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Publication Date
1990-05-01

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REPORT NO.
UCB/SEMM-90/06

STRUCTURAL ENGINEERING
MECHANICS AND MATERIALS

REDUCED VECTOR BASIS FOR DYNAMIC ANALYSIS OF LARGE DAMPED STRUCTURES

BY

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MAY 1990

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BERKELEY, CALIFORNIA

Reduced Vector Basis for Dynamic Analysis of Large Damped Structures

by

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Abstract

The reduced vector basis approach is used to unify the classical and modern methods to solve the dynamic responses of large damped linear structures. The classical eigenvector basis is re-examined. The generalized Ritz basis is used to provide a more direct and efficient solution to the forced vibration problem. The concept of load dependent and response dependent vector bases is developed. Physical interpretation, mathematical rigor and computational efficiency are emphasized in this reduced vector basis approach.

The solution technique is extended to a class of structures with positive semi-definite stiffness matrices. A self-equilibrating elastic force approach is used to extract basis vectors for structures with zero strain energy modes.

The steady state solution is used as the fundamental approach to establish the vector basis. Undamped basis vectors are derived based on the physics of the problem, numerical solution techniques and mathematical series expansion. Krylov subspace evolves as the logical vector subspace for the particular solution of the equation of motion. Static and harmonic correction vectors are included in the basis. The effect of coupled damping in the reduced vector space is corrected by iterative solution. By considering the first iteration, this approach provides a simple design procedure for passively damped structures.

Based on the same physical and mathematical principles, the damped vector basis is derived by using the in-phase and out-of phase displacement vectors formulation in lieu of the first order state space formulation. This solution subspace for the damped structure is equivalent to the Krylov subspace for the undamped structure.

Viscoelastic materials have frequency dependent properties. The frequency domain approach allows the solution of a class of viscoelastically damped structures to be analyzed in the reduced vector space. A comprehensive design procedure for both the viscously and viscoelastically damped structures can therefore be analyzed and designed efficiently.

A new dynamic substructure solution technique is formulated in the reduced vector basis. Load dependent and boundary motion vector bases define the vector subspace for the dynamic response of substructures. This enables the reduced vector basis technique to be effectively implemented for large linear structures with damping.

Approved by _____

E.L. Wilson, Chair

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NOMENCLATURE

Symbols

a,b,c	=	scalar coefficient
A	=	general coefficient matrix
	=	coefficient matrix of first order differential equation of motion
b,B	=	observation vector/vectors
B	=	coefficient matrix of first order differential equation of motion
C	=	material stress-strain constitutive tensor
c,**c**,C	=	viscous damping constant/matrix
D	=	energy dissipation per cycle
E	=	modulus of elasticity
$\underline{E}$	=	normalized frequency dependent modulus of elasticity
f	=	natural frequency, cycle per second (Hertz)
	=	function
F	=	force
g	=	forcing function in time domain
G	=	forcing function in frequency domain
i	=	index
	=	imaginary unit, $\sqrt{-1}$
I	=	identity matrix
j	=	index
k	=	index
k,**k**,**K**	=	stiffness constant/matrix
m	=	number of modes or vectors
m,**M**	=	mass constant/matrix

n	=	number of equations
	=	number of kinematic degrees of freedom
	=	rank of matrix when used with subscript, **M**, **C** and **K**
$p, \mathbf{p}, \mathbf{P}$	=	spatial load amplitude/vector/vectors
$\mathbf{q}, \mathbf{Q}$	=	modal coordinates in time and frequency domain
$\mathbf{r}$	=	residual error of the approximate solution to the differential equation
s	=	shift parameter of stiffness matrix
	=	cumulative sum of strain energy in elements
t	=	time, seconds
T	=	temperature
$T, \mathbf{T}$	=	transfer function/vector block
$u, \mathbf{u}, \mathbf{U}$	=	displacement , displacement vector in time and frequency domain
$\mathbf{v}, \mathbf{V}$	=	vector/vector block
w	=	strain energy
$\mathbf{w}$	=	vector consisting of in-phase and out-of-phase displacement components
x	=	variable
$\mathbf{x}, \mathbf{X}$	=	mass orthonormalized basis vector/vectors
$\mathbf{y}, \mathbf{Y}$	=	reduced coordinates in time and frequency domain
$\mathbf{z}$	=	state space coordinates vector
$\mathbf{Z}$	=	impedance matrix
$\dot{}$	=	$\frac{d}{dt}$
$\ddot{}$	=	$\frac{d^2}{dt^2}$
$\sim$	=	intermediate variable
$'$	=	load dependent solution of constrained substructure
$''$	=	boundary motion solution of substructure

α = scalar coefficient

β = scalar coefficient

= frequency ratio, loading frequency/natural frequency

ε = strain

= measure of error

= ratio of strain energy in selected group to complete structure

ξ = viscous damping ratio

ϕ = phase angle

ϕ = mode shape vector, eigenvector

Φ = matrix of eigenvectors

ψ = complex eigenvector in state space

Ψ = complex eigenvector block in state space

η = loss factor

$\mathfrak{n}$ = normalized frequency dependent loss factor

λ = eigenvalue or Rayleigh's quotient

Λ = matrix of eigenvalues

σ = stress

= general dynamic response

τ = modal participation factor

ζ = viscous modal damping

ω = loading frequency, radians/second

= natural frequency when used with subscript index

ϖ = intermediate frequency for fast Fourier transform computation

Subscripts

B	=	observation vector block
C	=	damping matrix
d	=	diagonal
	=	damped parameter
e	=	elastic
eq	=	equivalent
g	=	boundary degrees of freedom in global coordinate system
i,j,k,l	=	index
K	=	stiffness matrix
m	=	number of vectors
M	=	mass matrix
o	=	reference amplitude
r	=	reference point
	=	residual
	=	rigid body and/or mechanism modes
	=	relative coordinate system
s	=	shifted
	=	off diagonal
	=	statics
v	=	viscoelastic
X	=	the reduced subspace
w	=	number of elastic and viscoelastic elements
z	=	pertaining to zero strain energy
Φ	=	eigenvector space

ω = number of frequency points

= solution at frequency ω

Superscripts

b = boundary motion vector subspace

d = dynamic vector subspace

F = free-free boundary condition

I = imaginary part of a quantity

p = load dependent vector subspace

r = residual

R = real part of a quantity

T = matrix transpose

u = displacement

$\dot{u}$ = velocity

v = viscoelastic material

* = pertaining to quantity in the reduced vector space

Chapter 1

Introduction

The progress in computer hardware and software has tremendously enhanced the analytical ability of structural engineers. However, at the same time modern structures are becoming more complex and challenging to design. These structures are not only complex in geometric configurations but also the precision and performance requirements are becoming very stringent. As the structures are getting lighter, vibration control becomes very important. Vibration control can be achieved by passive structural design, active control systems or an optimal combination of both. Viscous damping by discrete dampers and hysteretic damping by viscoelastic materials are becoming a necessity in the next generation of structures in order to attain the required performance under dynamic loadings. The ability to achieve an optimum structural design is limited by the ability to understand the dynamics of the damped structure and the ability to analyze the complex structural system efficiently.

Conventional methods in structural dynamics, i.e. normal mode method and complex normal mode method, often demand excessive computing resources and are inefficient in solving these complex problems. In order to attain better design with a reasonable schedule and resources, new analytical techniques are needed. The purpose of this dissertation is to develop the mathematical basis for accurate and efficient analytical methods in order to understand the behavior of such complex structures and hence stride towards optimum designs of these systems. These methods are intended to be applicable to advanced civil, mechanical and aerospace structures, such as space structures, articulated structures, passively damped structures and actively controlled structures.

The most commonly used method in structural dynamics is the normal mode method which requires an eigensolution of the undamped system. A frequency cutoff is normally

specified based on the spectral content of the excitation. Basically, the eigen vector basis of the homogeneous differential equation is used to span the solution space. This is the most computationally intensive, time consuming and expensive phase of the whole solution process. While the eigensolution provides all the system modal characteristics, the information in the force vector is often ignored. However, the force vector provides some of the most important pieces of information describing the problem on hand. It determines the mesh size of the finite element model, the type of dynamic response analysis to be performed, the participation of each mode, and the residual high frequency spatial loading etc. The solution of the general eigenvalue problem is therefore often too general and provides superfluous information requiring excessive numerical operations with doubtful improvement on the accuracy of dynamic response.

Wilson et al[1] introduced the Ritz Vector Method for dynamic analysis of a class of problems with fixed spatial load distributions. Only modes with significant participation are represented in the Ritz vector basis. This solution subspace is small and provides an accurate approximation to the solution of the differential equation. In addition, the static correction vector is also represented in the solution space so that correction for modal truncation is provided. Wilson and Bayo[2] applied the Ritz Vector algorithm to soil-structure interaction problem and wave propagation problem with excellent results. Based on this algorithm, Arnold et al[3] reported a drastic improvement in CPU and throughput performance for dynamic analysis of large finite element models. Leger, Wilson and Clough[4] provided a thorough analysis of the Load Dependent Ritz Vector (LDRV) algorithm and introduced improved alternatives to the original method. The LDRV method is the foundation of this dissertation.

1.1 Scope

The scope of this study is to develop a sound mathematical basis for solution techniques in large complex structures with viscous and hysteretic damping. The structures are idealized by the linear elastic theory with small displacements and rotations. The governing differential equation of motion[5] in the time domain is:

$$\mathbf{M}\,\ddot{\mathbf{u}} + \mathbf{C}\,\dot{\mathbf{u}} + \mathbf{K}\,\mathbf{u} = \mathbf{p}\,g(t) \tag{1.1}$$

The dynamic behavior of the structure is described by this coupled matrix equation in n kinematic degrees of freedom which are represented by the displacement vector $\mathbf{u}$. $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, viscous damping and stiffness matrices of the structure. These structural characteristics are assumed to be time invariant. These matrices are derived from the finite element method and are symmetric. The ranks of the matrices are as follows:

$$\text{rank of } \mathbf{K} = n_K \leq n \tag{1.2a}$$

$$\text{rank of } \mathbf{M} = n_M \leq n_K \tag{1.2b}$$

$$\text{rank of } \mathbf{C} = n_C \leq n_K \tag{1.2c}$$

The stiffness matrix is positive semi-definite which can therefore represent a wide range of structures, e.g. grounded, articulated and space structures. The mass matrix is not required to be full rank so structures modelled by lumped translational masses can be solved directly without additional transformations. The damping matrix is assumed to be a general symmetric positive semi-definite matrix so that it represents both proportionally and non-proportionally damped structures.

Equation (1.1) can also be posed in the frequency domain as:

$$[-\omega^2\,\mathbf{M} + \mathbf{K}(\omega) + i\,\omega\,\mathbf{C}\,]\,\mathbf{U}(\omega) = \mathbf{p}\,G(\omega) \tag{1.3}$$

If hysteretic damping exists in the structure, it can be better represented in the frequency domain. The material properties of viscoelastic materials are typically frequency dependent and characterized in the frequency domain. The stiffness matrix becomes complex with the real and imaginary parts. With the inclusion of hysteretic damping, the governing equation in the frequency domain becomes:

$$\{[-\omega^2 \mathbf{M} + \mathbf{K}^R(\omega)] + i\,[\,\omega\,\mathbf{C} + \mathbf{K}^I(\omega)\,]\}\,\mathbf{U}(\omega) = \mathbf{p}\,G(\omega) \qquad (1.4)$$

The imaginary part of the stiffness matrix provides damping to the structural system. Depending on the structural materials used, the real part of the stiffness matrix can be frequency dependent. However, the viscous damping matrix is assumed to be frequency independent and not necessarily proportionally damped. Although the coefficient matrices are frequency dependent, the governing equation is in a linear algebraic form in the frequency domain. The behavior of the system governed by Equation (1.4) is considered to be linear.

The solution methods developed to solve Equations (1.1) and (1.4) provide dynamic responses to a class of problems with a variety of damping mechanisms:

1. conventional proportional viscous damping, which includes modal damping, Rayleigh damping and Caughey damping,
2. non-proportional damping, which includes structures constructed of a combination of materials with different damping characteristics, e.g. soil-concrete-steel, direct viscous dashpots and radiation damping of continuous media,
3. hysteretic damping of constant coefficient matrix (e.g. damping modelling of soil) and frequency dependent viscoelastic damping matrix (from frequency dependent complex modulus of viscoelastic materials).

1.2 Approach

The complexity in the geometry, materials, stress gradient, strain energy distribution and mass distribution often requires large finite element models[6] to represent the structures. Basically the goal of all the solution methods in structural dynamics is to reduce the degrees of freedom of the structure so that solution can be computed. The reduction can be a physical modelling process based on engineering judgement/intuition or consistent mathematical transformations. If simple convergence analysis or error estimate can be established, then the method is more reliable and robust.

When the original problem in n independent coordinates is reduced to m independent coordinates[7], i.e.

$$\mathbf{u} \approx \sum_{i=1}^{m} \mathbf{v}_i \, a_i \tag{1.5}$$

the solution is approximated and constrained in the *subspace* spanned by the coordinate vectors, $\mathbf{v}_i$, of the m reduced coordinates. Many, if not all, reduction techniques can be cast in this general form. This representation allows a better understanding of different methods in a consistent mathematical framework. In general, the Rayleigh-Ritz type formulation can be called the *subspace solution* to the Equations (1.1) and (1.3). Unlike the normal mode method, these basis vectors are not required to be orthogonal. They are more general and easy to compute. The challenge is to find the best subspace which can provide an accurate and efficient solution.

The Load Dependent Ritz Vector (LDRV) method provides an efficient method for dynamic analysis of a class of problems with fixed spatial load distributions. Unlike the normal mode method, one seeks only basis vectors to the particular solution of the differential equation of the undamped structure:

$$\mathbf{M}\,\ddot{\mathbf{u}} + \mathbf{K}\,\mathbf{u} = \mathbf{p}\,g(t) \tag{1.6}$$

are sought. Basis vectors with significant load participation are represented in the Ritz vector basis. This solution subspace so obtained is small and provides an accurate approximation to the solution of Equation (1.1) for structure with a positive definite stiffness matrix and modal damping. The static correction vector is also represented in the solution space so that correction for vector truncation is provided. An important feature is that a static strain energy convergence check is possible during the generation of basis vectors and a self termination criterion can be established. The mathematical basis of this method is reviewed in light of historical development of solution methods in structural dynamics. Sound mathematical and physical interpretations are developed such that a general development of this approach to solve more complex problems is possible. Methods to solve the dynamic response of structures with positive semi-definite stiffness matrix, viscous damping matrix and viscoelastically damping matrix are to be developed in a consistent mathematical form. Spatial and temporal convergence can also be established consistently.

Solution techniques are developed primarily through consideration in the frequency domain but the vector basis is applicable to both the time and frequency domain solution. Steady state solution can represent exact dynamic response by only a single vector which is the solution of a linear algebraic equation. The implication of this observation is that dynamic analysis can be accurately computed by solution method of static analysis. This concept is explored to develop dynamic vector basis for undamped and damped structures. The development in frequency domain allows this approach to be easily extended to frequency dependent viscoelastic damping and active structural control design (which is formulated primarily in the frequency domain).

Dynamic substructure method is a special form of the reduced vector space solution method. It provides a mathematically consistent way to reduce the dynamic degrees of freedom and also allows different substructure models and analyses to be performed by different individuals and organizations. The reduced vector basis solution method provides an efficient alternative dynamic substructure analysis of damped structures.

1.2.1 Transfer Function

In dynamic analysis, many different response quantities are often computed, e.g. stresses, member end forces, relative displacements etc. A generalized response quantity, σ, is related to the displacement vector by the following transformation:

$$\sigma = \mathbf{b}^T \mathbf{u} \tag{1.7}$$

where **b** is defined as the observation vector.

Therefore the corresponding modal response is given by:

$$\sigma_i = \mathbf{b}^T \phi_i \tag{1.8}$$

The characteristics of the response is best described by the transfer function relating the response to the excitation by superposition of modal responses as:

$$T_\sigma(\omega) = \sum_{i=1}^{m} \frac{\tau_i \, \sigma_i \, G(\omega)}{(1-\beta_i^2) + i \, 2\xi_i\omega_i} \tag{1.9}$$

$$\text{where} \quad \tau_i = \phi_i^T \mathbf{p} \tag{1.10}$$

$$\sigma_i = \mathbf{b}^T \phi_i = \phi_i^T \mathbf{b} \tag{1.11}$$

It is obvious that the modal response, σ_i, is as important as the modal force, τ_i.

The input-output transfer function relationship can also be considered in the physical coordinates. If a passively damped structure is excited by more than one force vector, i.e. $\mathbf{P} = [\mathbf{p}_1\ \mathbf{p}_2\ ]$, the equation of motion becomes:

$$\{[-\omega^2 \mathbf{M} + \mathbf{K}^R(\omega)] + i\,[\,\omega\,\mathbf{C} + \mathbf{K}^I(\omega)\,]\}\ \mathbf{U}(\omega) = \mathbf{P}\,\mathbf{G}(\omega) \tag{1.12}$$

Define the impedance of the structure as:

$$\mathbf{Z}(\omega) = \{[-\omega^2 \mathbf{M} + \mathbf{K}^R(\omega)] + i\,[\,\omega\,\mathbf{C} + \mathbf{K}^I(\omega)\,]\} \tag{1.13}$$

Symbolically, the solution in the frequency domain is given by:

$$\mathbf{U}(\omega) = \mathbf{Z}(\omega)^{-1}\,\mathbf{P}\,\mathbf{G}(\omega) \tag{1.14}$$

The transfer function matrix is given by:

$$\mathbf{T}_{\mathbf{BP}}(\omega) = \mathbf{B}^T\,\mathbf{Z}(\omega)^{-1}\,\mathbf{P} \tag{1.15}$$

The transfer function matrix is therefore contributed by the spatial load matrix, the observation matrix and the system impedance. Similar to the Betti's Reciprocity Law, if **B** is the load matrix and **P** is the observation matrix, then the transfer function matrix for the role reversal of **B** and **P** is:

$$\mathbf{T}_{\mathbf{PB}}(\omega) = \mathbf{P}^T\,\mathbf{Z}(\omega)^{-1}\,\mathbf{B} \tag{1.16}$$

But since $\mathbf{Z}(\omega)$ is symmetric,

$$\mathbf{T}_{\mathbf{PB}}(\omega)^T = \mathbf{T}_{\mathbf{BP}}(\omega) \tag{1.17}$$

Therefore, the Load Dependent Ritz Vectors should be generalized to include two effects:

1. the excitation spatial force vectors,
2. the observation vectors.

The "Load Dependent Ritz vectors" in this dissertation is generalized to include also the "Observation Dependent Ritz vectors". With other enhancements to the Load Dependent Ritz vector approach described in this dissertation, a more appropriate name for this class of methods, is the *Reduced Vector Basis Methods*.

1.3 Achievement

The Load Dependent Ritz Method is generalized into a general Reduced Vector Basis Method. An accurate and efficient method is developed based on the Load Dependent Ritz Vector concept for structures with positive semi-definite stiffness matrices. This technique also enables the development of static correction vectors when normal mode method is used for structures with singular stiffness matrices.

The steady state response, in lieu of the static response, provides a mathematically consistent way to generate basis vectors for dynamic analysis. Both the spatial and temporal characteristics of the excitation and observation can be considered in the vector basis. It also provides a better convergence for system behavior at frequencies of interest. This approach provides a physical meaning to a shifted vector basis similar to the spectral shifting method in eigensolution. Methods to define vector basis for the undamped, viscously damped and frequency dependent viscoelastically damped structures are also developed. In additional to the static correction vectors, harmonic correction vectors are introduced. A Reduced Vector Basis method for solving structures with frequency dependent stiffness and damping matrix is developed. Design methods consistent with the analytical procedures are developed for viscous and viscoelastic members to provide passive damping in the structure to control dynamic vibration.

The Reduced Vector Basis Method is extended to the dynamic substructure analysis. A new derivation of the dynamic substructure method is provided in a consistent mathematical

formulation and with a better physical interpretation than the conventional methods. The viscous and viscoelastic damping of substructures are treated in consistent manner as the mass and stiffness matrices. This method enables accurate analysis of substructures with passive damping design.

Chapter 2

Solution Methods for Dynamic Response Analysis

The governing partial differential equation of motion for dynamic response analysis is founded in classical mechanics of solids[8]. Modern solution techniques are based on discretization of continuous systems, typically by the finite element method[6]. The governing differential equation of motion is represented in matrix form in n dimensional space as:

$$\mathbf{M}\,\ddot{\mathbf{u}} + \mathbf{C}\,\dot{\mathbf{u}} + \mathbf{K}\,\mathbf{u} = \mathbf{p}\,g(t) \tag{2.1}$$

Conventional dynamic analysis[5] uses truncated eigenvectors of the structural system to span the solution space. For structural analysis, all of the extracted modes are normally used in the response evaluation even though only a small number of modes contribute to the response. For control analysis, normally only a few contributing structural modes are included in the analysis. In both cases, the effort expended in computing the non-participating modes is wasted. The widespread use of eigenvectors as the basis vectors for dynamic response computation sometimes misleads analysts to believe that an accurate eigensolution is essential to an accurate dynamic response solution.

The performance of engineering systems is often governed by a single load case. Extensive design iterations are required to understand the structural behavior and optimize the design. The LDRV method[1] which was introduced in 1982 is a significant milestone in dynamic analysis method. It provides an alternate vector basis which is more efficient than the eigenvector basis for dynamic response analysis. While the use of a truncated set of eigenvectors is applicable to dynamic analysis of arbitrary loadings, a reduced vector basis is more desirable to minimize the computer run time and mass storage for dynamic analyses.

In this chapter, the classical solution techniques in structural dynamics are reviewed. The mathematical basis of the LDRV method is explained in terms of the classical engineering analysis methods. Since new methods presented in this dissertation are developed as a natural extension of these classical methods, it is important to understand the underlying strength and weaknesses of these classical methods.

2.1 Eigenvector Basis

The classical method in structural dynamics requires an eigensolution of undamped free vibration problem:

$$\mathbf{M\ddot{u} + Ku = 0} \tag{2.2}$$

The modal extraction is normally truncated at specified frequency above which the dynamic response is expected to be negligible.

$$\mathbf{K\,\Phi = M\,\Phi\,\Lambda} \tag{2.3}$$

$$\text{where } \Phi = [\,\phi_1\ \phi_2 \cdots \phi_i \cdots \phi_m\,] \tag{2.4}$$

$$\Lambda = [\,\text{diag}\ \omega_i^2\,] \tag{2.5}$$

Φ, nxm, contains the eigenvectors and Λ, mxm, is the diagonal eigenvalue matrix. The eigensolution provides the natural resonance frequencies and mode shapes of the structure. The eigenvectors have the important orthogonality conditions as:

$$\Phi^T \mathbf{M}\,\Phi = \mathbf{I} \tag{2.6}$$

$$\Phi^T \mathbf{K}\,\Phi = \Lambda \tag{2.7}$$

2.1.1 Normal Mode Method

The normal mode method, also known as mode displacement method, uses a truncated set of eigenvectors to span the solution space of Equation (2.1):

$$\mathbf{u} \approx \sum_{i=1}^{m} \phi_i \; q_i$$

$$= \Phi \mathbf{q} \tag{2.8}$$

Equation (2.1) in the modal space is given by:

$$\ddot{\mathbf{q}} + \Phi^T \mathbf{C} \Phi \dot{\mathbf{q}} + \Lambda \mathbf{q} = \Phi^T \mathbf{p} \; g(t) \tag{2.9}$$

Since the damping matrix is normally not very well characterized, it is assumed to be diagonal and modal damping ratios are used (based on test results or experience with similar structures). Equation (2.1) is therefore uncoupled in the eigenspace

$$\ddot{q}_i + 2 \xi_i \omega_i \dot{q}_i + \omega_i^2 q_i = \tau_i \; g(t) \tag{2.10}$$

$$\text{where } \tau_i = \phi_i^T \mathbf{p} \tag{2.11}$$

Solutions to these single degree of freedom systems can be easily computed in the time domain by direct integration. Alternatively, the frequency domain solution, $Q_i(\omega)$, can be computed as:

$$Q_i(\omega) = \frac{\tau_i \, G(\omega)}{\omega_i^2 \left[(1-\beta_i^2) + i \; 2 \; \xi_i \; \beta_i\right]} \tag{2.12}$$

$$\text{where } \beta_i = \frac{\omega}{\omega_i} \tag{2.13}$$

$$G(\omega) = \int_0^{\infty} g(t) \; e^{-i\omega t} \, dt \tag{2.14}$$

The dynamic response of the system is then *approximated* by back transformation of the truncated series, Equation (2.8).

It is often taken for granted that an accurate dynamic analysis requires an accurate eigensolution. This is especially true when computers take over the difficult chore of performing the eigensolution. However, it cannot be proved that the eigenvectors basis is the best vector basis, in the sense of accuracy and efficiency, for the solution of dynamic response.

2.1.2 Mode Acceleration Method

The normal mode method often leads to undesirable errors due to modal truncation. The solution totally ignores the dynamic contribution due to truncated modes. These errors are especially important for accurate stress recovery and other response quantities.

The mode acceleration method[9] can be used to improve the accuracy of the normal mode method. A modified acceleration method[10] is presented below. The basic assumptions of the modified mode acceleration method are:

1. the dynamic effects are adequately captured by the modes included in the analysis,
2. since the higher modes have natural frequencies higher than the spectral content of the forcing function, these degrees of freedom respond only statically, i.e. the modal velocities and accelerations are essentially zero.

From Equation (2.12), the second condition is true if the ratio of the highest frequency of the forcing function (with significant magnitude) to the highest natural frequency included in the analysis satisfies:

$$| (1-\beta_m^2) + i\, 2\, \xi_m\, \beta_m | \approx 1.0 \qquad (2.15)$$

Using Equations (2.10) and (2.8), the solution is given by:

$$q_i = \frac{\tau_i}{\omega_i^2} g(t) - \frac{1}{\omega_i^2} \ddot{q}_i + 2 \frac{\xi_i}{\omega_i} \dot{q}_i \qquad \text{for } i \le m \tag{2.16a}$$

$$q_i = \frac{\tau_i}{\omega_i^2} g(t) \qquad \text{for } i > m \tag{2.16b}$$

$$\mathbf{u} = \sum_{i=1}^{m_M} \left[\phi_i \frac{1}{\omega_i^2} \phi_i^T \mathbf{p} \right] g(t) + \sum_{i=1}^{m} \phi_i \left[\frac{1}{\omega_i^2} \ddot{q}_i + \frac{2\xi_i}{\omega_i} \dot{q}_i \right] \tag{2.17}$$

It can be shown that the quantity in the bracket of the first summation sign is identical to the static response for a structure with positive definite stiffness matrix, and therefore,

$$\mathbf{u} = [\mathbf{K}^{-1} \mathbf{p}] g(t) + \sum_{i=1}^{m} \phi_i \left[\frac{1}{\omega_i^2} \ddot{q}_i + \frac{2\xi_i}{\omega_i} \dot{q}_i \right] \tag{2.18}$$

The mode acceleration method in the form presented in Equation (2.18) reveals that the corrected solution is spanned by two sets of basis vectors - the mode shape vectors, ϕ_i, and the static vectors, $\mathbf{K}^{-1}\mathbf{p}$. The solution is improved because of inclusion of static solution of higher modes. Mathematically speaking, the solution is improved because of the addition of static displacement vector to the solution vector basis. Straightly speaking, the solution of the mode acceleration method may not be in the truncated eigenspace.

With the increasing complexity and size of the finite element models, there is a need for accurate methods which are more efficient. A basic understanding of solution techniques to solve Equation (2.1) should be revisited. Classical Rayleigh-Ritz type methods are reviewed in the next section.

2.2 Rayleigh-Ritz Type Methods

Classical solution techniques to the partial differential equations are well known[7]. Energy methods and Rayleigh-Ritz type methods provide accurate solutions to many classical problems. These methods provide accurate solutions without solving large eigenvalue problem, and therefore bypasses the most computationally intensive phase of the solution process. Solutions to these problems were obtained in the generalized coordinates and not the modal coordinates. However, these classical methods are limited by the ability to handle a large number of unknowns and evaluate complex functions and integrals by hand. These methods reveals that computation of accurate natural frequencies and mode shapes is not essential to obtain accurate dynamic response and also the best subspace for frequency determination may not be the best subspace for dynamic solution. Engineering intuition, principles and experience are often used to develop these functions. The underlying principles of these classical methods should be exploited in modern computation on computers.

LDRV Algorithm is a natural outgrowth of some of the most important historical methods in the solution of dynamic response. These classical methods are reviewed in this chapter in light of the development towards the LDRV algorithm. Shortcomings of each method and contributions towards the LDRV method are highlighted.

2.2.1 Rayleigh's Method

Lord Rayleigh[11] solved the fundamental frequency of a system by using assumed displacement shapes. The solution to the undamped free vibration problem was sought and the following equation, in the equivalent discretized form, was solved:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0} \tag{2.7}$$

If the assumed displacement vector is **v**, then the Rayleigh's Quotient, λ, is :

$$\lambda = \frac{\mathbf{v}^T \mathbf{K} \mathbf{v}}{\mathbf{v}^T \mathbf{M} \mathbf{v}} \tag{2.19}$$

The approximate frequency of the structure is computed as :

$$f = \frac{\sqrt{\lambda}}{2\pi} \tag{2.20}$$

The selected vector, **v**, determines the accuracy of the natural frequency and the mode shape. The lower bound solution of λ provides the fundamental frequency of the structure[7] and the corresponding displacement shape is the fundamental mode shape of the structure.

One way to select **v** is by applying a static force, say **p**, which is judged to produce a displacement vector similar to the desired mode shape, so that

$$\mathbf{K}\ \mathbf{v}_s = \mathbf{p} \tag{2.21}$$

$$\lambda = \frac{\mathbf{v}_s^T\ \mathbf{p}}{\mathbf{v}_s^T\ \mathbf{M}\ \mathbf{v}_s} \tag{2.22}$$

Lord Rayleigh's interest, as discussed in Reference 11, was in sound vibration. This method was used only to compute the approximate fundamental frequency of a structure.

The same technique can be used to approximate dynamic response of a structural system evaluated as single mode response. The static solution to Equation (2.1) is used as the trial vector.

$$\mathbf{v} = \mathbf{K}^{-1}\ \mathbf{p} \tag{2.23}$$

The dynamic response is then obtained by assuming:

$$\mathbf{u}(t) \approx \mathbf{v}\, y(t) \tag{2.24}$$

This assumption will give an exact solution if the behavior of the system is static or quasi-static in nature. The residual force vector of this assumption can be expressed as :

$$\mathbf{r}(t) = \mathbf{M}\,\mathbf{v}\,\ddot{y} + \mathbf{C}\,\mathbf{v}\,\dot{y} + \mathbf{K}\,\mathbf{v}\,y - \mathbf{p}\,g(t) \tag{2.25}$$

By Galerkin's method[7], minimizing the projection of the residual on the trial vector v,

$$\mathbf{v}^T\,\mathbf{r}(t) = 0 \tag{2.26}$$

a single degree of freedom approximation to the system is obtained:

$$(\mathbf{v}^T\,\mathbf{M}\,\mathbf{v})\,\ddot{y} + (\mathbf{v}^T\,\mathbf{C}\,\mathbf{v})\,\dot{y} + (\mathbf{v}^T\,\mathbf{K}\,\mathbf{v})\,y = (\mathbf{v}^T\,\mathbf{p})\,g(t)$$

$$m^*\,\ddot{y} + c^*\,\dot{y} + k^*\,y = p^*\,g(t) \tag{2.27}$$

This is the rudiment of the reduced vector basis method. This provides a very simple approach to single mode dynamic response. This approach is extended in the LDRV method to a more general form.

2.2.2 Improved Rayleigh's Method

To improve Rayleigh's frequency determination, Stodola iteration[5] (inverse power method) is generally used.

$$\mathbf{v}_1 = \mathbf{K}^{-1}\,\mathbf{p} \tag{2.28}$$

$$\mathbf{v}_{i+1} = \mathbf{K}^{-1}\,\mathbf{M}\,\mathbf{v}_i \tag{2.29}$$

The vector sequence, $\mathbf{v}_i$, is known as a *Krylov Sequence*[12]. If j iterations are computed to obtain an accurate frequency, then j vectors are generated. Of all the computed vectors, the j-th vector will be the best approximation to the fundamental mode shape. Unlike the fundamental frequency, there is no guarantee that the dynamic response evaluated using $\mathbf{v}_j$

will give a more accurate result in dynamic response. If the force vector is nearly orthogonal to the fundamental mode, then Equation (2.27) will give a solution which is not representative of the physical behavior. Besides, the effort in computing all the $\mathbf{v}_i$s could be totally wasted. Therefore, the Improved Rayleigh's Method, in the form presented, is not suitable for dynamic response computation. The LDRV Method uses all the vectors computed, $\mathbf{v}_i$, to span the solution and consequently a more accurate solution can be obtained.

2.2.3 Ritz's Method

Ritz[7] assumed that the displacement function (vector) can be represented by a series of linearly independent functions (vectors) with undetermined coefficients.

$$\mathbf{u} \approx \Sigma \mathbf{v}_i y_i = \mathbf{V}\,\mathbf{y} \tag{2.30}$$

This allows the frequency determination to be performed in a vector subspace with a dimension larger than one but much small than n. One important feature of this method is that it provides an estimate for higher frequencies and mode shapes. Again, the accuracy of the solution is solely dependent on the selection of the basis vectors, v.

$$(\mathbf{V}^T \mathbf{M}\,\mathbf{V})\,\ddot{\mathbf{y}} + (\mathbf{V}^T \mathbf{K}\,\mathbf{V})\,\mathbf{y} = \mathbf{0} \tag{2.31}$$

$$\mathbf{M}^*\,\ddot{\mathbf{y}} + \mathbf{K}^*\,\mathbf{y} = \mathbf{0} \tag{2.32}$$

The eigensolution to this reduced problem,

$$\mathbf{K}^*\,\Phi^* = \mathbf{M}^*\,\Phi^*\,\Lambda^* \tag{2.33}$$

provides the approximate eigensolution to Equation (2.1):

$$\Lambda \approx \Lambda^* \tag{2.34}$$

$$\Phi \approx \mathbf{V}\,\Phi^{*} \tag{2.35}$$

Similar to the Improved Rayleigh's Method, a Stodola iteration (inverse iteration) can be used to improve the quality of **V** for eigensolution. This gives the famous *subspace* (simultaneous) *iteration* method[5].

$$\mathbf{V}_{i+1} = \mathbf{K}^{-1}\,\mathbf{M}\,\mathbf{V}_{i} \tag{2.36}$$

A set of random vectors is normally used as the starting vector block, $\mathbf{V}_1$. This is the Improved Rayleigh's method in *vector block form*. This method was primarily developed to compute only the eigensolution and not the dynamic response. The intermediate vector blocks are normally not retained.

The solution of dynamic response by using the Ritz vectors is similar to Rayleigh's approach. Again by using Galerkin's method and requiring the residual vector to have zero projection onto (i.e. orthogonal to) the basis vectors, i.e.

$$\mathbf{V}^{T}\,\mathbf{r}(t) = \mathbf{0} \tag{2.37}$$

Equation (2.1) is reduced to :

$$\mathbf{V}^{T}\,\mathbf{r}(t) = \mathbf{0} \tag{2.38}$$

$$(\mathbf{V}^{T}\mathbf{M}\mathbf{V})\,\ddot{\mathbf{y}} + (\mathbf{V}^{T}\mathbf{C}\,\mathbf{V})\,\dot{\mathbf{y}} + (\mathbf{V}^{T}\mathbf{K}\,\mathbf{V})\,\mathbf{y} = (\mathbf{V}^{T}\,\mathbf{p})\,g(t)$$

$$\mathbf{M}^{*}\,\ddot{\mathbf{y}} + \mathbf{C}^{*}\,\dot{\mathbf{y}} + \mathbf{K}^{*}\,\mathbf{y} = \mathbf{p}^{*}\,g(t) \tag{2.39}$$

In general, the coefficient matrices are coupled. The dimension of Equation (2.1) is reduced and the solution is constrained in the **V** subspace. The accuracy of the solution is dependent on the choice of **V**. Direct solution methods, e.g. step by step integration in the time domain or frequency by frequency solution in the frequency domain, can be used to provide the solution to Equation (2.37). However, if a modal extraction is performed on the reduced eigenvalue problem, ($\mathbf{M}^{*}$, $\mathbf{K}^{*}$), and modal damping is assumed, Equation

(2.37) is uncoupled. Conventional modal superposition analysis can be used. Normal mode method is a special case which uses the lowest eigenvectors, Φ, of the (M, K) system as the basis vectors.

In classical engineering analysis, functions (e.g. Fourier series, polynomials etc.) or vectors from a simpler or similar problem are selected as trial functions. Similar to Rayleigh's Method, the improved subspace $\mathbf{V}_1$ for frequency determination may not the best solution space for dynamic responses evaluation. The LDRV Method provides a recurrence relationship to generate a subspace vector basis from the given spatial load distribution and hence overcomes the difficult task of selecting a set of good trial vectors.

2.3 Modal Participation

It is well known that only a limited number of modes participate in most dynamic responses. For example, if a cantilever beam of square section is forced in an arbitrary direction, there is no dynamic response in the orthogonal lateral direction or the axial direction. It is desirable to extract only those modes which participate in the response and skip those modes which are insignificant to the response.

Equation (2.1) is uncoupled into scalar modal equations by the modal transformation:

$$\mathbf{u} = \Phi\, \mathbf{q} \tag{2.8}$$

If the rank of the mass matrix, M, is n_M, there are n_M modal coordinates. By assuming modal damping, the i-th uncoupled modal equation is :

$$\ddot{q}_i + 2\,\xi_i\,\omega_i\,\dot{q}_i + \omega_i^2\,q_i = \tau_i\,g(t) \tag{2.10}$$

This is equivalent to a single degree of freedom oscillator with a unit mass and a stiffness of ω_i^2. The response to g(t) is scaled by a modal participation factor, τ_i. τ_i can also be be

viewed as the generalized force, in the spatial sense, and is the projection of the spatial force vector, **p**, onto the i-th mode shape. If the dynamic response of interest is σ (= $\mathbf{b}^T\mathbf{u}$), then from Equation (1.9) the observation vector, **b**, is also as important as the load vector **p**. A reciprocity relationship exists between **p** and **b**. The modal participation factor can be generalized to include both the load modal participation and response participation factors.

2.3.1 Modal Static Solution

From Equation (2.10), the static solution can be represented by modal superposition.

$$q_i = \frac{\tau_i}{\omega_i^2} \tag{2.40}$$

$$\mathbf{u}_s = \sum_{i=1}^{m} \phi_i \frac{\tau_i}{\omega_i^2} \tag{2.41}$$

The modal contribution to the static solution is proportional to the modal static force, τ_i, and inversely proportional to the modal stiffness, ω_i^2. Unlike the dynamic response, quite a number of modes are required for convergence of the static solution because there is no dominant dynamic amplification. Of course, it is not prudent to compute the static solution by modal superposition. However, it is important to note that static vector, $\mathbf{u}_s$, contains the necessary eigenvector scaled by the participation factors which help to define a good vector basis for the dynamic solution. If the spatial force vector is orthogonal to the i-th mode, then τ_i is null. In this case, ϕ_i is not represented in the static vector.

2.3.2 Modal Dynamic Solution

The closed form dynamic response for the i-th mode, can be written in the frequency domain, with $\beta_i = \omega/\omega_i$, as :

$$Q_i(\omega) = \frac{\tau_i \, G(\omega)}{\omega_i^2 \, [(1-\beta_i^2) + i \, 2 \, \xi_i \, \beta_i]} \tag{2.12}$$

The closed form solution is therefore:

$$U(\omega) = \sum_{i=1}^{m} \left[\phi_i \, \frac{\tau_i}{\omega_i^2} \right] \frac{G(\omega)}{(1-\beta_i^2) + i \, 2 \, \xi_i \, \beta_i} \tag{2.42}$$

It is obvious that both the dynamic and static solutions are spanned by the vectors $\left[\phi_i \, \frac{\tau_i}{\omega_i^2} \right]$. The static solution is a special case of Equation (2.42) with $\omega=0$. The dynamic response is related to the modal static response by two factors:

1. the spectral amplitude of the loading, $G(\omega)$
2. the modal amplification due to the modal frequency and damping, $\left[\frac{1}{(1-\beta_i^2) + i \, 2 \, \xi_i \, \beta_i} \right]$

If a vector subspace can provide accurate static solution, the corresponding dynamic solution can also be found in the same vector subspace. The dynamic effect in the subspace is approximated through the solution to the second order differential equation, Equation (2.39).

Convergence of the dynamic solution is more rapid than the static solution because of the presence of dominant modes with large dynamic amplification factors. If a set of truncated modes is used, the response due to higher modes is not accounted for in the solution. In the frequency range of interest, the β_i of these modes is small and the residual response is static in nature. Therefore, static correction for modal truncation is appropriate.

The LDRV method exploits modal participation factors and ranks the modes intrinsically. Static vectors, which contain the modal participation factors, are used as the basis vectors to provide an approximation to the solution of Equation (2.1).

2.4 Gram-Schmidt Mass Orthonormalization

Given two arbitrary but independent vectors in the n dimensional space, two orthogonal unit basis vectors can be computed. The Gram-Schmidt (GS) orthonormalization procedure[13] provides a general computational procedure to orthonormalize basis vectors in the n dimensional space. A new orthogonal unit vector can be generated from a new independent vector by removing the projections of this new vector onto the previously computed unit vectors. The use of Gram-Schmidt orthonormalization can significantly improve the accuracy of numerical solution since the coordinate axes are orthogonal. Any vector basis **V** can be transformed into a set of orthogonal vector basis **X**, given a starting vector and a weighing matrix, **M**.

Given v_1,

Initialization,
$$x_1 = \frac{v_1}{(v_1^T M v_1)^{0.5}} \tag{2.43}$$

Orthogonalization,
$$\tilde{v}_i = v_i - \sum_{j=1}^{i-1} x_j c_j \tag{2.44}$$

$$c_j = (x_j^T M v_i)$$

Normalization,
$$x_i = \frac{\tilde{v}_i}{(\tilde{v}_i^T M \tilde{v}_i)^{0.5}} \tag{2.45}$$

The vector basis **X** so computed is mass orthonormalized:

$$\mathbf{X}^T \mathbf{M} \mathbf{X} = \mathbf{I} \tag{2.46}$$

X is a linear transform of **V** and spans the same vector space. Φ, which is a special case of **X**, is also orthogonal to **K**. The orthogonal expansion property of the orthonormal vector basis allows a check for convergence as each basis vector has been computed.

2.5 Properties of Mass Orthonormalized Vector Basis

Similar to the Fourier series and other orthogonal functions, the mass orthonormalized vector basis allows for a series representation of the applied force vector and the static displacement vector. Convergence of the solution can be checked as more vectors are included in the solution.

2.5.1 Expansion of Force Vector in Orthonormal Vector Basis

Any force vector **p** in the range of **M** can be expanded in terms of the inertial force vectors of the mass orthonormalized vector basis, i.e.

$$\mathbf{p} = \sum_{i=1}^{n_M} \mathbf{M}\ \mathbf{x}_i\ b_i \tag{2.47}$$

Pre-multiply Equation (2.47) by the basis vectors and use the orthonormal property of the basis vectors:

$$b_i = \mathbf{x}_i^T\ \mathbf{p} \tag{2.48}$$

$$\mathbf{p} = \mathbf{M}\ \mathbf{X}\ \mathbf{X}^T\ \mathbf{p} \tag{2.49}$$

However, **p** can also be approximated by an incomplete series, up to m terms, with a residual force vector, $\mathbf{p}^r_m$.

$$\mathbf{p}_m = \mathbf{M}\ \mathbf{X}_m\ \mathbf{X}_m^T\ \mathbf{p} \tag{2.50}$$

$$\begin{aligned} \mathbf{p}^r_m &= \mathbf{p} - \mathbf{M}\ \mathbf{X}_m\ \mathbf{X}_m^T\ \mathbf{p} \\ &= [\ \mathbf{I} - \mathbf{M}\ \mathbf{X}_m\ \mathbf{X}_m^T\ ]\ \mathbf{p} \\ &= \mathbf{A}_m\ \mathbf{p} \end{aligned} \tag{2.51}$$

Alternatively, a vector updating scheme can be used :

$$\mathbf{p}^r_m = \mathbf{p}^r_{m-1} - \mathbf{M}\,\mathbf{x}_m\,(\mathbf{x}_m^T\,\mathbf{p}^r_{m-1}) \tag{2.52}$$

2.5.2 Expansion of Displacement Vector in Orthonormalized Vector Basis

Any displacement vector **u** in the solution space of Equation (2.1) can be expanded in terms of the mass orthonormalized basis vectors.

$$\mathbf{u} = \sum_{i=1}^{n_M} \mathbf{x}_i\,c_i \tag{2.53}$$

$$c_i = \mathbf{x}_i^T\,\mathbf{M}\,\mathbf{u} \tag{2.54}$$

$$\mathbf{u} = \mathbf{X}\,\mathbf{X}^T\,\mathbf{M}\,\mathbf{u} \tag{2.55}$$

However, **u** can also be approximated by an incomplete series, up to m terms, and a residual displacement vector, $\mathbf{u}^r_m$.

$$\mathbf{u}_m = \mathbf{X}_m\,\mathbf{X}_m^T\,\mathbf{M}\,\mathbf{u} \tag{2.56}$$

$$\begin{aligned}\mathbf{u}^r_m &= \mathbf{u} - \mathbf{X}_m\,\mathbf{X}_m^T\,\mathbf{M}\,\mathbf{u} \\ &= [\,\mathbf{I} - \mathbf{X}_m\,\mathbf{X}_m^T\,\mathbf{M}\,]\,\mathbf{u} \\ &= \mathbf{A}_m^T\,\mathbf{u}\end{aligned} \tag{2.57}$$

Alternatively, a vector updating scheme can be used :

$$\mathbf{u}^r_m = \mathbf{u}^r_{m-1} - \mathbf{x}_m\,(\mathbf{x}_m^T\,\mathbf{M}\,\mathbf{u}^r_{m-1}) \tag{2.58}$$

The residual vector, $\mathbf{u}^r_j$, is mass orthogonal to the orthonormalized vector basis, $\mathbf{X}_m$, and is spanned by the complementary vector basis of the vector space.

2.6 Wilson's LDRV Method

Similar to single mode Rayleigh's method, Wilson[1] used the static solution to Equation (2.1), as the approximate solution.

$$\mathbf{v}_1 = \mathbf{K}^{-1}\,\mathbf{p} \tag{2.23}$$

The initial vector is mass normalized.

$$\mathbf{x}_1 = \frac{\mathbf{v}_1}{(\mathbf{v}_1^T\,\mathbf{M}\,\mathbf{v}_1)^{0.5}} \tag{2.43}$$

To correct for the effect of inertia force not accounted for in Equation (2.23), successive static correction vectors are computed as in the Improved Rayleigh's method.

$$\mathbf{v}_{i+1} = \mathbf{K}^{-1}\,\mathbf{M}\,\mathbf{x}_i \tag{2.36}$$

However, instead of using the latest computed vector, $\mathbf{v}_{i+1}$, as the best approximation, this vector is orthonormalized against the previous vectors and then becomes a new trial vector as in the Ritz's method.

$$\bar{\mathbf{v}}_i = \mathbf{v}_i - \sum_{j=1}^{i-1} \mathbf{x}_j\,(\mathbf{x}_j^T\,\mathbf{M}\,\mathbf{v}_i) \tag{2.44}$$

$$\mathbf{x}_i = \frac{\bar{\mathbf{v}}_i}{(\bar{\mathbf{v}}_i^T\,\mathbf{M}\,\bar{\mathbf{v}}_i)^{0.5}} \tag{2.45}$$

This method overcomes some of the major shortcomings of the classical methods and unifies all these methods into one. All the computed intermediate vectors are used efficiently to form a reduced vector basis. It provides a procedure to automatically generate trial vectors for the Ritz's method. Only vectors participating in the response are generated. The original Wilson's algorithm is presented in Reference 1. Gram-Schmidt orthogonal expansion of force vector was used for error analysis, Equation (2.51). Since the vector

basis is generated from the load vector, the method is called the Load Dependent Ritz Vector method. In the Numerical Analysis literature, this method is called the Lanczos algorithm with full re-orthogonalization and the vectors are called the Lanczos vectors.

Both the normal mode method and the LDRV method provide approximate solutions to Equation (2.1) but the LDRV method is computationally more efficient. Since some solution techniques for dynamic response analysis, e.g. response spectrum analysis, random vibration analysis and other forms of frequency domain analyses, work best with uncoupled single degree of freedom equations an eigenvalue problem of the Ritz subspace is solved. In spite of the eigensolution in the reduced space, the solution space is the same and the accuracy of the solution is not changed. Wilson's method combines all the advantages of the Rayleigh, Improved Rayleigh, Ritz and Gram-Schmidt methods into one single method.

An improved LDRV method is presented in Table 2.1[14]. It uses the displacement expansion of the Gram-Schmidt method to generate Ritz vectors, as shown in Equation (2.56). The static solution to Equation (2.1) is used as a root vector. When each new basis vector is computed, its projection onto the root vector is removed from the root vector. The residual root vector is used for providing the inertia correction. This method provides a better control over numerical round off errors and minimizes the possibility of generating basis vectors due to round off errors incurred at each step.

2.7 Error Analysis

One major advantage of the Gram-Schmidt method is that it allows static convergence to be checked when each basis vector becomes available during the computation.

$$\mathbf{p}^r_j = \mathbf{p}^r_{j-1} - \mathbf{M}\,\mathbf{x}_j\,\mathbf{x}_j^T\,\mathbf{p}^r_{j-1} \tag{2.51}$$

$$\mathbf{u}^r_j = \mathbf{u}^r_{j-1} - \mathbf{x}_j\,\mathbf{x}_j^T\,\mathbf{M}\,\mathbf{u}^r_{j-1} \tag{2.57}$$

Table 2.1
Algorithm for Generating Ritz Vectors
Based on Static Root Vector

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1. Given mass, stiffness matrices **M**, **K** and load vector **p**

2. Triangularize stiffness matrix - $\mathbf{K} = \mathbf{L}\,\mathbf{D}\,\mathbf{L}^T$

3. Solve for static vector - $\mathbf{K}\,\mathbf{u}(0) = \mathbf{p}$

4. Solve for additional vectors (i=1,...,N)
   a. Solve equations for new vectors
      $\mathbf{K}\,\mathbf{x}_i(1) = \mathbf{M}\,\mathbf{u}(i-1)$
   b. Make vector orthogonal to previously found vectors for j=1,...,i-1
      $\mathbf{x}_i(j+1) = \mathbf{x}_i(j) - c_j\,\mathbf{x}_j$ where $c_j = \mathbf{x}_j^T\,(\mathbf{M}\,\mathbf{x}_i)$
   c. Normalize vector - $\mathbf{x}_i^T\,\mathbf{M}\,\mathbf{x}_i = 1.0$
   d. Make static vector orthogonal
      $\mathbf{u}(i) = \mathbf{u}(i-1) - (\mathbf{x}_i^T\,\mathbf{M}\,\mathbf{u}(i-1))\,\mathbf{x}_i$
   e. Calculate static participation
      $e_i = \mathbf{p}^T\,\mathbf{u}(i) \,/\, \mathbf{p}^T\,\mathbf{u}(0)$
   f. If i = N, mass normalize and add static vector as last Ritz vector
      $\mathbf{x}_{N+1} = \mathbf{u}(i)$ (optional)

5. Orthogonalization of Ritz vectors with respect to stiffness matrix
   a. Solve small subspace eigenvalue problem
      $[\,\mathbf{A} - \omega_i^2\,\mathbf{I}\,]\,\mathbf{y}_i = \mathbf{0}$
      where
      $\mathbf{A} = \mathbf{X}^T\,\mathbf{K}\,\mathbf{X}$
      $\mathbf{I} = \mathbf{X}^T\,\mathbf{M}\,\mathbf{X}$
      ω_i = approximate frequencies (i=1,...,N)
   b. Compute final stiffness and mass orthogonal
      Ritz vectors
      $^{o}\mathbf{X} = \mathbf{X}\,\mathbf{Y}$

The error measure in the force vector with j basis vectors is the ratio of the residual force vector norm and the static force vector norm :

$$\varepsilon_j = \frac{\mathbf{p}^{r}_{j}{}^{T}\mathbf{p}^{r}_{j}}{\mathbf{p}^{T}\mathbf{p}} \tag{2.59}$$

The error measure in the displacement vector with j basis vectors is the ratio of the residual displacement vector norm and the static displacement vector norm :

$$\varepsilon_j = \frac{\mathbf{u}^{r}_{j}{}^{T}\mathbf{u}^{r}_{j}}{\mathbf{u}^{T}\mathbf{u}} \tag{2.60}$$

These error measures consist of mixed force and moment quantities or length and angle quantities. The units of these norms are inconsistent. The numerical magnitudes of displacement and angle (force and moment) are often quite different and a more meaningful convergence criterion cannot be set. If an energy unit is adapted by using the inner product of force vector and displacement vector, a more consistent and meaningful convergence criterion is possible. One of the following error measures, which correspond to some forms of residual strain energy, can be used as a termination criterion for generation of Ritz vectors.

$$\varepsilon_j = \frac{\mathbf{p}^{T}\mathbf{u}^{r}_{j}}{\mathbf{p}^{T}\mathbf{u}} = \frac{\mathbf{p}^{T}\mathbf{A}_j{}^{T}\mathbf{u}}{\mathbf{p}^{T}\mathbf{u}} \tag{2.61}$$

$$\varepsilon_j = \frac{\mathbf{p}^{r}_{j}{}^{T}\mathbf{u}}{\mathbf{p}^{T}\mathbf{u}} = \frac{\mathbf{p}^{T}\mathbf{A}_j{}^{T}\mathbf{u}}{\mathbf{p}^{T}\mathbf{u}} \tag{2.62}$$

$$\varepsilon_j = 1 - \frac{\mathbf{p}_j{}^{T}\mathbf{u}_j}{\mathbf{p}^{T}\mathbf{u}} \tag{2.63}$$

These error measures can be evaluated as each new basis vector has been computed. When the termination strain energy criterion is met, the residual displacement vector is mass orthonormalized and included in the vector basis. The inclusion of this vector enables the vector basis to span the exact static solution. When orthonormalized, this vector is treated

like other basis vectors and a dynamic contribution of this vector is obtained in the computation (mode acceleration method only provides a static correction by this vector).

2.8 Lanczos Method

The LDRV is very similar to the computational sequence of the Lanczos method[12,15]. If the same starting displacement vector is used, the Ritz Vectors as defined and the Lanczos Vectors span the same solution space. The Lanczos method only requires orthogonalization to two preceding vectors and hence incurs less numerical operations. The reduced matrix is tridiagonal if there are no round off errors. The original Lanczos algorithm was plagued by numerical round off errors. It has be revitalized by re-orthogonalization strategy[12]. Full re-orthogonalization is used in the LDRV method. A small amount of round off errors actually helps the Lanczos method to provide seeds to generate more vectors for eigenvector evaluation. The Lanczos algorithm is seeing by most people as a method for computing eigenvectors but in fact it should be regarded as a way of computing good basis for a low order subspace. The main purpose of LDRV method is to compute the smallest set of vectors for solution of the dynamic response. The improved LDRV method is particularly suitable for this task since it minimizes the effect of round off errors.

2.9 Analysis of LDRV Algorithm

One goal of the Ritz method is to diminish the error in the solution to the differential equation, Equation (2.1), caused by the subspace approximation. Equation (2.38) implies that for a given vector basis, the solution to Equation (2.39) has no error in the chosen subspace and hence the best solution in Galerkin's sense. The LDRV provides a small set

of vector which is particularly rich in the vectors necessary to span the solution space as shown in Equation (2.42). However, this also provides an excellent approximation to the modal frequencies and mode shapes of the participating modes. In other words, the algorithm selectively extracts the modes which contribute to the dynamic response. The accuracy of this approximation can be explained as follows.

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{K}^{-1}\,\mathbf{p} \\ &= \Sigma\,\phi_i\,\frac{\tau_i}{\omega_i^2} \\ &= \Phi\,\Lambda^{-1}\,\tau \end{aligned} \tag{2.64}$$

By using the orthogonality relationship,

$$\begin{aligned} \mathbf{K}\,\Phi &= \mathbf{M}\,\Phi\,\Lambda \\ \mathbf{K}^{-1}\,\mathbf{M}\,\Phi &= \Phi\,\Lambda^{-1} \end{aligned} \tag{2.65}$$

then,

$$\begin{aligned} \mathbf{v}_i &= \mathbf{K}^{-1}\,\mathbf{M}\,\mathbf{v}_{i-1} \\ &= \Phi\,\Lambda^{-i}\,\tau \end{aligned} \tag{2.66}$$

Now use $\mathbf{V}$ (= [$\mathbf{v}_i$]) as the vector basis and the reduced system is defined by Equation (2.39). By using modal force participation factors, τ_i, as shown in Equation (2.66), then

$$\begin{aligned} k^*_{ij} &= \mathbf{v}_i^T\,\mathbf{K}\,\mathbf{v}_j \\ &= \tau^T\,\Lambda^{-i-j+1}\,\tau \\ &= \Sigma\,\tau_i^2\,\omega_i^{-2(i+j-1)} \end{aligned} \tag{2.67}$$

$$\begin{aligned} m^*_{ij} &= \mathbf{v}_i^T\,\mathbf{M}\,\mathbf{v}_j \\ &= \tau^T\,\Lambda^{-i-j}\,\tau \\ &= \Sigma\,\tau_i^2\,\omega_i^{-2(i+j)} \end{aligned} \tag{2.68}$$

For a single mode approximation, the approximate eigenvalue is given by:

$$\lambda = \frac{\Sigma\, \tau_i^2\, \omega_i^{-2}}{\Sigma\, \tau_i^2\, \omega_i^{-4}} \tag{2.69}$$

The approximate eigenvalue is a weighted value of all the powers of eigenvalues of the system scaled by the square of the modal participation factors (generalized force). If the force vector has no projection onto the i-th mode, the i-th mode has no weight in the reduced mass and stiffness matrices.

Alternatively, the modal static displacement can be used to analyze the selectively extracted eigenvalues.

$$\begin{aligned} \mathbf{v}_s &= \Sigma\, \phi_i\, q_i \\ \mathbf{v} &= \Phi\, \mathbf{q} \end{aligned} \tag{2.41}$$

By using the orthogonality condition of Equation (2.65), then

$$\mathbf{v}_i = \Phi\, \Lambda^{-i+1}\, \mathbf{q} \tag{2.70}$$

$$\begin{aligned} k^*_{ij} &= \mathbf{v}_i^T\, \mathbf{K}\, \mathbf{v}_j \\ &= \mathbf{q}^T\, \Lambda^{-i-j+3}\, \mathbf{q} \\ &= \Sigma\, q_i^2\, \omega_i^{-2(i+j+3)} \end{aligned} \tag{2.71}$$

$$\begin{aligned} m^*_{ij} &= \mathbf{v}_i^T\, \mathbf{M}\, \mathbf{v}_j \\ &= \mathbf{q}^T\, \Lambda^{-i-j+2}\, \mathbf{q} \\ &= \Sigma\, q_i^2\, \omega_i^{-2(i+j-2)} \end{aligned} \tag{2.72}$$

For a single mode approximation, the approximate eigenvalue is given by:

$$\lambda = \frac{\Sigma\, q_i^2\, \omega_i^2}{\Sigma\, q_i^2} \tag{2.73}$$

The Ritz vectors used in the selective modal extraction are rich in modes contributing to the static displacement. As shown in Equation (2.42), these vectors provide a good vector basis for solution of dynamic response. The approximate eigenvalues of the reduced system are weighted averages of all the eigenvalues scaled by the spectral contributions to the static displacement.

Therefore, the natural frequencies and mode shapes of the reduced system are good approximations to the eigensolution of Equation (2.2). It is a smaller set than the eigen solution because of the intrinsic scaling by the modal participation factors.

2.10 Positive Definite and Positive Semi-definite Stiffness Matrix

The solution technique of the LDRV method presented in this chapter relies on the static displacement vectors to form the reduced basis. For structural systems with rigid body modes and mechanism modes, the stiffness matrix is positive semi-definite. Formal static solution is not possible. Solution techniques for this class of problems is explored in the next chapter.

Chapter 3

Reduced Vector Basis Method for Structures with Positive Semi-definite Stiffness Matrices

The LDRV method developed so far is applicable to grounded structures with positive definite stiffness matrices. Space structures, typically smaller components of large structures, sometimes are also analyzed as grounded structures. There is a large class of structures with positive semi-definite stiffness matrices, e.g. space structures, robotics, articulated structures etc. Particularly, large computational effort is often required for modern space structures because:

1. accurate stress analysis is required to minimize weight
2. stringent dynamic performance requires high fidelity structural models
3. light weight design and the presence of flexible appendages result in high modal density at a low frequency range
4. high frequency content of disturbances requires refined finite element mesh and modal extraction to high natural frequencies.

Extension of the LDRV method to the class of problems with semi-definite stiffness matrices provides an efficient and robust algorithm to solve dynamic response of these structures. This chapter generalizes the LDRV technique to positive semi-definite stiffness matrices and hence provide a robust algorithm to generate a reduced vector basis for dynamic response analysis of structures with rigid body and mechanism modes.

3.1 Characteristics of Structures with Positive Semi-definite Stiffness Matrices

A free-free structure is defined as a structure not connected to a structural support. It is characterized by rigid body motions. These motions do not cause any internal strain and does not induce any strain energy in the structure. An articulated structure is characterized by mechanism motions. These mechanism modes are also rigid body motions which are associated with no internal strain energy. A structure can have a combination of these two modes. This type of structures is intrinsically unstable (in the static sense) when subject to external forces. The stiffness matrices of this type of structures are singular. Consequently, formal static solution is not possible.

Since the stiffness matrix is singular, it does not have full rank. These rigid body and mechanism vectors are null space vectors of the stiffness matrix. These vectors are associated with zero natural frequencies and their shapes are non-unique. Any linear combination of these independent zero strain energy vectors also gives a zero strain energy vector. However, if there is numerical sensitivity problem in the stiffness matrix, non-zero strain energy will be associated with these vectors and the natural frequencies will not be zero. If these vectors are not computed correctly, the corresponding frequencies will also not be zero. If these vectors are subsequently treated as elastic body modes in the analysis, erroneous dynamic responses will be computed. This problem is especially acute when the structure is flexible and has plenty of low frequency modes. Sometimes, it is very difficult to distinguish which are the correct modes to use in the dynamic response analysis. It is essential to compute the zero energy vectors associated with zero frequencies as accurately as possible.

Associated with each zero strain energy mode is a force vector which induces only rigid body or mechanism motion. Therefore a general force vector can be decomposed into two

components, one that induces only rigid body/mechanism modes and one that induces only elastic deformation. Consequently the elastic body force is self-equilibrating.

Mathematically, the zero strain energy modes are a linear combination of eigenvectors with zero eigenvalues. They are orthogonal to the balance of the eigenvectors which can be classified as elastic modes. The elastic body force is the residual force vector after all the zero strain energy forces are removed from the force vector.

3.2 Preliminary Treatment of Positive Semi-definite Stiffness Matrices

In order to develop a LDRV algorithm for a structure with positive semi-definite stiffness matrix, it is obvious that the matrix must be made positive definite. In this section, a few methods are discussed to provide solution to generate basis vectors for dynamic response analysis. Methods independent of arbitrary selection of constants or degrees of freedom for transformation are preferred.

In a way, solution to this positive semi-definite system problem is similar to the eigenvalue problem of rank deficient matrix[5]. Many methods were developed to solve eigenvalue problem of rank deficient systems. These methods are briefly reviewed in an attempt to apply them to generate basis vectors. The most common way to solve this problem is by using a mass-shifted stiffness matrix which preserves the eigenvectors and only shifts the eigenvalues by a constant. A method based on this concept is developed and its performance is discussed. The advantages and disadvantages of other methods are also discussed. The best way to solve this problem is presented in the Section 3.3.

3.2.1 Shifted Stiffness Matrix Approach

Similar to the shifted stiffness method to compute eigensolution, the shifted stiffness matrix is non-singular and can be used to extract Ritz vectors.

$$\mathbf{K_s} = \mathbf{K} + s\ \mathbf{M} \tag{3.1}$$

By this approach, the shift, "s", is used to make the stiffness matrix non-singular and it has no physical meaning per se. If it is set too small, the ratio between the largest and smallest eigenvalues is large. In this case, $\mathbf{K_s}$ will be changed from a singular to a near singular or ill-conditioned matrix. Vectors obtained from this system will have numerical errors and therefore lead to wrong solution. From this point of view, "s" has to be a moderately large number.

The shifted stiffness matrix is equivalent to a mass proportional spring system superimposed on the original rank deficient elastic stiffness matrix. When "s" is positive and mass matrix is diagonal, this is similar to the classical beam on elastic foundation problem or shell type behavior. The effect of these springs can cause highly localized displacement and hence makes it very difficult to obtain accurate rigid body vectors and frequencies. In the presence of moderately large shifted mass spring, the Ritz vectors will not span the rigid body modes unless a large number of vectors are computed. Spurious low frequency modes will be extracted which have non-zero strain energy. To minimize this problem, a small "s" will therefore be preferred.

The success of mass shifted stiffness matrix approach depends on the selection of a small enough "s" to enable an accurate recovery of rigid body modes and frequencies. However, it also requires a large enough "s" to avoid ill-conditioning of shifted stiffness matrix. There is a range of "s" which provides a good solution. However, both trial and error are required to verify the accuracy and reliability of the solution. A rational method to select the "s" value is needed to make this approach robust.

Another problem associated with Ritz vectors obtained from the $\mathbf{K}_s$ system is that they contain arbitrarily large rigid body components, depending on the value of "s", that the elastic components are overwhelmed by the rigid body components and they are not adequately represented in the vectors in finite precision computation. This causes two problems. Firstly, using these vectors for dynamic response evaluation will not give accurate elastic body response. Secondly, the work done by rigid body forces dominates over the elastic strain energy and thus renders the energy norm quite useless as a convergence criterion.

A free-free beam was analyzed using the root vector Load Dependent Ritz Vector method as presented in Table 2.1. The exact eigenvalue solution of this example is shown in Table 3.1 for comparison purpose.

The free-free beam is subject to a lateral force at the tip in the y direction. The spectral content of the reduced vector basis is used as a comparative measure of the quality of the Ritz vector spaces from different shift values. Therefore, eigenvalues of this reduced vector basis were computed. A summary of the analyses is shown in Table 3.2. The algorithm extracts only modes in the y-z plane as expected.

When the shift is very small (case 1), the solution is totally wrong due to ill-conditioning of the matrix. When the shift is small (cases 2 to 7) the rigid body component dominates the static displacement solution and elastic modes are not adequately represented in the displacement vector even though the convergence tolerance is set to be very small. In this case, basically any loading will give the rigid body mode since the shift is near the zero eigenvalues of the system. When the shift is at 10 Hertz and tolerance is at 0.001 (case 8), only one elastic mode is extracted. When the shift is increased to a moderate level (cases 9 to 11), accurate solution can be obtained. However, case 10 gives two rigid body modes while all other cases give only one rigid body mode. When the shift is further increased (cases 12 to 17), rigid body modes become quit difficult to compute unless a

larger Ritz vector space is used. This is quite inefficient since the solution space becomes unnecessary large, not to improve the elastic representation, only to allow an accurate rigid body representation. When the shift is negative (cases 18 to 23), the algorithm has problem in recovering both the rigid body and elastic body modes.

While this approach is very effective for eigensolution pf structures with positive semi-definite stiffness matrix, it does not satisfy the requirements for a robust reduced vector basis method for dynamic response analysis. Other methods independent of the choice of an arbitrary constant is more desirable.

3.2.2 Retaining Same Number of Degrees of Freedom

One group of analysis methods to make the stiffness matrix positive definite is by augmenting the stiffness matrix. The shifted mass matrix approach as described in Subsection 3.3.1 is one of them. Other methods are discussed below.

1. Synthesis of full rank stiffness matrix by rigid body vectors

By the eigenvalue solution and orthogonality conditions,

$$\mathbf{K}\,\Phi = \mathbf{M}\,\Phi\,\Lambda \tag{3.2}$$

$$\Phi^{T}\,\mathbf{M}\,\Phi = \mathbf{I} \tag{3.3}$$

as shown in Appendix A that if M is full rank,

$$\mathbf{K} = \sum_{i=1}^{n_M} \omega_i^2\,(\,\mathbf{M}\,\phi_i\,)\,(\,\mathbf{M}\,\phi_i\,)^{T} \tag{3.4}$$

Table 3.1 Free-free Beam Example

Length = 10.0 inches

Section (bxd parallel to x,y respectively) = 0.5 in x 1.0 in

Weight Density = 0.1 lb/in^3

Young's Modulus = 100,000 psi

Poisson's Ratio = 0.3

Gravity = 386 in/sec^2

Number of nodes = 11

Number if elements = 10

Mode No	Freq (Hz)	Mode No	Freq (Hz)
1	0.0	15	1110.3
2	0.0	16	*1362.0
3	0.0	17	1452.4
4	0.0	18	1792.5
5	0.0	19	*1822.2
6	0.0	20	1932.5
7	97.6	21	2075.5
8	*193.0	22	*2241.1
9	260.6	23	2274.1
10	493.5	24	*2597.1
11	*500.6	25	2839.2
12	780.0	26	*2861.3
13	*908.2	27	*3025.8
14	978.3	28	3675.9

Note:

* Elastic body modes in y-z plane

Table 3.2 Frequencies in Shifted Subspace

CASE NO	SHIFT* (Hz)	TOL**	f_1 (Hz)	f_2 (Hz)	f_3 (Hz)	f_4 (Hz)	f_5 (Hz)	f_6 (Hz)	f_7 (Hz)	f_8 (Hz)
1	0.001	0.01	31.40	188.3						
2	0.01	0.01	0.00							
3	0.01	0.001	0.00							
4	0.01	0.0001	0.00							
5	0.1	0.01	0.00							
6	1.0	0.01	0.00							
7	10.0	0.01	0.00							
8	10.0	0.001	0.00	193.2						
9	100.0	0.01	0.00	193.0	507.7					
10	100.0	0.001	-0.01	0.03	193.0	500.6	909.2	1471.8		
11	350.0	0.01	-0.03	193.0	500.6	910.9	1507.9			
12	700.0	0.01	13.7	195.0	503.3	937.6	1661.4			
13	700.0	0.001	0.6	193.0	500.6	909.0	1392.5	2141.9		
14	1000.0	0.01	105.9	468.4	939.5	1744.0				
15	1000.0	0.001	38.5	205.5	505.2	922.2	1456.6	2273.6		
16	1000.0	0.0001	2.23	193.0	500.6	908.4	1367.5	1895.4	2586.0	
17	1000.0	0.00001	0.03	193.0	500.6	908.2	1362.1	1885.8	2292.8	2800.8
18	-1.0	0.001	0.00							
19	-10.0	0.001	0.00							
20	-100.0	0.001	0.08							
21	-350.0	0.001	2.46							
22	-700.0	0.01	3.78							
23	-700.0	0.001	3.78							

Notes :

* Actual shift in algorithm equals to $(2 \times \pi \times \text{shift})^2$ with sign as shown.

** TOL is the error measure defined in Equation (2.61).

If the structural system has n_z zero energy modes, there are n_z zero eigenvalues. Therefore, it is obvious that **K** is rank deficient by an order n_z. The vector $\mathbf{M}\phi$ is the zero strain energy force pattern associated with the zero strain energy mode. Since the natural frequency is zero, the contribution of this force into the stiffness matrix is lost as shown in Equation (3.4). One way to synthesize a full rank stiffness matrix is by providing stiffness (i.e. strain energy) to the zero strain energy modes, in the generalized sense, to render the stiffness matrix positive definite.

$$\mathbf{K_z} = \mathbf{K} + \sum_{i=1}^{n_z} \rho_i \, (\mathbf{M} \, \phi_i) \, (\mathbf{M} \, \phi_i)^T \tag{3.5}$$

It can be shown that when a rigid body force is applied to $\mathbf{K_z}$, it will only result in displacement of the "zero strain energy modes":

$$\mathbf{K_z} \, \mathbf{u} = \mathbf{M} \, \phi_i \, , \quad \text{for } i \leq n_z \tag{3.5}$$

By using the definition of $\mathbf{K_z}$, Equation (3.5), and the orthogonality conditions Equations (3.2) and (3.3), it can be shown that:

$$\mathbf{u} = \frac{1}{\rho_i} \, \phi_i \tag{3.6}$$

Therefore, the "zero strain energy component" of an applied force vector will produce "zero strain energy displacement vector" while the elastic body components will only produce elastic displacement. The best values for ρ_i are the values larger than the corresponding highest frequency of interest. The major disadvantage of this method, and indeed a serious one for large system, is $\left[\mathbf{M}\phi_i(\mathbf{M}\phi_i)^T\right]$ is in general a full matrix. Therefore $\mathbf{K_z}$ will be not be sparse and is bad news for matrix storage and factorization for static solution. This method can only be applied to smaller systems and not to any large finite element systems.

2. Addition of soft springs to the structure

Since a structure with positive semi-definite stiffness matrix is statically unstable, a minimum number of soft springs can be attached to the structure to make it statically determinate and stable[5]. The spring constants should be selected such that the pseudo-rigid body frequencies are, say two order of magnitude, smaller than the first elastic frequency of the structure. When these two groups of vibration modes are well separated, the interaction between these two groups is minimal. Hence the error introduced by the addition of soft springs is judged to be small. The pseudo-rigid body frequencies are then set to zero for analysis. This is a physical approach which may require some trial and error. The selection of degrees of freedom for spring attachment is arbitrary. The proper selection of spring constants is important since it affects the condition number of the resultant stiffness matrix. The use of this stiffness matrix will cause large pseudo-rigid body component in the displacement vector and diminishes the numerical quality of the elastic deformation component.

3.2.3 Reduction of Number of Degrees of Freedom

One group of methods to handle singular stiffness matrices is by reducing the total number of degrees of freedom by the zero strain energy modes and thereby render the structure stable.

1. Displacement Constraints

Displacement (physical) constraints can be applied to the structure with zero strain energy modes and this changes the structure to a static determinate one. This will reduce the total degrees of freedom by the number of rigid body modes. This effectively changes the structure to a grounded one. In general, when forces are applied to a grounded

structure, there will be reaction forces. Basically a vector basis of a different problem is computed. The solution to the original problem is approximated by including the zero strain energy modes into the vector basis. The advantage of this method is that the sparseness or the profile of the stiffness matrix will not be changed substantially. The selection of constrained degrees of freedom is arbitrary and should be supplied by the analyst. The dimensions of the stiffness matrix and Ritz vectors are not the same. The disadvantage is that the solution to the constrained problem actually may not be related to the original problem and may produce erroneous results.

2. Equilibrium Constraints

Equilibrium (generalized) constraints[5] can be applied to inertia load vector such that inertia forces will have no work done during a rigid body motion. By applying these constraints, a transformation matrix relating the dependent degrees of freedom to the independent degrees of freedom can be formulated. The reduced stiffness and mass matrices are formed by matrix triple product. The dimension of the matrices are reduced but the transformation introduces more coupling between degrees of freedom such that the initial sparseness or profile of the matrices are destroyed. Particularly, the diagonal mass matrix is now coupled. The total numerical operations and storage requirement may actual be increased by this procedure. Back transformation is also required to recover the full displacement vectors. This method therefore is not suitable for computation of large systems.

3.3 Self-equilibrating Force Method

The methods presented in Section 3.2 can accomplish the task of providing static vectors from a rank deficient stiffness matrix. However, they all lack of robustness and

cannot be used as a reliable method for all conditions. In view of these deficiencies, a more fundamental approach based on the behavior of structure and the properties of the stiffness matrix is developed[16] and is described in detail in the following subsections.

3.3.1 Zero Strain Energy (Null Space) Vector Basis

If a stiffness matrix, **K**, is positive semi-definite, then it is possible to have displacement vectors such that

$$\mathbf{K}\ \mathbf{u}_z = \mathbf{0} \tag{3.7}$$

If **K** has a null space of dimension n_z, it has n_z zero strain energy modes. Also, the structure can be made stable if n_z independent supports are provided. For analysis purpose, if the total degrees of freedoms are segregated into the elastic degrees of freedom, e, and the zero strain energy support degrees of freedom, z. The displacements at the z degrees of freedom are set to be unity,

$$\begin{bmatrix} \mathbf{K}_{ee} & \mathbf{K}_{ez} \\ \mathbf{K}_{ze} & \mathbf{K}_{zz} \end{bmatrix} \begin{bmatrix} \mathbf{U}_e \\ \mathbf{I}_z \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{3.8}$$

The zero strain energy displacement vectors, $\mathbf{U}_z$, are computed as :

$$\mathbf{U}_z = \begin{bmatrix} -\mathbf{K}_{ee}^{-1}\mathbf{K}_{ez} \\ \mathbf{I}_z \end{bmatrix} \tag{3.9}$$

$\mathbf{U}_z$ is mass orthonormalized to $\mathbf{X}_z$ by the Gram-Schmidt procedure. If the zero strain energy vectors, $\mathbf{U}_z$ consists of only rigid body modes, these vectors can be computed using the geometric coordinates of the structure only.

The mass orthonormalized vectors, $\mathbf{X}_z$, are the mass normalized eigenvectors because they satisfy both the mass and stiffness orthogonality conditions. The Ritz basis vectors, $\mathbf{X}_z$, and the eigenvectors, Φ_z, so computed are exact. The eigenvalues associated with $\mathbf{X}_z$

are identically zero or accurate to the precision of the machine. This provides a necessary and good check for the accuracy of the finite element model.

To each zero strain energy mode, there exists a force vector which excites the structure in this mode with no strain energy induced in the structure. The force vectors associated with these zero strain energy modes have the force distribution given by:

$$\mathbf{P}_z = \mathbf{M}\,\mathbf{X}_z \tag{3.10}$$

3.3.2 Elastic Vector Basis

Since the zero strain energy modes can be computed exactly, it is only necessary to compute a good vector basis for the elastic part of the response vector, $\mathbf{u}_e$. The force component associated with zero strain energy modes should also be removed from the load vector such that only the self-equilibrating elastic body force vector is used to extract elastic deformations.

If the self-equilibrating force is applied to the structure with rigid body and mechanism modes, the structure will be in neutral equilibrium. This is similar to applying a vertical force to a roller skate on a horizontal surface. The displacement solution is non-unique but the internal force distribution within the structure is unique. However, the stiffness matrix is singular and a formal non-unique solution cannot be computed. Based on this physical reasoning, a solution can be obtained by constraining (supporting) the structure at n_z degrees of freedom to render the structure statically determinate. Since the purpose of the LDRV method is to compute basis vectors for elastic deformation only, the non-unique solution is not a concern. When the static vector is used to compute a new basis vector, the Gram-Schmidt procedure removes its projections onto the zero energy modes. Therefore, a unique elastic displacement is extracted despite the arbitrariness of the selection of the constraining degrees of freedom and the non-uniqueness of the static solution.

A formal mathematical derivation of this approach is possible. The displacement of the structure is separated into the zero strain energy displacement and elastic body displacement,

$$\mathbf{u} = \mathbf{X}_z \mathbf{y}_z + \mathbf{u}_e \tag{3.11}$$

and also, by orthogonality of the zero strain energy vectors and elastic vectors,

$$\mathbf{X}_z^T \mathbf{M} \mathbf{u}_e = \mathbf{0} \tag{3.12}$$

$$\mathbf{y}_z = \mathbf{X}_z^T \mathbf{M} \mathbf{u} \tag{3.13}$$

Assuming that the damping forces associated with the zero strain energy modes are zero, then

$$\mathbf{M} \mathbf{X}_z \ddot{\mathbf{y}}_z + \mathbf{M} \ddot{\mathbf{u}}_e + \mathbf{C} \dot{\mathbf{u}}_e + \mathbf{K} \mathbf{u}_e = \mathbf{p}\, g(t) \tag{3.14}$$

By pre-multiplying Equation (3.14) by $\mathbf{X}_z^T$, imposing the orthogonality of $\mathbf{X}_z$ and $\mathbf{u}_e$ and assuming similar orthogonality condition for **C**, Equation (3.14) becomes :

$$\ddot{\mathbf{y}}_z = \mathbf{X}_z^T \mathbf{p}\, g(t) \tag{3.15}$$

Substitute Equation (3.15) back into Equation (3.14), then

$$\mathbf{M} \ddot{\mathbf{u}}_e + \mathbf{C} \dot{\mathbf{u}}_e + \mathbf{K} \mathbf{u}_e = (\mathbf{p} - \mathbf{M} \mathbf{X}_z \mathbf{X}_z^T \mathbf{p})\, g(t) \tag{3.16}$$

Define the spatial load vector on the right hand side of Equation (3.16) as the elastic load vector, $\mathbf{p}_e$, i.e.

$$\mathbf{p}_e = \mathbf{p} - \mathbf{M} \mathbf{X}_r \mathbf{X}_r^T \mathbf{p} \tag{3.17}$$

Equation (2.51) also gives the same result.

The stiffness matrix of Equation (3.16) is positive semi-definite and the flexibility matrix, or $\mathbf{K}^{-1}$, does not exist. Only non-unique solutions exist for the system, i.e.

$$\mathbf{K}\ (\mathbf{u}_e + \mathbf{u}_z) = \mathbf{p}_e = \begin{bmatrix} \mathbf{p}_{ee} \\ \mathbf{p}_{ez} \end{bmatrix} \tag{3.18}$$

The solutions to Equation (3.18) is non-unique and are within a rigid body transformation from each other. Zero displacements can be specified at the z degrees of freedom to compute the zero strain energy vectors. As such, a special solution to the Equation (3.18) is obtained.

$$\mathbf{u}_s = (\mathbf{u}_e + \mathbf{u}_z) = \begin{bmatrix} \mathbf{K}_{ee}^{-1}\,\mathbf{p}_{ee} \\ \mathbf{0} \end{bmatrix} \tag{3.19}$$

where $\mathbf{K}_{ee}$ is the stiffness submatrix of the e degrees of freedom.

The solution strategy to compute basis vectors for Equation (3.16) is therefore made identical to the structure with positive definite stiffness matrix. The major difference is that:

1. zero energy modes are computed initially,
2. orthonormalize these vectors and include in the vector basis
3. the starting load vector is made self-equilibrating via zero strain energy modes.

3.3.3 Selection of Constraining Degrees of Freedom

Theoretically, the selection of constraining degrees of freedom has no effect on quality of the Ritz vectors because of the self-equilibrating nature of the modified load vector. The only requirement is that the set of constraining degrees of freedom must render the structure statically determinate. Depending on the stiffness distribution of the structure, equation solver and machine precision, the effect on the quality of the numerical solution is normally small.

It is a good practice that the analyst has total control over the singularities of the finite element model. A good model should not contain extraneous singularities, e.g. the out-of-plane rotation degrees of freedom of flat plates. Some equation solvers in structural analysis codes have the capability to constrain singularities automatically. In this case, no specification of constraining degrees of freedom is required. If such a feature is used for convenience, the computer output should be carefully checked. This approach is particularly convenient if the structure does not have mechanism modes and the rigid body vectors are computed from the geometry.

When mechanism modes exist, boundary static vectors can be computed by specifying the constraining degrees of freedom to render the structure stable. The same factorized stiffness matrix can then be used to extract elastic vectors and thereby only one matrix factorization is required to compute both the zero strain energy and elastic basis vectors.

3.3.4 Convergence Criteria

By using the self-equilibrating force approach, a meaningful energy norm can be established for convergence criterion. The norm is based on the elastic strain energy only and it excludes any contribution due to rigid body components. The elastic force, $\mathbf{p_e}$, and the static displacement, $\mathbf{u_s}$, define the static strain energy in the structure.

$$\mathbf{p_e} = \mathbf{p} - \mathbf{M}\,\mathbf{X_r}\,\mathbf{X_r}^T\,\mathbf{p} \tag{3.17}$$

$$\mathbf{u_s} = (\mathbf{u_e} + \mathbf{u_z}) = \begin{bmatrix} \mathbf{K_{ee}}^{-1}\,\mathbf{p_{ee}} \\ \mathbf{0} \end{bmatrix} \tag{3.18}$$

The residual static displacement vector is computed at each step and an energy norm check is performed:

$$\mathbf{u^r_j} = \mathbf{u^r_{j-1}} - \mathbf{x_j}\,\mathbf{x_j}^T\,\mathbf{M}\,\mathbf{u^r_{j-1}} \tag{3.20}$$

$$\varepsilon_j = \frac{\mathbf{p_e}^T \mathbf{u}^r_j}{\mathbf{p_e}^T \mathbf{u}_s} \tag{3.21}$$

Implementation of this norm is very easy since it is compatible with the generation sequence of the basis vectors.

3.4 The Algorithm

A new algorithm was developed to extract basis vectors of structures with positive semi-definite stiffness matrices. The algorithm is shown in Table 3.3. This algorithm is based on the algorithm presented in Table 2.1. It provides a robust method for dynamic modal analysis of space structures. An initial set of Ritz vectors is allowed such that the zero strain energy vectors or previously computed Ritz vectors can be incorporated into the vector basis. The zero strain energy vectors are computed separately by static condensation or from the geometric coordinates. Only the elastic component of the load vector is used. The algorithm is also capable of handling multiple load vectors by using the vectors from previous load vectors as initial Ritz vectors. The algorithm has been implemented in EAL[17] computer code using its arithmetic and matrix utilities.

The algorithm as shown in Table 3.3 computes basis vectors for a force vector at a time.. However, if the algorithm is coded in a vector block form, the computer I/O time can be reduced[4]. A loop to check for the highest frequency cutoff can also be incorporated if needed. Different convergence criteria can also be implemented very easily.

Table 3.3

Reduced Vector Basis Algorithm for Structures with Positive Semi-definite Stiffness Matrices

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Equation of Motion $\mathbf{M}\ddot{u} + \mathbf{K}\,u = P(s)\,G(t)$

1. Given mass matrix **M** (nxn), stiffness matrix **K** (nxn) and load vector $\mathbf{P} = [\,\mathbf{p}_k\,]$ (nx$l$)
2. Set default parameters :
   number of rigid body modes = nr
   max number of vectors per load vector = nv
   tolerance for convergence = tol
   tolerance for linear dependence = dtol
3. Input initial set of Ritz vectors $\mathbf{R} = [\,\mathbf{r}_i\,]$ (nxq), if **R** exists
4. If **R** exists, perform mass orthonormalization on $\mathbf{r}_i$, for i=1, q
   a. Initialization
      $a_1 = \mathbf{r}_1^T \mathbf{M}\,\mathbf{r}_1$
      $\mathbf{x}_1 = \mathbf{r}_1 * a_1^{-0.5}$
   b. Mass orthogonalization
      $\mathbf{x}_i^* = \mathbf{r}_i - \Sigma\,\mathbf{r}_j^T\,(\mathbf{M}\,\mathbf{x}_i)\,\mathbf{x}_j \quad j=1, i-1$
   c. Normalization
      $a_i = \mathbf{x}_i^{*T}\,\mathbf{M}\,\mathbf{x}_i^*$
      $\mathbf{x}_i = \mathbf{x}_i^*\ a_i^{-0.5}$
5. Factorize stiffness matrix $\mathbf{K} = \mathbf{L}\,\mathbf{D}\,\mathbf{L}^T$
6. Perform steps 7 through 11 for each load pattern k=1, l
7. Remove rigid body force components from force vector $\mathbf{p}_k$.
   $\mathbf{p}_k^* = \mathbf{p}_k - \Sigma\,(\mathbf{M}\,\mathbf{x}_j)\,(\mathbf{x}_j^T\,\mathbf{p}_k) \quad j=1, nr$
   $\mathbf{p}_k(0) = \mathbf{p}_k^*$
8. Solve for static vector $\mathbf{u}_k(0)$
   $\mathbf{L}\,\mathbf{D}\,\mathbf{L}^T\,\mathbf{u}_k(0) = \mathbf{p}_k(0)$
9. Calculate energy norm $= \mathbf{p}_k^T(0)\,\mathbf{u}_k(0)$
10. Mass orthogonalize $\mathbf{u}_k(0)$ to $\mathbf{x}_j$
    $\mathbf{u}_k(q) = \mathbf{u}_k(0) - \Sigma\,\mathbf{u}_k(0)^T\,(\mathbf{M}\,\mathbf{x}_j)\,\mathbf{x}_j \quad j=1, q$

11. Solve for additional vectors i=1, nv
    a. Solve for new vector

       $q = q + 1$

       $a_i = (\mathbf{M}\, \mathbf{u}_k(i-1))^T (\mathbf{M}\, \mathbf{u}_k(i-1))$

       $z_i = \mathbf{M}\, \mathbf{u}_k(i-1) * a_i^{-0.5}$

       $\mathbf{L}\, \mathbf{D}\, \mathbf{L}^T\, x_q^* = z_i$

    b. Mass orthogonalization

       $x_q^{**} = x_q^* - \Sigma\, x_q^{*T} (\mathbf{M}\, x_j)\, x_j \quad j=1, q-1$

    c. Normalization

       $a_q = x_q^{**T}\, \mathbf{M}\, x_q^{**}$

       $x_q = x_q^{**} * a_q^{-0.5}$

    d. Check linear dependence of Ritz vectors

       $d_i = x_q^T\, \mathbf{M}\, x_q - 1.0$

       If $|d_i| >$ dtol, go to step 6.

    e. Mass orthogonalize static vector

       $\mathbf{u}(i) = \mathbf{u}(i-1) - \mathbf{u}(i-1)^T (\mathbf{M}\, x_q)\, x_q$

    f. Calculate residual energy norm

       $e_i = p_k^T(0)\, \mathbf{u}(i) \,/\, p_k^T(0)\, \mathbf{u}(0)$

    g. If $e_i >$ tol, go to a.

    h. Orthonormalize $\mathbf{u}(i)$ and include in the vector basis. $q = q + 1$

12. Stiffness orthogonalization by Rayleigh-Ritz Method to complete selective modal extraction
    a. Orthonormalization of Ritz Vectors with respect to stiffness matrix

       Solve standard eigenvalue problem:

       $[\, \mathbf{A} - \omega_i^2\, \mathbf{I}\,]\, y_i = \mathbf{0}$

       where

       $\mathbf{A} = \mathbf{X}^T\, \mathbf{K}\, \mathbf{X}$

       $\mathbf{I} = \mathbf{X}^T\, \mathbf{M}\, \mathbf{X}$

       ω_i = approximate frequencies

    b. Compute selective modes by back transformation

       $^o\mathbf{X} = \mathbf{X}\, \mathbf{Y}$

3.5 Graphic Representation of the Self-equilibrating Force Method

The free-free beam example in Tables 3.1 and 3.2 is used as an example to demonstrate the solution procedure graphically to provide a physical interpretation of the algorithm. Figure 3.1 shows all the rigid body vectors associated with the free-free beam which are computed based on the nodal coordinates. Figure 3.2 depicts how rigid body forces are removed from the applied force vector to form the self-equilibrating force vector. Figure 3.3 shows the constrained static solution, the newly generated mass orthonormalized basis vectors and the residual root vector. The data for Figures 3.2 and 3.3 are provided in Table 3.4.

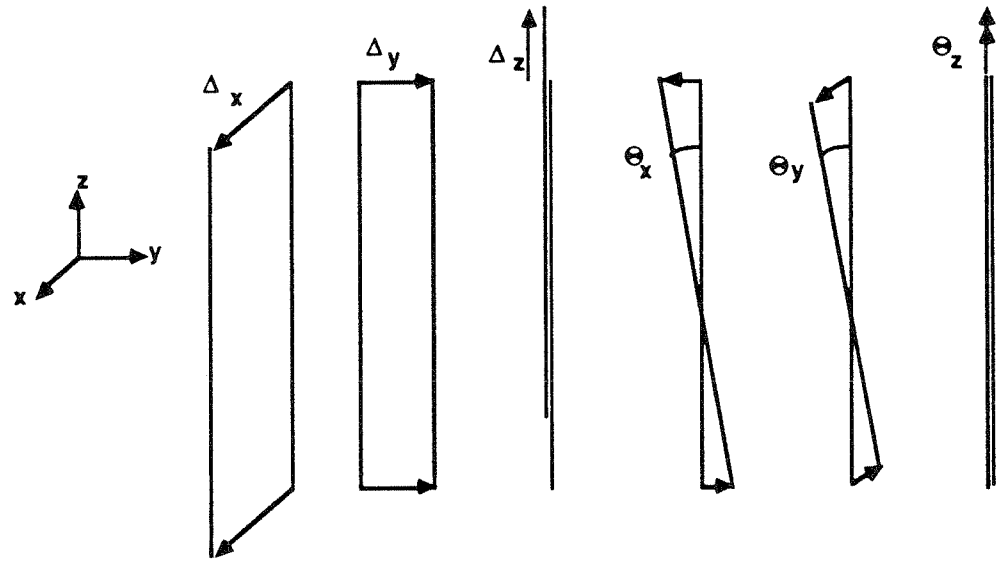

Figure 3.1 Rigid Body Vectors

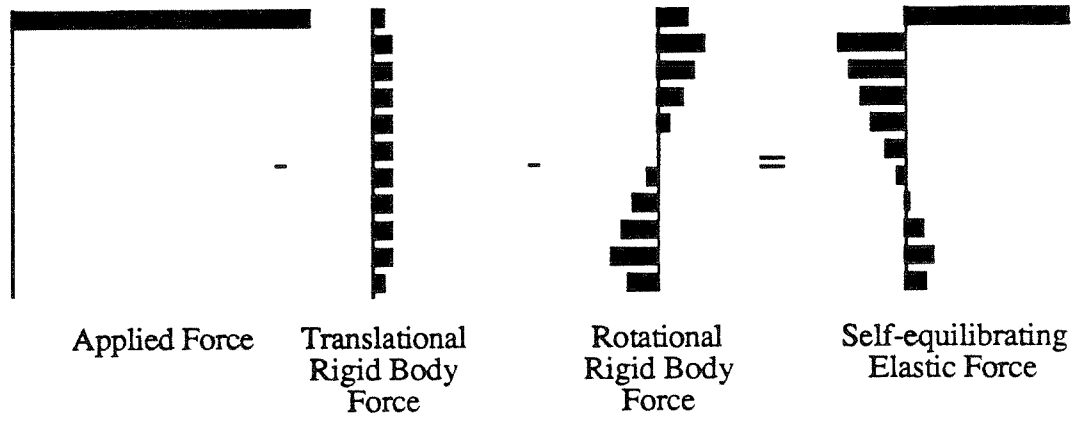

Figure 3.2 Self-equilibrating Elastic Force Vector

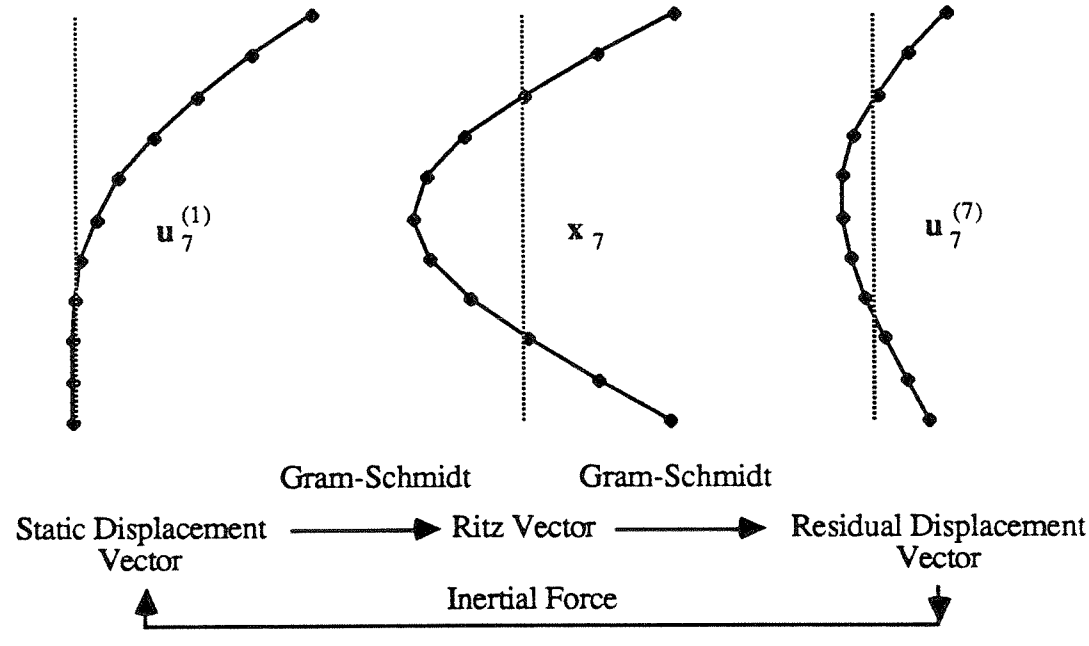

Figure 3.3 Ritz Vector Generating Sequence

Table 3.4 Force and Displacement Vectors of Example

Node No.	Force Vector	R. B. Force 1	R. B. Force 2	Elastic Force	Root Displ	First Vector	Root Vector
11	1000	50	147	803	8.04	54.5	2.3
10	0	100	235	-335	6.04	28.8	1.05
9	0	100	176	-276	4.24	4.35	0.414
8	0	100	118	-218	2.75	-15.9	-0.74
7	0	100	59	-159	1.62	-29	-1.12
6	0	100	0	-100	0.84	-33.2	-1.16
5	0	100	-59	-41	0.359	-27.9	-0.89
4	0	100	-118	18	0.111	-14.4	-0.4
3	0	100	-176	76	0.016	5.31	0.26
2	0	100	-235	135	0	28.4	0.985
1	0	50	-147	97	0	52.4	1.73

3.6 Examples

The same free-free beam problem was analyzed using the improved LDRV method. A summary of the analysis is listed in Table 3.5. Since the applied load can activate only two rigid body modes in the y-z plane, only two rigid body vectors are needed as initial vectors. However, all six rigid body modes are used in the example as a test case. For all positive shifts and negative shifts below the first natural frequency, the solutions are very accurate and stable. All difficulties associated with the previous analysis as shown in Table 3.2 are circumvented. The last frequency computed seems to carry a slight inaccuracy. If higher accuracy is required, the convergence tolerance can be reset. When the negative shift is beyond the first natural frequency, the solution of elastic modes start to show problem.

The shifting of stiffness matrix will tend to bias the displacement vector to the mode shape of the closest eigenvalue. This duplicates the zero frequency rigid body mode problem. The root vector will be overwhelmed by the eigenvector of the nearest eigenvalue in the same way as the rigid body modes dominate over the elastic modes at small shift values. This problem is solved in the case of zero frequency by using the *a priori* knowledge of the rigid body modes. However, the same information is not available at the shift point. The results indicate that it is best to use no shift or a small shift.

Table 3.5 Free-free Beam by Improved LDRV Algorithm

Case No.	Shift* (Hz)	Converg. Tolerance	f_7** (Hz)	f_8 (Hz)	f_9 (Hz)	f_{10} (Hz)	f_{11} (Hz)
Exact			193.0	500.6	908.2	1362.0	1822.2
1	0	0.01	193.0	500.6	942.7		
2	0	0.001	193.0	500.6	908.2	1367.7	2017.6
3	0.001	0.01	193.0	500.6	942.7		
4	0.1	0.01	193.0	500.6	942.7		
5	700.0	0.01	193.0	501.0	925.2	1618.0	
6	1000.0	0.01	195.1	515.0	989.7	1805.7	
7	1000.0	0.001	193.0	500.9	913.1	1427.9	2229.9
8	-1.0	0.01	193.0	500.6	942.7		
9	-1.0	0.001	193.0	500.6	908.2	1367.7	2017.6
10	-10.0	0.01	193.0	500.6	942.7		
11	-100.0	0.01	193.0	500.6	940.9		
12	-350.0	0.01	262.6				
13	-350.0	0.001	262.6				
14	-350.0	0.0001	262.6				
15	-700.0	0.01	578.3				
16	-700.0	0.001	578.3				
17	-1000.0	0.0001	442.7	1362.3			

Notes :

* Actual shift in algorithm equals to $(2 \times \pi \times \text{shift})^2$ with sign as shown.

** The first 6 modes are rigid body modes with 0.00 Hertz frequencies.

A free-free subassembly of a spacecraft is used to demonstrate the performance of algorithm to reduce the order of a large problem. The model contains flexible solar panels of high modal density at low frequencies. It consists of 4136 degrees of freedom. Four methods were used:

1. the LDRV method,
2. the eigensolution (EV) method (Lanczos eigensolver),
3. 96 boundary vectors (BV) and constrained LDRV, and
4. 96 boundary vectors and constrained normal modes (Lanczos solver).

In applying the LDRV method, a total of 15 load cases was used to excite the force and sensor locations. The results are summarized in Table 3.6.

Table 3.6

Comparison of Different Methods

Method	CPU (Cray-sec)	Cutoff Criterion	No. of Vectors	Highest Freq.(Hz)
LDRV	160	tol=0.01	88	446.
EV	335	100 Hz	386	99.4
BV+LDRV	134	tol=0.01	179	283.
BV+EV	487	150 Hz	560	1488.

The Lanczos eigensolver used in this example is very efficient and is coded in FORTRAN. It extracts all the modes below the specified frequency cutoff. The LDRV method used is coded in EAL (Engineering Analysis Language[17]) arithmetic and matrix utilities. It can be seen that the LDRV method performs very well in both CPU and mass

storage. Performance can be improved if the algorithm is optimized and the coding is written in FORTRAN or other computer languages.

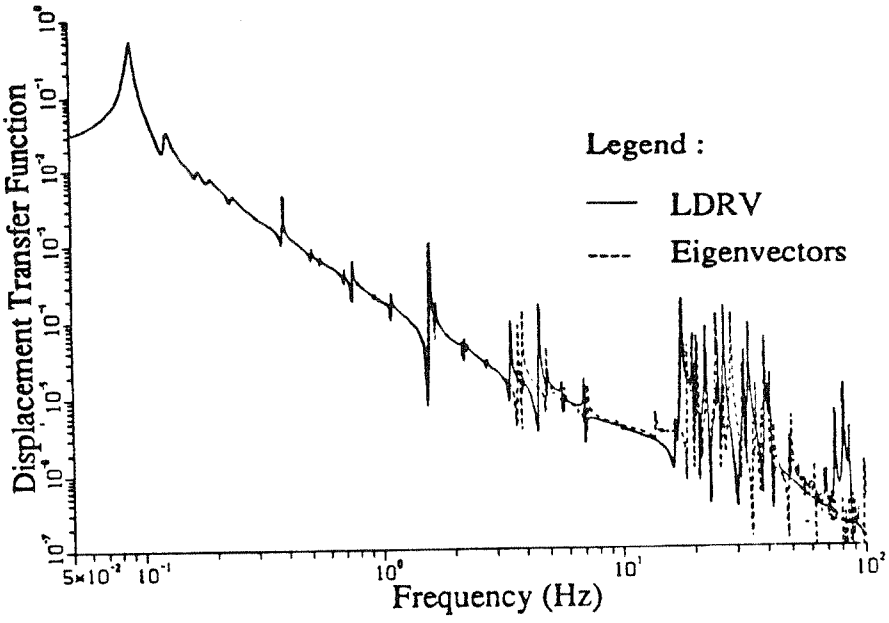

Figure 3.4. Comparison of Transfer Functions

To compare the accuracy of the solution, the transfer functions at a sensor location using the LDRV and eigenvectors are shown in Figure 3.4. It is interesting to note that practically all the peaks at lower frequencies are computed accurately. The comparison deteriorates in amplitude and frequency only at higher frequencies where the amplitude is lower by three to four orders of magnitude. The accuracy of these higher frequency modes is not important to the response. The model also does not have the fidelity to capture the mode shapes of higher frequency solar panel modes. If a transient response analysis is performed using these selectively extracted modes, the response can be shown to be practically identical.

3.7 Applicability to Normal Mode Method

The method developed in this chapter is equally applicable to dynamic response analysis of structures with semi-definite stiffness matrices by the normal mode method. The self-equilibrating force approach and statically constrained structure can used effectively to computed the elastic static displacement vector for static correction[18,19].

$$\mathbf{p}_e = \mathbf{p} - \mathbf{M}\,\mathbf{X}_r\,\mathbf{X}_r^T\,\mathbf{p} \tag{3.17}$$

$$\mathbf{u}_s = (\mathbf{u}_e + \mathbf{u}_z) = \begin{bmatrix} \mathbf{K}_{ee}^{-1}\,\mathbf{p}_{ee} \\ \mathbf{0} \end{bmatrix} \tag{3.19}$$

Projections of this vector onto the retained normal modes are removed from the vector.

$$\mathbf{u}_r = \mathbf{u}_s - \Phi_m\,(\Phi_m^T \mathbf{M}\,\mathbf{u}_s) \tag{3.22}$$

The mass orthonormalized static correction vector can then be computed as:

$$\phi_{m+1} = \frac{\mathbf{u}_r}{(\mathbf{u}_r^T\,\mathbf{M}\,\mathbf{u}_r)^{0.5}} \tag{3.23}$$

The associated Rayleigh-Ritz frequency is:

$$\omega_{m+1} = \sqrt{\phi_{m+1}^T\,\mathbf{K}\,\phi_{m+1}} \tag{3.24}$$

This static correction is used to complete the vector space for accurate static solution at low excitation frequency. The same concept can also be extended to Lanczos method for extraction of eigensolution with multiple roots at zero.

3.8 Remark

In general, this method can be used to compute dynamic responses of structures with positive semi-definite stiffness matrices. Therefore, in the later chapters, only positive definite stiffness matrices are discussed.

Chapter 4

Undamped Vector Basis for Dynamic Response Analysis

The success of load dependent Ritz vector concept motivates a more fundamental study of vector basis of undamped structures. In search of a more general concept in vector basis representation for dynamic response solution, the steady state harmonic solution provides an enlightening physical, mathematical and numerical interpretation of the vector basis concept. The dynamic steady state solution is represented exactly, in closed form, by one single vector. The solution process is also quite trivial. As a matter of fact, it basically solves the linear algebraic equation. This is identical to a static solution. The harmonic analysis technique also has the distinct advantage that it can be viewed as both the frequency domain analysis (transfer functions) and the time domain analysis (Fourier series). Another advantage of the frequency domain solution is that some materials are characterized in the frequency domain and others have frequency independent hysteretic damping. This concept is explored in this chapter to search for a more general and flexible method to generate reduced vector basis. The steady state displacement vectors from an undamped structure are used as the vector basis to evaluate the dynamic responses of the damped structure. The concept developed in this chapter is used in the next chapter to generate damped vector basis for the dynamic responses of damped structures.

4.1 Steady State Solution for Linear Dynamic System

The viscously damped structure subject to a harmonically varying force, $\mathbf{p}\, e^{i\omega t}$, is governed by the differential equation:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{p}\, e^{i\omega t} \tag{4.1}$$

For the undamped system, $\mathbf{C} = \mathbf{0}$, the governing equation of motion for the harmonic force is:

$$\mathbf{M}\,\ddot{\mathbf{u}} + \mathbf{K}\,\mathbf{u} = \mathbf{p}\,e^{i\omega t} \tag{4.2}$$

If the forcing frequency is not at any of a undamped natural frequency, then a unique solution exists. Assume the solution has the form of:

$$\mathbf{u}(t) = \mathbf{u}^{R}\,e^{i\omega t} \tag{4.3}$$

The real vector, $\mathbf{u}^{R}$, represents the magnitude of the harmonic response at each degree of freedom. Since there is no damping, the displacement solution is in phase with the force. A direct and exact solution can be obtained by solving the linear algebraic equation:

$$(\mathbf{K} - \omega^2\,\mathbf{M})\;\mathbf{u}^{R} = \mathbf{p} \tag{4.4}$$

Therefore the solution to Equation (4.2) is:

$$\mathbf{u}(t) = \left[(\mathbf{K} - \omega^2\,\mathbf{M})^{-1}\;\mathbf{p}\right] e^{i\omega t} \tag{4.5}$$

A study of the steady state response of a dynamic system due to harmonic loading is very important in the understanding dynamic behavior of structural systems. There are three reasons here to study this special forcing function in detail:

1. Only one vector, $(\mathbf{K}-\omega^2\mathbf{M})^{-1}\mathbf{p}$, is required to represent the exact solution of an undamped system while the corresponding normal mode solution provides an approximate solution only.
2. The behavior of structural system under harmonic loading can be solved by linear matrix equations, $(\mathbf{K}-\omega^2\mathbf{M})^{-1}\mathbf{p}$, rather than matrix differential equation. This offers an extremely important computational advantage.

3. The harmonic solution is fundamental to the dynamic response due to arbitrary loading, i.e. Equation (1.1), in both the frequency domain solution and time domain. In the frequency domain, by varying the forcing frequency, ω, the dynamic response, $(\mathbf{K} - \omega^2\mathbf{M})^{-1}\mathbf{p}$, is actually the transfer functions of the structural system, i.e.

$$\mathbf{U}(\omega) = \left[(\mathbf{K} - \omega^2\mathbf{M})^{-1}\mathbf{p}\right] G(\omega) \tag{4.6}$$

The vector transfer function completely describes the behavior of the system by characterizing the input (excitation) and output (response) relationship of the structural system. Solution to any type of forcing function (transient, random etc.) can be obtained from the transfer functions. Similarly, from the time domain point of view, if the forcing function is decomposed into a Fourier series (say N points consistent with the Fast Fourier Transform algorithm), i.e.

$$g(t) = \sum_{j=0}^{N-1} c_j \, e^{i\,(j\Delta\omega)t} \tag{4.7}$$

then the time domain solution is:

$$\mathbf{u}(t) = \sum_{j=0}^{N-1} \left[(\mathbf{K} - (j\Delta\omega)^2\mathbf{M})^{-1}\mathbf{p}\right] c_j \, e^{i\,(j\Delta\omega)t} \tag{4.8}$$

4. If materials in the structure has frequency dependent material properties, and frequency dependent or constant hysteretic damping, the governing equation in the frequency domain is given by:

$$\{[-\omega^2\mathbf{M} + \mathbf{K}^R(\omega)] + i\,[\,\omega\,\mathbf{C} + \mathbf{K}^I(\omega)\,]\}\,\mathbf{U}(\omega) = \mathbf{p}\,G(\omega) \tag{1.3}$$

The only good way to solve for the dynamic response is through harmonic solution (frequency domain solution).

4.2 Undamped Vector Basis by Steady Steady Solution

Different solution methods based on the response of an undamped structure subject to a harmonically varying force at a given frequency are analyzed in the following subsections. Clear physical and mathematical interpretations of these methods are developed which are essential to develop the reduced vector basis for more complex problems discussed in the later chapters. The study of these basic solution techniques provides insight into understanding the role of basis vectors in dynamic response and developing the best numerical solution to the problem.

4.2.1 Direct Solution

A unique and exact solution can be obtained if the given forcing frequency is not at any of the undamped natural frequencies:

$$\mathbf{u}^R = (\mathbf{K} - \omega^2 \mathbf{M})^{-1} \mathbf{p} \tag{4.4}$$

and the solution $\mathbf{u}^R$ is in phase with the forcing vector. It is important to note that from the solution vector space point of view, *only one vector* is necessary to span the exact solution. The knowledge of natural frequencies and mode shapes are irrelevant in computing the exact solution. Therefore no frequency and mode shape solution is required. The exact solution is computed very efficiently. Different input frequencies extract different response displacement shapes (basis vectors). It should also be noted that these shapes have the properties of mode shapes since the shapes remain the same during the dynamic event and only the amplitude varies with time. The phase of the velocity vector is 90° ahead of the displacement vector:

$$\dot{\mathbf{u}} = i\omega \, \mathbf{u}^R e^{i\omega t} \tag{4.9}$$

and the acceleration vector is of opposite phase to the displacement vector:

$$\ddot{\mathbf{u}} = -\omega^2 \mathbf{u}^R e^{i\omega t} \tag{4.10}$$

Even though the solution is exact for a given frequency, a linear solution is required for each new frequency.

Even though the exact solution to Equation (4.2) for a given frequency is obtained by a single vector representation, a single vector cannot provide an accurate solution for a more general forcing function. A series of displacement vectors at different forcing frequencies are required to form a vector basis for the dynamic solution. Then the solution will be dependent on the choice of forcing frequencies and also linear solutions for each frequency must be performed. This is not a preferable procedure.

4.2.2 Normal Mode Method

It is common to solve a dynamic problem by normal mode superposition. Of course, this will require a solution to the undamped eigenvalue problem and also a truncation criterion must be established.

$$\mathbf{u} \approx \Phi_m \mathbf{q} \tag{4.11}$$

$$[\Lambda_m - \omega^2 \mathbf{I}_m] \mathbf{q} = \Phi_m{}^T \mathbf{p} \tag{4.12}$$

$$q_i = \frac{\phi_i{}^T \mathbf{p}}{\omega_i{}^2 - \omega^2} \tag{4.13}$$

$$\mathbf{u} \approx \sum_{i=1}^{m} \phi_i \frac{\phi_i{}^T \mathbf{p}}{\omega_i{}^2 - \omega^2} \tag{4.14}$$

The advantage of this method over the direct solution is that dynamic response for different input frequencies can be obtained relatively easily from the uncoupled modal

equations. Unlike the direct solution, the solution so obtained contains error due to modal truncation. This error can be approximated by finding a vector to complete the vector span of the static solution. Projections of this static vector onto the eigenvectors can then be removed from the static vector, i.e.

$$\mathbf{u_s} = \mathbf{K^{-1}\, p} \tag{4.15}$$

$$\mathbf{u_r} = \mathbf{u_s} - \mathbf{\Phi_m \Phi_m^T M\, u_s} \tag{4.16}$$

Hence the residual vector is made orthogonal to the m eigenvectors. Furthermore, when this residual vector is mass normalized, it will behavior like an ordinary eigenvector with the frequency characterized by the Rayleigh's approximation.

$$\phi^*_{m+1} = \frac{\mathbf{u_r}}{\mathbf{u_r^T M\, u_r}} \tag{4.17}$$

$$\omega^*_{m+1} = \sqrt{\phi_{m+1}^T \mathbf{K}\, \phi_{m+1}} \tag{4.18}$$

The solution with static correction vector is given in the frequency domain by:

$$\mathbf{u} \approx \sum_{i=1}^{m} \phi_i \frac{\phi_i^T \mathbf{p}\, G(\omega)}{\omega_i^2 - \omega^2} + \phi^*_{m+1} \frac{\phi^*_{m+1}{}^T \mathbf{p}\, G(\omega)}{\omega^*_{m+1}{}^2 - \omega^2} \tag{4.19}$$

If the forcing frequency is known, it is more appropriate to compute:

$$\mathbf{u_\omega} = \mathbf{(K - \omega^2 M)^{-1}\, p} \tag{4.4}$$

and proceed to find the m+1 correction mode. This completes the span of the steady state solution at the given frequency. However, this requires the knowledge of the solution to the problem which is $\mathbf{(K - \omega^2 M)^{-1}\, p}$ and it is the also exact solution. The intent of the correction mode is to approximate the modal truncation error. In essence, rather than static correction, a *harmonic correction* is computed. Therefore, for the analysis of an undamped system with only one harmonic forcing frequency, there is really no justification

for computing the solution in the modal domain since the approximate solution in the modal space is numerically more intensive than the exact solution. This provides an excellent example that modal analysis should not be used indiscriminately to solve problems in structural dynamics. The eigenvector basis, even with modal truncation correction, is therefore not the best vector basis because of numerically intensive computation and relatively large vector space. It may only be justified when a transfer function or steady state response over a wide frequency band is required.

4.2.3 Iterative Method (Stodola Type Iteration)

If Equation (4.4) is solved iteratively, more vectors can be generated with only one matrix factorization. The superscript of $\mathbf{u}^R$ will be dropped for clarity and Equation (4.4) is rewritten for the iterative solution as:

$$\mathbf{K}\,\mathbf{u} = \mathbf{p} + \omega^2\,\mathbf{M}\,\mathbf{u} \tag{4.20}$$

Since the unknown displacement vector also appears on the right hand side of Equation (4.20), the iterative solution scheme is written as:

$$\mathbf{K}\,\mathbf{u}^{(k+1)} = \mathbf{p} + \omega^2\,\mathbf{M}\,\mathbf{u}^{(k)} \tag{4.21}$$

The solution is contributed by two parts, the static part, $\mathbf{K}^{-1}\mathbf{p}$, and the inertial part, $\left[\omega^2\mathbf{K}^{-1}\mathbf{M}\mathbf{u}\right]$. The static part of the solution remained unchanged in each iteration. The iteration is used only to obtain a better approximation of the inertial (i.e. dynamic) effect. Now assume that the starting condition is given by:

$$\mathbf{u}^{(0)} = \mathbf{0} \tag{4.22}$$

$$\mathbf{K}\,\mathbf{u}^{(1)} = \mathbf{p} \tag{4.23}$$

Therefore the solution is given by:

$$\mathbf{u}^{(2)} = \mathbf{K}^{-1}\mathbf{p} + \omega^2 (\mathbf{K}^{-1}\mathbf{M})(\mathbf{K}^{-1}\mathbf{p}) \tag{4.24}$$

$$\mathbf{u}^{(k+1)} = \sum_{i=0}^{k} \omega^{2i} (\mathbf{K}^{-1}\mathbf{M})^i (\mathbf{K}^{-1}\mathbf{p}) \tag{4.25}$$

It is obvious that if Equation (4.25) converges, the solution vector span is given by $[\mathbf{K}^{-1}\mathbf{p},\ (\mathbf{K}^{-1}\mathbf{M})(\mathbf{K}^{-1}\mathbf{p}),\ (\mathbf{K}^{-1}\mathbf{M})^2(\mathbf{K}^{-1}\mathbf{p}),\ (\mathbf{K}^{-1}\mathbf{M})^3(\mathbf{K}^{-1}\mathbf{p}),\ldots,(\mathbf{K}^{-1}\mathbf{M})^k(\mathbf{K}^{-1}\mathbf{p})]$. The span so computed is independent of the forcing frequency, ω. Therefore, this can form a good vector basis for dynamic analysis. This vector subspace is known in mathematics as the Krylov subspace[12].

The Cayley-Hamilton theorem[13] of matrix theory proves that every square matrix, $\mathbf{A}$, satisfies its own characteristic equation and there is a minimum k such that the vectors $\mathbf{A}^i\mathbf{u}$ are dependent, i.e.

$$\left[\sum_{i=0}^{k} a_i \mathbf{A}^i\right]\mathbf{u} = \mathbf{0}, \quad k \le n \tag{4.26}$$

This theorem states that the n-th power of any square matrix $\mathbf{A}$ can be expressed as a linear combination of lower powers of $\mathbf{A}$. Consequently, $(\mathbf{K}^{-1}\mathbf{M})^i(\mathbf{K}^{-1}\mathbf{p})$ are not all independent if $i \ge n_M$ The maximum number of independent vectors from the Krylov subspace is n_M. If $\mathbf{K}^{-1}\mathbf{p}$ has zero projections in the range of $\mathbf{M}$, the number of independent vectors is reduced.

The following illustrates this process for a single degree of freedom system with natural frequency, ω_1:

$$u^{(1)} = \frac{p}{k} \tag{4.27}$$

$$u^{(2)} = \frac{p}{k} + \frac{\omega^2}{\omega_1^2}\frac{p}{k} \tag{4.28}$$

$$u^{(3)} = \frac{p}{k}\left(1 + \frac{\omega^2}{\omega_1^2} + \frac{\omega^4}{\omega_1^4}\right) \tag{4.29}$$

This sequence forms a geometric series and it converges to:

$$u^{(\infty)} = \frac{p}{k\left(1 - \frac{\omega^2}{\omega_1^2}\right)} \tag{4.30}$$

The process is obviously not efficient for a single degree of freedom. When this method is applied to a multiple degrees of freedom system, a set of basis vectors which contains the modes participating in the dynamic response can be found.

Equation (4.20) indicates that the solution process starts from a static solution. By adding $\left[-\omega_o^2 \mathbf{M}\mathbf{u}\right]$ to both side of Equation (4.20), i.e.

$$(\mathbf{K} - \omega_o^2\,\mathbf{M})\,\mathbf{u} = \mathbf{p} + (\omega^2 - \omega_o^2)\,\mathbf{M}\,\mathbf{u} \tag{4.31}$$

$$\mathbf{u} = (\mathbf{K} - \omega_o^2\,\mathbf{M})^{-1}\,\mathbf{p} + (\omega^2 - \omega_o^2)\,(\mathbf{K} - \omega_o^2\,\mathbf{M})^{-1}\,\mathbf{M}\,\mathbf{u} \tag{4.32}$$

a starting harmonic solution is possible. Consequently the harmonic correction vector is made part of the basis. A similar iterative scheme can be set up to generate basis vectors using this frequency shift approach:

$$\mathbf{u}^{(k+1)} = (\mathbf{K} - \omega_o^2\,\mathbf{M})^{-1}\,\mathbf{p} + (\omega^2 - \omega_o^2)\,(\mathbf{K} - \omega_o^2\,\mathbf{M})^{-1}\,\mathbf{M}\,\mathbf{u}^{(k)} \tag{4.33}$$

Analyze this iterative scheme in the eigenspace by using the orthogonality condition given in Appendix A,

$$\mathbf{u}^{(k+1)} = \sum_{i=1}^{n_M} \phi_i \frac{\phi_i^T \mathbf{p}}{(\omega_i^2 - \omega_o^2)} + \sum_{i=1}^{n_M} \phi_i \frac{(\omega^2 - \omega_o^2)}{(\omega_i^2 - \omega_o^2)} \phi_i^T \mathbf{M}\,\mathbf{u}^{(k)}) \tag{4.34}$$

It can be seen that the vectors are scaled by $\frac{1}{(\omega_i^2 - \omega_o^2)}$, and the eigenvector closest to the shift frequency has the strongest convergence property. If static convergence is necessary,

an additional static solution is required.. A direct physical interpretation of this approach is not very clear or obvious.

4.2.4 Iterative Incremental Method

To improve the information obtained in each iteration, only the incremental improvement to the solution is computed. If the solution method is conceived as successive improvement of solution by adding corrections to the static solution, then

$$\mathbf{u} = \sum_{i=1}^{m} \Delta \mathbf{u}^{(i)} \tag{4.35}$$

and the incremental displacement vector is defined as:

$$\Delta \mathbf{u}^{(k+1)} = \mathbf{u}^{(k+1)} - \mathbf{u}^{(k)} \tag{4.36}$$

The iterative incremental solution can be derived by considering:

$$\mathbf{K}\, \mathbf{u}^{(k+1)} = \mathbf{p} + \omega^2 \mathbf{M}\, \mathbf{u}^{(k)} \tag{4.21}$$

and therefore:

$$\mathbf{K}\, \Delta \mathbf{u}^{(k+1)} = \omega^2 \mathbf{M}\, \Delta \mathbf{u}^{(k)} \tag{4.37}$$

The starting condition of the iterative method is:

$$\mathbf{u}^{(0)} = \mathbf{0} \tag{4.22}$$

and the solution to Equation (4.37) is given by:

$$\Delta \mathbf{u}^{(1)} = \mathbf{K}^{-1}\mathbf{p} \tag{4.38}$$

$$\Delta \mathbf{u}^{(2)} = \omega^2 (\mathbf{K}^{-1}\mathbf{M})\ (\mathbf{K}^{-1}\mathbf{p}) \tag{4.39}$$

$$\Delta \mathbf{u}^{(k)} = \omega^{2(k-1)} (\mathbf{K}^{-1}\mathbf{M})^{k-1}\ (\mathbf{K}^{-1}\mathbf{p}) \tag{4.40}$$

$$\mathbf{u}_m = \sum_{i=0}^{m-1} \omega^{2i} (\mathbf{K}^{-1}\mathbf{M})^i (\mathbf{K}^{-1}\mathbf{p}) \tag{4.41}$$

This solution technique has a very meaningful physical interpretation. The solution is initially approximated by the static displacement, $\Delta\mathbf{u}^{(1)} = \mathbf{K}^{-1}\mathbf{p}$, with no consideration of inertial force. The inertial force due to this static displacement is $\omega^2 \mathbf{M}\, \Delta\mathbf{u}^{(1)}$ and the corresponding displacement vector due to this inertial correction force is therefore $\Delta\mathbf{u}^{(2)} = \omega^2 \mathbf{K}^{-1} \mathbf{M}\, \Delta\mathbf{u}^{(1)}$. This correction process continues until the increment is small. The solution is the sum of all these incremental corrections. The solution consists of one static response vector and the rest are correction due to inertial (dynamic) effect. Also, the solution process to the dynamic problem is by a series of static analysis. This is a numerical solution technique formalized by the equations above and it is a mathematical presentation of the physical insight in the derivation of the original of Ritz vectors[1].

This vector space computed by this method is identical to the one presented in Section 4.2.3 except the computation sequence is different. Similar to the method presented in Section 4.2.3, a shifting frequency can also be applied to this method.

4.2.5 Taylor's Expansion Method

A more interesting but simple derivation of the reduced vector basis is by using the Taylor series expansion:

$$f(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!} \frac{d^i f}{dx^i}(0). \tag{4.42}$$

and therefore Equation (4.4) can be represented by:

$$\mathbf{u} = \sum_{i=0}^{\infty} \omega^{2i} (\mathbf{K}^{-1}\mathbf{M})^i \mathbf{K}^{-1} \mathbf{p} \tag{4.43}$$

Therefore, the steady state solution at a given frequency, ω, is spanned by the basis vectors, $(\mathbf{K}^{-1}\mathbf{M})^i\ \mathbf{K}^{-1}\ \mathbf{p}$.

However, the Taylor series can be expanded around any frequency of interest, and the resulting series is given by:

$$\mathbf{u} = \sum_{i=0}^{\infty} (\omega^2 - \omega_o^2)^i\ [(\mathbf{K} - \omega_o^2\ \mathbf{M})^{-1}\mathbf{M}]^i\ (\mathbf{K} - \omega_o^2\mathbf{M})^{-1}\ \mathbf{p} \qquad (4.44)$$

This provides a good mathematical and physical interpretation of the vector basis best suited for a given temporal/frequency characteristics of the forcing function. When the expansion is around zero frequency, the basis is identical to the LDRV basis which is based on the static solution. When the expansion is around a given frequency, then this is a harmonic vector basis which provides harmonic correction at that frequency. This series is also called the Maclaurin's series[20]. All the mathematical properties of convergence of Taylor's series are applicable to Equations (4.43) and (4.44). Since the radius of convergence of Taylor's series cannot be larger than the distance to the closest singularity point (undamped natural frequencies), an equivalent Laurent's expansion is more appropriate[20,21]. In this case, the expansion is basically the same but the convergence property is valid between singularity points. This concept provides a systematic way to generate higher order approximation to the harmonic solution. The purpose of showing this expansion is to establish a reduced vector basis for solution of general dynamic response and it should not be used to compute the steady state solution.

The Taylor's (Laurent's) expansion provides a rigorous mathematical explanation of the shifted stiffness matrix concept rather than the ad hoc explanation given in subsection 4.2.3. From the computational point of view, only one matrix factorization is performed for each shift to generate as many independent vectors as permitted by the spectral content of the force vector.

4.3 Formulation of Undamped Vector Basis Method for General Forcing Functions

It was shown that different methods can be used to derive a set of basis vectors for the steady state harmonic problem. Although the same vector basis is derived from these different methods, the presentation provides a sound physical, numerical and mathematical interpretation of the Krylov subspace. This enables the extension of the reduced vector basis to solve dynamic responses of general forcing functions.

From Section 4.1, if the forcing function is not harmonic, then the solution is given by:

$$\mathbf{u}(t) = \sum_{j=0}^{N-1} (\mathbf{K} - (j\Delta\omega)^2 \mathbf{M})^{-1}\mathbf{p}\, c_j e^{i(j\Delta\omega)t} \tag{4.8}$$

By Equation (4.43), if the matrix inverse of Equation (4.8) is represented by:

$$(\mathbf{K} - (j\Delta\omega)^2 \mathbf{M})^{-1}\mathbf{p} = \sum_{i=0}^{\infty} (j\Delta\omega)^{2i} (\mathbf{K}^{-1}\mathbf{M})^i \mathbf{K}^{-1}\mathbf{p} \tag{4.45}$$

then,

$$\mathbf{u}(t) = \sum_{j=0}^{N-1} \left[\sum_{i=0}^{\infty} (j\Delta\omega)^{2i} (\mathbf{K}^{-1}\mathbf{M})^i (\mathbf{K}^{-1}\mathbf{p}) \right] c_j e^{i(j\Delta\omega)t} \tag{4.46}$$

$$\mathbf{u}(t) = \sum_{i=0}^{\infty} \left[\sum_{j=0}^{N-1} (j\Delta\omega)^{2i} c_j e^{i(j\Delta\omega)t} \right] (\mathbf{K}^{-1}\mathbf{M})^i (\mathbf{K}^{-1}\mathbf{p}) \tag{4.47}$$

Equation (4.47) shows that the same vector basis for steady state solution can be used to compute dynamic response of a general forcing function. The basis vector is defined as:

$$\mathbf{v}_i = (\mathbf{K}^{-1}\mathbf{M})^i \mathbf{K}^{-1}\mathbf{p} \tag{4.48}$$

From this formulation, it becomes obvious that the Krylov subspace is more appropriate than the eigen subspace in solving the dynamic response of Equation (4.2). In lieu of computing the response as shown in Equation (4.47), the reduced vector basis solution can be represented by:

$$\mathbf{u} = \mathbf{V}\,\mathbf{y} \tag{4.49}$$

As discussed in Chapter 2, using Galerkin's method by requiring the residual vector in the subspace to have zero projection onto the subspace, the reduced vector basis solution is governed by the following differential equation:

$$(\mathbf{V}^T\,\mathbf{M}\,\mathbf{V})\,\ddot{\mathbf{y}} + (\mathbf{V}^T\,\mathbf{C}\,\mathbf{V})\,\dot{\mathbf{y}} + (\mathbf{V}^T\,\mathbf{K}\,\mathbf{V})\,\mathbf{y} = \mathbf{V}^T\mathbf{p}\;g(t) \tag{4.50}$$

Despite the different derivations in Section 4.2, the computational procedure for these vectors is the same. Mass orthonormalized vectors should be used to improve the quality of the computed basis vectors, i.e.

$$\mathbf{X}^T\,\mathbf{M}\,\mathbf{X} = \mathbf{I} \tag{4.51}$$

The solution to the governing differential equation is represented in the reduced vector space as:

$$\mathbf{u} = \mathbf{X}\,\mathbf{y} \tag{4.52}$$

$$(\mathbf{X}^T\,\mathbf{M}\,\mathbf{X})\,\ddot{\mathbf{y}} + (\mathbf{X}^T\,\mathbf{C}\,\mathbf{X})\,\dot{\mathbf{y}} + (\mathbf{X}^T\,\mathbf{K}\,\mathbf{X})\,\mathbf{y} = \mathbf{X}^T\mathbf{p}\;g(t) \tag{4.53}$$

Typical algorithms are presented in Tables 2.1 and 3.3. The Lanczos algorithm can also be used to generate the Krylov subspace basis vectors. Furthermore, if stiffness orthogonal vector basis is used, i.e.

$$\mathbf{X}^T\,\mathbf{K}\,\mathbf{X} = \Lambda_{\mathbf{X}} \tag{4.54}$$

then existing software based on uncoupled normal modes can be used without any modification.

$$\ddot{y} + (X^T\ C\ X)\ \dot{y} + \Lambda_X\ y\ =\ X^T p\ g(t) \tag{4.55}$$

The reduced vector basis method presented here can therefore be attributed to Rayleigh, Ritz, Taylor, Wilson, Krylov, Galerkin, Stodola, Lanczos etc. A proper treatment of the damping matrix is necessary for computing accurate dynamic response in this undamped reduced vector subspace.

4.4 Viscous Damping Modelling of Energy Dissipation

Viscous damping is an *idealized* concept which requires the damping force to be directly proportional to the velocity. It is often represented physically by an ideal dashpot.

$$F_d = c\,\dot{u} \tag{4.56}$$

For a single degree of freedom system, this simplifies the solution of the governing differential equation.

$$m\,\ddot{u} + c\,\dot{u} + k\,u = g(t) \tag{4.57}$$

$$\ddot{u} + 2\xi\omega_1\,\dot{u} + \omega_1^2\,u = \frac{g(t)}{m} \tag{4.58}$$

where, $\omega_1 = \sqrt{\frac{k}{m}}$ (4.59)

$$\xi = \frac{c}{2m\omega_1} \tag{4.60}$$

For most structures the inherent damping in the structural system cannot be easily attributed to any damping mechanism. However, material damping and structural damping in most structures are often quite low. Equivalent energy dissipation representation which has a *simple mathematical form for easy solution* will suffice. The overall damping of these lightly damped structures are often grossly but adequately represented by an equivalent modal viscous damping ratio. Modal testing of typical structures, actual structures, prototypes or scaled models will provide equivalent viscous damping ratios for design and analysis purposes. Modal damping is assigned at the modal analysis level. If necessary, the global damping matrix can be computed from these assigned modal damping values[22] as (see Appendix A):

$$\mathbf{C} = \mathbf{M}\Phi \, [\, 2\xi_i\omega_i \,] \, \Phi^T\mathbf{M} \tag{4.61}$$

This synthesized damping has the same rank as the number of damping ratios assigned. In this representation, modes which are orthogonal to the vibration modes included in Equation (4.61) have zero damping. Therefore this damping matrix should be used cautiously. Furthermore, in this explicit form, the damping matrix can be fully populated and not suitable for large finite element analysis.

Modal damping can be generalized to Rayleigh's damping[5] in the following form,

$$\mathbf{C} = a\,\mathbf{M} + b\,\mathbf{K} \tag{4.62}$$

By the orthogonal property of the mode shape vectors, this damping matrix can be diagonalized by the mode shape vectors. This results in a set of uncoupled differential equations which can be used to compute dynamic response. Given two frequencies and two damping ratios, coefficients a and b can be computed. This damping matrix provides

viscous modal damping to the normal modes. The Rayleigh's damping can be further generalized into the Caughey series[5] as:

$$\mathbf{C} = \mathbf{M} \sum a_i (\mathbf{M}^{-1}\mathbf{K})^i \tag{4.63}$$

Again, by the orthogonality condition, this results in a set of uncoupled differential equations for dynamic response analysis. However, Caughey series is very seldom used explicitly because of the large size of the matrix. The form presented in Equation (4.63) requires the mass matrix to be non-singular. As many coefficients as the number of specified modal damping can be computed. These types of damping are often called proportional damping or classical damping.

Proportional damping is commonly used because of its simplicity. However, non-proportional damping is becoming more commonplace and important. When damping is deliberately designed into a structure to control vibration, it is difficult to satisfy the proportional damping criterion. Actually this will impose a severe constraint on the passive damping design for the sake of ease in analysis.

Structures with widely different material damping characteristics result in non-proportional damping. Knowing the materials and construction of a specific part of the structure, these mass and stiffness coefficients can be used to compute the damping contribution at the finite element level. The damping matrix of the i-th element can be computed as:

$$\mathbf{c}_i = a_i \mathbf{m}_i + b_i \mathbf{k}_i \tag{4.64}$$

The global damping matrix can then assembled for a structure with different materials.

$$\mathbf{C} = \sum \mathbf{c}_i \tag{4.65}$$

This explicit damping matrix in general cannot be orthogonalized by the undamped mode shape vectors and the differential equations are coupled.

Other than using mass and stiffness in design to control dynamic vibrations, damping is a very effective way to suppress vibrations. One concept is to use internal or external direct dashpots. An explicit damping matrix can be formed easily for these damping elements. Again, in general the damping matrix cannot be orthogonalized by the mode shape vectors and the damping is non-classical. If damping is to be introduced into a structure by design, the solution method must accurately predict the dynamic behavior of the damped structure. Rigorous, reliable and accurate analysis methods must be used for this class of damped structures.

For analysis of large or semi-infinite media, e.g. soil and fluid, finite element model can only represent the critical portion of the structure. The energy radiating out of the boundary of the finite element model must be modelled correctly in order to obtain accurate results. Direct dashpot or coupled damping matrix can be used to model the energy transmitted through the boundary as viscous energy loss. This modelling technique results in a structural system with a non-proportional damping matrix.

The most important issues for design and analysis of structures with viscous damping representation are:

1. correction due to vector truncation
2. effect of damping coupling in generalized coordinates
3. how to design viscous damping into a structure

4.5 Dynamic Response of Classically Damped System

Classically damped systems are analyzed by the normal method by assigned modal viscous damping ratios. When direct time integration method is used to compute the response in the time domain, Rayleigh's damping is often used for simplicity. The treatment of damping in the normal mode and reduced vector bases are considered in this section.

4.5.1 Normal Mode Vector Basis

Static correction vector is assumed to be appended in the normal mode basis for correcting the modal truncation error (see Section 4.2.2). When the damping is assumed to be proportional, it results in uncoupled modal damping. Then the modal equation of motion is:

$$\ddot{q}_i + 2\,\xi_i\,\omega_i\,\dot{q}_i + \omega_i^2\,q_i = \phi_i^T\,\mathbf{p}\,e^{i\omega t} \tag{4.66}$$

$$Q_i(\omega) = \frac{\phi_i^T\,\mathbf{p}}{\omega_i^2}\;\frac{1}{(1-\beta_i^2) + i\,2\,\xi_i\,\beta_i} \tag{4.67}$$

$$Q_i(\omega) = \frac{\phi_i^T\,\mathbf{p}}{\omega_i^2}\left[\frac{1-\beta_i^2}{(1-\beta_i^2)^2 + (2\,\xi_i\beta_i)^2} - i\,\frac{2\,\xi_i\,\beta_i}{(1-\beta_i^2)^2 + (2\,\xi_i\beta_i)^2}\right] \tag{4.68}$$

$$\mathbf{u}^R(\omega) = \sum \phi_i \left[\frac{\phi_i^T\,\mathbf{p}}{\omega_i^2}\;\frac{1-\beta_i^2}{(1-\beta_i^2)^2 + (2\,\xi_i\beta_i)^2}\right] \tag{4.69}$$

$$\mathbf{u}^I(\omega) = \sum \phi_i \left[\frac{\phi_i^T\,\mathbf{p}}{\omega_i^2}\;\frac{-(2\,\xi_i\,\beta_i)}{(1-\beta_i^2)^2 + (2\,\xi_i\beta_i)^2}\right] \tag{4.70}$$

It can be seen that the the same vector basis, ϕ_i, is spanning the real part and imaginary part of the solution. In each mode the phase between the real and imaginary parts of each degree

of freedom is the same. However, this simple relationship only holds for the individual modes and not for the resulting displacement vector in the physical space.

4.5.2 Reduced Vector Basis

If the damping of a structure is assumed to be proportional, or in term of Rayleigh's or Caughey's damping, the analysis of this structure in the reduced vector basis is quite straight forward. Since the reduced vector basis is mass orthonormalized and stiffness orthogonalized, the reduced damping matrix is diagonal.

$$\mathbf{X}^T \mathbf{C} \mathbf{X} = [2 \xi_i \omega_{Xi}] \tag{4.71}$$

If constant modal damping ratios are used in the normal mode analysis, the same damping ratios can be used in the reduced vector basis (since the basis for modal damping is quite weak anyway). The damping matrices in the physical space from these two reduced vector spaces can be synthesized by orthogonal expansion (see Appendix A):

$$\mathbf{C}_\Phi \approx (\mathbf{M}\Phi) [2 \xi_i \omega_i] (\mathbf{M}\Phi)^T \tag{4.61}$$

$$\mathbf{C}_X \approx (\mathbf{MX}) [2 \xi_i \omega_{Xi}] (\mathbf{MX})^T \tag{4.72}$$

The small discrepancy is in the two approximations is therefore:

$$\Delta \mathbf{C} \approx (\mathbf{M}\Phi) [2 \xi_i \omega_i] (\mathbf{M}\Phi)^T - (\mathbf{MX}) [2 \xi_i \omega_{Xi}] (\mathbf{MX})^T \tag{4.73}$$

Since the original definition of modal damping is quite weak, the small difference does not have real physical meaning.

4.6 Dynamic Response of Non-classically Damped System

In reality, most structures with an appreciable amount of energy dissipation are not classically damped. Accurate and efficient, but physical, analysis methods are important

for better design of damped structures. If the viscous damping matrix can be defined in the physical space, the complex eigenvectors can provide uncoupled coordinates for dynamic analysis. The theoretical basis of the method is well developed[23]. This method is not only more expensive but it is quite abstract to be useful for structural design purposes. Accurate solution was developed using the undamped eigen and Ritz basis[24] by iterative analysis in the time domain. The results compared favorably with the complex eigenvector approach[25].

The solution methods presented in the following subsections also attempt to provide an accurate solution in the undamped vector basis. This allows the experience in dynamic analysis in the modal domain to be extended to the design of complex linearly damped systems. However, as described in Chapter 1, solutions are often required in the frequency domain because of the definition of the forcing function, the frequency dependent nature of material properties and interaction with the active control systems. Therefore, iterative solution methods in the frequency domain are developed. This solution method can be used effectively to design structures with viscous damping mechanisms as described in Section 4.7.

4.6.1 Normal Mode Vector Basis

Assuming that the solution is in the undamped normal mode vector basis, Φ, which includes the static correction vector:

$$\mathbf{u}(t) \approx \Phi\, \mathbf{q}(t) \tag{4.74}$$

The governing equation of motion in the normal mode vector basis in the frequency domain is reduced to:

$$[-\omega^2 \mathbf{I} + i\,\omega\Phi^T\mathbf{C}\Phi + \Lambda]\, \mathbf{Q}(\omega) = \Phi^T\mathbf{p}\, G(\omega) \tag{4.75}$$

Since the damping is non-proportional, the reduced damping matrix is not diagonal. Now, let the reduced damping matrix be partitioned into a matrix with only diagonal elements, $C_{\Phi d}$, and another matrix with only off-diagonal elements, $C_{\Phi s}$:

$$\Phi^T C \Phi = C_{\Phi d} + C_{\Phi s} \tag{4.76}$$

Then Equation (4.76) becomes :

$$[-\omega^2 I + i\,\omega(C_{\Phi d} + C_{\Phi s}) + \Lambda]\, Q(\omega) = \Phi^T p\, G(\omega) \tag{4.77}$$

The complex linear equation must be solved, frequency by frequency, in order to obtain the solution in the frequency domain. However, since the coupling is only through the off-diagonal damping matrix, and if the damping forces are small (lightly damped system, say $\xi < 0.20$), an accurate approximate solution can be obtained by the iterative solution method by putting the off-diagonal damping matrix to the right hand side of Equation (4.77) as:

$$[-\omega^2 I + i\,\omega\, C_{\Phi d} + \Lambda]\, Q(\omega) = \Phi^T p\, G(\omega) - i\,\omega\, C_{\Phi s}\, Q(\omega) \tag{4.78}$$

Define the leading coefficient matrix as the impedance matrix, $Z_{\Phi d}(\omega)$:

$$Z_{\Phi d}(\omega) = [-\omega^2 I + i\,\omega\, C_{\Phi d} + \Lambda] \tag{4.79}$$

An iterative solution scheme can then be written as:

$$Z_{\Phi d}(\omega)\, Q(\omega)^{(k+1)} = \Phi^T p\, G(\omega) - i\,\omega\, C_{\Phi s}\, Q(\omega)^{(k)} \tag{4.80}$$

Since the left hand side of Equation (4.80) is diagonal, solution can be computed relatively easily. Since it is a lightly damped system, a good starting solution is obtained by neglecting the coupling terms :

$$\mathbf{Z}_{\Phi d}(\omega)\,\mathbf{Q}^{(1)}(\omega) = \Phi^T \mathbf{p}\, G(\omega) \tag{4.81}$$

The equivalent scalar equation is simply:

$$Q_i^{(1)}(\omega) = \left[\frac{\phi_i^T \mathbf{p}}{\omega_i^2}\right]\left[\frac{G(\omega)}{(1 - \beta_i^2) + i\,\beta_i \left(\frac{\phi_i^T C \phi_i}{\omega_i}\right)}\right] \tag{4.82}$$

This is the solution of an equivalent system with diagonal damping. However, a corrected solution which includes the damping coupling can be obtained easily by :

$$\mathbf{Z}_{\Phi d}(\omega)\,\mathbf{Q}^{(2)}(\omega) = \Phi^T \mathbf{p}\, G(\omega) - i\,\omega\, C_{\Phi s}\, \mathbf{Q}^{(1)}(\omega) \tag{4.83}$$

$$\mathbf{Z}_{\Phi d}(\omega)\,\mathbf{Q}^{(k)}(\omega) = \Phi^T \mathbf{p}\, G(\omega) - i\,\omega\, C_{\Phi s}\, \mathbf{Q}^{(k-1)}(\omega) \tag{4.84}$$

Expressing this solution process in closed form:

$$\mathbf{Q}^{(k)}(\omega) = \mathbf{Z}_{\Phi d}(\omega)^{-1}\left\{\sum_{i=0}^{k-1}\left[-i\omega\, C_{\Phi s}\, \mathbf{Z}_d(\omega)^{-1}\right]^i\right\}\Phi^T \mathbf{p}\, G(\omega) \tag{4.85}$$

This solution technique is actual the same as some of the iterative (relaxation) solution techniques of matrix linear equations in the field of numerical analysis[26,27,28], e.g. the Jacobi's and Gauss-Seidel's methods. The physical properties of the structural matrices provide the positive definite requirement which is necessary for the proof of convergence of this type of methods. Therefore, basically all the iterative solution methods from numerical analysis can be tapped to provide the best solution to different forms of this problem. However, from an engineering intuition point of view, the solution will converge very fast, presumably within a few cycles. One additional feature of this method is that it also clearly demonstrates the effect of coupled damping on the behavior of the structural system. This provides the engineer with the vital physical information which can affect the design decision.

It is interesting to examine the vector basis of the solution obtained by the method described:

$$\mathbf{U}^{(k)}(\omega) = \Phi\, \mathbf{Z}_{\Phi d}(\omega)^{-1}\left\{ \sum_{i=0}^{k-1} \left[-i\omega\, \mathbf{C}_{\Phi s}\mathbf{Z}_{\Phi d}(\omega)^{-1}\right]^i \right\}\Phi^T\mathbf{p}G(\omega) \tag{4.86}$$

The displacement vector is a linear combination of ϕ_i. Therefore, the iterations basically improve the accuracy of the solution in the same vector space due to damping coupling in the generalized coordinates. The transfer function relating the input forcing function, $G(\omega)$, to the displacement vector, $\mathbf{U}(\omega)$ is also approximated by Equation (4.86). The transfer function also enables dynamic analysis of random vibration since $G(\omega)$ is only known as the power spectral density, $S(\omega)$.

4.6.2 Reduced Vector Basis

The reduced vector basis will perform the same task as the normal mode vector basis with a smaller set of basis vectors. The mathematics and computation are basically the same.

The solution can be assumed to be a linear combination of basis vectors, $\mathbf{x}_i$. Without loss of generality, $\mathbf{X}$ can be assumed to be mass orthonormalized and stiffness orthogonalized.

$$\mathbf{u}(t) \approx \mathbf{X}\,\mathbf{y}(t) \tag{4.52}$$

$$[-\omega^2\mathbf{I} + i\,\omega\mathbf{X}^T\mathbf{C}\mathbf{X} + \Lambda_X]\,\mathbf{Y}(\omega) = \mathbf{X}^T\mathbf{p}\,G(\omega) \tag{4.87}$$

Now, let the reduced damping matrix be partitioned into a matrix with only diagonal elements, $\mathbf{C}_{Xd}$, and another matrix with only off-diagonal elements, $\mathbf{C}_{Xs}$:

$$\mathbf{X}^T\mathbf{C}\mathbf{X} = \mathbf{C}_{Xd} + \mathbf{C}_{Xs} \tag{4.88}$$

Then Equation (4.87) becomes:

$$[-\omega^2 \mathbf{I} + i\,\omega(\mathbf{C_{Xd}} + \mathbf{C_{Xs}}) + \Lambda_X]\,\mathbf{Z}(\omega) = \mathbf{X}^T\mathbf{p}\,G(\omega) \tag{4.89}$$

A complex linear equation must be solved, frequency by frequency, in order to obtain the solution to Equation (4.89). However, since the coupling is only through the off-diagonal damping matrix, and if the damping forces are small (lightly damped system, say $\xi < 0.20$), an accurate approximate solution can be obtained if the off-diagonal damping matrix is put to the right hand side of Equation (4.89).

$$[-\omega^2 \mathbf{I} + i\,\omega\,\mathbf{C_{Xd}} + \Lambda_X]\,\mathbf{Z}(\omega) = \mathbf{X}^T\mathbf{p}\,G(\omega) - i\,\omega\,\mathbf{C_{Xs}}\,\mathbf{Z}(\omega) \tag{4.90}$$

Define the leading coefficient matrix as the impedance matrix, $\mathbf{Z_{Xd}}(\omega)$:

$$\mathbf{Z_{Xd}}(\omega) = [-\omega^2 \mathbf{I} + i\,\omega\,\mathbf{C_{Xd}} + \Lambda_X] \tag{4.91}$$

An iterative solution scheme can then be written as:

$$\mathbf{Z_{Xd}}(\omega)\,\mathbf{Y}(\omega)^{(k+1)} = \mathbf{X}^T\mathbf{p}\,G(\omega) - i\,\omega\,\mathbf{C_{Xs}}\,\mathbf{Y}(\omega)^{(k)} \tag{4.92}$$

Since the left hand side of Equation (4.92) is diagonal and solution can be computed relatively easily. Since it is a lightly damped system, a good starting approximate solution can be obtained by neglecting the coupling terms :

$$\mathbf{Z_{Xd}}(\omega)\,\mathbf{Y}(\omega)^{(1)} = \mathbf{X}^T\mathbf{p}\,G(\omega) \tag{4.93}$$

The equivalent scalar equation is simply:

$$Q_i^{(1)}(\omega) = \left[\frac{\mathbf{x}_i^T\mathbf{p}}{\omega_{Xi}^2}\right]\left[\frac{G(\omega)}{(1 - \beta_{Xi}^2) + i\,\beta_{Xi}\left(\dfrac{\mathbf{x}_i^T\mathbf{C}\mathbf{x}_i}{\omega_{Xi}}\right)}\right] \tag{4.94}$$

This is the solution of an equivalent system with diagonal damping. However, a corrected solution which includes the damping coupling can be obtained easily by :

$$\mathbf{Z_{Xd}}(\omega)\,\mathbf{Q}^{(2)}(\omega) = \mathbf{X}^{T}\mathbf{p}\,G(\omega) - i\,\omega\,\mathbf{C_{Xs}}\,\mathbf{Q}^{(1)}(\omega) \tag{4.95}$$

$$\mathbf{Z_{Xd}}(\omega)\,\mathbf{Q}^{(k)}(\omega) = \mathbf{X}^{T}\mathbf{p}\,G(\omega) - i\,\omega\,\mathbf{C_{Xs}}\,\mathbf{Q}^{(k-1)}(\omega) \tag{4.96}$$

Expressing this solution process in closed form:

$$\mathbf{Q}^{(k)}(\omega) = \mathbf{Z_{Xd}}(\omega)^{-1}\left\{\sum_{i=0}^{k-1}\left[-i\omega\,\mathbf{C_{Xs}}\mathbf{Z_{Xd}}(\omega)^{-1}\right]^{i}\right\}\mathbf{X}^{T}\mathbf{p}\,G(\omega) \tag{4.97}$$

The displacement solution is given by:

$$\mathbf{U}^{(k)}(\omega) = \mathbf{X}\,\mathbf{Z_{Xd}}(\omega)^{-1}\left\{\sum_{i=0}^{k-1}\left[-i\omega\,\mathbf{C_{Xs}}\mathbf{Z_{Xd}}(\omega)^{-1}\right]^{i}\right\}\mathbf{X}^{T}\mathbf{p}\,G(\omega) \tag{4.98}$$

The solution process in the reduced vector basis is therefore identical to that presented in Subsection 4.6.1. Therefore use of the reduced vector basis poses no conceptual or computational difficulty when compared with the normal mode basis.

Since the computation procedure in the reduced vector basis and normal mode basis is identical, from this point forward the "reduced basis vectors" are also referenced as "Ritz modes" or simply as "modes".

4.7 Design Method for Structures Damped by Viscous Dashpots

In order to attain passive vibration control of a structure, traditionally the structure is designed to be stiff, massive or a combination of both. The natural frequency of the system is designed to be away from the dominant forcing frequency range. The amount of damping inherited in the structure is often contented with. Damping treatments have been

regarded as an unreliable source of energy dissipation. With the new weight and dynamic performance requirements of modern structures, passive damping design is becoming a necessity. However, accurate analyses of structures with non-proportional damping are still quite complex. If dashpots are to be included in a structural design, the design procedure must be considerably simplified. The simplification must not be based on an ad hoc procedure but should rather be based on valid engineering assumptions and consistent mathematical derivation. It should also be flexible enough to allow structures to have complex geometric configurations.

Due to the materials used in the high performance precision structures and the tight joint construction, the inherent damping in these systems is often very low, often below 0.5%. The dynamic performance can be much enhanced if the system can be designed to have damping in the 5% to 25% range. This will reduce the resonant harmonic response by a factor of 10 to 50 and the expected random vibration response by a factor of 3 to 7. The benefit is tremendous. The real question is "how to do it?".

Within the range of practical interest, the damping forces are still relatively small compared to the inertial and elastic forces. If the system is non-proportionally damped, the damping coupling between the generalized coordinates is relatively small. The preliminary damping design can therefore be based on the uncoupled damping assumption. After the conceptual and preliminary design is complete, a more detailed analysis can be performed to predict the dynamic response and evaluate the effect of damping coupling.

The first step is to analyze a baseline structural design with finite element analysis. Mass and stiffness properties are all near the final stage of design. Undamped natural frequencies and mode shapes are computed. Dynamic response analysis of the structure with inherent damping is performed. Critical vibration modes are identified. Based on these results, the damping requirement for each mode can be established, e.g. the vibration of the third mode must be reduced by a factor of 5 by higher damping in order to meet the

performance objective. Now, the engineer must decide where to put dashpots, how many and also the magnitude of viscous damping.

In the generalized modal coordinates (normal mode or reduced vector), if the damping coupling terms are not considered, the modal differential equation of motion is reduced to:

$$\ddot{y}_i + \mathbf{x}_i^T \mathbf{C}\, \mathbf{x}_i \dot{y}_i + \omega_{Xi}^2\, y_i = \mathbf{x}_i^T \mathbf{p}\, g(t) \tag{4.99}$$

If a certain viscous modal damping value, ξ_i , is required for that particular vector basis, then:

$$\ddot{y}_i + 2\, \xi_i\, \omega_{Xi}\, \dot{y}_i + \omega_{Xi}^2\, y_i = \mathbf{x}_i^T \mathbf{p}\, g(t) \tag{4.100}$$

Therefore, the designed dampers should have a damping matrix satisfying:

$$2\, \xi_i\, \omega_{Xi} = \mathbf{x}_i^T \mathbf{C}\, \mathbf{x}_i \tag{4.101}$$

The contribution of each damper to the system level damping must be evaluated for a balanced structural design. If there are k dampers in the structure, the global damping matrix is assembled from the element damping matrices, $\mathbf{c}_j$, as:

$$\mathbf{C} = \sum_{j=1}^{k} \mathbf{c}_j \tag{4.102}$$

Then,

$$2\, \xi_i\, \omega_{Xi} = \sum_{j=1}^{k} \phi_i^T\, \mathbf{c}_j\, \phi_i \tag{4.103}$$

Defining the contribution of the j-th element to the i-th mode damping ratio as the *element modal damping ratio* , ζ_{ji}, as:

$$\zeta_{ji} = \frac{\phi_i^T\, \mathbf{c}_j\, \phi_i}{2\, \omega_{Xi}} \tag{4.104}$$

Then,

$$\xi_i = \sum_{j=1}^{k} \zeta_{ji} \tag{4.105}$$

Equations (4.104) and (4.105) are very important for the design of dashpots as they relates the effectiveness of the damping element to the modal response. It can be seen that the damping ratio contribution of a particular element in a given mode is proportional to the relative displacement between the ends of the damper, the magnitude of the damping coefficient, and inversely proportional to the natural frequency.

The most effective locations for dampers are identified by the following procedure. If the mode shapes are not significantly changed by the dampers, the shape can be used to guide the design. Finite element method can be used most effectively to find the sensitivity of all possible locations. A separate finite element model for damping elements is constructed based on the same finite element grid. The damping elements are placed at all the possible locations in this model. A unit damping coefficient is specified for all the dampers and the unit element damping matrix is $\mathbf{c}_j$. Using the same mode shapes from the undamped structure, compute element triple product. Actually, no new computer code is required for this operation. If the unit dampers are modelled as members with unit elastic axial stiffness, the element modal strain energy output will be computed as $\left(\frac{\phi_i^T \mathbf{c}_j \phi_i}{2}\right)$. Therefore, if the strain energy of these elements are ranked in descending order, it provides a list ranking the most efficient locations for the placement of dampers. A cumulative sum of the ranked elements is then computed:

$$s_{ki} = \sum_{j=1}^{k} \frac{\phi_i^T \mathbf{c}_j \phi_i}{2} \tag{4.106}$$

Then, if only one type of damper is used, i.e.

$$F_d = c\,\dot{u} \tag{4.54}$$

and the required modal damping and number of dampers are given, based on Equations (4.105) and (4.106), i.e.

$$\xi_i = c\frac{s_{ki}}{\omega_i} \tag{4.107}$$

the required damping coefficient of the dampers can be computed:

$$c = \frac{s_{ki}}{\xi_i\,\omega_i} \tag{4.108}$$

Alternatively, if the damping coefficient of the dampers is given, the number of dampers necessary to provide the desired level of damping can be computed easily based on the cumulative contribution of the dampers, Equation (4.106). If different dampers are used at different locations, the computation is similar and hence not shown in detail here.

The final step is to check the accuracy of the assumption of uncoupled modal response. The significance and extent of the coupling will be apparent if the iterative method, as shown in Equation (4.96), is used. This method for design of dampers to meet the overall structural performance, to the knowledge of the author, is not currently used in any design. This design method with the iterative analysis method is extremely flexible and powerful for this class of design problem. The appropriate name for this method is the "element modal damping ratio method".

Chapter 5

Damped Vector Basis for Dynamic Response Analysis

For non-classically damped structures, the governing differential equation is normally posed in the first order conical form, i.e. in the state space, with displacement and velocity as the variables. The conventional method for solving dynamic responses is by using the complex mode shapes[10,23,25] (complex eigenvectors), in the 2n dimensional space, as the basis vectors. However, this type of representation does not give too much physical meaning to structural engineers and therefore very difficult to gain physical insight into the problem for design purpose.

The use of undamped vector basis, as described in Chapter 4, provides a good physical interpretation and the solution method can be easily understood and applied to a design. The modal coupling is solved by iteration. As an extension of the concept developed in Chapter 4, a complementary set of reduced basis vectors is developed in the 2n dimensional space based on the the displacement vectors of steady state solution of a given forcing frequency. Solution to the second order differential equation is sought in terms of in-phase and out-of-phase displacement vectors in lieu of the solution in the first order state space.

Similar to the Stodola method for an undamped system iterative solution is used to establish the displacement vectors. These vectors are then used as the basis vectors for computing dynamic response of damped structure due to general forcing function. Since the basis vectors are derived from the mass, stiffness and damping matrices, the effect of damping should be better represented in this basis then the undamped basis. Similar to Chapter 4, the solution techniques are discussed primarily in the frequency domain. As such, the solution method can be easily extended to structures with hysteretic (viscoelastic) damping and control-structure interaction problem. The behavior of structural systems with damping, the effect of damping on the form of solution, and the computational

techniques are discussed. A logical derivation of the Krylov subspace for computation of dynamic responses due to harmonic loading is established for the viscously damped systems.

5.1 Steady State Solution

The equation of motion of a linear dynamic system under a harmonic force is represented by the following equation:

$$\mathbf{M}\,\ddot{\mathbf{u}} + \mathbf{C}\,\dot{\mathbf{u}} + \mathbf{K}\,\mathbf{u} = \mathbf{p}\,e^{i\omega t} \tag{5.1}$$

In this representation, the structure is actually excited by two harmonically varying forcing functions simultaneously

$$e^{i\omega t} = \cos\omega t + i \sin\omega t \tag{5.2}$$

The spatial force distribution is the same for these two forcing functions. The use of complex arithmetics allows a compact representation of the solution to Equation (5.1).

5.1.1 Exact Solution in Complex Arithmetics

Let the solution to Equation (5.1) be given by:

$$\mathbf{u}(\omega) = \mathbf{u}_{\omega}\,e^{i\omega t} \tag{5.3}$$

then, the equation of motion is reduced to:

$$[\,(\mathbf{K} - \omega^2\,\mathbf{M}) + i\,\omega\,\mathbf{C}\,]\,\mathbf{u}_{\omega}\,e^{i\omega t} = (\mathbf{p} + i\,\mathbf{0})\,e^{i\omega t} \tag{5.4}$$

Define the impedance matrix, $\mathbf{Z}(\omega)$, of the damped structure as:

$$\mathbf{Z}(\omega) = [\ (\mathbf{K} - \omega^2 \mathbf{M}) + i\ \omega\ \mathbf{C}\ ] \tag{5.5}$$

The solution to Equation (5.1) can be expressed as:

$$\mathbf{Z}(\omega)\ \mathbf{u}_\omega = (\mathbf{p} + i\ \mathbf{0}) \tag{5.6}$$

$$\mathbf{u}_\omega = \mathbf{Z}(\omega)^{-1}\ (\mathbf{p} + i\ \mathbf{0})$$

$$= \mathbf{u}^R + i\ \mathbf{u}^I \tag{5.7}$$

$$\text{where } \mathbf{u}^R = \mathbf{Re}\ [\mathbf{Z}(\omega)^{-1}\ \mathbf{p}] \tag{5.8}$$

$$\mathbf{u}^I = \mathbf{Im}\ [\mathbf{Z}(\omega)^{-1}\ \mathbf{p}] \tag{5.9}$$

Therefore at each degree of freedom, there are two response values, one real and one imaginary. It becomes apparent that at each physical degree of freedom, two coordinates are required to describe its motion. The two coordinates can be in the form of amplitude and phase angle pair, or real and imaginary pair. This solution offers very little insight into the behavior of the structure.

A single degree of freedom system is used as an example to give some insight into this solution method.

$$\ddot{u} + 2\xi_i\omega_i\ \dot{u} + \omega_i^2\ u = p_i\ e^{iwt} \tag{5.10}$$

$$Z(\omega) = (\omega_i^2 - \omega^2) + i\ (2\xi_i\omega_i\omega) \tag{5.11}$$

$$Z(\omega)^{-1} = \frac{1}{(\omega_i^2 - \omega^2) + i\ (2\xi_i\omega_i\omega)} \tag{5.12}$$

$$u^R = \frac{p_i}{\omega_i^2}\ \frac{1-\beta_i^2}{(1-\beta_i^2)^2 + (2\xi_i\beta_i)^2} \tag{5.13}$$

$$u^I = \frac{p_i}{\omega_i^2} \frac{-2\xi_i\beta_i}{(1-\beta_i^2)^2 + (2\xi_i\beta_i)^2} \tag{5.14}$$

$$u_\omega = \frac{p_i}{\omega_i^2} \frac{1}{(1-\beta_i^2)^2 + (2\xi_i\beta_i)^2} \left[(1-\beta_i^2) - i\ (2\xi_i\beta_i)\right] \tag{5.15}$$

Alternatively, the impedance can be expressed in terms of amplitude and phase angle:

$$\tan\phi = \left(\frac{2\xi_i\beta_i}{1-\beta_i^2}\right) \tag{5.16}$$

$$Z(\omega) = \omega_i^2 \sqrt{(1-\beta_i^2)^2 + (2\xi_i\beta_i)^2}\ e^{i\phi} \tag{5.17}$$

$$Z(\omega)^{-1} = \frac{1}{\omega_i^2 \sqrt{(1-\beta_i^2)^2 + (2\xi_i\beta_i)^2}}\ e^{-i\phi} \tag{5.18}$$

It can be seen that the solution of a large system can be quite difficult and computation is quite intensive and abstract.

5.1.2 Exact Solution in Real Arithmetics in 2n Dimensional Space

Despite the simplicity of the expression for solution, Equation (5.7), the complex arithmetic formulation does not offer much insight into the physical behavior of the structural system. In order to understand the relationship between the vector components in real number and the structural matrices, a more in depth study is required.

By grouping the real part and imaginary part of the solution, the displacement vector due to the harmonically varying force, Equations (5.2) and (5.7), becomes:

$$\mathbf{u} = (\mathbf{u}^R \cos\omega t - \mathbf{u}^I \sin\omega t) + i\,(\mathbf{u}^R \sin\omega t + \mathbf{u}^I \cos\omega t) \tag{5.19}$$

Therefore, the solution to the differential equation of motion with a cosine forcing function is:

$$\mathbf{u} = (\mathbf{u}^R \cos \omega t - \mathbf{u}^I \sin \omega t) \tag{5.20}$$

The solution to a sine forcing function is:

$$\mathbf{u} = (\mathbf{u}^R \sin \omega t + \mathbf{u}^I \cos \omega t) \tag{5.21}$$

The presence of the damping matrix introduces an out-of-phase harmonic component with a different basis vector. In the absence of the viscous damping matrix, the displacement vector is in-phase with the applied force and only one vector is required to describe the solution. It is also due to the presence of the damping matrix that the computation of the solution to a damped system becomes quite difficult.

From Equations (5.4) and (5.7), one can equate the real and imaginary parts using real arithmetics for solution as follows:

$$(\mathbf{K} - \omega^2 \mathbf{M})\ \mathbf{u}^R - \omega\ \mathbf{C}\ \mathbf{u}^I = \mathbf{p} \tag{5.21}$$

$$\omega\ \mathbf{C}\ \mathbf{u}^R + (\mathbf{K} - \omega^2 \mathbf{M})\ \mathbf{u}^I = \mathbf{0} \tag{5.22}$$

This can also be represented in the 2n space by the following matrix equation:

$$\begin{bmatrix} \mathbf{K}-\omega^2\mathbf{M} & -\omega\mathbf{C} \\ \omega\mathbf{C} & \mathbf{K}-\omega^2\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{u}^R \\ \mathbf{u}^I \end{bmatrix} = \begin{bmatrix} \mathbf{p} \\ \mathbf{0} \end{bmatrix} \tag{5.23}$$

The coefficient matrix of Equation (5.23) contains 2n equations. The matrix is not symmetric but can be solved directly by a general matrix linear equation solver. The computation effort will be very much dependent on the bandwidth of the damping matrix, **C**. Even if the bandwidth is small, the minimum bandwidth or profile of this equation is larger than the total number of degrees of freedom. Therefore this can be a significant effort for large system in the sense of static solution. The inclusion of the effect of damping significantly increases the computational effort.

5.1.3 Exact Solution in Real Arithmetics in n Dimensional Space

With lower part of the right hand side of Equation (5.23) is zero, the solution is similar to the equation of static condensation except that the coefficient matrix is not a symmetric stiffness matrix. If the forcing frequency is not at an undamped natural frequency, the relationship between $\mathbf{u}^R$ and $\mathbf{u}^I$ can be established using the second submatrix equation of Equation (5.23):

$$\mathbf{u}^I = - \omega (\mathbf{K} - \omega^2 \mathbf{M})^{-1} \mathbf{C} \mathbf{u}^R \tag{5.24}$$

The solution to Equation (5.23) is therefore given by:

$$\mathbf{u}^R = [(\mathbf{K} - \omega^2 \mathbf{M}) + \omega^2 \mathbf{C} (\mathbf{K} - \omega^2 \mathbf{M})^{-1} \mathbf{C}]^{-1} \mathbf{p} \tag{5.25}$$

Therefore the solution can be computed in two steps. $\mathbf{u}^R$ is computed first using Equation (5.25). $\mathbf{u}^I$ is then computed by substituting $\mathbf{u}^R$ into Equation (5.24).

Equation (5.25) can be rewritten as follows:

$$\mathbf{u}^R = [\mathbf{I} + \omega^2 (\mathbf{K} - \omega^2 \mathbf{M})^{-1} \mathbf{C} (\mathbf{K} - \omega^2 \mathbf{M})^{-1} \mathbf{C}]^{-1} (\mathbf{K} - \omega^2 \mathbf{M})^{-1} \mathbf{p} \tag{5.26}$$

$(\mathbf{K} - \omega^2 \mathbf{M})^{-1}\mathbf{p}$ is the undamped harmonic solution and $[\mathbf{I} + \omega^2(\mathbf{K}-\omega^2 \mathbf{M})^{-1}\mathbf{C}(\mathbf{K}-\omega^2 \mathbf{M})^{-1}\mathbf{C}]^{-1}$ is the matrix operator which augments the undamped solution to incorporate the contribution of damping. This operator requires intensive numerical operations and a full square matrix storage. From Equation (5.24), it can be seen that the matrix operator $[-\omega(\mathbf{K}-\omega^2\mathbf{M})^{-1}\mathbf{C}]$ transforms the real part of the solution into the imaginary part. The solution so obtained is exact if an explicit viscous damping matrix can be specified. Solution to the undamped or damped eigenvalue problem is not necessary. The solution can be interpreted as a vector basis with only one vector spanning the in-phase part of the solution and one vector spanning the 90° out-of-phase part of the solution. Unlike the undamped modal coordinates, the dynamic solution at each degree of freedom does not

have the same phase. The displacement shape is not constant with respect to time. However, the solution is periodic in nature and the same shape will be repeated periodically at the period of the forcing function.

At the undamped natural frequencies, $(\mathbf{K} - \omega^2\mathbf{M})$ becomes singular and a unique solution to Equation (5.25) cannot be found. But if $\mathbf{C}$ is non-singular, the solution to Equation (5.23) can be written as :

$$\mathbf{u}^I = -[\omega^{-1}(\mathbf{K} - \omega^2\mathbf{M})\mathbf{C}^{-1}(\mathbf{K} - \omega^2\mathbf{M}) + \omega\,\mathbf{C}]^{-1}\,\mathbf{p} \tag{5.27}$$

$$\mathbf{u}^R = -\omega^{-1}\,\mathbf{C}^{-1}(\mathbf{K} - \omega^2\mathbf{M})\,\mathbf{u}^I \tag{5.28}$$

However, if both $(\mathbf{K} - \omega^2\mathbf{M})$ and $\mathbf{C}$ are singular, a unique solution to Equation (5.23) is not possible. In this case, the system as modelled consists of zero strain energy modes and the techniques described in Chapter 3 should be used.

5.1.4 Single Degree of Freedom Systems

In order to comprehend the effect of damping on complex multiple degrees of freedom systems, it is important to understand the behavior a single degree of freedom system. For a single degree of freedom system with unit mass (a typical mass normalized uncoupled modal coordinate of multiple degrees of freedom structure), from Equation (5.25) the real part of the solution is :

$$\begin{aligned} u^R &= \left(1 + \frac{4\beta_i^2\xi_i^2}{(1-\beta_i^2)^2}\right)^{-1} \frac{1}{\omega_i^2\,(1-\beta_i^2)}\,p \\ &= \frac{1-\beta_i^2}{(1-\beta_i^2)^2 + 4\beta_i^2\xi_i^2}\,\frac{p}{\omega_i^2} \end{aligned} \tag{5.29}$$

$\left[1 + \dfrac{4\beta_i^2\xi_i^2}{(1-\beta_i^2)^2}\right]$ is the scalar equivalent of the operator matrix and is a measure of the contribution of damping to the real part of the solution. If the system has no damping, the

displacement is in phase with the force and the amplitude is simply given by $\left[\frac{p}{\omega_i^2 (1-\beta_i^2)}\right]$.

For a lightly damped system, the contribution due to damping is very small except near the undamped natural frequency where it becomes relatively large.

$$u^I = \frac{-2\beta_i\xi_i}{(1-\beta_i^2)} u^R$$

$$= \frac{-2\beta_i\xi_i}{(1-\beta_i^2)^2 + 4\beta_i^2\xi_i^2} \frac{p}{\omega_i^2} \qquad (5.30)$$

$\left[\frac{-2\beta_i\xi_i}{(1-\beta_i^2)}\right]$ is the scalar equivalent of the operator matrix which relates the real and imaginary parts of the solution.

It is important to note that at an undamped resonant frequency, the real part of the dynamic stiffness is zero and the solution is bounded only because of the presence of damping. In this case the real part of the solution is zero and the complex part has a large amplitude $\left[= \frac{1}{2\xi_i} \frac{p}{\omega_i^2}\right]$.

5.1.5 Classically Damped Systems

If direct frequency solution is used to compute the response, a damping matrix must be specified. The simplest form of classical damping is by using Rayleigh's damping with the damping matrix proportional to the mass and stiffness matrices.

$$\mathbf{C} = a\,\mathbf{M} + b\,\mathbf{K} \qquad (5.31)$$

$$\mathbf{u}^R = [\mathbf{I} + \omega^2 (\mathbf{K} - \omega^2 \mathbf{M})^{-1}\mathbf{C}(\mathbf{K} - \omega^2 \mathbf{M})^{-1}\mathbf{C}]^{-1}(\mathbf{K} - \omega^2 \mathbf{M})^{-1}\mathbf{p} \qquad (5.25)$$

The above equation cannot be readily solved without large computer storage and computation time. It is not obvious that by using Rayleigh's damping the solution to Equations (5.23) and (5.25) can be simplified.

A more general form of classical damping is the Caughey's damping[5] which is a series representation and provides more parameters to control the damping specification.

$$\mathbf{C} = \mathbf{M} \sum_{i=0}^{m} a_i (\mathbf{M}^{-1} \mathbf{K})^i \tag{5.32}$$

The Caughey's damping as presented in Equation (5.32) requires the mass matrix to be non-singular. This requires a static condensation step to be performed to reduce the degrees of freedom. The actual specification of the parameters are not easy unless the natural frequencies of the system are computed. The use of this damping matrix is not practical.

Another way to specify the damping matrix is by synthesis from the modal damping[22], $(c_i = 2\xi_i\omega_i)$, mode by mode.

$$\begin{aligned} \mathbf{C} &= \sum_{i=1}^{m} (\mathbf{M}\phi_i)(2\xi_i\omega_i)(\mathbf{M}\phi_i)^T \\ &= \mathbf{M}\Phi \, [2\, \xi_i\, \omega_i\,] \, (\mathbf{M}\Phi)^T \end{aligned} \tag{5.33}$$

The damping matrix so specified has the rank of the number of modes included. It can be as high as the mass matrix. If the mass matrix is rank deficient, the damping matrix will definitely be rank deficient. If modal damping is not specified for some selected modes, these modes will be undamped. The use of this damping matrix in Equation (5.25) is not practical.

Therefore, for conventional structures which dynamic responses can be adequately modelled by modal viscous damping ratios (classically damped), the assembly of damping matrix and the direct solution of damped harmonic response is quite a nuisance.

For structures designed dampers to suppress dynamic vibration, the damping contribution of the damper can be modelled as part of the damping matrix. Consequently, it is very unlikely that the final damping matrix is proportional. Only in these cases, the

computation of damped response vectors to establish a damped vector basis may be important. Since the direct solution is numerically not efficient in computing such a basis, alternate methods are studied.

5.2 Complex Modes Method

The governing differential equation, Equation (5.1) can be cast in the first order form[23] in the state space representation with the displacement and velocity as variables:

$$\begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{bmatrix} \dot{u} \\ \ddot{u} \end{bmatrix} + \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix} \begin{bmatrix} u \\ \dot{u} \end{bmatrix} = \begin{bmatrix} p \\ 0 \end{bmatrix} e^{i\omega t} \tag{5.34}$$

Redefine the vector and coefficient matrices as:

$$\mathbf{z} = \begin{bmatrix} u \\ \dot{u} \end{bmatrix} \tag{5.35}$$

$$\mathbf{A} = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \tag{5.36}$$

$$\mathbf{B} = \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix} \tag{5.37}$$

Rewrite the differential equation as:

$$\mathbf{A}\,\dot{\mathbf{z}} + \mathbf{B}\,\mathbf{z} = \begin{bmatrix} p \\ 0 \end{bmatrix} e^{i\omega t} \tag{5.38}$$

The homogeneous differential equation of Equation (5.34) has complex eigenvalues, λ_i, and complex eigenvectors, ψ_i.

$$\lambda_i \mathbf{A}\,\psi_i + \mathbf{B}\,\psi_i = \mathbf{0} \tag{5.39}$$

The complex Lanczos algorithm[29] can be used to solve the complex eigenvalue problem. The computation effort of these vectors is quite intensive. Similar to the real eigenvector to the undamped problem, by orthogonality condition of the complex eigenvectors, the

differential equation is uncoupled in the eigenspace. Hence dynamic responses of the damped structure in the eigenspace can be computed easily.

$$\mathbf{z} = \Psi\, \mathbf{q} \tag{5.40}$$

$$\Psi^T \mathbf{A}\, \Psi\, \mathbf{q} + \Psi^T \mathbf{B}\, \Psi\, \mathbf{q} = \Psi^T \begin{bmatrix} \mathbf{p} \\ \mathbf{0} \end{bmatrix} e^{i\omega t} \tag{5.41}$$

The complex eigenvector can be expressed as follows:

$$\psi_i = \begin{bmatrix} \psi^u_i \\ \psi^{\dot{u}}_i \end{bmatrix} \tag{5.42}$$

and

$$\psi^{\dot{u}}_i = \lambda_i\, \psi^u_i \tag{5.43}$$

Therefore, despite the formulation is in the state space form with displacement and velocity as variables, the eigenvector in the state space is defined by the complex displacement eigenvector and eigenvalue.

The complex displacement vector from a harmonic forcing function can be shown to have the same quality as the complex eigenvector. From Equation (5.7), the solution can be assumed to be:

$$\mathbf{u} = (\mathbf{u}^R + i\, \mathbf{u}^I)\, e^{i\omega t} \tag{5.44}$$

$$\dot{\mathbf{u}} = (i\omega \mathbf{u}^R - \omega\, \mathbf{u}^I)\, e^{i\omega t} \tag{5.45}$$

Therefore, in the state space representation,

$$\begin{bmatrix} \mathbf{u} \\ \dot{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{u}^R \\ -\omega \mathbf{u}^I \end{bmatrix} e^{i\omega t} + i \begin{bmatrix} \mathbf{u}^I \\ \omega \mathbf{u}^R \end{bmatrix} e^{i\omega t} \tag{5.51}$$

The velocity vector is a scalar multiple of the out-of-phase displacement vector. The vector basis for the out-of-phase displacement vectors and the velocity vectors is identical. The

vector basis for the state space representation is the same as the second order differential equation with in-phase and out-of-phase vector representation. Furthermore, if the second order differential equation solution is cast in the 2n displacement formulation, Equation (5.23), the two approaches look a lot more similar. The displacement approach can accommodate a frequency shift (Taylor series expansion around a selected point) easily. It also allows constant hysteretic damping and frequency dependent viscoelastic damping to be included easily. In this regard, the 2n displacement formulation has distinct advantages over the state space formulation.

5.3 Iterative Solution Method

The exact solution to Equation (5.1) requires a factorization of the 2nx2n unsymmetric matrix, Equation (5.23), or the computation and factorization of the nxn $\left[(K-\omega^2M)+\omega^2C(K-\omega^2M)^{-1}C\right]$ matrix. For problems of relatively small size, it provides a direct and exact solution. However, for large systems this may require large computational effort and computer storage space since the bandwidth (or profile) of the matrix, $\left[C(K-\omega^2M)^{-1}C\right]$, may be large or fully populated. However, if the damping of the system is small, as is normally the case (say $\xi < 0.3$), an approximate solution can be obtained quite accurately and efficiently. Also, when the forcing function is not exactly harmonic, the approximate solution will provide a vector basis which spans the solution space reasonable well and it also provides the harmonic correction. Inspired by the treatment of the undamped basis of Chapter 4, a damped vector basis is derived.

5.3.1. Static Correction Iteration Method (Stodola Type Iteration)

If the damping force and inertial force are treated as applied forces, similar to the D'Alembert Principle, Equations (5.23) is rewritten in the following form :

$$\begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} u^R \\ u^I \end{bmatrix} = \begin{bmatrix} p \\ 0 \end{bmatrix} + \begin{bmatrix} \omega^2 M & \omega C \\ -\omega C & \omega^2 M \end{bmatrix} \begin{bmatrix} u^R \\ u^I \end{bmatrix} \tag{5.47}$$

Since the displacement is unknown, the inertia and damping forces on the right hand side of Equation (5.47) is also unknown. The equation can be solved by the following iterative scheme.

$$\begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} u^R \\ u^I \end{bmatrix}^{(k+1)} = \begin{bmatrix} p \\ 0 \end{bmatrix} + \begin{bmatrix} \omega^2 M & \omega C \\ -\omega C & \omega^2 M \end{bmatrix} \begin{bmatrix} u^R \\ u^I \end{bmatrix}^{(k)} \tag{5.48}$$

$$\begin{bmatrix} u^R \\ u^I \end{bmatrix}^{(k+1)} = \begin{bmatrix} K^{-1}p \\ 0 \end{bmatrix} + \begin{bmatrix} \omega^2 K^{-1} M & \omega K^{-1} C \\ -\omega K^{-1} C & \omega^2 K^{-1} M \end{bmatrix} \begin{bmatrix} u^R \\ u^I \end{bmatrix}^{(k)} \tag{5.49}$$

Each iteration corrects the inaccuracy of the the dynamic effect due to damping and inertial forces. Even though Equation (5.49) is cast in the 2n dimensional space, the computation can be performed in the n dimensional space. The stiffness matrix is factorized only once. The analysis is reduced to simple static analyses of the applied force, inertia force and damping force only. This method is very similar to the Stodola's method which is used compute the undamped natural frequency and mode shape of a structure. Typically, all intermediate vectors are discarded (over written in the computer) at each iteration.

The system is represented in the 2n dimensional space for analysis purpose. The actual computation and storage can be in the n dimensional space, i.e. the factorized stiffness matrix, the damping and mass matrices.

$$u^{R(k+1)} = K^{-1}p + \omega^2 K^{-1} M u^{R(k)} + \omega K^{-1} C u^{I(k)} \tag{5.50}$$

$$u^{R(k+1)} = -\omega^2 K^{-1} C u^{R(k)} + \omega^2 K^{-1} M\, u^{I(k)} \tag{5.51}$$

In order to analyze this method, the following definitions are helpful:

$$\mathbf{A}_1 = \begin{bmatrix} \omega^2\mathbf{K}^{-1}\mathbf{M} & \omega\mathbf{K}^{-1}\mathbf{C} \\ -\omega\mathbf{K}^{-1}\mathbf{C} & \omega^2\mathbf{K}^{-1}\mathbf{M} \end{bmatrix} \tag{5.52}$$

$$\mathbf{w}^{(k)} = \begin{bmatrix} \mathbf{u}^R \\ \mathbf{u}^I \end{bmatrix}^{(k)} \tag{5.53}$$

$$\mathbf{w}^{(1)} = \begin{bmatrix} \mathbf{K}^{-1}\mathbf{p} \\ \mathbf{0} \end{bmatrix} \tag{5.54}$$

Then Equation (5.49) can be shown as:

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(1)} + \mathbf{A}_1\, \mathbf{w}^{(k)} \tag{5.55}$$

In order to provide a simple analysis of the method, a few iteration steps are shown as follows:

$$\mathbf{w}^{(2)} = (\mathbf{I} + \mathbf{A}_1)\, \mathbf{w}^{(1)} \tag{5.56}$$

$$\mathbf{w}^{(3)} = (\mathbf{I} + \mathbf{A}_1 + \mathbf{A}_1^2)\, \mathbf{w}^{(1)} \tag{5.57}$$

$$\mathbf{w}^{(k+1)} = \left(\sum_{j=0}^{k} \mathbf{A}_1^j\right) \mathbf{w}^{(1)} \tag{5.58}$$

Therefore, at each iteration, the coefficient matrix of a higher power is added to the matrix coefficient series.

5.3.2 Static Incremental Correction Method

The same iterative concept can be explored in a different form to understand the incremental improvement to the solution at each step. An incremental correction iterative solution method can be devised by defining:

$$\mathbf{w}^{(k)} = \begin{bmatrix} \mathbf{u}^R \\ \mathbf{u}^I \end{bmatrix}^{(k)} \tag{5.59}$$

$$\mathbf{w}^{(k)} = \sum_{j=1}^{k} \Delta \mathbf{w} \tag{5.60}$$

$$\Delta \mathbf{w}^{(k+1)} = \mathbf{w}^{(k+1)} - \mathbf{w}^{(k)} \tag{5.61}$$

The iterative equation for this static incremental correction method can be written as:

$$\begin{bmatrix} \Delta \mathbf{u}^R \\ \Delta \mathbf{u}^I \end{bmatrix}^{(k+1)} = \begin{bmatrix} \omega^2 \mathbf{K}^{-1}\mathbf{M} & \omega \mathbf{K}^{-1}\mathbf{C} \\ -\omega \mathbf{K}^{-1}\mathbf{C} & \omega^2 \mathbf{K}^{-1}\mathbf{M} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{u}^R \\ \Delta \mathbf{u}^I \end{bmatrix}^{(k)} \tag{5.62}$$

or,

$$\Delta \mathbf{w}^{(k+1)} = \mathbf{A}_1 \Delta \mathbf{w}^{(k)} \tag{5.63}$$

The initial vector is defined as:

$$\Delta \mathbf{w}^{(1)} = \begin{bmatrix} \mathbf{K}^{-1}\mathbf{p} \\ \mathbf{0} \end{bmatrix} \tag{5.64}$$

A few iteration steps are shown to gain insight into the solution process:

$$\Delta \mathbf{w}^{(2)} = \mathbf{A}_1 \Delta \mathbf{w}^{(1)} \tag{5.65}$$

$$\Delta \mathbf{w}^{(3)} = \mathbf{A}_1^2 \Delta \mathbf{w}^{(1)} \tag{5.66}$$

$$\Delta \mathbf{w}^{(k)} = \mathbf{A}_1^k \Delta \mathbf{w}^{(1)} \tag{5.67}$$

$$\mathbf{w}^{(k+1)} = \left(\sum_{j=0}^{k} \mathbf{A}_1^j \right) \mathbf{w}^{(1)} \tag{5.68}$$

Therefore, at each iteration, a higher power matrix is added to the matrix coefficient series. The result is identical to the Static Correction Iteration Method. Since the initial

vector is not included in the linear solution at every step, the computed vector has higher numerically quality.

5.3.3 Harmonic Correction Method

When a structure is in state steady harmonic vibration, the inertial forces is well defined. Therefore the inertia force of Equation (5.48) can remain on the left hand side. Now, rewrite the equation and assume $\mathbf{K}-\omega^2\mathbf{M}$ is not singular:

$$\begin{bmatrix} \mathbf{K}-\omega^2\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}-\omega^2\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{u}^R \\ \mathbf{u}^I \end{bmatrix} = \begin{bmatrix} \mathbf{p} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \omega\mathbf{C} \\ -\omega\mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}^R \\ \mathbf{u}^I \end{bmatrix} \tag{5.69}$$

An iterative solution to Equation (5.69) can be set up easily as follows :

$$\begin{bmatrix} \mathbf{K}-\omega^2\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}-\omega^2\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{u}^R \\ \mathbf{u}^I \end{bmatrix}^{(k+1)} = \begin{bmatrix} \mathbf{p} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \omega\mathbf{C} \\ -\omega\mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}^R \\ \mathbf{u}^I \end{bmatrix}^{(k)} \tag{5.70}$$

These 2nx2n coefficient matrices are actually composed of two nxn submatrices. It is obvious that $(\mathbf{K}-\omega^2\mathbf{M})$ must be non-singular and the first step of the iteration process is the undamped harmonic solution to the problem. All subsequent iterations are just improvements to correct for the neglected damping effect. Rewrite this equation for computation purpose:

$$(\mathbf{K}-\omega^2\mathbf{M})\ \mathbf{u}^{R(k+1)} = \mathbf{p} + \omega\ \mathbf{C}\ \mathbf{u}^{I\,(k)} \tag{5.71}$$

$$(\mathbf{K}-\omega^2\mathbf{M})\ \mathbf{u}^{I\,(k+1)} = -\omega\ \mathbf{C}\ \mathbf{u}^{R(k)} \tag{5.72}$$

The results of the first two iterations are given below:

$$\mathbf{u}^{R(1)} = (\mathbf{K}-\omega^2\mathbf{M})^{-1}\mathbf{p} \tag{5.73}$$

$$\mathbf{u}^{I\,(1)} = -\omega\ (\mathbf{K}-\omega^2\mathbf{M})^{-1}\mathbf{C}\ \mathbf{p} \tag{5.74}$$

$$\mathbf{u}^{R(2)} = (\mathbf{K}-\omega^2\mathbf{M})^{-1}\mathbf{p} - \omega^2\ (\mathbf{K}-\omega^2\mathbf{M})^{-1}\mathbf{C}(\mathbf{K}-\omega^2\mathbf{M})^{-1}\mathbf{C}\mathbf{p} \tag{5.75}$$

$$\mathbf{u}^{I(2)} = -\omega\, \mathbf{C}(\mathbf{K}-\omega^2\mathbf{M})^{-1}\mathbf{p} + \omega^3\, \mathbf{C}(\mathbf{K}-\omega^2\mathbf{M})^{-1}\mathbf{C}(\mathbf{K}-\omega^2\mathbf{M})^{-1}\mathbf{C}\mathbf{p} \tag{5.76}$$

However, the quality of the vector space is dependent of the choice of ω. However, it is very easy to include frequency dependent viscous and viscoelastic damping in the solution process.

Now by defining the incremental correction of each iteration as:

$$\Delta\mathbf{u}^{(k+1)} = \mathbf{u}^{(k+1)} - \mathbf{u}^{(k)} \tag{5.77}$$

$$\begin{bmatrix} \mathbf{K}-\omega^2\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}-\omega^2\mathbf{M} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u}^R \\ \Delta\mathbf{u}^I \end{bmatrix}^{(k+1)} = \begin{bmatrix} \mathbf{0} & \omega\mathbf{C} \\ -\omega\mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u}^R \\ \Delta\mathbf{u}^I \end{bmatrix}^{(k)} \tag{5.78}$$

The vectors so computed span the same space as those computed from Equation (5.70) and the vector space contains the exact harmonic solution. This is the increment harmonic correction formulation.

5.3.4 Taylor Series Expansion Method

Similar to the undamped vector basis, a Taylor series approach can be used to derive the basis vectors for the problem given by:

$$\begin{bmatrix} \mathbf{u}^R \\ \mathbf{u}^I \end{bmatrix} = \begin{bmatrix} \mathbf{K}-\omega^2\mathbf{M} & -\omega\mathbf{C} \\ \omega\mathbf{C} & \mathbf{K}-\omega^2\mathbf{M} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{p} \\ \mathbf{0} \end{bmatrix} \tag{5.23}$$

Similar to the development in Section 4.2.5, the expansion can be around the zero frequency or a selected frequency. The expansion is straight forward but tedious. The solution to Equation (5.23) is thus expressed as a linear combination of vectors which can then be used as the basis vectors for computation of dynamic responses as discussed in Section 5.4.

5.4 Reduced Vector Basis Dynamic Response

Once the vector basis is developed, the computation of dynamic response is relatively easy. The solution is constrained in the subspace defined by the basis and is expressed as a linear combination in the vector coordinates as:

$$\begin{bmatrix} \mathbf{u}^R \\ \mathbf{u}^I \end{bmatrix} = \begin{bmatrix} \mathbf{V}^R \\ \mathbf{V}^I \end{bmatrix} \mathbf{y} \tag{5.79}$$

In the frequency domain, the solution can be simplified to:

$$\{[\mathbf{V}^{R^T}(\mathbf{K}-\omega^2\mathbf{M})\mathbf{V}^R + \mathbf{V}^{I^T}(\mathbf{K}-\omega^2\mathbf{M})\mathbf{V}^I] + [\,\omega\mathbf{V}^{I^T}\mathbf{C}\mathbf{V}^R - \omega\mathbf{V}^{R^T}\mathbf{C}\mathbf{V}^I\,]\}\,\mathbf{Y}(\omega)$$

$$= \mathbf{V}^{R^T}\mathbf{p}\,G(\omega) \tag{5.80}$$

If the effect of damping is small, then the complex part of the basis vector is very small in comparison with the real part. Then Equation (5.80) degenerates to the solution to the undamped vector basis.

5.5 Comment

In order to passively control the vibration of a structure, damping must be introduced. This establishes the need for accurate analysis method of non-proportionally damped structures. This chapter presented a new angle in looking at the dynamic response of viscously damped structures. It is based on the same physical, mathematical and numerical concepts as the undamped vector basis. The theoretical framework of the vector space is sound and it enables accurate solution to be computed. The damped vector basis so derived is based on the mass, stiffness and damping matrices and is therefore more representation of the damped structure. This damped basis can be easily extended to evaluate dynamic response of structures with viscoelastic passive damping design. More work in the

implementation area, namely Gram-Schmidt orthogonalization and vector convergence criteria, is required to demonstrate that the effectiveness of this vector basis.

Chapter 6

Viscoelastically Damped Structures

Passive vibration control can be achieved through proper design of the three basic structural parameters, mass, stiffness and damping. Structural mass and stiffness are often used to attenuate dynamic responses. The use of discrete dampers and distributed damping materials is less common despite the benefit of damping is well known. The analyses of large damped systems are quite difficult. The damping characteristics, and mechanisms of materials and structural systems are often not very defined. Therefore, a systematic design approach to meet the design damping goal has been evolving very slowly. Most structures were designed with an amount of passive damping inherited from the selected materials and the method of construction/fabrication. Damping is seldom used as a controllable design parameter. However, with better characterization of viscoelastic materials and better analytical techniques, it has been possible to design a desirable level of damping into a specific mode of vibration and thereby achieved passive vibration control[30]. This passive damping design with viscoelastic materials has been demonstrated to be very successful through a series of dynamic tests[31]. The comparison between analytical predictions and test results is very favorably. Consequently, the possibility for innovative structural design has increased tremendously. Substantial savings in materials and cost are now possible. A new breed of structural systems is emerging.

In this chapter, the basic modelling techniques of viscoelastic materials are reviewed. The analytical methods currently used to design the viscoelastically damped structures are presented in a mathematical form which is consistent with the general theoretical development of this dissertation. New methods for broader application of viscoelastically damped structures, and assumptions and limitations are also thoroughly discussed.

6.1 Viscoelastic Material Characterization

The material properties of viscoelastic materials are rate dependent. This rate dependent property results in hysteretic behavior. By understanding the energy dissipation characteristics of these materials, damping can be more effectively designed into structures to control vibrations. Since viscoelastic materials are rate dependent, they are also frequency dependent. Most viscoelastic material properties are also strongly temperature dependent.

6.1.1 Uniaxial Strain-Strain Relationship

The uniaxial stress-strain mechanical properties of viscoelastic materials can be characterized by a standard linear model[32]:

$$\sigma + \alpha \frac{d\sigma}{dt} = E\varepsilon + \beta E \frac{d\varepsilon}{dt} \tag{6.1}$$

Kelvin and Voigt viscoelastic models are special cases of this model. In this simple form, the stress is dependent on the rate of the strain and vice verse. For a more general rheological model of viscoelastic materials, higher derivatives can be introduced into the mathematical representation of the standard model:

$$\sigma + \sum_{i=1}^{n} \alpha_i \frac{d^i \sigma}{dt^i} = E\varepsilon + E \sum_{i=1}^{n} \beta_i \frac{d^i \sigma}{dt^i} \tag{6.2}$$

For harmonic response, of the form $\sigma = \sigma_o e^{i\omega t}$, due to imposed strain $\varepsilon = \varepsilon_o e^{i\omega t}$, the stress-strain relationship can be simplified to:

$$\sigma_o = \frac{E\,\varepsilon_o\left[\sum_{i=1}^{n}\beta_i\,(i\omega)^i\right]}{\left[\sum_{i=1}^{n}\alpha_i\,(i\omega)^i\right]} \tag{6.3}$$

which can also be written in the form:

$$\sigma_o = E_o(\omega)\,\varepsilon_o \tag{6.4}$$

or,

$$\sigma_o = (E^R(\omega) + i\,E^I(\omega))\,\varepsilon_o \tag{6.5}$$

where E^R and E^I are now complex functions of ω. By proper choices of β_i and α_i, this material model can be tailored to fit measured data for E^R and E^I as a function of frequency. Alternatively, the material loss factor, η, is defined as the ratio of the complex modulus to the real modulus:

$$\eta(\omega) = \frac{E^I(\omega)}{E^R(\omega)} \tag{6.6}$$

The stress strain relationship can then be expressed as:

$$\sigma_o = E^R(\omega)\,(1 + i\,\eta(\omega))\,\varepsilon_o \tag{6.7}$$

This representation allows a more physical interpretation of the stiffness and damping characteristics of the viscoelastic materials. The loss factor, η, is related to the material dissipation as described in the next subsections.

The complex modulus can be measured experimentally, irrespective of the order of expansion of the material modelling. A typical plot of material properties, i.e. real modulus and loss factor, is shown in Figure 6-1.

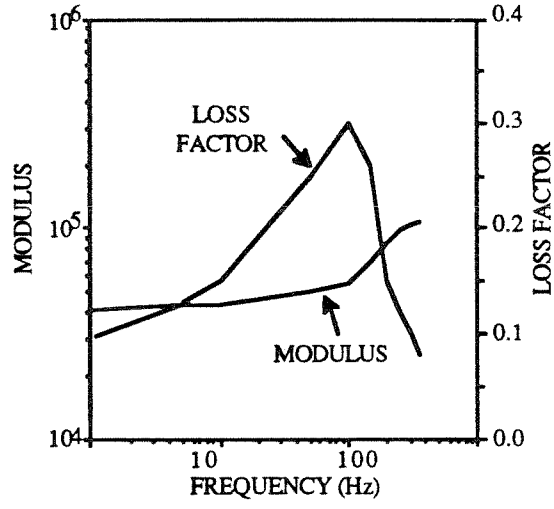

Figure 6-1. Typical Viscoelastic Material Properties

6.1.2 Dissipation

Equation (6.5) represents frequency dependent material viscoelastic damping. Unlike viscous damping, the viscoelastic damping in the material is proportional to the strain, and hence the displacement of the system. If a strain cycle, $\varepsilon = \varepsilon_o e^{i\omega t}$, is imposed, the energy dissipation per cycle is given by:

$$D = \int_0^{2\pi/\omega} \sigma \left(\frac{d\varepsilon}{dt}\right) dt \tag{6.8}$$

The terms inside the integral can be expressed as follows:

$$\begin{aligned} \varepsilon &= \varepsilon_o e^{i\omega t} \\ &= \varepsilon_o (\cos \omega t + i \sin \omega t) \end{aligned} \tag{6.9}$$

$$\frac{d\varepsilon}{dt} = \varepsilon_o (\omega \sin \omega t - i \omega \cos \omega t) \tag{6.10}$$

$$\begin{aligned} \sigma &= E^R (1 + i \eta) \varepsilon_o e^{i\omega t} \\ &= E^R \varepsilon_o (\cos \omega t - \eta \sin \omega t) + i E^R \varepsilon_o (\eta \sin \omega t + \cos \omega t) \end{aligned} \tag{6.11}$$

Take the real parts of the stress and strain and integrate in accordance with Equation (6.8):

$$D = \pi \eta E^R \varepsilon_o^2 \tag{6.12}$$

Therefore materials with higher modulus, higher loss factor and higher strain will give higher energy dissipation and hence higher damping.

6.2 Single Degree of Freedom System

Since most multiple degrees of freedom system can be transformed into or approximated by uncoupled single degree of freedom system, understanding of the dynamic behavior of a damped single degree of freedom system is very important. A study of the single degree of freedom oscillator is presented in the following subsections to provide insight into the dynamic behavior of viscously damped and viscoelastically damped systems.

6.2.1 Viscously Damped System

A dashpot is an idealized model with the damping force proportional to the velocity. The advantage of this type of damping modelling in vibration is that the mathematical solution is very simple. It is therefore often used to model (or approximate the effect of) energy dissipation in structures. Other forms of damping are often expressed as an equivalent viscous damping for ease of solution.

A typical viscously damped single degree of freedom oscillator is shown in Figure 6-2. The governing differential equation of motion is:

$$m\ddot{u} + c\dot{u} + ku = p \cos \omega t \tag{6.13}$$

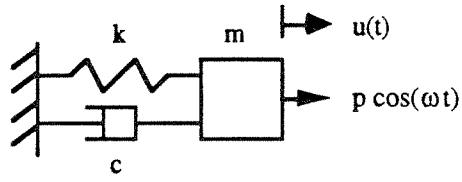

Figure 6-2. Viscously Damped System

The static displacement of the system is given by:

$$u_s = \frac{p}{k} \tag{6.14}$$

The undamped natural frequency, ω_1, is:

$$\omega_1 = \sqrt{\frac{k}{m}} \tag{6.15}$$

The steady state response of the system is:

$$u = A\cos(\omega t - \phi) \tag{6.16}$$

$$\text{where amplitude, } A = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}} \frac{p}{k} \tag{6.17}$$

$$\text{Phase angle, } \phi = \tan^{-1}\left(\frac{2\xi\beta}{1-\beta^2}\right) \tag{6.18}$$

$$\text{damping ratio, } \xi = \frac{c}{2m\omega_1} \tag{6.19}$$

$$\text{frequency ratio, } \beta = \frac{\omega}{\omega_1} \tag{6.20}$$

The peak response is:

$$A_{max} = \frac{1}{2\xi\sqrt{1-\xi^2}} \frac{p}{k} \tag{6.21}$$

For lightly damped system (say $\xi \lesssim 0.3$), the peak response is:

$$A_{max} \approx \frac{1}{2\xi} \frac{p}{k} \tag{6.22}$$

The energy dissipation per cycle in the system is:

$$D = \pi c \omega A^2 \ (= 2 \pi \xi \omega_1 \omega m A^2) \tag{6.23}$$

For a viscously damped system, the energy dissipation is a linear function of the forcing frequency. This may be undesirable for modelling some physical systems.

When the system is under free vibration, the damped natural frequency of the system, ω_d, is:

$$\omega_d = \sqrt{1-\xi^2}\ \omega_1 \tag{6.24}$$

The natural frequency of a system is lowered by the damping present in the system.

6.2.2 Viscoelastically Damped System

The viscoelastically damped system is characterized by a spring with complex stiffness as shown in Figure 6-3.

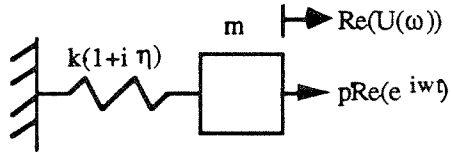

Figure 6-3. Viscoelastically Damped System

The system is better described in the frequency domain as:

$$[-\omega^2 m + k (1+i\eta)] U = p e^{i\omega t} \tag{6.25}$$

$$U = \frac{p}{k [(1-\beta^2) + i\eta]} e^{i\omega t}$$

$$U = \frac{p[(1-\beta^2) - i\eta]}{k\,[(1-\beta^2)^2 + \eta^2]}\, e^{i\omega t} \tag{6.26}$$

The dynamic amplification of a viscoelastically damped system is compared with a comparable viscously damped system as shown in Figure 6-4.

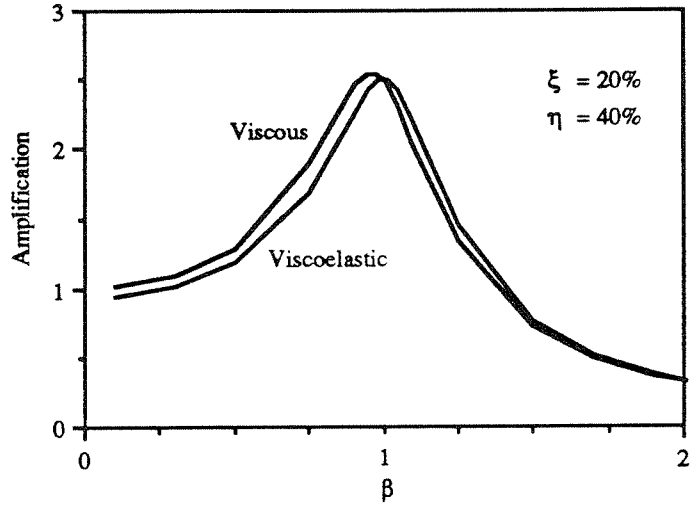

Figure 6-4. Dynamic Amplification of Viscously and Viscoelastically Damped Systems

The static displacement of this system can be found by setting the forcing frequency to zero,

$$u_s = \frac{1}{\sqrt{1+\eta^2}} \frac{p}{k} \tag{6.27}$$

This static displacement of this model is therefore dependent on the loss factor. This relationship is non-causual and non-physical. This is attributed to the assumption of constant loss factor over the entire frequency range. However, if actual measured (frequency dependent) loss factor is used, this inconsistency can be resolved. For the level of viscoelastic damping normally exists in structures, this error is not significant.

The steady state response of the system is:

$$u = A\cos(\omega t - \phi) \tag{6.28}$$

$$\text{where amplitude, } A = \frac{1}{\sqrt{(1 - \beta^2)^2 + \eta^2}} \frac{p}{k} \tag{6.29}$$

$$\text{Phase angle, } \phi = \tan^{-1}\left(\frac{\eta}{1 - \beta^2}\right) \tag{6.30}$$

The energy dissipation per cycle in the system is:

$$D = \pi \eta k A^2 \tag{6.31}$$

Unlike the viscously damped system the energy dissipation of viscoelastically damped system is frequency independent (if the stiffness and loss factor are frequency independent).

The equivalent viscous damping of viscoelastically damped system can be found by imposing equal response amplitude and equating the energy dissipation per cycle as:

$$D = \pi \eta k A^2 = \pi c_{eq} \omega A^2 \tag{6.32}$$

$$c_{eq} = \frac{\eta k}{\omega} \tag{6.33}$$

$$\xi_{eq} = \frac{\eta k}{2m\omega_1\omega} \tag{6.34}$$

$$\xi_{eq} = \frac{\eta}{2\beta} \tag{6.35}$$

Since the viscous energy dissipation is a function of the loading frequency, the equivalent viscous damping of a viscoelastically damped system is also frequency dependent. Since

the energy dissipation is maximum at resonance, the equivalent damping is normally evaluated at the resonant frequency. Therefore at the resonant frequency,

$$\xi_{eq} = \frac{\eta}{2} \tag{6.36}$$

At the resonant frequency, this relationship is exact and the phase angle due to this equivalent relationship is also exact.

Unlike the viscously damped system, the viscoelastic damping has no effect on resonant frequency of the system, i.e.

$$\omega_d = \omega_1 \tag{6.37}$$

6.2.3 Viscously and Viscoelastically Damped System

If viscoelastic materials are used in a structure, the structure is probably subject to both viscous damping and viscoelastic damping. Figure 6-5 depicts a viscously and viscoelastically damped single degree of freedom oscillator.

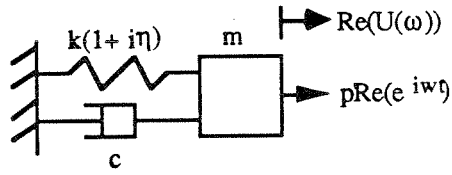

Figure 6-5. Viscously and Viscoelastically Damped System

The system is better described in the frequency domain as:

$$[(-\omega^2 m + k) + i(\omega c + \eta k)]\, U = p\, e^{i\omega t} \tag{6.38}$$

$$U = \frac{p}{k\,[(1-\beta^2) + i\,(2\xi\beta+\eta)]}\, e^{i\omega t} \tag{6.39}$$

$$U = \frac{p[(1-\beta^2) - i\,(2\xi\beta+\eta)]}{k\,[(1-\beta^2)^2 + (2\xi\beta+\eta)^2]}\, e^{i\omega t} \tag{6.40}$$

The static displacement of this system can be found by setting the forcing frequency to zero,

$$u_s = \frac{1}{\sqrt{1+\eta^2}}\,\frac{p}{k} \tag{6.41}$$

The static displacement of this model is also dependent on the loss factor as expected. Again for the level of hysteretic damping normal exists in structures, this error is not significant.

The steady state response of the system is:

$$u = A\cos(\omega t - \phi) \tag{6.42}$$

$$\text{where amplitude, } A = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta+\eta)^2}}\,\frac{p}{k} \tag{6.43}$$

$$\text{phase angle, } \phi = \tan^{-1}\left(\frac{2\xi\beta+\eta}{1-\beta^2}\right) \tag{6.44}$$

The equivalent viscous damping of viscoelastically damped system can be determined by adding the contribution due to viscous damping and viscoelastic damping as:

$$\xi_{eq} = \xi + \frac{\eta}{2\beta} \tag{6.45}$$

The damped natural frequency of the system is influenced by the viscous damping only:

$$\omega_d = \sqrt{1-\xi^2}\;\omega_1 \tag{6.46}$$

The conclusion and concept drawn from this single degree of freedom system pave the way for the design and analysis of more complex structures which are described in detail in the following sections.

6.3 Generalization of Constitutive Relationship

It has been demonstrated that the most effective way of using viscoelastic materials to introduce energy dissipation and damping into a structure is through shear deformation in the viscoelastic materials[32,33]. Shear moduli and loss factors of these materials have been extensively tested and documented. In order to analyze a structure with viscoelastic materials, the general constitutive relationship must be known. However, due to the complexity of the material properties in relationship to temperature and frequency, they are seldom completely measured. It is assumed that the general constitutive relationship can be characterized by the same complex modulus representation as the shear deformation. If it is further assumed that the material in use is isotropic and the same loss factor is applicable to other stiffness moduli. Therefore,

$$\sigma_{ij} = C^*_{ijkl}\, \varepsilon_{kl} \tag{6.47}$$

$$C^*_{ijkl} = C_{ijkl}(T, \omega)\, [\, 1 + i\, \eta^v(T, \omega)\,] \tag{6.48}$$

where η^v = material loss factor

If the viscoelastic materials are designed to act primarily through shear deformation to provide damping and stiffness, then the errors introduced by these assumptions are small. These assumptions allow conventional finite elements to be used to model structures with viscoelastic materials. Complex viscoelastically damped structures can therefore be designed with the help of finite element computer codes.

6.4 Formulation of Finite Elements with Viscoelastic Material Properties

The basic finite element formulations of viscoelastic elements are similar to elastic elements. The only difference is in the constitutive relationship. Since the material properties are generally represented in complex numbers, the stiffness matrix is also complex. Finite element library of existing computer codes can be used to compute viscoelastic element stiffness matrices without any modification. The element stiffness matrix has two parts[30,34]:

$$\mathbf{k}(\omega) = \mathbf{k}^{R}(\omega) + i\,\mathbf{k}^{I}(\omega) \tag{6.49}$$

Since the imaginary component is assumed to be a scalar multiple of the real component, the complex element stiffness matrix at a constant temperature is represented by:

$$\mathbf{k}(\omega) = \mathbf{k}^{R}(\omega) + i\,\eta^{v}(\omega)\,\mathbf{k}^{R}(\omega) \tag{6.50}$$

Consequently, the global structural stiffness matrix is also complex. The complex global stiffness matrix consisting of the viscoelastic elements is symbolically represented by:

$$\mathbf{K}_{v}(\omega) = \sum_{j=1}^{n_v} \mathbf{k}^{R}{}_{j}(\omega) + i \sum_{j=1}^{n_v} \mathbf{k}^{I}{}_{j}(\omega) \tag{6.51}$$

where n_v = number of viscoelastic elements

and therefore,

$$\mathbf{K}_{v}(\omega) = \mathbf{K}_{v}{}^{R}(\omega) + i\,\mathbf{K}_{v}{}^{I}(\omega) \tag{6.52}$$

The imaginary component of the global stiffness matrix may be contributed by element stiffness matrices with different material loss factors. If the global stiffness matrix of the

elastic elements is given by $\mathbf{K}_e$, the real part of the global stiffness matrix of the structure can be defined as:

$$\mathbf{K}^R(\omega) = \mathbf{K}_e + \mathbf{K}_v{}^R(\omega) \tag{6.53}$$

The governing equation of the viscoelastically damped structure in the frequency domain is:

$$[(\mathbf{K}^R(\omega) - \omega^2 \mathbf{M}) + i\,\mathbf{K}^I(\omega)]\,\mathbf{U}(\omega) = \mathbf{p}\,G(\omega) \tag{6.54}$$

If viscous damping also exists in the structure, the governing equation is:

$$\{(\mathbf{K}^R(\omega) - \omega^2 \mathbf{M}) + i\,[\omega C + \mathbf{K}^I(\omega)]\}\,\mathbf{U}(\omega) = \mathbf{p}\,G(\omega) \tag{6.55}$$

6.5 Finite Element Modelling of Structure with Viscoelastic Materials

Similar to the design of a structure with complex geometric configuration, the design of a viscoelastically damped structure with complex geometric configuration must be modelled by finite element method. Currently, viscoelastic materials are used in very thin layers. The modelling of material with thin layer is difficult because of the bad aspect ratio associated with the finite element. However, experience shows that reasonable accuracy can be expected from such modelling technique[30]. This is because the stress field in the material is normally quite simple. The design of the viscoelastic structure requires the material to be stressed in shear only. Since most plate elements does not have adequate out-of-plane shear representation, solid elements are normally used. Most solid finite elements have interpolation functions which can adequately describe this mode of stress and strain distribution provided that the matrix conditioning with respect to the machine precision is not too bad. Figure 6-6 illustrates the modelling of a typical constrained layer by a mixture of plate elements and solid elements. Plate elements are modelled at the centerline of plates. As such, rigid offset elements, constrained equations or master-slave

type modelling is necessary to define the geometric relationship between the plate centerline and the interface of plate and solid to avoid additional matrix ill conditioning and also reduce the total number of unknowns.

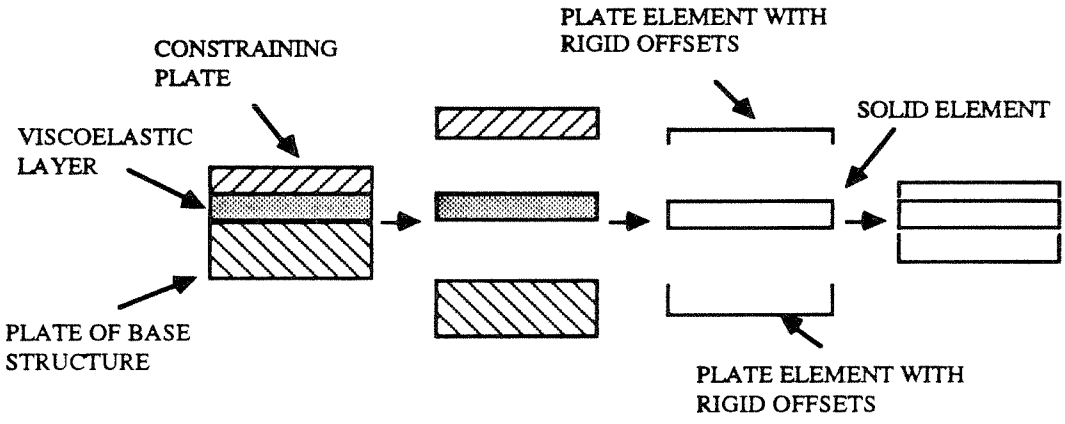

Figure 6-6. Modelling of Constrained Layer

A simple viscoelastic strut design is shown in Figure 6-7. A thin layer of viscoelastic material is sandwiched between the inner and outer thin. The axial force in the tube is transmitted through the shear deformation in the viscoelastic layer[34,35]. The finite element model of this design is shown in Figure 6-8.

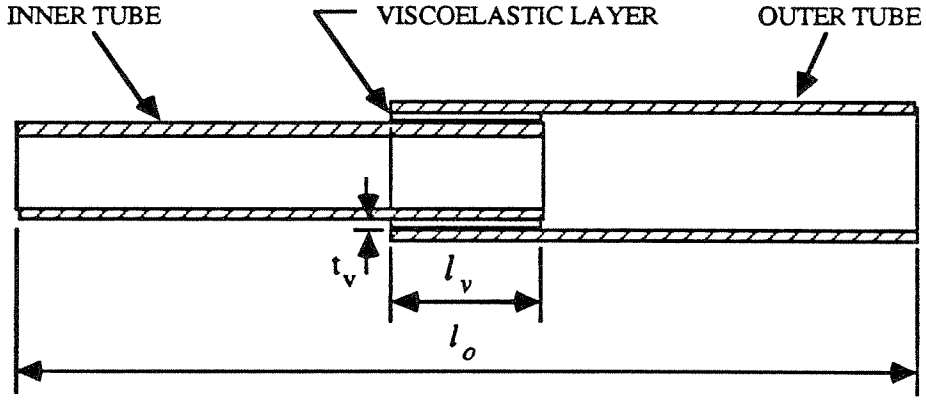

Figure 6-7. Conceptual Design of Viscoelastic Strut

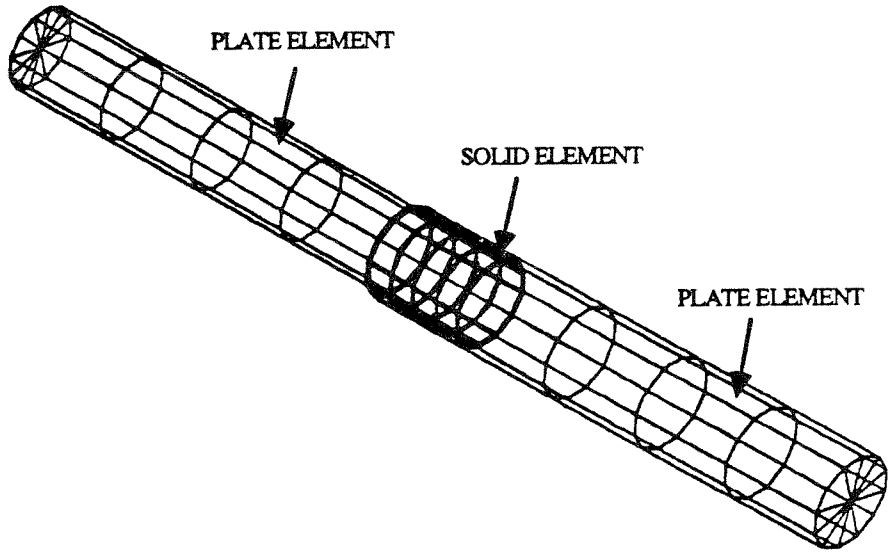

Figure 6-8. Finite Element Model of Viscoelastic Strut

6.6 Analysis Method for Viscoelastically Damped Structures

Analysis methods similar to those used in viscously damped structures can be used. For most structures, the explicit viscous damping matrix is never computed. Modal damping ratios are assigned to the uncoupled modal differential equation. However, when the structure is designed to use viscoelastic materials to enhance the damping characteristic, the conventional method of assigning modal damping ratios cannot be used. Complex stiffness matrix which characterizes the viscoelastic damping of the structure must be computed from the material properties. The exact solution to Equation (5.55) is rather complicated and computationally intensive. Assumptions must be made to simplify the analysis. These assumptions should simplify the mathematics for computational efficiency

without significant sacrifice of accuracy. The following three assumptions are therefore important in computing a solution to the problem:

1. Frequency dependence is not significant over the range of dynamic response. The material property can be taken as constant at the dominate frequency. If necessary, adjustment to solution can be made to correct for any frequency dependence.
2. Undamped vector basis is used such that the behavior of the structure can be conceptualized and understood in the undamped vector space.

$$\mathbf{u} = \mathbf{X}\,\mathbf{y} \tag{6.56}$$

$$\mathbf{X}^T\mathbf{M}\mathbf{X} = \mathbf{I} \tag{5.57}$$

$$\mathbf{X}^T\mathbf{K}^R\mathbf{X} = \Lambda_X \tag{6.58}$$

Correction is to be made to account for the damping coupling in the undamped vector space.

3. Inherent passive damping contributed by the elastic materials and the joints of the structure is adequately described by modal viscous damping ratios and the damping matrix in the reduced vector space is diagonal:

$$\mathbf{X}^T\,\mathbf{C}\,\mathbf{X} = \mathbf{C}_{Xd} = \text{Diagonal} = [2\xi_i\omega_{Xi}] \tag{6.59}$$

Consequently, the governing equation of the viscoelastically damped structure, Equation (6.55), is simplified to the following form:

$$\left[(\mathbf{K}^R-\omega^2\mathbf{M}) + i\,(\omega\mathbf{C}+\mathbf{K}^I)\right]\,\mathbf{U}(\omega) = \mathbf{p}\,G(\omega) \tag{6.60}$$

In the reduced vector space, Equation (6.60) is simplified to the following form:

$$[(\Lambda_X - \omega^2\,\mathbf{I}) + i\,(\,\omega\mathbf{X}^T\mathbf{C}\mathbf{X} + \mathbf{X}^T\mathbf{K}^I\mathbf{X})\,]\,\mathbf{X}\,\mathbf{Y}(\omega) = \mathbf{X}^T\mathbf{p}\,G(\omega) \tag{6.61}$$

In general, the complex stiffness matrix cannot be diagonalized by the basis vectors. The reduced complex stiffness matrix is defined as:

$$\mathbf{K_X^I} = \mathbf{X^T K^I X} \tag{6.62}$$

This reduced complex stiffness matrix is partitioned into two matrices, one containing the diagonals, $\mathbf{K_{Xd}^I}$, and one containing the off diagonals, $\mathbf{K_{Xs}^I}$.

$$\mathbf{K_X^I} = \mathbf{K_{Xd}^I} + \mathbf{K_{Xs}^I} \tag{6.63}$$

Also, the diagonal terms of the reduced complex stiffness matrix are defined as:

$$\mathbf{x_i^T K^I x_i} = \eta_i \omega_{Xi}^2 \tag{6.64}$$

Then, Equation (6.61) can be written as:

$$\left\{ (\Lambda - \omega^2 \mathbf{I}) + i \left[2\omega\xi_i\omega_{Xi} + \eta_i \omega_{Xi}^2 \right] \right\} \mathbf{Y}(\omega) = \mathbf{X^T p}\, G(\omega) - i\, \mathbf{K_{Xs}^I}\, \mathbf{X}\, \mathbf{Y}(\omega) \tag{6.65}$$

Define the leading coefficient matrix of Equation (6.65) as the impedance matrix:

$$\mathbf{Z_{Xd}} = (\Lambda - \omega^2 \mathbf{I}) + i \left[2\omega\xi_i\omega_{Xi} + \eta_i \omega_{Xi}^2 \right] \tag{6.66}$$

The impedance matrix as defined is diagonal so that no matrix linear solution is required to compute the response. The modal damping coefficient has contribution from both viscous and viscoelastic damping. However, since the unknown also appears on the right hand side of Equation (6.65), the solution is to be solved iteratively. Diagonal damping terms are considered on the left hand side of Equation (6.65). Since even at the increased level of damping, the damping forces are smaller than the inertial and elastic forces. Therefore a rather accurate solution can be computed by assuming that the contribution due to damping coupling is negligible. Once the approximate response is computed, a correction step can

be taken to improve the solution. This iterative solution also reveals the extent of damping coupling in the x_i coordinates.

$$\mathbf{Z}_{Xd}\, \mathbf{Y}^{(1)}(\omega) = \mathbf{X}^T\mathbf{p}\, G(\omega) \tag{6.67}$$

$$\mathbf{Z}_{Xd}\, \mathbf{Y}^{(2)}(\omega) = \mathbf{X}^T\mathbf{p}\, G(\omega) - i\, \mathbf{K}_X^I{}_s\, \mathbf{Y}^{(1)}(\omega) \tag{6.68}$$

The uncoupled scalar equation in the i-th vector coordinate can be expressed as:

$$[(1 - \beta_{Xi}{}^2) + i\,(2\xi_i\beta_{Xi} + \eta_i)]\; Y_i{}^{(1)}(\omega) = \frac{\mathbf{x}_i{}^T\mathbf{p}}{\omega_{Xi}{}^2} G(\omega) \tag{6.69}$$

$$[(1 - \beta_{Xi}{}^2) + i\,(2\xi_i\beta_{Xi} + \eta_i)]\; Y_i{}^{(2)}(\omega) = \frac{\mathbf{x}_i{}^T\mathbf{p}}{\omega_{Xi}{}^2} G(\omega) - i \sum_{j\neq i} \frac{\mathbf{x}_i{}^T\mathbf{K}^I\mathbf{x}_j}{\omega_{Xi}{}^2}\, Y_j{}^{(1)}(\omega) \tag{6.70}$$

6.7 Design Method for Structures Damped by Viscoelastic Materials

In order to design structures damped by the application of viscoelastic materials, a set of simplified equations consistent with the governing equation must be derived. Assumptions must be made based on physical insight and experience to simplify the mathematics but the main physical behavior must still be captured. This would allow quick evaluation of different design alternatives. Preliminary sensitivity and trade studies can also be performed. Once the design concept is selected, the design can be checked by more rigorous analytical procedures.

Besides the assumptions stated in Subsection 6.6, two additional assumptions can be made to facilitate the design process. These assumptions will capture the primary structural behavior, simplify the mathematics and also enable the use of results from finite element analysis efficiently:

1. modal shapes are not change significantly by the introduction of viscoelastic materials

2. the damping is primarily contributed by the diagonal terms and the modal damping coupling has only secondary effect , i.e.

$$\left| \frac{\mathbf{x}_i^T \mathbf{K}^I \mathbf{x}_j}{\omega_i \omega_j} \right| \leq \varepsilon \qquad \text{for } i \neq j \tag{6.71}$$

say $\varepsilon = 0.2$, then simple algebraic expressions can be derived.

Now, the diagonal term of the reduced complex stiffness can be computed from each finite element with viscoelastic materials as:

$$\begin{aligned} \mathbf{x}_i^T \mathbf{K}^I \mathbf{x}_i &= \sum_{j=1}^{n_v} \mathbf{x}_i^T \mathbf{k}^I{}_j \mathbf{x}_i \\ &= \sum_{j=1}^{n_v} \eta^v{}_j \mathbf{x}_i^T \mathbf{k}^R{}_j \mathbf{x}_i \end{aligned} \tag{6.72}$$

But, by using the definition of structural modal loss factor:

$$\mathbf{x}_i^T \mathbf{K}^I \mathbf{x}_i = \eta_i \omega_i^2 \tag{6.64}$$

the modal loss factor in the i-th coordinate is given by:

$$\begin{aligned} \eta_i &= \frac{\sum_{j=1}^{n_v} \eta^v{}_j \mathbf{x}_i^T \mathbf{k}^R{}_j \mathbf{x}_i}{\mathbf{x}_i^T \mathbf{K}^R \mathbf{x}_i} \\ &= \sum_{j=1}^{n_v} \eta^v{}_j \left(\frac{\mathbf{x}_i^T \mathbf{k}^R{}_j \mathbf{x}_i}{\mathbf{x}_i^T \mathbf{K}^R \mathbf{x}_i} \right) \end{aligned} \tag{6.73}$$

Equation (6.73) can be simplified by considering the modal strain energy of each finite element. The strain energy in the j-th element is defined as:

$$w_j = \frac{1}{2} \mathbf{x}_i^T \mathbf{k}^R{}_j \mathbf{x}_i, \tag{6.74}$$

Then,

$$\eta_i = \sum_{j}^{n_v} \eta^v{}_j \left(\frac{(w_j)_i}{\sum_{k}^{n_w} (w_k)_i} \right) \tag{6.75}$$

where n_w = total number of elastic and viscoelastic elements

Define the modal strain energy ratio of the j-th element in the i-th mode as:

$$\varepsilon^v{}_{ji} = \frac{(w_j)_i}{\sum_{k=1}^{n_w} (w_k)_i} \tag{6.76}$$

By the orthogonality condition of the stiffness matrix and basis vector:

$$\omega_{Xi}{}^2 = \mathbf{x}_i{}^T \mathbf{K}^R \mathbf{x}_i \tag{6.58}$$

the sum of strain energy of all the elements is conveniently computed as:

$$\sum_{k}^{n_w} (w_k)_i = \frac{\omega_{Xi}{}^2}{2} \tag{6.77}$$

Defining the contribution of the j-th element to the i-th mode damping ratio as the *element modal damping ratio*, η_{ji}, as:

$$\eta_{ji} = \eta^v{}_j \varepsilon^v{}_{ji} \tag{6.78}$$

Then, Equation (6.75) becomes:

$$\eta_i = \sum_{j=1}^{n_v} \eta_{ji} \tag{6.79}$$

If the sum of strain energy ratios of all the viscoelastic elements is given by ε_i, then

$$\varepsilon^v_i = \sum_j \varepsilon^v_{ji} \tag{6.80}$$

If only one material is used, the modal loss factor is:

$$\eta_i = \eta^v \varepsilon^v_i \tag{6.81}$$

By using Equations (6.78) through (6.81), it is possible to design damping into the structure by knowing the modal strain energy distribution of each basis vector. In order to introduce damping into a specific vibration mode, viscoelastic material should be applied to area with high strain energy density. The system damping is proportional to the material loss factor and strain energy ratio in the viscoelastic material. The analysis and design procedure described provides an efficient method to extract the relevant and concise information of a complex structure subject to dynamic forces. This method has been implemented for the design and analysis of viscoelastic struts for large space structures[34]. However, if the dynamic response of the structures is over a wide frequency range and the material properties exhibit significant frequency dependence, this effect must be properly included in the design/analysis procedure such that the the structural design will meet the design requirements.

6.8 Analysis of Frequency Dependent Viscoelastic Structures

The governing equation of motion for structure with both viscous and viscoelastic damping is:

$$\left\{ (\mathbf{K}^R(\omega) - \omega^2 \mathbf{M}) + i \left[\omega \mathbf{C} + \mathbf{K}^I(\omega) \right] \right\} \mathbf{U}(\omega) = \mathbf{p}\, G(\omega) \tag{6.55}$$

When material properties exhibit significant frequency dependence but only constant material moduli and loss factors are used in the analysis, the computed resonant frequencies and structural damping may contain significant errors. Consequently, the predicted dynamic responses may not be accurate.

When eigenvectors are used as the vector basis, an iterative eigensolution procedure is used[36]. The natural frequencies are dependent on the material properties used in the stiffness matrix which is frequency dependent. The frequency used in computing the material properties is matched with the natural frequency, mode by mode. The process is not only tedious but expensive. Also, the final set of eigenvectors are not mass or stiffness orthogonal. This significantly reduces the applicability of this approach.

Two methods are discussed in the following subsections. The first method is the exact direct solution and the second method is by the use of reduced vector basis.

6.8.1 Direct Frequency Response Solution

If the viscous damping matrix is explicitly defined, an exact solution to Equation (6.55) is possible. At each frequency, Equation (6.55) is a set of linear complex algebraic equations which can be solved using a complex matrix equation solver.

$$\mathbf{U}(\omega) = [(\mathbf{K}^R(\omega) - \omega^2 \mathbf{M}) + i\,(\omega \mathbf{C} + \mathbf{K}^I(\omega)]^{-1}\,\mathbf{p}\,G(\omega) \tag{6.82}$$

The solution can also be obtained using real arithmetic in the 2n dimensional space as described in Chapter 5 except the damping matrix is augmented by the complex stiffness matrix.

$$\begin{bmatrix} \mathbf{K}^R(\omega)-\omega^2\mathbf{M} & -\omega\mathbf{C}-\mathbf{K}^I(\omega) \\ \omega\mathbf{C}+\mathbf{K}^I(\omega) & \mathbf{K}^R(\omega)-\omega^2\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{u}^R(\omega) \\ \mathbf{u}^I(\omega) \end{bmatrix} = \begin{bmatrix} \mathbf{p} \\ \mathbf{0} \end{bmatrix} G(\omega) \tag{6.83}$$

Vector transfer function can be computed based on Equations (6.82) and (6.83). It is useful to compute the dynamic response in the frequency domain, e.g. random vibration described by power spectral density. The solution in the time domain is then obtained by:

$$\mathbf{u}(t) = \sum_{j=0}^{n_\omega} (\mathbf{u}^R(\varpi_j) \cos(\varpi_j t) - \mathbf{u}^I(\varpi_j) \sin(\varpi_j t)) \tag{6.84}$$

$$\text{where } \varpi_j = j\,\Delta\omega \tag{6.85}$$

$$n_\omega = (\text{number of Fourier terms} - 1) \tag{6.86}$$

or by the inverse fast Fourier transform.

The complexity of solution is basically the same if the complex stiffness matrix is frequency dependent or frequency independent. However, this direct frequency response solution is very computational intensive because of the number of frequencies required for an accurate solution.

6.8.2 Reduced Vector Basis Frequency Response Solution

Even though the material properties are frequency dependent, the governing equation of motion is linear in the frequency domain. Therefore, if a vector subspace is properly selected, the size of the problem can be substantially reduced. However, the frequency dependent properties must be included to compute the effect due to the change in stiffness and damping.

The vector basis for the governing equation of motion is derived in the frequency domain in Chapter 4. It was shown that if a harmonic frequency is specified, the vector basis computed represents the undamped steady state solution at that frequency exactly. This observation can be used effectively to solve the dynamic response of structures with

the frequency dependent material properties. Undamped basis vectors of the following problem are computed:

$$(\mathbf{K}^{R}(\omega_{(i)}) - \omega_{(i)}^{2}\,\mathbf{M})\;\mathbf{U}(\omega) \;=\; \mathbf{p} \tag{6.87}$$

Therefore, the solution basis vectors can be composed of load dependent Ritz vectors from suitable number of selected frequencies, say two to three. The frequencies can be selected based on Rayleigh's frequency. When the first group of basis vectors are computed, a response analysis can be performed to identify the frequency range where the dynamic responses are most dominant. Then a second frequency is selected to generate basis vectors which best span the solution near that particular frequency. The previously computed vectors are retained in the Ritz basis. As more vectors are computed, the vector space so computed will give an accurate basis over the frequency range of interest. The correct flexibility of the structure due to frequency dependent material properties is therefore logically incorporated into the basis. The solution is computed in a smaller subspace even though the complex stiffness matrix is frequency dependent.

After the basis is defined, frequency dependent solution to Equation (6.55) is formulated in the reduced vector basis. In order to maintain computational efficiency, the following steps are followed to minimize the number of matrix factorizations at every frequency. A reference frequency, ω_r, is selected such that material properties are computed with respect to this reference point:

$$E^{R}(\omega) = E^{R}(\omega_r)\left[1+\underline{E}(\omega)\right] \tag{6.88}$$

$$E^{I}(\omega) = E^{R}(\omega_r)\,\underline{\eta}(\omega) \tag{6.89}$$

The complex modulus can therefore be expressed as:

$$E(\omega) = E^{R}(\omega_r)\left[1+ \underline{E}(\omega) + i\,\underline{\eta}(\omega)\right] \tag{6.90}$$

Assuming only one type of viscoelastic material is used in the structure (similar derivation if more than one material is used), the stiffness matrix due to the viscoelastic elements is

$$\begin{aligned} \mathbf{K}_v(\omega) &= \mathbf{K}^R(\omega_r)\left[1+\underline{E}(\omega)+i\,\underline{\eta}(\omega)\right] \\ &= \mathbf{K}_r\left[1+\underline{E}(\omega)+i\,\underline{\eta}(\omega)\right] \end{aligned} \tag{6.91}$$

$$\text{where } \mathbf{K}_r = \mathbf{K}^R(\omega_r) \tag{6.92}$$

The global stiffness matrix of the structure is contributed by two parts, the stiffness from elastic material, $\mathbf{K}_e$, and the stiffness from viscoelastic material, $\mathbf{K}_v(\omega)$.

$$\begin{aligned} \mathbf{K}(\omega) &= \mathbf{K}_e + \mathbf{K}_v(\omega) \\ &= \mathbf{K} + \mathbf{K}_r\left[\underline{E}(\omega)+i\,\underline{\eta}(\omega)\right] \end{aligned} \tag{6.93}$$

$$\text{where, } \mathbf{K} = \mathbf{K}_e + \mathbf{K}_r \tag{6.94}$$

The governing equation can therefore be expressed as:

$$\left\{\left[-\omega^2\mathbf{M}+\mathbf{K}+\underline{E}(\omega)\mathbf{K}_r\right] + i\left[\omega\mathbf{C}+\underline{\eta}(\omega)\mathbf{K}_r\right]\right\}\mathbf{U}(\omega) = \mathbf{p}G(\omega) \tag{6.95}$$

If the vector basis $\mathbf{X}$ has the following orthogonal properties:

$$\mathbf{X}^T\,\mathbf{M}\,\mathbf{X} = \mathbf{I} \tag{6.57}$$

$$\mathbf{X}^T\,\mathbf{K}\,\mathbf{X} = \Lambda_X \tag{6.96}$$

and express the subspace solution as:

$$\mathbf{u} \approx \mathbf{X}\,\mathbf{y} \tag{6.56}$$

then the size of Equation (6.95) can be reduced to the dimension of $\mathbf{X}$ by substituting Equation (6.56) into Equation (6.95) and pre-multiplying by $\mathbf{X}^T$:

$$\left\{\left[-\omega^2\mathbf{I}+\Lambda_X+\underline{E}(\omega)\mathbf{X}^T\mathbf{K}_r\mathbf{X}\right] + i\left[\omega\mathbf{X}^T\mathbf{C}\mathbf{X}+\underline{\eta}(\omega)\mathbf{X}^T\mathbf{K}_r\mathbf{X}\right]\right\}\mathbf{Y}(\omega)$$

$$= \mathbf{X}^T\mathbf{p}\, G(\omega) \tag{6.97}$$

In general, $\mathbf{X}^T\mathbf{K}_r\mathbf{X}$ and $\mathbf{X}^T\mathbf{C}\mathbf{X}$ are not diagonal. Separate the matrices into diagonal and off diagonal parts as follows:

$$\mathbf{X}^T\mathbf{K}_r\mathbf{X} = \mathbf{K}_{Xd} + \mathbf{K}_{Xs} \tag{6.98}$$

$$\mathbf{X}^T\mathbf{C}\mathbf{X} = \mathbf{C}_{Xd} + \mathbf{C}_{Xs} \tag{6.99}$$

Then the governing equation can be simplified to:

$$\begin{aligned}&\left\{\left[-\omega^2\mathbf{I}+\Lambda_X+\underline{E}(\omega)\mathbf{K}_{Xd}\right] + i\left[\omega\mathbf{C}_{Xd}+\eta(\omega)\mathbf{K}_{Xd}\right]\right\}\mathbf{Y}(\omega)\\ &= \mathbf{X}^T\mathbf{p}\, G(\omega) - \left\{\underline{E}(\omega)\mathbf{K}_{Xs}+i\left[\omega\mathbf{C}_{Xd}+\eta(\omega)\mathbf{K}_{Xs}\right]\right\}\mathbf{Y}(\omega)\end{aligned} \tag{6.100}$$

Define the impedance matrix of the system as:

$$\mathbf{Z}_{Xd} = \left\{\left[-\omega^2\mathbf{I}+\Lambda_X+\underline{E}(\omega)\mathbf{K}_{Xd}\right] + i\left[\omega\mathbf{C}_{Xd}+\eta(\omega)\mathbf{K}_{Xd}\right]\right\} \tag{6.101}$$

The impedance matrix so defined is diagonal and solution can be obtained easily. However, the unknown also appears on the right hand side of Equation (6.100). Therefore, solve this equation iteratively,

$$\mathbf{Z}_{Xd}\,\mathbf{Y}^{(1)}(\omega) = \mathbf{X}^T\mathbf{p}\, G(\omega) \tag{6.102}$$

$$\mathbf{Z}_{Xd}\,\mathbf{Y}^{(2)}(\omega) = \mathbf{X}^T\mathbf{p}\, G(\omega) - \left\{\underline{E}(\omega)\mathbf{K}_{Xs}+i\left[\omega\mathbf{C}_{Xd}+\eta(\omega)\mathbf{K}_{Xs}\right]\right\}\mathbf{Y}^{(1)}(\omega) \tag{6.103}$$

In this form, the frequency dependent material properties of the viscoelastic material can be included in the analysis effectively and efficiently.

In scalar form, the i-th generalized degree of freedom can be expressed as:

$$\left\{\left[-\omega^2 + \lambda_{Xi} + \underline{E}(\omega)k_{Xii}\right] + i\left[\omega\, c_{Xii} + \eta(\omega)k_{Xii}\right]\right\} Y_j^{(1)} = \mathbf{x}_i^T\mathbf{p}\, G(\omega) \tag{6.104}$$

$$\left\{\left[-\omega^2 + \lambda_{Xi} + \underline{E}(\omega)k_{Xii}\right] + i\left[\omega\, c_{Xii} + \eta(\omega)k_{Xii}\right]\right\} Y_i^{(2)}$$

$$= \mathbf{x}_i^T \mathbf{p}\, G(\omega) - \sum_{j \neq i} \{ \underline{E}(\omega) k_{Xij} + i [\omega c_{Xij} + \underline{\eta}(\omega) k_{Xij}] \} Y_j^{(1)}(\omega) \qquad (6.105)$$

In this manner, the solution to a structure with frequency dependent viscoelastic materials and viscous dampers can be computed efficiently in the reduced vector space. The numerical operations required are limited to static solution at a few selected frequencies and one real eigenvalue problem in the reduced space. The rest of the computations can be performed in the uncoupled coordinates in scalar complex arithmetics.

Chapter 7

Reduced Vector Basis for Dynamic Substructure Analysis of Damped Structures

Dynamic substructure methods[37,38] are important alternate methods to solve the equation of motion of a complete structure. These methods can be systematically categorized as reduced vector basis methods which utilize special vector bases to solve the equation of motion in the reduced coordinates of individual substructures in lieu of the coordinates of the complete structure. Dynamic substructure methods are important to dynamic analyses of large finite element models because:

1. they reduce a large and complex problem to practical size for computation, and thus allow accurate and efficient dynamic analysis of large structures
2. they allow different engineers and/or engineering organizations to work on structural components simultaneously and independently, and considerably simplifies the logistic of interface and scheduling
3. during analysis/design cycle, only the substructure models which are modified have to be updated and the only the reduced system level model (not the complete explicit model) has to be re-analyzed
4. minimum amount of data characterizing the dynamics of the substructures are to be transferred between organizations, and no explicit model of proprietary structural design is required to be transferred.

Different substructure methods are characterized by the use of different vector bases to approximate the solution. Conventional dynamic substructure method uses boundary node static displacement vectors to enforce compatibility between substructures and constrained normal modes to represent the elastic deformation of the interior nodes. While compati-

bility is best described by the boundary node static displacement vectors, the use of constrained normal modes to span the elastic deformation of interior nodes is merely a mathematical convenience.

If the spatial distribution of the dynamic force is well defined, the load dependent Ritz vectors[1] provide a smaller set of basis vectors for the solution of dynamic response. A special substructure method based on the Ritz vector concept was presented by Wilson and Bayo[39]. This substructure method generates the Ritz vector basis from the applied force vectors and three translational inertial force vectors. Compatibility at the substructure boundary degrees of freedom is enforced by the boundary static displacement vectors.

For a substructure method to provide a reliable and accurate solution, it should satisfy the following requirements:

1. geometric compatibility between substructures
2. dynamic equilibrium within the substructure
3. dynamic equilibrium of the complete structure

An alternate formulation of dynamic substructure method which satisfies all these requirements is presented in the following subsections. A general reduced vector basis approach which allows substructures to be analyzed in the local reduced coordinates is used to reduce the order of the finite element model of the complete structure. This formulation is consistent with the development of the earlier chapters. The derivation presented provides a clear physical comprehension to the solution process and a consistent mathematical foundation to the method.

The undamped vector basis approach is used for the substructure analysis. A consistent treatment of viscous and viscoelastic damping matrix for dynamic substructure method is presented.

7.1 General Formulation of Dynamic Substructure Problem

The governing differential equation of motion for the complete structure subject to external dynamic forces is given by:

$$\mathbf{M}\,\ddot{\mathbf{u}} + \mathbf{C}\,\dot{\mathbf{u}} + \mathbf{K}\,\mathbf{u} = \mathbf{p}\,g(t) \tag{7.1}$$

The goal of substructure method is to provide solution to the complete structure in local reduced substructure coordinates. Therefore the substructure displacement degrees of freedom are grouped into two sets, i.e. the internal degrees of freedom, $\mathbf{u}_i$, and boundary degrees of freedom in the global coordinate system, $\mathbf{u}_g$. Without loss of generality, all the interface degrees of freedom are assumed to be in the global coordinate system and are grouped into a single set, $\mathbf{u}_g$. The displacement vector is therefore given by:

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_g \end{bmatrix} \tag{7.2}$$

The mass, damping, and stiffness matrices can be expressed in the following matrix form:

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_1 & \mathbf{0} & \cdots & \mathbf{M}_{1g} \\ \mathbf{0} & \mathbf{M}_2 & \cdots & \mathbf{M}_{2g} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{M}_{g1} & \mathbf{M}_{g2} & \cdots & \mathbf{M}_{gg} \end{bmatrix} \tag{7.3}$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} & \cdots & \mathbf{C}_{1g} \\ \mathbf{0} & \mathbf{C}_2 & \cdots & \mathbf{C}_{2g} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{C}_{g1} & \mathbf{C}_{g2} & \cdots & \mathbf{C}_{gg} \end{bmatrix} \tag{7.4}$$

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_1 & \mathbf{0} & . & . & \mathbf{K}_{1g} \\ \mathbf{0} & \mathbf{K}_2 & . & . & \mathbf{K}_{2g} \\ . & . & . & . & . \\ . & . & . & . & . \\ \mathbf{K}_{g1} & \mathbf{K}_{g2} & . & . & \mathbf{K}_{gg} \end{bmatrix} \tag{7.5}$$

The force vector is given by:

$$\mathbf{p} = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ . \\ . \\ \mathbf{p}_g \end{bmatrix} \tag{7.6}$$

The dynamic equilibrium equations of a typical substructure, including the global displacements, can be expressed by the following matrix equation:

$$\begin{bmatrix} \mathbf{M}_i & \mathbf{M}_{ig} \\ \mathbf{M}_{gi} & \mathbf{M}_g^{(i)} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_i \\ \ddot{\mathbf{u}}_g \end{bmatrix} + \begin{bmatrix} \mathbf{C}_i & \mathbf{C}_{ig} \\ \mathbf{C}_{gi} & \mathbf{C}_g^{(i)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}}_i \\ \dot{\mathbf{u}}_g \end{bmatrix} + \begin{bmatrix} \mathbf{K}_i & \mathbf{K}_{ig} \\ \mathbf{K}_{gi} & \mathbf{K}_g^{(i)} \end{bmatrix} \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_g \end{bmatrix} = \begin{bmatrix} \mathbf{p}_i \\ \mathbf{p}_g^{(i)} \end{bmatrix} g(t) \tag{7.7}$$

The global mass, damping and stiffness submatrices and force vector for the global degrees of freedom are given by:

$$\mathbf{M}_{gg} = \Sigma\, \mathbf{M}_g^{(i)} \tag{7.8}$$

$$\mathbf{C}_{gg} = \Sigma\, \mathbf{C}_g^{(i)} \tag{7.9}$$

$$\mathbf{K}_{gg} = \Sigma\, \mathbf{K}_g^{(i)} \tag{7.10}$$

$$\mathbf{p}_g = \Sigma\, \mathbf{p}_g^{(i)} \tag{7.11}$$

For clarity of the derivation of this method, suppose the complete structural system is assembled as in Equation (7.1) and the solution to the matrix differential equation is sought. The substructure analysis can be conceptualized as the summation of the solutions of two separate analyses, $\mathbf{u}'$ and $\mathbf{u}''$. The concept is similar to the moment distribution method (or Hardy Cross method). In the first analysis it is assumed that the degrees of freedom at the boundary degrees of freedom are locked.

$$\mathbf{u}' = \begin{bmatrix} \mathbf{u}'_1 \\ \mathbf{u}'_2 \\ \cdot \\ \cdot \\ \mathbf{0} \end{bmatrix} \tag{7.12}$$

The structures is subject to interior forces on the substructures.

$$\mathbf{M}\,\ddot{\mathbf{u}}' + \mathbf{C}\,\dot{\mathbf{u}}' + \mathbf{K}\,\mathbf{u}' = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \cdot \\ \cdot \\ \mathbf{0}_g \end{bmatrix} g(t) + \begin{bmatrix} \mathbf{0}_1 \\ \mathbf{0}_2 \\ \cdot \\ \cdot \\ \mathbf{I}_g \end{bmatrix} \mathbf{R}(t) \tag{7.13}$$

The reaction forces at the boundary degrees of freedom, $\mathbf{R}(t)$, are required for the dynamic equilibrium of the structure. A second analysis is required to account for the dynamics due to the applied forces at the boundary degrees of freedom and to negate the reaction forces from the first analysis. The structure is not subject to any force at the substructure interior degrees of freedom.

$$\mathbf{u}'' = \begin{bmatrix} \mathbf{u}''_1 \\ \mathbf{u}''_2 \\ \cdot \\ \cdot \\ \mathbf{u}''_g \end{bmatrix} \tag{7.14}$$

$$\mathbf{M}\,\ddot{\mathbf{u}}'' + \mathbf{C}\,\dot{\mathbf{u}}'' + \mathbf{K}\,\mathbf{u}'' = \begin{bmatrix} \mathbf{0}_1 \\ \mathbf{0}_2 \\ \cdot \\ \cdot \\ \mathbf{p}_g \end{bmatrix} g(t) - \begin{bmatrix} \mathbf{0}_1 \\ \mathbf{0}_2 \\ \cdot \\ \cdot \\ \mathbf{I}_g \end{bmatrix} \mathbf{R}(t) \tag{7.15}$$

It is obvious that due to linearity the sum of the two solutions will give the solution to the original problem.

$$\mathbf{u} = \mathbf{u}' + \mathbf{u}'' \tag{7.16}$$

$$\begin{aligned} \mathbf{M}\,\ddot{\mathbf{u}} + \mathbf{C}\,\dot{\mathbf{u}} + \mathbf{K}\,\mathbf{u} &= \mathbf{M}\,(\ddot{\mathbf{u}}' + \ddot{\mathbf{u}}'') + \mathbf{C}\,(\dot{\mathbf{u}}' + \dot{\mathbf{u}}'') + \mathbf{K}\,(\mathbf{u}' + \mathbf{u}'') \\ &= (\mathbf{M}\,\ddot{\mathbf{u}}' + \mathbf{C}\,\dot{\mathbf{u}}' + \mathbf{K}\,\mathbf{u}') + (\mathbf{M}\,\ddot{\mathbf{u}}'' + \mathbf{C}\,\dot{\mathbf{u}}'' + \mathbf{K}\,\mathbf{u}'') \end{aligned}$$

$$= \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \cdot \\ \cdot \\ \mathbf{0}_g \end{bmatrix} g(t) + \begin{bmatrix} \mathbf{0}_1 \\ \mathbf{0}_2 \\ \cdot \\ \cdot \\ \mathbf{I}_g \end{bmatrix} \mathbf{R}(t) + \begin{bmatrix} \mathbf{0}_1 \\ \mathbf{0}_2 \\ \cdot \\ \cdot \\ \mathbf{p}_g \end{bmatrix} g(t) - \begin{bmatrix} \mathbf{0}_1 \\ \mathbf{0}_2 \\ \cdot \\ \cdot \\ \mathbf{I}_g \end{bmatrix} \mathbf{R}(t)$$

$$= \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \cdot \\ \cdot \\ \mathbf{p}_g \end{bmatrix} g(t) \qquad (7.17)$$

Therefore, the problem of finding the approximate solution to **u** is reduced to finding the approximate solutions to **u**' and **u**". In other words, the problem is reduced to finding the best vector bases for these two separate problems. It is shown in the following sections that this approach allows the computation of vector basis of a substructure to be independent of other substructures. It is also demonstrated that two sets of Ritz bases for each substructure will provide the vector bases for an accurate solution to the complete structure. These two vector bases are the load dependent Ritz vectors, $\mathbf{X}^p$, and boundary motion Ritz vectors, $\mathbf{X}^b$.

7.2 Load Dependent Vector Basis

The solution to the first problem of the Hardy Cross approach is quite simple. Rewrite Equation (7.13) explicitly:

$$\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} & \cdot\cdot & \mathbf{M}_{1g} \\ \mathbf{0} & \mathbf{M}_2 & \cdot\cdot & \mathbf{M}_{2g} \\ \cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot & \cdot\cdot & \cdot \\ \mathbf{M}_{g1} & \mathbf{M}_{g2} & \cdot\cdot & \mathbf{M}_{gg} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}'_1 \\ \ddot{\mathbf{u}}'_2 \\ \cdot \\ \cdot \\ \mathbf{0}_g \end{bmatrix} + \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} & \cdot\cdot & \mathbf{C}_{1g} \\ \mathbf{0} & \mathbf{C}_2 & \cdot\cdot & \mathbf{C}_{2g} \\ \cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot & \cdot\cdot & \cdot \\ \mathbf{C}_{g1} & \mathbf{C}_{g2} & \cdot\cdot & \mathbf{C}_{gg} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}}'_1 \\ \dot{\mathbf{u}}'_2 \\ \cdot \\ \cdot \\ \mathbf{0}_g \end{bmatrix} + \begin{bmatrix} \mathbf{K}_1 & \mathbf{0} & \cdot\cdot & \mathbf{K}_{1g} \\ \mathbf{0} & \mathbf{K}_2 & \cdot\cdot & \mathbf{K}_{2g} \\ \cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot & \cdot\cdot & \cdot \\ \mathbf{K}_{g1} & \mathbf{K}_{g2} & \cdot\cdot & \mathbf{K}_{gg} \end{bmatrix} \begin{bmatrix} \mathbf{u}'_1 \\ \mathbf{u}'_2 \\ \cdot \\ \cdot \\ \mathbf{0}_g \end{bmatrix}$$

$$= \begin{bmatrix} p_1 \\ p_2 \\ \cdot \\ \cdot \\ 0_g \end{bmatrix} g(t) + \begin{bmatrix} 0_1 \\ 0_2 \\ \cdot \\ \cdot \\ I_g \end{bmatrix} R(t) \qquad (7.18)$$

It is obvious from the physics of the problem and the mathematical form of Equation (7.18) that each substructure is not interacting with other substructures. Therefore, a local structural analysis can be performed for each substructure:

$$\begin{bmatrix} M_i & M_{ig} \\ M_{gi} & M_g^{(i)} \end{bmatrix} \begin{bmatrix} \ddot{u}'_i \\ 0_g \end{bmatrix} + \begin{bmatrix} C_i & C_{ig} \\ C_{gi} & C_g^{(i)} \end{bmatrix} \begin{bmatrix} \dot{u}'_i \\ 0_g \end{bmatrix} + \begin{bmatrix} K_i & K_{ig} \\ K_{gi} & K_g^{(i)} \end{bmatrix} \begin{bmatrix} u'_i \\ 0_g \end{bmatrix}$$

$$= \begin{bmatrix} p_i \\ 0_g^{(i)} \end{bmatrix} g(t) + \begin{bmatrix} 0_i \\ I_g^{(i)} \end{bmatrix} R_i(t) \qquad (7.19)$$

Alternatively, Equation (7.19) can be written as:

$$M_i \ddot{u}_i' + C_i \dot{u}_i' + K_i u'_i = p_i \qquad (7.20)$$

$$M_{gi} \ddot{u}_i' + C_{gi} \dot{u}_i' + K_{gi} u'_i = R_i(t) \qquad (7.21)$$

It is appropriate to use the Load Dependent Ritz Method to provide solution to Equation (7.20). As described in Chapter 4, this set of vectors provides a small set of basis vectors which fully characterizes the local static and dynamic behavior of the substructure constrained at the interface degrees of freedom and subject to the force vector, $\mathbf{p}_i$. If there is no external force on the substructure, then this set is null.

The mass orthonormalized Load Dependent Ritz vectors, X^P_i, are computed by using the algorithm presented in Chapter 4. The reduction of substructure degrees of freedom due to forces on the substructure is represented by the following transformation with y^P_i as the generalized coordinates:

$$u'_i = X^P_i \, y^P_i \qquad (7.22)$$

7.3 Boundary Motion Vector Basis

The substructure load dependent Ritz bases provide the solution for the local non-interacting dynamic behavior of the substructures. The geometric compatibility, and static and dynamic interaction between the substructures are provided by the boundary motion Ritz vectors. The effects of the local forces and dynamics of boundary constraints on the complete structure are contained in the reaction forces at the interface degrees of freedom which is expressed as:

$$\mathbf{R}(t) = \Sigma \mathbf{R}_i(t) \tag{7.23}$$

Although these reaction forces are conceptually important to the development and analysis of the substructure method, there is no need to compute them explicitly.

The solution to the second problem, $\mathbf{u}''$, of the Hardy Cross approach is quite simple. Rewrite Equation (7.15) explicitly:

$$\begin{bmatrix} M_1 & 0 & \cdots & M_{1g} \\ 0 & M_2 & \cdots & M_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ M_{g1} & M_{g2} & \cdots & M_{gg} \end{bmatrix} \begin{bmatrix} \ddot{u}''_1 \\ \ddot{u}''_2 \\ \vdots \\ \ddot{u}''_g \end{bmatrix} + \begin{bmatrix} C_1 & 0 & \cdots & C_{1g} \\ 0 & C_2 & \cdots & C_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ C_{g1} & C_{g2} & \cdots & C_{gg} \end{bmatrix} \begin{bmatrix} \dot{u}''_1 \\ \dot{u}''_2 \\ \vdots \\ \dot{u}''_g \end{bmatrix} + \begin{bmatrix} K_1 & 0 & \cdots & K_{1g} \\ 0 & K_2 & \cdots & K_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ K_{g1} & K_{g2} & \cdots & K_{gg} \end{bmatrix} \begin{bmatrix} u''_1 \\ u''_2 \\ \vdots \\ u''_g \end{bmatrix}$$

$$= \begin{bmatrix} 0_1 \\ 0_2 \\ \vdots \\ p_g \end{bmatrix} g(t) - \begin{bmatrix} 0_1 \\ 0_2 \\ \vdots \\ I_g \end{bmatrix} R(t) \tag{7.24}$$

Again, the Load Dependent Ritz Vector Method is used to extract the basis vectors to this problem. Compatibility at the substructure interface is ensured if the vector basis is derived

from this equation of motion. This basis also enforces global static and dynamic equilibrium of the complete structure.

When there are only limited interface degrees of freedom, the substructure method is very efficient. This is normally the case when a physical structural component is attached to another in a statically or slightly indeterminate manner. However, if the substructures are defined only for computational reason, it may result in large number of vectors required to represent the boundary motion. In this case, care are must be exercised to minimize the interface degrees of freedom to increase the efficiency of the analysis.

7.3.1 Boundary Motion Static Vector Basis

Since the first solution, **u'**, derives the vector basis for the local forces at the interior degrees of freedom of the substructure, the second problem is not subject to any external force at the interior degrees of freedom, i.e.:

$$\begin{bmatrix} \mathbf{K}_1 & \mathbf{0} & \cdots & \mathbf{K}_{1g} \\ \mathbf{0} & \mathbf{K}_2 & \cdots & \mathbf{K}_{2g} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \mathbf{K}_{g1} & \mathbf{K}_{g2} & \cdots & \mathbf{K}_{gg} \end{bmatrix} \begin{bmatrix} \mathbf{u''}_1 \\ \mathbf{u''}_2 \\ \cdot \\ \cdot \\ \mathbf{u''}_g \end{bmatrix}_s = \begin{bmatrix} \mathbf{0}_1 \\ \mathbf{0}_2 \\ \cdot \\ \cdot \\ \mathbf{R''}_g \end{bmatrix} \tag{7.25}$$

This feature is identical to the static condensation problem. Omitting the last submatrix block of Equation (7.25), i.e.

$$\begin{bmatrix} \mathbf{K}_1 & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{K}_2 & \cdots \\ \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdots \end{bmatrix} \begin{bmatrix} \mathbf{u''}_1 \\ \mathbf{u''}_2 \\ \cdot \\ \cdot \end{bmatrix}_s + \begin{bmatrix} \mathbf{K}_{1g} \\ \mathbf{K}_{2g} \\ \cdot \\ \cdot \end{bmatrix} \mathbf{u''}_{gs} = \begin{bmatrix} \mathbf{0}_1 \\ \mathbf{0}_2 \\ \cdot \\ \cdot \end{bmatrix} \tag{7.26}$$

and taking advantage of the diagonal blocks, the internal displacements of the substructures can be expressed as functions of the interface displacements:

$$\mathbf{u}''_s = \begin{bmatrix} -\mathbf{K}_1^{-1}\,\mathbf{K}_{1g} \\ -\mathbf{K}_2^{-1}\,\mathbf{K}_{2g} \\ \cdot \\ \cdot \\ \mathbf{I}_g \end{bmatrix} \mathbf{u}''_{gs} \tag{7.27}$$

Therefore it is obvious that the basis vectors for the global static solution are given by:

$$\mathbf{T} = \begin{bmatrix} -\mathbf{K}_1^{-1}\,\mathbf{K}_{1g} \\ -\mathbf{K}_2^{-1}\,\mathbf{K}_{2g} \\ \cdot \\ \cdot \\ \mathbf{I}_g \end{bmatrix} \tag{7.28}$$

Therefore, the substructure level static basis vectors which are consistent with the global static basis vectors are given by:

$$\mathbf{T}_i = -\mathbf{K}_i^{-1}\,\mathbf{K}_{ig} \tag{7.29}$$

$$\mathbf{u}''_{is} = \mathbf{T}_i\,\mathbf{u}''_{gs} \tag{7.30}$$

The static basis vectors for the substructure can be computed independently. This set of vectors can be computed based on unit displacements at the interface degrees of freedom. This computation is similar to the element stiffness matrix except the element stiffness matrix is derived from continuous differential equation governing the behavior of the element while the substructure stiffness is derived from the discrete matrix equation governing the behavior of the substructure. The principle of superposition of substructure matrices and element matrices is the same. The displacement compatibility between substructures at the interface degrees of freedom is automatically ensured by this approach.

In fact, if only these static vectors are included in the solution space, it is the same as the static substructure method or the Guyan Reduction Method[40] with the interface degrees of freedom retained as active degrees of freedom. The inclusion of this set of static

displacement vectors in the solution space with the Load Dependent Ritz vectors enables the static displacements of the complete structure to be represented exactly in the solution. Therefore the static correction to the dynamic problem of the complete structure is also included in the solution basis.

The reduction in degrees of freedom of a substructure due to the effect of boundary static displacement is summarized by the following transformation:

$$\mathbf{u}''_i = \mathbf{T}_i \mathbf{u}_g \tag{7.31}$$

7.3.2 Boundary Motion Dynamic Vector Basis

The boundary motion static Ritz vectors only provides the vector basis for the static solution of the second problem. Vector basis for the dynamic response must also be established in order to provide an accurate solution to Equation (7.15). Again, the Load Dependent Ritz Method can be used effectively. In fact, this step is not included in the conventional substructure method. In the conventional substructure method, there is no distinction between the vector bases due to forces on the interior degrees of freedom or the boundary motions. The dynamics of the boundary motions must be represented by the eigenvectors with the boundary degrees of freedom totally constrained. In fact, when there is no external force on the substructure, which is a very common case, the dynamics of the substructure is totally excited by the boundary motions. This constrained eigenvector representation can be a very inefficient. Many eigenvectors are required to provide a good basis to represent the dynamic behavior. The solutions seem to be acceptable in most cases because the interface of most structures are quite stiff. This may be sufficiently accurate for many cases if eigenvectors are used. However, the vector basis derived herein provides a method which is totally general. This vector basis is actually derived based on how the substructure will response dynamically to the boundary motion.

The substructures are excited inertially due to boundary motions. A specific set of vectors can be generated to represent the dynamic loading. Since the boundary degrees of freedom are represented as the physical degrees of freedom in the reduced equation, the dynamic effects of these degrees of freedom are represented and computed explicitly in the differential equation. As for the interior degrees of freedom of the substructures, the local dynamic effects due to boundary motions must be represented by proper choice of solution basis. This is accomplished by separating the effect of displacements at the boundary degrees of freedom which are computed explicitly and the additional dynamic displacements relative to the static displacements[41].

$$\begin{bmatrix} \mathbf{u}''_i \\ \mathbf{u}''_g \end{bmatrix} = \begin{bmatrix} \mathbf{u}''_i \\ \mathbf{u}''_g \end{bmatrix}_b + \begin{bmatrix} \mathbf{u}''_i \\ \mathbf{0}''_g \end{bmatrix}_r \tag{7.32}$$

$$\begin{bmatrix} \mathbf{u}''_i \\ \mathbf{u}''_g \end{bmatrix} = \begin{bmatrix} \mathbf{T}_i \\ \mathbf{I}_g \end{bmatrix} \mathbf{u}''_g + \begin{bmatrix} \mathbf{u}''_i \\ \mathbf{0}''_g \end{bmatrix}_r \tag{7.33}$$

Substitute this into the governing differential equation of the undamped substructure:

$$\begin{bmatrix} \mathbf{M}_i & \mathbf{M}_{ig} \\ \mathbf{M}_{gi} & \mathbf{M}_g^{(i)} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}''_i \\ \ddot{\mathbf{u}}''_g \end{bmatrix} + \begin{bmatrix} \mathbf{K}_i & \mathbf{K}_{ig} \\ \mathbf{K}_{gi} & \mathbf{K}_g^{(i)} \end{bmatrix} \begin{bmatrix} \mathbf{u}''_i \\ \mathbf{u}''_g \end{bmatrix} = \begin{bmatrix} \mathbf{0}_i \\ \mathbf{I}_g \end{bmatrix} \mathbf{R}_i(t) \tag{7.34}$$

By using the properties of the transformation matrix, $\mathbf{T}_i$, the first submatrix equation of Equation (7.34) becomes:

$$\mathbf{M}_i (\ddot{\mathbf{u}}''_i)_r + \mathbf{K}_i(\mathbf{u}''_i)_r = -(\mathbf{M}_i\mathbf{T}_i + \mathbf{M}_{ig}) \ddot{\mathbf{u}}''_g \tag{7.35}$$

If the substructure is an explicit finite element model with a diagonal mass matrix, the coupled mass submatrix, $\mathbf{M}_{ig}$, is null and Equation (7.35) can be simplified to:

$$\mathbf{M}_i (\ddot{\mathbf{u}}''_i)_r + \mathbf{K}_i(\mathbf{u}''_i)_r = -\mathbf{M}_i \mathbf{T}_i \ddot{\mathbf{u}}''_g \tag{7.36}$$

Therefore, the dynamics of the interior degrees of freedom of the substructure is reduced to a forced vibration problem in the relative displacement coordinate system with the interior

masses, $\mathbf{M}_i$, subject to acceleration fields which have the same spatial distributions of the static boundary node displacement vectors, $-\mathbf{T}_i \ddot{\mathbf{u}}''_g$. Therefore, this becomes an excellent case for application of the Load Dependent Ritz method. The flexibility and rigid body mass convergence criteria[41] can be used to terminate the generation of the dynamic boundary motion Ritz vectors.

The mass orthonormalized basis vectors for the dynamics of the substructure interior degrees of freedom due to boundary motions, $\mathbf{X}^b_i$, can be computed using the algorithm presented in Chapter 4 with $-\mathbf{M}_i\mathbf{T}_i$ as the starting force vectors. The computed basis vectors are to be orthogonalized to $\mathbf{X}^p_i$ and $\mathbf{T}_i$ which are already made part of the vector basis.

The reduction of substructure degrees of freedom due to boundary motions is represented by the following transformation:

$$\mathbf{u}''_i = \mathbf{X}^b_i \mathbf{y}^b_i + \mathbf{T}_i \mathbf{u}_g \tag{7.37}$$

7.4 Undamped Reduced Vector Basis of Substructures

If the undamped reduced vector basis is used to characterize the dynamics of the complete structure, the undamped substructure vector basis can be used effectively. As shown in Section 7.3, the undamped vector basis of each substructure can be computed independently without the knowledge of other substructures. The reduced vector basis is represented by:

$$\mathbf{X}_i = [\, \mathbf{X}^p \; \mathbf{X}^b \; \mathbf{T} \,]_i \begin{bmatrix} \mathbf{y}^p \\ \mathbf{y}^b \\ \mathbf{u}_g \end{bmatrix}_i \tag{7.38}$$

Combine the load dependent vector basis and dynamic boundary motion basis into a dynamic vector basis:

$$\mathbf{X}^{d}_{i} = [\, \mathbf{X}^{p}\ \mathbf{X}^{b} \,]_{i} \tag{7.39}$$

The solution basis is represented by:

$$\mathbf{X}_{i} = [\, \mathbf{X}^{d}\ \mathbf{T} \,]_{i} \begin{bmatrix} \mathbf{y}^{d} \\ \mathbf{u}_{g} \end{bmatrix}_{i} \tag{7.40}$$

From Equation (7.40), it can therefore be generalized that the starting vectors of the Load Dependent Ritz Vector algorithm for the substructure can be the vector block consisting of the local force vector/vectors and the inertia force vectors due to boundary motions:

$$\mathbf{P}_{i} = [\mathbf{p}_{i}\ \ -\mathbf{M}_{i}\mathbf{T}_{i}] \tag{7.41}$$

with the $\mathbf{T}_{i}$ already included in the vector basis.

The formal matrix transformation to the reduced coordinates of the complete substructure is simplified to:

$$\begin{bmatrix} \mathbf{u}_{1} \\ \mathbf{u}_{2} \\ \cdot \\ \cdot \\ \mathbf{u}_{g} \end{bmatrix} = \begin{bmatrix} \mathbf{X}^{d}_{1} & \mathbf{0} & \cdot\ \cdot & \mathbf{T}_{1g} \\ \mathbf{0} & \mathbf{X}^{d}_{2} & \cdot\ \cdot & \mathbf{T}_{2g} \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \mathbf{0} & \mathbf{0} & \cdot\ \cdot & \mathbf{I}_{g} \end{bmatrix} \begin{bmatrix} \mathbf{y}_{1} \\ \mathbf{y}_{2} \\ \cdot \\ \cdot \\ \mathbf{u}_{g} \end{bmatrix} \tag{7.42}$$

Equation (7.42) defines the solution vector space of the substructure method and is the most important equation describing the substructure method. The rest of the development is simply an application of consistent mathematical transformations. By Galerkin's method as described in Chapter 2, the governing differential equation of motion in the reduced coordinates of the complete structure is given by:

$$\mathbf{M}^{*}\ddot{\mathbf{y}} + \mathbf{C}^{*}\dot{\mathbf{y}} + \mathbf{K}^{*}\mathbf{y} = \mathbf{p}^{*} \tag{7.43}$$

$$\text{where, } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ u_g \end{bmatrix} \tag{7.44}$$

$$\mathbf{M}^* = \begin{bmatrix} {X^d}_1{}^T M_1 X^d{}_1 & 0 & . . & {X^d}_1{}^T M_1 T_{1g} + {X^d}_1{}^T M_{1g} \\ 0 & {X^d}_2{}^T M_2 X^d{}_2 & . . & {X^d}_2{}^T M_2 T_{2g} + {X^d}_2{}^T M_{2g} \\ . & . & . . & . \\ . & . & . . & . \\ T_{g1} M_1 X^d{}_1 + M_{g1} X^d{}_1 & T_{g2} M_2 X^d{}_2 + M_{g2} X^d{}_2 & . . & \Sigma(M^{(i)}{}_g + T_{gi} M_i T_{ig} + T_{gi} M_{ig} + M_{gi} T_{ig}) \end{bmatrix} \tag{7.45}$$

$$\mathbf{C}^* = \begin{bmatrix} {X^d}_1{}^T C_1 X^d{}_1 & 0 & . . & {X^d}_1{}^T C_1 T_{1g} + {X^d}_1{}^T C_{1g} \\ 0 & {X^d}_2{}^T C_2 X^d{}_2 & . . & {X^d}_2{}^T C_2 T_{2g} + {X^d}_2{}^T C_{2g} \\ . & . & . . & . \\ . & . & . . & . \\ T_{g1} C_1 X^d{}_1 + C_{g1} X^d{}_1 & T_{g2} C_2 X^d{}_2 + C_{g2} X^d{}_2 & . . & \Sigma(C^{(i)}{}_g + T_{gi} C_i T_{ig} + T_{gi} C_{ig} + C_{gi} T_{ig}) \end{bmatrix} \tag{7.46}$$

$$\mathbf{K}^* = \begin{bmatrix} {X^d}_1{}^T K_1 X^d{}_1 & 0 & . . & {X^d}_1{}^T K_1 T_{1g} + {X^d}_1{}^T K_{1g} \\ 0 & {X^d}_2{}^T K_2 X^d{}_2 & . . & {X^d}_2{}^T K_2 T_{2g} + {X^d}_2{}^T K_{2g} \\ . & . & . . & . \\ . & . & . . & . \\ T_{g1} K_1 X^d{}_1 + K_{g1} X^d{}_1 & T_{g2} K_2 X^d{}_2 + K_{g2} X^d{}_2 & . . & \Sigma(K^{(i)}{}_g + T_{gi} K_i T_{ig} + T_{gi} K_{ig} + K_{gi} T_{ig}) \end{bmatrix} \tag{7.47}$$

$$\mathbf{p}^* = \begin{bmatrix} {X^d}_1{}^T p_1 \\ {X^d}_2{}^T p_2 \\ . \\ . \\ p_g + \Sigma T_{gi} p_i \end{bmatrix} \tag{7.48}$$

The sizes of these matrices and vector are substantially reduced by the Ritz vector representation. Since in generating Ritz vectors, the vectors are normally mass orthonormalized, the mass matrix is simplified to:

$$M^* = \begin{bmatrix} I_1 & 0 & \cdot\cdot & X^{d}{}_1{}^{T}M_1T_{1g}+X^{d}{}_1{}^{T}M_{1g} \\ 0 & I_2 & \cdot\cdot & X^{d}{}_2{}^{T}M_2T_{2g}+X^{d}{}_2{}^{T}M_{2g} \\ \cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot & \cdot\cdot & \cdot \\ T_{g1}M_1X^{d}{}_1+M_{g1}X^{d}{}_1 & T_{g2}M_2X^{d}{}_2+M_{g2}X^{d}{}_2 & \cdot\cdot & \Sigma(M^{(i)}{}_g+T_{gi}M_iT_{ig}+T_{gi}M_{ig}+M_{gi}T_{ig}) \end{bmatrix}$$

(7.49)

If the vectors are also stiffness orthogonalized, the stiffness matrix is simplified to:

$$K^* = \begin{bmatrix} \Lambda_1 & 0 & \cdot\cdot & X^{d}{}_1{}^{T}K_1T_{1g}+X^{d}{}_1{}^{T}K_{1g} \\ 0 & \Lambda_2 & \cdot\cdot & X^{d}{}_2{}^{T}K_2T_{2g}+X^{d}{}_2{}^{T}K_{2g} \\ \cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot & \cdot\cdot & \cdot \\ T_{g1}K_1X^{d}{}_1+K_{g1}X^{d}{}_1 & T_{g2}K_2X^{d}{}_2+K_{g2}X^{d}{}_2 & \cdot\cdot & \Sigma(K^{(i)}{}_g+T_{gi}K_iT_{ig}+T_{gi}K_{ig}+K_{gi}T_{ig}) \end{bmatrix}$$

(7.50)

Λ_i is the diagonal matrix which contains the approximate eigenvalues of the substructure constrained at the boundary. The stiffness orthogonalization process performs linear combination of the basis vectors only and does not change the vector space. There is no particular computational advantage to perform this step at the substructure level other than just knowing the approximate natural frequencies of the constrained substructures. However, if damping ratios are to be specified at the substructure level, it is easier to work with the uncoupled substructure vector basis.

Again, it can be seen that the assembly of the structural matrices and force vectors is modular in nature. There is no cross coupling between substructures. The coupling only exists between the substructure and the interface degrees of freedom. The procedure is very similar to the conventional substructure method. This works exactly the same way as

the direct stiffness method in assembling element matrices into global matrices. The substructure matrices are computed at the substructure level without the knowledge of other substructures and are given by:

$$\mathbf{M}^*_i = \begin{bmatrix} \mathbf{I}_i & \mathbf{X}^{d\,T}_i\mathbf{M}_i\mathbf{T}_{ig}+\mathbf{X}^{d\,T}_i\mathbf{M}_{ig} \\ \mathbf{T}_{gi}\mathbf{M}_i\mathbf{X}^d_i+\mathbf{M}_{gi}\mathbf{X}^d_i & \mathbf{M}^{(i)}_g+\mathbf{T}_{gi}\mathbf{M}_i\mathbf{T}_{ig}+\mathbf{T}_{gi}\mathbf{M}_{ig}+\mathbf{M}_{gi}\mathbf{T}_{ig} \end{bmatrix} \tag{7.51}$$

$$\mathbf{C}^*_i = \begin{bmatrix} \mathbf{X}^{d\,T}_i\mathbf{C}_i\mathbf{X}_i & \mathbf{X}^{d\,T}_i\mathbf{C}_i\mathbf{T}_{ig}+\mathbf{X}^{d\,T}_i\mathbf{C}_{ig} \\ \mathbf{T}_{gi}\mathbf{C}_i\mathbf{X}^d_i+\mathbf{C}_{gi}\mathbf{X}^d_i & \mathbf{C}^{(i)}_g+\mathbf{T}_{gi}\mathbf{C}_i\mathbf{T}_{ig}+\mathbf{T}_{gi}\mathbf{C}_{ig}+\mathbf{C}_{gi}\mathbf{T}_{ig} \end{bmatrix} \tag{7.52}$$

$$\mathbf{K}^*_i = \begin{bmatrix} \Lambda_i & \mathbf{0} \\ \mathbf{0} & \mathbf{K}^{(i)}_g+\mathbf{T}_{gi}\mathbf{K}_i\mathbf{T}_{ig}+\mathbf{T}_{gi}\mathbf{K}_{ig}+\mathbf{K}_{gi}\mathbf{T}_{ig} \end{bmatrix} \tag{7.53}$$

$$\mathbf{p}^*_i = \begin{bmatrix} \mathbf{X}_i^T\mathbf{p}_i \\ \mathbf{T}_i^T\mathbf{p}_i \end{bmatrix} \tag{7.54}$$

The solution method to Equation (7.43) in the reduced substructure coordinates is essentially identical to the solution to the governing differential equation of a complete structure. The reduced vector basis methods described in Chapters 2 through 6 are applicable.

7.5 Substructure Viscous Damping Matrix

A few methods for dynamic analysis of damped structures are described in Chapters 4 through 6. Most dynamic substructure methods totally neglect the definition of damping at the substructure level and viscous damping ratios are assigned to the normal modes at the final system level response analysis. This method cannot account for the effect of the damping mechanisms which are deliberately designed into the structure for vibration control. For these structures with specific damping design, the mathematical treatment of damping must be quite accurate to predict the effect of such a design. If the structure is

actually damped by discrete dampers only, the damping matrix is well defined. However, this is normally not the case. The application of dynamic substructure method introduces complication in the definition of the structural damping characteristics. It is most important not to simplify the damping characterization too early in the computation. It is advisable to carry the consistent damping model to the last step to retain the accuracy and traceability of the damping characteristics of the structure.

In the event that physical dashpots are used to control vibrations, it is quite straight forward to compute a consistent reduced substructure damping matrix in the undamped reduced vector basis. The reduced substructure damping matrix is:

$$C^*_i = \begin{bmatrix} X^{d}_i{}^T C_i X^d_i & X^{d}_i{}^T C_i T_{ig} + X^{d}_i{}^T C_{ig} \\ T_{gi} C_i X^d_i + C_{gi} X^d_i & C^{(i)}_g + T_{gi} C_i T_{ig} + T_{gi} C_{ig} + C_{gi} T_{ig} \end{bmatrix} \tag{7.55}$$

The intrinsic damping in the structure due to materials and connections is also often represented by viscous damping. The detail mechanisms of this type of damping are not well characterized. Typically, a substructure (physical structural subassembly) is designed, fabricated and tested separately. The mass, stiffness and damping characteristics are tested and correlated to the structural subassembly model. The substructure model is then integrated into the complete structure for performance/response analysis/prediction. In general, there are two ways to test the structural subassembly, i.e. fixed base and free-free tests. Each test method has its own merits and is applicable under specific conditions.

The fixed base test is very similar to the substructure analysis method with vector basis derived from constrained boundary. If there is no specific damping mechanism in the structure, the intrinsic damping of structural component is normally described by modal damping ratios, ξ_i. These damping ratios can be obtained from actual test data of the component or based on experience from similar components. Since each substructure can

be made of different materials or fabrication methods, the damping characteristics of each substructure can be quite different.

Suppose the substructure has intrinsic damping contributed from the structural materials and construction and is modelled by Rayleigh's damping:

$$\mathbf{C}_i = a_i \mathbf{M}_i + b_i \mathbf{K}_i \tag{7.56}$$

The reduced substructure damping matrix is therefore:

$$\begin{aligned} \mathbf{C}^*_i &= a_i \mathbf{M}^*_i + b_i \mathbf{K}^*_i \\ &= \begin{bmatrix} a_i \mathbf{I}_i + b_i \Lambda_i & a_i(\mathbf{X}^d{}_i^T \mathbf{M}_i \mathbf{T}_{ig} + \mathbf{X}^d{}_i^T \mathbf{M}_{ig}) \\ a_i(\mathbf{T}_{gi} \mathbf{M}_i \mathbf{X}^d{}_i + \mathbf{M}_{gi} \mathbf{X}^d{}_i) & a_i \mathbf{M}^{*(i)}{}_g + b_i \mathbf{K}^{*(i)}{}_g \end{bmatrix} \end{aligned} \tag{7.57}$$

The damping in the reduced vector basis can therefore be represented by uncoupled diagonal damping ratios.

$$\mathbf{C}^*_i = \begin{bmatrix} [\, 2\xi_i \omega_i \,] & a_i(\mathbf{X}^d{}_i^T \mathbf{M}_i \mathbf{T}_{ig} + \mathbf{X}^d{}_i^T \mathbf{M}_{ig}) \\ a_i(\mathbf{T}_{gi} \mathbf{M}_i \mathbf{X}^d{}_i + \mathbf{M}_{gi} \mathbf{X}^d{}_i) & a_i \mathbf{M}^{*(i)}{}_g + b_i \mathbf{K}^{*(i)}{}_g \end{bmatrix} \tag{7.58}$$

However, there is damping coupling between the boundary coordinates and the reduced vector coordinates. Unless boundary motions are also applied during the test, the rest of the damping submatrices are quite difficult to evaluate. In the case of Rayleigh's damping representation, the coupled damping is a function of the mass matrix. Rayleigh's damping may be a good representation if:

1. the boundary is only a very small local area of the substructure,
2. only low intrinsic damping exists in this area,
3. the boundary is quite stiff and does not contribute much to the overall dynamics of the system.

In these cases, it may be sufficient to use Rayleigh's damping to model the damping contribution at the boundary degrees of freedom. In any event, the reduced substructure damping matrix is not diagonal and should be assembled into a global damping matrix.

However, if the most appropriate structural test to the substructure is a simulated free-free test, the measured damping will correspond to the free-free modes of the substructure. The damping ratios so measured will characterize the entire substructure, i.e. both the constrained vectors and boundary motion vectors. However, since the analysis is normally performed in the constrained vector and boundary motion vector bases, a transformation of the damping from the free-free coordinates into the analysis coordinates is required. From Appendix A, the synthesized damping matrix in the substructure physical coordinates, C^F_i, due to the free-free modes, Φ^F, is given by:

$$C^F_i = M\Phi^F_i[2\xi_i\omega_i](M\Phi^F_i)^T \tag{7.59}$$

where, Φ^F_i = mass orthonormalized (measured or analytical free-free) vectors of the i-th substructure

This synthesized damping matrix has the rank of the number of damping ratios included and this matrix is used to approximate the substructure damping matrix:

$$C_i \approx C^F_i \tag{7.60}$$

Therefore the reduced substructure damping matrix is given by:

$$C^*_i = [X^d_i\ T_i]^T M\Phi^F_i\,[2\xi_i\omega_i]\,(M\Phi^F_i)^T[X^d_i\ T_i]$$

$$C^*_i = \begin{bmatrix} X^d{}_i{}^T M\Phi^F_i[2\xi_i\omega_i]\Phi^F{}_i{}^T MX^d_i & X^d{}_i{}^T M\Phi^F_i[2\xi_i\omega_i]\Phi^F{}_i{}^T MT_i{}^T \\ T_i{}^T M\Phi^F_i[2\xi_i\omega_i]\Phi^F{}_i{}^T MX^d_i & T_i{}^T M\Phi^F_i[2\xi_i\omega_i]\Phi^F{}_i{}^T MT_i{}^T \end{bmatrix} \tag{7.61}$$

Conventional substructure analysis method does not prescribe energy dissipation due to boundary motions at the substructure level. However, with this formulation, dissipation

due to boundary motions is modelled. This requires damping to be specified as a matrix rather than damping ratios of the constrained normal modes. However, since the modal damping characterization is not too precise anyway, simplification of this damping matrix should be implemented to reduce computational requirements. As more structures are designed with specific damping mechanisms for vibration control, the mathematical representation of damping requires the same level of detail as the mathematical representations of the mass and stiffness matrices.

7.6 Reduced Viscoelastic Damping Matrix

If viscoelastic materials are used in a substructure for passive vibration control, the complex stiffness matrix must be reduced consistently with the mass and stiffness matrices to synthesize into the system level model. Since the models of structures with viscoelastic materials are normally quite large, the application of dynamic substructure method is essential to reduce the computation. Dynamic substructure method of viscoelastic structure with constant material shear modulus and loss factor utilizing the static boundary vectors was implemented[34]. The solution was very satisfactory for the problem demonstrated and was used effectively in conjunction with the design of viscoelastic struts. A more general formulation of the substructure method for viscoelastic structure is presented below.

From Chapter 6, since the viscoelastic materials are modelled by complex number, the substructure stiffness matrix is also complex and composed of the real part, and the imaginary part.

$$\mathbf{K}_i(\omega) = \mathbf{K}_i^R(\omega) + i\,\mathbf{K}_i^I(\omega) \qquad (6.)$$

The governing equation of the substructure is given by:

$$\left\{ \begin{bmatrix} K_i & K_{ig} \\ K_{gi} & K_g^{(i)} \end{bmatrix} - \omega^2 \begin{bmatrix} M_i & M_{ig} \\ M_{gi} & M_g^{(i)} \end{bmatrix} + i\omega \begin{bmatrix} C_i & C_{ig} \\ C_{gi} & C_g^{(i)} \end{bmatrix} + i \begin{bmatrix} K^I_i & K^I_{ig} \\ K^I_{gi} & K^I_{g(i)} \end{bmatrix} \right\} \begin{bmatrix} U_i \\ U_g \end{bmatrix}$$

$$= \begin{bmatrix} p_i \\ p_g^{(i)} \end{bmatrix} G(\omega) \tag{7.7}$$

The method presented in Chapter 6 can be used to extract the Ritz vector basis of the frequency dependent substructure. The transformation between the physical and reduced space can therefore be expressed as:

$$X_i = [\, X^d \; T \,]_i \begin{bmatrix} y^d \\ u_g \end{bmatrix}_i \tag{7.40}$$

The consistent complex part of the stiffness matrix can be computed as:

$$K^{I*}{}_i = \begin{bmatrix} X^d{}_i{}^T K^I{}_i X^d{}_i & X^d{}_i{}^T K^I{}_i T_{ig} + X^d{}_i{}^T K^I{}_{ig} \\ T_{gi} K^I{}_i X^d{}_i + K^I{}_{gi} X^d{}_i & K^{I(i)}{}_g + T_{gi} K^I{}_i T_{ig} + T_{gi} K^I{}_{ig} + K^I{}_{gi} T_{ig} \end{bmatrix} \tag{7.52}$$

This substructure complex stiffness matrix must then be assembled into the global complex stiffness matrix as follows:

$$K^* = \begin{bmatrix} X^d{}_1{}^T K^I{}_1 X^d{}_1 & 0 & \cdots & X^d{}_1{}^T K^I{}_1 T_{1g} + X^d{}_1{}^T K^I{}_{1g} \\ 0 & X^d{}_2{}^T K^I{}_2 X^d{}_2 & \cdots & X^d{}_2{}^T K^I{}_2 T_{2g} + X^d{}_2{}^T K^I{}_{2g} \\ \vdots & \vdots & \cdots & \vdots \\ T_{g1} K^I{}_1 X^d{}_1 + K^I{}_{g1} X^d{}_1 & T_{g2} K^I{}_2 X^d{}_2 + K^I{}_{g2} X^d{}_2 & \cdots & \Sigma (K^{I(i)}{}_g + T_{gi} K^I{}_i T_{ig} + T_{gi} K^I{}_{ig} + K^I{}_{gi} T_{ig}) \end{bmatrix}$$

(7.47)

At this level, the analysis method presented in Chapter 6 can be used to compute the solution of a viscoelastic structure in the frequency domain. If only one material is used in a substructure or the complete structure, some of the computation may be simplified so that

some of the submatrices may not be re-computed at each frequency. In this case, only scalar multiplication is sufficient and hence the computation effort is reduced.

Chapter 8

Conclusion

A class of methods based on the reduced vector basis was presented to solve the dynamic responses of large damped linear structures. This approach was traced back to classical methods in solving continuous differential equations by trial functions. The same approach was adapted to modern computation methods in solving discretized matrix differential equations. While structures are best modelled in the physical coordinate system with large number of kinematic degrees of freedom, dynamic responses are best characterized and computed in the reduced coordinates. The reduced vector basis representation of a structure reduces of order of the differential equation for practical computation.

The concept of the reduced vector basis has been quite obscure to many individuals who understand structural dynamic only in terms of eigenvectors or normal modes. Many individuals concluded that the use of a vector basis smaller than the eigenvector basis is neither mathematically correct nor rigorous. This dissertation provides clear physical interpretations and rigorous mathematical derivations of the reduced vector basis. The most important concept in reducing the order of the differential equation is by finding the vector basis which spans the particular solution rather than the normal mode basis which consists of the eigenvectors of the homogeneous differential equation.

The steady state solution to harmonic excitation is the key to the understanding of the reduced vector space concept. The solution to the steady state solution provides the natural vector basis for solution of general dynamic forcing functions. The reduced vector subspace is derived based on iterative (relaxation) solution methods, and Taylor's and Laurent's series expansion. The concept of static correction which may be intriguing in the

eigenvector space is very clear and natural in the reduced vector basis. In addition, the harmonic correction vector can be derived easily using this approach. Solution to a damped structure is computed in the reduced vector basis by iterative solution methods. Uncoupled responses (one cycle iterative solution) can be used during the initial design. For analysis of the preliminary design, additional iterations will properly account for the effects of damping coupling. Therefore, this provides a unified method for both design and analysis of damped structures in the reduced vector basis.

Since the concept of steady state vibration is a very general one, it can easily be applied to generate a damped vector basis. In this case the vector basis consists of both in-phase and out-of-phase components. In addition, the effects of constant hysteretic damping and frequency dependent viscoelastic damping in the frequency domain can be considered easily. Finite element analysis with complex stiffness matrix representation is shown to be handled efficiently in the reduced vector basis. Efficient design and analysis methods for the frequency dependent structures were derived in the reduced vector basis.

Dynamic substructure method is essential to the design and analysis of large structures. Based on the reduced vector basis approach, an accurate dynamic substructure method was derived. Consistent transformation of the viscous damping matrix and the imaginary part of the stiffness matrix allows the damping of structures to be modelled efficiently and accurately.

The class of methods presented can be used to design large flexible precision structures for passive vibration control. Damping can be introduced by either discrete dashpots or advanced viscoelastic materials. It also allows controls-structures interaction problems to be solved efficiently in the reduced vector basis in the frequency domain. Furthermore, since the mathematical approach in deriving these methods is very general, new methods can easily be derived to solve new problems of large damped structures.

This class of reduced vector basis methods is extremely powerfully in providing solution to large problems. All conventional methods in structural dynamics can be re-visited and revised to enhance computational efficiency. Additional research, implementation and application should be conducted to gain maturity of this class of methods toward eventual acceptance as the standard methods for solution. The following are a few areas where additional work is necessary:

1. The applicability of the undamped and damped vector bases to compute dynamic responses of damped structures should be studied for passively damped structures. The numerical computation algorithm which includes the Gram-Schmidt orthonormalization process and rational termination criteria should be implemented.
2. The reduced vector basis approach should be implemented and applied to the design and analysis of structures with frequency dependent complex viscoelastic materials.
3. The load dependent and boundary motion Ritz bases for dynamic substructure method should be implemented to gain maturity and confidence of the method in practical design of complex structures with passive damping.

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APPENDIX

A.1 Orthogonal Expansion in Eigenvector Basis

Orthogonal expansions of various vectors and matrices in terms of the eigenvectors are important to the understanding of the structural behavior. These orthogonal expansions enable syntheses of the structural vectors or matrices from modal representation of the structure or modal test data. They also provide error estimates of the residual effects due to modal truncation. These properties are frequently used in this dissertation and are summarized in the following subsections.

The expansions in terms of the mode shapes (eigenvectors, ϕ_i) or modal inertial force vectors ($\mathbf{M}\,\phi_i$) are based on the orthogonality properties of the eigenvectors:

$$\Phi^T \mathbf{M} \Phi = \mathbf{I} \tag{A.1}$$

$$\Phi^T \mathbf{K} \Phi = \Lambda \tag{A.2}$$

A.1.1 Displacement Vector Expansion

$$\mathbf{u} \approx \sum_{i=1}^{m} \phi_i \, a_i \tag{A.3}$$

$$a_i = \phi_i^T \mathbf{M} \mathbf{u} \tag{A.4}$$

$$\mathbf{u} \approx \Phi \Phi^T \mathbf{M} \mathbf{u} \tag{A.5}$$

$$\mathbf{u}_r = [\mathbf{u} - \Phi \Phi^T \mathbf{M} \mathbf{u}] \tag{A.6}$$

A.1.2 Force Vector Expansion

$$\mathbf{p} \approx \sum_{i=1}^{m} \mathbf{M} \, \phi_i \, \tau_i \tag{A.7}$$

$$\tau_i = \phi_i^T \mathbf{p} \tag{A.8}$$

$$\mathbf{p} \approx \mathbf{M}\,\Phi\,\Phi^T\,\mathbf{p} \tag{A.9}$$

$$\mathbf{p}_r = [\mathbf{p} - \mathbf{M}\,\Phi\,\Phi^T\,\mathbf{p}] \tag{A.10}$$

A.1.3 Stiffness Matrix Expansion

$$\mathbf{K} \approx \sum_{i=1}^{m} [\,(\mathbf{M}\,\phi_i)\,\omega_i^2\,(\mathbf{M}\,\phi_i)^T\,] \tag{A.11}$$

$$\mathbf{K} \approx (\mathbf{M}\,\Phi)\,\Lambda\,(\mathbf{M}\,\Phi)^T \tag{A.12}$$

$$\mathbf{K}_r = [\mathbf{K} - (\mathbf{M}\,\Phi)\,\Lambda\,(\mathbf{M}\,\Phi)^T] \tag{A.13}$$

$$\mathbf{K}^{-1} \approx \sum_{i=1}^{m} \phi_i^T\,\omega_i^{-2}\,\phi_i \tag{A.14}$$

$$\mathbf{K}^{-1} \approx \Phi\,\Lambda^{-1}\,\Phi^T \tag{A.15}$$

$$\mathbf{K}^{-1}_r = [\mathbf{K}^{-1} - \Phi\,\Lambda^{-1}\,\Phi^T] \tag{A.16}$$

If $\mathbf{K}$ is rank deficient, the contributions from the zero strain energy modes are not synthesized into $\mathbf{K}$ or $\mathbf{K}^{-1}$, i.e. only the elastic modes are included in the expansion.

A.1.4 Mass Matrix Expansion

$$\mathbf{M} \approx \sum_{i=1}^{m} [\,(\mathbf{M}\,\phi_i)\,(\mathbf{M}\,\phi_i)^T\,] \tag{A.17}$$

$$\mathbf{M} \approx (\mathbf{M}\,\Phi)\,(\mathbf{M}\,\Phi)^T \tag{A.18}$$

$$\mathbf{M}_r = [\mathbf{M} - (\mathbf{M}\,\Phi)\,(\mathbf{M}\,\Phi)^T] \tag{A.19}$$

A.1.5 Damping Matrix Expansion

$$\mathbf{C} \approx \sum_{i=1}^{m} [\,(\mathbf{M}\,\phi_i)\,(2\,\xi_i\,\omega_i)\,(\mathbf{M}\,\phi_i)^T\,] \tag{A.20}$$

$$\mathbf{C} \approx (\mathbf{M}\,\Phi)\,[2\,\xi_i\,\omega_i]\,(\mathbf{M}\,\Phi)^T \tag{A.21}$$

$$\mathbf{C}_r = [\mathbf{C} - (\mathbf{M}\,\Phi)\,[2\,\xi_i\,\omega_i]\,(\mathbf{M}\,\Phi)^T] \tag{A.22}$$

$$\mathbf{K}^I \approx \sum_{i=1}^{m} [\,(\mathbf{M}\,\phi_i)\,(\eta_i\,\omega_i^2)\,(\mathbf{M}\,\phi_i)^T\,] \tag{A.23}$$

$$\mathbf{K}^{I} \approx (\mathbf{M}\,\Phi)\,[\eta_i\,\omega_i^2]\,(\mathbf{M}\,\Phi)^T \tag{A.24}$$

$$\mathbf{K}^{I}_r = [\mathbf{K}^{I} - (\mathbf{M}\,\Phi)\,[\eta_i\,\omega_i^2]\,(\mathbf{M}\,\Phi)^T] \tag{A.25}$$

A.1.6 Static Strain Energy Expansion

$$\begin{aligned} 2\varepsilon &= \mathbf{u}^T\mathbf{K}\mathbf{u} \\ &\approx (\Phi\Phi^T\mathbf{M}\mathbf{u})^T\,\mathbf{K}\,(\Phi\Phi^T\mathbf{M}\mathbf{u}) \\ &= (\Phi^T\mathbf{M}\mathbf{u})^T\,\Lambda\,(\Phi^T\mathbf{M}\mathbf{u}) \\ &= \sum_{i=1}^{m} a_i^2\,\omega_i^2 \end{aligned} \tag{A.26}$$

Alternatively,

$$\begin{aligned} 2\varepsilon &= \mathbf{p}^T\mathbf{u} \\ &\approx (\mathbf{M}\Phi\Phi^T\mathbf{p})^T\,\mathbf{K}^{-1}\,(\mathbf{M}\Phi\Phi^T\mathbf{p}) \\ &= (\Phi^T\mathbf{p})^T\,\Lambda^{-1}\,(\Phi^T\mathbf{p}) \\ &= \sum_{i=1}^{m} \frac{\tau_i^2}{\omega_i^2} \end{aligned} \tag{A.27}$$

$$2\varepsilon_r = \mathbf{p}^T\mathbf{u} - \sum_{i=1}^{m} \frac{\tau_i^2}{\omega_i^2} \tag{A.28}$$

A.2 Orthogonal Expansion in Mass Orthonormalized Reduced Vector Basis

Since the dynamic response solution can be more efficiently computed in the reduced vector basis, orthogonal expansions of various vectors and matrices in terms of the mass orthonormalized basis vectors are important to the understanding of the structural behavior as related to the vector basis. These orthogonal expansions enable syntheses of the structural vectors or matrices from the basis vectors. They also provide error estimates of the residual effects due to vector truncation. These expansions are almost identical to eigenvector expansions. The concept of orthogonal expansions also reinforces the ease of applicability and physical interpretation of these vectors. These properties are frequently used in this dissertation and are summarized in the following subsections.

The expansions in terms of mass orthonormalized basis vectors, $\mathbf{x}_i$, or inertial force vectors ($\mathbf{M}\,\mathbf{x}_i$) are based on the orthogonality properties of the basis vectors:

$$\mathbf{X}^T\,\mathbf{M}\,\mathbf{X} = \mathbf{I} \tag{A.29}$$

$$\mathbf{X}^T\,\mathbf{K}\,\mathbf{X} = \Lambda_x \tag{A.30}$$

A.2.1 Displacement Vector Expansion

$$\mathbf{u} \approx \sum_{i=1}^{m} \mathbf{x}_i\, a_i \tag{A.31}$$

$$a_i = \mathbf{x}_i^T\,\mathbf{M}\,\mathbf{u} \tag{A.32}$$

$$\mathbf{u} \approx \mathbf{X}\,\mathbf{X}^T\,\mathbf{M}\,\mathbf{u} \tag{A.33}$$

$$\mathbf{u}_r = [\mathbf{u} - \mathbf{X}\,\mathbf{X}^T\,\mathbf{M}\,\mathbf{u}] \tag{A.34}$$

A.2.2 Force Vector Expansion

$$\mathbf{p} \approx \sum_{i=1}^{m} \mathbf{M}\,\mathbf{x}_i\,\tau_i \tag{A.35}$$

$$\tau_i = x_i^T p \tag{A.36}$$

$$p \approx M X X^T p \tag{A.37}$$

$$p_r = [p - M X X^T p] \tag{A.38}$$

A.2.3 Stiffness Matrix Expansion

$$K \approx \sum_{i=1}^{m} [(M x_i) \omega_{xi}^2 (M x_i)^T] \tag{A.39}$$

$$K \approx (M X) \Lambda_x (M \Phi)^T \tag{A.40}$$

$$K_r = [K - (M X) \Lambda_x (M X)^T] \tag{A.41}$$

$$K^{-1} \approx \sum_{i=1}^{m} x_i^T \omega_i^{-2} x_i \tag{A.42}$$

$$K^{-1} \approx X \Lambda_x^{-1} X^T \tag{A.43}$$

$$K^{-1}_r = [K^{-1} - X \Lambda_x^{-1} X^T] \tag{A.44}$$

If K is rank deficient, the contributions from the zero strain energy modes are not synthesized into K or K^{-1}, i.e. only the elastic vectors are included in the expansion.

A.2.4 Mass Matrix Expansion

$$M \approx \sum_{i=1}^{m} [(M x_i) (M x_i)^T] \tag{A.45}$$

$$M \approx (M X) (M X)^T \tag{A.46}$$

$$M_r = [M - (M X) (M X)^T] \tag{A.47}$$

A.2.5 Damping Matrix Expansion

$$C \approx \sum_{i=1}^{m} [(M x_i) (2 \xi_i \omega_i) (M x_i)^T] \tag{A.48}$$

$$C \approx (M X) [2 \xi_i \omega_i] (M X)^T \tag{A.49}$$

$$C_r = [C - (M X) [2 \xi_i \omega_i] (M X)^T] \tag{A.50}$$

$$\mathbf{K}^{I} \approx \sum_{i=1}^{m} [\ (\mathbf{M}\ \mathbf{x}_i\ )\ (\eta_i\ \omega_i^2\ )\ (\mathbf{M}\ \mathbf{x}_i\ )^T\ ] \tag{A.51}$$

$$\mathbf{K}^{I} \approx (\mathbf{M}\ \mathbf{X})\ [\eta_i\ \omega_i^2\ ]\ (\mathbf{M}\ \mathbf{X}\ )^T \tag{A.52}$$

$$\mathbf{K}^{I}_r = [\mathbf{K}^{I} - (\mathbf{M}\ \mathbf{X})\ [\eta_i\ \omega_i^2\ ]\ (\mathbf{M}\ \mathbf{X}\ )^T] \tag{A.53}$$

A.2.6 Static Strain Energy Expansion

$$\begin{aligned} 2\varepsilon &= \mathbf{u}^T\mathbf{K}\mathbf{u} \\ &\approx (\mathbf{X}\mathbf{X}^T\mathbf{M}\mathbf{u})^T\ \mathbf{K}\ (\mathbf{X}\mathbf{X}^T\mathbf{M}\mathbf{u}) \\ &= (\mathbf{X}^T\mathbf{M}\mathbf{u})^T\ \Lambda_x\ (\mathbf{X}^T\mathbf{M}\mathbf{u}) \\ &= \sum_{i=1}^{m} a_i^2\ \omega_i^2 \end{aligned} \tag{A.54}$$

Alternatively,

$$\begin{aligned} 2\varepsilon &= \mathbf{p}^T\mathbf{u} \\ &\approx (\mathbf{M}\mathbf{X}\mathbf{X}^T\mathbf{p})^T\ (\mathbf{X}\mathbf{X}^T\mathbf{M}\mathbf{u}) \\ &= (\mathbf{X}^T\mathbf{p})^T(\mathbf{X}^T\mathbf{M}\mathbf{u}) \\ &= \sum_{i=1}^{m} \tau_i\ a_i \end{aligned} \tag{A.55}$$

$$2\varepsilon_r = \mathbf{p}^T\mathbf{u} - \sum_{i=1}^{m} \tau_i\ a_i \tag{A.56}$$