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The Hilbert basis method for D-flat directions and the superpotential

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Abstract: We discuss, using the Hilbert basis method, how to efficiently construct a complete basis for D-flat directions in supersymmetric Abelian and non-Abelian gauge theories. We extend the method to discrete (R and non-R) symmetries. This facilitates the construction of a basis of all superpotential terms in a theory with given symmetries.

Contents

1   Introduction

Holomorphic gauge invariant monomials play an important role in the understanding of supersymmetric theories and phenomenological applications. They represent D-flat directions in supersymmetric gauge theories [1] and constitute possible superpotential terms. However, in somewhat complex theories the explicit constructions of these monomials can be quite cumbersome in practice. For instance, already the construction of all gauge invariant monomials for the minimal supersymmetric extension of the standard model (MSSM) is rather involved [2] (see also [3] for the discussion in stringy extensions of the MSSM).

In [4] it was shown how to construct the so-called Hilbert basis for holomorphic gauge invariant monomials $\mathscr{M}$, which are known to be in one-to-one correspondence with the D-flat directions. The Hilbert basis allows us to write every monomial $\mathscr{M}$ in the form

$$\mathscr{M} \; = \; \prod_{i=1}^{H} \mathscr{M}_i^{\eta_i} \qquad \text{with } \eta_i \; \in \; \mathbb{N}_0 \; . \tag{1.1}$$

Here H is the number of independent basis monomials $\mathscr{M}_i$ which can only be determined algorithmically.

The purpose of this note is to extend the notion of the Hilbert basis such as to include discrete R and non-R symmetries as well. In section 2 we start by reviewing the Hilbert basis method for continuous gauge symmetries. Section 3 is devoted to the extension to discrete symmetries. The general case is discussed in section 4. In section 5 we comment on potential applications, and finally, section 6 contains our conclusions.

2 Review of Hilbert bases for continuous gauge symmetries

Let us briefly review the Hilbert basis method for (continuous) gauge symmetries [4]. We start by looking at a theory with a single U(1) factor and then extend the discussion to L different U(1) factors.

2.1 Warm-up example: a single U(1)

Consider a U(1) gauge theory and fields $\phi^{(f)}$ ($1 \le f \le F$) with charges $q^{(f)}$. A monomial

$$\mathscr{M} \; = \; \left(\phi^{(1)}\right)^{n_1} \cdots \left(\phi^{(F)}\right)^{n_F} \tag{2.1}$$

is gauge invariant if

$$q^{(1)} \, n_1 + \cdots + q^{(F)} \, n_F \; = \; 0 \; . \tag{2.2a}$$

This condition may be recast as

$$q^T \cdot n \; = \; 0 \; , \tag{2.2b}$$

$q^T = (q^{(1)}, \ldots, q^{(F)})$ and $n^T = (n_1, \ldots, n_F)$. That is, the vector n has to be orthogonal to the charge vector q. The requirement that $\mathscr{M}$ be holomorphic amounts to demanding that $n_i \in \mathbb{N}_0$. The solutions are the intersection of the hyperplane perpendicular to q and the lattice points in $\mathbb{N}_0^F$. The so-called Hilbert basis

$$\mathscr{H} \; = \; \left\{ h^{(1)}, \ldots h^{(H)} \right\} , \tag{2.3}$$

is a complete set of vectors $h^{(i)}$ with the property that each solution n of (2.2) can be written as

$$n \; = \; \sum_{i=1}^{H} \eta_i \, h^{(i)} \qquad \text{with } \eta_i \; \in \; \mathbb{N}_0 \; . \tag{2.4}$$

Every element $h^{(i)}$ of the Hilbert basis is in one-to-one correspondence with a gauge invariant monomial

$$\mathscr{M}_i \; = \; \left(\phi^{(1)}\right)^{h_1^{(i)}} \cdots \left(\phi^{(F)}\right)^{h_F^{(i)}} \tag{2.5}$$

such that every gauge invariant monomial is given by (1.1).

2.2 Generalization to L U(1) factors

In the case of L U(1) factors the condition (2.2) can be rewritten as

$$Q \cdot n = 0, \quad n \in \mathbb{N}_0^F \tag{2.6}$$

with the charge matrix

$$Q = \begin{pmatrix} q_1^{(1)} & \cdots & q_1^{(F)} \\ \vdots & & \vdots \\ q_L^{(1)} & \cdots & q_L^{(F)} \end{pmatrix}, \tag{2.7}$$

where $q_\ell^{(f)}$ denotes the ℓth U(1) charge of the fth field $\phi^{(f)}$.

An important fact about the above problem is that it is well-known in the mathematical literature [5–7]. There are efficient algorithms such as the ones provided by [8, 9], allowing us to compute the Hilbert basis for a given matrix Q very efficiently.

2.3 Non-Abelian symmetries

In the case of a non-Abelian symmetry G, gauge invariance of a monomial composed of G representations $\boldsymbol{r}^{(i)}$ is equivalent to gauge invariance w.r.t. the r U(1) factors generated by the Cartan generators of G. That is, the Hilbert basis method allows us immediately to construct G invariant monomials. Some of the resulting monomials are zero and others are redundant. Examples for vanishing monomials include the baryons of SU(N_c) theories, $\varepsilon^{\alpha_1 \alpha_2 \cdots \alpha_{N_c}} \phi_{\alpha_1}^{(i_1)} \phi_{\alpha_2}^{(i_2)} \cdots \phi_{\alpha_{N_c}}^{(i_{N_c})}$, which vanish if, say, $i_1 = i_2$. In order to construct only non-vanishing and inequivalent monomials, there are various methods available. One of them is 'consecutive basis building' and the other is to systematically remove redundant monomials from the outcome of the Hilbert basis construction. Both methods are briefly reviewed in appendix A. A `mathematica` package allowing for an automatized computation of the independent monomials can be found at [10].

3 Discrete symmetries

In what follows, we generalize the Hilbert basis method such as to applicable to discrete symmetries also. This allows us, in particular, to identify a complete basis for allowed superpotential terms.

3.1 Discrete non-R symmetries

We now discuss how to construct the Hilbert basis for discrete non-R symmetries. We illustrate our method by a simple example, which we then generalize.

3.1.1 Warm-up example

We start by a single $\mathbb{Z}_M$ symmetry under which the $\phi^{(f)}$ have charges $p^{(f)}$. The requirement of $\mathbb{Z}_M$ invariance of the monomial (2.1) translates into

$$p^{(1)} n_1 + \cdots + p^{(F)} n_F = 0 \mod M . \tag{3.1}$$

Without loss of generality we can assume that all discrete charges $p^{(f)}$ are non-negative. Equation (3.1) is equivalent to

$$\left(-M, p^{(1)}, \ldots p^{(F)}\right) \cdot \begin{pmatrix} m \\ n_1 \\ \vdots \\ n_F \end{pmatrix} \;=\; 0 \qquad \text{with } m \;\in\; \mathbb{N}_0 \tag{3.2}$$

and $n_f \in \mathbb{N}_0$, as before. One can now compute the Hilbert basis for the above problem. The basis monomials are then given by

$$\mathscr{M}_i \;=\; \left(\phi^{(1)}\right)^{h_2^{(i)}} \cdots \left(\phi^{(F)}\right)^{h_{F+1}^{(i)}} \tag{3.3}$$

after truncation of the first element $h_1^{(i)}$ which represents not a field $\phi^{(f)}$ but m.

3.1.2 Multiple discrete non-R symmetries

The extension to $\mathbb{Z}_{M_1} \times \cdots \times \mathbb{Z}_{M_K}$ is straightforward. We assume that $\mathbb{Z}_{M_1} \times \cdots \times \mathbb{Z}_{M_K}$ is already the smallest irreducible symmetry of a possible larger discrete group (cf. [11]). We define the charge matrix

$$C \;=\; (-M \mid P) \;, \tag{3.4}$$

where

$$M \;=\; \text{diag}\,(M_1, \ldots, M_K) \tag{3.5}$$

and

$$P \;=\; \begin{pmatrix} p_1^{(1)} & \cdots & p_1^{(F)} \\ \vdots & & \vdots \\ p_K^{(1)} & \cdots & p_K^{(F)} \end{pmatrix} \tag{3.6}$$

with $p_k^{(f)}$ denoting the $\mathbb{Z}_{M_k}$ charge of $\phi^{(f)}$. We can again assume that all charges $p_k^{(f)}$ are non-negative.

3.2 Discrete R symmetries

Let us now turn to discrete R symmetries. We start by discussing a single $\mathbb{Z}_N^R$ and then generalize the setting to more than one discrete R symmetry.

3.2.1 Warm-up example

We start by a single $\mathbb{Z}_N^R$ symmetry under which the $\phi^{(f)}$ have charges $r^{(f)}$, which, again, can all be chosen non-negative. The requirement of $\mathbb{Z}_N^R$ invariance of the monomial (2.1) translates into

$$r^{(1)} n_1 + \cdots + r^{(F)} n_F \;=\; 2 \mod N \;. \tag{3.7}$$

Here we have adopted the convention that the superpotential has R charge 2. The above equation is equivalent to

$$\left(-2, -N, p^{(1)}, \ldots p^{(F)}\right) \cdot \begin{pmatrix} \ell \\ m \\ n_1 \\ \vdots \\ n_F \end{pmatrix} = 0 \qquad \text{with } \ell = 1 \,, \tag{3.8}$$

$m \in \mathbb{N}_0$ and $n_f \in \mathbb{N}_0$, as before. One can now compute the solution of the above problem. First, one identifies a basis of all vectors orthogonal to $\left(-2, -N, p^{(1)}, \ldots p^{(F)}\right)^T$. To deal with the restriction $\ell = 1$ we identify two subsets of the Hilbert basis. One, where the first entry equals zero, which we call the homogeneous solution space. And one where the first entry equals one, which we call the inhomogeneous solution space. If a homogeneous solution is labeled by $n_{\text{hom}}^{(h)} \in \mathscr{H}$ and a inhomogeneous solution by $n_{\text{inhom}}^{(i)} \in \mathscr{H}$ we find the general solution to (3.8) to be

$$n = n_{\text{inhom}}^{(i)} + \sum_{h=1}^{H_0} \eta_h \, n_{\text{hom}}^{(h)} \tag{3.9}$$

with fixed i and $\eta_h \in \mathbb{N}_0$. That is, there are H_1 branches of solutions, where H_1 denotes the number of inhomogeneous solutions. Each branch consists of H_0 solutions with H_0 denoting the number of homogeneous solutions. We can find all monomials spanning the superpotential by truncating the vectors n accordingly.

To see what that means in practice, consider a setting with a $\mathbb{Z}_4^R$ symmetry and fields ψ and ϕ with R charges $r_\psi = 1$ and $r_\phi = 0$. The generalized charge matrix for this example reads

$$C = (-2, -4, 1, 0) \,, \tag{3.10}$$

where the last two entries are the R charges of ψ and ϕ. There is only one vector orthogonal to C with first component equal to 1, namely $(1, 0, 2, 0)^T$, such that the unique inhomogeneous solution is given by $n_{\text{inhom}} = (1, 0, 2, 0)^T$. Similarly, we obtain two vectors orthogonal to C with vanishing first component, namely $(0, 1, 4, 0)^T$ and $(0, 0, 0, 1)^T$. The entries of the vectors represent the exponents of the fields in the corresponding monomials. We hence have found that all allowed superpotential terms are of the form

$$\mathscr{M} = \psi^2 \, \psi^{4\eta_1} \, \phi^{\eta_2} \tag{3.11}$$

with $\eta_i \in \mathbb{N}_0$. Of course, one could have obtained the result without the Hilbert basis method; however, for more complex systems this method is highly advantageous.

3.2.2 Multiple discrete R symmetries

Let us extend the discussion to $\mathbb{Z}_{N_1}^R \times \cdots \times \mathbb{Z}_{N_J}^R$. Define the charge matrix

$$C = \left(\begin{array}{c|c|c} \text{-2} & & \\ \vdots & -N & R \\ \text{-2} & & \end{array} \right) \,, \tag{3.12}$$

with

$$N = \operatorname{diag}(N_1, \ldots, N_J) \tag{3.13}$$

and

$$R = \begin{pmatrix} r_1^{(1)} & \cdots & r_1^{(F)} \\ \vdots & & \vdots \\ r_J^{(1)} & \cdots & r_J^{(F)} \end{pmatrix}. \tag{3.14}$$

Here $r_j^{(f)}$ denotes the jth R charge of the fth field. We compute the kernel of the matrix C. The last F components of the kernel vectors with the first entry equal to 1 or 0 (before truncation) define the inhomogeneous or homogeneous solutions, respectively. The general solution will again be of the form (3.9). As before, the identification of the (last F) entries of the vectors with the exponents will then give us the desired invariant monomials, i.e. allowed superpotential terms.

4 Putting all together

4.1 Charge matrix for $\mathrm{U}(1)^L$ symmetry with discrete R and non-R symmetries

We consider now the general $\mathrm{U}(1)^L \times \mathbb{Z}_{M_1} \times \cdots \times \mathbb{Z}_{M_K} \times \mathbb{Z}_{N_1}^R \times \cdots \times \mathbb{Z}_{N_J}^R$ case. The charge matrix is

$$C = \left(\begin{array}{ccc|c|c|c} -2 & & & & & \\ & \vdots & & -N & 0 & R \\ -2 & & & & & \\ \hline 0 & & 0 & 0 & -M & P \\ \hline 0 & & 0 & 0 & 0 & Q \end{array} \right) \tag{4.1}$$

with Q, P, R, N and M defined in equations (2.7), (3.6), (3.14), (3.13) and (3.5), respectively. We compute the kernel of C, and, again, we decompose these vectors in those with first components equal to 1 or 0. From those we obtain the inhomogeneous or homogeneous solutions, respectively, by projecting on the last F components. As before, the general solution is of the form (3.9), and identifying the entries of the vectors with the exponents will then give us the desired invariant monomials, i.e. allowed superpotential terms.

4.2 A stringy example

We base our example on the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold discussed in [4]. We consider the $\mathbb{Z}_4^R$ vacuum discussed there and construct the superpotential for the standard model singlets with R charges 0 or 2. At the orbifold point, the symmetry seen by these fields is of the type

$$G_{\text{symm}} = \mathrm{U}(1)^L \times (\mathbb{Z}_2)^6 \times (\mathbb{Z}_4^R)^3 \tag{4.2}$$

with $L = 8$. Here we have already eliminated the non-Abelian symmetries by forming $\mathrm{SU}(N)$ gauge invariant monomials (cf. our discussion in 2.3). The discrete symmetries follow from the space-group selection rules and H-momentum conservation [12] (for the rules in this specific geometry see [13]), and are partially redundant, e.g. there are in

fact only two independent $\mathbb{Z}_4^R$ factors. These symmetries and selection rules constrain the holomorphic correlators of the theory [12, 14], which can also partially come from the Kähler potential in an appropriate description (cf. the discussion in [15, section 4]). In what follows we will not distinguish between such correlators and allowed superpotential terms.

In the vacuum discussed in [4], there is a residual $\mathbb{Z}_4^R$ symmetry, which forbids the superpotential at the perturbative level. If we switch on an additional field with R charge 2 we will break the $\mathbb{Z}_4^R$ and obtain a non-zero superpotential in the vacuum. We will switch on the additional field $\bar{\phi}_1$. That is, the fields

$$
\begin{aligned}
\tilde{\phi}^{(i)} \;=\; & \{\bar{\phi}_1, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_8, \phi_9, \phi_{10}, \phi_{11}, \phi_{12}, \phi_{13}, \phi_{14}, \\
& x_1, x_2, x_3, x_4, x_5, \bar{x}_1, \bar{x}_3, \bar{x}_4, \bar{x}_5, y_3, y_4, y_5, y_6 \}
\end{aligned}
\tag{4.3}
$$

will now acquire vacuum expectation values (VEVs). The Hilbert basis contains 15408 elements for this choice. The superpotential starts at lowest order with 4 Hilbert basis elements

$$
\mathscr{W} \;=\; (x_4 \bar{x}_4 + x_5 \bar{x}_5 + \phi_9 \phi_{13} + \phi_{10} \phi_{14}) \, \bar{\phi}_1 + \cdots \; .
\tag{4.4}
$$

We also considered the appearance of the proton decay operator $QQQ\ell$. The Hilbert basis involving $QQQ\ell$ consists of 4284 elements where we focus only on first generation quarks and leptons. The lowest order $QQQ\ell$ operator occurs at order 11 in the field VEVs. An example is given by

$$
\mathscr{W} \;\supset\; Q_1 \, Q_2 \, Q_2 \, \ell_1 \, \bar{\phi}_1 \, x_1 \, x_2 \, x_3 \, x_4 \, \bar{x}_3 \, \phi_2 \, \phi_4 \, \phi_9 \, \phi_{12}^2 \; .
\tag{4.5}
$$

These examples show that the Hilbert basis method is powerful enough to handle very complex examples with many fields.

5 Applications and speculations

Defining the subsets $\mathscr{H}_{\text{inhom}} = \{\mathscr{M}_{\text{inhom}}^{(1)}, \ldots, \mathscr{M}_{\text{inhom}}^{(H_1)}\}$ and $\mathscr{H}_{\text{hom}} = \{\mathscr{M}_{\text{hom}}^{(1)}, \ldots, \mathscr{M}_{\text{hom}}^{(H_0)}\}$ which are constructed from $n_{\text{inhom}}^{(i)}$ and $n_{\text{hom}}^{(h)}$, the full superpotential to all orders is

$$
\mathscr{W} \;=\; \sum_{i=1}^{H_1} \sum_{n_1, \ldots, n_{H_0}} \lambda_{n_1 \cdots n_{H_0}}^{(i)} \mathscr{M}_{\text{inhom}}^{(i)} \left(\mathscr{M}_{\text{hom}}^{(1)}\right)^{\eta_1} \cdots \left(\mathscr{M}_{\text{hom}}^{(H_0)}\right)^{\eta_{H_0}} \; .
\tag{5.1}
$$

We can further speculate that the structure of the superpotential is

$$
\mathscr{W}_{\text{structure}} \;=\; \frac{\sum\limits_{i=1}^{H_1} \lambda_i \, \mathscr{M}_{\text{inhom}}^{(i)}}{1 - \sum\limits_{h=1}^{H_0} \kappa_h \, \mathscr{M}_{\text{hom}}^{(h)}} \; .
\tag{5.2}
$$

However, the relations between the couplings implied by $\mathscr{W}_{\text{structure}}$ will in general be incorrect. Yet $\mathscr{W}_{\text{structure}}$ can be used quickly to answer certain questions such as at which order some combination of fields appears first. Such questions arise in (generalized) Froggatt-Nielsen model building [16] with many symmetries and where the order in which a given

term appears is a measure for the suppression of the corresponding effective coupling. Another, very similar application concerns the identification of approximate continuous R and non-R symmetries, where one is interested in the lowest order terms that explicitly break such symmetries [17, 18]. Given (5.2) one immediately reads off the lowest order at which perturbative superpotential terms arise.

Whether or not the relations between the coefficients implied by (5.2) can be obtained in some interesting setting needs still to be explored. However, it is tempting to speculate that in certain highly symmetric string compactifications, such as orbifolds, the superpotential (and holomorphic terms in the Kähler potential) will be some known function of the building blocks. It will be interesting to study this question in more detail.

6   Conclusions

We have described a simple method that allows us, given the symmetries of the theory, to construct the building blocks of the D-flat directions and the superpotential. This basis is given by the basis of non-negative integer solutions n of the simple matrix equation $C \cdot n = 0$, which has been extensively studied in the mathematical literature. Publicly available codes allow us then to compute the basis very efficiently. We have discussed a specific example, based on a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold model, in which we computed the superpotential basis for certain singlet fields. The successful construction of this basis demonstrates that our methods can be used to efficiently compute the superpotential to all orders even in rather complex systems.

Acknowledgments

A   Gauge invariant monomials for SU(N)

We review the construction of gauge invariant monomials for SU(N) with matter fields ϕ_i and $\overline{\phi_j}$.

A.1   Consecutive basis building

Let us consider a general example of L SU(N_i) gauge groups, SU(N_1) $\times \cdots \times$ SU(N_L). In order to construct the Hilbert basis $\mathscr{H}$ we proceed as follows: in a first step, we construct a basis $\mathscr{H}_1$ of SU(N_1) singlets, consisting of elementary SU(N_1) singlets and SU(N_1) invariant monomials. As is well known, the latter will be given by the 'mesons' and 'baryons', which

in SU(3) would look like,

$$(\phi_i\overline{\phi}_j) \equiv \phi_i^\alpha\overline{\phi}_j^{\dot\alpha}\delta_{\alpha\dot\alpha} \,, (\phi_i\phi_j\phi_k) \equiv \phi_i^\alpha\phi_j^\beta\phi_k^\gamma\varepsilon_{\alpha\beta\gamma} \,, (\overline{\phi}_i\overline{\phi}_j\overline{\phi}_k) \equiv \overline{\phi}_i^{\dot\alpha}\overline{\phi}_j^{\dot\beta}\overline{\phi}_k^{\dot\gamma}\varepsilon_{\dot\alpha\dot\beta\dot\gamma} \,. \tag{A.1}$$

These monomials will transform as singlets under $SU(N_1)$ and together with the singlet fields they build the basis $\mathscr{H}_1$.

We now use $\mathscr{H}_1$ to build the basis $\mathscr{H}_{1,2}$, which will be the basis of $SU(N_1) \times SU(N_2)$ singlets. Obviously, $\mathscr{H}_{1,2}$ will contain terms of $\mathscr{H}_1$ which are already $SU(N_2)$ singlets and also monomials which are constructed in a similar way as in (A.1). The only difference is that terms in $\mathscr{H}_1$ can have more than one $SU(N_2)$ index, i.e. they can furnish higher representations.

From here on the course of action is always the same: we use the previous basis $\mathscr{H}_{1,\ldots,k}$ to build $\mathscr{H}_{1,\ldots,k,k+1}$, the basis of $SU(N_1) \times \cdots \times SU(N_k) \times SU(N_{k+1})$ singlets until we eventually find the basis $\mathscr{H} \equiv \mathscr{H}_{1,\ldots,L}$ of monomials invariant under the full gauge group. One can find an explicit example of this method for $SU(3) \times SU(2) \times U(1)$ in [2].

A.2 Cartan subalgebras

Alternatively, one can construct the monomials for non-Abelian symmetry groups by using the Hilbert basis method for the $U(1)^L$ symmetry defined by the Cartan subalgebras. For this strategy, we split all fields in their tensor components and assign them a charge according to the Cartan charges. Let us consider some fields from [2] to explain this procedure. Assume, we have an $SU(3) \times SU(2)$ gauge group. $SU(3)$ has rank 2 and its Cartan subalgebra can be taken to be the one generated by the two diagonal Gell-Mann matrices, λ_3 and λ_8, whereas $SU(2)$ has rank 1, hence we use the diagonal Pauli matrix σ_3.

Let us assume we have three fields, $Q(\mathbf{3},\mathbf{2})$, $\overline{u}(\overline{\mathbf{3}},\mathbf{1})$ and $\ell(\mathbf{1},\mathbf{2})$. Q has six tensor components $Q^{\alpha i}$, where $\alpha = 1,2,3$ is the $SU(3)$ index and $i = 1,2$ the $SU(2)$ index. Now we can assign charges to each component under the respective gauge groups, using the eigenvalues of the generators of the Cartan subalgebras, which is particularly easy when using the diagonal matrices λ_3, λ_8 and σ_3. Therefore, $Q_2 \equiv Q^{\alpha=2,i=1}$ will be assigned the charges

$$q_1^{Q_2} = -1 \,, \quad q_2^{Q_2} = 1 \,, \quad q_3^{Q_2} = 1 \,. \tag{A.2}$$

Another example is $\overline{u}^{\dot\alpha}$, which is an $SU(2)$ singlet and an $SU(3)$ anti-triplet, thus carrying zero charge under the $SU(2)$ gauge group and opposite $SU(3)$ charges. Taking the component $\overline{u}_2 \equiv \overline{u}^{\dot\alpha=\dot 2}$, we get

$$q_1^{\overline{u}_2} = 1 \,, q_2^{\overline{u}_2} = -1 \,, q_3^{\overline{u}_2} = 0 \,. \tag{A.3}$$

In this way we can split each field into its components (six for Q, three for $\overline{u}$ and two for ℓ) and build a 3×11 charge matrix Q, which in our example will look like

$$Q = \begin{pmatrix} 1 & -1 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -2 & 1 & 1 & -2 & -1 & -1 & 2 & 0 & 0 \\ 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \,. \tag{A.4}$$

– 9 –

Using algorithms like [8, 9], we can now find the Hilbert basis of all solutions for the charge matrix Q, the same way as before in equation (2.7). Solutions would for example be $Q^{11} Q^{21} Q^{32} \ell^2$, $\overline{u}^{\dot{1}} \overline{u}^{\dot{2}} \overline{u}^{\dot{3}}$, $Q^{11} \overline{u}^{\dot{1}} \ell^2$ or many more.

In order to associate these solutions to proper gauge invariant monomials, we define a prescription of how to translate such expressions to objects in which the (generalized) color indices are contracted appropriately. An increasing series will be contracted with the total antisymmetric Levi-Civita tensor, indices which have the same value but are dotted and undotted will be contracted with the Kronecker delta. Using this procedure, our examples would look like

$$Q^{11} Q^{21} Q^{32} \ell^2 \quad \Longleftrightarrow \quad Q^{\alpha a} Q^{\beta b} Q^{\gamma c} \ell^d \, \varepsilon_{\alpha\beta\gamma} \, \varepsilon_{ac} \, \varepsilon_{bd} \,, \tag{A.5a}$$

where we had to contract antisymmetrically several times,

$$\overline{u}^{\dot{1}} \overline{u}^{\dot{2}} \overline{u}^{\dot{3}} \quad \Longleftrightarrow \quad \overline{u}^{\dot{\alpha}} \overline{u}^{\dot{\beta}} \overline{u}^{\dot{\gamma}} \, \varepsilon_{\dot{\alpha}\dot{\beta}\dot{\gamma}} \,, \tag{A.5b}$$

which is very similar to the one above and

$$Q^{11} \overline{u}^{\dot{1}} \ell^2 \quad \Longleftrightarrow \quad Q^{\alpha a} \overline{u}^{\dot{\alpha}} \ell^b \, \delta_{\alpha\dot{\alpha}} \, \varepsilon_{ab} \,, \tag{A.5c}$$

where we were able to see when to use the Kronecker delta.

A caveat of this procedure is that we will end up with many monomials occurring more than once, e.g. (A.5b) will appear six times. Therefore, we have to remove the redundant ones. Furthermore, one has the possibility to end up with monomials which will vanish due to antisymmetric contraction. Take (A.5b) again: if $\overline{u}$ were a field with only one generation, the monomial would clearly vanish. This means that one has to check the Hilbert basis for zero-valued monomials and remove them, which will leave the basis nonetheless intact, since the contribution of these monomials would be zero in any case.

We see that all these caveats are manageable. Furthermore, this procedure has a big advantage compared to the method described in the previous subsection A.1. Using the Cartan subalgebras allows us to quickly and fully automated build monomials for several $SU(N)$ and $U(1)$ gauge groups, which would get very tedious (in certain cases impossible), especially for $N > 3$, more than three gauge groups or too many fields. We have created a `mathematica` package ourselves, based on [8, 9], which automatizes this procedure [10].

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