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Spectral analysis of quasiperiodic
Schrödinger operators

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Shiwen Zhang

Dissertation Committee:
Professor Svetlana Jitomirskaya, Chair
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2016

Dedication

This thesis is dedicated to my beloved parents.

For their endless love, support and encouragement.

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Abstract of the Dissertation

Spectral analysis of quasiperiodic Schrödinger operators

By

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Doctor of Philosophy in Mathematics
University of California, Irvine, 2016
Professor Svetlana Jitomirskaya, Chair

We consider discrete quasiperiodic Schrödinger operators with analytic sampling functions. The thesis has two main themes: first, to provide a sharp arithmetic criterion of full spectral dimensionality for analytic quasiperiodic Schrödinger operators in the positive Lyapunov exponent regime. Second, to provide a concrete example of Schrödinger operator with mixed spectral types.

For the first theme, we introduce a notion of β -almost periodicity and prove quantitative lower spectral/quantum dynamical bounds for general bounded β -almost periodic potentials. Applications include the sharp arithmetic criterion in the positive Lyapunov exponent regime and arithmetic criteria for families with zero Lyapunov exponents, with applications to Sturmian potentials and the critical almost Mathieu operator.

For the second part, we consider a family of one frequency discrete analytic quasiperiodic Schrödinger operators which appear in [18]. We show that this family provides an example of coexistence of absolutely continuous and point spectrum for some parameters as well as coexistence of absolutely continuous and singular continuous spectrum for some other parameters.

Introduction

0.1 Discrete Schrödinger operators

Let $\Omega = \mathbb{R}^{\mathbb{Z}}$ and $T : \Omega \mapsto \Omega$ is given by $(T\theta)(n) = \theta(n + 1)$. Let $f(\theta) := \theta(0)$ and $V_{\theta}(n) := \theta(n) = f(T^n\theta)$. Consider the following 1-dimensional discrete Schrödinger operator on $l^2(\mathbb{Z})$ given by

$$(1) \quad (Hu)_n = u_{n+1} + u_{n-1} + V_{\theta}(n)u_n.$$

Our main interest is the following situation, when V is actually dynamically defined with a certain underlying ergodic base dynamics (Ω, T, ν) . In particular, when T is a rotation on the torus equipped with Lebesgue measure, (1) is called quasiperiodic Schrödinger operator.

0.2 Motivation and Background

Singular continuous spectral measures of Schrödinger operators, usually defined by what they are not, are still not very well understood. The aim of direct spectral theory

is to obtain properties of spectral measures/spectra and associated quantum dynamics based on the properties of the potential. In the context of 1D operators this is most often done via the study of solutions/transfer matrices/dynamics of transfer-matrix cocycles. Indeed, there are many beautiful results linking the latter to either dimensional properties of spectral measures (going back to [51]) or directly to quantum dynamics (e.g. [61, 33]). There is also a long thread of results relating dimensional properties of spectral measures to quantum dynamics (e.g. [13, 11] and references therein) as well as results connecting spectral/dynamical properties to some further aspects (e.g. [62, 20]). Many of those have been used to obtain dimensional/quantum dynamical results (sometimes sharp) for several concrete families (e.g. [31]). However, there were no results directly linking easily formulated properties of the potential to dimensional/quantum dynamical results, other than for specific families or a few that ensure either the mere singularity or continuity of spectral measures (and their immediate consequences). In particular, we don't know of any quantitative results of this type. In [60], we consider discrete Schrödinger operator and prove quantitative lower spectral/quantum dynamical bounds for general potentials. The result is the first one with such direct link.

The study of one-dimensional one-frequency quasiperiodic operators with general analytic potentials has seen remarkable advances in the last two decades, from the Eliasson's KAM point spectrum proof for the general class [36], to Bourgain-Goldstein's non-perturbative method [16], to Avila's global theory [4]. In particular, many results have been obtained in the regime of positive Lyapunov exponents (dubbed supercritical in [4]). They can be divided into two classes

- Those that hold for all frequencies (e.g. [52, 17, 33, 55, 56, 57])
- Those that have arithmetic (small denominator type) obstructions preventing their holding for all frequencies, thus requiring a Diophantine type condition (e.g. [16, 46, 30])¹

¹Not all results can currently be classified this way, most notable example being the Cantor structure of the spectrum [47], currently proved for a non-arithmetically-defined full measure set

Results of the first kind often (but not always [17, 74]) do not require analyticity and hold in higher generality. Results of the second kind describe phenomena where there is a transition in the arithmetics of the frequency, thus an extremely interesting question is to determine where does this transition happen and to understand the neighborhood of the transition. However, even though some improvements on the frequency range of some results above have been obtained (e.g. [75]), most existing proofs often require a removal of a non-arithmetically defined measure zero set of frequencies, thus cannot be expected to work up to the transition. There have been remarkable recent advances in obtaining complete arithmetic criteria in presence of transitions [10, 53, 54] or non-transitions [5] for explicit popular Hamiltonians: almost Mathieu operator and Maryland model, but there have been no such results that work for large families of potentials. In the first part of the thesis, we prove a sharp arithmetic criterion of full spectral dimensionality in the positive Lyapunov exponent regime for analytic quasiperiodic potential. The criterion links in a sharp way a purely analytic property of the spectral measure to arithmetic property of the frequency. Such transition in the arithmetics of the frequency is extremely interesting e.g. in popular Hamiltonians: almost Mathieu operator.

The quasiperiodic Schrödinger operators with intermediate sized potentials are less understood. Recent work of Avila [4] explains the structure of the spectrum for typical potentials. But still there are very few concrete examples where one really can say that there is a mixed spectrum. There are previous works of Bourgain [15], Fedotov and Klopp [39] that provide such mixed spectra examples in other regimes. Recently Bjerklöv and Krikorian [19] announced an example of this nature. In [78], we consider a family of one frequency discrete analytic quasiperiodic Schrödinger operators which appear in [18]. We show that this family provides an example of 1-d discrete Schrödinger operator that has mixed spectral types: coexistence of p.p. spec-

of frequencies, while the statement has no known arithmetic obstructions. Theoretically there may also be results such as [5] which formally should belong to the first group but the proof requires argument that highly depends on the arithmetics, so they must be in the second group, in spirit. In some sense [17] is a result of this type.

trum and a.c. spectrum, as well as coexistence of s.c. spectrum and a.c. spectrum. We consider the class of potentials introduced in [18]. By combining several recent results on localization, reducibility and continuity, we provide such concrete examples of mixed spectra.

Chapter 1

Quantitative continuity of singular continuous spectral measures and arithmetic criteria.

1.1 Introduction

Consider Schrödinger operator on $l^2(\mathbb{Z})$ given by

$$(1.1) \quad (Hu)_n = u_{n+1} + u_{n-1} + V(n)u_n$$

For $\beta > 0$, we say a real sequence $\{V(n)\}_{n \in \mathbb{Z}}$ has β -repetitions if there is a sequence of positive integers $q_n \rightarrow \infty$ such that

$$(1.2) \quad \max_{1 \leq j \leq q_n} |V(j) - V(j \pm q_n)| \leq e^{-\beta q_n}$$

We will say that $\{V(n)\}_{n \in \mathbb{Z}}$ has ∞ -repetitions if (1.2) holds for any $\beta > 0$. For $\beta < \infty$, we will say that $\{V(n)\}_{n \in \mathbb{Z}}$ is β -almost periodic, if, for some $\epsilon > 0$, $V(\cdot + kq_n)$ satisfies (1.2) for any $|k| \leq e^{\epsilon \beta q_n}/q_n$, i.e.,

$$(1.3) \quad \max_{1 \leq j \leq q_n, |k| \leq e^{\epsilon \beta q_n}/q_n} |V(j + kq_n) - V(j + (k \pm 1)q_n)| \leq e^{-\beta q_n}$$

for any n . We will say that $\{V(n)\}_{n \in \mathbb{Z}}$ is ∞ -almost periodic, if it is β -almost periodic for any $\beta < \infty$. We note that β and even ∞ -almost periodicity does not imply almost

periodicity in the usual sense. In particular, it is easily seen that there is an explicit set of generic skew shift potentials that satisfy this condition.

We will prove

Theorem 1.1.1 *Let H be given by (1.1) and V is bounded and β -almost periodic. Then, for an explicit $C = C(\epsilon, V) > 0$, for any*

$$(1.4) \quad \gamma < 1 - C/\beta$$

the spectral measure is γ -spectral continuous.

For the definition of spectral continuity (a property that also implies packing continuity and thus lower bounds on quantum dynamics) see Section 1.1.1. We formulate a more precise (specifying the dependence of C on ϵ, V) version in Theorem 1.1.6.

Our result can be viewed as a quantitative version simultaneously of two well known statements

- *Periodicity implies absolute continuity.* Indeed, we prove that a quantitative weakening (β -almost periodicity) implies quantitative continuity of the (fractal) spectral measure.
- *Gordon condition (a single/double almost repetition) implies continuity of the spectral measure.* Indeed, we prove that a quantitative strengthening (multiple almost repetitions) implies quantitative continuity of the spectral measure.

Potentials with ∞ -repetitions are known in the literature as Gordon potentials¹. This property has been used fruitfully in the spectral theory in various situations, see reviews [25, 27] and references therein. In many cases those potentials were automatically β or even ∞ -almost periodic, so satisfied almost repetitions over sufficiently many periods. However, even in such cases, what all those papers used was the strength of the approximation over one-two (almost) periods based on Gordon Lemma type arguments. Our main technical accomplishment here is that we find a

¹While ∞ -repetitions are usually used in the definition of Gordon potentials, typically β -repetitions for sufficiently large β are enough for the applications

new algebraic argument and develop technology that allows to obtain quantitative corollaries from the fact that the approximation stays strong over many periods, thus exploring this feature analytically for the first time.

Lower bounds on spectral dimension lead to lower bounds on packing dimension, thus also for the packing/upper box counting dimensions of the spectrum as a set and for the upper rate of quantum dynamics. Therefore we obtain corresponding non-trivial results for all above quantities.

It is clear that our general result only goes in one direction, as even absolute continuity of the spectral measures does not imply β -almost periodicity for $\beta > 0$.

However, in the important context of analytic quasiperiodic operators this leads to a sharp if-and-only-if result.

Let $H = H_{\theta, \alpha, V}$ be a Schrödinger operator on $l^2(\mathbb{Z})$ given by

$$(1.5) \quad (Hu)_n = u_{n+1} + u_{n-1} + V(\theta + n\alpha)u_n, \quad n \in \mathbb{Z}, \quad \theta \in \mathbb{T}$$

where V is the potential, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is the frequency and $\theta \in \mathbb{T}$ is the phase. Let $\mu = \mu_{\theta, \alpha}$ be the spectral measure associated with vectors $\delta_0, \delta_1 \in l^2(\mathbb{Z})$ in the usual sense.

Given $\alpha \in (0, 1)$, let p_n/q_n be the continued fraction approximants to α . Define

$$(1.6) \quad \beta(\alpha) := \limsup_n \frac{\log q_{n+1}}{q_n} \in [0, \infty].$$

Let $S := \{E \in \sigma(H) : L(E) > 0\}$, where $\sigma(H)$ is the spectrum of H and $L(E)$ is the Lyapunov exponent, be the set of supercritical energies (or, equivalently, the set of E such that the corresponding transfer-matrix cocycle is non-uniformly hyperbolic). S depends on α and V but not on θ .

Our main application is

Theorem 1.1.2 *For any analytic V and any θ , the spectral measure μ restricted to S is of full spectral dimension if and only if $\beta(\alpha) = \infty$.*

Full spectral dimensionality is defined through the boundary behavior of Borel transform of the spectral measure (see details in Section 1.1.1). It implies a range

of properties, in particular, maximal packing dimension and quasiballistic quantum dynamics. Thus our criterion links in a sharp way a purely analytic property of the spectral measure to arithmetic property of the frequency. The result is local (so works for any subset of the supercritical set, see Theorem 1.1.4 for more detail) and quantitative (so we obtain separately quantitative spectral singularity and spectral continuity statements for every finite value of β , see Theorems 1.1.5 and 1.1.6).

A natural way to distinguish between different singular continuous spectral measures is by their Hausdorff dimension. However Hausdorff dimension is a poor tool for characterizing the singular continuous spectral measures arising in the regime of positive Lyapunov exponents, as it is always equal to zero (for a.e. phase for any ergodic case [72], and for every phase for one frequency analytic potentials [52]²). Similarly, the lower transport exponent is always zero for piecewise Lipschitz potentials [33, 57]. Thus those two quantities don't even distinguish between pure point and singular continuous situations. In contrast, our quantitative version of Theorem 1.1.2, contained in Theorems 1.1.5 and 1.1.6, shows that spectral dimension is a good tool to finely distinguish between different kinds of singular continuous spectra appearing in the supercritical regime for analytic potentials.

The continuity part of Theorem 1.1.2 is robust and only requires some regularity of V . Besides the mentioned criterion, Theorem 1.1.1 allows us to obtain new results for other popular models, such as the critical almost Mathieu operator, Sturmian potentials, and others.

Indeed, our lower bounds are effective for $\beta > C \sup_{E \in \sigma(H)} L(E)$ where $L(E)$ is the Lyapunov exponent (see Theorem 1.1.6) thus the range of β is increased for smaller Lyapunov exponents, and in particular, we obtain full spectral dimensionality (and therefore quasiballistic motion) as long as $\beta(\alpha) > 0$, when Lyapunov exponents are zero on the spectrum. This applies, in particular, to Sturmian potentials and the critical almost Mathieu operator.

As an example, setting $S_0 = \{E : L(E) = 0\}$ we have

²The result of [52] is formulated for trigonometric polynomial v . However it extends to the analytic case - and more - by the method of [57].

Theorem 1.1.3 *For Lipshitz V , the quantum dynamics is quasiballistic*

1. *for any $\beta(\alpha) > 0$, if $S_0 \neq \emptyset$*
2. *for $\beta(\alpha) = \infty$, otherwise*

A similar statement also holds for full spectral dimensionality or packing/box counting dimension one. The Lipshitz condition can be relaxed to piecewise Lipshitz (or even Hölder), leading to part 1 also holding for Sturmian potentials. This in turn leads to first explicit examples of operators whose integrated density of state has different Hausdorff and packing dimensions, within both the critical almost Mathieu and Sturmian families.

The fact that quantum motion can be quasiballistic for highly Liouville frequencies was first realized by Last [64] who proved that almost Mathieu operator with an appropriate (constructed step by step) Liouville frequency is quasiballistic. Quasiballistic property is a G_δ in any regular (a-la Simon's Wonderland theorem [71]) space [42, 23], thus this was known for (unspecified) topologically generic frequencies. Here we show a precise arithmetic condition on α depending on whether or not Lyapunov exponent vanishes somewhere on the spectrum. Thus, in the regime of positive Lyapunov exponents, the quantum motion is very interesting, with dynamics almost bounded along some scales [57] (this property is sometimes called quasilocalization) and almost ballistic along others. For finite values of $\beta(\alpha)$ in this regime our result also yields power-law quantum dynamics along certain scales while bounded along others.

1.1.1 Main application

Fractal properties of Borel measures on \mathbb{R} are linked to the boundary behavior of their Borel transforms [34]. Let

$$(1.7) \quad M(E + i\varepsilon) = \int \frac{d\mu(E')}{E' - (E + i\varepsilon)}$$

be the Borel transform of measure μ . Fix $0 < \gamma < 1$. If for μ a.e. E ,

$$(1.8) \quad \liminf_{\varepsilon \downarrow 0} \varepsilon^{1-\gamma} |M(E + i\varepsilon)| < \infty,$$

we say measure μ is (upper) γ -spectral continuous. Note that spectral continuity (and singularity) captures the \liminf power law behavior of $M(E + i\varepsilon)$, while the corresponding \limsup behavior is linked to the Hausdorff dimension [34]. Define the (upper) spectral dimension of μ to be

$$(1.9) \quad s(\mu) = \sup \{ \gamma \in (0, 1) : \mu \text{ is } \gamma\text{-spectral continuous} \}.$$

For a Borel subset $S \subset \mathbb{R}$, let μ_S be the restriction of μ on S . A reformulation of Theorem 1.1.2 is

Theorem 1.1.4 *Suppose V is real analytic and $L(E) > 0$ for every E in some Borel set $S \subset \mathbb{R}$. Then for any $\theta \in \mathbb{T}$, $s(\mu_S) = 1$ if and only if $\beta(\alpha) = +\infty$.*

Remark 1 *If for μ a.e. E ,*

$$(1.10) \quad \liminf_{\varepsilon \downarrow 0} \varepsilon^{1-\gamma} |M(E + i\varepsilon)| = +\infty,$$

we say measure μ is (upper) γ -spectral singular. We can also consider

$$(1.11) \quad \tilde{s}(\mu) = \inf \{ \gamma \in (0, 1) : \mu \text{ is } \gamma\text{-spectral singular} \}.$$

Obviously, $s(\mu) \leq \tilde{s}(\mu)$. The main theorem also holds for $\tilde{s}(\mu)$.

1.1.2 Spectral singularity, continuity and proof of Theorem 1.1.4

We first study γ -spectral singularity of μ . We are going to show that under the assumption of Theorem 1.1.4 we have:

Theorem 1.1.5 *Assume $L(E) > a > 0$ for $E \in S$. There exists $c = c(a) > 0$ such that for any α, θ , if*

$$(1.12) \quad \gamma > \frac{1}{1 + \frac{c}{\beta(\alpha)}},$$

then μ_S is γ -spectral singular.

Obviously, Theorem 1.1.5 implies that if $\beta < +\infty$, then

$$(1.13) \quad s(\mu_S) \leq \tilde{s}(\mu_S) \leq \frac{1}{1 + c/\beta} < 1.$$

The analyticity of potential and positivity of Lyapunov exponent are only needed for spectral singularity. We now formulate a more precise version of the general spectral continuity result, Theorem 1.1.1.

For $S \subset \sigma(H)$ assume there are constants $\Lambda > 0$ and $n_0 \in \mathbb{N}$ such that for any $k \in \mathbb{Z}, E \in S$ and $n \geq n_0$

$$(1.14) \quad \left\| \begin{pmatrix} E - V(n+k) & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} E - V(k) & -1 \\ 1 & 0 \end{pmatrix} \right\| \leq e^{\Lambda n}$$

Clearly, such Λ always exists for bounded V , with $n_0 = 1$.

As before we denote μ_S the spectral measure of H restricted to a Borel set $S \subset \sigma(H)$.

Theorem 1.1.6 *Let H be given by (1.1) and V satisfies (1.14) and is β -almost periodic with $\epsilon > 0$. Then, for a $C(\epsilon) = C_0(1 + 1/\epsilon)$, with Λ given by (1.14), if*

$$(1.15) \quad \beta > C(\epsilon) \frac{\Lambda}{1 - \gamma}$$

then μ_S is γ -spectral continuous. Here C_0 is a universal constant. Consequently, we have

$$(1.16) \quad \tilde{s}(\mu_S) \geq s(\mu_S) \geq 1 - C(\epsilon) \frac{\Lambda}{\beta}.$$

Proof of Theorem 1.1.4: Under the assumption of Theorem 1.1.4, if $\beta < +\infty$, Theorem 1.1.5 provides the upper bound (1.13) for the spectral dimension.

We will now get the lower bound using Theorem 1.1.6. Let $V_\theta(n) := V(\theta + n\alpha)$. By boundedness of V and compactness of the spectrum, there is a constant $\Lambda_V < \infty$ such that (1.14) holds uniformly for $E \in \sigma(H_\theta), \theta \in \mathbb{T}$. In order to apply Theorem 1.1.6, it is enough to show that for any $\beta < \beta(\alpha)$, $V(\theta + j\alpha)$ has β -repetitions for any $\theta \in \mathbb{T}, j \in \mathbb{Z}$. Indeed, by (1.6), there is a subsequence q_{n_k} such that

$$\log q_{n_k+1}/q_{n_k} > \beta$$

Since V is analytic, for any θ, j and $1 \leq n \leq q_{n_k}$

$$|V(\theta + j\alpha + n\alpha) - V(\theta + j\alpha + n\alpha \pm q_{n_k}\alpha)| \leq C \|q_{n_k}\alpha\| \leq C \frac{1}{q_{n_k+1}} \leq C e^{-\beta q_{n_k}}$$

Thus if $\beta(\alpha) = \infty$, $\tilde{s}(\mu_S) = s(\mu_S) = 1$. ■

Property (1.14) naturally holds in a *sharp* way in the context of ergodic potentials with uniquely ergodic underlying dynamics. Assume the potential $V = V_\theta$ is generated by some homeomorphism T of a compact metric space Ω and a function $f : \Omega \rightarrow \mathbb{R}$ by

$$(1.17) \quad V_\theta(n) = f(T^n\theta), \quad \theta \in \Omega, \quad n \in \mathbb{Z}.$$

Assume (Ω, T) is uniquely ergodic with an ergodic measure ν . It is known that the spectral type of H_θ is ν -almost surely independent of θ (e.g [21]). In general, however, the spectral type (locally) does depend on θ ([58]). If f is continuous then, by uniform upper-semicontinuity (e.g. [38])

$$(1.18) \quad \limsup_n \sup_\theta \frac{1}{n} \log \left\| \begin{pmatrix} E - V_\theta(n) & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} E - V_\theta(1) & -1 \\ 1 & 0 \end{pmatrix} \right\| \leq L(E), \quad \forall E$$

This was recently extended in [57] to almost continuous f . Following [57], we will say a function f is almost continuous if it is bounded and its set of discontinuities has a closure of ν measure zero. By Corollary 3.2 in [57], if f is bounded and almost continuous then (1.18) also holds for every E . Moreover, if the Lyapunov exponent $L(E)$ is continuous on some compact set S , then, by compactness and subadditivity, the lim sup in (1.18) will be also uniform in $E \in S$. Since by upper semicontinuity $L(E)$ is continuous on the set where it is zero, as a consequence of Theorem 1.1.6, we obtain

Corollary 1.1.1 *Assume the function f in (1.17) is bounded and almost continuous and $L(E) = 0$ on some Borel subset S of $\sigma(H_\theta)$. If $V_\theta(n)$ is β -almost periodic for some $\beta > 0, \epsilon > 0$, then $s(\mu_S^\theta) = 1$.*

Proof: For any $0 < \gamma < 1$, set $\Lambda' = \beta(1 - \gamma)/2C$ where $C = C(\epsilon)$ is given in Theorem 1.1.6.³ Since $L(E) = 0$ on S , by the arguments above, there is $n_0 = n_0(\Lambda')$ independent of θ and E such that

$$\left\| \begin{pmatrix} E - V_\theta(n) & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} E - V_\theta(1) & -1 \\ 1 & 0 \end{pmatrix} \right\| \leq e^{\Lambda'n}, \quad n \geq n_0, \quad E \in S, \theta \in \Omega$$

Obviously, $\beta > C\Lambda'/(1 - \gamma)$, so Theorem 1.1.6 is applicable and (1.16) holds. Therefore, $s(\mu_{S,\theta}) \geq 1 - \frac{C\Lambda'}{\beta} > \gamma$. . ■

Let $S_0 = \{E : L(E) = 0\}$ and $S_+ = \{E : L(E) > 0\}$.

As an immediate consequence we obtain

Theorem 1.1.7 *If $V_\theta(n)$ is given by (1.17) with uniquely ergodic (Ω, T) and almost continuous f , then, for every θ we have*

1. $s(\mu_{S_0}) = 1$, as long as V is β -almost periodic with $\beta > 0$.
2. $s(\mu_{S_+}) = 1$, as long as V is β -almost periodic with $\beta = \infty$.

Remark 2 1. $\beta > 0$ is not a necessary condition in general for $s(\mu_{S_0}) = 1$, for $s(\mu^{ac}) = 1$ even if V is not β -almost periodic for any β , and the support of the absolutely continuous spectrum is contained in (and may coincide with) S_0 . It is a very interesting question to specify a quantitative almost periodicity condition for $s(\mu_{S_0}^{sing}) = 1$, in particular, find an arithmetic criterion for analytic one frequency potentials for $s(\mu_{S_{cr}}) = 1$ where $S_{cr} \subset S_0$ is the set of critical energies in the sense of Avila's global theory.

2. According to Theorem 1.1.4, $\beta = \infty$ is also necessary if f is analytic and T is an irrational rotation of the circle (β will depend on T). In case f has lower regularity, it is an interesting question to determine optimal condition on β .

³If $\beta = \infty$ take any finite β instead.

1.1.3 Relation with other dimensions; Corollaries for the AMO, Sturmian potentials, and Transport exponents.

If we replace the \liminf by \limsup in the definition of upper spectral dimension, we will define correspondingly the lower spectral dimension which will coincide with the Haurdorff dimension $dim_H(\mu)$ of a measure μ .

Also one can consider the packing dimension of μ , denoted by $dim_P(\mu)$. The packing dimension can be defined in a similar way as in (1.9) through the γ -dimensional lower derivative $\underline{D}^\gamma \mu(E)$. It can be easily shown that $\underline{D}^\gamma \mu(E) \leq \liminf_{\varepsilon \downarrow 0} \varepsilon^{1-\gamma} |M(E+i\varepsilon)|$. Thus the relation between packing dimension and upper spectral dimension is $dim_P(\mu) \geq \tilde{s}(\mu)$.⁴ Therefore, the lower bound we get in Theorem 1.1.6 also holds for the packing dimension.

Lower bounds on spectral dimension also have immediate applications to the lower bounds on quantum dynamics. Denote by δ_j be the vector $\delta_j(n) = \chi_j(n)$. For $p > 0$, define

$$(1.19) \quad \langle |X|_{\delta_0}^p \rangle(T) = \frac{2}{T} \int_0^\infty e^{-2t/T} \sum_n |n|^p |\langle e^{-itH} \delta_0, \delta_n \rangle|^2$$

The growth rate of $\langle |X|_{\delta_0}^p \rangle(T)$ characterizes how fast does $e^{-itH} \delta_0$ spread out. In order to get the power law bounds for $\langle |X|_{\delta_0}^p \rangle(T)$, it is natural to define the following upper $\beta_{\delta_0}^+(p)$ and lower $\beta_{\delta_0}^-(p)$ dynamical exponents as

$$(1.20) \quad \beta_{\delta_0}^+(p) = \limsup_{T \rightarrow \infty} \frac{\log \langle |X|_{\delta_0}^p \rangle(T)}{p \log T}, \quad \beta_{\delta_0}^-(p) = \liminf_{T \rightarrow \infty} \frac{\log \langle |X|_{\delta_0}^p \rangle(T)}{p \log T}$$

The dynamics is called ballistic if $\beta_{\delta_0}^-(p) = 1$ for all $p > 0$, and quasiballistic if $\beta_{\delta_0}^+(p) = 1$ for all $p > 0$. We will also say that the dynamics is quasilocalized if $\beta_{\delta_0}^-(p) = 0$ for all $p > 0$.

In [48], it is shown that the upper and lower transport exponents of a discrete Schrödinger operator (1.1) can be bounded from below by the packing and Hausdorff dimension of its spectral measure respectively. Therefore, by [48] we have $\beta_{\delta_0}^+(p) \geq s(\mu)$, $\forall p$. As a direct consequence of Theorem 1.1.6 we have

⁴In contrast with the Hausdorff dimension, the relation for the packing dimension only goes in one direction, in general, unlike what is claimed in [22].

Corollary 1.1.2 *If $V(n)$ is bounded and ∞ -almost periodic, the upper dynamical exponent $\beta_{\delta_0}^+(p)$ of the operator (1.5) is one for any $p > 0$, and the associated dynamics is quasiballistic.*

This has nice immediate consequences. In particular, consider the almost Mathieu operator:

$$(1.21) \quad (H_{\lambda, \theta, \alpha} u)_n = u_{n+1} + u_{n-1} + 2\lambda \cos 2\pi(\theta + n\alpha)u_n, \quad \lambda > 0.$$

As a consequence of the formula for the Lyapunov exponent and Theorem 1.1.4, one has:

Corollary 1.1.3 *The almost Mathieu operator (1.21) is quasiballistic⁵ for any (and all) $\theta \in \mathbb{T}$*

1. For $\lambda < 1$, for all α
2. For $\lambda = 1$, as long as $\beta(\alpha) > 0$
3. For $\lambda > 1$, as long as $\beta(\alpha) = \infty$.

Statement 1 is a corollary of absolute continuity [62, 1] and is listed here for completeness only. Statements 2,3 are direct corollaries of Theorem 1.1.7.

For $\lambda > 1$, Hausdorff dimension of the spectral measure of the almost Mathieu operator is equal to zero [52] and $\beta^-(p) = 0$ for all $p > 0$ [33]. Thus almost Mathieu operators with $\lambda > 1$ and $\beta(\alpha) = \infty$ provide a family of *explicit* examples of operators that are simultaneously quasilocalized and quasiballistic and whose spectral measures satisfy

$$0 = \dim_H(\mu) < \dim_P(\mu) = 1.$$

The same holds of course for \cos replaced with any almost continuous f as long as Lyapunov exponent is positive everywhere on the spectrum, in particular for $f = \lambda g$ where g is either bi-Lipshitz (as in [50]) or analytic, and $\lambda > \lambda(g)$ is sufficiently large.

Let dN be the density states measure of the almost Mathieu operator and Σ be the spectrum. It is well known that in the critical case, $\lambda = 1$, Σ has Lebesgue

⁵And has spectral dimension one and packing dimension one of the spectral measure

measure zero ([9, 63]). It is then interesting to consider the fractal dimension of the spectrum (as a set). Since $dN = \mathbb{E}(d\mu_\theta)$ and $\text{supp}_{\text{top}}(dN) = \Sigma$, by the discussion above we have

Corollary 1.1.4 *For the critical almost Mathieu operator, $\lambda = 1$, and $\beta(\alpha) > 0$ we have $\dim_P(dN) = \dim_P(\Sigma) = 1$.*

Last and Shamis proved in [66] (see also [68]) that for a dense G_δ set of α (which therefore has a generic intersection with the set $\{\alpha : \beta(\alpha) > 0\}$), the Hausdorff dimension of the spectrum is equal to zero. Thus the spectrum of the critical almost Mathieu operator with a topologically generic frequency is an example of a *set* such that

$$0 = \dim_H(\Sigma) < \dim_P(\Sigma) = 1.$$

Moreover, Last [63] showed that if $q_{n+1} > Cq_n^3$ for all n , (which is the set containing $\{\alpha : \beta(\alpha) > 0\}$) then $\dim_H(\Sigma) \leq 1/2$. Thus critical almost Mathieu operators with $\beta(\alpha) > 0$ and any θ provide an explicit family of operators that all have spectra satisfying $\dim_H(\Sigma) \leq 1/2 < \dim_P(\Sigma) = \dim_B(\Sigma) = 1$.

We note that the question of fractal dimension of the critical almost Mathieu operator attracted a lot of attention in Physics literature, with many numerical and heuristic results. In particular, Wilkinson-Austin [73] conjectured that $\dim_B(\Sigma) < 1/2$ for *all* α and there were many results rigorously or numerically confirming this for certain α . Our corollary 1.1.4 provides an explicit example disproving this conjecture.

Another well known family are Sturmian Hamiltonians given by

$$(1.22) \quad (Hu)_n = u_{n+1} + u_{n-1} + \lambda \chi_{[1-\alpha, 1)}(n\alpha + \theta \bmod 1)u_n,$$

where $\lambda > 0$, $\alpha = \mathbb{R} \setminus \mathbb{Q}$. If $\alpha = \frac{\sqrt{5}-1}{2}$, it is called the Fibonacci Hamiltonian. The spectral properties of the Fibonacci Hamiltonian have been thoroughly studied in a series of papers in the past three decades, see [26, 28] for more references. Recently, Damanik, Gorodetski, Yessen proved in [31] that for every $\lambda > 0$, the density of states

measure dN_λ is exact-dimensional (the Hausdorff and upper box counting dimension are the same) and $\dim_H(dN_\lambda) < \dim_H(\Sigma_\lambda)$.

Our results show that the exact dimensionality properties of Sturmian Hamiltonians strongly rely on the arithmetic properties of α . It was shown in [12] that if α is irrational, the Lyapunov exponent of Sturmian operator restricted to the spectrum is zero. Also the spectrum of Sturmian Hamiltonian $\Sigma_{\lambda,\alpha}$ is always a Cantor set with Lebesgue measure zero. Moreover, for Sturmian potentials results similar to those for the critical Mathieu operator in Corollary 1.1.4 also hold. Let μ_θ be the spectral measure of Sturmian operator (1.22) and let $dN_{\lambda,\alpha}$ be the density states of measure and $\Sigma_{\lambda,\alpha}$ be the spectrum. We say that phase θ is α -Diophantine if there exist $\gamma < \infty, \tau > 1$ such that $\|\theta + m\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{\gamma}{(|m|+1)^\tau}$ for all $m \in \mathbb{Z}$. Clearly, this is a full measure condition. We have

Theorem 1.1.8 *For Sturmian operator $H_{\theta,\lambda,\alpha}$ with $\beta(\alpha) > 0$ and $\lambda > 0$, if θ is α -Diophantine, the spectral dimension of μ_θ is one.*

As a consequence, if $\beta(\alpha) > 0$ and $\lambda > 0$, then the packing dimension of $dN_{\lambda,\alpha}$ and $\Sigma_{\lambda,\alpha}$ are both equal to one.

Previously, Liu, Qu and Wen [70, 69] studied the Hausdorff and upper box counting dimension of $\Sigma_{\lambda,\alpha}$ of Sturmian operators. For large couplings, they gave a criterion on $\alpha \in (0, 1)$ for the Hausdorff dimension of the spectrum to be equal to one. Combining Theorem 1.1.8 with their results, we have

Corollary 1.1.5 *Let $\Sigma_{\lambda,\alpha}$ be the spectrum of the Sturmian Hamiltonian with $\lambda > 20$. There are explicit α such that for a.e. θ ,*

$$(1.23) \quad \dim_H(\mu_{\lambda,\alpha}^\theta) < s(\mu_{\lambda,\alpha}^\theta) = \dim_P(\mu_{\lambda,\alpha}^\theta) = 1$$

$$(1.24) \quad \dim_H(dN_{\lambda,\alpha}) < s(dN_{\lambda,\alpha}) = \dim_P(dN_{\lambda,\alpha}) = 1$$

The proof for the Sturmian case is given in Section 1.4.

The rest of this paper is organized in the following way. After giving the preliminaries in Section 1.1.4 we proceed to the proof of the general continuity statement in Section 1.2. First we quickly reduce Theorem 1.1.6 to Lemma 1.2.1 where we also specify the constant C_0 appearing in Theorem 1.1.6. We note that we do not aim to optimize the constants here and many of our arguments have room for corresponding improvement. Lemma 1.2.1 is further reduced to the estimate on the traces of the transfer matrices over eventual almost periods, Theorem 1.2.1, through its corollaries, Lemmas 1.2.2 and 1.2.3. Theorem 1.2.1 is the key element and the most technical part of the proof. It is of interest in its own right as can be viewed as the quantitative version of the fact that period length transfer matrices of periodic operators are elliptic: it provides quantitative bounds on the traces of transfer matrices over almost periods based on quantitative almost periodicity, for spectrally a.e. energy. In section 1.2.2 we separate this statement into hyperbolic and almost parabolic parts, correspondingly Lemmas 1.2.4 and 1.2.5. In section 1.2.3 we use the extended Schnol's Theorem to study the hyperbolic case and in section 1.2.4 we combine estimates on level sets of the polynomials, power-law subordinacy bounds, and an elementary but very useful algebraic representation of matrix powers (Lemma 1.2.9) to study the almost parabolic case. Lemmas 1.2.2 and 1.2.3 are proved in Section 1.2.5, completing the continuity part. In Section 1.3 we focus on the analytic quasiperiodic potentials and prove Theorem 1.1.5. The proof is based on a lemma about density of localized blocks (Lemma 1.3.3). Finally, we discuss Sturmian potentials in Section 1.4, proving Theorem 1.1.8 and then providing explicit examples for Corollary 1.1.5.

1.1.4 Preliminaries

m-function and subordinacy theory

In this part, we will briefly introduce the power-law extension of the Gilbert-Pearson subordinacy theory [43, 44], developed in [51]. We will also list the necessary related facts on the Weyl-Titchmarsh m -function. More details can be found, e.g., in [21].

Let H be as in (1.5) and $z = E + i\varepsilon \in \mathbb{C}$. Consider equation

$$(1.25) \quad Hu = zu.$$

with the family of normalized phase boundary conditions:

$$(1.26) \quad u_0^\varphi \cos \varphi + u_1^\varphi \sin \varphi = 0, \quad -\pi/2 < \varphi < \pi/2, \quad |u_0^\varphi|^2 + |u_1^\varphi|^2 = 1.$$

Let $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ and $\mathbb{Z}^- = \{\dots, -2, -1, 0\}$. Denote by $u^\varphi = \{u_j^\varphi\}_{j \geq 0}$ the right half line solution on \mathbb{Z}^+ of (1.25) with boundary condition (1.26) and by $u^{\varphi,-} = \{u_j^{\varphi,-}\}_{j \leq 0}$ the left half line solution on \mathbb{Z}^- of the same equation. Also denote by v^φ and $v^{\varphi,-}$ the right and left half line solutions of (1.25) with the orthogonal boundary conditions to u^φ and $u^{\varphi,-}$, i.e., $v^\varphi = u^{\varphi+\pi/2}, v^{\varphi,-} = u^{\varphi+\pi/2,-}$. For any function $u : \mathbb{Z}^+ \rightarrow \mathbb{C}$ we denote by $\|u\|_l$ the norm of u over a lattice interval of length l ; that is

$$(1.27) \quad \|u\|_l = \left[\sum_{n=1}^{[l]} |u(n)|^2 + (l - [l])|u([l] + 1)|^2 \right]^{1/2}$$

Similarly, for $u : \mathbb{Z}^- \rightarrow \mathbb{C}$, we define

$$(1.28) \quad \|u\|_l = \left[\sum_{n=1}^{[l]-1} |u(-n)|^2 + (l - [l])|u(-[l])|^2 \right]^{1/2}$$

Now given any $\varepsilon > 0$, we define lengths $l = l(\varphi, \varepsilon, E)$, by requiring the equality

$$(1.29) \quad \|u^\varphi\|_{l(\varphi, \varepsilon)} \|v^\varphi\|_{l(\varphi, \varepsilon)} = \frac{1}{2\varepsilon}$$

We also define $l^-(\varphi)$ by $u^{\varphi,-}, v^{\varphi,-}$ through the same equation. Direct computation shows that

$$(1.30) \quad \|u^\varphi\|_l \cdot \|v^\varphi\|_l \geq \frac{1}{2}([l] - 1)$$

Denote by $m_\varphi(z) : \mathbb{C}^+ \mapsto \mathbb{C}^+$ and $m_\varphi^-(z) : \mathbb{C}^+ \mapsto \mathbb{C}^+$ the right and left Weyl-Titchmarsh m-functions associated with the boundary condition (1.26). Let $m = m_0$ and $m^- = m_0^-$ be the half line m-functions corresponding to the Dirichlet boundary conditions. The following key inequality [51] relates $m_\varphi(E + i\varepsilon)$ to the solutions u^φ and v^φ given by (1.25),(1.26).

Lemma 1.1.1 (J-L inequality, Theorem 1.1 in [51]) For $E \in \mathbb{R}$ and $\varepsilon > 0$, the following inequality holds for any $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2}]$:

$$(1.31) \quad \frac{5 - \sqrt{24}}{|m_\varphi(E + i\varepsilon)|} < \frac{\|u^\varphi\|_{l(\varphi, \varepsilon)}}{\|v^\varphi\|_{l(\varphi, \varepsilon)}} < \frac{5 + \sqrt{24}}{|m_\varphi(E + i\varepsilon)|}$$

We need to study the whole-line m-function which is given by the Borel transform of the spectral measure μ of operator H (see e.g., [21]). The following relation between whole line m-function M and half line m-function m_φ was first shown in [29] as a corollary of the maximal modulus principle. One can also find a different proof based on a direct computation in the hyperbolic plane in [7].

Proposition 1.1.1 (Corollary 21 in [29]) Fix $E \in \mathbb{R}$ and $\varepsilon > 0$,

$$(1.32) \quad |M(E + i\varepsilon)| \leq \sup_{\varphi} |m_\varphi(E + i\varepsilon)|$$

This proposition implies that in order to obtain an upper bound for the whole line m-function, namely, the continuity of whole line spectrum, it is enough to obtain a uniform upper bound of the half line m-function for any boundary condition.

On the other hand, consider a unitary operator $U : l^2(\mathbb{Z}) \rightarrow l^2(\mathbb{Z})$, defined by $(U\psi)_n = \psi_{-n+1}$, $n \in \mathbb{Z}$. For any operator H on $l^2(\mathbb{Z})$, we define an operator \tilde{H} on $l^2(\mathbb{Z})$ by $\tilde{H} = UHU^{-1}$. Denote by $\tilde{m}, \tilde{m}_\varphi, \tilde{u}^\varphi$ and $\tilde{l}(\varphi)$, correspondingly, m, m_φ, u^φ and $l(\varphi)$ of the operator \tilde{H} . We will need the following well known facts (see e.g. Section 3, [52]). For any $\varphi \in (-\pi/2, \pi/2]$ we have

$$(1.33) \quad M(z) = \frac{m_\varphi(z)\tilde{m}_{\pi/2-\varphi} - 1}{m_\varphi(z) + \tilde{m}_{\pi/2-\varphi}}$$

and

$$(1.34) \quad \tilde{l}(\pi/2 - \varphi) = l^-(\varphi), \quad \|u\|_l = \|Uu\|_l$$

Similar to Lemma 5 in [52], a direct consequence of relation (1.33) is the following result.

Lemma 1.1.2 For any $0 < \gamma < 1$, suppose that there exists a $\varphi \in (-\pi/2, \pi/2]$ such that for μ -a.e. E in some Borel set S , we have that $\liminf_{\varepsilon \rightarrow 0} \varepsilon^{1-\gamma} |m_\varphi(E +$

$i\varepsilon| = \infty$ and $\liminf_{\varepsilon \rightarrow 0} \varepsilon^{1-\gamma} |\tilde{m}_{\pi/2-\varphi}(E + i\varepsilon)| = \infty$. Then for μ -a.e. E in S , $\liminf_{\varepsilon \rightarrow 0} \varepsilon^{1-\gamma} |M(E + i\varepsilon)| = \infty$, namely, the restriction $\mu(S \cap \cdot)$ is γ -spectral singular.

Transfer matrices and Lyapunov exponents

Although Theorem 1.1.6 does not involve any further conditions on the potential, it will be convenient in what follows to use the dynamical notations. Let $\Omega = \mathbb{R}^{\mathbb{Z}}$ and $T : \Omega \mapsto \Omega$ is given by $(T\theta)(n) = \theta(n+1)$. Let $f(\theta) := \theta(0)$. Then any potential V can be written in the way (1.17), $V_{\theta}(n) := \theta(n) = f(T^n\theta)$. Thus for a fixed $\{\theta_n\}_{n \in \mathbb{Z}} = \theta \in \Omega$, we will rewrite the potential V as $V_{\theta}(n) = f(T^n\theta)$ as in (1.17). For our general theorem we do not introduce any topology, etc; this is being done purely for the notational convenience. Denote the n -step transfer-matrix by $A_n(\theta, E)$:

$$(1.35) \quad A_n(\theta, E) = A(T^n\theta, E)A(T^{n-1}\theta, E) \cdots A(T\theta, E), \quad n > 0$$

and

$$A_0 = Id, \quad A_n(\theta, E) = A_{-n}^{-1}(T^n\theta, E), \quad n < 0,$$

where

$$(1.36) \quad A(\theta, E) = \begin{pmatrix} E - f(\theta) & -1 \\ 1 & 0 \end{pmatrix}.$$

The connection to Schrödinger operators is clear since a solution of $Hu = Eu$ can be reformulated as

$$(1.37) \quad A_n(\theta, E) \begin{pmatrix} u_1 \\ u_0 \end{pmatrix} = \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix}, \quad n \in \mathbb{Z}.$$

In other words, the spectral properties of Schrödinger operators H are closely related to the dynamics of the family of skew product $(T, A(\theta, E))$ over $\Omega \times \mathbb{R}^2$. We will often suppress either θ or E or both from the notations if corresponding parameters are fixed through the argument.

If V is actually dynamically defined by (1.17) with a certain underlying ergodic base dynamics (Ω, T, ν) then, by the general properties of subadditive ergodic cocycles, we can define the Lyapunov exponent

$$(1.38) \quad L(E, T) = \lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\Omega} \log \|A_n(\theta, E)\| d\nu = \inf_{n > 0} \frac{1}{n} \int_{\Omega} \log \|A_n(\theta, E)\| d\nu$$

1.2 Spectral Continuity

1.2.1 Proof of Theorem 1.1.6

Throughout this section we assume (1.14) is satisfied uniformly for $E \in S, \theta \in \Omega$ (see Section 1.4.2). Assume V is β -almost periodic for some $\epsilon > 0$. The proof of Theorem 1.1.6 is based on the following estimates on the growth of the l -norm of the half line solutions. Let u^φ, v^φ be given as in (1.25)-(1.27).

Lemma 1.2.1 *For $0 < \gamma < 1$, assume $\beta > 100(1 + 1/\epsilon)\Lambda/(1 - \gamma)$. For μ -a.e. E , there is a sequence of positive numbers $\eta_k \rightarrow 0$ so that for any φ*

$$(1.39) \quad 1/16(L_k)^\gamma \leq \|v^\varphi\|_{L_k}^2 \leq (L_k)^{2-\gamma}$$

where $L_k = \ell_k(\varphi, \eta_k, E)$ is given as in (1.29).

Proof of Theorem 1.1.6: Fix $0 < \gamma < 1$. Set $C(\epsilon) := 100(1 + 1/\epsilon)$. Lemma 1.2.1 can be applied to any $\beta > C(\epsilon)\frac{\Lambda}{1-\gamma}$. According to (1.39) and the J-L inequality (1.31), for μ -a.e. E and any φ

$$\begin{aligned} \eta_k^{1-\gamma} |m_\varphi(E + i\eta_k)| &\leq \frac{1}{\left(2\|u^\varphi\|_{L_k}\|v^\varphi\|_{L_k}\right)^{1-\gamma}} \cdot (5 + \sqrt{24}) \frac{\|v^\varphi\|_{L_k}}{\|u^\varphi\|_{L_k}} \\ &\leq C_\gamma \cdot \frac{(L_k^{(2-\gamma)/2})^\gamma}{(1/4L_k^{\gamma/2})^{2-\gamma}} = C < \infty \end{aligned}$$

Since η_k is independent of φ , for a fixed E and η_k , we can take the supremum w.r.t. φ . By (1.32) in Proposition 1.1.1, we have for μ -a.e. E ,

$$\eta_k^{1-\gamma} |M(E + i\eta_k)| < C$$

i.e.,

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^{1-\gamma} |M(E + i\varepsilon)| < \infty, \quad \mu\text{-a.e. } E,$$

which proves the γ -spectral continuity of Theorem 1.1.6. The lower bound (1.16) comes from the definition of spectral dimensionality. \blacksquare

The proof of Lemma 1.2.1 follows from the following estimates on the trace of the transfer matrix. Let q_k be the sequence given in (1.3).

Theorem 1.2.1 *If*

$$(1.40) \quad \beta > (37 + 11/\epsilon)\Lambda,$$

then for μ a.e. E , there is $K(E)$ such that

$$(1.41) \quad |\text{Trace}A_{q_k}(E)| < 2 - e^{-10\Lambda q_k}, \quad k \geq K(E).$$

This theorem is the key estimate of the spectral continuity. It can be viewed as a quantitative version of the classical fact that period-length transfer matrices of periodic operators are elliptic on the spectrum. Indeed, we prove that β -almost periodicity implies quantitative bounds on ellipticity. The proof will be given in the following two subsections. The direct consequence of Theorem 1.2.1 are the following estimates on the norm of the transfer matrices. They show that if the trace of the transfer matrix over an almost period is strictly less than 2, then there is a sub-linearly bounded subsequence. We will use this result to prove Lemma 1.2.1 first. The proof of Lemma 1.2.2 will be left to Section 1.2.5. Let $K(E)$ be given by Theorem 1.2.1.

Lemma 1.2.2 *For any $\xi > 0$ set $N_k = \lfloor e^{\xi q_k} \rfloor$ and suppose that, in addition to the conditions of Theorem 1.2.1*

$$(1.42) \quad \beta > 15\Lambda + (2 + 1/\epsilon)\xi.$$

Then for μ -a.e. E , the following estimate holds:

$$(1.43) \quad \sum_{n=1}^{N_k \cdot q_k} \|A_n(E)\|^2 \leq e^{(\xi+15\Lambda)q_k}, \quad k \geq K(E)$$

Additionally,

Lemma 1.2.3 For $0 < \gamma < 1$, assume that in addition to the conditions of Lemma 1.2.2

$$(1.44) \quad \xi > \frac{16\Lambda}{1-\gamma}.$$

Then

$$(1.45) \quad \sum_{n=1}^{N_k \cdot q_k} \|A_n(E)\|^2 \leq (N_k \cdot q_k)^{2-\gamma}, \quad k \geq K(E).$$

Proof of Lemma 1.2.1: It is enough to prove r.h.s. of (1.39) since then the l.h.s. of (1.39) follows from $\|u^\varphi\|_{L_k} \|v^\varphi\|_{L_k} \geq 1/4L_k$.

For any $0 < \gamma < 1$, set $\beta_0 = 100(1 + 1/\epsilon)\frac{\Lambda}{1-\gamma}$, $\xi = \frac{17\Lambda}{1-\gamma}$. Then (1.40),(1.42) and (1.44) are satisfied for all $\beta > \beta_0$. Therefore, (1.45) holds with above choice of parameters. Let $l_k = [e^{\xi q_k}] \cdot q_k$. Rewrite (1.45) as $\sum_{n=1}^{l_k} \|A_n(E)\|^2 < l_k^{2-\gamma}$. Thus for any φ , $\|v^\varphi\|_{l_k}^2 \leq 4l_k^{2-\gamma}$. By (1.30), we have

$$(1.46) \quad \frac{1}{4} \cdot l_k \leq \|u^\varphi\|_{l_k} \|v^\varphi\|_{l_k} \leq 4 \cdot l_k^{2-\gamma}$$

Set

$$(1.47) \quad \varepsilon_k(\varphi) := \frac{1}{2\|u^\varphi\|_{l_k} \|v^\varphi\|_{l_k}}$$

Then,

$$(1.48) \quad \eta_k = \inf_{\varphi} \varepsilon_k(\varphi) \geq \frac{1}{8 \cdot l_k^{2-\gamma}} > 0$$

is well defined. Set $L_k(\varphi) := \ell(\varphi, \eta_k, E)$, the length scale satisfies

$$(1.49) \quad \eta_k = \frac{1}{2\|u^\varphi\|_{L_k(\varphi)} \cdot \|v^\varphi\|_{L_k(\varphi)}}$$

By (1.48),

$$L_k(\varphi) \leq 4\|u^\varphi\|_{L_k} \|v^\varphi\|_{L_k} = \frac{2}{\eta_k} \leq 16 \cdot l_k^{2-\gamma}$$

Since $\eta_k \leq \varepsilon_k(\varphi)$ and $\|u^\varphi\|_l \|v^\varphi\|_l$ is monotone increasing in l , we obtain for any φ

$$(1.50) \quad l_k \leq L_k(\varphi) \leq 16 \cdot l_k^{2-\gamma}$$

By the definition of l_k , for large k ,

$$(1.51) \quad e^{(\xi - \frac{\Lambda}{200(1-\gamma)})q_k} \cdot q_k \leq L_k(\varphi) \leq e^{((2-\gamma)\xi + \Lambda/200)q_k} \cdot q_k$$

Write $L_k(\varphi) = [L_k(\varphi)] + \tilde{L}_k(\varphi)$ and

$$(1.52) \quad [L_k(\varphi)] = (N_k(\varphi) - 1) \cdot q_k + r_k(\varphi), \quad N_k(\varphi) \in \mathbb{N}, \quad 0 \leq r_k(\varphi) < q_k$$

where $[L_k(\varphi)]$ is the integer part of $L_k(\varphi)$. Define

$$(1.53) \quad \xi_k(\varphi) = \frac{\log N_k(\varphi)}{q_k}$$

We have

$$(1.54) \quad [L_k(\varphi)] = (e^{\xi_k(\varphi) \cdot q_k} - 1) \cdot q_k + r_k(\varphi), \quad e^{\xi_k(\varphi) \cdot q_k} \in \mathbb{N}, \quad 0 \leq r_k(\varphi) < q_k$$

For large q_k , it is easy to check

$$e^{(\xi_k(\varphi) - \Lambda/200) \cdot q_k} \cdot q_k \leq (e^{\xi_k(\varphi) \cdot q_k} - 1) \cdot q_k \leq L_k(\varphi) \leq e^{\xi_k(\varphi) \cdot q_k} \cdot q_k$$

Using (1.51), we have for any φ

$$(1.55) \quad \xi - \Lambda/200 \leq \xi_k(\varphi) \leq (2 - \gamma)\xi + \Lambda/100 \leq 2\xi + \Lambda/100$$

Together with the choice of β and ξ , we have

$$\beta > \beta_0 > 15\Lambda + (2 + 1/\epsilon)\xi_k(\varphi)$$

Now we can again apply Lemma 1.2.2 with parameters β , $\xi_k(\varphi)$ and the length scale $N_k(\varphi) = e^{\xi_k(\varphi) \cdot q_k}$, to get

$$(1.56) \quad \sum_{n=1}^{N_k(\varphi) \cdot q_k} \|A_n(E)\|^2 \leq e^{(\xi_k(\varphi) + 15\Lambda)q_k}$$

Notice $L_k(\varphi) \geq e^{(\xi_k(\varphi) - \Lambda/200) \cdot q_k}$ implies that

$$\frac{1}{(L_k)^{2-\gamma}} \sum_{n=1}^{N_k(\varphi) \cdot q_k} \|A_n(E)\|^2 \leq e^{(-(1-\gamma)\xi_k + 16\Lambda)q_k}$$

By the l.h.s. of (1.55), we obtain

$$(1 - \gamma)(\xi_k(\varphi) + \Lambda/200) > (1 - \gamma)\xi = 17\Lambda$$

which implies

$$(1 - \gamma)\xi_k(\varphi) > 17\Lambda - (1 - \gamma)\Lambda/200 > 16.5\Lambda,$$

and

$$(1.57) \quad \frac{1}{(L_k)^{2-\gamma}} \sum_{n=1}^{N_k \cdot q_k} \|A_n(E)\|^2 \leq e^{-\Lambda q_k/2} \leq 1$$

Finally, by Lemma 1.2.3 we have

$$\|v^\varphi\|_{L_k}^2 \leq \sum_{n=1}^{[L_k]+1} |v_n^\varphi|^2 \leq \sum_{n=1}^{N_k(\varphi) \cdot q_k} \left(|v_n^\varphi|^2 + |v_{n+1}^\varphi|^2 \right) \leq \sum_{n=1}^{N_k(\varphi) \cdot q_k} \|A_n(E)\|^2 \leq (L_k)^{2-\gamma}$$

■

1.2.2 Proof of Theorem 1.2.1

The proof of Theorem 1.2.1 will be divided into two cases. We will first exclude the energies where the trace is much greater than 2 infinitely many times using extended Schnol's Theorem (Lemma 1.2.6). Then we will estimate the measure of energies where the trace is close to 2 through subordinacy theory. The conclusion consists of the following two lemmas. Again let q_k be the sequence given by (1.3) with certain $\beta, \epsilon > 0$.

Lemma 1.2.4 *For any $\tau > 0$, if*

$$(1.58) \quad \beta > (3 + 1/\epsilon)\tau + (7 + 1/\epsilon)\Lambda,$$

then for spectrally a.e. E , there is $K_1(E)$ such that,

$$(1.59) \quad |\text{Trace } A_{q_k}(E)| < 2 + e^{-\tau q_k}, \quad \forall k \geq K_1(E)$$

Lemma 1.2.5 *If*

$$(1.60) \quad \beta > (25 + 1/\epsilon)\Lambda,$$

then for spectrally a.e. E , there is $K_2(E)$ such that

$$(1.61) \quad |\text{Trace } A_{q_k}(E) \pm 2| > e^{-10\Lambda q_k}, \quad \forall k \geq K_2(E)$$

With these two lemmas, we will first have the

Proof of Theorem 1.2.1:

Follows immediately by combining Lemma 1.2.4 with $\tau = 10\Lambda$ and Lemma 1.2.5. ■

Remark 3 *It may be interesting to compare Theorem 1.2.1 with the technique Last used in his proof of zero Hausdorff dimensionality of the spectral measures of supercritical Liouville almost Mathieu operators [64]. An important step there was using Schnol's theorem to show that eventually spectrally almost every energy is in the union of the spectral bands of the periodic approximants **enlarged** by a factor of q_k^2 . Here we show that spectrally almost every energy is in the union of the **shrunked** spectral bands of the periodic approximants, a much more delicate statement, technically, thus with more powerful consequences.*

1.2.3 The hyperbolic case: Proof of Lemma 1.2.4

We are going to show that if q is an ‘approximate’ period as in (1.3) with certain $\beta, \epsilon > 0$ and satisfies

$$(1.62) \quad |\text{Trace } A_q(E)| \geq 2 + e^{-\tau q}$$

then the trace of the transfer matrix at the scale $e^{\tau q/2}$ will be very large and any generalized eigenfunction of $Hu = Eu$ will be bounded from below at the scale $e^{\tau q/2}$. If this happens for infinitely many q , then any generalized eigenfunction will have at least a larger than $1/2$ power law growth (in index) along some fixed subsequence. By the extended Schnol's Theorem, such E must belong to a set of spectral measure zero.

Claim 1.2.1 *Suppose $q \rightarrow \infty$ satisfy $|\text{Trace } A_q(E)| \geq 2 + e^{-\tau q}$ and*

$$(1.63) \quad \max_{1 \leq j \leq q, |k| \leq e^{\epsilon \beta q}/q} |V(j+kq) - V(j+(k \pm 1)q)| \leq e^{-\beta q}, \quad \epsilon > 0.$$

Assume further that

$$(1.64) \quad \beta > (3 + 1/\epsilon)\tau + (7 + 1/\epsilon)\Lambda$$

Then there is $x_q^i \in \mathbb{Z}, i = 1, \dots, 4$, independent of E , such that $|x_q^i| \rightarrow \infty$ as $q \rightarrow \infty$ and for any $|u_0|^2 + |u_1|^2 = 1$, $\max_{i=1, \dots, 4} |u_{x_q^i}^E| > 1/16e^q$, where u_n^E is a solution with boundary values (u_0, u_1) .

Lemma 1.2.6 (Extended Schnol's Theorem) Fix any $y > 1/2$. For any sequence $|x_k| \rightarrow \infty$ (where the sequence is independent of E), for spectrally a.e. E , there is a generalized eigenvector u^E of $Hu = Eu$, such that

$$|u_{x_k}^E| < C(1 + |k|)^y$$

We can now finish the proof of Lemma 1.2.4.

Proof of Lemma 1.2.4:

Let q_k be given as in (1.3). Combining Claim 1.2.1 with Lemma 1.2.6 with $\{x_k\} = \cup_{i=1}^4 \{x_{q_k}^i\}$ we obtain that the set of E such that there are infinitely many q_{k_j} with $|\text{Trace}A_{q_{k_j}}(E, \alpha)| \geq 2 + e^{-\tau q_{k_j}}$ has spectral measure zero. ■

Claim 1.2.1 is based on the following results. First, we need to estimate the norm of the conjugation matrix for any hyperbolic $SL(2, \mathbb{R})$ matrix w.r.t. the distance between its trace and 2:

Lemma 1.2.7 Suppose $G \in SL(2, \mathbb{R})$ with $2 < |\text{Trace}G| \leq 6$. The invertible matrix B such that

$$(1.65) \quad G = B \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} B^{-1}$$

where $\rho^{\pm 1}$ are the two conjugate real eigenvalues of G with $|\det B| = 1$ satisfies

$$(1.66) \quad \|B\| = \|B^{-1}\| < \frac{\sqrt{\|G\|}}{\sqrt{|\text{Trace}G| - 2}}$$

If $|\text{Trace}G| > 6$, then $\|B\| \leq \frac{2\sqrt{\|G\|}}{\sqrt{|\text{Trace}G| - 2}}$.

The proof is based on a direct computation of the conjugate matrices. For the sake of completeness, we present it in Appendix A.1.

Fix $\tau > 0$ and apply Lemma 1.2.7 to A_q satisfying

$$(1.67) \quad |\text{Trace } A_q| > 2 + e^{-\tau q}.$$

We then have

Claim 1.2.2 *For large q ,*

$$(1.68) \quad A_q = B \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} B^{-1}$$

where $\rho^{\pm 1}$ are the two conjugate real eigenvalues of A_q with $\rho > 1$ and B satisfies $|\det B| = 1$ and

$$(1.69) \quad \|B\| = \|B^{-1}\| < e^{(\tau/2 + \Lambda + \Lambda/200)q}$$

Second, we need to use the almost periodicity (1.63) of the potential to obtain approximation statements for the transfer matrices. Set

$$(1.70) \quad N = \lceil e^{(\tau/2 + \Lambda/100)q} \rceil.$$

Under the assumption (1.63) and (1.64) on V as in Claim 1.2.1, we have for q large enough (the largeness depend on Λ and n_0),

Claim 1.2.3 *Under the conditions of Claim 1.2.1,*

$$(1.71) \quad \|A_{Nq} - A_q^N\| \leq 2e^{-\Lambda q} |\rho|^N \leq 2e^{-\Lambda q} |\text{Trace } A_q^N|$$

and

$$(1.72) \quad \|[A_{Nq}]^{-1} - A_{-Nq}\| \leq 4e^{-\Lambda q} |\rho|^N \leq 4e^{-\Lambda q} |\text{Trace } A_q^N|$$

Let us now finish the proof of Claim 1.2.1.

Proof of Claim 1.2.1: Decomposing A_q as in (1.68), we obtain $|\rho| > 1 + e^{-\tau q/2}$.

Obviously, $|\text{Trace } A_q^N| \geq |\rho|^N$. By (1.70), $N > 2e^{\tau q/2} \cdot q$, thus

$$|\text{Trace } A_q^N| \geq (1 + e^{-\tau q/2})^{2e^{\tau q/2} \cdot q} \geq e^q.$$

Assume q is large enough so that $2e^{-\Lambda q} \leq 1/10$. By (1.71), we have,

$$(1.73) \quad |\text{Trace}A_{Nq}| > (1 - 2e^{-\Lambda q})|\text{Trace}A_q^N| \geq \frac{9}{10}e^q.$$

Now consider solution u of $Hu = Eu$ with normalized initial value

$$X = \begin{pmatrix} u_1 \\ u_0 \end{pmatrix}, \quad \|X\| = 1$$

Then by (1.37):

$$(1.74) \quad A_{Nq} \cdot X = \begin{pmatrix} u_{Nq+1} \\ u_{Nq} \end{pmatrix}, \quad A_{-Nq} \cdot X = \begin{pmatrix} u_{-Nq+1} \\ u_{-Nq} \end{pmatrix}.$$

By the Cayley-Hamilton theorem combined with (1.72) and (1.73), we have

$$\begin{aligned} \frac{9}{10}|\text{Trace}A_q^N| \cdot \|X\| &\leq \|\text{Trace}A_{Nq}X\| \\ &= \|A_{Nq} \cdot X + [A_{Nq}]^{-1} \cdot X\| \\ &\leq \|A_{Nq} \cdot X\| + \|A_{-Nq} \cdot X\| + \frac{2}{10}|\text{Trace}A_q^N| \cdot \|X\| \end{aligned}$$

Then

$$\|A_{Nq} \cdot X\| + \|A_{-Nq} \cdot X\| \geq \frac{7}{10}|\text{Trace}A_q^N| \cdot \|X\| \geq \frac{1}{2}|\text{Trace}A_q^N|$$

which is equivalent to

$$\max \left\{ \left\| \begin{pmatrix} u_{Nq+1} \\ u_{Nq} \end{pmatrix} \right\|, \left\| \begin{pmatrix} u_{-Nq+1} \\ u_{-Nq} \end{pmatrix} \right\| \right\} \geq 1/4|\text{Trace}A_q^N|.$$

Therefore

$$\max \left\{ |u_{Nq+1}|, |u_{Nq}|, |u_{-Nq+1}|, |u_{-Nq}| \right\} \geq 1/16e^q.$$

Let $x_q^i = (-1)^i Nq + 1 - [i/3]$, $i = 1, \dots, 4$. Then for every q and one of $i = 1, \dots, 4$, $|u_{x_q^i}| > 1/16e^q$. ■

It now remains to prove (1.71),(1.72) in Claim 1.2.3. Set

$$(1.75) \quad \Delta_i = A_q(T^{(i-1)q}\theta, E) - A_q(\theta, E), \quad i = -N + 1, \dots, N.$$

Claim 1.2.4 Suppose (1.14) holds for $n \geq n_0$ and is uniform in $E \in S$. Fix $E \in S, \theta \in \Omega$. If V_θ satisfies (1.63) with $\epsilon > 0$, then there is a constant C_{n_0} (depends only on n_0 and upper bound of $\|V\|_\infty$), such that

$$(1.76) \quad \|\Delta_i(\theta, E)\| \leq |i-1|qC_{n_0}e^{(\Lambda-\beta)q}, \quad |i| = 1, \dots, [e^{\epsilon\beta q}/q]$$

Proof: The proof is quite standard. Suppose $1 \leq i \leq [e^{\epsilon\beta q}/q]$. Then for $|k| < i$, $|k|q < e^{\epsilon\beta q}$. Since $V_{T^{kq}\theta}(n) = V_\theta(n+kq)$, (1.63) implies that for $|k| < i$ the following holds:

$$|V_{T^{kq}\theta}(j) - V_{T^{(k+1)q}\theta}(j)| \leq e^{-\beta q}, \quad 1 \leq j \leq q$$

which implies

$$\|A(T^{kq+j}\theta) - A(T^{(k+1)q+j}\theta)\| \leq e^{-\beta q}, \quad 1 \leq j \leq q, \quad |k| < i$$

By a standard telescoping argument, for any $\theta' = T^{kq}\theta, |k| < i$,

$$\|A_q(T^q\theta') - A_q(\theta')\| \leq qC_V^{m_0}e^{(\Lambda-\beta)q} = qC_{n_0}e^{(\Lambda-\beta)q}$$

where C_V is such that $\|A(\theta', E)\| \leq C_V, \forall \theta', E$. In the above estimate, if $n > n_0$, we use the bound (1.14). When $n \leq n_0$, we use the trivial bound $\|A_n\| \leq C_V^{m_0}$, We have

$$A_{iq}(\theta, E) = A_q(T^{(i-1)q}\theta) \cdots A_q(T^q\theta)A_q(\theta).$$

Therefore for $1 \leq i \leq [e^{\epsilon\beta q}/q]$

$$\|\Delta_i\| \leq \sum_{k=1}^{i-1} \|A_q(T^{kq}\theta) - A_q(T^{(k-1)q}\theta)\| \leq (i-1)qC_{n_0}e^{(\Lambda-\beta)q}$$

Since (1.63) is symmetric w.r.t. $T \rightarrow T^{-1}$, (1.76) for $i \leq 0$ follows by taking $T' = T^{-1}$. ■

Proof of Claim 1.2.3: Write for any i , $A_q^i = B^{-1} \begin{pmatrix} \rho^i & 0 \\ 0 & \rho^{-i} \end{pmatrix} B$ so $\|A_q^i\| \leq \|B\|^2|\rho|^i$. Set $G(\theta) = \frac{1}{\rho}A_q(\theta)$ and $G_j = G(T^{(j-1)q}\theta)$. By (1.69), we have $\|G^i\| \leq \|B\|^2 \leq e^{(\tau+2\Lambda+\Lambda/100)q}$. Under the assumption (1.64), we have $\tau/2 + \Lambda/100 < \epsilon\beta$ so by (1.70), $Nq < e^{\epsilon\beta q}$. Then Claim 1.2.4 implies, for $j = -N, \dots, N$ and large q , that

$$\|G_j - G\| = \frac{1}{\rho} \|\Delta_j\| \leq NqC_{n_0}e^{(\Lambda-\beta)q} \leq e^{(-\beta+\tau/2+\Lambda+\Lambda/50)q}$$

Now we want to apply Lemma C.0.1 to these G_j , with $M = e^{(\tau+2\Lambda+\Lambda/100)q}$ and $\delta = e^{(-\beta+\tau/2+\Lambda+\Lambda/50)q}$. Direct computation gives

$$NM^2\delta < e^{(-\beta+3\tau+5\Lambda+\Lambda/20)q}$$

By (1.64) we have $\beta - (3\tau + 5\Lambda + \Lambda/20) > \Lambda$. Therefore, for q large enough (the largeness only depends on n_0), we have $NM\delta < NM^2\delta < e^{-\Lambda q}$. Then Lemma C.0.1 implies that

$$\left\| \prod_{j=1}^N G_j - G^N \right\| \leq 2NM^2\delta \leq 2e^{-\Lambda q}$$

and

$$\left\| \prod_{j=1}^N G_{-N+j} - G^N \right\| \leq 2NM^2\delta \leq 2e^{-\Lambda q}$$

Therefore

$$(1.77) \quad \|A_{Nq} - A_q^N\| = |\rho|^N \cdot \left\| \prod_{j=1}^N G_j - G^N \right\| \leq 2e^{-\Lambda q} |\text{Trace} A_q^N|.$$

establishing (1.71) , and

$$(1.78) \quad \|A_{Nq}(T^{-Nq}\theta) - A_q^N(\theta)\| = |\rho|^N \cdot \left\| \prod_{j=1}^N G_{-N+j} - G^N \right\| \leq 2e^{-\Lambda q} |\text{Trace} A_q^N|.$$

Since

$$\|A_{-Nq}(\theta) - [A_q^{-1}(\theta)]^N\| = \|[A_{Nq}]^{-1}(T^{-Nq}\theta) - [A_q^N(\theta)]^{-1}\| = \|[A_{Nq}](T^{-Nq}\theta) - [A_q^N(\theta)]\|$$

this implies

$$\|A_{-Nq}(\theta) - [A_q^{-1}(\theta)]^N\| \leq 2e^{-\Lambda q} |\text{Trace} A_q^N|.$$

Also,

$$(1.79) \quad \|[A_{Nq}]^{-1} - [A_q^{-1}]^N\| = \|[A_{Nq}] - [A_q]^N\|,$$

therefore, by (1.77), we obtain (1.72). ■

Lemma 1.2.6 is proved in the same way as the standard Schnol's Lemma, however the statement in this form, while very useful, does not seem to be in the literature (we learned it from S. Molchanov, see the Acknowledgement). For the sake of completeness, we include a short proof in the Appendix.

1.2.4 Energies with Trace close to 2: Proof of Lemma 1.2.5

All throughout this section, we will assume again that all q are large enough and satisfy (1.3) with certain $\beta, \epsilon > 0$, i.e.,

$$(1.80) \quad \max_{1 \leq j \leq q, |k| \leq e^{\epsilon \beta q}/q} |V(j+kq) - V(j+(k \pm 1)q)| \leq e^{-\beta q}, \quad \epsilon > 0.$$

We are going to show that spectrally almost surely, there are only finitely many q such that $\text{Trace}A_q$ is close to 2.

In fact, we are going to prove the following quantitative estimate on the measure of energies where the trace of the associated transfer matrix is close to 2.

Lemma 1.2.8 *Let Λ be given by (1.14) on some set $S \subset \sigma(H)$. Let*

$$(1.81) \quad S_q = \{E : 0 < |\text{Trace}A_q \pm 2| < e^{-10\Lambda q}\}.$$

Assume (1.80) holds and

$$(1.82) \quad \beta > (25 + 11/\epsilon)\Lambda.$$

Then

$$(1.83) \quad \mu(S_q) < 4q \cdot e^{-\Lambda q/15} < e^{-\Lambda q/20}$$

where $\mu = \mu_S$ is the spectral measure restricted to S .

Once we have Lemma 1.2.8, Borel Cantelli lemma immediately implies Lemma 1.2.5. So the main task is to prove (1.83).

In order to estimate the spectral measure of S_q , first we recall the following results on the structure of S_q . Let $\mathcal{P}_n(\mathbb{R})$ denote the polynomials over \mathbb{R} of exact degree n . Let the class $\mathcal{P}_{n;n}(\mathbb{R})$ be elements in $\mathcal{P}_n(\mathbb{R})$ with n *distinct* real zeros.

Proposition 1.2.1 (Theorem 6.1,[55]) *Let $p \in \mathcal{P}_{n;n}(\mathbb{R})$ with $y_1 < \dots < y_{n-1}$ the local extrema of p . Let*

$$(1.84) \quad \zeta(p) := \min_{1 \leq j \leq n-1} |p(y_j)|$$

and $0 \leq a < b$. Then,

$$(1.85) \quad |p^{-1}(a, b)| \leq 2 \operatorname{diam}(z(p - a)) \max \left\{ \frac{b - a}{\zeta(p) + a}, \left(\frac{b - a}{\zeta(p) + a} \right)^{\frac{1}{2}} \right\}$$

where $z(p)$ is the zero set of p and $|\cdot|$ denotes the Lebesgue measure.

Fix any $\tau > 0$. We apply Proposition 1.2.1 to polynomial $\operatorname{Trace}A_q(E) \in \mathcal{P}_{q,q}(\mathbb{R})$, with $a = 2$, $b = 2 + e^{-\tau q}$. Clearly, $\operatorname{diam}(z(\operatorname{Trace}A_q - 2))$ is bounded from above by some constant that only depends on $\|V\|_\infty$. We also have $|\zeta(\operatorname{Trace}A_q)| \geq 2$. Since $b - a < 1$, we have $(\operatorname{Trace}A_q)^{-1}(a, b) \leq C_V \sqrt{b - a} = C_V e^{-\tau q/2}$ where C_V is some constant that only depends on $\|V\|_\infty$. Since $(\operatorname{Trace}A_q)^{-1}(a, b)$ contains at most q bands, setting $S_q = \{E : 2 < \operatorname{Trace}A_q < 2 + e^{-\tau q}\}$, we have

$$(1.86) \quad S_q = \bigcup_{j=1}^q I_j, \quad |I_j| \leq |S_q| \leq C_V e^{-\tau q/2}.$$

The same analysis works for $(a, b) = (2 - e^{-\tau q}, 2), (-2 - e^{-\tau q}, -2), (-2, -2 + e^{-\tau q})$. Thus the structure (1.86) also holds for the other three cases.

Denote by

$$(1.87) \quad \varepsilon_q^j = |I_j| < e^{(-\tau/2 + \Lambda/200)q}.$$

If $I_j \cap \Sigma \neq \emptyset$, pick $E_j \in I_j \cap \Sigma$ where $\Sigma = \sigma(H)$ is the spectrum. Set $\tilde{I}_j = (E_j - \varepsilon_q^j, E_j + \varepsilon_q^j)$. Then $I_j \subset \tilde{I}_j$, so it is enough to estimate the spectral measure of $\cup \tilde{I}_j$.

Set $N_q = [e^{(\tau/2 - \Lambda/200)q}]$. For any $\varepsilon_q > 0$, define $l_q = l(\varphi, \varepsilon_q, E), u^\varphi, v^\varphi$ as in (1.29). Write $l_q = [l_q] + l_q - [l_q]$, and $[l_q] = K_q \cdot q + r_q$, where $0 \leq r_q = [l_q] \bmod q < q$ and $0 \leq l_q - [l_q] < 1$. We need the following power law estimate, which is the key part to the proof of Lemma 1.2.8.

Claim 1.2.5 *Suppose $E \in S_q \cap \Sigma$ and $0 < \varepsilon_q < e^{(-\tau/2 + \Lambda/200)q}$. Suppose (1.80) holds. Assume that $\beta > (2 + 1/\varepsilon)\tau + (5 + 1/\varepsilon)\Lambda$ and $\tau \geq 10\Lambda$. Then for every initial condition φ ,*

$$(1.88) \quad \|u^\varphi\|_{l_q}^2 \geq e^{\frac{1}{10}\Lambda q}$$

Combining (1.88) with the subordinacy theory, we are ready to estimate the m -function and the spectral measure.

Proof of Lemma 1.2.8: Take $\tau = 10\Lambda$. Then $\beta > (25 + 11/\epsilon)\Lambda$ satisfies the requirement in Claim 1.2.5. Let $E_j \in I_j \cap \Sigma \subset S_q \cap \Sigma$. For any φ , let $u^{\varphi, E_j}, v^{\varphi, E_j}$ be the right half line solution associated with the energy E_j . According to (1.87), Claim 1.2.5 can be applied to all u^{φ, E_j} .

We have for any φ ,

$$\|u^{\varphi, E_j}\|_{l_q(j)}^2 \geq e^{\frac{1}{10}\Lambda q}, \quad j = 1, \dots, q$$

where $l_q(j) = l(\varphi, E_j, \varepsilon_q^j)$.

Then by the J-L inequality (1.31) and the definition of $l_q(j)$, we have

$$\varepsilon_q^j \cdot |m_\varphi(E_j + i\varepsilon_q^j)| < \frac{5 + \sqrt{24}}{2\|u^{\varphi, E_j}\|_{l_q} \cdot \|v^{\varphi, E_j}\|_{l_q}} \cdot \frac{\|v^{\varphi, E_j}\|_{l_q}}{\|u^{\varphi, E_j}\|_{l_q}} < \frac{5 + \sqrt{24}}{2} \cdot e^{-\Lambda q/10}$$

Notice that the interval I_j is independent of the boundary condition φ , and so is ε_q^j . Therefore, we can take the supremum w.r.t. φ on both sides of the above inequality. By Proposition 1.1.1, we have

$$\varepsilon_q^j \cdot |M(E_j + i\varepsilon_q^j)| \leq \frac{5 + \sqrt{24}}{2} \cdot e^{-\Lambda q/10}$$

On the other hand, by the definition of $M(z)$ in (1.7), we have

$$\Im M(E + i\varepsilon) \geq \frac{1}{2\varepsilon} \mu(E - \varepsilon, E + \varepsilon), \quad E \in \mathbb{R}, \quad \varepsilon > 0$$

Therefore,

$$\mu(E_j - \varepsilon_q^j, E_j + \varepsilon_q^j) \leq 2\varepsilon_q^j \cdot |M(E_j + i\varepsilon_q^j)| \leq (5 + \sqrt{24})e^{-\Lambda q/10}$$

which implies

$$\mu(I_j) \leq \mu(\tilde{I}_j) \leq e^{-\Lambda q/15}$$

Since in (1.86) there are four cases for S_q and each of them satisfies the previous estimates, the spectral measure of S_q will be bounded by $4qe^{-\Lambda q/15} \leq e^{-\Lambda q/20}$. \blacksquare

The proof of Claim 1.2.5 relies on the following estimates on the transfer matrices. The first one is a formula for the power of a general $SL(2, \mathbb{R})$ matrix. It is elementary but turned out particularly useful and will be an important part of our quantitative estimates in both hyperbolic and nearly parabolic cases. As we did not find it in the literature, we will provide a proof of it as well as of the next Lemma, in the Appendix.

Lemma 1.2.9 *Suppose $A \in SL(2, \mathbb{R})$ has eigenvalues $\rho^{\pm 1}$. For any $k \in \mathbb{N}$, if $\text{Trace}A \neq 2$, then*

$$(1.89) \quad A^k = \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} \cdot \left(A - \frac{\text{Trace}A}{2} \cdot I \right) + \frac{\rho^k + \rho^{-k}}{2} \cdot I$$

Otherwise, $A^k = k(A - I) + I$.

The key to the estimates in the nearly parabolic case is then the following simple

Lemma 1.2.10 *There are universal constants $1 < C_1 < \infty, c_1 > 1/3$ such that for $E \in S_q$ and $1 \leq k \leq N_q$, we have*

$$(1.90) \quad c_1 < \frac{\rho^k + \rho^{-k}}{2} < C_1, \quad c_1 k < \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} < C_1 k$$

Second, since $A_q(\theta)$ is almost periodic (with an exponential error), the iteration of $A_q(\theta)$ along the orbit will be close to its power. The argument is similar to what we used in the proof of (1.71) in the previous part.

Claim 1.2.6 *Fix $\theta \in \Omega$, $E \in S_q \cap \Sigma$ and $\tau > 0$. Suppose (1.80) holds with $\beta > (2 + 1/\epsilon)\tau + (5 + 1/\epsilon)\Lambda$. Then for any $1 \leq k \leq N_q$, we have*

$$(1.91) \quad \|A_{kq} - A_q^k\| \leq 2e^{-\Lambda q}.$$

Proof: Set $\Delta_j = A_q(T^{j-1}\theta) - A_q(\theta)$. By the Claim 1.2.4, $\|\Delta_j\| \leq jqCe^{(-\beta+\Lambda)q}$, $j < [e^{\epsilon\beta q}/q]$. Recall that $N_q = [e^{(\tau/2-\Lambda/200)q}]$. The condition on β guarantees $N_q < [e^{\epsilon\beta q}/q]$, therefore we have $\|\Delta_j\| \leq e^{(-\beta+\tau/2+\Lambda+\Lambda/100)q}$ for all $j = 1, \dots, N_q$. We need to check the other requirements of Lemma C.0.1. According to Lemmas 1.2.9, 1.2.10,

$$\|A_q^j\| < C_1 j \|A_q - \frac{\text{Trace}A_q}{2} \cdot I\| + C_1 < 3C_1 N_q \|A_q\| < e^{(\tau/2+\Lambda+\Lambda/100)q}.$$

Now apply Lemma C.0.1 to the sequence $A_q(\theta), \dots, A_q(T^{j-1}\theta), \dots, A_q(T^{k-1}\theta)$, with $M = e^{(\tau/2+\Lambda+\Lambda/100)q}$ and $\delta = e^{(-\beta+\tau/2+\Lambda+\Lambda/100)q}$. We have $N_q M^2 \delta < e^{(-\beta+2\tau+3\Lambda+\Lambda/40)q}$. Since $\beta > (2 + 1/\epsilon)\tau + (5 + 1/\epsilon)\Lambda > 2\tau + 5\Lambda$ we have $\beta - (2\tau + 3\Lambda + \Lambda/40) > \Lambda$. Therefore, for q large enough, $N_q M \delta < N_q M^2 \delta < e^{-\Lambda q}$. Thus, by Lemma C.0.1, we have $\|A_{kq} - A_q^k\| = \|\prod_{j=1}^k A_q(T^{j-1}\theta) - A_q^k(\theta)\| \leq 2e^{-\Lambda q}$. ■

Now we are ready to finish the proof of the most technical part.

Proof of Claim 1.2.5: We first show the following lower bound for $K_q = \lfloor \frac{[l_q]}{q} \rfloor$:

$$(1.92) \quad K_q > e^{\Lambda q/6} > 18C_1 \cdot e^{\Lambda q/8}$$

Actually, if $K_q \geq N_q$, (1.92) is automatically satisfied since $\tau \geq 10\Lambda$.

Now assume $K_q < N_q$. For any $n \leq [l_q] + 1$, write $n = kq + r$, where $0 \leq k \leq K_q$, $0 \leq r \leq q$. Set $X_\varphi = \begin{pmatrix} \cos \varphi \\ -\sin \varphi \end{pmatrix}$. According to (1.89), (1.90), we have for any φ , $1 \leq k \leq K_q < N_q$,

$$\|A_q^k \cdot X_\varphi\| < C_1 k \|A_q - \frac{\text{Trace} A_q}{2} \cdot I\| + C_1 < C_1 k (\|A_q\| + 3/2) + C_1$$

and by (1.91),

$$\|A_{kq} \cdot X_\varphi\| \leq \|A_q^k \cdot X_\varphi\| + \|(A_{kq} - A_q^k) \cdot X_\varphi\| \leq C_1 k (\|A_q\| + 3/2) + C_1 + 1$$

For $n_0 < r \leq q$, and for any $\theta' \in \Omega$, $\|A_r(\theta')\| \leq e^{\Lambda q}$. For $1 \leq r \leq n_0$, we bound $\|A_r(\theta')\|$ by C^{m_0} as in the proof of Claim 1.2.4. Therefore $\|A_r(\theta')\| \leq e^{\Lambda q}$ for all $1 \leq r \leq q$ with q large. Thus,

$$\begin{aligned} \|A_{kq+r}(\theta) \cdot X_\varphi\| &\leq \|A_r(T^{kq}\theta)\| \cdot \|A_{kq}(\theta) \cdot X_\varphi\| \\ &\leq e^{\Lambda q} \left(C_1 k (\|A_q\| + 3/2) + C_1 + 1 \right) \\ &\leq e^{\Lambda q} \cdot \left(C_1 k (e^{\Lambda q} + 3/2) + C_1 + 1 \right) \\ &\leq k \cdot e^{(2\Lambda+\Lambda/200)q} \end{aligned}$$

Recall that $\begin{pmatrix} u_{n+1}^\varphi \\ u_n^\varphi \end{pmatrix} = A_n \cdot X_\varphi$, direct computation shows

$$\begin{aligned}
\|u^\varphi\|_{l_q}^2 &\leq \sum_{n=1}^{[l_q]+1} \|A_n \cdot X_\varphi\|^2 \\
&\leq \sum_{r=1}^q \|A_r \cdot X_\varphi\|^2 + \sum_{k=1}^{K_q} \sum_{r=1}^q \|A_{kq+r} \cdot X_\varphi\|^2 \\
&\leq q \cdot e^{2\Lambda q} + \sum_{k=1}^{K_q} \sum_{r=1}^q k^2 \cdot e^{(4\Lambda+\Lambda/100)q} \\
&\leq q \cdot e^{2\Lambda q} + K_q^3 \cdot q \cdot e^{(4\Lambda+\Lambda/100)q} \\
&\leq K_q^3 \cdot e^{(4\Lambda+\Lambda/20)q}
\end{aligned}$$

Since φ is arbitrary, and $\begin{pmatrix} v_{n+1}^\varphi \\ v_n^\varphi \end{pmatrix} = A_n \cdot X_{\varphi+\pi/2}$, $\|v^\varphi\|_{l_q}^2$ has the same upper bound. Therefore, $\|u^\varphi\|_{l_q} \|v^\varphi\|_{l_q} \leq K_q^3 \cdot e^{(4\Lambda+\Lambda/20)q}$. On the other hand, since $\varepsilon_q < e^{(-\tau/2+\Lambda/200)q}$, we have

$$(1.93) \quad \|u^\varphi\|_{l_q} \|v^\varphi\|_{l_q} = \frac{1}{2\varepsilon_q} \geq e^{(\tau/2-\Lambda/100)q}$$

With $\tau \geq 10\Lambda$, this implies that $K_q^3 \geq e^{q\Lambda/2}$. Therefore,

$$(1.94) \quad K_q > e^{\Lambda q/6} > 18C_1 \cdot e^{\Lambda q/8}$$

as claimed.

In order to get the lower bound on $\|u^\varphi\|_{l_q}^2$ in (1.88), we need to consider two cases.

case I: For φ such that

$$(1.95) \quad \|(A_q - \frac{\text{Trace}A_q}{2} \cdot I) \cdot X_\varphi\| \geq e^{-\Lambda q/8},$$

by (1.89),(1.90), for any $1 \leq k \leq 18C_1 \cdot e^{\Lambda q/8} \leq N_q$, we have

$$\begin{aligned}
\|A_q^k \cdot X_\varphi\| &= \left\| \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} \cdot \left(A_q - \frac{\text{Trace}A_q}{2} \cdot I \right) X_\varphi + \frac{\rho^k + \rho^{-k}}{2} \cdot X_\varphi \right\| \\
&\geq \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} \cdot \left\| \left(A_q - \frac{\text{Trace}A_q}{2} \cdot I \right) X_\varphi \right\| - \frac{\rho^k + \rho^{-k}}{2} \cdot \|X_\varphi\| \\
&\geq \frac{1}{3} k \cdot e^{-\Lambda q/8} - C_1
\end{aligned}$$

Due to (1.91), we have then

$$\|A_{kq} \cdot X_\varphi\| \geq \|A_q^k \cdot X_\varphi\| - \|(A_{kq} - A_q^k) \cdot X_\varphi\| \geq \frac{1}{3}k \cdot e^{-\Lambda q/8} - 2C_1.$$

Therefore, for $9C_1 \cdot e^{\Lambda q/8} \leq k \leq 18C_1 \cdot e^{\Lambda q/8}$, we have

$$(1.96) \quad \|A_{kq} \cdot X_\varphi\| \geq C_1 > 1$$

By (1.94) and (1.96) we obtain

$$\|u^\varphi\|_{l_q}^2 \geq \frac{1}{2} \sum_{n=1}^{[l_q]-1} \|A_n \cdot X_\varphi\|^2 \geq \frac{1}{2} \sum_{k=[9C_1 \cdot e^{\Lambda q/8}]+1}^{[18C_1 \cdot e^{\Lambda q/8}]} \|A_{kq} \cdot X_\varphi\|^2 \geq e^{\Lambda q/10}$$

case II: For φ such that

$$(1.97) \quad \|(A_q - \frac{\text{Trace}A_q}{2}I) \cdot X_\varphi\| < e^{-\Lambda q/8},$$

by (1.89),(1.90), for any $1 \leq k \leq N_q$ we get

$$\begin{aligned} \|A_q^k \cdot X_\varphi\| &= \left\| \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} \cdot \left(A_q - \frac{\text{Trace}A_q}{2}I \right) X_\varphi + \frac{\rho^k + \rho^{-k}}{2} \cdot X_\varphi \right\| \\ &\geq \frac{\rho^k + \rho^{-k}}{2} \cdot \|X_\varphi\| - \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} \cdot \left\| \left(A_q - \frac{\text{Trace}A_q}{2}I \right) X_\varphi \right\| \\ &\geq 1/2 - C_1 k \cdot e^{-\Lambda q/8} \end{aligned}$$

Combining with (1.91), we have

$$\|A_{kq} \cdot X_\varphi\| \geq \|A_q^k \cdot X_\varphi\| - \|(A_{kq} - A_q^k) \cdot X_\varphi\| \geq \frac{1}{4} - C_1 k \cdot e^{-\Lambda q/8}$$

Then for $1 \leq k \leq \frac{1}{8C_1} \cdot e^{\Lambda q/8} \leq K_q \leq N_q$, we obtain $\|A_{kq} \cdot X_\varphi\| \geq \frac{1}{8}$. This implies

$$\|u^\varphi\|_{l_q}^2 \geq \frac{1}{2} \sum_{k=1}^{[\frac{1}{8C_1} \cdot e^{\Lambda q/8}]} \|A_{kq} \cdot X_\varphi\|^2 \geq e^{\Lambda q/10}.$$

1.2.5 Proof of Lemmas 1.2.2 and 1.2.3

Assume that $|\text{Trace}A_q| < 2 - e^{-\tau q} < 2$. Thus in the expression in Lemma 1.2.9, $\rho = e^{i\psi}$, $\psi \in (-\pi, \pi)$. We have for any j ,

$$(1.98) \quad A_q^j = \frac{\sin j\psi}{\sin \psi} \cdot \left(A_q - \frac{\text{Trace}A_q}{2} \cdot I \right) + \frac{\cos j\psi}{2} \cdot I, \quad \psi \in (-\pi, \pi)$$

Then $|2 \cos \psi| = |\text{Trace} A_q| < 2 - e^{-\tau q}$ implies $|\sin \psi| > \sqrt{1 - (1 - \frac{1}{2}e^{-\tau q})^2} > C e^{-\tau q/2}$.
Therefore,

$$\|A_q^j\| \leq C \cdot e^{\tau q/2} \cdot (\|A_q\| + 1) + 1$$

(here q is large enough so that $\|A_q\| \leq e^{\Lambda q}$.) If $\tau = 10\Lambda$, we obtain

$$\|A_q^j\| \leq e^{(6\Lambda + \Lambda/100)q}.$$

Now let $N = \lceil e^{\xi q} \rceil$. By the same argument as used for the proof of (1.71) and (1.91) (based on Lemma C.0.1), if $\beta > 15\Lambda + (2 + 1/\epsilon)\xi$, then for any $j \leq N^\xi$,

$$\|A_q^j - A_{jq}\| < e^{(-\beta + 13\Lambda + 2\xi + \Lambda/20)q} < e^{-\Lambda q}.$$

As a consequence, we have $\|A_{jq}\| \leq e^{(6\Lambda + \Lambda/50)q}$, and $\|A_{jq+r}\| \leq e^{(7\Lambda + \Lambda/50)q}$ for all $0 \leq r \leq q$, $0 \leq j \leq N^\xi$. Therefore

$$\sum_{n=1}^{Nq} \|A_n(E)\|^2 \leq \sum_{k=0}^{N^\xi} \sum_{r=1}^q \|A_{kq+r}(\theta, E)\|^2 \leq Nq \cdot e^{(14\Lambda + \Lambda/25)q} \leq e^{(\xi + 15\Lambda)q}$$

■

Proof of Lemma 1.2.3 Since $N > e^{(\xi - \Lambda/200)q}$ for q large, then for any $\gamma < 1$,

$$\frac{1}{(Nq)^{2-\gamma}} \sum_{n=1}^{Nq} \|A_n(E)\|^2 < e^{-(1-\gamma)\xi + 15\Lambda} q$$

If $\xi > \frac{16\Lambda}{1-\gamma}$, then $(1-\gamma)\xi - (15\Lambda) \geq 1/2\Lambda$. Therefore,

$$\frac{1}{(Nq)^{2-\gamma}} \sum_{n=1}^{Nq} \|A_n(E)\|^2 \leq e^{-1/2\Lambda q} \leq 1$$

■

1.3 Spectral Singularity

1.3.1 Power law estimates and proof of Theorem 1.1.5

Throughout this section, our potential will be given by $V_\theta(n) = V(\theta + n\alpha)$, $n \in \mathbb{Z}$, where $V(\theta)$ is a real analytic function defined on the torus with analytic extension to the strip $\{z : |\text{Im}z| < \rho\}$.

According to Lemma 1.1.2, it is enough to find a φ such that both m_φ and $\tilde{m}_{\pi/2-\varphi}$ are γ -spectral singular. The main technical tool to estimate m -function is the subordinacy theory Lemma 1.1.1. We also need one more general statement about the existence of generalized eigenfunctions with sub-linear growth in its l -norm (see [65]). That is, for μ_θ -a.e. E , there exists $\varphi \in (-\pi/2, \pi/2]$ such that u^φ and $u^{\varphi,-}$ both obey

$$(1.99) \quad \limsup_{l \rightarrow \infty} \frac{\|u\|_l}{l^{1/2} \log l} < \infty$$

This inequality provides us an upper bound for the l -norm of the solution. To apply subordinacy theory, we also need a lower bound for the l -norm. It will be derived from the following lower bounds on the norm of the transfer matrices. Denote

$$(1.100) \quad \tilde{A}_n(\theta, E, \alpha) = A_n(\theta - \alpha, E, -\alpha)$$

We have

Lemma 1.3.1 *Fix $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ with $\beta = \beta(\alpha) < \infty$. Assume that $L(E) \geq a > 0, E \in S$. There is $c = c(a, \rho) > 0$ such that for $l > l(E, \beta, \rho)$, and any $\theta \in \mathbb{T}$, the following hold:*

$$(1.101) \quad \sum_{k=1}^l \|A_k(\theta, E, \alpha)\|^2 \geq l^{1+\frac{2c}{\beta}}$$

and

$$(1.102) \quad \sum_{k=1}^l \|\tilde{A}_k(\theta, E, \alpha)\|^2 \geq l^{1+\frac{2c}{\beta}}$$

Proof of Theorem 1.1.5: For any φ , we have

$$(1.103) \quad \|u^\varphi\|_l^2 + \|v^\varphi\|_l^2 \geq \frac{1}{2} \sum_{k=1}^l \|A_k(\theta)\|^2$$

and

$$(1.104) \quad \|u^{\varphi,-}\|_l^2 + \|v^{\varphi,-}\|_l^2 \geq \frac{1}{2} \sum_{k=1}^l \|\tilde{A}_k(\theta)\|^2$$

Therefore, a direct consequence of (1.101) is the power law estimate for the left hand side of (1.103), i.e., $\|u^\varphi\|_l^2 + \|v^\varphi\|_l^2 \geq l^{1+\frac{2c}{\beta}}$ for l large.

On the other hand, according to (1.99), for μ_θ -a.e. E , there exist $\varphi(E)$ and $C = C(E) < \infty$, such that for large l ,

$$(1.105) \quad \|u^\varphi\|_l \leq Cl^{1/2} \log l, \quad \|u^{\varphi,-}\|_l \leq Cl^{1/2} \log l$$

Let us consider the right half line estimates for u^φ, m_φ first. From (1.103) and (1.105), we have

$$\|v^\varphi\|_l^2 \geq l^{1+\frac{2\varepsilon}{\beta}} - Cl(\log l)^2$$

and then

$$(1.106) \quad \|v^\varphi\|_l \geq l^{1/2+c/\beta}$$

provided $\beta < \infty$ and $l > l(\beta, E, \rho)$.

Applying subordinacy theory (1.31) to (1.105),(1.106), one has for any $\gamma \in (0, 1)$, any $\varepsilon > 0$

$$\begin{aligned} \varepsilon^{1-\gamma} |m_\varphi(E + i\varepsilon)| &\geq \frac{1}{\left(2\|u^\varphi\|_{l(\varepsilon)}\|v^\varphi\|_{l(\varepsilon)}\right)^{1-\gamma}} \cdot (5 - \sqrt{24}) \frac{\|v^\varphi\|_{l(\varepsilon)}}{\|u^\varphi\|_{l(\varepsilon)}} \\ &\geq c_\gamma \frac{\|v^\varphi\|_l^\gamma}{\|u^\varphi\|_l^{2-\gamma}} \\ &\geq c_\gamma l^{(1+c/\beta)\gamma-1} \cdot \log^{-2} l \end{aligned}$$

where $c_\gamma > 0$ may denote different constants that only depend on γ . Set $\gamma_0 = \gamma_0(\beta) = \frac{1}{1+c/\beta} < 1$, since $\beta < \infty$. We have for any $\gamma > \gamma_0$,

$$\varepsilon^{1-\gamma} |m_\varphi(E + i\varepsilon)| \geq c_\gamma l^{\gamma/\gamma_0-1} \cdot \log^{-2} l \rightarrow \infty$$

as $\varepsilon \rightarrow 0$.

Using (1.104) and (1.105), the same argument works for $u^{\varphi,-}, v^{\varphi,-}$ and m_φ^- . Therefore, for spectrally a.e. E $\liminf_{\varepsilon \rightarrow 0} \varepsilon^{1-\gamma} |m_\varphi(E + i\varepsilon)| = \infty$ and $\liminf_{\varepsilon \rightarrow 0} \varepsilon^{1-\gamma} |m_\varphi^-(E + i\varepsilon)| = \infty$. According to Lemma 1.1.2, μ is γ -spectral singular for any $\gamma > \gamma_0$. The conclusion for the spectral dimension follows from the definition directly. \blacksquare

The proof of Lemma 1.3.1 depends on the following lemmas about the localization density of the half line solution. The key observation is that in the regime of positive Lyapunov exponents we can guarantee transfer matrix growth at scale q_n somewhere within any interval of length q_n , giving a contribution to (1.101).

Lemma 1.3.2 *Assume that $L(E) \geq a > 0, \alpha \in \mathbb{R} \setminus \mathbb{Q}$. There are $c_2 = c_2(a, \rho) > 0$ and a positive integer $d = d(\rho)$ such that for $E \in S$ and $n > n(E, \rho)$, there exists an interval Δ_n such that*

$$(1.107) \quad \text{Leb}(\Delta_n) \geq \frac{c_2}{4dn}$$

and for any $\theta \in \Delta_n$, we have ⁶

$$(1.108) \quad \|A_n(\theta, E, \alpha)\|_{HS}^2 > e^{nL(E)/8}$$

In the following, we will use $\|\cdot\|$ for the HS norm $\|\cdot\|_{HS}$. Now let c_2 and d be given as in Lemma 1.3.2. Denote

$$(1.109) \quad k_n = \left\lceil \frac{c_2 q_n}{4d} \right\rceil - 1$$

where, as before q_n are the denominators of the continued fraction approximants to α . Based on Lemma 1.3.2, one can show that

Lemma 1.3.3 *Fix $E \in S$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Let k_n be given as in (1.109). Suppose q_n is large enough so that (1.107) holds for Δ_{k_n} . Then for any θ , and any $N \in \mathbb{N}$, there is $j_N(\theta) \in [2Nq_n, 2(N+1)q_n)$ such that*

$$(1.110) \quad \|A_{j_N}(\theta, E, \alpha)\| > e^{k_n L(E)/16}$$

We first use Lemma 1.3.2 and Lemma 1.3.3 to finish the proof of Lemma 1.3.1. The proofs of these two lemmas are left to next section.

⁶We denote by $\|\cdot\|_{HS}$ the Hilbert-Smith norm of a $SL(2, \mathbb{R})$ matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \|A\|_{HS} = \sqrt{a^2 + b^2 + c^2 + d^2}$$

Proof of Lemma 1.3.1: For l sufficiently large, there is q_n such that, $l \in [2q_n, 2q_{n+1})$.

Write l as

$$l = 2Nq_n + r,$$

where $0 \leq r < 2q_n$, $1 \leq N < \frac{q_{n+1}}{q_n}$. Suppose q_n is large enough so that (1.107) holds for Δ_{k_n} . Then Lemma 1.3.3 is applicable. Fix θ . Consider $A_n(\theta, E, \alpha)$ first. Let $j_n(\theta) \in [2nq_n, 2(n+1)q_n)$, $n = 0, 1, \dots, N$, be given as in (1.110). Direct computation shows that

$$\begin{aligned} \sum_{k=1}^l \|A_k(\theta)\|^2 &\geq \|A_{j_0}(\theta)\|^2 + \|A_{j_1}(\theta)\|^2 + \dots + \|A_{j_{N-1}}(\theta)\|^2 \\ &\geq Ne^{k_n L(E)/16} \end{aligned}$$

We have $l = 2Nq_n + r < 4Nq_n$, i.e., $N > l/(4q_n)$. (1.109) implies $c_2q_n/(5d) < k_n < c_2q_n/(4d)$ for q_n large, so we have

$$\sum_{k=1}^l \|A_k(\theta)\|^2 \geq \frac{l}{4q_n} e^{k_n L(E)/16} \geq \frac{l}{4q_n} e^{16cq_n}$$

where $c = c(c_2, d, a)$. Then for sufficiently large l , we have

$$\sum_{k=1}^l \|A_k(\theta)\|^2 > le^{8cq_n}$$

We also assume l large enough (meaning q_n large enough), so that $\frac{\log q_{n+1}}{q_n} < 2\beta$, i.e., $e^{q_n} > q_{n+1}^{\frac{1}{2\beta}}$. Then

$$\sum_{k=1}^l \|A_k(\theta)\|^2 \geq l \cdot q_{n+1}^{\frac{4c}{\beta}} \geq l \cdot \left(\frac{l}{2}\right)^{\frac{4c}{\beta}} \geq l^{1+\frac{2c}{\beta}}.$$

For the same θ , repeat the above procedure for $A_n(\theta - \alpha, -\alpha, E)$. Notice that $\tilde{A}_n(\theta, E, \alpha) = A_n(\theta - \alpha, E, -\alpha)$. Therefore, we have a sequence of positive integers $\tilde{j}_N(\theta - \alpha) \in [2Nq_n, 2(N+1)q_n)$ for any $N \in \mathbb{N}$ such that

$$(1.111) \quad \|\tilde{A}_{\tilde{j}_N}(\theta, E, \alpha)\| > e^{k_n L(E)/16}$$

The rest of the computations are exactly the same as for $A_n(\theta, E, \alpha)$. Notice that the constants c_2 and d in Lemma 1.3.2 are independent of the choice of α or $-\alpha$ and θ . So k_n and c will be the same for A_n and \tilde{A}_n ■

1.3.2 Proof of the density lemmas

Proof of Lemma 1.3.2: Denote

$$(1.112) \quad f_n(\theta) = \|A_n(\theta)\|_{HS}^2$$

Obviously, $f_n(\theta)$ is a real analytic function with analytic extension to the strip $\{z : |\operatorname{Im}z| < \rho\}$. For bounded S we have

$$(1.113) \quad \|f_n\|_\rho := \sup_{|\operatorname{Im}z| < \rho} |f_n(z)| < e^{C_1 n}, \quad E \in S$$

where $C_1 = C_1(S, \|V\|_\rho)$ can be taken uniform for all $E \in S$. Expand $f_n(\theta)$ into its Fourier series on \mathbb{T} as

$$(1.114) \quad f_n(\theta) = \sum_{k \in \mathbb{Z}} b_n(k) e^{2\pi i k \theta}$$

where $b_n(k)$ is the k -th Fourier coefficients of $f_n(\theta)$ so satisfies

$$(1.115) \quad |b_n(k)| < \|f_n\|_\rho \cdot e^{-2\pi\rho|k|}, \quad \forall k \in \mathbb{Z}$$

We split $f_n(\theta)$ into two parts, for some positive integer d which will be specified a little bit later

$$f_n(\theta) = g_n(\theta) + R_n(\theta), \quad g_n(\theta) = \sum_{|k| \leq d \cdot n} b_n(k) e^{2\pi i k \theta}, \quad R_n(\theta) = \sum_{|k| > d \cdot n} b_n(k) e^{2\pi i k \theta}$$

For any $\theta \in \mathbb{T}$

$$\begin{aligned} |R_n(\theta)| &\leq \sum_{|k| > d \cdot n} |b_n(k)| \leq \sum_{|k| > d \cdot n} \|f_n\|_\rho \cdot e^{-2\pi\rho|k|} \\ &\leq \frac{2}{1 - e^{-2\pi\rho}} e^{C_1 n} e^{-2\pi\rho d n} \end{aligned}$$

Now pick

$$(1.116) \quad d = \left\lceil \frac{C_1}{2\pi\rho} \right\rceil + 2$$

With this choice of d , we have $2\pi\rho d > C_1 + 1$, so for any $\theta \in \mathbb{T}$

$$(1.117) \quad |R_n(\theta)| \leq \frac{2}{1 - e^{-2\pi\rho}} e^{-n} < 1, \quad n > n_0(\rho)$$

Now we assume that the Lyapunov exponent $L(E)$ of $A(\theta, E)$ is positive. Denote

$$\begin{aligned}\Theta_n^1 &= \{\theta : f_n(\theta) > e^{nL(E)/8}\} \\ \Theta_n^2 &= \{\theta : g_n(\theta) > e^{nL(E)/4}\} \\ \Theta_n^3 &= \{\theta : f_n(\theta) > e^{nL(E)/2}\}\end{aligned}$$

According to (1.117), we see that if $f_n(\theta) > e^{nL(E)/2}$, then

$$g_n(\theta) > f_n(\theta) - |R_n(\theta)| > e^{nL(E)/2} - 1 > e^{nL(E)/4}, \quad n > n(E)$$

and if $g_n(\theta) > e^{nL(E)/4}$, then

$$f_n(\theta) > g_n(\theta) - |R_n(\theta)| > e^{nL(E)/8}, \quad n > n(E)$$

Therefore, we have for large n ,

$$(1.118) \quad \Theta_n^3 \subseteq \Theta_n^2 \subseteq \Theta_n^1$$

On the other hand,

$$\begin{aligned}2nL(E) &\leq 2 \int_{\mathbb{T}} \log \|A_n(\theta)\|_{HS} d\theta = \int_{\mathbb{T}} \log f_n(\theta) d\theta \\ &\leq \text{Leb}(\Theta_n^3) \log \|f_n\|_{\rho} + (1 - \text{Leb}(\Theta_n^3)) \log e^{nL(E)/2} \\ &\leq \text{Leb}(\Theta_n^3) \cdot C_1 n + (1 - \text{Leb}(\Theta_n^3)) \cdot nL(E)/2\end{aligned}$$

which implies $\text{Leb}(\Theta_n^3) \geq \frac{3L(E)}{2C_1 - L(E)}$. Since $L(E) \geq a > 0, E \in S$, we have

$$(1.119) \quad \text{Leb}(\Theta_n^3) \geq \frac{3a}{2C_1 - a} := c_2(a, \rho) > 0$$

Thus

$$(1.120) \quad \text{Leb}(\Theta_n^2) \geq c_2(a, \rho) > 0$$

Since $g_n(\theta)$ is a trigonometric polynomial of degree $2dn$, the set Θ_n^2 consists of no more than $4dn$ intervals. Therefore, there exists a segment, $\Delta_n \subset \Theta_n^2$, with $\text{Leb}(\Delta_n) > \frac{c_2}{4dn}$. Obviously, Δ_n is also contained in Θ_n^1 , i.e., for any $\theta \in \Delta_n$,

$$\|A_n(\theta)\|_{HS}^2 > e^{nL(E)/8}$$

and

$$\text{Leb}(\Delta_n) > \frac{c_2}{4dn}, \quad n > n(E, \rho)$$

where d only depends on ρ and is independent of n . ■

The following standard lemma is proved e.g. in [52]

Lemma 1.3.4 (Lemma 9, [52]) *Let $\Delta \subset [0, 1]$ be an arbitrary segment. If $|\Delta| > \frac{1}{q_n}$. Then, for any θ ; there exists a j in $\{0, 1, \dots, q_n + q_{n-1} - 1\}$ such that $\theta + j\alpha \in \Delta$.*

Proof of Lemma 1.3.3: The case $N = 0$ is already covered by Lemma 1.3.4. The proof for the case $N > 0$ follows the same strategy. Notice that (1.109) implies $|\Delta_{k_n}| > \frac{c_2}{4dk_n} > \frac{1}{q_n}$ for large q_n . Applying Lemma 1.3.4 to $\theta + 2Nq_n$, we have that there exists a j in $\{0, 1, \dots, q_n + q_{n-1} - 1\}$ such that $\theta + 2Nq_n\alpha + j\alpha \in \Delta_{k_n}$, i.e.,

$$\|A_{k_n}(\theta + 2Nq_n\alpha + j\alpha)\| > e^{k_n L(E)/32}$$

Since

$$A_{2Nq_n+j+k_n}(\theta) = A_{k_n}(\theta + 2Nq_n\alpha + j\alpha)A_{2Nq_n+j}(\theta)$$

and A_i is unimodular, we have that either

$$\|A_{2Nq_n+j}(\theta)\| \geq e^{k_n L(E)/32} \quad \text{or} \quad \|A_{2Nq_n+j+k_n}(\theta)\| \geq e^{k_n L(E)/32}$$

Let j_N be $2Nq_n + j$ or $2Nq_n + j + k_n$, so that j_N satisfies (1.110). Clearly,

$$2Nq_n \leq 2Nq_n + j < 2Nq_n + j + k_n < 2Nq_n + 2q_n$$

Therefore, $j_N \in [2Nq_n, 2(N+1)q_n)$. ■

1.4 Sturmian Hamiltonian

Liu, Qu and Wen [70, 69] studied the Hausdorff and upper box counting dimension of $\Sigma_{\lambda, \alpha}$ with general irrational frequencies. For any irrational $\alpha \in (0, 1)$ with continued fraction expansion $[0; a_1, a_2, \dots]$, define

$$(1.121) \quad K_*(\alpha) = \liminf_{k \rightarrow \infty} \left(\prod_{i=1}^k a_i \right)^{1/k} \quad \text{and} \quad K^*(\alpha) = \limsup_{k \rightarrow \infty} \left(\prod_{i=1}^k a_i \right)^{1/k}.$$

Then (Theorem 1 [70], Theorem 1.1 [69]) for large coupling constant λ , $\dim_H \Sigma_{\alpha, \lambda} = 1$ iff $K_*(\alpha) = \infty$ and $\overline{\dim}_B \Sigma_{\alpha, \lambda} = 1$ iff $K^*(\alpha) = \infty$.

The usual way to study Sturmian Hamiltonian is to decompose Sturmian potentials into canonical words, which obey recursive relations. Here we present an alternative approach to study spectral dimension properties of Sturmian Hamiltonian based on the technics we developed in Theorem 1.1.6.

We will first prove Theorem 1.1.8. Set

$$(1.122) \quad V_\theta(n) = \lambda \chi_{[1-\alpha, 1)}(\theta + n\alpha \bmod 1)$$

It is well known that for Sturmian H_θ , the restriction of Lyapunov exponent on the spectrum is zero (see Theorem 1, [32]). By the discussion after (1.18) (see [57]) or else, specifically for Sturmian potentials, by [67], for arbitrarily small $\Lambda > 0$ and $n \geq n_0(\Lambda)$, $\|A_n(\theta, E)\| \leq e^{\Lambda n}$ uniformly in θ and $E \in \sigma(H_\theta)$. Here we will apply Corollary 1.1.1 directly.

Let q_k be the subsequence of denominators of the continued fraction approximants of α such that $\|q_k \alpha\| < e^{-\beta q_k/2}$. In order to apply Corollary 1.1.1, it is enough to verify that $V_\theta(n)$ given by (1.122) is $\beta(\alpha)$ -almost periodic for α -Diophantine $\theta \in \mathbb{T}$. Fix $\tau > 1$. If θ is α -Diophantine there is $\gamma > 0$ such that $\|\theta + m\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{\gamma}{(|m|+1)^\tau}$ for any $m \in \mathbb{Z}$. Then for $|m| \leq q$

$$\text{dist}(\theta + m\alpha, \{\mathbb{Z}, 1 - \alpha + \mathbb{Z}\}) \geq \min_{|m| \leq q+1} \|\theta + m\alpha\|_{\mathbb{R}/\mathbb{Z}}.$$

Therefore,

$$\min_{|m| \leq q} \text{dist}(\theta + m\alpha, \{\mathbb{Z}, 1 - \alpha + \mathbb{Z}\}) \geq \min_{|m| \leq q+1} \frac{\gamma}{(|m|+1)^\tau} \geq \frac{\gamma}{(q+2)^\tau}.$$

Let $N = \lceil e^{\beta q/4} \rceil$. Then for $|j| \leq N$, $q > q_0(\gamma, \beta)$ and any $|m| \leq q$, we have

$$\|jq\alpha\| \leq |j| \cdot \|q\alpha\| \leq e^{-\beta q/4} \leq \frac{\gamma}{10(q+2)^\tau} \leq \frac{1}{10} \text{dist}(\theta + m\alpha, \{\mathbb{Z}, 1 - \alpha + \mathbb{Z}\})$$

Therefore, for any $|m| \leq q$ and $|j| \leq N$, $\theta + m\alpha \bmod 1$ and $\theta + m\alpha + jq\alpha \bmod 1$ belong to one of the same open intervals $\{(0, 1 - \alpha), (1 - \alpha, 1)\}$, which implies that

$$\chi_{[1-\alpha, 1)}(\theta + m\alpha \bmod 1) = \chi_{[1-\alpha, 1)}(\theta + m\alpha + jq\alpha \bmod 1), \quad |m| \leq q, |j| \leq N$$

Therefore, for $0 \leq m \leq q$,

$$V_\theta(m) = V_\theta(m + q) = \cdots = V_\theta(m + Nq)$$

which immediately implies $\beta(\alpha)$ -almost periodicity for the sequence q_k with $\epsilon = 1/4$.

Since the set of α -Diophantine θ has full Lebesgue measure, the conclusion for the density of states follows directly from $dN = \mathbb{E}(d\mu_\theta)$.

Next we will construct α to prove Corollary 1.1.5. We will define inductively the continued fraction coefficients $a_n, n \geq 1$, so $\alpha = [a_1, \cdots, a_n, \cdots]$. Fix $\beta > 0$. Start with some n_0 large. For $1 \leq i \leq n_0$, set $a_i = 1$. Set $[a_1, \cdots, a_n] = p_n/q_n$. Now, for $k = 1, \cdots$ define $n_k = q_{n_0} + q_{n_1} + \cdots + q_{n_{k-1}}$ and $a_n = \begin{cases} e^{\beta q_{n_k}}; & n = n_k + 1, \\ 1; & n_k + 2 \leq n \leq n_{k+1}, \end{cases}$ for $k = 0, 1, \cdots$

Set $\alpha = [a_1, \cdots, a_n, \cdots]$. It is easy to check that

•

$$\begin{aligned} \beta + \frac{\log q_{n_k}}{q_{n_k}} &= \frac{\log a_{n_k+1} q_{n_k}}{q_{n_k}} < \frac{\log q_{n_k+1}}{q_{n_k}} < \beta + \frac{\log 2q_{n_k}}{q_{n_k}} \\ &\implies \frac{\log q_{n_k+1}}{q_{n_k}} \rightarrow \beta \end{aligned}$$

•

$$\begin{aligned} (a_1 a_2 \cdots a_{n_k})^{1/n_k} &= (a_{n_0+1} a_{n_1+1} \cdots a_{n_{k-1}+1})^{1/n_k} \\ &= (e^{\beta q_{n_0}} e^{\beta q_{n_1}} \cdots e^{\beta q_{n_{k-1}}})^{1/(q_{n_0} + q_{n_1} + \cdots + q_{n_{k-1}})} \\ &= e^\beta < \infty \end{aligned}$$

Therefore, α constructed in the above way satisfies $\beta(\alpha) > 0$ while $K_*(\alpha) < \infty$. Then Corollary 1.1.5 follows from [70] and Theorem 1.1.8. ■

On the other hand, if we take $\alpha = [0; 1, 2, 3, \cdots, k, \cdots]$, then $K_*(\alpha) = \infty$ while $\beta(\alpha) = 0$. By [70], for Sturmian Hamiltonian with frequencies α such that $K_*(\alpha) = \infty$, $\dim_H \Sigma_{\alpha, \lambda} = \dim_P \Sigma_{\alpha, \lambda} = 1$.

Chapter 2

Mixed spectral types for the one frequency discrete quasi-periodic Schrödinger operator

2.1 Introduction

In [18], Bjerklöv considers the following discrete quasi-periodic Schrödinger operator on $l^2(\mathbb{Z})$

$$(2.1) \quad (H_{K,\theta,\omega}u)_n = -u_{n+1} - u_{n-1} + V(\theta + n\omega)u_n, \quad n \in \mathbb{Z}$$

where

$$(2.2) \quad V(\theta) = \exp(Kf(\theta + \omega)) + \exp(-Kf(\theta))$$

$\theta, \omega \in \mathbb{T}^b$, $f : \mathbb{T}^b \rightarrow \mathbb{R}$, is assumed to be a non-constant real-analytic function with zero mean, $\int_{\mathbb{T}^b} f(\theta) d\theta = 0$ and $K \in \mathbb{R}$ is any constant. Consider the Lyapunov exponent $L(E)$ defined as in (1.38). In this explicit example, Bjerklöv shows that for large K we have a situation with mixed dynamics: zero Lyapunov exponent in a region close to $E = 0$ and positive for larger E .

In this chapter, we are going to show that for one frequency case, in Bjerklöv's example (2.1), mixed dynamics actually lead to mixed spectra: for some parameters

(θ, ω) , $H_{K,\theta,\omega}$ has mixed absolutely continuous and point spectrum, and for some other (θ, ω) , $H_{K,\theta,\omega}$ has mixed absolutely continuous and singular continuous spectrum.

Without loss of generality, we assume that $\|f\|_{C^1(\mathbb{T})} = 1$ and f has analytic extension to the strip $|Imz| < h$, where $h \gg K$ (e.g., f can be taken as any entire function). It follows from [18] that $\min\{E \in \sigma(H_{K,\theta,\omega})\} = 0$ for any K, θ, ω . And also it is not hard to show that $\max\{E \in \sigma(H_{K,\theta,\omega})\} \asymp e^{K\|f\|_\infty}$. For any $\epsilon > 0$ small (w.l.g. we assume $0 < \epsilon < 1$), denote $I_{\epsilon,K} = [\epsilon, 4e^{K\|f\|_\infty}]$. We have $I_{\epsilon,K} \cap \sigma(H_{K,\theta,\omega}) \neq \emptyset$. We say the frequency $\omega \in \mathbb{T}$ satisfies the Diophantine Condition (denote by $\omega \in DC(\kappa, \tau)$) if

$$\|\omega \cdot n\| \geq \frac{\kappa}{|n|^\tau}, \quad \forall n \in \mathbb{Z} \setminus \{0\}$$

for some $\kappa > 0, \tau > 0$. And denote $DC = \bigcup_\kappa DC(\kappa, \tau)$ ¹ for some fixed $\tau > 1$. It is well known that DC has full Lebesgue measure in any box.

The main results are as follows.

Theorem 2.1.1 *Let V be given as in (2.2). Fix $\omega_0 \in DC(\kappa, \tau)$. For any $\epsilon > 0$, there are $K = K(\omega_0, \epsilon, f) > 0$, $\delta = \delta(\omega_0, \epsilon, K) > 0$, and for any $\omega \in B_\delta(\omega_0) := \{\omega \in \mathbb{T} : |\omega - \omega_0| < \delta\}$, there is $0 < \epsilon_0 = \epsilon_0(\omega, K, h, \|f\|_h) < \epsilon$ such that*

- (a) *for a.e. $\omega \in B_\delta(\omega_0)$ and a.e. $\theta \in \mathbb{T}$, $H_{K,\theta,\omega}$ has pure point spectrum in $I_{\epsilon,K}$ with exponentially decaying eigenvectors and has purely absolutely continuous spectrum in $[0, \epsilon_0]$.*
- (b) *for a.e. $\omega \in B_\delta(\omega_0)$, there is a dense G_δ set of θ , such that $H_{K,\theta,\omega}$ has purely singular continuous spectrum in $I_{\epsilon,K}$ and has purely absolutely continuous spectrum in $[0, \epsilon_0]$.*
- (c) *for ω in a dense subset of $B_\delta(\omega_0)$ and for any θ , $H_{K,\theta,\omega}$ has purely singular continuous spectrum in $I_{\epsilon,K}$ and has purely absolutely continuous spectrum in*

¹Here $\|\cdot\|$ means the distance to the closest integer. If

$$\|\omega \cdot n\| \geq \frac{\kappa}{|n|(\log(|n| + 1))^2}, \quad \forall n \in \mathbb{Z} \setminus \{0\}$$

for some $\kappa > 0$, we say ω satisfies the Strong Diophantine Condition (denote by $\omega \in SDC(\kappa)$). Denote $SDC = \bigcup_\kappa SDC(\kappa)$ which also has full Lebesgue measure.

$[0, \epsilon_0]$.

Previously, Bourgain [15] constructed quasi-periodic operator with two frequencies which has coexistence of absolutely continuous and point spectrum. While mixed spectra are expected to occur for generic one-frequency operators, such examples for the discrete case have been considered difficult to construct explicitly. Recently Bjerklöv and Krikorian [19] announced an example of this nature. For continuous model, Fedotov and Klopp [39] showed coexistence of absolutely continuous and singular spectrum for a family of quasi-periodic operators and also gave a criterion for the existence of absolutely continuous and singular spectrum in the semi-classical regime.

Here we give a short proof which shows that operator (2.1) with potential (2.2) has mixed spectral types. The mixed nature of spectrum follows from coexistence of positive Lyapunov exponent and zero Lyapunov exponent which was obtained in [18] and a combination of several recent results on localization, reducibility and continuity of Lyapunov exponent..

2.2 Singular spectrum in the positive Lyapunov exponent region

The transfer matrix $A_n(\theta, E)$ and the Lyapunov exponent $L(E)$ are defined as usual in (1.35),(1.38).

In the following, we would like to fix f and consider Lyapunov exponent $L(E, \omega, K)$ as function of energy E , frequency ω , and parameter K . In [18], Bjerklöv proved that:

Theorem 2.2.1 ([18]) *Assume that V is as in (2.2), and that $\omega \in DC(\kappa, \tau)$. Then for any $\epsilon > 0$ there is a $K_0 = K_0(\epsilon, f, \kappa, \tau) > 0$ and $c = c(f) > 0$ such that for all $K > K_0$, we have*

$$L(E, \omega, K) \geq cK, \quad \text{for all } E \notin [0, \epsilon].$$

The proof is based on Large Deviation Theorem (LDT)-Avalanche Principle(AvP) scheme developed by Bourgain, Goldstein, Schlag [16, 45]. Due to some technical

reasons, the largeness of K depends on the Diophantine Conditions of ω in this theorem, which means the positivity is not uniform for all ω . However, we can get the following local non-perturbative positivity. Bourgain and Jitomirskaya showed that Lyapunov exponent is jointly continuous in (ω, E) at any irrational frequency (Theorem 1, [17]). The following result is obvious:

Proposition 2.2.1 *Fix any $\epsilon > 0$ and $\omega_0 \in DC(\kappa, \tau)$, let $K_0 = K_0(\epsilon, f, \omega_0) > 0$ be given as in Theorem 2.2.1, then for any $K > K_0$, there is $\delta = \delta(\omega_0, \epsilon, K) > 0$, such that for any $\omega \in B_\delta(\omega_0)$, $L(E, \omega, K) > 0$ on $I_{\epsilon, K}$, where the lower bound only depends on ω_0, ϵ, K, f and is uniform in E and ω .*

The absence of a.c. spectrum on $I_{\epsilon, K}$ is therefore obvious due to Kotani theory. What we want to claim is the pure point spectrum or purely singular continuous spectrum in this region.

Anderson Localization (part (a)) Let $\Omega = SDC \cap B_\delta(\omega_0)$, which is a full measure subset of $B_\delta(\omega_0)$. Notice that the positivity of $L(E, \omega, K)$ is uniform for $E \in I_{\epsilon, K}$ and $\omega \in \Omega$. Then according to the non-perturbative localization result in [16], for any $\theta \in \mathbb{T}$, a.e. $\omega \in \Omega$, $H_{K, \theta, \omega}$ exhibits A.L. in $I_{\epsilon, K}$. Thus by Fubini's theorem, $H_{K, f, \theta, \omega}$ has A.L. for a.e. $\omega \in \Omega$ and a.e. $\theta \in \mathbb{T}$.

Purely s.c. spectrum (part (b)) Let Ω be the same as in the previous part. Goldstein and Schlag [47] show that for a.e. $\omega \in \Omega$, the intersection $\sigma(H_{K, f, \theta, \omega}) \cap I_{\epsilon, K}$ is a Cantor set. Then according to a theorem of Gordon [41], nowhere dense structure of the spectrum implies the absence of point spectrum for a dense G_δ set of θ (see Theorem 6 in [41]). Therefore, for a.e. $\omega \in \Omega$, there is a dense G_δ set of θ such that $H_{K, \theta, \omega}$ has purely singular continuous spectrum in $I_{\epsilon, K}$.

Purely s.c. spectrum (part (c)) Absence of point spectrum in this part is based on rational approximation. More precisely, denote by

$$\beta(\omega) := \limsup_n \frac{\log q_{n+1}}{q_n}$$

where $\frac{p_n}{q_n}$ is the n^{th} rational approximation of ω . Notice that

$$\sup_n \sup_{(\theta, \omega) \in \mathbb{T}^2, E \in I_{\epsilon, K}} \frac{1}{|n|} \log \|A^n(\theta, E)\| \leq 10K.$$

Then by standard Gordon type argument (see e.g. [40, 24]), if $\beta(\omega) > 40K$, then for any θ , $H_{K, \theta, \omega}$ does not have any point spectrum. Combine with the positivity of Lyapunov exponent in $I_{\epsilon, K}$, the proof for purely s.c. spectrum of part (c) is completed. Notice that for any $\beta_0 \in [0, \infty]$, the level set $\Omega_{\beta_0} := \{\omega : \beta(\omega) = \beta_0\}$ is a dense set. For later purpose, we would like to pick the dense subset $\Omega_{\beta_0} \cap B_\delta(\omega_0)$ with $40K < \beta_0 < h/2$.

2.3 Absolutely continuous spectrum near the bottom

Next we are going to show that for any $\omega \in \mathbb{T}$ with finite $\beta(\omega)$, if the energy E is sufficiently small (depends on ω), then the Schrödinger cocycle is almost reducible. This will imply purely a.c. spectrum near the bottom of the spectrum for any phase. To complete the proof of the main theorem, we first pick some ω near ω_0 and some θ which give us point spectrum or singular continuous spectrum as in the previous part. Then for these pairs of (ω, θ) , we apply the almost reducibility result to get the coexistence of two types of spectrum.

The key step to find purely a.c. spectrum near the bottom is the following reducibility result at $E = 0$, which generalizes Lemma 5.1 in [18] to the case $0 < \beta(\omega) < \infty$.

Proposition 2.3.1 *For any frequency ω with $\beta(\omega) < \infty$, if $h > 2\beta$, then there exists analytic transformation $C : \mathbb{T} \rightarrow SL(2, \mathbb{R})$ such that*

$$C(\theta + \omega)A(\theta, 0)C(\theta)^{-1} = A_0,$$

where

$$A_0 = \begin{pmatrix} 1 & \hat{k} \\ 0 & 1 \end{pmatrix}, \quad \hat{k} \in \mathbb{R}$$

Remark 4 *If ω is Diophantine or $\beta = 0$, the proposition has been proved in Lemma 5.1 [18]. If $\beta > 0$, we can still find such a transformation C provided h is large. The only loss is the decrease of the width of the analytic strip. Also the analytic norm of the transformation and the constant could be very large. Actually, C has an analytic extension to the strip $|Imz| < h - 2\beta$, with $\|C\|_{h-2\beta} \sim e^{K\|f\|_h}$. We also have $|\hat{k}| \sim e^{K\|f\|_h}$*

Proof: Recall the main steps in the proof of Lemma 5.1 [18], if there are $g, h : \mathbb{T} \rightarrow \mathbb{R}$ satisfying the following equations

$$(2.3) \quad \begin{aligned} g(\theta + \omega) - g(\theta) &= f(\theta + \omega) \\ k(\theta) &= -e^{-Kg(\theta-\omega)-Kg(\theta)} \quad , \quad \hat{k} = \int_{\mathbb{T}} k(\theta) d\theta \end{aligned}$$

$$(2.4) \quad h(\theta + \omega) - h(\theta) = \hat{k} - k(\theta)$$

then set

$$C(\theta) = \begin{pmatrix} 1 & h(\theta) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & \exp(-Kg(\theta-\omega)) \\ -\exp(Kg(\theta-\omega)) & \exp(Kg(\theta)) \end{pmatrix} \in SL(2, \mathbb{R})$$

Direct computation shows that

$$C(\theta + \omega)A(\theta, 0)C(\theta)^{-1} = \begin{pmatrix} 1 & \hat{k} \\ 0 & 1 \end{pmatrix}$$

which is the desired form.

For real analytic f with zero average, if ω is Diophantine, equations (2.3),(2.4) always have real analytic solutions g, h , which is the case in [18].

If $\beta > 0$, recall for f analytic in the strip $|Imz| < h$, the Fourier coefficients of f satisfy $|\hat{f}_k| \leq \|f\|_h e^{-h|k|}$, therefore, from Fourier series expansion, equation (2.3) has an analytic solution g in the strip $|Imz| < h - \beta$ provided $h > \beta$. From the definition of k , $k(\theta)$ also has analytic extension to the strip $|Imz| < h - \beta$ with $\|k\|_{h-\beta} \sim e^{K\|g\|_{h-\beta}} \sim e^{K\|f\|_h}$. Then for the same reason, equation (2.4) also has an analytic solution h in the strip $|Imz| < h - 2\beta$ provided $h - \beta > \beta$. ■

Then it is easy to see that by applying C to $A(\theta, E)$, we have

$$\begin{aligned} C(\theta + \omega)A(\theta, E)C(\theta)^{-1} &= A_0 + C(\theta + \omega) \begin{pmatrix} -E & 0 \\ 0 & 0 \end{pmatrix} C(\theta)^{-1} \\ (2.5) \qquad \qquad \qquad &= A_0 + EF(\theta) \end{aligned}$$

$$(2.6) \qquad \qquad \qquad := G(\theta, E) \in SL(2, \mathbb{R})$$

where

$$F(\theta) = C(\theta + \omega) \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} C(\theta)^{-1}.$$

From the proof of Proposition 2.3.1, we see that F has analytic extension to the strip $|Imz| < h - 2\beta$ and the largeness of $\|A_0\|, \|F\|_{h-2\beta}$ depend on $\omega, K, \|f\|_h$.

For Diophantine frequency, as Bjerklöv mentioned in Remark 2 in [18], one can show purely absolutely continuous spectrum for sufficient small E based on KAM approach as in [35]. Here since we also need to deal with Liouvillean frequency, we want to prove all cases together with the almost reducibility concept.

We say the skew product system (ω, A) is *almost reducible* if there exist $\eta > 0$ and a sequence of analytic maps $B^{(n)} : \mathbb{T} \rightarrow PSL(2; \mathbb{R})$, admitting holomorphic extensions to the common strip $|Imz| < \eta$ such that $B^{(n)}(z + \omega)A(z)B^{(n)}(z)^{-1}$ converges to a constant uniformly in $|Imz| < \eta$. We need the following result about almost reducibility:

Proposition 2.3.2 (Corollary 1.2, [2]) *Any one-frequency analytic quasi-periodic $SL(2, \mathbb{R})$ cocycle close to constant is analytically almost reducible.*

Proof of purely a.c. spectrum near the bottom. According to Proposition 2.3.2, $(\omega, A_0 + EF(\theta))$ is almost reducible for small E . More precisely, consider G in (2.6) with form (2.5). There exists $\epsilon_0 = \epsilon_0(\omega, \|A_0\|, h, \|F\|_{h-2\beta}) < \epsilon$ such that for $0 < E < \epsilon_0$, $(\omega, G(\theta, E))$ is almost reducible. (Such a quantitative version can be found in [49, 77].) Therefore, $(\omega, A(\theta, E))$ is also almost reducible for $0 < E < \epsilon_0$. As

a corollary of almost reducibility [2, 3], we have that for any θ , $H_{K,f,\theta,\omega}$ has purely absolutely continuous spectrum in $[0, \epsilon_0]$. ■

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APPENDICES

Appendix A

Proof of Lemma 1.2.7:

Suppose that u, v are the two normalized eigenvectors of G such that

$$Gu = \rho u, \quad Gv = \rho^{-1}v, \quad \|u\| = \|v\| = 1$$

Denote the angle between u and v by θ . Without loss of generality we assume further that $|\theta| < \pi/2$. Set $\tilde{B} = (u, v)$, $B = \frac{\tilde{B}}{\sqrt{|\det \tilde{B}|}}$. Obviously, $\|\tilde{B}\| \leq 1$, $|\det B| = 1$, and $\det \tilde{B} = \|u\| \cdot \|v\| \cdot \sin \theta$. Therefore,

$$\|B\| \leq \frac{1}{\sqrt{|\sin \theta|}}$$

On the other hand, $G(u - v) = \rho u - \rho^{-1}v$, which implies that

$$\rho - \rho^{-1} = \rho\|u\| - \rho^{-1}\|v\| \leq \|\rho u - \rho^{-1}v\| = \|G(u - v)\| \leq \|G\| \cdot \|u - v\|$$

By the law of cosines, $\|u - v\| = 2 \sin \frac{\theta}{2}$. Then

$$2 \sin \frac{\theta}{2} \geq \frac{\rho - \rho^{-1}}{\|G\|} = \frac{\sqrt{(|\text{Trace}G| + 2)(|\text{Trace}G| - 2)}}{\|G\|}$$

$|\text{Trace}G| \leq 6$ implies that $|\text{Trace}G| + 2 \geq 2(|\text{Trace}G| - 2)$, then $2 \sin \frac{\theta}{2} \geq \frac{\sqrt{2}(|\text{Trace}G| - 2)}{\|G\|}$.

Therefore,

$$\sin \theta \geq 2 \sin \frac{\theta}{2} \cdot \frac{1}{\sqrt{2}} \geq \frac{|\text{Trace}G| - 2}{\|G\|}$$

and

$$\|B\| \leq \frac{\sqrt{\|G\|}}{\sqrt{|\text{Trace}G| - 2}}$$

It is also easy to see that if $|\text{Trace}G| > 6$, $\|B\| \leq \frac{2\sqrt{\|G\|}}{\sqrt{|\text{Trace}G| - 2}}$. ■

Appendix B

Proof of Lemma 1.2.9 and Lemma 1.2.10

Proof of Lemma 1.2.9: Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$ has eigenvalues ρ and ρ^{-1} .

Case I: $\text{Trace} A \neq 2$. Obviously, $\rho \neq 1$ and

$$(B.1) \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = B \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} B^{-1}$$

where B is the conjugation matrix. Suppose $\rho \neq d$. We can pick the conjugation matrix as

$$(B.2) \quad B = \begin{pmatrix} 1 & \frac{b}{\rho^{-1}-a} \\ \frac{c}{\rho-d} & 1 \end{pmatrix}, \quad B^{-1} = \frac{\rho-d}{\rho-\rho^{-1}} \begin{pmatrix} 1 & -\frac{b}{\rho^{-1}-a} \\ -\frac{c}{\rho-d} & 1 \end{pmatrix}.$$

If $\rho = d$, it is easy to see that $bc = 0$. Without loss of generality, we assume $c = 0, b \neq 0$. We can pick the conjugation matrix as

$$(B.3) \quad B = \begin{pmatrix} 1 & 1 \\ \frac{d-d^{-1}}{b} & 0 \end{pmatrix}, \quad B^{-1} = \frac{b}{d^{-1}-d} \begin{pmatrix} 0 & -1 \\ -\frac{d-d^{-1}}{b} & 1 \end{pmatrix}.$$

Direct computation using (B.1),(B.2),(B.3) shows that for any $k \in \mathbb{N}$,

$$(B.4) \quad A^k = \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} \cdot \left(A - \frac{a+d}{2} \cdot I \right) + \frac{\rho^k + \rho^{-k}}{2} \cdot I$$

Case II: $\text{Trace} A = 2$. Also follows by a (simpler) direct computation, considering separately $a = 1$ and $a \neq 1$. ■

Proof of Lemma 1.2.10: Now assume $E \in S_q$ and $1 \leq k \leq N_q$. Apply (B.4) to $A_q(E)$. First, suppose $2 < \text{Trace}A_q(E) < 2 + e^{-\tau q}$. Then

$$1 < \rho = \frac{\text{Trace}A_q(E) + \sqrt{(\text{Trace}A_q(E))^2 - 4}}{2} < \frac{2 + e^{-\tau q} + \sqrt{(2 + e^{-\tau q})^2 - 4}}{2} < 1 + e^{(-\tau/2 + \Lambda/200)q}$$

There is a universal constant C , such that for any $1 \leq k \leq N_q < e^{(\tau/2 - \Lambda/200)q}$,

$$1 < \rho^k < (1 + e^{(-\tau/2 + \Lambda/200)q})^{N_q} < C.$$

Therefore,

$$(B.5) \quad 1 < \frac{\rho^k + \rho^{-k}}{2} < C.$$

On the other hand,

$$\frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} = \sum_{i=1}^k \rho^{k-2i+1}$$

therefore,

$$(B.6) \quad k \leq \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} < C_1 k$$

Now assume $2 - e^{-\tau q} < \text{Trace}A_q(E) < 2$. Then $\rho = e^{i\psi}$ and (B.4) can be expressed as

$$(B.7) \quad A_q^k = \frac{\sin k\psi}{\sin \psi} \cdot \left(A_q - \frac{\text{Trace}A_q}{2} \cdot I \right) + \frac{\cos k\psi}{2} \cdot I$$

We have $1 - \frac{1}{2}e^{-\tau q} < \cos \psi < 1$. Then $|\sin \psi| < e^{-\tau q/2}$, and $|\psi| < \frac{\pi}{2} |\sin \psi| < 2e^{-\tau q/2}$.

As in the hyperbolic case, we set $N_q = \lceil e^{(\tau/2 - \Lambda/200)q} \rceil$. For $k \leq N_q$,

$$|k\psi| < 2e^{-\Lambda q/200}$$

Then for q large enough, we have $\frac{2}{\pi}|k\psi| \leq |\sin k\psi| \leq |k\psi| \leq \sqrt{3}/2$. Therefore $\frac{2}{\pi}k \leq \left| \frac{\sin k\psi}{\sin \psi} \right| < \frac{\pi}{2}k$ and $1 \geq \cos k\psi > 1/2$.

Exactly the same argument works for the case $\{E : -2 < \text{Trace}A_q < -2 + e^{-\tau q}\}$ and $\{E : -2 - e^{-\tau q} < \text{Trace}A_q < -2\}$. ■

Appendix C

Some estimates on matrix products

Lemma C.0.1 *Suppose G is a two by two matrix satisfying*

$$(C.1) \quad \|G^j\| \leq M < \infty, \quad \text{for all } 0 < j \leq N,$$

where $M \geq 1$ only depends on N . Let $G_j = G + \Delta_j$, $j = 1, \dots, N$, be a sequence of two by two matrices with

$$(C.2) \quad \delta = \max_{1 \leq j \leq N} \|\Delta_j\|.$$

If

$$(C.3) \quad NM\delta < 1/2,$$

then for any $n \leq N$

$$(C.4) \quad \left\| \prod_{j=1}^n G_j - G^n \right\| \leq 2NM^2\delta$$

Proof of Lemma C.0.1: Denote by

$$D = \max_{1 \leq k_1, k_2 \leq N} \left\| \prod_{j=k_1}^{k_2} G_j \right\|.$$

Then a simple perturbation argument, as in e.g. [59], one can show that $D \leq M(\delta DN + 1)$. Thus $D \leq \frac{M}{1 - M\delta N}$. Direct computation shows that for any $1 \leq n \leq N$,

$$\prod_{j=1}^n G_j - G^n = \sum_{k=0}^{n-1} \left(\prod_{j=k+2}^n G_j \right) \Delta_{k+1} G^k$$

Therefore,

$$\left\| \prod_{j=1}^n G_j - G^n \right\| \leq ND\delta M \leq \frac{M^2\delta N}{1 - M\delta N}$$

Clearly, if $M\delta N < 1/2$, then $\left\| \prod_{j=1}^n G_j - G^n \right\| \leq 2NM^2\delta$. ■

Appendix D

Extended Schnol's Theorem (Lemma 1.2.6)

Let $y > 1/2$ and x_k be any sequence such that $|x_k| \rightarrow \infty$ as $k \rightarrow \infty$. For a Borel set $B \in \mathbb{R}$, denote

$$\mu_{n,m}(B) = \langle \delta_n, \chi_B(H) \delta_m \rangle$$

and

$$\rho(B) = \sum_n a_n (\mu_{n,n}(B) + \mu_{n+1,n+1}(B))$$

where

$$a_n = \begin{cases} c(1 + |k|)^{-2y}, & n = x_k \\ c(1 + |n|)^{-2y}, & \text{else} \end{cases}$$

with $c > 0$ chosen so that $\sum_n a_n = 1/2$.

Then, ρ is a Borel probability measure with $\rho(B) = 0$ if and only if $\mu(B) = 0$, i.e., ρ and μ are mutually absolutely continuous. By the Cauchy-Schwarz inequality,

$$|\mu_{n,m}(B)| \leq \mu_{n,n}(B)^{\frac{1}{2}} \mu_{m,m}(B)^{\frac{1}{2}}.$$

therefore $\mu_{n,m}$ is absolutely continuous with respect to ρ . By the Radon-Nikodym Theorem, there exists a measurable density

$$F_{n,m}(E) = \left[\frac{d\mu_{n,m}}{d\rho} \right](E), \quad \rho - a.e. E$$

with

$$\mu_{n,m}(B) = \int \chi_B(E) F_{n,m}(E) d\rho(E).$$

Then for every bounded measurable function f , we have that

$$\langle \delta_n, f(H)\delta_m \rangle = \int f(E) F_{n,m}(E) d\rho(E)$$

In particular, if g is compactly supported and bounded, we may set $f(E) = Eg(E)$ and have

$$\begin{aligned} & \int g(E) \left(EF_{n,m}(E) \right) d\rho(E) \\ &= \langle \delta_n, Hg(H)\delta_m \rangle \\ &= \langle \delta_{n+1} + \delta_{n-1} + V_n\delta_n, g(H)\delta_m \rangle \\ &= \int g(E) F_{n+1,m}(E) d\rho(E) + \int g(E) F_{n-1,m}(E) d\rho(E) + \int g(E) V_n F_{n,m}(E) d\rho(E) \\ &= \int g(E) \left[F_{n+1,m}(E) + F_{n-1,m}(E) + V_n F_{n,m}(E) \right] d\rho(E) \end{aligned}$$

For any fixed $m \in \mathbb{Z}$, let $u^E(n) = F_{n,m}(E)$. Thus we have for any g

$$\int g(E) \left((H - E)u^E \right)(n) d\rho(E) = 0$$

i.e., $\{u^E(n)\}_{n \in \mathbb{Z}}$ is a generalized eigenfunction of $Hu = Eu$ for ρ a.e. E .

On the other hand, let

$$B_n = \left\{ E : F_{n,n} \geq \frac{1}{a_n} \right\}$$

Then

$$\rho(B_n) = \sum_k a_k \mu_{k,k}(B_n) \geq a_n \mu_{n,n}(B_n) = a_n \int_{B_n} F_{n,n}(E) d\rho(E)$$

While

$$\int_{B_n} F_{n,n}(E) d\rho(E) \geq \frac{1}{a_n} \rho(B_n)$$

Therefore,

$$\int_{B_n} \left(a_n F_{n,n}(E) - 1 \right) d\rho(E) \leq 0$$

Therefore, $\rho(B_n) = 0$, i.e., for ρ -a.e. E , $F_{n,n}(E) \leq \frac{1}{a_n}$, thus

$$|F_{n,m}| \leq a_n^{-\frac{1}{2}} a_m^{-\frac{1}{2}}$$

Fix $m = 0$, and let $u^E(n) = F_{n,0}$, then according to the previous proof, $\rho - a.e.$ E , u^E is generalized eigenfunction of $Hu = Eu$ and obey the estimate

$$|u^E(n)| \leq a_0^{-\frac{1}{2}} a_n^{-\frac{1}{2}}$$

By the choice of a_n , we have

$$|u^E(x_k)| \leq (1 + |k|)^y.$$

■