $\mathcal{E}\cong\mathcal{O}(k+1)\oplus\mathcal{O}(k)\oplus\mathcal{O}(-2k-1)$, for some $k\geq 1$. There is a bijection of points

$$\text{Bun}^{\lambda}_{\underline{G}}(\mathbb{P}^1,S)\leftrightarrow (P_{s_1}\backslash G/B)^S\times\text{Bun}^1_{\text{PGL}(2)}(\mathbb{P}^1,S)$$

Eis_λ is the constant function on the point corresponding to $c_1(S)\times(1,1,1)$ and $\text{Eis}_{s_1(\lambda)}$ is the constant function on the point corresponding to $c_1(\varnothing)\times(1,1,1)$. By the PGL(2) calculation, for every point, pt, of $\text{Bun}^1_{\text{PGL}(2)}(\mathbb{P}^1,S)$, C^{λ}_{Eis} contains the constant function on the point corresponding to $\text{pt}\times(1,1,1)$. Furthermore, for $w_0,w_1,w_\infty\in\{1,s_2,s_1s_2\}\cong P_{s_1}\backslash G/B$,

$$T^0_{w_0}T^1_{w_1}T^\infty_{w_\infty}\underline{1}_{\text{pt}\times(1,1,1)} = \underline{1}_{\text{pt}\times(w_0,w_1,w_\infty)}.$$

Therefore, C^{λ}_{Eis} consists of all functions taking nonzero value only on the points of $\text{Bun}^{\lambda}_{\underline{G}}(\mathbb{P}^1,S)$, so

$$\dim_{\mathbb{C}}(C^{\lambda}_{\text{Eis}}) = \left|(P_{s_1}\backslash G/B)^S\times\text{Bun}^1_{\text{PGL}(2)}(\mathbb{P}^1,S)\right| = 3^3\cdot (2^3+1)$$

$\mathcal{E} \cong \mathcal{O}(\lambda)$, $\lambda \in 2\rho + \Lambda_+$

There is $\underline{B}$-bundle stable under $\mathrm{Aut}(\mathcal{E})$. There is a map $\mathrm{fib}_s : \mathrm{Bun}^{\lambda}_{\underline{G}}(\mathbb{P}^1, S) \to B\backslash G/B$ given by the relative position $(F^{\mathrm{stab}}_s, F_s)$ of the flag F_s in the fiber at s to the stable $\underline{B}$-bundle. The points of the locus $\mathrm{Bun}^{\lambda}_{\underline{G}}(\mathbb{P}^1, S)$ are identified with $(B\backslash G/B)^S$. Moreover, Eis_λ is identified with $\underline{1}_{(1,1,1)}$ and $T^0_{w_0} T^1_{w_1} T^{\infty}_{w_\infty} \mathrm{Eis}_\lambda$ is identified with $\underline{1}_{(w_0,w_1,w_\infty)}$. Therefore, there is an isomorphism of $(\mathcal{H}_{\mathrm{fin}})^{\otimes S}$ modules $(\mathcal{H}_{\mathrm{fin}})^{\otimes S} \to C^{\lambda}_{\mathrm{Eis}}$ given by $1 \mapsto \mathrm{Eis}_\lambda$. $\dim_{\mathbb{C}}(C^{\lambda}_{\mathrm{Eis}}) = |W|^3$.

Proof of Equation 4.15

First, check that $C^0_{\mathrm{Eis}} \cap C^{\rho}_{\mathrm{Eis}} = 0$. Indeed, every function in C^0_{Eis} takes nonzero values only on points of $\mathrm{Bun}^0_{\underline{G}}(\mathbb{P}^1, S)$, but every nontrivial function in C^{ρ}_{Eis} takes nonzero value on some point of $\mathrm{Bun}^{\rho}_{\underline{G}}(\mathbb{P}^1, S)$. Then, observe that the spaces $\{C^{\lambda}_{\mathrm{Eis}}\}_{\lambda \in \Lambda_+ \smallsetminus \{0,\rho\}} \cup \{C^0_{\mathrm{Eis}} \oplus C^{\rho}_{\mathrm{Eis}}\}$ are pairwise orthogonal. This is because functions in $C^0_{\mathrm{Eis}} \oplus C^{\rho}_{\mathrm{Eis}}$ take nonzero values only on points of $\mathrm{Bun}^0_{\underline{G}}(\mathbb{P}^1, S) \cup \mathrm{Bun}^{\rho}_{\underline{G}}(\mathbb{P}^1, S)$, whereas functions in $C^{\lambda}_{\mathrm{Eis}}$ for $\lambda \in \Lambda_+ \smallsetminus \{0, \rho\}$ only take nonzero value on points of $\mathrm{Bun}^{\lambda}_{\underline{G}}(\mathbb{P}^1, S)$.

Remark 4.5.1. The space of cusp forms $C_{cusp} \subset C_{\mathrm{Aut}}$ is the space orthogonal to C_{Eis}. See that cusp forms are functions taking nonzero values only on the generic locus of $c_0(s_3, s_3, s_3; \varnothing)$, as well as the following points of $\mathrm{Bun}^0_{\underline{G}}(\mathbb{P}^1, S) \cup \mathrm{Bun}^{\rho}_{\underline{G}}(\mathbb{P}^1, S)$:

1. $c_0(s_3, s_3, s_3; \delta)$ for $\delta \subset \{s_1, s_2\}$ nonempty
2. $c^L_0(R)$ for $R \subset S$ nonempty
3. $c^R_0(R)$ for $R \subset S$ nonempty
4. $c_\rho(s_3, s_3, s_3; \{s_1, s_2\})$

The space of cusp forms is given by the following equations.

$$f(c_\rho(s_3,s_3,s_3;\{s_1,s_2\}) = -\sum_{\mathrm{pt}\in c_0(s_3,s_3,s_3;\{s_1,s_3\})} f(\mathrm{pt})$$

$$f(c_0^L(S)) = -(q-1)f(c_0^L(0)) = -(q-1)f(c_0^L(1)) = -(q-1)f(c_0^L(\infty)) = (q-1)^2 f(c_0^L(01))$$
$$= (q-1)^2 f(c_0^L(0\infty)) = (q-1)^2 f(c_0^L(1\infty))$$

$$f(c_0^R(S)) = -(q-1)f(c_0^R(0)) = -(q-1)f(c_0^R(1)) = -(q-1)f(c_0^R(\infty)) = (q-1)^2 f(c_0^R(01))$$
$$= (q-1)^2 f(c_0^R(0\infty)) = (q-1)^2 f(c_0^R(1\infty))$$

$$f(c_0(s_3,s_3,s_3;\{s_1,s_2\}) = -(q-1)f(c_0(s_3,s_3,s_3;\{s_1\}) = -(q-1)f(c_0(s_3,s_3,s_3;\{s_2\})$$

$$\sum_{\mathrm{pt}\in c_0(s_3,s_3,s_3;\{s_1,s_3\})} f(\mathrm{pt}) + f(c_0(s_3,s_3,s_3;\{s_1\}) + f(c_0^L(01)) + f(c_0^R(01)) = 0$$

Counting points and constraints shows $\dim_{\mathbb{C}}(C_{cusp}) = q$.

Proof of Proposition 4.5.2

$\lambda = 0$

$\widetilde{C}^0$ is generated over $(\mathcal{H}_{\mathrm{fin}})^{\otimes S}$ by J_0. Using Equation 4.9 one can always write any monomial $T^0_{w_0}T^1_{w_1}T^\infty_{w_\infty}$, $w_s \in W$, as a sum of monomials where for any $i \in \{1,2\}$

$$\ell(w_\infty s_i) < \ell(w_\infty) \implies \ell(w_0 s_i) > \ell(w_0),\ \ell(w_1 s_i) > \ell(w_1).$$

Let us list the triples (w_0,w_1,w_∞) that satisfy this condition.

1. $(w_0,w_1,1),\ w_0,w_1 \in W$
2. $(w_0,w_1,w_\infty),\ w_\infty \in \{s_1,s_2s_1\},\ w_0,w_1 \in \{1,s_2,s_1s_2\}$
3. $(w_0,w_1,w_\infty),\ w_\infty \in \{s_2,s_1s_s\},\ w_0,w_1 \in \{1,s_1,s_2s_1\}$
4. $(1,1,s_3)$

There are $|W|^2 + 2\cdot 3^2 + 2\cdot 3^2 + 1 = 73$ such triples. Let M be the set of 69 monomials formed from excluding the following four from the 73 listed monomials:

$$T^0_{s_1}T^1_{s_1}T^\infty_{s_1s_2}, T^0_{s_2s_1}T^1_{s_1}T^\infty_{s_1s_2}, T^0_{s_1}T^1_{s_2s_1}T^\infty_{s_1s_2}, T^0_{s_2s_1}T^1_{s_2s_1}T^\infty_{s_1s_2}$$

$\widetilde{C}^0$ is spanned over $\mathbb{C}$ by M. This follows from two Lemmas.

Lemma 4.5.2. $T^0_{s_1}T^1_{s_1}T^\infty_{s_1s_2} \in \mathrm{Span}_{\mathbb{C}}(M)$

Proof. Explicitly,

$$\begin{aligned} T^0_{s_1}T^1_{s_1}T^\infty_{s_1s_2} &= -T^\infty_{s_1s_2} - T^0_{s_1}T^\infty_{s_1s_2} - T^1_{s_1}T^\infty_{s_1s_2} + q^{-1}(T^0_{s_1s_2} + T^0_{s_2})(T^1_{s_1s_2} + T^1_{s_2})(T^\infty_{s_2s_1} + T^\infty_{s_1}) \\ &- q^{-1}(T^0_{s_2s_1} + T^0_{s_3})(T^1_{s_2s_1} + T^1_{s_3}) \end{aligned} \tag{4.17}$$

To prove Equation 4.17, observe that it rearranges to Equation 4.18, which I will prove in Section 4.7.

$$\mathrm{Avg}^{01}_1(T^\infty_{s_1s_2} + q^{-1}T^{01}_{s_2}(T^{01}_{s_1} - T^\infty_{s_1}) - q^{-1}T^S_{s_2}T^\infty_{s_1}) = 0 \tag{4.18}$$

□

Lemma 4.5.3. $\mathrm{Span}_{\mathbb{C}}(M)$ is closed under $T^0_{s_2}$ and $T^1_{s_2}$.

Proof. It is sufficient to check closure under $T^0_{s_2}$. Consider a monomial $m = T^0_{w_0}T^1_{w_1}T^\infty_{w_\infty} \in M$. $T^0_{s_2}m \in M$ unless (w_0, w_∞) is one of the following

- $(1, s_2), (1, s_1s_2), (1, s_3)$
- $(s_1s_2, s_1), (s_1s_2, s_2s_1)$

In each case, it is straighforward calculation to check that $T^0_{s_2} \in \mathrm{Span}_{\mathbb{C}}(M)$.

□

$\lambda = \rho$

$\widetilde{C}^\rho$ is generated over $(\mathcal{H}_{\mathrm{fin}})^{\otimes S}$ by $J_\rho, J_{\alpha_1}, J_{\alpha_2}, J_{-\rho}$. I'll filter $\widetilde{C}^\rho$ by subsets of $\{s_1, s_2\}$. $F^\varnothing(\widetilde{C}^\rho)$ is the submodule generated by J_ρ. For simple reflection s_i, $F^{\{s_i\}}$ is the submodule generated by J_ρ and $J_{s_i\cdot\rho}$. $F^{\{s_1,s_2\}} = \widetilde{C}^\rho$. By Equations 4.10, 4.11, and 4.12, the following are true in the associated graded:

$$\mathrm{Avg}^s_1 J_{\alpha_2} = \mathrm{Avg}^s_2 J_{\alpha_1} = 0 \text{ for } s \in S$$

$$\mathrm{Avg}^s_i J_{-\rho} = 0 \text{ for } i \in \{1, 2\},\ s \in S$$

Therefore,

$$\dim_{\mathbb{C}}(\mathrm{Gr}^\varnothing(\widetilde{C}^\rho)) \le \dim_{\mathbb{C}}((\mathcal{H}_{\mathrm{fin}})^{\otimes S}) = |W|^3$$

$$\dim_{\mathbb{C}}(\mathrm{Gr}^{\{s_1\}}(\widetilde{C}^\rho)) \le \dim_{\mathbb{C}}((\mathcal{H}_{\mathrm{fin}})^{\otimes S}/\langle \mathrm{Avg}_1\rangle_{s\in S}) = \dim_{\mathbb{C}}((\mathcal{H}_{\mathrm{fin}}/\mathrm{Avg}_1)^{\otimes S}) = 3^3$$

$$\dim_{\mathbb{C}}(\mathrm{Gr}^{\{s_2\}}(\widetilde{C}^\rho)) \le \dim_{\mathbb{C}}((\mathcal{H}_{\mathrm{fin}})^{\otimes S}/\langle \mathrm{Avg}_2\rangle_{s\in S}) = \dim_{\mathbb{C}}((\mathcal{H}_{\mathrm{fin}}/\mathrm{Avg}_2)^{\otimes S}) = 3^3$$

$$\dim_{\mathbb{C}}(\mathrm{Gr}^{\{s_1,s_2\}}(\widetilde{C}^\rho)) \leq \dim_{\mathbb{C}}((\mathcal{H}_{\text{fin}})^{\otimes S}/\langle \mathrm{Avg}_1, \mathrm{Avg}_2\rangle_{s\in S}) = 1^3$$

$$\dim_{\mathbb{C}}(\widetilde{C}^\rho) \leq 6^3 + 3^3 + 3^3 + 1^3$$

Remark 4.5.2. For $\delta \subset \{s_1, s_2\}$, pick an additive character

$$\psi_\delta : N(\mathbb{F}_q)/[N(\mathbb{F}_q), N(\mathbb{F}_q] \cong \oplus_{\{s_1,s_2\}}\mathbb{F}_q \to \mathbb{C}^\times,$$

that is generic in the arguments δ. One can identify the graded component of $\widetilde{C}^\rho$ with the Whittaker module for the finite Hecke algebra.

$$\mathrm{Gr}^\delta(\widetilde{C}^\rho) \cong (C^{(N,\psi_\delta)}[\mathcal{B}])^{\otimes S}.$$

The Whittaker module is the the space of $(N(\mathbb{F}_q), \psi_\delta)$ equivariant functions on the points of the flag variety. It is a finite Hecke module by convolution after identifying $\mathcal{B} \cong G/B$.

$\langle \check{\alpha}_i, \lambda\rangle = 0$, $\lambda \neq 0$

$\widetilde{C}^\lambda$ is generated over $(\mathcal{H}_{\text{fin}})^{\otimes S}$ by J_λ. Using Equation 4.13 one can always write any monomial $T^0_{w_0} T^1_{w_1} T^\infty_{w_\infty} J_\lambda$, $w_s \in W$, as a sum of monomials where

$$\ell(w_\infty s_i) < \ell(w_\infty) \implies \ell(w_0 s_i) > \ell(w_0),\ \ell(w_1 s_i) > \ell(w_1)$$

Let us count how many triples (w_0, w_1, w_∞) satisfy this condition. There are three $w \in W$ such that $\ell(w s_i) < \ell(w)$ and three such that $\ell(w s_i) > \ell(w)$. The set of $s \in S$ such that $\ell(w_s s_i) < \ell(w)$ is exactly one of the following five: $\varnothing, \{0\}, \{1\}, \{\infty\}, \{0,1\}$.

$\langle \check{\alpha}_i, \lambda\rangle = 1$, $\lambda \neq \rho$

$\widetilde{C}^\lambda$ is generated over $(\mathcal{H}_{\text{fin}})^{\otimes S}$ by J_λ and $J_{s_i\cdot\lambda}$. Let F^0 be the submodule generated by J_λ. By Equation 4.14, in the quotient $\widetilde{C}^\lambda/F^0$, $\mathrm{Avg}_i J_{s_i\cdot\lambda} = 0$. Therefore,

$$\dim_{\mathbb{C}}(\widetilde{C}^\lambda) \leq \dim_{\mathbb{C}}(F^0) + \dim_{\mathbb{C}}(\widetilde{C}^\lambda/F_0)) \leq \dim_{\mathbb{C}}((\mathcal{H}_{\text{fin}})^{\otimes S}) + \dim_{\mathbb{C}}((\mathcal{H}_{\text{fin}}/\langle \mathrm{Avg}_i\rangle)^{\otimes S}) = |W|^3 + 3^3$$

$\lambda =\in 2\rho + \Lambda_+$

$\widetilde{C}^\lambda$ is generated by J_λ under $(\mathcal{H}_{\text{fin}})^{\otimes S}$, so $\dim_{\mathbb{C}}(\widetilde{C}^\lambda) \leq \dim_{\mathbb{C}}((\mathcal{H}_{\text{fin}})^{\otimes S}) = |W|^3$

4.6 Directions: Functional Equation, Many Points

Many Points of Tame Ramification

I state the following natural generalization of Conjecture 1.2.1 to $\mathbb{P}^1$ with several points of tame ramification $S \subset \mathbb{P}^1(\mathbb{F}_q)$, $S \neq \varnothing$.

Conjecture 4.6.1. If ρ is integral then C_{Eis} is the $\mathcal{H}^{\otimes S}$ module generated by Eis_0 with the following relations

1. (Translation Relation) For any $\lambda \in \Lambda$ and $p, q \in S$,
$$(J^p_\lambda - J^q_\lambda)\mathrm{Eis}_0 = 0$$
2. (Reflection Relation) For any simple reflection, $s_\alpha \in W$ and $p, q \in S$
$$\left(\prod_{s\in S\smallsetminus\{p\}} \mathrm{Avg}^s_{s_\alpha} - \prod_{s\in S\smallsetminus\{q\}} \mathrm{Avg}^s_{s_\alpha}\right)\mathrm{Eis}_0 = 0$$

When S consists of two points, the quotient of $\mathcal{H}^{\otimes S}$ by the translation and reflection relation is identified with the regular bimodule of $\mathcal{H}$. In this case, Conjecture 4.6.1 amounts to identifying C_{Eis} with the regular bimodule. This is done in the categorical geometric setting in Section 2.6 of [21]. A similar argument works in the arithmetic function field setting. I am not aware of any reference but would be grateful to be referred to one.

Reflection Relation as a Functional Equation

The idea to identify the reflection relation as a form of the functional equation for Eisenstein series is a suggestion of Zhiwei Yun. It did not end up playing a prominent role in this thesis, but I hope to clarify this connection in future work.

I propose that the reflection relation from Conjecture 1.2.1 could be related to the functional equation for Eisenstein series. Let $\mathrm{Bun}_{\underline{T}}(\mathbb{P}^1, S)$ be the space classifying pairs $(\mathcal{E}, \{(V_s, F^0_s, F^1_s, \tau_s)\}_{s\in S})$, where $\mathcal{E}$ is a $\underline{T}$-bundle on $\mathbb{P}^1$, V_s is a vector space, and $F^0_s, F^s_1 \subset V_s$ are flags with an identification $\tau_s : \mathrm{Gr}(F^0_s) \cong \mathcal{E}|_s$. The constant term space is the space of compactly supported functions on the rational points of $\mathrm{Bun}_{\underline{T}}(\mathbb{P}^1, S)$.

$$\mathrm{CT} := C[\mathrm{Bun}_{\underline{T}}(\mathbb{P}^1, S)]$$

The functional equation for Eisenstein series expresses that parabolic induction $\mathrm{Eis} : \mathrm{CT} \to C_{\mathrm{Aut}}$ intertwines an action of the Weyl group on the constant term space. CT is identified with the quotient of $\mathcal{H}^{\otimes S}$ by the translation relation.

$$\begin{array}{ccc} \mathcal{H}^{\otimes S}/\text{translation} & \xrightarrow{\cong} & \text{CT} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \text{Eis}} \\ \widetilde{C_{\text{Eis}}} & \xrightarrow{\overset{?}{\cong}} & C_{\text{Eis}} \end{array}$$

The constant term space is free of rank $|W|^{|S|}$ over $\mathbb{C}[\Lambda]$. In light of the functional equation it is natural to conjecture that C_{Eis} is free of rank $|W|^{|S|-1}$. See Figure 4.2 for inspiration in the case $\underline{G} = \text{PGL}_2$.

4.7 Appendix: Proof of Equation 4.18

I first found Equation 4.18 and its proof with algebra software. It is helpful to first verify Equation 4.18 in C^0_{Eis} to see why it could be true in $\widetilde{C}^0$.

Geometric Intepretation of Equation 4.18

Define the function $f \in C^0_{\text{Eis}}$.

$$f \coloneqq T^{\infty}_{s_1 s_2}\text{Eis}_0 + q^{-1}T^{01}_{s_2}(T^{01}_{s_1} - T^{\infty}_{s_1})\text{Eis}_0 - q^{-1}T^{S}_{s_2}T^{\infty}_{s_1}\text{Eis}_0$$

Consider the projection forgetting the lines ℓ_s at $s \in \{0,1\}$:

$$\pi : \text{Bun}_{\underline{G}}(\mathbb{P}^1, S) \to \text{Bun}_{\underline{G}}(\mathbb{P}^1, S, \{0,1\}, s_1)$$

$\text{Avg}^{01}_1 = \pi^*\pi_!$. Verifying Equation 4.18 in C^0_{Eis} is equivalent to checking that $\pi_! f = 0$.

$$T^{\infty}_{s_1 s_2}\text{Eis}_0 = c_0(1, s_1 s_2, s_1 s_2)$$

$$T^{01}_{s_2}(T^{01}_{s_1} - T^{\infty}_{s_1})\text{Eis}_0 = T^{01}_{s_2}c_0(s_1, s_1, s_1) = c_0(s_3, s_1 s_2, s_1 s_2)$$

$$T^{S}_{s_2}T^{\infty}_{s_1} = c_0(s_2, s_3, s_3) + c_0(1, s_3, s_3)$$

The generic locus of $c_0^{01,s_1}(\varnothing) \subset \text{Bun}_{\underline{G}}(\mathbb{P}^1, S, \{0,1\}, s_1)$ is where the bundle is trivial and all parabolic data are pairwise transverse. Define $c_0^{01,s_1}(01)$ as the locus where the bundle is trivial and p_0, p_1 coincide but are transverse to (ℓ_∞, p_∞). Comparing stabilizers,

$$\pi_! c_0(1, s_1 s_2, s_1 s_2) = q^{-1}\pi_! c_0(1, s_3, s_3) = c_0^{01,s_1}(01)$$

$$\pi_! c_0(s_3, s_1 s_2, s_1 s_2) = \pi_! c_0(s_2, s_3, s_3) = c_0^{01,s_1}(\varnothing)$$

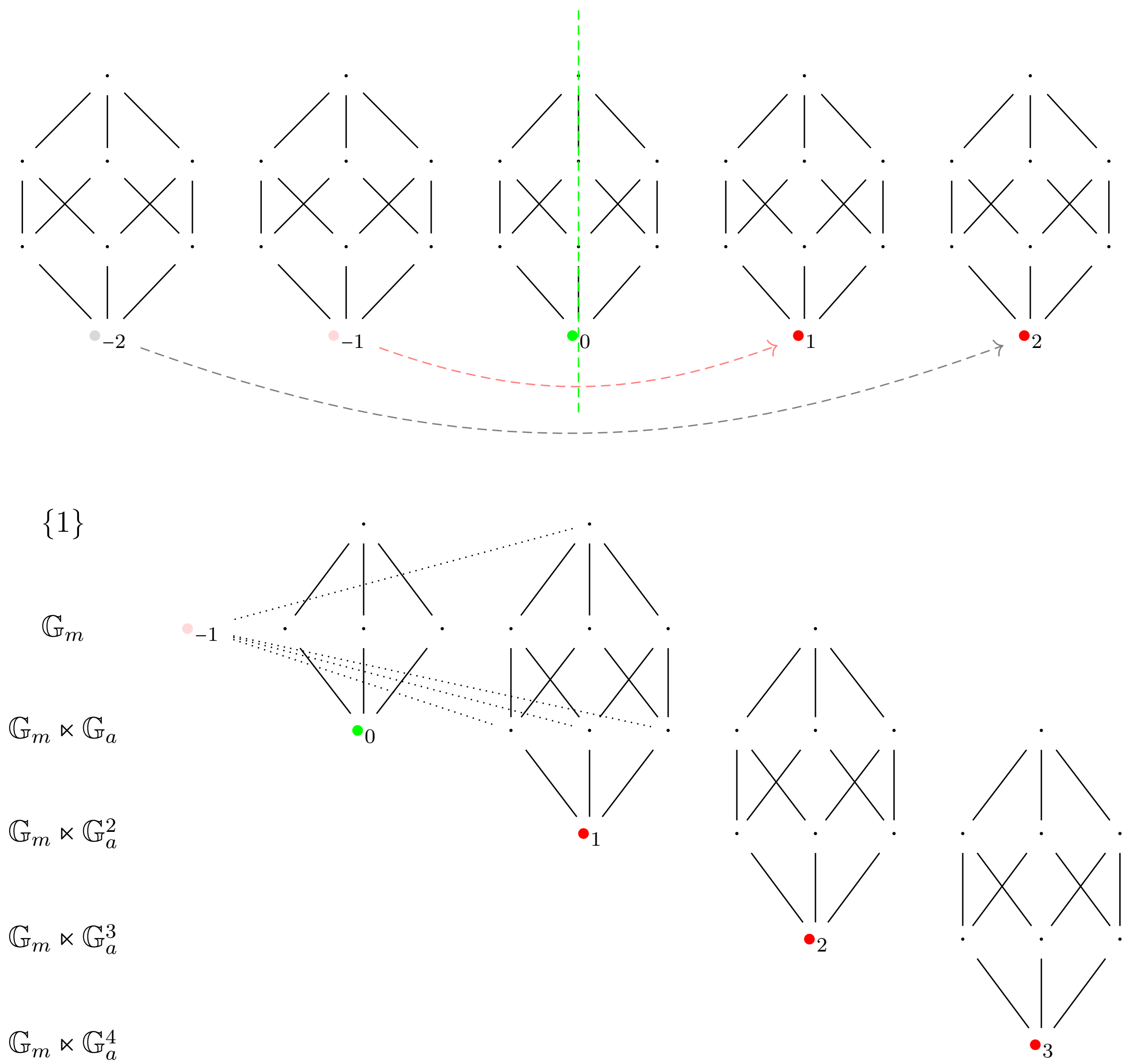


Figure 4.2: This figure depicts the reflection relation when $\underline{G} = \mathrm{PGL}_2$. The upper figure shows the structure of $\mathcal{H}^{\otimes S}$/translation and evokes the functional equation of Langlands. The lower figure shows the structure of C_{Eis}, which I have shown is obtained by quotient of the upper figure by the reflection relation. The point is to imagine the lower figure as a "folding" of the upper one. In the upper figure k is short for J_k and in the lower figure k is short for Eis_k. The colors are matched to those of Figure 4.1.

Proof of Equation 4.18

Returning to the proof of Equation 4.18, from the reflection relation

$$\mathrm{Avg}_2^0\mathrm{Avg}_2^1 = \mathrm{Avg}_2^\infty\mathrm{Avg}_2^1$$

$$\implies T_{s_2}^0\mathrm{Avg}_2^1 = T_{s_2}^\infty\mathrm{Avg}_2^1$$

$$\implies T_{s_2s_1}^\infty T_{s_2}^0\mathrm{Avg}_2 = T_{s_2s_1}^\infty T_{s_2}^\infty\mathrm{Avg}_2^1$$

$$\implies T_{s_2}^S T_{s_1}^\infty = T_{s_3}^\infty + (T_{s_2}^1 T_{s_3}^\infty - T_{s_2}^0 T_{s_2s_1}^\infty)$$

Therefore,

$$\begin{aligned}\mathrm{Avg}_1^{01}(T_{s_1s_2}^\infty + q^{-1}T_{s_2}^{01}(T_{s_1}^{01} - T_{s_1}^\infty) - q^{-1}T_{s_2}^S T_{s_1}^\infty) &= \mathrm{Avg}_1^{01}(T_{s_1s_2}^\infty - q^{-1}T_{s_3}^\infty)\\ &+ q^{-1}\mathrm{Avg}_1^{01}\left(T_{s_2}^{01}(T_{s_1}^{01} - T_{s_1}^\infty) - (T_{s_2}^1 T_{s_3}^\infty - T_{s_2}^0 T_{s_2s_1}^\infty)\right)\end{aligned}$$

Equation 4.18 follows from the following two lemmas.

Lemma 4.7.1.

$$\mathrm{Avg}_1^{01}T_{s_1s_2}^\infty = q^{-1}\mathrm{Avg}_1^{01}T_{s_3}^\infty$$

Proof. From the reflection relation

$$\begin{aligned}&\mathrm{Avg}_1^\infty\mathrm{Avg}_1^1 = \mathrm{Avg}_1^0\mathrm{Avg}_1^1\\ &\implies T_{s_1}^\infty\mathrm{Avg}_1^1 = T_{s_1}^0\mathrm{Avg}_1^1\\ &\implies q^{-1}\mathrm{Avg}_1^{01}T_{s_3}^\infty = q^{-1}T_{s_1s_2}^\infty\mathrm{Avg}_1^0 T_{s_1}^\infty\mathrm{Avg}_1^1\\ &= q^{-1}T_{s_1s_2}^\infty\mathrm{Avg}_1^0 T_{s_1}^0\mathrm{Avg}_1^1 = q^{-1}T_{s_1s_2}^\infty(q\mathrm{Avg}_1^0)\mathrm{Avg}_1^1 = \mathrm{Avg}_1^{01}T_{s_1s_w}^\infty\end{aligned}$$

□

Lemma 4.7.2.

$$\mathrm{Avg}_1^{01}T_{s_2}^{01}(T_{s_1}^{01} - T_{s_1}^\infty) = \mathrm{Avg}_1^{01}(T_{s_2}^1 T_{s_3}^\infty - T_{s_2}^0 T_{s_2s_1}^\infty)$$

Proof. First, observe that by the reflection relation $T_{s_2}^1 T_{s_3}^\infty - T_{s_2}^0 T_{s_2s_1}^\infty = T_{s_2s_1}^\infty(T_{s_2}^{01} - T_{s_2}^\infty)$. Then, check that

$$\begin{aligned}&q\mathrm{Avg}_1^{01}T_{s_2}^{01}(T_{s_1}^{01} - T_{s_1}^\infty) - q\mathrm{Avg}_1^{01}T_{s_2s_1}^\infty(T_{s_2}^{01} - T_{s_2}^\infty) =\\ &A\cdot\mathrm{Avg}_1^0(\mathrm{Avg}_1^1 - \mathrm{Avg}_1^\infty) + B\cdot\mathrm{Avg}_1^1(\mathrm{Avg}_1^0 - \mathrm{Avg}_1^\infty)\\ &+ C\cdot\mathrm{Avg}_2^0(\mathrm{Avg}_2^1 - \mathrm{Avg}_2^\infty) + D\cdot\mathrm{Avg}_2^1(\mathrm{Avg}_2^0 - \mathrm{Avg}_2^\infty) = 0,\end{aligned}$$

where

$$A = qT^1_{s_2} + T^1_{s_2}T^\infty_{s_1s_2} + T^1_{s_2}T^\infty_{s_3} + (q-1)T^1_{s_1s_2} - T^1_{s_1s_2}T^\infty_{s_1} - T^1_{s_1s_2}T^\infty_{s_2} - T^1_{s_1s_2}T^\infty_{s_2s_1} - qT^0_{s_2} - qT^{0\infty}_{s_2}$$
$$- T^0_{s_2}T^\infty_{s_3} + qT^{01}_{s_2} + T^{01}_{s_2}T^\infty_{s_1s_2} + (q-1)T^0_{s_2}T^1_{s_1s_2} - T^{0\infty}_{s_2}T^1_{s_1s_2} - qT^0_{s_1s_2} - qT^0_{s_1s_2}T^\infty_{s_2} - T^0_{s_1s_2}T^\infty_{s_3}$$
$$+ qT^0_{s_1s_2}T^1_{s_2} + T^{0\infty}_{s_1s_2}T^1_{s_2} + (q-1)T^{01}_{s_1s_2} - T^{01}_{s_1s_2}T^\infty_{s_2}$$

$$B = (q-1)T^0_{s_2}T^\infty_{s_1s_2} - T^{01}_{s_2}T^\infty_{s_1s_2} + T^0_{s_2}T^1_{s_1s_2} + T^{0\infty}_{s_2}T^1_{s_1s_2} + (q-1)T^{0\infty}_{s_1s_2} - T^\infty_{s_1s_2}T^1_{s_2} + T^{01}_{s_1s_2} + T^{01}_{s_1s_2}T^\infty_{s_2}$$

$$C = qT^\infty_{s_1} - qT^1_{s_1} - qT^1_{s_1}T^\infty_{s_2s_1} - (q-1)T^1_{s_2s_1} + T^1_{s_2s_1}T^\infty_{s_1} + T^1_{s_2s_1}T^\infty_{s_2} + T^{1\infty}_{s_2s_1} + qT^{0\infty}_{s_1} - qT^{01}_{s_1}$$
$$- qT^{01}_{s_1}T^\infty_{s_2s_1} - (q-1)T^0_{s_1}T^1_{s_2s_1} + T^{0\infty}_{s_1}T^1_{s_2s_1} + T^0_{s_1}T^1_{s_2s_1}T^\infty_{s_2} + T^0_{s_1}T^{1\infty}_{s_2s_1} + qT^0_{s_2s_1}T^\infty_{s_1} + T^0_{s_2s_1}T^\infty_{s_1s_2}$$
$$+ T^0_{s_2s_1}T^\infty_{s_3} - T^0_{s_2s_1}T^1_{s_1} + (q-1)T^0_{s_2s_1}T^{1\infty}_{s_1} - T^0_{s_2s_1}T^1_{s_1}T^\infty_{s_2} - T^{0\infty}_{s_2s_1}T^1_{s_1}$$

$$D = -qT^\infty_{s_1} - qT^\infty_{s_2s_1} + qT^1_{s_1} - qT^{0\infty}_{s_1} - qT^0_{s_1}T^\infty_{s_2s_1} + qT^{01}_{s_1}$$

□

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