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Matroid polytope subdivisions and valuations

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Matroid polytope subdivisions and valuations

by

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Abstract

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Many important invariants for matroids and polymatroids are *valuations* (or are *valuative*), which is to say they satisfy certain relations imposed by subdivisions of matroid polytopes. These include the Tutte polynomial, the Billera-Jia-Reiner quasi-symmetric function, Derksen's invariant $\mathcal{G}$, and (up to change of variables) Speyer's invariant h.

We prove that the ranks of the subsets and the activities of the bases of a matroid define valuations for the subdivisions of a matroid polytope into smaller matroid polytopes; this provides a more elementary proof that the Tutte polynomial is a valuation than previously known.

We proceed to construct the $\mathbb{Z}$-modules of all $\mathbb{Z}$-valued valuative functions for labeled matroids and polymatroids on a fixed ground set, and their unlabeled counterparts, the $\mathbb{Z}$-modules of valuative invariants. We give explicit bases for these modules and for their dual modules generated by indicator functions of polytopes, and explicit formulas for their ranks. This confirms Derksen's conjecture that $\mathcal{G}$ has a universal property for valuative invariants.

We prove also that the Tutte polynomial can be obtained by a construction involving equivariant K-theory of the Grassmannian, and that a very slight variant of this construction yields Speyer's invariant h. We also extend results of Speyer concerning the behavior of such classes under direct sum, series and parallel connection and two-sum; these results were previously only established for realizable matroids, and their earlier proofs were more difficult.

We conclude with an investigation of a generalisation of matroid polytope subdivisions from the standpoint of tropical geometry, namely subdivisions of *Chow polytopes*. The Chow polytope of an algebraic cycle in a torus depends only on its tropicalisation. Generalising this, we associate a Chow polytope subdivision to any abstract tropical variety in $\mathbb{R}^n$. Several significant polyhedra associated to tropical varieties are special cases of our Chow subdivision. The Chow subdivision of a tropical variety X is given by a simple combinatorial

construction: its normal subdivision is the Minkowski sum of X and an upside-down tropical linear space.

Contents

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Chapter 1

Overview

1.1 Matroids and their polytopes

The *matroid* is a combinatorial structure with many faces. Not only is it widely applicable in mathematics, it can be formalised in a large number of ways which, on the surface, don't look equivalent. Basically, a matroid abstracts the properties of an *independence* relation of whatever nature on a finite set. Let E be a finite set, of cardinality n. We will often take $\mathsf{E} = [n] := \{1, 2, \ldots, n\}$, which is a completely general choice: we only care about E up to bijections of sets. For simplicity we give only one definition of a matroid M on the *ground set* E, that in terms of bases. We say that a set $\mathcal{B}$ of subsets of E is the set of *bases* $\mathcal{B}(M)$ of a matroid M if it satisfies the following axioms:

- $\mathcal{B} \neq 0$;
- For any $B, B' \subseteq \mathcal{B}$ and $b' \in B' \setminus b$, there exists $b \in B \setminus B'$ such that $B \cap \{b\} \setminus \{b'\} \in \mathcal{B}$ (the *exchange* axiom).

The axioms could also be cast to capture other data associated to M, such as the *independent sets* $\mathcal{I}(M)$, the set of all subsets of E contained in a basis, or the *rank function* rk_M on subsets $A \subseteq \mathsf{E}$ such that $\mathrm{rk}_M(A)$ is the size of the largest intersection of A with a basis.

For instance, the first and paramount example of an independence relation is *linear independence* of vectors in a vector space. Let $\mathsf{E} = \{v_1, \ldots, v_n\}$ be a subset of a vector space V. Then there is a matroid M on E whose bases $\mathcal{B}(M(V))$ are the subsets of E forming vector space bases for $\operatorname{span} \mathsf{E}$, whose independent sets $\mathcal{I}(M)$ are the linearly independent subsets of E, and whose rank function is $\mathrm{rk}_M(A) = \dim \operatorname{span} A$. We say that a matroid is *representable* over a given field $\mathbb{K}$ if and only if it has this form for some V a $\mathbb{K}$-vector space. Representable matroids are a class of fundamental importance to matroid theory — but equally one could say that it's the frequency with which combinatorial situations with

underlying vector arrangements extend to non-representable matroids that makes matroid theory shine. We'll see two examples of this in later sections.

We will be particularly concerned here with certain functions on matroids. Before developing the conditions characterising these, we discuss an example of such a function: the *Tutte polynomial*, probably the best-known function on matroids. The Tutte polynomial is an *invariant*: it takes equal values on isomorphic matroids. We'll talk about it at some length, both on account of its own importance (several of our results concern it) and because its story parallels one of our main themes.

The Tutte polynomial $t(M)$ of a matroid M is a bivariate polynomial,

$$t(M) = \sum_{A \subseteq \mathsf{E}} (x-1)^{\mathrm{rk}_M(\mathsf{E}) - \mathrm{rk}_M(A)} (y-1)^{|A| - \mathrm{rk}_M(A)} \ .$$

This presentation is known as the *rank generating function*; the two quantities in the exponent can be taken as measuring how far off A is from a basis in two different senses, given that $|B| = \mathrm{rk}(B) = \mathrm{rk}(\mathsf{E})$ when B is a basis. There is another presentation in terms of *internal* and *external activities*; we defer to Section 2.5 for the statement.

Example 1.1.1. The bases of the rank r *uniform matroid* $U_{\mathsf{E},r}$ on E are all subsets of E of size r. Its Tutte polynomial is

$$\begin{aligned} t(U_{\mathsf{E},r}) = (x-1)^r + \binom{n}{1}(x-1)^{r-1} + \cdots + \binom{n}{r-1}(x-1) \\ + \binom{n}{r} + \binom{n}{r+1}(y-1) + \cdots + \binom{n}{n-1}(y-1)^{n-r-1} + (y-1)^{n-r}. \quad \diamondsuit \end{aligned}$$

Tutte took interest in the fact that the number of spanning trees of a graph could be computed recursively in terms of certain graph minors. In fact the set of spanning trees of a graph G are the bases of a matroid on the ground set $\mathrm{Edges}(G)$. In this light the statement becomes the more transparent one that the number of bases of M satisfies the *deletion-contraction recurrence*,

$$f(M) = \begin{cases} f(M \setminus e) + f(M/e) & e \text{ not a loop or coloop}, \\ f(M \setminus e) f(e) & e \text{ a loop}, \\ f(M/e) f(e) & e \text{ a coloop}. \end{cases} \tag{1.1.1}$$

A *loop* in M is an element $e \in \mathsf{E}$ contained in no basis; a *coloop* is one contained in every basis. $M \setminus e$ is the matroid on $E \setminus e$ whose bases are the bases of M not containing e; M/e is the matroid on the same set $E \setminus e$ whose bases are the bases of M containing e, with e removed.

Several other invariants $f : \{\text{matroids}\} \to R$ of interest, for a ring R, also satisfy the deletion-contraction recurrence, among them other basic properties like number of independent sets, and invariants with nontrivial graph-theoretic content like the chromatic polynomial. Tutte defined the Tutte polynomial and showed that it had a universal property

([88] but there only for graphs; Crapo in [23] does the general case), solving the problem of classifying all invariants satisfying the deletion-contraction recurrence.

Theorem 1.1.2. *Let f be a matroid invariant satisfying* (1.1.1)*. Then $f = f' \circ t$ for some ring homomorphism f'. Explicitly, $f(M) = t(M)(f(\text{coloop}), f(\text{loop}))$.*

(By 'coloop' and 'loop' here we refer to matroids on a singleton ground set whose only element is a coloop, respectively a loop.)

The main perspective on matroids adopted by this thesis takes them as polytopes. (A *polyhedron* is an intersection of half-spaces; a *polytope* is a bounded polyhedron, equivalently the convex hull of finitely many points. A *lattice polytope* or *polyhedron* is one whose vertices are contained in a fixed lattices.)

Fix a real vector space $\mathbb{R}^{\mathsf{E}}$ with a distinguished basis $\{e_i : i \in \mathsf{E}\}$ and its dual basis $\{e^i : i \in \mathsf{E}\}$. For a set $I \subseteq \mathsf{E}$, we write $e_I = \sum_{i \in I} e_i$ for the zero-one indicator vector of I. Given a matroid M on E, its matroid polytope is

$$\text{Poly}(M) = \text{conv}\{e_B : B \in \mathcal{B}(M)\}.$$

The next theorem gives a pleasant intrinsic characterisation of matroid polytopes, which can be taken as another axiom system for matroids.

Theorem 1.1.3 (Gelfand-Goresky-MacPherson-Serganova [37])**.** *A polytope $Q \subseteq \mathbb{R}^{\mathsf{E}}$ is a matroid polytope if and only if*

- *every vertex of Q is of the form e_I for $I \subseteq \mathsf{E}$, and*
- *every edge of Q is of the form $\{e_{I \cup i}, e_{I \cup j}\}$ for $i, j \notin I \subseteq \mathsf{E}$.*

The second of these conditions of course implies the first, unless Q is a point.

Matroid polytopes were first studied in connection with optimisation and with the machinery of linear programming, introduced there by Edmonds [33] (who in fact treated a mild variant of $\text{Poly}(M)$, defined as the convex hull of all independent sets). A second line of inquiry regarding matroid polytopes springs from the observation that their edge vectors are the roots of the A_n root system. Generalisations to other root systems are then studied: these are the *Coxeter matroids* [14]. Most recently, matroid polytopes have appeared in a number of related algebraic-geometric contexts. We go into this in more detail in Section 1.3.

1.2 Subdivisions and valuations

The central construction in which matroid polytopes are involved in this thesis are subdivisions. Let $\mathcal{P}$ be a set of closed convex sets in a real vector space V; in our case $\mathcal{P}$ is the

set of matroid polytopes, together with the empty set. A *subdivision* of sets in $\mathcal{P}$ is a cell complex Σ in V whose underlying space Σ is in $\mathcal{P}$ and such that, if $P_1, \ldots, P_k$ are the maximal cells of V, every intersection $P_K := \bigcap_{k\in K} P_k$ ($\emptyset \neq K \subseteq [k]$) of some of these cells is in $\mathcal{P}$. For uniformity we put $P_\emptyset = |\Sigma|$. Since faces of a matroid polytope are matroid polytopes by Theorem 1.1.3, Σ is a matroid polytope subdivision as soon as $|\Sigma|$ and the maximal cells P_i are matroid polytopes. For example, Figure 1.1 portrays the simplest nontrivial matroid polytope subdivision.

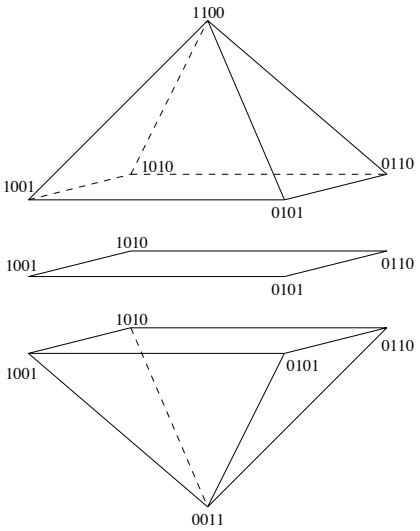


Figure 1.1: The matroid subdivision of a regular octahedron into two square pyramids.

Let G be an abelian group. A function $f : \mathcal{P} \to G$ is a *valuation* if, for any subdivision Σ of sets in $\mathcal{P}$ with the P_K defined as above, we have

$$\sum_{K\subseteq[k]} (-1)^{|K|} f(P_K) = 0. \tag{1.2.1}$$

That is, valuations are functions that add in subdivisions, taking account of the overlaps at the boundaries of the maximal cells. We require that valuations take value 0 on $\emptyset$.

Valuations are a topic of classical interest; their defining property is similar to that of measures, and they're a basic tool in convexity. Among their uses is that they help us gain control over the possible structures of subdivisions, and thus over the structures subdivisions characterise. For instance, if an integer-valued function f is positive and is zero on polytopes not of full dimension, then any full-dimensional polytope P can be subdivided into at most $f(P)$ pieces in $\mathcal{P}$. (This is essentially the situation with Speyer's invariant h introduced at the end of Section 1.3.)

One fundamental valuation is the indicator function: indeed, the way (1.2.1) accounts for overlaps is exactly what's needed for the indicator function to be valuative. As usual, the

indicator function of a set $P \subseteq \mathbb{R}^{\mathsf{E}}$ is the map of sets $\mathbf{1}(P) : \mathbb{R}^{\mathsf{E}} \to \mathbb{Z}$ for which $\mathbf{1}(P)(x) = 1$ if $x \in P$ and $\mathbf{1}(P)(x) = 0$ otherwise; by the unadorned name "the indicator function", referring to a function on $\mathcal{P}$, we mean $\mathbf{1} : \mathcal{P} \to \mathrm{Hom}_{\mathrm{Set}}(\mathbb{R}^{\mathsf{E}}, \mathbb{Z})$.

Just as for the deletion-contraction recurrence (1.1.1), it turns out that several interesting functions of matroids are valuations on matroid polytopes. The Tutte polynomial is one. Some others have been recently introduced by Derksen [29], Billera, Jia and Reiner [11], and Speyer [83]. In addition there is interest in certain properties of matroid polytopes which are self-evidently valuations, such as their volume [5] or their Ehrhart polynomial [30], but which are otherwise little understood.

If all sets in $\mathcal{P}$ are bounded, another basic valuation is the function χ taking the value 1 on every nonempty set in $\mathcal{P}$. This can be thought of as the Euler characteristic, and from this perspective checking that χ is a valuation essentially amounts to repeated application of the Mayer-Vietoris sequence. Chapter 2 is an investigation of valuations that can be constructed from χ. The flexibility in this situation is this: if X is a closed convex set, and Σ is a subdivision, then $\Sigma \cap X$ with the cell structure given by intersecting cells of Σ with X is a subdivision too. So $M \mapsto \chi(\mathrm{Poly}(M) \cap X)$ is a valuation. These are a generalisation of evaluations of $\mathbf{1}$ (take X to be a point).

In Chapter 2, after setting this machinery up, we use it to build some families of valuations which encapsulate a lot of information about matroids. The first family is the characteristic functions of ranks of sets, or more generally of chains of sets.

Theorem 2.5.1, Proposition 2.6.1. *The function*

$$s_{A,r}(M) = \begin{cases} 1 & \mathrm{rk}_M(A_i) = r_i \text{ for all } i \\ 0 & \text{otherwise} \end{cases} \tag{1.2.2}$$

is a matroid valuation, for any chain $\emptyset \subsetneq A_1 \subseteq \cdots \subseteq A_k \subseteq \mathsf{E}$ and $\underline{r} = (r_1, \ldots, r_k) \in \mathbb{Z}^k$.

The second family is the characteristic function of internal and external activities of bases.

Theorem 2.5.4. *The function*

$$f_{B,I,E}(M) = \begin{cases} 1 & B \in \mathcal{B}(M) \text{ has internal activity } I \text{ and external activity } E \\ 0 & \text{otherwise} \end{cases}$$

is a matroid valuation, for any subsets $B, I, E \subseteq \mathsf{E}$.

One can construct the Tutte polynomial T as a linear combination of either of these families, yielding more elementary proofs that T is valuative than previously known (Corollary 2.5.7).

We return to the question of characterising all valuations. Chapter 3 is dedicated to this question in a range of settings. One generalisation we make throughout that chapter

is to *polymatroids*, another structure of general interest introduced by Edmonds. We also invoke *megamatroids*, which are primarily a technical tool. The polytopes of polymatroids are characterised by a close variant of Theorem 1.1.3: they are those lattice polytopes in the positive orthant whose edges are parallel to vectors of the form $e_i - e_j$ but have no restrictions on length. Rank functions can also be defined for polymatroids. Finally, megamatroid polyhedra are simply the unbounded analogues of polymatroid polytopes.

If f is a valuation, then so is $g \circ f$ for any group homomorphism g. Thus one could ask for a *universal* valuation which all valuative functions of matroids factor through, parallel to Theorem 1.1.2. One natural candidate for a universal valuation is $\mathbf{1}$. (In the language with which Chapter 3 begins, valuations which factor through $\mathbf{1}$ are *strong* valuations, and functions satisfying (1.2.1) are *weak* valuations.) This works in several classical cases: every valuation does factor through $\mathbf{1}$ when $\mathcal{P}$ is the set of all convex bodies [41], and when $\mathcal{P}$ is a set of convex bodies closed under intersection [91]. But these do not subsume the case of matroid polytopes.

Example 1.2.1. Half a regular octahedron as in Figure 1.1 is a matroid polytope, as is any of its images under the $\mathfrak{S}_4$ symmetries of the octahedron, where $\mathfrak{S}$ denotes the symmetric group. But a quarter octahedron can be obtained as the intersection of two of these images, and this is not a matroid polytope. ♢

The first substantive result of Chapter 3 establishes that $\mathbf{1}$ is indeed a universal valuation for matroids and polymatroids.

Theorem 3.3.5. *Any valuation f of matroids or polymatroids is of the form $f' \circ \mathbf{1}$ for some group homomorphism f'.*

Valuativity is a linear condition (unlike (1.1.1)), and so the set of valuative functions on (poly)matroids on $[n]$ of rank D, valued in a ring R, is an R-module. A few conditions on R become necessary; everything will work as stated if $R = \mathbb{Q}$. To keep the notation light, we call the module of valuations $V(n, r)$ for the moment, and ignore notationally the question of whether it's matroids or polymatroids we're dealing with. (In the more fastidious notation of Chapter 3, we use the symbol $P_{(\mathrm{P})\mathrm{M}}(n, r)^\vee$.) By definition, $V(n, r)$ is dual to the quotient of the free module on the set of (poly)matroids by relations like (1.2.1). Theorem 3.3.5 above gives us a set of generators of $V(n, r)$, namely the indicator functions of points. For greater control we might ask for a presentation in generators and relations. In fact, in the cases we care about, $V(n, r)$ will be a free module and we can find a basis. This is the aim of the rest of Chapter 3.

It turns out that the $s_{A,\underline{r}}$ from (1.2.2) generate $V(n, r)$ for matroids and polymatroids, and it's easy to get a basis with a little care paid to the index set.

Corollary 3.5.5. *A basis for $V(n, r)$ consists of those $s_{A,\underline{r}}$ such that $s_{A,\underline{r}}$ is not identically zero and is not equal to any $s_{A',\underline{r}'}$ for A', r' subsequences of A, r (i.e. none of the equations $\mathrm{rk}_M(A_i) = r_i$ are redundant).*

A near-exact analogue is true for the module of valuative *invariants*, which could be written $V(n,r)^{\mathfrak{S}_n}$: we could modify Corollary 3.5.5 by imposing only the condition that A is a subsequence of a fixed maximal chain of sets. But in this case the statement is nicer in a variant:

Corollary 3.6.4. *A basis for $V(n,r)^{\mathfrak{S}_n}$ consists of those $s_{A,\underline{r}}$ such that $s_{A,\underline{r}}$ is not identically zero and A is the chain $[1] \subseteq [2] \subseteq \cdots \subseteq [n]$.*

The chapter culminates in a very clean algebraic description of the modules of valuative invariants on *all* (poly)matroids, where now n and r are allowed to vary, and we consider $V^{\mathfrak{S}_\infty}$, the direct sum of the $V(n,r)^{\mathfrak{S}_n}$ over all n and r. There is also a clean structure for *additive* valuations, those valuations whose values are zero on matroid polytopes of non-maximal dimension.

Theorem 3.1.7. *The module $V^{\mathfrak{S}_\infty}$ is a free associative algebra over R, in the generators $s_{\{\mathrm{pt}\},r}$. The submodule of additive valuations is the free Lie algebra on these generators, included in $V^{\mathfrak{S}_\infty}$ by $[a,b] \mapsto ab - ba$.*

The multiplication invoked here is essentially such that the $s_{A,r}$ multiply by catenating subscripts. This is, moreover, the natural product structure that is associated with valuations in the *Hopf algebra* structures on (poly)matroids and their valuations which we introduce in section 3.7. (Poly)matroids thus have a place in the study of combinatorial Hopf algebras, a recent active area of research [1].

1.3 Matroids and the Grassmannian

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0.[1] The Grassmannian $\mathrm{Gr}(d, \mathsf{E})$ is the space parametrising d-dimensional vector subspaces of $\mathbb{K}^{\mathsf{E}}$. It is a projective variety: it embeds into the projectivisation of $\bigwedge^d \mathbb{K}^{\mathsf{E}}$ by the *Plücker embedding*, which sends a subspace V to the wedge product of a basis of V. The *Plücker coordinates* are the corresponding projective coordinates. There are $\binom{n}{d}$ Plücker coordinates $p_{i_1 \cdots i_d}$, one for each basis element $e_{i_1} \wedge \cdots \wedge e_{i_d}$ of $\bigwedge^d \mathbb{K}^{\mathsf{E}}$. (Recall that $n = |\mathsf{E}|$.)

By the Gelfand-MacPherson correspondence, the Grassmannian is the parameter space for a second kind of object, namely arrangements of n vectors in $\mathbb{K}^d$ modulo the GL_d action. The correspondence has an elementary description: given an $d \times n$ matrix A over $\mathbb{K}$ whose rowspan is a d-dimensional subspace $V \subseteq \mathbb{K}^n$, the corresponding vector arrangement consists of the columns of A. So, given a point $x \in \mathrm{Gr}(d, \mathsf{E})$, consider its support $\mathcal{B}$ in the Plücker coordinates. Using the hyperplane arrangement description, we see that this $\mathcal{B}$ is the set of bases of a representable matroid $M(x)$.

[1] In other words we may as well choose $\mathbb{K} = \mathbb{C}$.

Here is a more geometric setting in which $M(x)$ is encountered. Consider the n-dimensional algebraic torus $T := (\mathbb{K}^*)^{\mathsf{E}}$. There is a natural action of T on $\mathrm{Gr}(d, \mathsf{E})$, by which the ith coordinate of T scales the ith vector in the arrangement (observe that this scaling does not change $M(x)$). The same action can be defined by letting T act on $\mathbb{K}^{\mathsf{E}}$ coordinatewise and carrying out the constructions in the first paragraph equivariantly; in particular this action is a restriction from a T-action on $\mathbb{P}(\bigwedge^d \mathbb{K}^{\mathsf{E}})$. If x is a point of a (projectivised) $\mathbb{K}$-vector space with a T-action, there is an associated *weight polytope*, given as the convex hull of the characters T acts by on the smallest T-invariant vector subspace containing x. In the case at hand. the weight polytope of $x \in \mathrm{Gr}(d, \mathsf{E})$ is $\mathrm{Poly}(M(x))$. A related way to say this: given x, the variety $\overline{Tx}$, the closure of the orbit of x under the T-action is a *toric variety*. Toric varieties are a very tractable class of algebraic varieties, due in no small part to their combinatorial nature. In particular, to every lattice polytope is associated a toric variety, and in our case, $\overline{Tx}$ is the variety associated to the polytope $\mathrm{Poly}(M(x))$.

Matroid subdivisions have made prominent appearances in algebraic geometry, such as in compactifying the moduli space of hyperplane arrangements (Hacking, Keel and Tevelev [42] and Kapranov [46]), compactifying fine Schubert cells in the Grassmannian (Lafforgue [55, 56]), as well as Speyer's h below. Lafforgue's work implies, for instance, that a matroid whose polytope has no subdivisions is representable in at most finitely many ways, up to the actions of the obvious groups.

Underlying these appearances are certain degenerations which can be constructed over a field with a valuation ν. (We have here an unfortunate collision of terminology: a valuation on a field, i.e. a homomorphism from its group of units to an ordered group, has nothing to do with a valuation of polytopes.) Our variety $\overline{Tx} \subseteq \mathrm{Gr}(d, \mathsf{E})$ degenerates into reducible varieties $Y_1 \cup \cdots \cup Y_k$ such that each nonempty intersection $\bigcap_{i \in I} Y_i$ of components is again a torus orbit closure of form $\overline{Ty_I}$. Associated to this degeneration is a subdivision of the weight polytope of $\overline{Tx}$ into the weight polytopes of the various $\overline{Ty_I}$, with identical combinatorics: i.e. the facets are the $\mathrm{Poly}(M(y_{\{i\}}))$, and we have $\bigcap_{i \in I} \mathrm{Poly}(M(y_{\{i\}})) = \mathrm{Poly}(M(y_I))$. The subdivision is constructed by lifting the vertices of the weight polytope into one more dimension, in a fashion encoding certain values of ν. The faces of the lifted polytope which are visible from the direction of lifting form the subdivision in question. This is what's known as a *regular subdivision*. Figure 1.2 is an example of a regular subdivision.

Starting with a representable matroid $M(x)$, we have constructed a subvariety $\overline{Tx}$ of $\mathrm{Gr}(d, \mathsf{E})$. This opens a set of avenues for the study of matroids by geometric means, through working with the classes of the $\overline{Tx}$ in cohomology theories of $\mathrm{Gr}(d, \mathsf{E})$ (for instance the Chow cohomology). Given our concern with valuations, the natural cohomology theory to use is *algebraic K-theory*: more precisely we only need the 0th K-theory functor K^0. This is defined in Section 4.2. For the present purposes we need only a few facts. When X is a sufficiently nice variety, like all those we will be working with, $K^0(X)$ is a ring. To every subvariety $Y \subseteq X$ corresponds an element $[Y] \in K^0(X)$. The key property is this: when a

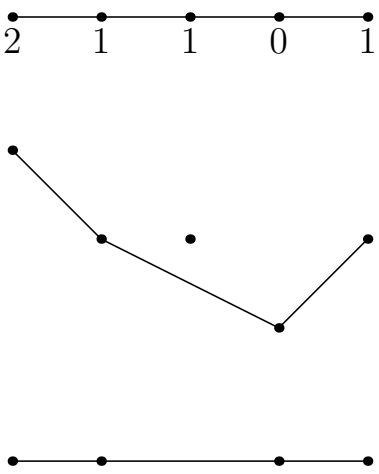


Figure 1.2: The regular subdivision of a 1-dimensional polytope with specified lifting heights.

variety $Y = Y_1 \cup Y_2$ is reducible, the relation

$$[Y] - [Y_1] - [Y_2] + [Y_1 \cup Y_2] = 0$$

holds in $K^0[X]$. Note the analogy with (1.2.1). We have similar relations for reducible varieties $Y \subseteq X$ with any number of components. The degeneration of $\overline{Tx}$ we describe in the last paragraph doesn't affect K-theory class. Altogether, this shows that the map $M(x) \mapsto y(M(x)) := [\overline{Tx}] \in K^0(\mathrm{Gr}(d, \mathsf{E}))$ is a valuation of representable matroids. Multiplication being a linear map, the same is true of $M(x) \mapsto Cy(M(x))$ for any class $C \in K^0(\mathrm{Gr}(d, \mathsf{E}))$.

Actually, we mostly use a richer functor, the *equivariant* K-theory $K^0_T(X)$, which retains some information about the way the torus T acts on all the varieties in question. The properties above also hold of $K^0_T(X)$. Our main technical tool in Chapter 4 is *equivariant localisation*. This reduces computations in the ring $K^0(\mathrm{Gr}(d, \mathsf{E}))$ to computations in a direct sum of copies of $K^0_T(\mathrm{point}) = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, the ring of Laurent polynomials in n variables (or, canonically, the group ring of the characters of T); there is no parallel to this reduction in the non-equivariant case. This is of great computational utility. The computations of equivariant localisation have a very strong polyhedral flavour, and our proofs end up mainly dealing in lattice point generating functions of the matroid polytope and related polyhedra.

We would of course like to be able to work with all matroids, not just the representable ones $M(x)$ which we started with above. It is a remarkable fact, proved in [83], that even though nonrepresentable matroids M have no corresponding subvarieties they still have associated classes $y(M)$ in $K^0_T(\mathrm{Gr}(d, \mathsf{E}))$, which can be computed from their polytopes. We can treat valuations on the set of all matroids this way.

Following the track of the previous chapters, we might ask whether the valuations of the form $M(x) \mapsto \mathcal{C}[\overline{Tx}]$ generate all valuations. It turns out they don't (Example 4.4.4). However, we can construct two important valuations in terms of these, and these are the

main results of Chapter 4. The first is the Tutte polynomial. The second is Speyer's invariant h which is the subject of [83]. Speyer conjectures (and has proven in several cases) that, up to a change of variable, h is positive; if this were true, it would impose sharp upper bounds on the number of faces possible of each dimension in a matroid polytope subdivision.

Though we won't define all the necessary objects in this exposition, we state the main theorems here to highlight the striking formal similarity between the two invariants.

Theorems 4.7.1, 4.8.5. *There exist maps and varieties* $\mathrm{Gr}(d, \mathsf{E}) \xleftarrow{\pi} \mathrm{F}\ell \xrightarrow{\pi'} (\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$ *such that*

$$t(M)(\alpha, \beta) = (\pi')_* \pi^* (y(M) \cdot [\mathcal{O}(1)])$$
$$h(M)(\alpha, \beta) = (\pi')_* \pi^* (y(M))$$

where $(\pi')_*$ *and* π^* *are respectively pushforward and pullback in* K*-theory, and* $K^0(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}) = \mathbb{Z}[\alpha, \beta]/(\alpha^n, \beta^n)$.

1.4 Tropical geometry

Our last chapter, Chapter 5, lies outside the thread of the other chapters regarding matroid valuations. Instead it builds off the appearances of matroids in tropical geometry, and consists of some first steps towards a generalisation.

Tropical geometry is a relatively new field of study. At its beginnings lies the following observation. Let X be a complex affine or projective variety, and consider its image under taking the coordinatewise real part of the logarithm: this image is known as the *amoeba* of X. As the base of the logarithm tends to infinity, the amoeba shrinks and in the limit approaches a polyhedral complex. Algebraically we can get the same limiting polyhedral complex directly by working over an algebraically closed field with a valuation, and replacing the logarithm by the valuation (and taking the closure of the resulting set). The polyhedral complex obtained this way is a *tropical variety*, called the *tropicalisation* $\operatorname{Trop} X$ of the algebraic variety X we began with.[2] When we want to refer to expressly non-tropical objects we will call them *classical*.

In the words of Maclagan [60], tropical varieties are combinatorial shadows of algebraic varieties: that is, much information about classical varieties is retained in their tropicalisation. It can be a profitable attack on a classical geometric problem to consider the tropical analogue, reducing hard algebra to hopefully easier combinatorics. We limit ourselves to one quick example here: Mikhalkin used tropical techniques to count the *rational* curves of

[2] In particular our tropical varieties are all embedded in a real vector space (or near enough). Another body of work in tropical geometry attempts to define a notion of abstract tropical variety, dissociated from any particular embedding; this is not our concern.

degree d through $3d-1$ points in the plane, for all d [67]; the number was only known for $d \leq 4$ until around fifteen years ago.

That said, the combinatorics of tropical varieties, especially the global combinatorics, is far from trivial. One of the better understood cases is that of linear spaces: these are very closely related to matroids and matroid subdivisions. Let X be a linear subspace of complex projective space. Its tropicalisation $\operatorname{Trop} X$ depends only on the valuations of the Plücker coordinates of X, and therefore on the regular subdivision of its matroid polytope described last section. We can compute $\operatorname{Trop} X$ from the matroid subdivision. If the subdivision is trivial (there is only one piece) then $\operatorname{Trop} X$ is the *Bergman fan* of [7]. Given a polytope P in a real vector space V, its *normal fan* is the polyhedral complex on the dual space V^*, whose (closed) faces consist of all linear functionals maximised at a given face of P. The Bergman fan is a certain subcomplex of the normal fan. For a regular subdivision of polytopes one can define an analogue of the normal fan, and then an analogue of the Bergman complex, and the analogue holds true. The situation is bijective: the Plücker coordinates can also be recovered from the tropical linear space. Indeed, the Grassmannian in its Plücker embedding tropicalises to the *tropical Grassmannian*, which is a parameter space for tropicalised linear spaces [82].

It is a natural pursuit to develop a formalisation of tropical geometry that doesn't depend on classical algebraic geometry. From this perspective, the presence of classical varieties in the explanation of tropical varieties opening this section is an unsatisfying feature, as it's hard to get a handle on. Given a polyhedral complex C, suppose we wish to determine whether $C = \operatorname{Trop} X$ for an algebraic variety X. There are a set of necessary conditions on the local combinatorics that are easy to test, but there are (usually) no good sufficient conditions: indeed, Sturmfels proved [86] that determining realizability of a matroid in characteristic 0 is equivalent to determining solvability of a system of Diophantine equations over $\mathbb{Q}$. This informs us that, even for linear spaces, we can't expect sufficient conditions for being a tropicalisation. For this and other reasons, we define a *tropical variety* in general to be a polyhedral complex with the aforementioned local conditions on its combinatorics.

Accepting this latter definition, we find that tropical linear spaces are exactly in bijection with regular matroid subdivisions. This is one of three classes of tropical variety that have been studied which are in combinatorial bijection with classes of subdivisions of polytopes. There exist bijections between

(1) tropical linear spaces and matroid polytope subdivisions;

(2) tropical hypersurfaces and lattice polytope subdivisions;

(3) tropical zero-dimensional varieties and *fine mixed subdivisions* of simplices [6].

(Regarding case 3, it is conventionally the tropical hyperplane arrangement dual to the zero-dimensional variety which is associated to a fine mixed subdivision in the literature.)

An effective algebraic cycle on a variety Y is a formal positive $\mathbb{Z}$-linear combination of irreducible subvarieties of Y. There is a classical construction, due to Chow and van der Waerden [20], of a parameter space for effective algebraic cycles of given dimension and degree in projective space, the *Chow variety*. For example, the Grassmannian is the Chow variety for degree 1 cycles, these being linear spaces. The Chow variety is projective and has a torus action, so as in the last section we can associate a weight polytope to any of its points, i.e. any cycle X in Y; this is called the *Chow polytope* of X. We get more: working over a valued field, the regular subdivision construction associates a subdivision of the Chow polytope to X.

It turns out that, in the bijections (1)–(3) above, if any of the tropical varieties is of the form $\operatorname{Trop} X$ for a classical algebraic cycle X, then the associated polytopal subdivision is the Chow polytope subdivision. This makes it natural to consider the Chow polytope of any tropicalisation. The main result of Chapter 5 extends this to all tropical varieties, providing a simple combinatorial construction of a "Chow subdivision" for any tropical variety.

The *k-skeleton* of a cell complex is the subcomplex of all faces of dimension $\leq k$. Ignoring the issue of multiplicity, we have the following result.

Theorem 5.4.1. *Let C be a tropical variety of dimension e. Let L be the $(e-1)$-skeleton of the simplex* $\operatorname{conv}\{-e_1, \ldots, -e_n\}$. *Then the Minkowski sum $C + L$ is the codimension 1 skeleton of the normal fan to a subdivision of polytopes. If $C = \operatorname{Trop} X$ is a tropicalisation, then this subdivision is the Chow polytope subdivision of X.*

Unlike (1)–(3), this general Chow polytope construction does not afford a bijection: Chapter 5 closes with an example of two tropical varieties which have equal Chow polytope. In any event, the polytopes obtained from Theorem 5.4.1 lack a clean characterisation on the model of Theorem 1.1.3, and there is certainly much to be done to obtain a combinatorial description of all tropical varieties along these lines.

Chapter 2

Valuations for matroid polytope subdivisions

This chapter is joint work with Federico Ardila and Edgard Felipe Rincón. It is to appear in the *Canadian Mathematical Bulletin* with the same title. (This version incorporates some minor changes, largely for consistency with other chapters.)

2.1 Introduction

Aside from its wide applicability in many areas of mathematics, one of the pleasant features of matroid theory is the availability of a vast number of equivalent points of view. Among many others, one can think of a matroid as a notion of independence, a closure relation, or a lattice. One point of view has gained prominence due to its applications in algebraic geometry, combinatorial optimization, and Coxeter group theory: that of a matroid as a polytope. This chapter is devoted to the study of functions of a matroid which are amenable to this point of view.

To each matroid M one can associate a (basis) *matroid polytope* $\text{Poly}(M)$, which is the convex hull of the indicator vectors of the bases of M. One can recover M from $\text{Poly}(M)$, and in certain instances $\text{Poly}(M)$ is the fundamental object that one would like to work with. For instance, matroid polytopes play a crucial role in the matroid stratification of the Grassmannian [37]. They allow us to invoke the machinery of linear programming to study matroid optimization questions [78]. They are also the key to understanding that matroids are just the type A objects in the family of Coxeter matroids [14].

The subdivisions of a matroid polytope into smaller matroid polytopes have appeared prominently in different contexts: in compactifying the moduli space of hyperplane arrangements (Hacking, Keel and Tevelev [42] and Kapranov [46]), in compactifying fine Schubert cells in the Grassmannian (Lafforgue [55, 56]), and in the study of tropical linear spaces

(Speyer [82]).

Billera, Jia and Reiner [11] and Speyer [82, 83] have shown that some important functions of a matroid, such as its quasisymmetric function and its Tutte polynomial, can be thought of as nice functions of their matroid polytopes: they act as valuations on the subdivisions of a matroid polytope into smaller matroid polytopes.

The purpose of this chapter is to show that two much stronger functions are also valuations. Consider the matroid functions

$$f_1(M) = \sum_{A \subseteq [n]} (A, \mathrm{rk}_M(A)) \quad \text{and} \quad f_2(M) = \sum_{B \text{ basis of } M} (B, E(B), I(B)),$$

regarded as formal sums in the free group with basis all triples in $\times$. Here rk_M denotes matroid rank, and $E(B)$ and $I(B)$ denote the sets of externally and internally active elements of B.

Theorems 2.5.1 and 2.5.4. *The functions f_1 and f_2 are valuations for matroid polytope subdivisions: for any subdivision of a matroid polytope* $\mathrm{Poly}(M)$ *into smaller matroid polytopes* $\mathrm{Poly}(M_1), \ldots, \mathrm{Poly}(M_m)$*, these functions satisfy*

$$f(M) = \sum_i f(M_i) - \sum_{i<j} f(M_{ij}) + \sum_{i<j<k} f(M_{ijk}) - \cdots,$$

where $M_{ab\ldots c}$ is the matroid whose polytope is $\mathrm{Poly}(M_a) \cap \mathrm{Poly}(M_b) \cap \cdots \cap \mathrm{Poly}(M_c)$.

The chapter is organized as follows. In Section 2.2 we present some background information on matroids and matroid polytope subdivisions. In Section 2.3 we define valuations under matroid subdivisions, and prove an alternative characterization of them. In Section 2.4 we present a useful family of valuations, which we use to prove Theorems 2.5.1 and 2.5.4 in Section 2.5. Finally in section 2.6 we briefly discuss some previously known matroid valuations.

2.2 Preliminaries on matroids and matroid subdivisions

A *matroid* is a combinatorial object which unifies several notions of independence. We start with basic definitions; for more information on matroid theory we refer the reader to [72]. There are many equivalent ways of defining a matroid. We will adopt the basis point of view, which is the most convenient for the study of matroid polytopes.

Definition 2.2.1. A *matroid* M is a pair $(\mathsf{E}, \mathcal{B})$ consisting of a finite set E and a collection of subsets $\mathcal{B}$ of E, called the *bases* of M, which satisfies the *basis exchange axiom:* If $B_1, B_2 \in \mathcal{B}$ and $b_1 \in B_1 - B_2$, then there exists $b_2 \in B_2 - B_1$ such that $(B_1 \setminus b_1) \cup b_2 \in \mathcal{B}$.

We will find it convenient to allow $(\mathsf{E}, \emptyset)$ to be a matroid; this is not customary.

A subset $A \subseteq \mathsf{E}$ is *independent* if it is a subset of a basis. All the maximal independent sets contained in a given set $A \subseteq \mathsf{E}$ have the same size, which is called the *rank* $\mathrm{rk}_M(A)$ of A. In particular, all the bases have the same size, which is called the rank $r(M)$ of M.

Example 2.2.2. If E is a finite set of vectors in a vector space, then the maximal linearly independent subsets of E are the bases of a matroid. The matroids arising in this way are called *representable*, and motivate much of the theory of matroids. ♢

Example 2.2.3. If $k \leq n$ are positive integers, then the subsets of size k of $[n] = \{1, \ldots, n\}$ are the bases of a matroid, called the *uniform matroid* $U_{k,n}$. ♢

Example 2.2.4. Given positive integers $1 \leq s_1 < \ldots < s_r \leq n$, the sets $\{a_1, \ldots, a_r\}$ such that $a_1 \leq s_1, \ldots, a_r \leq s_r$ are the bases of a matroid, called the *Schubert matroid* $SM_n(s_1, \ldots, s_r)$. These matroids were discovered by Crapo [23] and rediscovered in various contexts; they have been called shifted matroids [4, 52], PI-matroids [11], generalized Catalan matroids [13], and freedom matroids [26], among others. We prefer the name Schubert matroid, which highlights their relationship with the stratification of the Grassmannian into Schubert cells [12, Section 2.4]. ♢

The following geometric representation of a matroid is central to our study.

Definition 2.2.5. Given a matroid $M = ([n], \mathcal{B})$, the (basis) *matroid polytope* $\mathrm{Poly}(M)$ of M is the convex hull of the indicator vectors of the bases of M:

$$\mathrm{Poly}(M) = \mathrm{convex}\{e_B \,:\, B \in \mathcal{B}\}.$$

For any $B = \{b_1, \ldots, b_r\} \subseteq [n]$, by e_B we mean $e_{b_1} + \cdots + e_{b_r}$, where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

When we speak of "a matroid polytope", we will refer to the polytope of a specific matroid, in its specific position in $\mathbb{R}^n$. The following elegant characterization is due to Gelfand, Goresky, MacPherson, and Serganova [37]:

Theorem 2.2.6. *Let $\mathcal{B}$ be a collection of r-subsets of $[n]$ and let $\mathrm{Poly}(\mathcal{B}) = \mathrm{convex}\{e_B : B \in \mathcal{B}\}$. The following are equivalent:*

1. *$\mathcal{B}$ is the collection of bases of a matroid.*
2. *Every edge of $\mathrm{Poly}(\mathcal{B})$ is a parallel translate of $e_i - e_j$ for some $i, j \in [n]$.*

When the statements of Theorem 2.2.6 are satisfied, the edges of $\mathrm{Poly}(\mathcal{B})$ correspond exactly to the pairs of different bases B, B' such that $B' = (B \setminus i) \cup j$ for some $i, j \in [n]$. Two such bases are called *adjacent bases*.

A *subdivision* of a polytope P is a set of polytopes $\Sigma = \{P_1, \ldots, P_m\}$, whose vertices are vertices of P, such that

- $P_1 \cup \cdots \cup P_m = P$, and
- for all $1 \leq i < j \leq m$, if the intersection $P_i \cap P_j$ is nonempty, then it is a proper face of both P_i and P_j.

The *faces* of the subdivision Σ are the faces of the P_i; it is easy to see that the interior faces of Σ (i.e. faces not contained in the boundary of P) are exactly the non-empty intersections between some of the P_i.

Definition 2.2.7. A *matroid polytope subdivision* is a subdivision of a matroid polytope $Q = \text{Poly}(M)$ into matroid polytopes $Q_1 = \text{Poly}(M_1), \ldots, Q_m = \text{Poly}(M_m)$. We will also refer to this as a *matroid subdivision* of the matroid M into $M_1, \ldots, M_m$.

The lower-dimensional faces of the subdivision, which are intersections of subcollections of the Q_i, are also of interest. Given a set of indices $A = \{a_1, \ldots, a_s\} \subseteq [m]$, we will write $Q_A = Q_{a_1 \cdots a_s} := \bigcap_{a \in A} Q_a$. By convention, $Q_\emptyset = Q$. Since any face of a matroid polytope is itself a matroid polytope, it follows that any nonempty Q_A is the matroid polytope of a matroid, which we denote M_A.

Because of the small number of matroid polytopes in low dimensions, there is a general lack of small examples of matroid subdivisions. In two dimensions the only matroid polytopes are the equilateral triangle and the square, which have no nontrivial matroid subdivisions. In three dimensions, the only nontrivial example is the subdivision of a regular octahedron (with bases $\{12, 13, 14, 23, 24, 34\}$) into two square pyramids (with bases $\{12, 13, 14, 23, 24\}$ and $\{13, 14, 23, 24, 34\}$, respectively); this subdivision is shown in Figure 2.1.

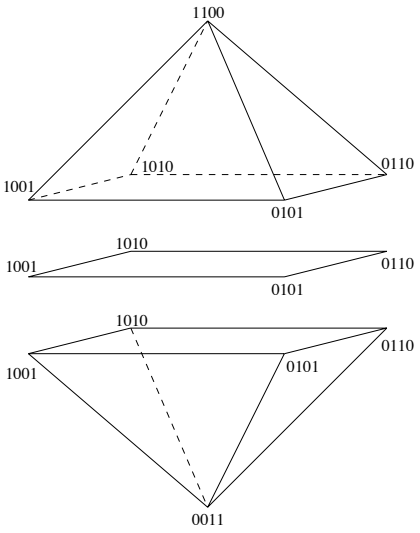


Figure 2.1: The matroid subdivision of a regular octahedron into two square pyramids.

Example 2.2.8. A more interesting example is the following subdivision [11, Example 7.13]: Let $M_1 = SM_6(2,4,6)$ be the Schubert matroid whose bases are the sets $\{a,b,c\} \subseteq [6]$ such that $a \leq 2, b \leq 4$, and $c \leq 6$. The permutation $\sigma = 345612$ acts on the ground set $[6]$ of M_1, thus defining the matroids $M_2 = \sigma M_1$ and $M_3 = \sigma^2 M_1$. (Note that σ^3 is the identity.) Then $\{M_1, M_2, M_3\}$ is a subdivision of $M = U_{3,6}$. One can easily generalize this construction to obtain a subdivision of $U_{a,ab}$ into a isomorphic matroids.

Under the projection $(x_1, \ldots, x_6) \mapsto (x_1+x_2, x_3+x_4, x_5+x_6)$, $U_{3,6}$ is taken to the hexagon of Figure 2.2, and the M_i are the preimages of the three parallelograms of that figure. Notice that Figure 2.2 is also a polymatroid subdivision, as in Chapter 3. ♢

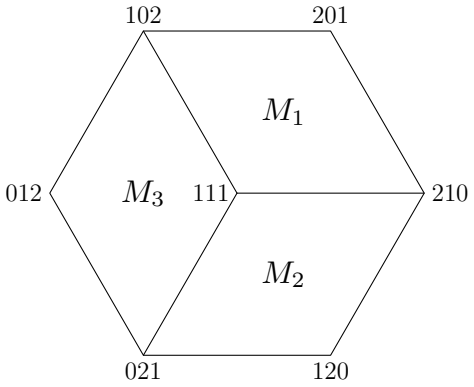


Figure 2.2: A projection of the subdivision of Example 2.2.8.

2.3 Valuations under matroid subdivisions

We now turn to the study of matroid functions which are valuations under the subdivisions of a matroid polytope into smaller matroid polytopes. Throughout this section, $\mathsf{Mat} = \mathsf{Mat}_n$ will denote the set of matroids with ground set $[n]$, and G will denote an arbitrary abelian group. As before, given a subdivision $\{M_1, \ldots, M_m\}$ of a matroid M and a subset $A \subseteq [m]$, M_A is the matroid whose polytope is $\bigcap_{a \in A} \mathrm{Poly}(M_a)$.

Definition 2.3.1. A function $f : \mathsf{Mat} \to G$ is a *valuation under matroid subdivision*, or simply a *valuation*[1], if for any subdivision $\{M_1, \ldots, M_m\}$ of a matroid $M \in \mathsf{Mat}$, we have

$$\sum_{A \subseteq [m]} (-1)^{|A|} f(M_A) = 0 \tag{2.3.1}$$

[1]This use of the term *valuation* is standard in convex geometry [63]. It should not be confused with the unrelated notion of a matroid valuation found in the theory of valuated matroids [32].

or, equivalently,

$$f(M) = \sum_i f(M_i) - \sum_{i<j} f(M_{ij}) + \sum_{i<j<k} f(M_{ijk}) - \cdots \tag{2.3.2}$$

Recall that, contrary to the usual convention, we have allowed $\emptyset = ([n], \emptyset)$ to be a matroid. We will also adopt the convention that $f(\emptyset) = 0$ for all the matroid functions considered in this chapter.

Many important matroid functions are well-behaved under subdivision. Let us start with some easy examples.

Example 2.3.2. The function vol, which assigns to each matroid $M \in \mathsf{Mat}$ the n-dimensional volume of its polytope $\mathrm{Poly}(M)$, is a valuation. This is clear since the lower-dimensional faces of a matroid subdivision have volume 0. ♢

Example 2.3.3. The *Ehrhart polynomial* $\ell_P(x)$ of a lattice polytope P in $\mathbb{R}^d$ is the polynomial such that, for a positive integer n, $\ell_P(n) = |nP \cap \mathbb{Z}^d|$ is the number of lattice points contained in the n-th dilate nP of P [84, Section 4.6]. By the inclusion-exclusion formula, the function $\ell : \mathsf{Mat} \to \mathbb{R}[x]$ defined by $\ell(M) = \ell_{\mathrm{Poly}(M)}(x)$ is a valuation. ♢

Example 2.3.4. The function $b(M) = (\text{number of bases of } M)$ is a valuation. This follows from the fact that the only lattice points in $\mathrm{Poly}(M)$ are its vertices, which are the indicator vectors of the bases of M; so $b(M)$ is the evaluation of $\ell(M)$ at $x = 1$. ♢

Before encountering other important valuations, let us present an alternative way of characterizing them. This result may be known, but we have been unable to locate the precise statement that we need in the literature, so we include a proof for completeness.

Theorem 2.3.5. *A function $f : \mathsf{Mat} \to G$ is a valuation if and only if, for any matroid subdivision Σ of $Q = \mathrm{Poly}(M)$,*

$$f(M) = \sum_{F \in \mathrm{int}(\Sigma)} (-1)^{\dim(Q)-\dim(F)} f(M(F)), \tag{2.3.3}$$

where the sum is over the interior faces of the subdivision Σ, and $M(F)$ denotes the matroid whose matroid polytope is F.

To prove Theorem 2.3.5 we first need to recall some facts from topological combinatorics. These can be found, for instance, in [84, Section 3.8].

Definition 2.3.6. A *regular cell complex* is a finite set $C = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$ of pairwise disjoint and nonempty *cells* $\sigma_i \subseteq \mathbb{R}^d$ such that for any $i \in [s]$:

1. $\overline{\sigma_i} \approx \mathbb{B}^{m_i}$ and $\overline{\sigma_i} \setminus \sigma_i \approx \mathbb{S}^{m_i-1}$ for some nonnegative integer m_i, called the *dimension* of σ_i.

2. $\overline{\sigma_i} \setminus \sigma_i$ is the union of some other σ_js.

Here $\overline{\sigma_i}$ denotes the topological closure of σ_i and $\approx$ denotes homeomorphism. Also $\mathbb{B}^l$ and $\mathbb{S}^l$ are the l-dimensional closed unit ball and unit sphere, respectively. The *underlying space* $|C|$ of C is the topological space $\sigma_1 \cup \cdots \cup \sigma_s$.

Definition 2.3.7. Let C be a regular cell complex, and let c_i be the number of i-dimensional cells of C. The *Euler characteristic* of C is:

$$\chi(C) = \sum_{\sigma \in C} (-1)^{\dim(\sigma)} = \sum_{i \in \mathbb{N}} (-1)^i c_i = c_0 - c_1 + c_2 - c_3 \cdots .$$

The *reduced Euler characteristic* of C is $\tilde{\chi}(C) = \chi(C) - 1$. A fundamental fact from algebraic topology is that the Euler characteristic of C depends solely on the homotopy type of the underlying space $|C|$.

Definition 2.3.8. For a regular cell complex C, let $P(C)$ be the poset of cells of C, ordered by $\sigma_i \leq \sigma_j$ if $\overline{\sigma_i} \subseteq \overline{\sigma_j}$. Let $\widehat{P}(C) = P(C) \cup \{\hat{0}, \hat{1}\}$ be obtained from $P(C)$ by adding a minimum and a maximum element.

Definition 2.3.9. The *Möbius function* $\mu : \mathrm{Int}(P) \to \mathbb{Z}$ of a poset P assigns an integer to each closed interval of P, defined recursively by

$$\mu_P(x, x) = 1, \qquad \sum_{x \leq a \leq y} \mu(x, a) = 0 \text{ for all } x < y.$$

It can equivalently be defined in the following dual way:

$$\mu_P(x, x) = 1, \qquad \sum_{x \leq a \leq y} \mu(a, y) = 0 \text{ for all } x < y.$$

The following special case of Rota's Crosscut Theorem is a powerful tool for computing the Möbius function of a lattice.

Theorem 2.3.10 ([76])**.** *Let L be any finite lattice. Then for all $x \in L$,*

$$\mu(\hat{0}, x) = \sum_{B} (-1)^{|B|},$$

where the sum is over all sets B of atoms of L such that $\bigvee B = x$.

Finally, we recall an important theorem which relates the topology and combinatorics of a regular cell complex.

Theorem 2.3.11 ([84, Proposition 3.8.9])**.** *Let C be a regular cell complex such that $|C|$ is a manifold, with or without boundary. Let $P = \widehat{P}(C)$. Then*

$$\mu_P(x,y) = \begin{cases} \tilde{\chi}(|C|) & \text{if } x = \hat{0} \text{ and } y = \hat{1}, \\ 0 & \text{if } x \neq \hat{0}, y = \hat{1}, \text{ and } x \text{ is on the boundary of } |C|, \\ (-1)^{l(x,y)} & \text{otherwise,} \end{cases}$$

where $l(x,y)$ is the number of elements in a maximal chain from x to y.

We are now in a position to prove Theorem 2.3.5.

Proof of Theorem 2.3.5. Let $\Sigma = \{M_1, \ldots, M_m\}$ be a matroid subdivision of M. Let $\{Q_1, \ldots, Q_m\}$ and Q be the corresponding polytopes. Notice that the (relative interiors of the) faces of the subdivision Σ form a regular cell complex whose underlying space has closure Q. Additionally, the poset $\widehat{P}(\Sigma)$ is a lattice, since it has a meet operation $\sigma_i \wedge \sigma_j = \text{int}(\overline{\sigma}_i \cap \overline{\sigma}_j)$ and a maximum element.

We will show that

$$\sum_{F \in \text{int}(\Sigma)} (-1)^{\dim(Q)-\dim(F)} f(M(F)) = \sum_i f(M_i) - \sum_{i<j} f(M_{ij}) + \sum_{i<j<k} f(M_{ijk}) - \cdots \quad (2.3.4)$$

which will establish the desired result in view of (2.3.2). In the right hand side, each term is of the form $f(M(F))$ for an interior face F of the subdivision Σ and moreover, all interior faces F appear. The term $f(M(F))$ appears with coefficient

$$\sum_{A \subseteq [m] \,:\, M_A = M(F)} (-1)^{|A|+1}.$$

This is equivalent to summing over the sets of coatoms of the lattice $\widehat{P}(\Sigma)$ whose meet is F. By Rota's Crosscut Theorem 2.3.10, when applied to the poset $\widehat{P}(\Sigma)$ turned upside down, this sum equals $-\mu_{\widehat{P}(\Sigma)}(F, \hat{1})$. Theorem 2.3.11 tells us that this is equal to $(-1)^{l(F,\hat{1})-1} = (-1)^{\dim(Q)-\dim(F)}$, as desired. □

2.4 A powerful family of valuations

Definition 2.4.1. Given $X \subseteq \mathbb{R}^n$, let $i_X : \mathsf{Mat} \to \mathbb{Z}$ be defined by

$$i_X(M) = \begin{cases} 1 & \text{if } \text{Poly}(M) \cap X \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Our interest in these functions is that, under certain hypotheses, they are valuations under matroid subdivisions. Many valuations of interest, in particular those of Section 2.5, can be obtained as linear combinations of evaluations of these valuations, i.e. of compositions $f \circ i_X$ for some group homomorphism f. It is in this sense that we regard the family as powerful.

Theorem 2.4.2. *If $X \subseteq \mathbb{R}^n$ is convex and open, then i_X is a valuation.*

Proof. Let $M \in \mathsf{Mat}$ be a matroid and Σ be a subdivision of $Q = \mathrm{Poly}(M)$. We can assume that $Q \cap X \neq \emptyset$, or else the result is trivial. We can also assume that X is bounded by replacing X with its intersection with a bounded open set containing $[0,1]^n$.

We will first reduce the proof to the case when X is an open polytope in $\mathbb{R}^n$. By the Hahn-Banach separation theorem [77, Theorem 3.4], for each face F of Σ such that $F \cap X = \emptyset$ there exists an open halfspace H_F containing X and disjoint from F. Let

$$X' = \bigcap_{F \cap X = \emptyset} H_F$$

be the intersection of these halfspaces. Then $X' \supseteq X$ and $X' \cap F = \emptyset$ for each face F not intersecting X, so $i_{X'}$ and i_X agree on all the matroids of this subdivision. If we define X'' as the intersection of X' with some open cube containing Q, then $i_{X''}$ and i_X agree on this subdivision and X'' is an open polytope.

We can therefore assume that X is an open polytope in $\mathbb{R}^n$; in particular it is full-dimensional. Note that $X \cap \mathrm{int}(Q)$ is the interior $\mathrm{int}(R)$ of some polytope $R \subseteq Q$. Since R and Q have the same dimension, $R \approx \mathbb{B}^{\dim(Q)}$ and $\partial R \approx \mathbb{S}^{\dim(Q)-1}$. If F is a face of the subdivision Σ and σ is a face of the polytope R, let $c_{F,\sigma} = \mathrm{int}(F) \cap \mathrm{int}(\sigma)$. Since $c_{F,\sigma}$ is the interior of a polytope, it is homeomorphic to a closed ball and its boundary to the corresponding sphere. Define

$$\begin{aligned} C &= \{c_{F,\sigma} : c_{F,\sigma} \neq \emptyset\} \\ \partial C &= \{c_{F,\sigma} : c_{F,\sigma} \neq \emptyset \text{ and } \sigma \neq R\}. \end{aligned}$$

The elements of C form a partition of R and in this way C is a regular cell complex whose underlying space is R. Similarly, ∂C is a regular subcomplex whose underlying space is ∂R. Note that if F is an interior face of Σ, $c_{F,R} = \mathrm{int}(F) \cap \mathrm{int}(R) \neq \emptyset$ if and only if $F \cap X \neq \emptyset$, and in this case $\dim(c_{F,R}) = \dim(F)$.

We then have

$$\begin{aligned}
\sum_{F\in\mathrm{int}(\Sigma)} (-1)^{\dim(F)} i_X(M(F)) &= \sum_{\substack{F\in\mathrm{int}(\Sigma)\\ F\cap X\neq\emptyset}} (-1)^{\dim(F)} \\
&= \sum_{\substack{F\in\mathrm{int}(\Sigma)\\ c_{F,R}\neq\emptyset}} (-1)^{\dim(c_{F,R})} \\
&= \sum_{c_{F,R}\neq\emptyset} (-1)^{\dim(c_{F,R})} \\
&= \sum_{c\in C}(-1)^{\dim(c)} - \sum_{c\in\partial C}(-1)^{\dim(c)} \\
&= \chi(R) - \chi(\partial R) \\
&= 1 - \left(1 + (-1)^{\dim(Q)-1}\right) \\
&= (-1)^{\dim(Q)} \\
&= (-1)^{\dim(Q)} i_X(M),
\end{aligned}$$

which finishes the proof in view of Theorem 2.3.5. □

Corollary 2.4.3. *If $X \subseteq \mathbb{R}^n$ is convex and closed, then i_X is a valuation.*

Proof. As before, we can assume that X is bounded since $i_X = i_{X\cap[0,1]^n}$. Now let Σ be a subdivision of $Q = \mathrm{Poly}(M)$ into m parts. For all $A \subseteq [m]$ such that $X \cap Q_A = \emptyset$, the distance $d(X, Q_A)$ is positive since X is compact and Q_A is closed. Let $\epsilon > 0$ be smaller than all those distances, and define the convex open set

$$U = \{x \in \mathbb{R}^n : d(x, X) < \epsilon\}.$$

For all $A \subseteq [m]$ we have that $X \cap Q_A \neq \emptyset$ if and only if $U \cap Q_A \neq \emptyset$. By Theorem 2.4.2,

$$\sum_{A\subseteq[m]} (-1)^{|A|} i_X(M_A) = \sum_{A\subseteq[m]} (-1)^{|A|} i_U(M_A) = 0$$

as desired. □

In particular, i_P is a valuation for any polytope $P \subseteq \mathbb{R}^n$.

Proposition 2.4.4. *The constant function $c(M) = 1$ for $M \in \mathsf{Mat}$ is a valuation.*

Proof. This follows from $c(M) = i_{[0,1]^n}$. □

Proposition 2.4.5. *If $X \subseteq \mathbb{R}^n$ is convex, and is either open or closed, then the function $\overline{i_X} : \mathsf{Mat} \to \mathbb{Z}$ defined by*

$$\overline{i_X}(M) = \begin{cases} 0 & \text{if } \mathrm{Poly}(M) \cap X \neq \emptyset, \\ 1 & \text{otherwise,} \end{cases}$$

is a valuation.

Proof. Notice that $\overline{i_X} = 1 - i_X$, which is the sum of two valuations. □

2.5 Subset ranks and basis activities are valuations

We now show that there are two surprisingly fine valuations of a matroid: the ranks of the subsets and the activities of the bases.

2.5.1 Rank functions

Theorem 2.5.1. *Let G be the free abelian group on symbols of the form (A, s), $A \subseteq [n]$, $s \in \mathbb{Z}_{\geq 0}$. The function $F : \mathsf{Mat} \to G$ defined by*

$$F(M) = \sum_{A \subseteq [n]} (A, \mathrm{rk}_M(A))$$

is a valuation.

Proof. It is equivalent to show that the function $f_{A,s} : \mathsf{Mat} \to \mathbb{Z}$ defined by

$$f_{A,s}(M) = \begin{cases} 1 & \text{if } \mathrm{rk}_M(A) = s, \\ 0 & \text{otherwise,} \end{cases}$$

is a valuation. Define the polytope

$$P_{A,s} = \left\{ x \in [0,1]^n : \sum_{i \in A} x_i \geq s \right\}.$$

A matroid M satisfies that $\mathrm{rk}_M(A) = s$ if and only if it has a basis B with $|A \cap B| \geq s$, and it has no basis B such that $|A \cap B| \geq s + 1$. This is equivalent to $\mathrm{Poly}(M) \cap P_{A,s} \neq \emptyset$ and $\mathrm{Poly}(M) \cap P_{A,s+1} = \emptyset$. It follows that $f_{A,s} = i_{P_{A,s}} - i_{P_{A,s+1}}$, which is the sum of two valuations. □

2.5.2 Basis activities

One of the most powerful standard invariants of a matroid is its *Tutte polynomial*:

$$t_M(x,y) = \sum_{A \subseteq [n]} (x-1)^{r(M)-r(A)}(y-1)^{|A|-r(A)}.$$

Its importance stems from the fact that many interesting invariants of a matroid satisfy the *deletion-contraction recursion*, and every such invariant is an evaluation of the Tutte polynomial [19].

Definition 2.5.2. Let B be a basis of the matroid $M = ([n], \mathcal{B})$. An element $i \in B$ is said to be *internally active* with respect to B if $i < j$ for all $j \notin B$ such that $(B \setminus i) \cup j \in \mathcal{B}$. Similarly, an element $i \notin B$ is said to be *externally active* with respect to B if $i < j$ for all $j \in B$ such that $(B \setminus j) \cup i \in \mathcal{B}$. Let $I(B)$ and $E(B)$ be the sets of internally and externally active elements with respect to B.

Theorem 2.5.3. *(Tutte, Crapo [19]) The Tutte polynomial of a matroid is*

$$t_M(x,y) = \sum_{B \text{ basis of } M} x^{|I(B)|}y^{|E(B)|}.$$

Theorem 2.5.4. *Let G be the free abelian group generated by the triples (B,E,I), where $B \subseteq [n]$, $E \subseteq [n] \setminus B$ and $I \subseteq B$. The function $F : \mathsf{Mat} \to G$ defined by*

$$F(M) = \sum_{B \text{ basis of } M} (B, E(B), I(B)) \tag{2.5.1}$$

is a valuation.

Before proving this result, let us illustrate its strength with an example. Consider the subdivision of $M = U_{3,6}$ into three matroids M_1, M_2, and M_3 described in Example 2.2.8. Table 2.1 shows the external and internal activity with respect to each basis in each one of the eight matroids M_A arising in the subdivision. The combinatorics prescribed by Theorem 2.5.4 is extremely restrictive: in any row, any choice of (E,I) must appear the same number of times in the M_As with $|A|$ even and in the M_As with $|A|$ odd.

	M		M_1		M_2		$M_{1,2}$		M_3		$M_{1,3}$		$M_{2,3}$		$M_{1,2,3}$	
B	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$
123	$\emptyset$	123	$\emptyset$	123												
124	$\emptyset$	12	$\emptyset$	12												
125	$\emptyset$	12	$\emptyset$	12					$\emptyset$	125	$\emptyset$	125				
126	$\emptyset$	12	5	12					$\emptyset$	12	5	12				
134	$\emptyset$	1	$\emptyset$	1	$\emptyset$	134	$\emptyset$	134								
135	$\emptyset$	1	$\emptyset$	1	$\emptyset$	13	$\emptyset$	13	$\emptyset$	15	$\emptyset$	15	$\emptyset$	135	$\emptyset$	135
136	$\emptyset$	1	5	1	$\emptyset$	13	5	13	$\emptyset$	1	5	1	$\emptyset$	13	5	13
145	$\emptyset$	1	3	1	$\emptyset$	1	3	1	3	15	3	15	3	15	3	15
146	$\emptyset$	1	35	1	$\emptyset$	1	35	1	3	1	35	1	3	1	35	1
156	$\emptyset$	1							$\emptyset$	1						
234	1	$\emptyset$	1	$\emptyset$	1	34	1	34								
235	1	$\emptyset$	1	$\emptyset$	1	3	1	3	1	5	1	5	1	35	1	35
236	1	$\emptyset$	15	$\emptyset$	1	3	15	3	1	$\emptyset$	15	$\emptyset$	1	3	15	3
245	1	$\emptyset$	13	$\emptyset$	1	$\emptyset$	13	$\emptyset$	13	5	13	5	13	5	13	5
246	1	$\emptyset$	135	$\emptyset$	1	$\emptyset$	135	$\emptyset$	13	$\emptyset$	135	$\emptyset$	13	$\emptyset$	135	$\emptyset$
256	1	$\emptyset$							1	$\emptyset$						
345	12	$\emptyset$			12	$\emptyset$										
346	12	$\emptyset$			12	$\emptyset$										
356	12	$\emptyset$			12	3			12	$\emptyset$			12	3		
456	123	$\emptyset$			123	$\emptyset$			123	$\emptyset$			123	$\emptyset$		

Table 2.1: External and internal activities for a subdivision of $U_{3,6}$

We will divide the proof of Theorem 2.5.4 into a couple of lemmas.

Lemma 2.5.5. *Let $B \subseteq [n]$, $E \subseteq [n] \setminus B$ and $I \subseteq B$. Let*

$$V(B,E,I) = \{A \subseteq [n] : e_A - e_B = e_a - e_b \text{ with } a \in E \text{ and } a > b, \\ \text{or with } b \in I \text{ and } a < b\}$$

and

$$P(B,E,I) = \text{convex}\left\{ \frac{e_A + e_B}{2} : A \in V(B,E,I) \right\}.$$

Then for any matroid $M \in \mathsf{Mat}$, we have that $\text{Poly}(M) \cap P(B,E,I) = \emptyset$ *if and only if*

- *B is not a basis of M, or*
- *B is a basis of M with $E \subseteq E(B)$ and $I \subseteq I(B)$.*

To illustrate this lemma with an example, consider the case $n = 4$, $B = \{1,3\}$, $E = \{2\}$ and $I = \{3\}$. Then $V(B,E,I) = \{\{1,2\},\{2,3\}\}$. Figure 2.3 shows the polytope $P = P(B,E,I)$ inside the hypersimplex, whose vertices are the characteristic vectors of the 2-subsets of $[4]$. The polytope of the matroid M_1 with bases $\mathcal{B}_1 = \{\{1,2\},\{1,4\},\{2,3\},\{3,4\}\}$ does not intersect P because B is not a basis of M_1. The polytope of the matroid M_2 with bases $\mathcal{B}_2 = \{\{1,3\},\{1,4\},\{3,4\}\}$ does not intersect P either, because B is a basis of M_2, but 2 is externally active with respect to B and 3 is internally active with respect to B. Finally, the polytope of the matroid M_3 with bases $\mathcal{B}_3 = \{\{1,3\},\{2,3\},\{3,4\}\}$ does intersect P, since B is a basis of M_3 and 2 is not externally active with respect to B; the intersection point $\frac{1}{2}(0110 + 1010)$ "certifies" this.

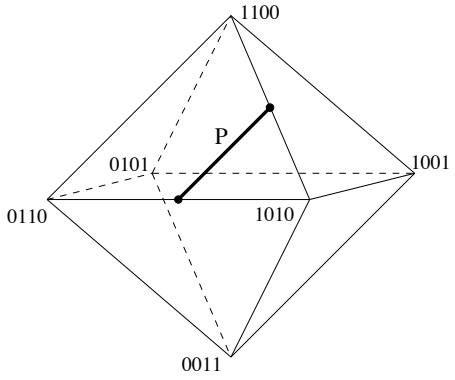


Figure 2.3: The polytope $P = P(B,E,I)$ inside $\text{Poly}(U_{2,4})$

Proof. Assume B is a basis of M. For $a \notin B$, a is externally active with respect to B if and only if there are no edges in $\mathrm{Poly}(M)$ which are translates of $e_a - e_b$ with $a > b$ which are incident to e_B. In the same way, for $b \in B$, b is internally active with respect to B if and only if there are no edges in $\mathrm{Poly}(M)$ which are translates of $e_a - e_b$ with $a < b$ which are incident to e_B. Since the vertices of $P(B, E, I)$ are precisely the midpoints of these edges when $a \in E$ and $b \in I$, if $\mathrm{Poly}(M) \cap P(B, E, I) = \emptyset$ then $E \subseteq E(B)$ and $I \subseteq I(B)$.

To prove the other direction, suppose that $\mathrm{Poly}(M) \cap P(B, E, I) \neq \emptyset$. First notice that, since $P(B, E, I)$ is on the hyperplane $x_1 + x_2 + \cdots + x_n = |B|$ and $\mathrm{Poly}(M)$ is on the hyperplane $x_1 + x_2 + \cdots + x_n = r(M)$, we must have $|B| = r(M)$. Moreover, since the vertices v of $P(B, E, I)$ satisfy $e_B \cdot v = r(M) - 1/2$ it follows that B must be a basis of M, or else the vertices w of $\mathrm{Poly}(M)$ would all satisfy $e_B \cdot w \leq r(M) - 1$.

Now let $q \in \mathrm{Poly}(M) \cap P(B, E, I)$. Since $q \in \mathrm{Poly}(M)$, we know that q is in the cone with vertex e_B generated by the edges of $\mathrm{Poly}(M)$ incident to e_B. In other words, if $A_1, A_2, \ldots, A_m$ are the bases adjacent to B,

$$q = e_B + \sum_{i=1}^{m} \lambda_i (e_{A_i} - e_B),$$

where the λ_i are all nonnegative. If we let $e_{c_i} - e_{d_i} = e_{A_i} - e_B$ for c_i and d_i elements of $[n]$, then

$$q = e_B + \sum_{i=1}^{m} \lambda_i (e_{c_i} - e_{d_i}).$$

On the other hand, since $q \in P(B, E, I)$,

$$q = \sum_{A \in V(B,E,I)} \gamma_A \frac{e_A + e_B}{2},$$

where the γ_A are nonnegative and add up to 1. Setting these two expressions equal to each other we obtain

$$q = e_B + \sum_{i=1}^{m} \lambda_i (e_{c_i} - e_{d_i}) = \sum_{A \in V(B,E,I)} \gamma_A \frac{e_A + e_B}{2}$$

and therefore

$$r = q - e_B = \sum_{i=1}^{m} \lambda_i (e_{c_i} - e_{d_i}) = \sum_{A \in V(B,E,I)} \gamma_A \frac{e_A - e_B}{2}.$$

For $A \in V(B, E, I)$ we will let $e_{a_A} - e_{b_A} = e_A - e_B$ for a_A and b_A again elements of $[n]$. We have

$$r = \sum_{i=1}^{m} \lambda_i (e_{c_i} - e_{d_i}) = \sum_{A \in V(B,E,I)} \gamma_A \frac{e_{a_A} - e_{b_A}}{2}. \tag{2.5.2}$$

Notice that there is no cancellation of terms in either side of (2.5.2), since the d_is and the b_As are elements of B, while the c_is and the a_As are not. Let $r = (r_1, r_2, \ldots, r_n)$ and let k be the largest integer for which r_k is nonzero.

Assume that $k \notin B$. From the right hand side of (2.5.2) and taking into account the definition of $V(B, E, I)$, we have that $k \in E$. From the left hand side we know there is an i such that $c_i = k$. But then $e_{c_i} - e_{d_i}$ is an edge of $\mathrm{Poly}(M)$ incident to e_B, and $d_i < k = c_i$ by our choice of k. It follows that k is not externally active with respect to B. In the case that $k \in B$, we obtain similarly that $k \in I$, and that $d_j = k$ for some j. Thus $e_{c_j} - e_{d_j}$ is an edge of $\mathrm{Poly}(M)$ incident to e_B and $c_j < k = d_j$, so k is not internally active with respect to B. In either case we conclude that $E \nsubseteq E(B)$ or $I \nsubseteq I(B)$, which finishes the proof. □

Lemma 2.5.6. *Let B be a subset of $[n]$, and let $E \subseteq [n] \setminus B$ and $I \subseteq B$. The function $G_{B,E,I} : \mathsf{Mat} \to \mathbb{Z}$ defined by*

$$G_{B,E,I}(M) = \begin{cases} 1 & \text{if } B \text{ is a basis of } M, E = E(B) \text{ and } I = I(B), \\ 0 & \text{otherwise,} \end{cases}$$

is a valuation.

Proof. To simplify the notation, we will write $\overline{i_B}$ instead of $\overline{i_{\{e_B\}}}$. We will prove that $G(B, E, I) = G'(B, E, I)$ where

$$G'_{B,E,I}(M) = (-1)^{|E|+|I|} \cdot \sum_{\substack{E \subseteq X \subseteq [n] \\ I \subseteq Y \subseteq [n]}} (-1)^{|X|+|Y|} \left(\overline{i_{P(B,X,Y)}}(M) - \overline{i_B}(M) \right), \tag{2.5.3}$$

which is a sum of valuations.

Let $M \in \mathsf{Mat}$. If B is not a basis of M then $\overline{i_B}(M) = 1$, and by Lemma 2.5.5 we have $\overline{i_{P(B,X,Y)}}(M) = 1$ for all X and Y. Therefore $G'_{B,E,I}(M) = 0 = G_{B,E,I}(M)$ as desired. If B is a basis of M then $\overline{i_B}(M) = 0$; and we use Lemma 2.5.5 to rewrite (2.5.3) as

$$\begin{aligned} G'_{B,E,I}(M) &= (-1)^{|E|+|I|} \cdot \sum_{\substack{E \subseteq X \subseteq E(B) \\ I \subseteq Y \subseteq I(B)}} (-1)^{|X|+|Y|} \\ &= (-1)^{|E|+|I|} \cdot \sum_{E \subseteq X \subseteq E(B)} (-1)^{|X|} \cdot \sum_{I \subseteq Y \subseteq I(B)} (-1)^{|Y|} \\ &= \begin{cases} 1 & \text{if } E = E(B) \text{ and } I = I(B), \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

as desired. □

Proof of Theorem 2.5.4. The coefficient of (B, E, I) in the definition of (2.5.1) is $G_{B,E,I}(M)$, so the result follows from Lemma 2.5.6. □

Theorem 2.5.4 is significantly stronger than the following result of Speyer which motivated it:

Corollary 2.5.7. *(Speyer, [82]) The Tutte polynomial (and therefore any of its evaluations) is a valuation under matroid subdivisions.*

Proof. By Theorem 2.5.3, $t_M(x, y)$ is the composition of the homomorphism $h : G \to \mathbb{Z}[x, y]$ defined by $h(B, E, I) = x^{|I|}\, y^{|E|}$ with the function F of Theorem 2.5.4. □

2.6 Related work

Previous to our work, Billera, Jia and Reiner [11] and Speyer [82, 83] had studied various valuations of matroid polytopes. A few months after the initial submission of the paper this chapter represents, we learned about Derksen's results on this topic [29], which were obtained independently and roughly simultaneously. Their approaches differ from ours in the basic fact that we have considered general matroid functions which are valuations, whereas they have been concerned with matroid invariants which are valuations; however there are similarities. We outline their main invariants here. See also Chapter 3 which takes up Derksen's approaches in considerable detail.

In his work on tropical linear spaces [82], Speyer shows that the Tutte polynomial is a valuative invariant. He also defines in [83] a polynomial invariant $g_M(t)$ of a matroid M which arises in the K-theory of the Grassmannian. It is not known how to describe $g_M(t)$ combinatorially in terms of M.

Given a matroid $M = (\mathsf{E}, \mathcal{B})$, a function $f : \mathsf{E} \to \mathbb{Z}_{>0}$ is said to be *M-generic* if the minimum value of $\sum_{b \in B} f(b)$ over all bases $B \in \mathcal{B}$ is attained just once. Billera, Jia, and Reiner study the valuation

$$QS(M) = \sum_{f \ M\text{-generic}} \prod_{b \in \mathsf{E}} x_{f(b)}$$

which takes values in the ring of *quasi-symmetric functions* in the variables x_i, i.e. the ring generated by

$$\sum_{i_1 < \ldots < i_r} x_{i_1}^{\alpha_1} \cdots x_{i_r}^{\alpha_r}$$

for all tuples $(\alpha_1, \ldots, \alpha_r)$ of positive integers.

Derksen's invariant is given by

$$G(M) := \sum_{\mathbf{A}} U(\mathrm{rk}_M(A_1) - \mathrm{rk}_M(A_0), \ldots, \mathrm{rk}_M(A_n) - \mathrm{rk}_M(A_{n-1}))$$

where $\mathbf{A} = (A_0, \ldots, A_n)$ ranges over all maximal flags of M, and $\{U(\mathbf{r}) : \mathbf{r}$ a finite sequence of nonnegative integers$\}$ is a particular basis for the ring of quasi-symmetric functions. (We

won't define the $U(\mathbf{r})$ more precisely, but we define their dual basis $u(\mathbf{r})$ in Section 3.6.) Derksen's invariant can be defined more generally on polymatroids. He shows that the Tutte polynomial and the quasisymmetric function of Billera, Jia and Reiner are specialisations of $G(M)$, and asks whether $G(M)$ is universal for valuative invariants in this setting. Chapter 3 answers this question in the affirmative.

For the remainder of this section, $F(M)$ will denote the function of our Theorem 2.5.1. Since $F(M)$ is not a matroid invariant, it cannot be a specialisation of $g_M(t), QS(M)$, or $G(M)$. As one would expect, $G(M)$ and $QS(M)$ are not specialisations of $F(M)$. One linear combination that certifies this is set out in Table 2.2, in which, to facilitate carrying out the relevant checks for $F(M)$, the relevant matroids are specified via their rank functions.

However, one can give a valuation which is similar in spirit to our $F(M)$ and specialises to Derksen's $G(M)$. This valuation will play a significant role in Chapter 3, where it is shown universal for matroid valuations. (It will be handled not as a single function s as below, but as its coordinates. The $s_{A,\underline{r}}$ of Chapter 3 is the coefficient of $\big((A_1, r(A_1)), \ldots, (A_n, r(A_n))\big)$ below.)

Proposition 2.6.1. *The function $s : \mathsf{Mat} \to G^n$ defined by*

$$s(M) = \sum_{\mathbf{A}} \big((A_1, r(A_1)), \ldots, (A_n, r(A_n))\big),$$

where $\mathbf{A} = (A_1, \ldots, A_n)$ ranges over all maximal flags of M, is a valuation.

Proof. The proof is a straightforward extension of our argument for Theorem 2.5.1. With the notation of that proof, checking whether a matroid M satisfies $\mathrm{rk}_M(A_i) = r_i$ for some fixed vector $\mathbf{r} = (r_i)$, *i.e.* whether the term $((A_1, r_1), \ldots, (A_n, r_n))$ is present in $s(M)$, is equivalent to checking that $\mathrm{Poly}(M)$ intersects P_{A_i,r_i} and not P_{A_i,r_i+1} for each i.

Observe that if $\mathrm{Poly}(M)$ intersects P_{A_i,r_i} for all i then $r(A_i) \geq r_i$ and, since $\mathbf{A}$ is a flag, we can choose a single basis of M whose intersection with A_i has at least r_i elements for each i. Therefore $\mathrm{Poly}(M)$ intersects $P_{A_1,r_1} \cap \cdots \cap P_{A_n,r_n}$.

Consider the sum

$$\sum (-1)^{e_1+\ldots+e_n} i_{P_{\mathbf{A},\mathbf{r}+\mathbf{e}}}(M) \tag{2.6.1}$$

where the sum is over all $\mathbf{e} = (e_1, ..., e_n) \in \{0,1\}^n$, and where $P_{\mathbf{A},\mathbf{r}+\mathbf{e}}$ is the intersection $P_{A_1,r_1+e_1} \cap \cdots \cap P_{A_n,r_n+e_n}$. By our previous observation this sum equals

$$\left(\sum_{e_1} (-1)^{e_1} i_{P_{A_1,r_1+e_1}}(M)\right) \cdots \left(\sum_{e_n} (-1)^{e_n} i_{P_{A_n,r_n+e_n}}(M)\right),$$

which is 1 if the term $((A_1, r_1), \ldots, (A_n, r_n))$ is present in $s(M)$, and is 0 otherwise. All the terms in (2.6.1) are valuations, hence s is a valuation. □

A	$\emptyset$	1	2	12	3	13	23	123
$\mathrm{rk}_{M_1}(A)$	0	1	1	1	0	1	1	1
$\mathrm{rk}_{M_2}(A)$	0	1	0	1	1	1	1	1
$\mathrm{rk}_{M_3}(A)$	0	0	1	1	1	1	1	1
$\mathrm{rk}_{M_4}(A)$	0	1	1	1	1	1	1	1
$\mathrm{rk}_{M_5}(A)$	0	1	1	2	0	1	1	2
$\mathrm{rk}_{M_6}(A)$	0	1	0	1	1	2	1	2
$\mathrm{rk}_{M_7}(A)$	0	0	1	1	1	1	2	2
$\mathrm{rk}_{M_8}(A)$	0	1	1	2	1	2	2	2
$\mathrm{rk}_{M_9}(A)$	0	0	1	1	1	1	2	2
$\mathrm{rk}_{M_{10}}(A)$	0	1	1	2	1	2	2	3
$\mathrm{rk}_{M_{11}}(A)$	0	1	1	2	1	1	2	2
$\mathrm{rk}_{M_{12}}(A)$	0	1	1	2	1	2	2	3

A	4	14	24	124	34	134	234	1234
$\mathrm{rk}_{M_1}(A)$	0	1	1	1	0	1	1	1
$\mathrm{rk}_{M_2}(A)$	1	1	1	1	1	1	1	1
$\mathrm{rk}_{M_3}(A)$	1	1	1	1	1	1	1	1
$\mathrm{rk}_{M_4}(A)$	1	1	1	1	1	1	1	1
$\mathrm{rk}_{M_5}(A)$	0	1	1	2	0	1	1	2
$\mathrm{rk}_{M_6}(A)$	1	2	1	2	1	2	1	2
$\mathrm{rk}_{M_7}(A)$	1	1	2	2	1	1	2	2
$\mathrm{rk}_{M_8}(A)$	1	2	2	2	1	2	2	2
$\mathrm{rk}_{M_9}(A)$	1	1	2	2	2	2	3	3
$\mathrm{rk}_{M_{10}}(A)$	1	1	2	2	2	2	3	3
$\mathrm{rk}_{M_{11}}(A)$	1	2	2	3	2	2	3	3
$\mathrm{rk}_{M_{12}}(A)$	1	2	2	3	2	2	3	3

i	1	2	3	4	5	6	7	8	9	10	11	12
c_i	-1	1	-1	1	1	-1	-1	1	2	-2	-2	2

Table 2.2: The top table contains the rank functions of twelve matroids M_i on $[4]$, $i = 1, \ldots, 12$. The bottom table shows coefficients c_i such that $\sum c_i F(M_i) = 0$ but $\sum c_i G(M_i) \neq 0$, $\sum c_i QS(M_i) \neq 0$.

Chapter 3

Valuative invariants for polymatroids

This chapter is joint work with Harm Derksen. It is to appear in *Advances in Mathematics* with the same title, as doi:10.1016/j.aim.2010.04.016. (This version incorporates some minor changes, largely for consistency with other chapters.)

3.1 Introduction

Matroids were introduced by Whitney in 1935 (see [94]) as a combinatorial abstraction of linear dependence of vectors in a vector space. Some standard references are [92] and [72]. Polymatroids are multiset analogs of matroids and appeared in the late 1960s (see [33, 44]). There are many distinct but equivalent definitions of matroids and polymatroids, for example in terms of bases, independent sets, flats, polytopes or rank functions. For polymatroids, the equivalence between the various definitions is given in [44]. Here is the definition in terms of rank functions:

Definition 3.1.1. Suppose that X is a finite set (the *ground set*) and $\mathrm{rk} : 2^X \to \mathbb{N} = \{0, 1, 2, \dots\}$, where 2^X is the set of subsets of X. Then (X, rk) is called a *polymatroid* if:

1. $\mathrm{rk}(\emptyset) = 0$;
2. rk is *weakly increasing*: if $A \subseteq B$ then $\mathrm{rk}(A) \leq \mathrm{rk}(B)$;
3. rk is *submodular*: $\mathrm{rk}(A \cup B) + \mathrm{rk}(A \cap B) \leq \mathrm{rk}(A) + \mathrm{rk}(B)$ for all $A, B \subseteq X$.

If moreover, $\mathrm{rk}(\{x\}) \leq 1$ for every $x \in X$, then (X, rk) is called a *matroid*.

An *isomorphism* $\varphi : (X, \mathrm{rk}_X) \to (Y, \mathrm{rk}_Y)$ is a bijection $\varphi : X \to Y$ such that $\mathrm{rk}_Y \circ \varphi = \mathrm{rk}_X$. Every polymatroid is isomorphic to a polymatroid with ground set $[d] = \{1, 2, \dots, d\}$ for some nonnegative integer d. The rank of a polymatroid (X, rk) is $\mathrm{rk}(X)$.

Let $S_{\mathrm{PM}}(d,r)$ be the set of all polymatroids with ground set $[d]$ of rank r, and $S_{\mathrm{M}}(d,r)$ be the set of all matroids with ground set $[d]$ of rank r. We will write $S_{(\mathrm{P})\mathrm{M}}(d,r)$ when we want to refer to $S_{\mathrm{PM}}(d,r)$ or $S_{\mathrm{M}}(d,r)$ in parallel. A function f on $S_{(\mathrm{P})\mathrm{M}}(d,r)$ is a (poly)matroid *invariant* if $f\big(([d],\mathrm{rk})\big) = f\big(([d],\mathrm{rk}')\big)$ whenever $([d],\mathrm{rk})$ and $([d],\mathrm{rk}')$ are isomorphic. Let $S^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)$ be the set of isomorphism classes in $S_{(\mathrm{P})\mathrm{M}}(d,r)$. Invariant functions on $S_{(\mathrm{P})\mathrm{M}}(d,r)$ correspond to functions on $S^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)$. Let $Z_{(\mathrm{P})\mathrm{M}}(d,r)$ and $Z^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)$ be the $\mathbb{Z}$-modules freely generated by $S_{(\mathrm{P})\mathrm{M}}(d,r)$ and $S^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)$ respectively. For an abelian group A, every function $f : S^{(\mathrm{sym})}_{(\mathrm{P})\mathrm{M}}(d,r) \to A$ extends uniquely to a group homomorphism $Z^{(\mathrm{sym})}_{(\mathrm{P})\mathrm{M}}(d,r) \to A$.

One of the most important matroid invariants is the Tutte polynomial. It was first defined for graphs in [88] and generalized to matroids in [18, 24]. This bivariate polynomial is defined by

$$\mathcal{T}\big((X,\mathrm{rk})\big) = \sum_{A\subseteq X} (x-1)^{\mathrm{rk}(X)-\mathrm{rk}(A)}(y-1)^{|A|-\mathrm{rk}(A)}.$$

Regarded as a polynomial in $x-1$ and $y-1$, $\mathcal{T}$ is also known as the *rank generating function*. The Tutte polynomial is universal for all matroid invariants satisfying a deletion-contraction formula. Speyer defined a matroid invariant in [83] using K-theory. Billera, Jia and Reiner introduced a quasi-symmetric function $\mathcal{F}$ for matroids in [11], which is a matroid invariant. This quasi-symmetric function is a powerful invariant in the sense that it can distinguish many pairs of non-isomorphic matroids. However, it does not specialize to the Tutte polynomial. Derksen introduced in [29] another quasi-symmetric function $\mathcal{G}$ which specializes to $\mathcal{T}$ and $\mathcal{F}$. Let $\{U_\alpha\}$ be the basis of the ring of quasi-symmetric functions defined in [29]. $\mathcal{G}$ is defined by

$$\mathcal{G}\big((X,\mathrm{rk})\big) = \sum_{\underline{X}} U_{r(\underline{X})},$$

where

$$\underline{X} : \emptyset = X_0 \subset X_1 \subset \cdots \subset X_d = X$$

runs over all $d!$ maximal chains of subsets in X, and

$$r(\underline{X}) = (\mathrm{rk}(X_1)-\mathrm{rk}(X_0), \mathrm{rk}(X_2)-\mathrm{rk}(X_1), \ldots, \mathrm{rk}(X_d)-\mathrm{rk}(X_{d-1})).$$

To a (poly)matroid $([d],\mathrm{rk})$ one can associate its *base polytope* $\mathrm{Poly}(\mathrm{rk})$ in $\mathbb{R}^d$ (see Definition 3.2.2). For $d \geq 1$, the dimension of this polytope is $\leq d-1$. The indicator function of a polytope $\Pi \subseteq \mathbb{R}^d$ is denoted by $\mathbf{1}(\Pi) : \mathbb{R}^d \to \mathbb{Z}$. Let $P_{(\mathrm{P})\mathrm{M}}(d,r)$ be the $\mathbb{Z}$-module generated by all $\mathbf{1}(\mathrm{Poly}(\mathrm{rk}))$ with $([d],\mathrm{rk}) \in S_{(\mathrm{P})\mathrm{M}}(d,r)$.

Definition 3.1.2. Suppose that A is an abelian group. A function $f : S_{(\mathrm{P})\mathrm{M}}(d,r) \to A$ is *strongly valuative* if there exists a group homomorphism $\widehat{f} : P_{(\mathrm{P})\mathrm{M}}(d,r) \to A$ such that

$$f\big(([d],\mathrm{rk})\big) = \widehat{f}(\mathbf{1}(\mathrm{Poly}(\mathrm{rk})))$$

for all $([d],\mathrm{rk}) \in S_{(\mathrm{P})\mathrm{M}}(d,r)$.

In Section 3.3 we also define a *weak valuative property* in terms of base polytope decompositions. Although seemingly weaker, we will show that the weak valuative property is equivalent to the strong valuative property.

Definition 3.1.3. Suppose that $d > 0$. A valuative function $f : S_{(\mathrm{P})\mathrm{M}}(d,r) \to A$ is said to be *additive*, if $f\big(([d], \mathrm{rk})\big) = 0$ whenever the dimension of Poly(rk) is $< d-1$.

Most of the known (poly)matroid invariants are *valuative*. For example, $\mathcal{T}$, $\mathcal{F}$ and $\mathcal{G}$ all have this property in common. Speyer's invariant is not valuative, but does have a similar property, which we will call the *covaluative* property. Valuative invariants and additive invariants can be useful for deciding whether a given matroid polytope has a decomposition into smaller matroid polytopes (see the discussion in [11, Section 7]). Decompositions of polytopes and their valuations are fundamental objects of interest in discrete geometry in their own right (see for instance the survey [65]). Matroid polytope decompositions appeared in the work of Lafforgue ([55, 56]) on compactifications of a fine Schubert cell in the Grassmannian associated to a matroid. The work of Lafforgue implies that if the base polytope of a matroid does not have a proper decomposition, then the matroid is rigid, i.e., it has only finitely many nonisomorphic realizations over a given field.

Main results

The following theorem proves a conjecture stated in [29]:

Theorem 3.1.4. *The $\mathcal{G}$-invariant is universal for all valuative (poly)matroid invariants, i.e., the coefficients of $\mathcal{G}$ span the vector space of all valuative (poly)matroid invariants with values in $\mathbb{Q}$.*

From $\mathcal{G}$ one can also construct a universal invariant for the covaluative property which specializes to Speyer's invariant.

It follows from the definitions that the dual $P_{(\mathrm{P})\mathrm{M}}(d,r)^\vee = \mathrm{Hom}_{\mathbb{Z}}(P_{(\mathrm{P})\mathrm{M}}(d,r), \mathbb{Z})$ is the space of all $\mathbb{Z}$-valued valuative functions on $S_{(\mathrm{P})\mathrm{M}}(d,r)$. If $P^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)$ is the push-out of the diagram

$$\begin{array}{ccc} Z_{(\mathrm{P})\mathrm{M}}(d,r) & \xrightarrow{\pi_{(\mathrm{P})\mathrm{M}}} & Z^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r) \\ {\scriptstyle \mathbf{1}_{(\mathrm{P})\mathrm{M}}}\downarrow & & \downarrow{\scriptstyle \mathbf{1}^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}} \\ P_{(\mathrm{P})\mathrm{M}}(d,r) & \xrightarrow[\rho_{(\mathrm{P})\mathrm{M}}]{} & P^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r) \end{array} \tag{3.1.1}$$

then the dual space $P^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)^\vee$ is exactly the set of all $\mathbb{Z}$-valued valuative (poly)matroid *invariants*. Let $p^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)$ be the rank of the free $\mathbb{Z}$-module $P^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)$, and $p_{(\mathrm{P})\mathrm{M}}(d,r)$ be the rank of the free $\mathbb{Z}$-module $P_{(\mathrm{P})\mathrm{M}}(d,r)$. Then $p^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)$ is the number of independent $\mathbb{Z}$-valued valuative (poly)matroid invariants, and $p_{(\mathrm{P})\mathrm{M}}(d,r)$ is the number of independent $\mathbb{Z}$-valued valuative functions on (poly)matroids. We will prove the following formulas:

Theorem 3.1.5.

a. $p^{\mathrm{sym}}_{\mathrm{M}}(d,r) = \binom{d}{r}$ *and* $\displaystyle\sum_{0\le r\le d} p^{\mathrm{sym}}_{\mathrm{M}}(d,r)x^{d-r}y^r = \frac{1}{1-x-y}$,

b. $p^{\mathrm{sym}}_{\mathrm{PM}}(d,r) = \begin{cases} \binom{r+d-1}{r} & \text{if } d\ge 1 \text{ or } r\ge 1; \\ 1 & \text{if } d=r=0 \end{cases}$ *and*
$\displaystyle\sum_{r=0}^{\infty}\sum_{d=0}^{\infty} p^{\mathrm{sym}}_{\mathrm{PM}}(d,r)x^d y^r = \frac{1-x}{1-x-y}$,

c. $\displaystyle\sum_{0\le r\le d} \frac{p_{\mathrm{M}}(d,r)}{d!}x^{d-r}y^r = \frac{x-y}{xe^{-x}-ye^{-y}}$,

d. $p_{\mathrm{PM}}(d,r) = \begin{cases} (r+1)^d - r^d & \text{if } d\ge 1 \text{ or } r\ge 1; \\ 1 & \text{if } d=r=0, \end{cases}$ *and*
$\displaystyle\sum_{d=0}^{\infty}\sum_{r=0}^{\infty} \frac{p_{\mathrm{PM}}(d,r)x^d y^r}{d!} = \frac{e^x(1-y)}{1-ye^x}$.

We also will give **explicit bases** for each of the spaces $P_{(\mathrm{P})\mathrm{M}}(d,r)$ and $P^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)$ and their duals (see Theorems 3.5.4, 3.6.3, Corollaries 3.5.5, 3.5.6, 3.6.6, 3.6.5).

The bigraded module

$$Z_{(\mathrm{P})\mathrm{M}} = \bigoplus_{d,r} Z_{(\mathrm{P})\mathrm{M}}(d,r)$$

has the structure of a Hopf algebra. Similarly, each of the bigraded modules $Z^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}$, $P_{(\mathrm{P})\mathrm{M}}$ and $P^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}$ has a Hopf algebra structure. The module $Z^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}$ is the usual Hopf algebra of (poly)matroids, where multiplication is given by the direct sum of matroids.

In Sections 3.8 and 3.9 we construct bigraded modules $T_{(\mathrm{P})\mathrm{M}}$ and $T^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}$ so that $T_{(\mathrm{P})\mathrm{M}}(d,r)^\vee$ is the space of all additive functions on $S_{(\mathrm{P})\mathrm{M}}(d,r)$ and $T^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)^\vee$ is the space of all additive invariants. Let $t_{(\mathrm{P})\mathrm{M}}(d,r)$ be the rank of $T_{(P)M}(d,r)$ and $t^{\mathrm{sym}}_{(P)M}(d,r)$ be the rank of $T^{\mathrm{sym}}_{(P)M}(d,r)$. Then $t_{(\mathrm{P})\mathrm{M}}(d,r)$ is the number of independent additive functions on (poly)matroids, and $t^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)$ is the number of independent additive invariants for (poly)matroids. We will prove the following formulas:

Theorem 3.1.6.

a. $\displaystyle\prod_{0\le r\le d} (1-x^{d-r}y^r)^{t^{\mathrm{sym}}_{\mathrm{M}}(d,r)} = 1-x-y$,

b. $\displaystyle\prod_{r,d}(1-x^d y^r)^{t^{\mathrm{sym}}_{\mathrm{PM}}(d,r)} = \frac{1-x-y}{1-y}$,

c. $\displaystyle\sum_{r,d} \frac{t_{\mathrm{M}}(d,r)}{d!} x^{d-r} y^r = \log\left(\frac{x-y}{xe^{-x}-ye^{-y}}\right)$,

d. $t_{\mathrm{PM}}(d,r) = \begin{cases} r^{d-1} & \text{if } d \geq 1 \\ 0 & \text{if } d = 0, \end{cases}$ *and*
$\displaystyle\sum_{r,d} \frac{t_{\mathrm{PM}}(d,r)}{d!} x^d y^r = \log\big(\frac{e^x(1-y)}{1-ye^x}\big)$.

We will also give **explicit bases** for the the spaces $T_{\mathrm{M}}(d,r)$ and $T_{\mathrm{PM}}(d,r)$ in Theorem 3.8.6, and of the dual spaces $T_{\mathrm{M}}^{\mathrm{sym}}(d,r)^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$, $T_{\mathrm{PM}}^{\mathrm{sym}}(d,r)^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$ in Theorem 3.10.2.

For $\mathbb{Q}$-valued functions we will prove the following isomorphisms in Section 3.10.

Theorem 3.1.7. *Let $u_0, u_1, u_2, \ldots$ be indeterminates, where u_i has bidgree $(1,i)$. We have the following isomorphisms of bigraded associative algebras over $\mathbb{Q}$:*

a. *The space $(P_M^{\mathrm{sym}})^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$ of $\mathbb{Q}$-valued valuative invariants on matroids is isomorphic to $\mathbb{Q}\langle\langle u_0, u_1\rangle\rangle$, the completion (in power series) of the free associative algebra generated by u_0, u_1.*

b. *The space $(P_{\mathrm{PM}}^{\mathrm{sym}})^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$ of $\mathbb{Q}$-valued valuative invariants on polymatroids is isomorphic to $\mathbb{Q}\langle\langle u_0, u_1, u_2, \ldots\rangle\rangle$.*

c. *The space $(T_{\mathrm{M}}^{\mathrm{sym}})^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$ of $\mathbb{Q}$-valued additive invariants on matroids is isomorphic to $\mathbb{Q}\{\{u_0, u_1\}\}$, the completion of the free Lie algebra generated by u_0, u_1.*

d. *The space $(T_{\mathrm{PM}}^{\mathrm{sym}})^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$ of $\mathbb{Q}$-valued additive invariants on polymatroids is isomorphic to $\mathbb{Q}\{\{u_0, u_1, u_2, \ldots\}\}$.*

Tables for $p_{(\mathrm{P})\mathrm{M}}$, $p_{(\mathrm{P})\mathrm{M}}^{\mathrm{sym}}$, $t_{(\mathrm{P})\mathrm{M}}$, $t_{(\mathrm{P})\mathrm{M}}^{\mathrm{sym}}$ are given in Appendix 3.B to this chapter.

An index of notations used in this chapter appears in Appendix 3.C. To aid the reader in keeping them in mind we present an abridged table here. In a notation of the schematic form $\mathrm{Letter}_{\mathrm{sub}}^{\mathrm{super}}(d,r)$:

The letter	S	refers to	the set of *-matroids
	Z		the $\mathbb{Z}$-module with basis all *-matroids
	P		the $\mathbb{Z}$-module of indicator functions of *-matroids
	T		the $\mathbb{Z}$-module of indicator functions of *-matroids, modulo changes on subspaces of dimension $< d-1$

with ground set $[d]$ of rank r. Here $*$ stands for one of the prefixes "" or "poly" or "mega". If the letter is lowercase, we refer not to the $\mathbb{Z}$-module but to its rank.

The subscript	M	means the *-matroids are	matroids
	PM		polymatroids
	MM		megamatroids (Def. 3.2.1);

additionally, when we want to refer to multiple cases in parallel,

the subscript	(P)M	covers	matroids and polymatroids
	*M		matroids and poly- and mega-matroids.

The superscript sym means that we are only considering *-matroids up to isomorphism.

3.2 Polymatroids and their polytopes

For technical reasons it will be convenient to have an "unbounded" analogue of polymatroids, especially when we work with their polyhedra. So we make the following definition.

Definition 3.2.1. A function $2^X \to \mathbb{Z} \cup \{\infty\}$ is called a *megamatroid* if it has the following properties:

1. $\mathrm{rk}(\emptyset) = 0$;
2. $\mathrm{rk}(X) \in \mathbb{Z}$;
3. rk is *submodular*: if $\mathrm{rk}(A), \mathrm{rk}(B) \in \mathbb{Z}$, then $\mathrm{rk}(A \cup B), \mathrm{rk}(A \cap B) \in \mathbb{Z}$ and $\mathrm{rk}(A \cup B) + \mathrm{rk}(A \cap B) \leq \mathrm{rk}(A) + \mathrm{rk}(B)$.

Obviously, every matroid is a polymatroid, and every polymatroid is a megamatroid. The *rank* of a megamatroid (X, rk) is the integer $\mathrm{rk}(X)$.

By a *polyhedron* we will mean a finite intersection of closed half-spaces. A *polytope* is a bounded polyhedron.

Definition 3.2.2. For a megamatroid $([d], \mathrm{rk})$, we define its *base polyhedron* $\mathrm{Poly}(\mathrm{rk})$ as the set of all $(y_1, \ldots, y_d) \in \mathbb{R}^d$ such that $y_1 + y_2 + \cdots + y_d = \mathrm{rk}(X)$ and $\sum_{i \in A} y_i \leq \mathrm{rk}(A)$ for all $A \subseteq X$.

If rk is a polymatroid then $\mathrm{Poly}(\mathrm{rk})$ is a polytope, called the *base polytope* of rk. In [33], Edmonds studies a similar polytope for a polymatroid $([d], \mathrm{rk})$ which contains $\mathrm{Poly}(\mathrm{rk})$ as a facet.

Lemma 3.2.3. *If* $([d], \mathrm{rk})$ *is a megamatroid, then* $\mathrm{Poly}(\mathrm{rk})$ *is nonempty.*

Proof. First, assume that rk is a megamatroid such that $r_i := \mathrm{rk}([i])$ is finite for $i = 0, 1, \ldots, d$. We claim that

$$y = (r_1 - r_0, r_2 - r_1, \ldots, r_d - r_{d-1}) \in \mathrm{Poly}(\mathrm{rk}).$$

Indeed, if $A = \{i_1, \ldots, i_k\}$ with $1 \leq i_1 < \cdots < i_k \leq d$ then, by the submodular property, we have

$$\sum_{i \in A} y_i = \sum_{j=1}^{k} (\mathrm{rk}([i_j]) - \mathrm{rk}([i_{j-1}])) \leq$$
$$\leq \sum_{j=1}^{k} \Big(\mathrm{rk}(\{i_1, \ldots, i_j\}) - \mathrm{rk}(\{i_1, \ldots, i_{j-1}\}) \Big) = \mathrm{rk}(\{i_1, \ldots, i_k\}) = \mathrm{rk}(A)$$

where the inequality holds even if the right hand side is infinite.

Now, assume that rk is any megamatroid. Define rk^N by

$$\mathrm{rk}^N(A) = \min_{X \subseteq A} \mathrm{rk}(X) + N(|A| - |X|). \tag{3.2.1}$$

Let N be large enough such that $\mathrm{rk}^N([d]) = \mathrm{rk}([d])$. It may help the reader's visualization to note that, in this case, $\mathrm{Poly}(\mathrm{rk}^N)$ is the intersection of $\mathrm{Poly}(\mathrm{rk})$ with the orthant $(-\infty, N]^d$. If $A, B \subseteq [d]$, then we have

$$\mathrm{rk}^N(A) = \mathrm{rk}(X) + N(|A| - |X|), \quad \mathrm{rk}^N(B) = \mathrm{rk}(Y) + N(|A| - |Y|)$$

for some $X \subseteq A$ and some $Y \subseteq B$. It follows that

$$\begin{aligned}
&\mathrm{rk}^N(A \cap B) + \mathrm{rk}^N(A \cup B) \\
&\quad \leq \mathrm{rk}(X \cap Y) + N(|A \cap B| - |X \cap Y|) + \mathrm{rk}(X \cup Y) + N(|A \cup B| - |X \cup Y|) \\
&\quad\quad = \mathrm{rk}(X \cap Y) + \mathrm{rk}(X \cup Y) + N(|A| + |B| - |X| - |Y|) \\
&\quad\quad\quad \leq \mathrm{rk}(X) + \mathrm{rk}(Y) + N(|A| + |B| - |X| - |Y|) = \mathrm{rk}^N(A) + \mathrm{rk}^N(B).
\end{aligned}$$

This shows that rk^N is a megamatroid. Since $\mathrm{rk}^N(A) \leq \mathrm{rk}(A)$ for all $A \subseteq [d]$, we have $\mathrm{Poly}(\mathrm{rk}^N) \subseteq \mathrm{Poly}(\mathrm{rk})$. Since $\mathrm{rk}^N(A) < \infty$ for all $A \subseteq [d]$, we have that $\mathrm{Poly}(\mathrm{rk}^N) \neq \emptyset$. We conclude that $\mathrm{Poly}(\mathrm{rk}) \neq \emptyset$. □

A megamatroid $([d], \mathrm{rk})$ of rank r is a polymatroid if and only if its base polytope is contained in the simplex

$$\Delta_{\mathrm{PM}}(d, r) = \{(y_1, \ldots, y_d) \in \mathbb{R}^d \mid y_1, \ldots, y_d \geq 0,\ y_1 + y_2 + \cdots + y_d = r\}$$

and it is a matroid if and only if its base polytope is contained in the hypersimplex

$$\Delta_{\mathrm{M}}(d, r) = \{(y_1, \ldots, y_d) \in \mathbb{R}^d \mid 0 \leq y_1, \ldots, y_d \leq 1,\ y_1 + y_2 + \cdots + y_d = r\}.$$

If $([d], \mathrm{rk})$ is a matroid, then a subset $A \subseteq [d]$ is a *basis* when $\mathrm{rk}(A) = |A| = \mathrm{rk}([d])$. In this case, the base polytope of $([d], \mathrm{rk})$ is the convex hull of all $\sum_{i \in A} e_i$ where $A \subseteq [d]$ is a basis (see [37]). The base polytope of a matroid was characterized in [37]:

Theorem 3.2.4. *A polytope* Π *contained in* $\Delta_{\mathrm{M}}(d,r)$ *is the base polytope of a matroid if and only if it has the following properties:*

1. *The vertices of* Π *have integral coordinates;*
2. *every edge of* Π *is parallel to* $e_i - e_j$ *for some* i, j *with* $i \neq j$.

We will generalize this characterization to megamatroids. The remainder of this section builds up to proving Proposition 3.2.9, showing that the following definition captures exactly the polyhedra Poly(rk) for rk a megamatroid.

Definition 3.2.5. A convex polyhedron contained in $y_1 + \cdots + y_d = r$ is called a *megamatroid polyhedron* if for every face F of Π, the linear hull $\mathrm{lhull}(F)$ is of the form $z + W$ where $z \in \mathbb{Z}^d$ and W is spanned by vectors of the form $e_i - e_j$.

The bounded megamatroid polyhedra are exactly the lattice polytopes among the *generalized permutohedra* of [73] or the *submodular rank tests* of [69]. General megamatroid polyhedra are the natural unbounded generalizations.

Faces of megamatroid polyhedra are again megamatroid polyhedra. If we intersect a megamatroid polyhedron Π with the hyperplane $y_d = s$, we get again a megamatroid polyhedron. For a megamatroid polyhedron Π, define $\mathrm{rk}_\Pi : 2^{[d]} \to \mathbb{Z} \cup \{\infty\}$ by

$$\mathrm{rk}_\Pi(A) := \sup\{\textstyle\sum_{i \in A} y_i \mid y \in \Pi\}.$$

Lemma 3.2.6. *Suppose that* Π *is a megamatroid polyhedron,* $A \subseteq B$ *and* $\mathrm{rk}_\Pi(A) < \infty$. *Let* F *be the face of* Π *on which* $\sum_{i \in A} y_i$ *is maximal. Then*

$$\mathrm{rk}_\Pi(B) = \mathrm{rk}_F(B).$$

Proof. If $\mathrm{rk}_F(B) = \infty$ then $\mathrm{rk}_\Pi(B) = \infty$ and we are done. Otherwise, there exists a face F' of F on which $\sum_{i \in B} y_i$ is maximal. Suppose that $\mathrm{rk}_F(B) < \mathrm{rk}_\Pi(B)$. Define $g(y) := \sum_{i \in B} y_i - \mathrm{rk}_F(B)$. Then g is constant 0 on F', and $g(y) > 0$ for some $y \in \Pi$. Therefore, there exists a face F'' of Π containing F', such that $\dim F'' = \dim F' + 1$ and $g(z) > 0$ for some $z \in F''$. Clearly, $z \notin F$ and F does not contain F''. We have $\mathrm{lhull}(F'') = \mathrm{lhull}(F') + \mathbb{R}(e_k - e_j)$ for some $k \neq j$. By possibly exchanging j and k, we may assume that F'' is contained in $\mathrm{lhull}(F') + \mathbb{R}_+(e_k - e_j)$, where $\mathbb{R}_+$ denotes the nonnegative real numbers. Since $z \in \mathrm{lhull}(F') + \mathbb{R}_+(e_k - e_j)$ and $g(z) > 0$ we have $k \in B$ and $j \notin B$. In particular $j \notin A$. For all $y'' \in F''$ we can write $y'' = y + r(e_k - e_j)$ with $y \in F$ and $r > 0$, and it follows that $\sum_{i \in A} y''_i \geq \sum_{i \in A} y_i = \mathrm{rk}_\Pi(A)$ since the j-th coordinate of y'' cannot contribute. So $F'' \subseteq F$. But this is a contradiction. We conclude that $\mathrm{rk}_F(B) = \mathrm{rk}_\Pi(B)$. □

Lemma 3.2.7. *Suppose that* $f(y) = \sum_{j=1}^{d} \alpha_j \sum_{i \in X_j} y_i$ *where*

$$\underline{X} : \emptyset \subset X_1 \subset X_2 \subset \cdots \subset X_d = [d]$$

is a maximal chain, and $\alpha_1 \ldots, \alpha_{d-1} \geq 0$. *For a megamatroid polyhedron* Π *we have*

$$\sup_{y \in \Pi} f(y) = \sum_{j=1}^{d} \alpha_j \operatorname{rk}_{\Pi}(X_j).$$

Proof. First, assume that Π is bounded. Define $F_0 = \Pi$, and for $j = 1, 2, \ldots, d$, let F_j be the face of F_{j-1} for which $\sum_{i \in X_j} y_i$ is maximal. By induction on j and Lemma 3.2.6, we have that $\operatorname{rk}_{F_j}(X_i) = \operatorname{rk}_{\Pi}(X_i)$ for all $j < i$. Also, F_j is contained in the hyperplane defined by the equation $\sum_{i \in X_j} y_i = \operatorname{rk}_{F_{j-1}}(X_j) = \operatorname{rk}_{\Pi}(X_j)$. We have $F_d = \{z\}$ where $z = (z_1, \ldots, z_d)$ is defined by the equations

$$\sum_{i \in X_j} z_i = \operatorname{rk}_{\Pi}(X_j), \quad j = 1, 2, \ldots, d.$$

It follows that

$$f(z) = \sum_{j=1}^{d} \alpha_j \sum_{i \in X_j} z_j = \sum_{j=1}^{d} \alpha_j \operatorname{rk}_{\Pi}(X_j).$$

Suppose that Π is unbounded. Let Π_N be the intersection of Π with the set $\{y \in \mathbb{R}^d \mid y_i \leq N,\ i = 1, 2, \ldots, d\}$. Now Π_N is a bounded megamatroid polyhedron for large positive integers N. (For small N, Π_N might be empty.) We have

$$\sup_{y \in \Pi} f(y) = \sup_{N} \sup_{y \in \Pi_N} f(y) = \sup_{N} \sum_{j=1}^{d} \alpha_j \operatorname{rk}_{\Pi_N}(X_j) = \sum_{j=1}^{d} \alpha_j \operatorname{rk}_{\Pi}(X_j).$$

□

Corollary 3.2.8. *If* Π *is a megamatroid polyhedron, then* rk_{Π} *is a megamatroid.*

Proof. For subsets $A, B \subseteq [d]$, choose a maximal chain $\underline{X}$ such that $X_j = A \cap B$ and $X_k = A \cup B$ for some j and k, and let

$$f_A(y) = \sum_{i \in A} y_i, \quad f_B(y) = \sum_{i \in B} y_i, \quad f(y) = \sum_{i \in A \cap B} y_i + \sum_{i \in A \cup B} y_i = f_A(y) + f_B(y).$$

By Lemma 3.2.7,

$$\operatorname{rk}_{\Pi}(A) + \operatorname{rk}_{\Pi}(B) = \sup_{y \in \Pi} f_A(y) + \sup_{y \in \Pi} f_B(y) \geq \sup_{y \in \Pi} f(y) = \operatorname{rk}_{\Pi}(A \cap B) + \operatorname{rk}_{\Pi}(A \cup B). \quad \square$$

Proposition 3.2.9. *A convex polyhedron* Π *in the hyperplane* $y_1 + y_2 + \cdots + y_d = r$ *is a megamatroid polyhedron if and only if* $\Pi = \mathrm{Poly}(\mathrm{rk})$ *for some megamatroid* rk.

Proof. Suppose that Π is a megamatroid polyhedron. Then rk_Π is a megamatroid by Corollary 3.2.8. Clearly we have $\Pi \subseteq \mathrm{Poly}(\mathrm{rk}_\Pi)$. Suppose that $f(y) = \sum_{i=1}^d \alpha_i y_i$ is a linear function on the hyperplane $y_1 + \cdots + y_d = r$. Let σ be a permutation of $[d]$ such that $\alpha_{\sigma(i)} \geq \alpha_{\sigma(j)}$ for $i < j$. Define $X_k = \{\sigma(1), \ldots, \sigma(k)\}$ for $k = 1, 2, \ldots, d$. We can write

$$f(y) = \sum_{j=1}^{d} \beta_j \sum_{i \in X_j} y_i,$$

where $\beta_j := \alpha_{\sigma(j)} - \alpha_{\sigma(j+1)} \geq 0$ for $j = 1, 2, \ldots, d-1$ and $\beta_d = \alpha_{\sigma(d)}$.

By Lemma 3.2.7 we have

$$\sup_{y \in \Pi} f(y) = \sum_{j=1}^{d} \beta_j \, \mathrm{rk}_\Pi(X_j) \geq \sup_{z \in \mathrm{Poly}(\mathrm{rk}_\Pi)} \sum_{j=1}^{d} \beta_j \sum_{i \in X_j} z_i = \sup_{z \in \mathrm{Poly}(\mathrm{rk}_\Pi)} f(z).$$

Since Π is defined by inequalities of the form $f(y) \leq c$, where f is a linear function and $c = \sup_{y \in \Pi} f(y)$, we see that $\mathrm{Poly}(\mathrm{rk}_\Pi) \subseteq \Pi$. We conclude that $\mathrm{Poly}(\mathrm{rk}_\Pi) = \Pi$.

Conversely, suppose that rk is a megamatroid, and that F is a face of $\mathrm{Poly}(\mathrm{rk})$. Choose y in the relative interior of F. Let S_F denote the set of all subsets A of $[d]$ for which $\sum_{i \in A} y_i = \mathrm{rk}(A)$. Note that $\emptyset, [d] \in S_F$. The linear hull of F is given by the equations

$$\sum_{i \in A} y_i = \mathrm{rk}(A), \quad A \in S_F.$$

We claim that S_F is closed under intersections and unions. If $A, B \in S_F$, then we have

$$\begin{aligned} \Big(\sum_{i \in A \cap B} y_i - \mathrm{rk}(A \cap B) \Big) + \Big(\sum_{i \in A \cup B} y_i - \mathrm{rk}(A \cup B) \Big) = \\ = \sum_{i \in A} y_i + \sum_{i \in B} y_i - \mathrm{rk}(A \cap B) - \mathrm{rk}(A \cup B) = \\ \mathrm{rk}(A) + \mathrm{rk}(B) - \mathrm{rk}(A \cap B) - \mathrm{rk}(A \cup B) \geq 0 \end{aligned}$$

by the submodular property. Since $\sum_{i \in A \cap B} y_i - \mathrm{rk}(A \cap B)$ and $\sum_{i \in A \cup B} y_i - \mathrm{rk}(A \cup B)$ are nonpositive, we conclude that $A \cap B, A \cup B \in S_F$ and

$$\mathrm{rk}(A) + \mathrm{rk}(B) = \mathrm{rk}(A \cap B) + \mathrm{rk}(A \cup B).$$

Let us call $A \in S_F$ *prime* if A is nonempty and not the union of two proper subsets in S_F. Let P_F be the set of primes in S_F. If $C = A \cup B$, then

$$\sum_{i \in C} y_i = \mathrm{rk}(C)$$

follows from the equations

$$\sum_{i\in A} y_i = \mathrm{rk}(A), \quad \sum_{i\in B} y_i = \mathrm{rk}(B), \quad \sum_{i\in A\cap B} y_i = \mathrm{rk}(A\cap B).$$

Let $C_1, C_2, \ldots, C_k$ be all prime sets in S_F. It follows that the linear hull of F is defined by all the equations

$$\sum_{i\in C_j} y_i = \mathrm{rk}(C_j), \quad j = 1, 2, \ldots, k.$$

Every element of S_F is a union of some of the C_j's. For every j, let B_j be the largest proper subset of C_j which lies in S_F. Define $A_j = C_j \setminus B_j$ and $r_j = \mathrm{rk}(C_j) - \mathrm{rk}(B_j)$. Then $A_1 \cup \cdots \cup A_k = [d]$ is a partition of $[d]$, and every element of S_F is a union of some of the A_j's. The linear hull of F is defined by the equations

$$\sum_{i\in A_j} y_i = r_j, \quad j = 1, 2, \ldots, k.$$

Clearly, $\mathrm{lhull}(F)$ contains some integral vector $z \in \mathbb{Z}^d$ and $\mathrm{lhull}(F)$ is equal to $z + W$ where W is the space spanned by all $e_i - e_j$ where i, j are such that $i, j \in A_k$ for some k. □

3.3 The valuative property

There are essentially two definitions of the valuative property in the literature, which we will refer to as the *strong valuative* and the *weak valuative* properties. The equivalence of these definitions is shown in [41] and [91] when valuations are defined on sets of polyhedra closed under intersection. In this section we will show that the two definitions are equivalent for valuations defined on megamatroid polytopes. Note that the class of megamatroid polytopes is not closed under intersection: see Example 1.2.1.

Definition 3.3.1. A *megamatroid polyhedron decomposition* is a decomposition

$$\Pi = \Pi_1 \cup \Pi_2 \cup \cdots \cup \Pi_k$$

such that $\Pi, \Pi_1, \ldots, \Pi_k$ are megamatroid polyhedra, and $\Pi_i \cap \Pi_j$ is empty or contained in a proper face of Π_i and of Π_j for all $i \neq j$.

Let $S_{\mathrm{MM}}(d, r)$ be the set of megamatroids on $[d]$ of rank r. Let $Z_{\mathrm{MM}}(d, r)$ be the $\mathbb{Z}$-module whose basis is given by all $\langle \mathrm{rk} \rangle$ where $\mathrm{rk} \in S_{\mathrm{MM}}(d, r)$.

For a megamatroid polyhedron decomposition

$$\Pi = \Pi_1 \cup \Pi_2 \cup \cdots \cup \Pi_k$$

we define $\Pi_I = \bigcap_{i\in I} \Pi_i$ if $I \subseteq \{1, 2, \ldots, k\}$. We will use the convention that $\Pi_\emptyset = \Pi$. Define

$$m_{\mathrm{val}}(\Pi; \Pi_1, \ldots, \Pi_k) = \sum_{I\subseteq\{1,2,\ldots,k\}} (-1)^{|I|} m_I \in Z_{\mathrm{MM}}(d, r),$$

where $m_I = \langle \mathrm{rk}^I \rangle$ if rk^I is the megamatroid with $\mathrm{Poly}(\mathrm{rk}^I) = \Pi_I$, and $m_I = 0$ if $\Pi_I = \emptyset$. We also define

$$m_{\mathrm{coval}}(\Pi; \Pi_1, \ldots, \Pi_k) = \langle \mathrm{rk}_\Pi \rangle - \sum_F \langle \mathrm{rk}_F \rangle.$$

where F runs over all interior faces of the decomposition.

Definition 3.3.2. A homomorphism of abelian groups $f : Z_{\mathrm{MM}}(d, r) \to A$ is called *weakly valuative*, if for every megamatroid polyhedron decomposition

$$\Pi = \Pi_1 \cup \Pi_2 \cup \cdots \cup \Pi_k$$

we have $f(m_{\mathrm{val}}(\Pi; \Pi_1, \ldots, \Pi_k)) = 0$. We say it is *weakly covaluative*, if for every megamatroid polyhedron decomposition

$$\Pi = \Pi_1 \cup \Pi_2 \cup \cdots \cup \Pi_k$$

we have $f(m_{\mathrm{coval}}(\Pi; \Pi_1, \ldots, \Pi_k)) = 0$.

We define a group homomorphism

$$E : Z_{\mathrm{MM}}(d, r) \to Z_{\mathrm{MM}}(d, r)$$

by

$$E(\langle \mathrm{rk} \rangle) = \sum_F \langle \mathrm{rk}_F \rangle$$

where F runs over all faces of $\mathrm{Poly}(\mathrm{rk})$ and rk_F is the megamatroid with $\mathrm{Poly}(\mathrm{rk}_F) = F$. For a polytope Π, we denote the set of faces of Π by $\mathrm{face}(\Pi)$.

Lemma 3.3.3. *The homomorphism $f : Z_{\mathrm{MM}}(d, r) \to A$ of abelian groups is weakly valuative if and only if $f \circ E$ is weakly covaluative.*

Proof. We have

$$E\big(m_{\mathrm{val}}(\Pi; \Pi_1, \ldots, \Pi_k)\big) = \sum_{I\subseteq\{1,2,\ldots,k\}} (-1)^{|I|} E(m_I) =$$
$$= \sum_{I\subseteq\{1,2,\ldots,k\}} (-1)^{|I|} \sum_{F\in\mathrm{face}(\Pi_I)} \langle \mathrm{rk}_F \rangle = \sum_F \langle \mathrm{rk}_F \rangle \sum_{\substack{I\subseteq\{1,2,\ldots,k\};\\ F\in\mathrm{face}(\Pi_I)}} (-1)^{|I|}. \quad (3.3.1)$$

Let $J(F)$ be the set of all indices i such that F is a face of Π_i. Suppose that F is a face of Π. Then $J(F) = \emptyset$ if and only if $F = \Pi$. We have

$$\sum_{\substack{I\subseteq\{1,2,\ldots,k\};\\ F\in\mathrm{face}(\Pi_I)}} (-1)^{|I|} = \sum_{I\subseteq J(F)} (-1)^{|I|} = \begin{cases} 1 & \text{if } F = \Pi; \\ 0 & \text{if } F \neq \Pi. \end{cases}$$

If F is an interior face, then $J(F) \neq \emptyset$ and

$$\sum_{\substack{I\subseteq\{1,2,\ldots,k\};\\ F\in\mathrm{face}(\Pi_I)}} (-1)^{|I|} = \sum_{I\subseteq J(F); I\neq\emptyset} (-1)^{|I|} = -1.$$

We conclude that

$$E(m_{\mathrm{val}}(\Pi; \Pi_1, \ldots, \Pi_k)) = \langle \mathrm{rk}_\Pi \rangle - \sum_F \langle \mathrm{rk}_F \rangle = m_{\mathrm{coval}}(\Pi; \Pi_1, \ldots, \Pi_k)$$

where the sum is over all interior faces F. The lemma follows. □

For a polyhedron Π in $\mathbb{R}^d$, let $\mathbf{1}(\Pi)$ denote its indicator function. Define $P_{\mathrm{MM}}(d, r)$ as the $\mathbb{Z}$-module generated by all $\mathbf{1}(\mathrm{Poly}(\mathrm{rk}))$, where rk lies in $S_{\mathrm{MM}}(d, r)$.

There is a natural $\mathbb{Z}$-module homomorphism

$$\mathbf{1}_{\mathrm{MM}} : Z_{\mathrm{MM}}(d, r) \to P_{\mathrm{MM}}(d, r)$$

such that

$$\mathbf{1}_{\mathrm{MM}}(\langle \mathrm{rk} \rangle) = \mathbf{1}(\mathrm{Poly}(\mathrm{rk}))$$

for all $\mathrm{rk} \in S_{\mathrm{MM}}(d, r)$.

Definition 3.3.4. A homomorphism of groups $f : Z_{\mathrm{MM}}(d, r) \to A$ is *strongly valuative* if there exists a group homomorphism $\widehat{f} : P_{\mathrm{MM}}(d, r) \to A$ such that $f = \widehat{f} \circ \psi_{\mathrm{MM}}$.

Suppose that $\Pi = \Pi_1 \cup \cdots \cup \Pi_k$ is a megamatroid decomposition. Then by the inclusion-exclusion principle, we have

$$\mathbf{1}_{\mathrm{MM}}(m_{\mathrm{val}}(\Pi; \Pi_1, \ldots, \Pi_k)) = \mathbf{1}_{\mathrm{MM}}\Big(\sum_{I\subseteq\{1,2,\ldots,k\}} (-1)^{|I|} m_I \Big) =$$

$$= \sum_{I\subseteq\{1,2,\ldots,k\}} (-1)^{|I|} \prod_{i\in I} \mathbf{1}(\Pi_i) = \sum_{I\subseteq\{1,2,\ldots,k\}} (-1)^{|I|} \mathbf{1}(\Pi_I) = \prod_{i=1}^{k} (\mathbf{1}(\Pi) - \mathbf{1}(\Pi_i)) = 0.$$

This shows that every homomorphism $f : Z_{\mathrm{MM}}(d, r) \to A$ of abelian groups with the strong valuative property has the weak valuative property. In fact the two valuative properties are equivalent by the following theorem:

Theorem 3.3.5. *A homomorphism $f : Z_{\mathrm{MM}}(d,r) \to A$ of abelian groups is weakly valuative if and only if it is strongly valuative.*

The proof of Theorem 3.3.5 is in Appendix 3.A. In view of this theorem, we will from now on just refer to the valuative property when we mean the weak or the strong valuative property.

For a megamatroid polytope Π, let Π° be the relative interior of Π. Define a homomorphism $\mathbf{1}^\circ_{\mathrm{MM}} : Z_{\mathrm{MM}}(d,r) \to P_{\mathrm{MM}}(d,r)$ by $\mathbf{1}^\circ_{\mathrm{MM}}(\langle \mathrm{rk} \rangle) = \mathbf{1}(Q^\circ(\mathrm{rk}))$.

Definition 3.3.6. Suppose that $f : Z_{\mathrm{MM}}(d,r) \to A$ is a homomorphism of abelian groups. We say that f is *strongly covaluative* if f factors through $\mathbf{1}^\circ_{\mathrm{MM}}$, i.e., there exists a group homomorphism $\widehat{f}$ such that $f = \widehat{f} \circ \mathbf{1}^\circ_{\mathrm{MM}}$.

Corollary 3.3.7. *A homomorphism $f : Z_{\mathrm{MM}}(d,r) \to A$ of abelian groups is weakly covaluative if and only if it is strongly covaluative.*

Proof. If $\Pi = \Pi_1 \cup \cdots \cup \Pi_k$ is a megamatroid polytope decomposition, then

$$\mathbf{1}^\circ_{\mathrm{MM}}(m_{\mathrm{coval}}(\Pi; \Pi_1, \ldots, \Pi_k)) = \mathbf{1}^\circ_{\mathrm{MM}}(\langle \mathrm{rk}_\Pi \rangle) - \sum_F \mathbf{1}^\circ_{\mathrm{MM}}(\langle \mathrm{rk}_F \rangle) = \mathbf{1}(\Pi^\circ) - \sum_F \mathbf{1}(F^\circ) = 0,$$

where F runs over all interior faces. This shows that if f has the strong covaluative property, then it has the weak covaluative property.

It is easy to verify that $\mathbf{1}^\circ_{\mathrm{MM}} \circ E = \mathbf{1}_{\mathrm{MM}}$. Suppose that f is weakly covaluative. By Lemma 3.3.3, $f \circ E^{-1}$ is weakly valuative. By Theorem 3.3.5, $f \circ E^{-1}$ is strongly valuative, so $f \circ E^{-1} = \widehat{f} \circ \mathbf{1}_{\mathrm{MM}}$ for some group homomorphism $\widehat{f}$, and $f = \widehat{f} \circ \mathbf{1}_{\mathrm{MM}} \circ E = \widehat{f} \circ \mathbf{1}^\circ_{\mathrm{MM}}$. This implies that f is strongly covaluative. □

Definition 3.3.8. Suppose that $d \geq 1$. A valuative group homomorphism $f : Z_{\mathrm{MM}}(d,r) \to A$ is *additive* if $f(\langle \mathrm{rk} \rangle) = 0$ for all megamatroids $([d], \mathrm{rk})$ for which Poly(rk) has dimension $< d-1$.

If $f : Z_{\mathrm{MM}}(d,r) \to A$ is additive, then for a megamatroid polyhedron decomposition

$$\Pi = \Pi_1 \cup \Pi_2 \cup \cdots \cup \Pi_k$$

we have

$$f(\mathrm{rk}_\Pi) = \sum_{i=1}^{k} f(\langle \mathrm{rk}_{\Pi_i} \rangle).$$

A megamatroid polyhedron decomposition $\Pi = \Pi_1 \cup \cdots \cup \Pi_k$ is a *(poly)matroid polytope decomposition* if $\Pi, \Pi_1, \ldots, \Pi_k$ are (poly)matroid polytopes. Let $S_{(\mathrm{P})\mathrm{M}}(d,r)$ be the set of (poly)matroids, and let $Z_{(\mathrm{P})\mathrm{M}}(d,r)$ be the free abelian group generated by $S_{(\mathrm{P})\mathrm{M}}(d,r)$. We

say that $f : Z_{(\mathrm{P})\mathrm{M}}(d,r) \to A$ has the weak valuative property if $f(m_{\mathrm{val}}(\Pi;\Pi_1,\ldots,\Pi_k)) = 0$ for every (poly)matroid polytope decomposition. We define the weak covaluative property for such homomorphisms f in a similar manner. The group homomorphism $E : Z_{\mathrm{MM}}(d,r) \to Z_{\mathrm{MM}}(d,r)$ restricts to homomorphisms $Z_{(\mathrm{P})\mathrm{M}}(d,r) \to Z_{(\mathrm{P})\mathrm{M}}(d,r)$. A group homomorphism $f : Z_{(\mathrm{P})\mathrm{M}}(d,r) \to A$ is weakly valuative if and only if $f \circ E$ is weak covaluative. Let $P_{(\mathrm{P})\mathrm{M}}(d,r) = \mathbf{1}_{\mathrm{MM}}(Z_{(\mathrm{P})\mathrm{M}}(d,r))$ and define $\mathbf{1}_{(\mathrm{P})\mathrm{M}} : Z_{(\mathrm{P})\mathrm{M}}(d,r) \to P_{(\mathrm{P})\mathrm{M}}(d,r)$ as the restrictions of $\mathbf{1}_{\mathrm{MM}}$. A homomorphism $f : Z_{(\mathrm{P})\mathrm{M}}(d,r) \to A$ is strongly valuative if and only if it factors through $\mathbf{1}_{(\mathrm{P})\mathrm{M}}$.

Corollary 3.3.9. *A homomorphism $f : Z_{(\mathrm{P})\mathrm{M}}(d,r) \to A$ is weakly valuative if and only if it is strongly valuative.*

Proof. We need to show that $\ker \mathbf{1}_{(\mathrm{P})\mathrm{M}}(d,r) = W_{(\mathrm{P})\mathrm{M}}(d,r)$. Clearly $W_{(\mathrm{P})\mathrm{M}}(d,r) \subseteq \ker \mathbf{1}_{(\mathrm{P})\mathrm{M}}(d,r)$. By Theorem 3.3.5, we have that $\ker \mathbf{1}_{\mathrm{MM}}(d,r) = W_{\mathrm{MM}}(d,r)$, so $\ker \mathbf{1}_{(\mathrm{P})\mathrm{M}}(d,r) = W_{\mathrm{MM}}(d,r) \cap Z_{(\mathrm{P})\mathrm{M}}(d,r)$. Define $\pi_{(\mathrm{P})\mathrm{M}} : Z_{\mathrm{MM}}(d,r) \to Z_{(\mathrm{P})\mathrm{M}}(d,r)$ by $\pi_{(\mathrm{P})\mathrm{M}}(\langle \mathrm{rk} \rangle) = \langle \mathrm{rk}' \rangle$ where $\mathrm{Poly}(\mathrm{rk}') = \mathrm{Poly}(\mathrm{rk}) \cap \Delta_{(\mathrm{P})\mathrm{M}}(d,r)$ if this intersection is nonempty and $\pi_{(\mathrm{P})\mathrm{M}}(\langle \mathrm{rk} \rangle) = 0$ otherwise. Note that $\pi_{(\mathrm{P})\mathrm{M}}$ is a projection of $Z_{\mathrm{MM}}(d,r)$ onto $Z_{(\mathrm{P})\mathrm{M}}(d,r)$. We have

$$\pi_{(\mathrm{P})\mathrm{M}}(m_{\mathrm{val}}(\Pi;\Pi_1,\ldots,\Pi_k)) = m_{\mathrm{val}}(\Pi \cap \Delta; \Pi_1 \cap \Delta, \ldots, \Pi_k \cap \Delta) \in W_{(\mathrm{P})\mathrm{M}}(d,r),$$

where $\Delta = \Delta_{(\mathrm{P})\mathrm{M}}(d,r)$. This shows that $\pi_{(\mathrm{P})\mathrm{M}}(W_{\mathrm{MM}}(d,r)) \subseteq W_{(\mathrm{P})\mathrm{M}}(d,r)$. We conclude that

$$\ker \mathbf{1}_{(\mathrm{P})\mathrm{M}}(d,r) = W_{\mathrm{MM}}(d,r) \cap Z_{(\mathrm{P})\mathrm{M}}(d,r) \subseteq \pi_{(\mathrm{P})\mathrm{M}}(W_{\mathrm{MM}}(d,r)) \subseteq W_{(\mathrm{P})\mathrm{M}}(d,r). \quad \square$$

The strong covaluative property for a group homomorphism $f : Z_{(\mathrm{P})\mathrm{M}}(d,r) \to A$ can also be defined. The proof of Corollary 3.3.7 generalizes to (poly)matroids and f is weakly covaluative if and only if f is strongly covaluative.

3.4 Decompositions into cones

A chain of length k in $[d]$ is

$$\underline{X} : \emptyset \subset X_1 \subset \cdots \subset X_{k-1} \subset X_k = [d]$$

(here $\subset$ denotes proper inclusion). We will write $\ell(\underline{X}) = k$ for the length of such a chain. If $d > 0$ then every chain has length ≥ 1, but for $d = 0$ there is exactly one chain, namely

$$\emptyset = [0]$$

and this chain has length 0. For a chain $\underline{X}$ of length k and a k-tuple $\underline{r} = (r_1, r_2, \ldots, r_k) \in (\mathbb{Z} \cup \{\infty\})^k$, we define a megamatroid polyhedron

$$R_{\mathrm{MM}}(\underline{X}, \underline{r}) = \Big\{ (y_1,\ldots,y_d) \in \mathbb{R}^d \;\Big|\; \sum_{i=1}^{d} y_i = r_k,\ \forall j \ \sum_{i \in X_j} y_i \leq r_j \Big\}.$$

We will always use the conventions $r_0 = 0$, $X_0 = \emptyset$. The megamatroid $\mathrm{rk}_{\underline{X},\underline{r}}$ is defined by $\mathrm{Poly}(\mathrm{rk}_{\underline{X},\underline{r}}) = R_{\mathrm{MM}}(\underline{X}, \underline{r})$.

For a megamatroid rk and a chain $\underline{X}$ of length k we define

$$R_{\mathrm{MM}}(\underline{X}, \mathrm{rk}) = R_{\mathrm{MM}}(\underline{X}, (\mathrm{rk}(X_1), \mathrm{rk}(X_2), \ldots, \mathrm{rk}(X_k))).$$

Suppose that Π is a polyhedron in $\mathbb{R}^d$ defined by $g_i(y_1, \ldots, y_d) \leq c_i$ for $i = 1, 2, \ldots, n$, where $g_i : \mathbb{R}^d \to \mathbb{R}$ is linear and $c_i \in \mathbb{R}$. For every face F of Π, the tangent cone Cone_F of F is defined by the inequalities

$$g_i(y_1, \ldots, y_d) \leq c_i$$

for all i for which the restriction of g_i to F is constant and equal to c_i.

Theorem 3.4.1 (Brianchon-Gram Theorem [15, 40])**.** *We have the following equality*

$$\mathbf{1}(\Pi) = \sum_F (-1)^{\dim F} \mathbf{1}(\mathrm{Cone}_F)$$

where F runs over all the bounded faces of Π.

For a proof, see [61].

Theorem 3.4.2. *For any megamatroid* $\mathrm{rk} : 2^{[d]} \to \mathbb{Z} \cup \{\infty\}$ *we have*

$$\mathbf{1}(\mathrm{Poly}(\mathrm{rk})) = \sum_{\underline{X}} (-1)^{d-\ell(\underline{X})} \mathbf{1}(R_{\mathrm{MM}}(\underline{X}, \mathrm{rk})).$$

Proof. Assume first that $\mathrm{rk}(X)$ is finite for all $X \subseteq [d]$. We define a convex polyhedron $Q_\varepsilon(\mathrm{rk})$ by the inequalities

$$\sum_{i \in A} y_i \leq \mathrm{rk}(A) + \varepsilon(d^2 - |A|^2)$$

for all $A \subseteq [d]$ and the equality $y_1 + \cdots + y_d = r$, where $r = \mathrm{rk}([d])$.

Faces of $Q_\varepsilon(\mathrm{rk})$ are given by intersecting $Q_\varepsilon(\mathrm{rk})$ with hyperplanes of the form

$$H_A = \Big\{(y_1, \ldots, y_d) \in \mathbb{R}^d \mid \sum_{i \in A} y_i = \mathrm{rk}(A) + \varepsilon(d^2 - |A|^2)\Big\}.$$

If $A, B \subseteq [d]$, and A and B are incomparable, and $(y_1, \ldots, y_d) \in H_A \cap H_B \cap Q_\varepsilon(\mathrm{rk})$, then

$$\begin{aligned}
\sum_{i \in A} y_i + \sum_{i \in B} y_i &= \mathrm{rk}(A) + \mathrm{rk}(B) + \varepsilon((d^2 - |A|^2) + (d^2 - |B|^2)) \\
&> \mathrm{rk}(A) + \mathrm{rk}(B) + \varepsilon((d^2 - |A \cup B|^2) + (d^2 - |A \cap B|^2)) \\
&\geq \mathrm{rk}(A \cup B) + \mathrm{rk}(A \cap B) + \varepsilon((d^2 - |A \cup B|^2) + (d^2 - |A \cap B|^2)) \\
&\geq \sum_{i \in A \cup B} y_i + \sum_{i \in A \cap B} y_i = \sum_{i \in A} y_i + \sum_{i \in B} y_i
\end{aligned}$$

This contradiction shows that $H_A \cap H_B \cap Q_\varepsilon(\mathrm{rk}) = \emptyset$. It follows that all faces are of the form

$$F_\varepsilon(\underline{X}) = Q_\varepsilon(\mathrm{rk}) \cap H_{X_1} \cap \cdots \cap H_{X_{k-1}}$$

where $k \geq 1$ and

$$X_0 = \emptyset \subset X_1 \subset \cdots \subset X_{k-1} \subset X_k = [d].$$

Also, all these faces are distinct.

Let us view $Q_\varepsilon(\mathrm{rk})$ as a bounded polytope in the hyperplane $y_1 + y_2 + \cdots + y_d = r$. For a face $F_\varepsilon(\underline{X})$, its tangent cone $\mathrm{Cone}_{F_\varepsilon(\underline{X})}$ is defined by the inequalities

$$\sum_{i \in X_j} y_i \leq \mathrm{rk}(X_j) + \varepsilon(d^2 - |X_j|^2)$$

(and the equality $\sum_{i=1}^d y_i = r$). If $\underline{X}$ has length k, then the dimension of $F_\varepsilon(\underline{X})$ is $d - k$. Theorem 3.4.1 implies that

$$\mathbf{1}(Q_\varepsilon(\mathrm{rk})) = \sum_{\underline{X}} (-1)^{d-\ell(\underline{X})} \mathbf{1}(\mathrm{Cone}_{F_\varepsilon(\underline{X})}).$$

When we take the limit $\varepsilon \downarrow 0$, then $\mathbf{1}(Q_\varepsilon(\mathrm{rk}))$ converges pointwise to $\mathbf{1}(\mathrm{Poly}(\mathrm{rk}))$, and $\mathbf{1}(\mathrm{Cone}_{F_\varepsilon(\underline{X})})$ converges pointwise to $\mathbf{1}(R_{\mathrm{MM}}(\underline{X}, \mathrm{rk}))$.

Finally, for a general polymatroid rk, we have $\mathrm{rk} = \lim_{N\to\infty} \mathrm{rk}^N$, where rk^N is as in the proof of Lemma 3.2.3, and rk^N has all ranks finite, and likewise

$$\lim_{N\to\infty} \mathbf{1}(R_{MM}(\underline{X}, \mathrm{rk}^N)) = \mathbf{1}(R_{MM}(\underline{X}, \mathrm{rk})).$$

So the result follows by taking limits. □

Example 3.4.3. To illustrate the proof of Theorem 3.4.2, consider the case where $d = 3$ and $r = 3$, and rk is defined by $\mathrm{rk}(\{1\}) = \mathrm{rk}(\{2\}) = \mathrm{rk}(\{3\}) = 2$, $\mathrm{rk}(\{1,2\}) = \mathrm{rk}(\{2,3\}) = \mathrm{rk}\{(1,3\}) = 3$, $\mathrm{rk}(\{1,2,3\}) = 4$. The decomposition of $Q_\varepsilon(\mathrm{rk})$ using the Brianchon-Gram theorem is depicted in Figure 3.1. Note how the summands in the decomposition correspond to the faces of $Q_\varepsilon(\mathrm{rk})$. The dashed triangle is the triangle defined by $y_1, y_2, y_3 \geq 0$, $y_1 + y_2 + y_3 = 4$. Instead of getting cones in the decomposition, we get polygons because we intersect with this triangle.

In the limit where ε approaches 0 we obtain Figure 3.2. This is exactly the decomposition in Theorem 3.4.2. In this decomposition, the summands do not correspond to the faces of Poly(rk). ♢

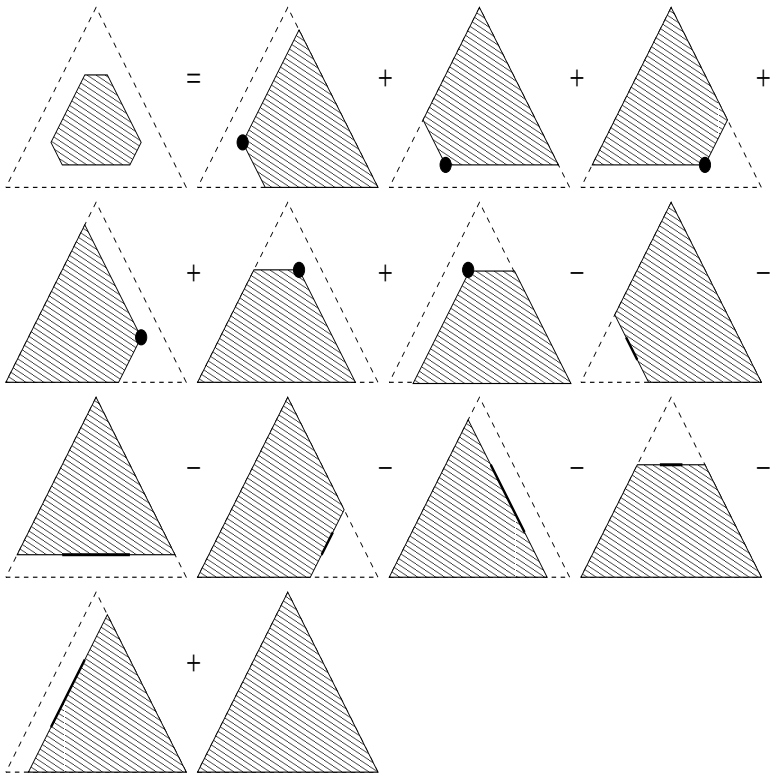


Figure 3.1: A decomposition of $Q_\epsilon(\mathrm{rk})$, as in Theorem 3.4.2.

3.5 Valuative functions: the groups $P_{\mathrm{M}}, P_{\mathrm{PM}}, P_{\mathrm{MM}}$

Lemma 3.5.1. *The function $\chi : Z_{MM}(d,r) \to \mathbb{Z}$ such that $\chi(\langle \mathrm{rk} \rangle) = 1$ for every megamatroid* rk *has the valuative property.*

Proof. Let

$$\Pi = \Pi_1 \cup \Pi_2 \cup \cdots \cup \Pi_k$$

be a megamatroid polyhedron decomposition. By Rota's crosscut theorem [76],

$$\chi(m_{\mathrm{val}}(\Pi; \Pi_1, \ldots, \Pi_k)) = \sum_F \mu(\Pi, F) = 0,$$

where F runs over the faces of the decomposition, and μ is the Möbius function. □

Lemma 3.5.2. *Let $H \subseteq \mathbb{R}^d$ be a closed halfspace. Define $j_H : Z_{MM}(d,r) \to \mathbb{Z}$ by*

$$j_H(\langle \mathrm{rk} \rangle) = \begin{cases} 1 & \text{if } \mathrm{Poly}(\mathrm{rk}) \subseteq H, \\ 0 & \text{otherwise.} \end{cases}$$

Then j_H is valuative.

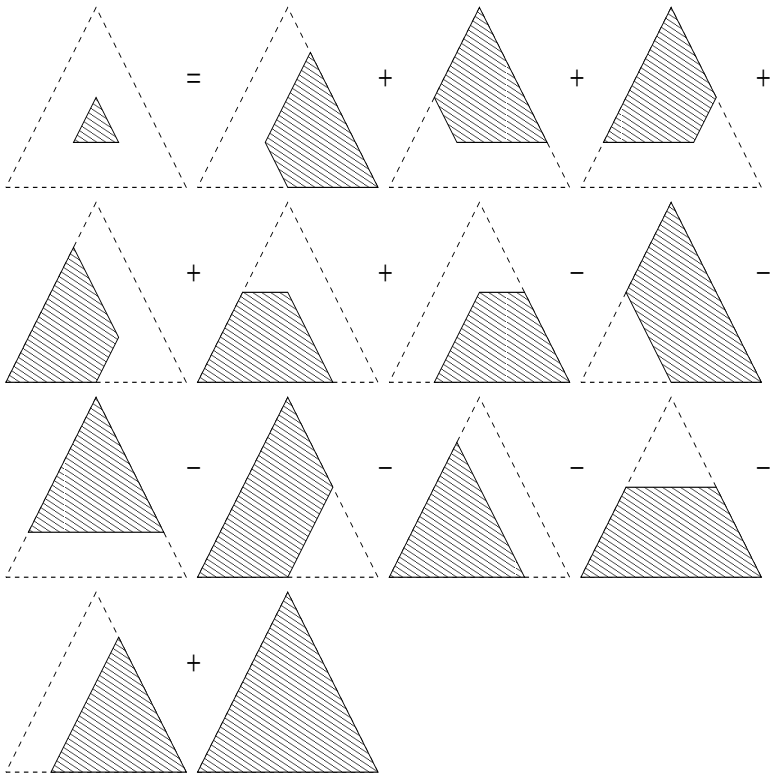


Figure 3.2: The limiting decomposition of Poly(rk) corresponding to Figure 3.1.

Proof. Let

$$\Pi = \Pi_1 \cup \Pi_2 \cup \cdots \cup \Pi_k$$

be a megamatroid polyhedron decomposition. The intersections of the faces of this decomposition with $\mathbb{R}^d \setminus H$ establish a regular cell complex structure on $\Pi \setminus H$, and a face F of the decomposition meets $\mathbb{R}^d \setminus H$ if and only if $(\chi - j_H)(\mathrm{rk}_F) = 1$. It follows that $\chi - j_H$ is valuative, by the argument of the previous proof applied to this complex. □

Lemma 3.5.2 can also be deduced from the fact that the indicator function of the polar dual has the valuative property (see [57]).

Suppose that $\underline{X}$ is a chain of length k and $\underline{r} = (r_1, \ldots, r_k)$ is an integer vector with $r_k = r$. Define a homomorphism $s_{\underline{X},\underline{r}} : Z_{\mathrm{MM}}(d, r) \to \mathbb{Z}$ by

$$s_{\underline{X},\underline{r}}(\mathrm{rk}) = \begin{cases} 1 & \text{if } \mathrm{rk}(X_j) = r_j \text{ for } j = 1, 2, \ldots, k, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.5.3. *The homomorphism $s_{\underline{X},\underline{r}}$ is valuative.*

Proof. For $\varepsilon > 0$, define the halfplane $H_1(\varepsilon)$ by the inequality

$$\sum_{j=1}^{k} \varepsilon^{j-1} \sum_{i \in X_j} y_i \leq \sum_{j=1}^{k} \varepsilon^{j-1} r_j$$

and define $H_2(\varepsilon)$ by

$$\sum_{j=1}^{k} \varepsilon^{j-1} \sum_{i \in X_j} y_i \leq \sum_{j=1}^{k} \varepsilon^{j-1} r_j - \varepsilon^k.$$

By Lemma 3.2.7 and Lemma 3.5.2, $(j_{H_1(\varepsilon)} - j_{H_2(\varepsilon)})(\mathrm{rk}) = 1$ if and only if

$$\sum_{j=1}^{k} \varepsilon^{j-1} r_j - \varepsilon^k < \sum_{j=1}^{k} \varepsilon^{j-1} \operatorname{rk}(X_j) = \max_{y \in \mathrm{Poly(rk)}} \sum_{j=1}^{k} \varepsilon^{j-1} \sum_{i \in X_j} y_i \leq \sum_{j=1}^{k} \varepsilon^{j-1} r_j \tag{3.5.1}$$

If (3.5.1) holds for arbitrary small ε, then it is easy to see (by induction on j) that $\mathrm{rk}(X_j) = r_j$ for $j = 1, 2, \ldots, k$. From this follows that $\lim_{\varepsilon \to 0} j_{H_1(\varepsilon)} - j_{H_2(\varepsilon)} = s_{\underline{X},\underline{r}}$. So $s_{\underline{X},\underline{r}}$ is valuative. □

Suppose that $d \geq 1$. Let $\mathfrak{p}_{\mathrm{MM}}(d, r)$ be the set of all pairs $(\underline{X}, \underline{r})$ such that $\underline{X}$ is a chain of length k $(1 \leq k \leq d)$ and $\underline{r} = (r_1, r_2, \ldots, r_k)$ is an integer vector with $r_k = r$. We define $R_{\mathrm{(P)M}}(\underline{X}, \underline{r}) = R_{\mathrm{MM}}(\underline{X}, \underline{r}) \cap \Delta_{\mathrm{(P)M}}(d, r)$. If $R_{\mathrm{(P)M}}(\underline{X}, \underline{r})$ is nonempty, then it is a (poly)matroid base polytope. Define $\mathfrak{p}_{\mathrm{PM}}(d, r) \subseteq \mathfrak{p}_{\mathrm{MM}}(d, r)$ as the set of all pairs $(\underline{X}, \underline{r})$ with $0 \leq r_1 < \cdots < r_k = r$. Let $\mathfrak{p}_{\mathrm{M}}(d, r)$ denote the set of all pairs $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{MM}}(d, r)$ such that $\underline{r} = (r_1, \ldots, r_k)$ for some k $(1 \leq k \leq d)$,

$$0 \leq r_1 < r_2 < \cdots < r_k = r$$

and

$$0 < |X_1| - r_1 < |X_2| - r_2 < \cdots < |X_{k-1}| - r_{k-1} \leq |X_k| - r_k = d - r.$$

For $d = 0$, we define $\mathfrak{p}_{\mathrm{MM}}(0, r) = \mathfrak{p}_{\mathrm{PM}}(0, r) = \mathfrak{p}_{\mathrm{M}}(0, r) = \emptyset$ for $r \neq 0$ and $\mathfrak{p}_{\mathrm{MM}}(0, 0) = \mathfrak{p}_{\mathrm{PM}}(0, 0) = \mathfrak{p}_{\mathrm{M}}(0, 0) = \{(\emptyset \subseteq [0], ())\}$.

Theorem 3.5.4. *The group $P_{*\mathrm{M}}(d, r)$ is freely generated by the basis*

$$\big\{\mathbf{1}(R_{*\mathrm{M}}(\underline{X}, \underline{r})) \;\big|\; (\underline{X}, \underline{r}) \in \mathfrak{p}_{*\mathrm{M}}(d, r)\big\}.$$

Proof. The case $d = 0$ is easy, so assume that $d \geq 1$.

For megamatroids. If rk is a megamatroid, then $\mathbf{1}(\mathrm{Poly(rk)})$ is an integral combination of functions $\mathbf{1}(R_{\mathrm{MM}}(\underline{X}, \underline{r}))$, $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{MM}}(d, r)$ by Theorem 3.4.2. This shows that $\mathbf{1}(R_{\mathrm{MM}}(\underline{X}, \underline{r}))$, $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{MM}}(d, r)$ generate $P_{\mathrm{MM}}(d, r)$. If $s_{\underline{X},\underline{r}}(R_{\mathrm{MM}}(\underline{X}', \underline{r}')) \neq 0$ then $\mathrm{rk}_{\underline{X}',\underline{r}'}(X_j) = r_j$ for all j, and $R_{\mathrm{MM}}(\underline{X}', \underline{r}') \subseteq R_{\mathrm{MM}}(\underline{X}, \underline{r})$. Suppose that

$$\sum_{i=1}^{k} a_i \mathbf{1}(R_{\mathrm{MM}}(\underline{X}^{(i)}, \underline{r}^{(i)})) = 0$$

with $k \geq 1$, $a_1, \ldots, a_k$ nonzero integers, and $(\underline{X}^{(i)}, \underline{r}^{(i)})$, $i = 1, 2, \ldots, k$ distinct. Without loss of generality we may assume that $R_{\mathrm{MM}}(\underline{X}^{(1)}, \underline{r}^{(1)})$ does not contain $R_{\mathrm{MM}}(\underline{X}^{(i)}, \underline{r}^{(i)})$ for any $i > 1$. We have

$$0 = s_{\underline{X}^{(1)}, \underline{r}^{(1)}}\Big(\sum_{i=1}^{k} a_i R_{\mathrm{MM}}(\underline{X}^{(i)}, \underline{r}^{(i)})\Big) = a_1.$$

Contradiction.

For polymatroids. It is clear that $P_{\mathrm{PM}}(d, r)$ is generated by all $\mathbf{1}(R_{\mathrm{PM}}(\underline{X}, \underline{r}))$, with $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{MM}}(d, r)$. If $r_1 < 0$ then $R_{\mathrm{PM}}(\underline{X}, \underline{r})$ is empty. Suppose that $r_{i+1} \leq r_i$. It is obvious that

$$R_{\mathrm{PM}}(\underline{X}, \underline{r}) = R_{\mathrm{PM}}(\underline{X}', \underline{r}')$$

where

$$\underline{X}' : \emptyset = X_0 \subset X_1 \subset \cdots \subset X_{i-1} \subset X_{i+1} \subset \cdots \subset X_k = [d]$$

and

$$\underline{r}' = (r_1, r_2, \ldots, r_{i-1}, r_{i+1}, \ldots, r_k).$$

Therefore, $P_{\mathrm{PM}}(d, r)$ is generated by all $\mathbf{1}(R_{\mathrm{PM}}(\underline{X}, \underline{r}))$ where $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{PM}}(d, r)$. If $\Pi = R_{\mathrm{PM}}(\underline{X}, \underline{r})$ with $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{PM}}(d, r)$, then $(\underline{X}, \underline{r})$ is completely determined by the polytope Π. For $1 \leq i \leq d$, define $a_i = \max\{y_i \mid y \in \Pi\}$. Then $\underline{r}$ is determined by $0 \leq r_1 < \cdots < r_k$ and

$$\{r_1, \ldots, r_k\} = \{a_1, \ldots, a_d\}.$$

The sets X_j, $j = 1, 2, \ldots, k$ are determined by $X_j = \{i \mid a_i \leq r_j\}$. This shows that the polytopes $R_{\mathrm{PM}}(\underline{X}, \underline{r})$, $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{PM}}(d, r)$ are distinct. A similar argument as in the megamatroid case shows that $\mathbf{1}(R_{\mathrm{PM}}(\underline{X}, \underline{r}))$, $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{PM}}(d, r)$ are linearly independent.

For matroids. From the polymatroid case it follows that $P_{\mathrm{M}}(d, r)$ is generated by all $\mathbf{1}(R_{\mathrm{M}}(\underline{X}, \underline{r}))$, where $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{PM}}(d, r)$. Suppose that $|X_{i-1}| - r_{i-1} \geq |X_i| - r_i$ for some i with $1 \leq i \leq k$ (with the convention that $r_0 = 0$). Then we have

$$\mathbf{1}(R_{\mathrm{M}}(\underline{X}, \underline{r})) = \mathbf{1}(R_{\mathrm{M}}(\underline{X}', \underline{r}'))$$

where

$$\underline{X}' : \emptyset = X_0 \subset X_1 \subset \cdots \subset X_{i-1} \subset X_{i+1} \subset \cdots \subset X_k = [d].$$

and

$$\underline{r} = (r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_k).$$

This shows that $P_{\mathrm{M}}(d, r)$ is generated by all $\mathbf{1}(R_{\mathrm{M}(\underline{X},\underline{r})})$ where $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{M}}(d, r)$. If $\Pi = R_{\mathrm{M}}(\underline{X}, \underline{r})$ with $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{M}}(d, r)$, then $(\underline{X}, \underline{r})$ is completely determined by the polytope Π. Note that $\mathrm{rk}_\Pi(A) = \min_j\{\mathrm{rk}_\Pi(X_j) + |A| - |A \cap X_j|\}$. If $\emptyset \subset A \subset [d]$ then $A = X_j$ for some j if and only if $\mathrm{rk}_\Pi(A) < \mathrm{rk}_\Pi(B)$ for all B with $A \subset B \subseteq [d]$ and $|A| - \mathrm{rk}_\Pi(A) > |B| - \mathrm{rk}_\Pi(B)$ for all B with $\emptyset \subseteq B \subset A$. So $X_1, \ldots, X_k$ are determined by Π, and $r_i = \mathrm{rk}_\Pi(X_j)$, $j = 1, 2, \ldots, k$ are determined as well. This shows that the polytopes $R_{\mathrm{M}}(\underline{X}, \underline{r})$, $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{M}}(d, r)$, are distinct. A similar argument as in the megamatroid case shows that $\mathbf{1}(R_{\mathrm{M}}(\underline{X}, \underline{r}))$, $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{M}}(d, r)$, are linearly independent. □

Let $(\underline{X},\underline{r}) \in \mathfrak{p}_{\mathrm{MM}}(d,r)$. Consider the homomorphism $s^{\leq}_{\underline{X},\underline{r}} : Z_{MM}(d,r) \to \mathbb{Z}$ defined by

$$s^{\leq}_{\underline{X},\underline{r}}(\mathrm{rk}) = \begin{cases} 1 & \text{if } \mathrm{rk}(X_j) \leq r_j \text{ for } j = 1,2,\ldots,k, \\ 0 & \text{otherwise.} \end{cases}$$

This homomorphism $s^{\leq}_{\underline{X},\underline{r}}$ is a (convergent infinite) sum of several homomorphisms of the form $s_{\underline{X}',\underline{r}'}$, so by Proposition 3.5.3 it is valuative.

In view of Theorem 3.5.4, if $f : Z_{(\mathrm{P})\mathrm{M}}(d,s) \to \mathbb{Z}$ is valuative, f is determined by its values on the (poly)matroids $R_{(\mathrm{P})\mathrm{M}}$, since the spaces $P_{(\mathrm{P})\mathrm{M}}(d,r)$ are finite-dimensional. For a (poly)matroid rk, $s^{\leq}_{\underline{X},\underline{r}}(\mathrm{rk}) = 1$ if and only if $\mathrm{Poly}(\mathrm{rk})$ is contained in $\mathrm{Poly}(R_{(\mathrm{P})\mathrm{M}}(\underline{X},\underline{r}))$. Therefore, the matrix specifying the pairing $P_{(\mathrm{P})\mathrm{M}}(r,d) \otimes P_{(\mathrm{P})\mathrm{M}}(r,d)^{\vee} \to \mathbb{Z}$ whose rows correspond to the polytopes $\mathrm{Poly}(R_{(\mathrm{P})\mathrm{M}}(\underline{X},\underline{r}))$, in some linear extension of the order of these polytopes by containment, and whose columns correspond in the same order to $s^{\leq}_{\underline{X},\underline{r}}$, is triangular. The next corollary follows.

Corollary 3.5.5. *The group $P_{(\mathrm{P})\mathrm{M}}(d,r)^{\vee}$ of valuations $Z_{(\mathrm{P})\mathrm{M}}(d,r) \to \mathbb{Z}$ has the two bases*

$$\left\{ s_{\underline{X},\underline{r}} : (\underline{X},\underline{r}) \in \mathfrak{p}_{(\mathrm{P})\mathrm{M}}(d,r) \right\}$$

and

$$\left\{ s^{\leq}_{\underline{X},\underline{r}} : (\underline{X},\underline{r}) \in \mathfrak{p}_{(\mathrm{P})\mathrm{M}}(d,r) \right\}.$$

If $\underline{X}$ is not a maximal chain, then $s_{\underline{X},\underline{r}}$ is a linear combination of functions of the form $s_{\underline{X}',\underline{r}'}$ where $\underline{X}'$ is a maximal chain. The following corollary follows from Corollary 3.5.5.

Corollary 3.5.6. *The group $P_{\mathrm{PM}}(d,r)^{\vee}$ of valuations $Z_{\mathrm{PM}}(d,r) \to \mathbb{Z}$ is generated by the functions $s_{\underline{X},\underline{r}}$ where $\underline{X}$ is a chain of subsets of $[d]$ of length d and $r = (r_1,\ldots,r_d)$ is an integer vector with $0 \leq r_1 \leq \cdots \leq r_d = r$.*

The generating set of this corollary appeared as the coordinates of the function H defined in §6 of Chapter 2, which was introduced there as a labeled analogue of Derksen's $\mathcal{G}$.

Proof of Theorem 3.1.5(d). Let $\mathfrak{a}(d,r)$ be the set of all sequences $(a_1,\ldots,a_d)$ with $0 \leq a_i \leq r$ for all i and $a_i = r$ for some i. Clearly $|\mathfrak{a}(d,r)| = (r+1)^d - r^d$. We define a bijection $f : \mathfrak{p}_{\mathrm{PM}}(d,r) \to \mathfrak{a}(d,r)$ as follows. If $(\underline{X},\underline{r}) \in \mathfrak{p}_{\mathrm{PM}}(d,r)$, then we define

$$f(\underline{X},\underline{r}) = (a_1,a_2,\ldots,a_d)$$

where $a_i = r_j$ and j is minimal such that $i \in X_j$.

Suppose that $(a_1,\ldots,a_d) \in \mathfrak{a}(d,r)$. Let k be the cardinality of $\{a_1,\ldots,a_d\}$. Now $r_1 < r_2 < \cdots < r_k$ are defined by

$$\{r_1,r_2,\ldots,r_k\} = \{a_1,\ldots,a_d\}$$

and for every j, we define

$$X_j = \{i \in [d] \mid a_i \le r_j\}.$$

Then we have

$$f^{-1}(a_1, \ldots, a_d) = (\underline{X}, \underline{r}).$$

A generating function for $p_{\mathrm{PM}}(d, r)$ is

$$\sum_{d=0}^{\infty}\sum_{r=0}^{\infty} \frac{p_{\mathrm{PM}}(d,r)x^d y^r}{d!} = 1 + \sum_{d=1}^{\infty}\sum_{r=0}^{\infty} \frac{(r+1)^d - r^d}{d!} x^d y^r =$$
$$= 1 + \sum_{r=0}^{\infty}\sum_{d=0}^{\infty} \frac{(r+1)^d - r^d}{d!} x^d y^r = 1 + \sum_{r=0}^{\infty} (e^{(r+1)x} - e^{rx}) y^r = 1 + \frac{e^x - 1}{1 - ye^x} = \frac{e^x(1-y)}{1-ye^x}.$$

□

Proof of Theorem 3.1.5(c). Suppose that $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{M}}(d, r)$ has length k. Define $u_1, u_2, \ldots, u_k$ by

$$u_1 = r_1, \quad u_i = r_i - r_{i-1} - 1 \ (2 \le i \le k).$$

Define $v_1, v_2, \ldots, v_k$ by

$$\begin{aligned} v_i &= (|X_i| - r_i) - (|X_{i-1}| - r_{i-1}) - 1 \ (1 \le i \le k-1), \\ v_k &= (|X_k| - r_k) - (|X_{k-1}| - r_{k-1}) = d - r - |X_{k-1}| + r_{k-1}. \end{aligned}$$

If $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{M}}(d, r)$, then we have that $u_1, \ldots, u_k, v_1, \ldots, v_k$ are nonnegative, and

$$u_1 + \cdots + u_k = r - k + 1, \quad v_1 + \cdots + v_k = d - r - k + 1.$$

Let $Y_i = X_i \backslash X_{i-1}$ for $i = 1, 2, \ldots, k$. If $k \ge 2$, then we have $u_1 + v_1 + 1 = |Y_1|$, $u_k + v_k + 1 = |Y_k|$ and $u_i + v_i + 2 = |Y_i|$ for $i = 2, 3, \ldots, k-1$. There are

$$\frac{d!}{(u_1+v_1+1)!(u_2+v_2+2)!(u_3+v_3+2)!\cdots(u_{k-1}+v_{k-1}+2)!(u_k+v_k+1)!}$$

partitions of $[d]$ into the subsets $Y_1, Y_2, \ldots, Y_k$, such that $(\underline{X}, \underline{r})$ has the given u and v values. If $k = 1$, then $u_1 + v_1 = d$ and $d!/(u_1+v_1)! = 1$ so there is just one pair $(\underline{X}, \underline{r})$ with given u and v values.

This yields the generating function

$$\sum_{d=0}^{\infty}\sum_{r=0}^{d} \frac{p_{\mathrm{M}}(d,r)}{d!} x^{d-r} y^r = \sum_{u_1, v_1 \ge 0} \frac{t^{u_1} s^{v_1}}{(u_1+v_1)!} +$$
$$+ \sum_{\substack{u_1,\ldots,u_k \ge 0 \\ v_1,\ldots,v_k \ge 0}} \frac{x^{u_1+u_2+\cdots+u_k+k-1} y^{v_1+v_2+\cdots+v_k+k-1}}{(u_1+v_1+1)!(u_2+v_2+2)!\cdots(u_{k-1}+v_{k-1}+2)!(u_k+v_k+1)!} \quad (3.5.2)$$

We have that

$$\sum_{u,v\geq 0}\frac{x^u y^v}{(u+v)!}=\sum_{d=0}^{\infty}\sum_{u+v=d}\frac{t^u s^v}{d!}=\sum_{d=0}^{\infty}\frac{x^{d+1}-y^{d+1}}{(x-y)d!}=\frac{xe^x-ye^y}{x-y}, \tag{3.5.3}$$

$$\sum_{u,v\geq 0}\frac{t^u s^v}{(u+v+1)!}=\sum_{d=0}^{\infty}\sum_{u+v=d}\frac{x^u y^v}{(d+1)!}=\sum_{d=0}^{\infty}\frac{x^{d+1}-y^{d+1}}{(x-y)(d+1)!}=$$
$$=\sum_{d=1}^{\infty}\frac{x^d-y^d}{(x-y)d!}=\sum_{d=0}^{\infty}\frac{x^d-y^d}{(x-y)d!}=\frac{e^x-e^y}{x-y}, \tag{3.5.4}$$

and

$$\sum_{u,v\geq 0}\frac{x^u y^v}{(u+v+2)!}=\sum_{d=0}^{\infty}\sum_{u+v=d}\frac{x^u y^v}{(d+2)!}=\sum_{d=0}^{\infty}\frac{x^{d+1}-y^{d+1}}{(x-y)(d+2)!}=$$
$$=\sum_{d=1}^{\infty}\frac{x^d-y^d}{(x-y)(d+1)!}=\frac{(e^x-1)/x-(e^y-1)/y}{x-y}=\frac{ye^x-y-xe^y+x}{(x-y)xy}. \tag{3.5.5}$$

Using (3.5.3), (3.5.4) and (3.5.5) with (3.5.2) yields

$$\sum_{d=0}^{\infty}\sum_{r=0}^{d}\frac{p_{\mathrm{M}}(d,r)}{d!}x^{d-r}y^r=$$
$$=\frac{xe^x-ye^y}{x-y}+xy\left(\frac{e^x-e^y}{x-y}\right)^2\sum_{k=2}^{\infty}\left(\frac{ye^x-y-xe^y+x}{x-y}\right)^{k-2}=$$
$$=\frac{xe^x-ye^y}{x-y}+\left(\frac{e^x-e^y}{x-y}\right)^2\frac{xy}{1-\dfrac{ye^x-y-xe^y+x}{x-y}}=$$
$$=\frac{xe^x-ye^y}{x-y}+\frac{xy(e^x-e^y)^2}{(x-y)(xe^y-ye^x)}=\frac{x-y}{xe^{-x}-ye^{-y}}. \tag{3.5.6}$$

□

The values of $p_{(\mathrm{P})\mathrm{M}}(d,r)$ for small d and r can be found in Appendix 3.B.

3.6 Valuative invariants: the groups $P^{\mathrm{sym}}_{\mathrm{M}}$, $P^{\mathrm{sym}}_{\mathrm{PM}}$, $P^{\mathrm{sym}}_{\mathrm{MM}}$

Let $Y_{\mathrm{MM}}(d,r)$ be the group generated by all formal differences $\langle \mathrm{rk}\rangle - \langle \mathrm{rk}\circ\sigma\rangle$ where $\mathrm{rk}: 2^{[d]}\to\mathbb{Z}\cup\{\infty\}$ is a megamatroid of rank r and σ is a permutation of $[d]$. We define

$Z^{\mathrm{sym}}_{\mathrm{MM}}(d,r) = Z_{\mathrm{MM}}(d,r)/Y_{\mathrm{MM}}(d,r)$. Let $\pi_{\mathrm{MM}} : Z_{\mathrm{MM}}(d,r) \to Z^{\mathrm{sym}}_{\mathrm{MM}}(d,r)$ be the quotient homomorphism. If $\mathrm{rk}_X : 2^X \to \mathbb{Z}\cup\{\infty\}$ is any megamatroid, then we can choose a bijection $\varphi : [d] \to X$, where d is the cardinality of X. Let $r = \mathrm{rk}_X(X)$. The image of $\langle \mathrm{rk}_X \circ\varphi\rangle$ in $Z^{\mathrm{sym}}_{\mathrm{MM}}(d,r)$ does not depend on φ, and will be denoted by $[\mathrm{rk}_X]$. The megamatroids (X,rk_X) and (Y,rk_Y) are isomorphic if and only if $[\mathrm{rk}_X] = [\mathrm{rk}_Y]$. So we may think of $Z^{\mathrm{sym}}_{\mathrm{MM}}(d,r)$ as the free group generated by all isomorphism classes of rank r megamatroids on sets with d elements.

Let $B_{\mathrm{MM}}(d,r)$ be the group generated by all $\mathbf{1}(\mathrm{Poly}(\mathrm{rk})) - \mathbf{1}(\mathrm{Poly}(\mathrm{rk}\circ\sigma))$ where $\mathrm{rk} : 2^{[d]} \to \mathbb{Z}\cup\{\infty\}$ is a megamatroid of rank r and σ is a permutation of $[d]$. Define $P^{\mathrm{sym}}_{\mathrm{MM}}(d,r) = P_{\mathrm{MM}}(d,r)/B_{\mathrm{MM}}(d,r)$ and let $\rho_{\mathrm{MM}} : P_{\mathrm{MM}}(d,r) \to P^{\mathrm{sym}}_{\mathrm{MM}}(d,r)$ be the quotient homomorphism. From the definitions it is clear that $\mathbf{1}_{\mathrm{MM}}(Y_{\mathrm{MM}}(d,r)) = B_{\mathrm{MM}}(d,r)$. Therefore, there exists a unique group homomorphism

$$\mathbf{1}^{\mathrm{sym}}_{\mathrm{MM}} : Z^{\mathrm{sym}}_{\mathrm{MM}}(d,r) \to P^{\mathrm{sym}}_{\mathrm{MM}}(d,r)$$

such that the following diagram commutes:

$$\begin{array}{ccc} Z_{\mathrm{MM}}(d,r) & \xrightarrow{\mathbf{1}_{\mathrm{MM}}} & P_{\mathrm{MM}}(d,r) \\ \downarrow{\scriptstyle \pi_{\mathrm{MM}}} & & \downarrow{\scriptstyle \rho_{\mathrm{MM}}} \\ Z^{\mathrm{sym}}_{\mathrm{MM}}(d,r) & \xrightarrow[\mathbf{1}^{\mathrm{sym}}_{\mathrm{MM}}]{} & P^{\mathrm{sym}}_{\mathrm{MM}}(d,r). \end{array} \tag{3.6.1}$$

This diagram is a push-out. Define $Y_{(\mathrm{P})\mathrm{M}}(d,r) = Y_{\mathrm{MM}}(d,r) \cap Z_{(\mathrm{P})\mathrm{M}}(d,r)$. The group $Y_{(\mathrm{P})\mathrm{M}}(d,r)$ is the group generated by all $\langle\mathrm{rk}\rangle - \langle\mathrm{rk}\circ\sigma\rangle$ where $\mathrm{rk} : 2^{[d]} \to \mathbb{N}$ is a (poly)matroid of rank r and σ is a permutation of $[d]$. Define $Z^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r) = Z_{(\mathrm{P})\mathrm{M}}(d,r)/Y_{(\mathrm{P})\mathrm{M}}(d,r)$. The group $Z^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)$ is freely generated by all $[\mathrm{rk}]$ where $\mathrm{rk} : X \to \mathbb{N}$ is a $*$matroid of rank r and $d = |X|$.

Define $B_{*\mathrm{M}}(d,r)$ as the group generated by all $\mathbf{1}(\mathrm{Poly}(\mathrm{rk})) - \mathbf{1}(\mathrm{Poly}(\mathrm{rk}\circ\sigma))$ where $\mathrm{rk} : 2^{[d]} \to \mathbb{N}$ is a $*$matroid of rank r and σ is a permutation of $[d]$. Let $P^{\mathrm{sym}}_{*\mathrm{M}}(d,r) = P_{*\mathrm{M}}(d,r)/B_{*\mathrm{M}}(d,r)$.

Lemma 3.6.1. *We have*

$$B_{(\mathrm{P})\mathrm{M}}(d,r) = B_{\mathrm{MM}}(d,r) \cap P_{(\mathrm{P})\mathrm{M}}(d,r).$$

Proof. Define $q_{(\mathrm{P})\mathrm{M}} : P_{\mathrm{MM}}(d,r) \to P_{(\mathrm{P})\mathrm{M}}(d,r)$ by $q_{(\mathrm{P})\mathrm{M}}(f) = f \cdot \mathbf{1}(\Delta_{(\mathrm{P})\mathrm{M}}(d,r))$. This is well defined because for any megamatroid $\mathrm{rk} : 2^{[d]} \to \mathbb{Z}\cup\{\infty\}$ of rank r, we have $q_{(\mathrm{P})\mathrm{M}}(\mathbf{1}(\mathrm{Poly}(\mathrm{rk}))) = \mathbf{1}(\mathrm{Poly}(\mathrm{rk}))\cdot\mathbf{1}(\Delta_{(\mathrm{P})\mathrm{M}}(d,r)) = \mathbf{1}(\mathrm{Poly}(\mathrm{rk})\cap\Delta_{(\mathrm{P})\mathrm{M}}(d,r))$ and $\mathrm{Poly}(\mathrm{rk})\cap\Delta_{(\mathrm{P})\mathrm{M}}(d,r)$ is either empty or a polymatroid polyhedron. Clearly, $q_{(\mathrm{P})\mathrm{M}}$ is a projection of $P_{\mathrm{MM}}(d,r)$ onto $P_{(\mathrm{P})\mathrm{M}}(d,r)$. Since $q_{(\mathrm{P})\mathrm{M}}(B_{\mathrm{MM}}(d,r)) \subseteq B_{(\mathrm{P})\mathrm{M}}(d,r)$, it follows that

$$B_{\mathrm{MM}}(d,r)\cap P_{(\mathrm{P})\mathrm{M}}(d,r) = q_{\mathrm{PM}}(B_{\mathrm{MM}}(d,r)\cap P_{(\mathrm{P})\mathrm{M}}(d,r)) \subseteq B_{(\mathrm{P})\mathrm{M}}(d,r).$$

It follows that $B_{\mathrm{MM}}(d,r)\cap P_{(\mathrm{P})\mathrm{M}}(d,r) = B_{(\mathrm{P})\mathrm{M}}(d,r)$. □

By restriction, we get also the commutative push-out diagrams (3.1.1) from the introduction. Define $\mathfrak{p}^{\mathrm{sym}}_{*\mathrm{M}}(d,r)$ as the set of all pairs $(\underline{X},\underline{r}) \in \mathfrak{p}_{*\mathrm{M}}(d,r)$ such that for every j, there exists an i such that

$$X_j = [i] = \{1, 2, \ldots, i\}.$$

Define $A_{*\mathrm{M}}(d,r)$ as the $\mathbb{Z}$ module generated by all $\mathbf{1}(R_{*\mathrm{M}}(\underline{X},\underline{r}))$ with $(\underline{X},\underline{r}) \in \mathfrak{p}^{\mathrm{sym}}_{*\mathrm{M}}(d,r)$.

Lemma 3.6.2. *We have*

$$P_{*\mathrm{M}}(d,r) = A_{*\mathrm{M}}(d,r) \oplus B_{*\mathrm{M}}(d,r).$$

Proof. By the definitions of $A_{*\mathrm{M}}(d,r)$ and $B_{*\mathrm{M}}(d,r)$ it is clear that $P_{*\mathrm{M}}(d,r) = A_{*\mathrm{M}}(d,r) + B_{*\mathrm{M}}(d,r)$. Consider the homomorphism $\tau : P_{*\mathrm{M}}(d,r) \to P_{*\mathrm{M}}(d,r)$ defined by $\tau(f) = \sum_\sigma f \circ \sigma$ where σ runs over all permutations of $[d]$. Clearly, $B_{*\mathrm{M}}(d,r)$ is contained in the kernel of τ. Recall that $\mathbf{1}(R_{*\mathrm{M}}(\underline{X},\underline{r}))$, $(\underline{X},\underline{r}) \in \mathfrak{p}_{*\mathrm{M}}(d,r)$ is a basis of $P_{*\mathrm{M}}(d,r)$. From this it easily follows that the set $\tau(\mathbf{1}(R_{*\mathrm{M}}(d,r)))$, $(\underline{X},\underline{r}) \in \mathfrak{p}^{\mathrm{sym}}_{*\mathrm{M}}(d,r)$ is independent over $\mathbb{Q}$. Therefore the restriction of τ to $A_{*\mathrm{M}}(d,r)$ is injective and $A_{*\mathrm{M}}(d,r) \cap B_{*\mathrm{M}}(d,r) = \{0\}$. □

Theorem 3.6.3. *The $\mathbb{Z}$-module $P^{\mathrm{sym}}_{\star\mathrm{M}}(d,r)$ is freely generated by all $\rho_{\star\mathrm{M}}(\mathbf{1}(R_{\star\mathrm{M}}(\underline{X},\underline{r})))$ with $(\underline{X},\underline{r})$ in $\mathfrak{p}^{\mathrm{sym}}_{\star\mathrm{M}}(d,r)$.*

Proof. It is clear that $\rho_{*\mathrm{M}}(A_{*\mathrm{M}}(d,r)) = P^{\mathrm{sym}}_{*\mathrm{M}}(d,r)$. So the restriction is surjective. It is also injective by Lemma 3.6.2. So the restriction of $\rho_{*\mathrm{M}} : P_{*\mathrm{M}}(d,r) \to P^{\mathrm{sym}}_{*\mathrm{M}}(d,r)$ to $A_{*\mathrm{M}}(d,r)$ is an isomorphism. From the definition of $A_{\star M}(d,r)$ it follows that the given set generates $P^{\mathrm{sym}}_{\star\mathrm{M}}(d,r)$, and the set is independent because of Theorem 3.5.4. □

The matroid polytopes $R_{\mathrm{M}}(\underline{X},\underline{r})$ are the polytopes of *Schubert matroids* and their images under relabeling the ground set. Schubert matroids were first described by Crapo [23], and have since arisen in several contexts. So Theorem 3.6.3 says that the indicator functions of Schubert matroids form a basis for $P^{\mathrm{sym}}_{\mathrm{M}}(d,r)$.

Recall that $Z^{\mathrm{sym}}_{*\mathrm{M}}$ can be viewed as the free $\mathbb{Z}$-module generated by all isomorphism classes of $*$matroids on a set with d elements of rank r. We say that a group homomorphism $f : Z^{\mathrm{sym}}_{\mathrm{MM}}(d,r) \to A$ is valuative if and only if $f \circ \pi_{\mathrm{MM}}$ is valuative. For any $(\underline{X},\underline{r}) \in \mathfrak{p}_{\mathrm{MM}}(d,r)$ and σ a permutation of $[d]$, we have $s_{\underline{X},\underline{r}}(\mathrm{rk} \circ \sigma) = s_{\sigma\underline{X},\underline{r}}(\mathrm{rk})$, where σ acts on $\underline{X}$ by permuting each set in the chain. So the symmetric group $\mathfrak{S}_d$ acts naturally on $P_{*\mathrm{M}}(d,r)$. It is easy to see that

$$P^{\mathrm{sym}}_{\star M}(d,r)^\vee \cong \left(P_{\star M}(d,r)^\vee\right)^{\mathfrak{S}_d},$$

where the right-hand side is the set of $\mathfrak{S}_d$-invariant elements of $P_{\star\mathrm{M}}(d,r)^\vee$.

For $(\underline{X},\underline{r}) \in \mathfrak{p}^{\mathrm{sym}}_{\mathrm{MM}}(d,r)$, define a homomorphism $s^{\mathrm{sym}}_{\underline{X},\underline{r}} : Z_{\mathrm{MM}}(d,r) \to \mathbb{Z}$ by

$$s^{\mathrm{sym}}_{\underline{X},\underline{r}} = \sum_{\sigma\underline{X}} s_{\sigma\underline{X},\underline{r}}$$

where the sum is over all chains $\sigma\underline{X}$ in the orbit of $\underline{X}$ under the action of the symmetric group. Then Corollary 3.5.5 implies the following.

Corollary 3.6.4. *The $\mathbb{Q}$-vector space $P^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$ of valuations $Z^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r) \to \mathbb{Q}$ has a basis given by the functions $s^{\mathrm{sym}}_{\underline{X},\underline{r}}$ for $(\underline{X},\underline{r}) \in \mathfrak{p}^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)$.*

For a sequence $\alpha = (\alpha_1, \ldots, \alpha_d)$ of nonnegative integers with $|\alpha| = \sum_i \alpha_i = r$, we define

$$u_\alpha = s_{\underline{X},\underline{r}} : Z^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r) \to \mathbb{Z},$$

where $\underline{X}_i = [i]$ for $i = 1, 2, \ldots, r$ and $\underline{r} = (\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_d)$. Parallel to Corollary 3.5.6 we also have the following.

Corollary 3.6.5. *The $\mathbb{Q}$-vector space $P^{\mathrm{sym}}_{\mathrm{PM}}(d,r)^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$ of valuations $Z^{\mathrm{sym}}_{\mathrm{PM}}(d,r) \to \mathbb{Q}$ has a $\mathbb{Q}$-basis given by the functions u_α, where α runs over all sequences $(\alpha_1, \ldots, \alpha_d)$ of nonnegative integers with $|\alpha| = r$.*

Corollary 3.6.6. *The $\mathbb{Q}$-vector space $P^{\mathrm{sym}}_{\mathrm{M}}(d,r)^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$ of valuations $Z^{\mathrm{sym}}_{\mathrm{M}}(d,r) \to \mathbb{Q}$ has a $\mathbb{Q}$-basis given by all functions u_α where α runs over all sequences $(\alpha_1, \ldots, \alpha_d) \in \{0,1\}^d$ with $|\alpha| = r$.*

Proof of Theorem 3.1.4. From the definitions of the U_α and the u_α, it follows that $u_\alpha(\langle \mathrm{rk} \rangle)$ is the coefficient of U_α in $\mathcal{G}(\langle \mathrm{rk} \rangle)$. In other words, $\{u_\alpha\}$ is a dual basis to $\{U_\alpha\}$. The universality follows from Corollary 3.6.6. □

The rank of $P^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)$ is equal to the cardinality of $\mathfrak{p}^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)$. If $(\underline{X},\underline{r})$ and $\ell(\underline{X}) = k$ lies in $\mathfrak{p}^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)$ then $\underline{X}$ is completely determined by the numbers $s_i := |X_i|$, $1 \le i \le k$.

Proof of Theorem 3.1.5(b). Given k, there are $\binom{r}{k-1}$ ways of choosing $\underline{r} = (r_1, \ldots, r_k)$ with $0 < r_1 < r_2 < \cdots < r_k = r$ and $\binom{d-1}{k-1}$ ways of choosing $(s_1, \ldots, s_k)$ with $0 < s_1 < s_2 < \cdots < s_k = d$. So the cardinality of $\mathfrak{p}^{\mathrm{sym}}_{\mathrm{PM}}(d,r)$ is

$$\sum_{k \ge 1} \binom{r}{k-1}\binom{d-1}{k-1} = \sum_{k \ge 0} \binom{r}{k}\binom{d-1}{k} = \binom{r+d-1}{r}.$$

$$\sum_{r,d} p^{\mathrm{sym}}_{\mathrm{PM}}(d,r) x^d y^r = \sum_{r,d} \tbinom{r+d-1}{r} x^d y^r = \sum_d (1-x)^{-d} y^d = \frac{1}{1 - \frac{y}{1-x}} = \frac{1-x}{1-x-y}.$$

□

Proof of Theorem 3.1.5(a). Let $t_i = s_i - r_i$. Then we have $0 < t_1 < t_2 < \cdots < t_{k-1} \leq t_k = d - r$. Given k, there are $\binom{r}{k-1}$ ways of choosing $\underline{r}$ such that $0 \leq r_1 < \cdots < r_k = r$ and $\binom{d-r}{k-1}$ ways of choosing $(t_1, \ldots, t_k)$ with $0 < t_1 < \cdots < t_{k-1} \leq t_k = d - r$. So the cardinality of $\mathfrak{p}_{\mathrm{M}}^{\mathrm{sym}}(d, r)$ is

$$\sum_{k \geq 1} \binom{r}{k-1}\binom{d-r}{k-1} = \sum_{k \geq 0} \binom{r}{k}\binom{d-r}{k} = \binom{d}{r}.$$

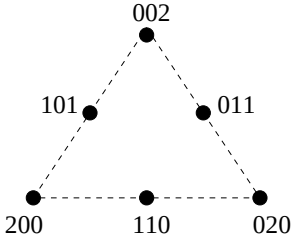
So we have

$$\sum_{r,d} p_{\mathrm{M}}^{\mathrm{sym}}(d, r) x^{d-r} y^r = \sum_{d} (x+y)^d = \frac{1}{1-x-y}.$$

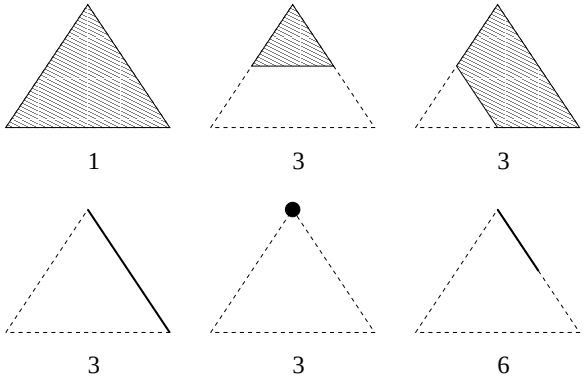
□

Example 3.6.7. Consider polymatroids for $r = 2$ and $d = 3$. All polymatroid base polytopes are contained in the triangle

$$\{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_1 + y_2 + y_3 = 2,\ y_1, y_2, y_3 \geq 0\}.$$



There are $\binom{d-1+r}{r} = \binom{4}{2}$ elements in our distinguished basis $\mathfrak{p}_{\mathrm{PM}}^{\mathrm{sym}}(3, 2)$ and the polytopes $R(\underline{X}, \underline{r})$, $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{PM}}^{\mathrm{sym}}(3, 2)$ are given by:



These 6 polytopes correspond to the following pairs $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{PM}}(3, 2)$.

$\underline{X}: \{1,2,3\}$ $\underline{r} = (2)$	$\underline{X}: \{1,2\} \subset \{1,2,3\}$ $\underline{r} = (1,2)$	$\underline{X}: \{1\} \subset \{1,2,3\}$ $\underline{r} = (1,2)$
$\underline{X}: \{1\} \subset \{1,2,3\}$ $\underline{r} = (0,2)$	$\underline{X}: \{1,2\} \subset \{1,2,3\}$ $\underline{r} = (0,2)$	$\underline{X}: \{1\} \subset \{1,2\} \subset \{1,2,3\}$ $\underline{r} = (0,1,2)$

The symmetric group $\mathfrak{S}_3$ acts on the triangle by permuting the coordinates y_1, y_2, y_3.

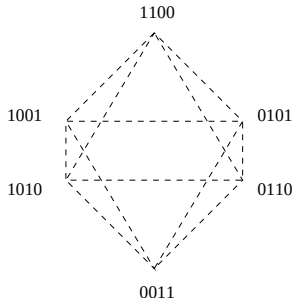
If $\mathfrak{S}_3$ acts on the generators $R(\underline{X}, \underline{r})$ with $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{PM}}(3,2)$, then we get all $R(\underline{X}, \underline{r})$ with $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{PM}}(3,2)$. In the figure, we wrote for each polytope the cardinality of the orbit under $\mathfrak{S}_3$. The cardinality of $\mathfrak{p}_{\mathrm{PM}}(3,2)$ is $1+3+3+3+3+6 = 19$. This is consistent with Theorem 3.1.5, because the cardinality is $(r+1)^d - r^d = 3^3 - 2^3 = 19$.

♢

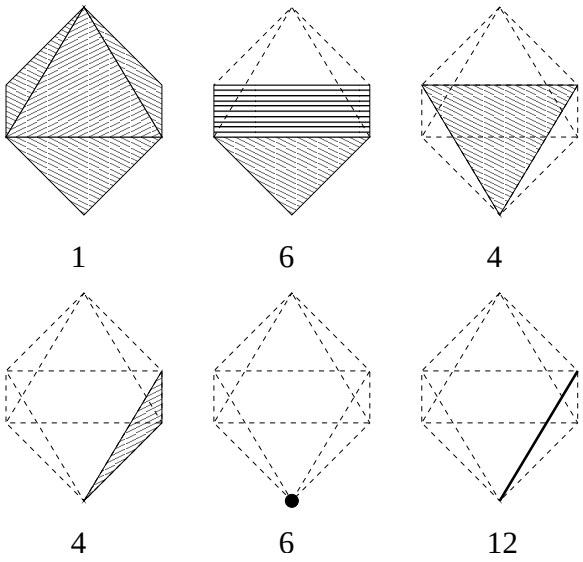
Example 3.6.8. Consider matroids for $r = 2$ and $d = 4$. All matroid base polytopes are contained in the set

$$\{(y_1, y_2, y_3, y_4) \in \mathbb{R}^4 \mid y_1 + y_2 + y_3 + y_4 = 2,\ \forall i\ 0 \le y_i \le 1\}.$$

This set is an octahedron:



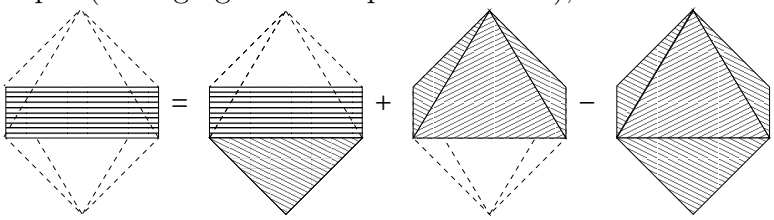
There are $\binom{d}{r} = \binom{4}{2}$ elements in $\mathfrak{p}_{\mathrm{M}}(4,2)$, and the polytopes $R_{\mathrm{M}}(\underline{X}, \underline{r})$, $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{M}}(4,2)$ are given by:



These 6 polytopes correspond to the following pairs $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{M}}(4,2)$.

$\underline{X}: \{1,2,3,4\}$ $\underline{r} = (2)$	$\underline{X}: \{1,2\} \subset \{1,2,3,4\}$ $\underline{r} = (1,2)$	$\underline{X}: \{1,2,3\} \subset \{1,2,3,4\}$ $\underline{r} = (1,2)$
$\underline{X}: \{1\} \subset \{1,2,3,4\}$ $\underline{r} = (0,2)$	$\underline{X}: \{1,2\} \subset \{1,2,3,4\}$ $\underline{r} = (0,2)$	$\underline{X}: \{1\} \subset \{1,2,3\} \subset \{1,2,3,4\}$ $\underline{r} = (0,1,2)$

The symmetric group $\mathfrak{S}_4$ acts by permuting the coordinates y_1, y_2, y_3, y_4. This group acts on the octahedron, but it is not the full automorphism group of the octahedron. Also note that not all elements of $\mathfrak{S}_4$ preserve the orientation. If $\mathfrak{S}_4$ acts on the generators $R_{\mathrm{M}}(\underline{X}, \underline{r})$ with $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{M}}(4,2)$, then we get all $R(\underline{X}, \underline{r})$ with $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{M}}(4,2)$. In the figure, we write for each polytope the cardinality of the orbit under $\mathfrak{S}_4$. The cardinality of $\mathfrak{p}_{\mathrm{M}}(4,2)$ is $1+6+4+4+6+12 = 33$, which is compatible with Theorem 3.1.5 and the table in Appendix 3.B. Besides the polytopes $R(\underline{X}, \underline{r})$, $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{M}}(4,2)$, there are 3 more matroid base polytopes (belonging to isomorphic matroids), but these decompose as follows.



$\diamondsuit$

3.7 Hopf algebra structures

Define $Z_{*\mathrm{M}} = \bigoplus_{d,r} Z_{*\mathrm{M}}(d,r)$, and in a similar way define $Z^{\mathrm{sym}}_{*\mathrm{M}}$, $P_{*\mathrm{M}}$, and $P^{\mathrm{sym}}_{*\mathrm{M}}$. We can view $Z_{*\mathrm{M}}$ as the $\mathbb{Z}$-module freely generated by all isomorphism classes of $*$matroids.

If $\mathrm{rk}_1 : 2^{[d]} \to \mathbb{Z} \cup \{\infty\}$ and $\mathrm{rk}_2 : 2^{[e]} \to \mathbb{Z} \cup \{\infty\}$ then we define

$$\mathrm{rk}_1 \boxplus \mathrm{rk}_2 : 2^{[d+e]} \to \mathbb{Z} \cup \{\infty\}$$

by

$$(\mathrm{rk}_1 \boxplus \mathrm{rk}_2)(A) = \mathrm{rk}_1(A \cap [d]) + \mathrm{rk}_2(\{i \in [e] \mid d+i \in A\})$$

for any set $A \subseteq [d+e]$. Note that $\boxplus$ is not commutative. We have a homomorphism

$$\nabla : Z_{\mathrm{MM}}(d,r) \otimes_{\mathbb{Z}} Z_{\mathrm{MM}}(e,s) \to Z_{\mathrm{MM}}(d+e, r+s).$$

defined by

$$\nabla(\langle \mathrm{rk}_1 \rangle \otimes \langle \mathrm{rk}_2 \rangle) = \langle \mathrm{rk}_1 \boxplus \mathrm{rk}_2 \rangle.$$

The multiplication $\nabla : Z_{\mathrm{MM}} \otimes_{\mathbb{Z}} Z_{\mathrm{MM}} \to Z_{\mathrm{MM}}$ makes $Z_{\mathrm{MM}}(d,r)$ into an associative (non-commutative) ring with 1. The unit $\eta : \mathbb{Z} \to Z_{\mathrm{MM}}(d,r)$ is given by $1 \mapsto \langle \mathrm{rk}_0 \rangle$ where $\mathrm{rk}_0 : 2^{[0]} \to \mathbb{Z} \cup \{\infty\}$ is the unique megamatroid defined by $\mathrm{rk}(\emptyset) = 0$. With this multiplication, $Z_{\mathrm{M}}(d,r)$ and $Z_{\mathrm{PM}}(d,r)$ are subrings of $Z_{\mathrm{MM}}(d,r)$. The multiplication also respects the bigrading of $Z_{\mathrm{MM}}(d,r)$.

Next, we define a comultiplication for Z_{MM}. Suppose that $X = \{i_1, i_2, \ldots, i_d\}$ is a set of integers with $i_1 < \cdots < i_d$ and $\mathrm{rk} : 2^X \to \mathbb{Z} \cup \{\infty\}$ is a megamatroid. We define a megamatroid $\widehat{\mathrm{rk}} : 2^{[d]} \to \mathbb{Z} \cup \{\infty\}$ by $\widehat{\mathrm{rk}}(A) = \mathrm{rk}(\{i_j \mid j \in A\})$. If $\mathrm{rk} : 2^X \to \mathbb{Z} \cup \{\infty\}$ is

a megamatroid and $B \subseteq A \subseteq X$ then we define $\mathrm{rk}_{A/B} : 2^{A \setminus B} \to \mathbb{Z} \cup \{\infty\}$ by $\mathrm{rk}_{A/B}(C) = \mathrm{rk}(B \cup C) - \mathrm{rk}(B)$ for all $C \subseteq A \setminus B$. We also define $\mathrm{rk}_A := \mathrm{rk}_{A/\emptyset}$ and $\mathrm{rk}_{/B} = \mathrm{rk}_{X/B}$.

We now define

$$\Delta : Z_{\mathrm{MM}} \to Z_{\mathrm{MM}} \otimes_{\mathbb{Z}} Z_{\mathrm{MM}}$$

by

$$\Delta(\langle \mathrm{rk} \rangle) = \sum_{A \subseteq [d];\ \mathrm{rk}(A) < \infty} \widehat{\langle \mathrm{rk}_A \rangle} \otimes \widehat{\langle \mathrm{rk}_{/A} \rangle}.$$

where A runs over all subsets of $[d]$ for which $\mathrm{rk}(A)$ is finite. This comultiplication is coassociative, but not cocommutative. If $\mathrm{rk} : 2^{[d]} \to \mathbb{Z} \cup \{\infty\}$ is a megamatroid, then the counit is defined by

$$\epsilon(\langle \mathrm{rk} \rangle) = \begin{cases} 1 & \text{if } d = 0; \\ 0 & \text{otherwise.} \end{cases}$$

The reader may verify that the multiplicative and comultiplicative structures are compatible, making Z_{MM} into a bialgebra. Note that Δ also restricts to comultiplications for Z_{PM} and Z_{M}, and Z_{PM} and Z_{M} are sub-bialgebras of Z_{MM}.

We define a group homomorphism $S : Z_{\mathrm{MM}} \to Z_{\mathrm{MM}}$ by

$$S(\langle \mathrm{rk} \rangle) = \sum_{r=1}^{d} (-1)^r \sum_{\substack{\underline{X};\ \ell(\underline{X}) = r, \\ \mathrm{rk}(X_1) < \infty, \ldots, \mathrm{rk}(X_r) < \infty}} \prod_{i=1}^{r} \widehat{\langle \mathrm{rk}_{X_i / X_{i-1}} \rangle}.$$

Here we use the convention $X_0 = \emptyset$. One can check that S makes Z_{MM} into a Hopf algebra. Restriction of S makes Z_{M} and Z_{PM} into sub-Hopf algebras of Z_{MM}. We conclude that $Z_{*\mathrm{M}}$ has the structure of bigraded Hopf algebras over $\mathbb{Z}$.

It is well known that $Z_{\mathrm{M}}^{\mathrm{sym}}$ has the structure of a Hopf algebra over $\mathbb{Z}$. Similarly we have that $Z_{\mathrm{MM}}^{\mathrm{sym}}$ and $Z_{\mathrm{PM}}^{\mathrm{sym}}$ have a Hopf algebra structure. The multiplication

$$\nabla : Z_{\mathrm{MM}}^{\mathrm{sym}} \otimes_{\mathbb{Z}} Z_{\mathrm{MM}}^{\mathrm{sym}} \to Z_{\mathrm{MM}}^{\mathrm{sym}}$$

is defined by

$$\nabla([\mathrm{rk}_1] \otimes [\mathrm{rk}_2]) = [\mathrm{rk}_1 \oplus \mathrm{rk}_2].$$

The comultiplication is defined by

$$\Delta([\mathrm{rk}]) = \sum_{A \subseteq X; \mathrm{rk}(A) < \infty} [(A, \mathrm{rk}_A)] \otimes [(X \setminus A, \mathrm{rk}_{/A})]$$

for any megamatroid $\mathrm{rk} : 2^X \to \mathbb{Z} \cup \{\infty\}$. The unit $\eta : \mathbb{Z} \to Z_{\mathrm{MM}}^{\mathrm{sym}}$ is given by $1 \mapsto [(\emptyset, \mathrm{rk}_0)]$ and the counit $\epsilon : Z_{\mathrm{MM}}^{\mathrm{sym}} \to \mathbb{Z}$ is defined by

$$\epsilon([(X, \mathrm{rk})]) = \begin{cases} 1 & \text{if } X = \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we define the antipode $S : Z_{\mathrm{MM}}^{\mathrm{sym}} \to Z_{\mathrm{MM}}^{\mathrm{sym}}$ by

$$S([\mathrm{rk}]) = \sum_{r=1}^{d} (-1)^r \sum_{\substack{\underline{X};\ \ell(\underline{X})=r,\\ \mathrm{rk}(X_1)<\infty,\ldots,\mathrm{rk}(X_r)<\infty}} \prod_{i=1}^{r} [(X_i \setminus X_{i-1}, \mathrm{rk}\, X_i/X_{i-1})].$$

From the definitions, it is clear that the $\pi_{*\mathrm{M}}$ are Hopf algebra morphisms.

The space P_{MM} inherits a Hopf algebra structure from Z_{MM}. We define the multiplication $\nabla : P_{\mathrm{MM}} \otimes P_{\mathrm{MM}} \to P_{\mathrm{MM}}$ by

$$\nabla(\mathbf{1}(\Pi_1) \otimes \mathbf{1}(\Pi_2)) = \mathbf{1}(\Pi_1 \times \Pi_2). \tag{3.7.1}$$

It is easy to verify that $\nabla \circ (\mathbf{1}_{\mathrm{MM}} \otimes \mathbf{1}_{\mathrm{MM}}) = \mathbf{1}_{\mathrm{MM}} \circ \nabla$.

To define the comultiplication $\Delta : P_{\mathrm{MM}} \to P_{\mathrm{MM}} \otimes P_{\mathrm{MM}}$, we would like to have that $(\psi_{\mathrm{MM}} \otimes \psi_{\mathrm{MM}}) \otimes \Delta = \Delta \circ \psi_{\mathrm{MM}}$. So for a megamatroid polytope $\mathrm{rk} : 2^{[d]} \to \mathbb{Z} \cup \{\infty\}$ we would like to have

$$\begin{aligned}\Delta(\mathbf{1}(\mathrm{Poly}(\mathrm{rk}))) = \Delta(\psi_{\mathrm{MM}}(\langle \mathrm{rk} \rangle)) = \sum_{A \subseteq [d]; \mathrm{rk}(A)<\infty} \psi_{\mathrm{MM}}(\widehat{\mathrm{rk}_A}) \otimes \psi_{\mathrm{MM}}(\widehat{\mathrm{rk}_{/A}}) = \\ = \sum_{A \subseteq [d]; \mathrm{rk}(A)<\infty} \mathbf{1}(\mathrm{Poly}(\mathrm{rk}_A)) \otimes \mathbf{1}(\mathrm{Poly}(\mathrm{rk}_{/A})).\end{aligned}$$

A basis of P_{MM} is given by all $R_{\mathrm{MM}}(\underline{X}, \underline{r})$, with $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{MM}} = \bigcup_{d,r} \mathfrak{p}_{\mathrm{MM}}(d, r)$. Recall that the rank function $\mathrm{rk}_{\underline{X},\underline{r}}$ is defined such that $\mathrm{Poly}(\mathrm{rk}_{\underline{X},\underline{r}}) = R_{\mathrm{MM}}(\underline{X}, \underline{r})$. We have that $\mathrm{rk}_{\underline{X},\underline{r}}(A) < \infty$ if and only if $A = X_i$ for some i. In this case we have

$$\Delta(\langle \mathrm{rk}_{\underline{X},\underline{r}} \rangle) = \sum_{i=0}^{k} \langle \widehat{\mathrm{rk}_{\underline{X}_i,\underline{r}_i}} \rangle \otimes \langle \widehat{\mathrm{rk}_{\underline{X}^i,\underline{r}^i}} \rangle,$$

where

$$\underline{X}_i : \emptyset \subset X_1 \subset \cdots \subset X_i, \quad \underline{r}_i = (r_1, r_2, \ldots, r_i)$$
$$\underline{X}^i : \emptyset \subset X_{i+1} \setminus X_i \subset \cdots \subset X_k \setminus X_i, \quad \underline{r}^i = (r_{i+1} - r_i, \ldots, r_k - r_i).$$

We define Δ by

$$\Delta(\mathbf{1}(R_{\mathrm{MM}}(\underline{X}, \underline{r}))) = \sum_{i=0}^{k} \mathbf{1}(R_{\mathrm{MM}}(\widehat{\underline{X}_i, \underline{r}_i})) \otimes \mathbf{1}(R_{\mathrm{MM}}(\widehat{\underline{X}^i, \underline{r}^i})).$$

From this definition and Theorem 3.4.2 follows that

$$\Delta(\mathbf{1}(\mathrm{Poly}(\mathrm{rk}))) = \sum_{\underline{X}} (-1)^{d-\ell(\underline{X})} \Delta\mathbf{1}(R_{\mathrm{MM}}(\underline{X}, \mathrm{rk})) =$$

$$= \sum_{\underline{X}} \sum_{i=0}^{\ell(\underline{X})} (-1)^{|X_i|-i} \mathbf{1}(R_{\mathrm{MM}}(\underline{X}_i, \widehat{\mathrm{rk}_{X_i}})) \otimes (-1)^{d-|X_i|-\ell(\underline{X})+i} \mathbf{1}(R_{\mathrm{MM}}(\underline{X}^i, \widehat{\mathrm{rk}_{/X_i}})) =$$

$$= \sum_{A \subseteq [d]; \mathrm{rk}(A) < \infty} \mathbf{1}(\mathrm{Poly}(\widehat{\mathrm{rk}}_A)) \otimes \mathbf{1}(\mathrm{Poly}(\widehat{\mathrm{rk}}_{/A})). \quad (3.7.2)$$

In a similar fashion we can define the antipode $S : P_{\mathrm{MM}} \to P_{\mathrm{MM}}$.

The Hopf algebra structure on P_{MM} naturally induces a Hopf algebra structure on $P_{\mathrm{MM}}^{\mathrm{sym}}$ such that ρ_{MM} and $\mathbf{1}_{\mathrm{MM}}^{\mathrm{sym}}$ are Hopf algebra homomorphisms. Also P_{PM} is a Hopf subalgebra of P_{MM} and P_{M} is a Hopf subalgebra of P_{PM}. Similarly $P_{\mathrm{PM}}^{\mathrm{sym}}$ is a Hopf subalgebra of $P_{\mathrm{MM}}^{\mathrm{sym}}$, and $P_{\mathrm{M}}^{\mathrm{sym}}$ is a Hopf subalgebra of $P_{\mathrm{PM}}^{\mathrm{sym}}$.

As a first observation to motivate the consideration of these Hopf algebra structures, we consider multiplicative invariants.

Definition 3.7.1. A multiplicative invariant for $*$matroids with values in a commutative ring A (with 1) is a ring homomorphism $f : Z_{*\mathrm{M}}^{\mathrm{sym}} \to A$.

That is to say, f is multiplicative if $f(\mathrm{rk}_1 \oplus \mathrm{rk}_2) = f(\mathrm{rk}_1) f(\mathrm{rk}_2)$. This is exactly the condition that f be a group-like element of the graded dual algebra $P_{\mathrm{M}}^{\mathrm{sym}}(d,r)^{\#}$. Many (poly)matroid invariants of note have this property, for instance the Tutte polynomial.

Proposition 3.7.2. *The Tutte polynomial $\mathcal{T} \in P_{\mathrm{M}}^{\mathrm{sym}}(d,r)^{\#}$ is given by*

$$\mathcal{T} = e^{(y-1)u_0+u_1} e^{u_0+(x-1)u_1}. \quad (3.7.3)$$

Proof. Recall the definition of u_α in terms of rank conditions on a chain of sets. In view of (3.7.2), we have that the multiplication in $(P_{*\mathrm{M}}^{\mathrm{sym}})^{\#}$ is given by $u_\alpha \cdot u_\beta = \binom{d+e}{d} u_{\alpha\beta}$, where α has length d and β has length e. Denote the right side of (3.7.3) by f. We have

$$f = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(i+j)!}{i!j!} ((y-1)u_0 + u_1)^i (u_0 + (x-1)u_1)^j$$

$$= \sum_i \sum_j \frac{(i+j)!}{i!j!} \sum_{\alpha \in \{0,1\}^{i+j}} (x-1)^{r_{i+j}-r_i} (y-1)^{i-r_i} \ (i+j)! u_\alpha$$

where $r_i = \sum_{k=1}^{i} \alpha_k$, so that $i - r_i$ is the number of indices $1 \le k \le i$ such that $\alpha_k = 0$, and $r_{i+j} - r_i$ is the number of indices $i+1 \le k \le j$ such that $\alpha_k = 1$.

Let $d = i + j$. For a matroid rk on $[d]$ of rank r, the elements rk and $1/d! \sum_{\sigma \in \mathfrak{S}_d} \mathrm{rk} \circ \sigma$ of $Z_{\mathrm{M}}(d, r)$ have equal image under π_{M}. Therefore

$$\begin{aligned} f(\mathrm{rk}) &= \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} f(\mathrm{rk} \circ \sigma) \\ &= \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \sum_{i+j=d} \frac{d!}{i!j!} \sum_{\alpha \in \{0,1\}^d} (x-1)^{r_d - r_i} (y-1)^{i - r_i} \; u_\alpha(\mathrm{rk} \circ \sigma) \\ &= \sum_{\sigma \in \mathfrak{S}_d} \sum_{i+j=d} \frac{1}{i!j!} (x-1)^{r - \mathrm{rk}(\sigma([i]))} (y-1)^{i - \mathrm{rk}(\sigma([i]))}. \end{aligned}$$

The set $\sigma([i])$ takes each value $A \subseteq [d]$ in $|A|!(d - |A|)!$ ways, so

$$f(\mathrm{rk}) = \sum_{A \subseteq [d]} (x-1)^{r - \mathrm{rk}(A)} (y-1)^{|A| - \mathrm{rk}(A)} = \mathcal{T}(\mathrm{rk}).$$

□

3.8 Additive functions: the groups $T_{\mathrm{M}}, T_{\mathrm{PM}}, T_{\mathrm{MM}}$

For $0 \leq e \leq d$ we define $P_{*\mathrm{M}}(d, r, e) \subseteq P_{*\mathrm{M}}(d, r)$ as the span of all $\mathbf{1}(\Pi)$ where $\Pi \subseteq \mathbb{R}^d$ is a $*$matroid polytope of dimension $\leq d - e$. We have $P_{*\mathrm{M}}(0, r, 0) = P_{*\mathrm{M}}(0, r)$ and $P_{*\mathrm{M}}(d, r, 1) = P_{*\mathrm{M}}(d, r)$ for $d \geq 1$. These subgroups form a filtration

$$\cdots \subseteq P_{*\mathrm{M}}(d, r, 2) \subseteq P_{*\mathrm{M}}(d, r, 1) \subseteq P_{*\mathrm{M}}(d, r, 0) = P_{*\mathrm{M}}(d, r).$$

Define $\overline{P}_{*\mathrm{M}}(d, r, e) := P_{*\mathrm{M}}(d, r, e) / P_{*\mathrm{M}}(d, r, e+1)$. If Π_1 and Π_2 are polytopes of codimension e_1 and e_2 respectively, then $\Pi_1 \times \Pi_2$ has codimension $e_1 + e_2$. It follows from (3.7.1) that the multiplication ∇ respects the filtration. Since $\mathrm{Poly}(\mathrm{rk}_A) \times \mathrm{Poly}(\mathrm{rk}_{/A})$ is contained in $\mathrm{Poly}(\mathrm{rk})$, it follows from (3.7.2) that the comultiplication Δ also respects the filtration:

$$\Delta(P_{*\mathrm{M}}(d, r, e)) \subseteq \sum_{i,j,k} P_{*\mathrm{M}}(i, j, k) \otimes P_{*\mathrm{M}}(d - i, r - j, e - k)$$

Similarly, the antipode S respects the grading. The associated graded algebra

$$\overline{P}_{*\mathrm{M}} = \bigoplus_{d,r,e} \overline{P}_{*\mathrm{M}}(d, r, e)$$

has an induced Hopf algebra structure. We define $T_{\star\mathrm{M}}(d, r) = \overline{P}_{\star\mathrm{M}}(d, r, 1)$.

For every partition $\underline{X} : [d] = \coprod_{i=1}^{e} X_i$ into nonempty subsets there exists a natural map

$$\Phi_{\underline{X}} : \prod_i \mathbb{R}^{X_i} \to \mathbb{R}^d$$

Define

$$P_{*\mathrm{M}}(\underline{X}) = \bigoplus_{r_1,r_2,\ldots,r_e\in\mathbb{Z}} P_{*\mathrm{M}}(|X_1|,r_1)\otimes\cdots\otimes P_{*\mathrm{M}}(|X_e|,r_e)$$

and

$$\overline{P}_{*\mathrm{M}}(\underline{X}) = \bigoplus_{r_1,r_2,\ldots,r_e\in\mathbb{Z}} T_{*\mathrm{M}}(|X_1|,r_1)\otimes\cdots\otimes T_{*\mathrm{M}}(|X_e|,r_e).$$

The map $\Phi_{\underline{X}}$ induces a group homomorphism

$$\phi_{\underline{X}} : P_{*\mathrm{M}}(\underline{X},e) \to P_{*\mathrm{M}}(d,r,e)$$

defined by

$$\phi_{\underline{X}}(\mathbf{1}(\Pi_1)\otimes\mathbf{1}(\Pi_2)\otimes\cdots\otimes\mathbf{1}(\Pi_e)) = \mathbf{1}(\Phi_{\underline{X}}(\Pi_1\times\Pi_2\times\cdots\times\Pi_e)).$$

The map $\phi_{\underline{X}}$ induces a group homomorphism

$$\overline{\phi}_{\underline{X}} : \overline{P}_{*\mathrm{M}}(\underline{X},e) \to \overline{P}_{*\mathrm{M}}(d,r,e).$$

A vector $y=(y_1,\ldots,y_d)\in\mathbb{R}^d$ is called $\underline{X}$-integral if $\sum_{i\in X_j} y_i\in\mathbb{Z}$ for $j=1,2,\ldots,e$. An $\underline{X}$-integral vector y is called $\underline{X}$-regular, if for every j and every $Y\subseteq X_j$ we have: if $\sum_{i\in Y} y_i\in\mathbb{Z}$, then $Y=\emptyset$ or $Y=X_j$. In other words, an $\underline{X}$-integral vector y is called $\underline{X}$-regular if it is not integral for any refinement of $\underline{X}$. We call y $\underline{X}$-balanced if $\sum_{i\in S} y_i = 0$ holds if and only if S is a union of some of the X_j's.

Choose an $\underline{X}$-balanced vector $y_{\underline{X}}$ for every $\underline{X}$. For $f\in P_{*\mathrm{M}}(d,r)$ we define

$$\gamma_{\underline{X}}(f)(x) := \lim_{\varepsilon\downarrow 0} f(x+\varepsilon y_{\underline{X}}).$$

If Π is a $*$matroid base polytope, then $\gamma_{\underline{X}}(\mathbf{1}(\Pi))(x)$ is constant on faces of Π. This shows that $\gamma_{\underline{X}}(\mathbf{1}(\Pi))\in P_{*\mathrm{M}}(d,r)$. So $\gamma_{\underline{X}}$ is an endomorphism of $P_{*\mathrm{M}}(d,r)$. Now $\gamma_{\underline{X}}$ also induces an endomorphism $\overline{\gamma}_{\underline{X}}$ of $\overline{P}_{*\mathrm{M}}(d,r)$.

Lemma 3.8.1. *We have that* $\gamma_{\underline{X}}\circ\gamma_{\underline{X}} = \gamma_{\underline{X}}$.

Proof. Suppose that $x\in\mathbb{R}^d$. Consider the set S of all $x+\varepsilon y_{\underline{X}}$ with $\varepsilon\in\mathbb{R}$. There exist a partition $\underline{Y}$ of $[d]$ and a dense open subset U of S such that all points in U are $\underline{Y}$-regular. Then there exists a $\delta>0$ such that $T=\{x+\varepsilon y \mid 0<\varepsilon<\delta\}$ has only $\underline{Y}$-regular points. For every $*$matroid base polytope Π, we have that $T\cap\Pi=\emptyset$ or $T\subseteq\Pi$. It follows that for every $f\in P_{\mathrm{M}*}(d,r)$ there exists a constant c such that f is equal to c on T. Therefore $\gamma_{\underline{X}}(f)(x)=c$ and $\gamma_{\underline{X}}(f)$ is constant and equal to c on T. We conclude that $\gamma_{\underline{X}}(\gamma_{\underline{X}}(f))(x)=c=\gamma_{\underline{X}}(f)(x)$. □

Lemma 3.8.2. *Suppose that $\underline{X}, \underline{Y}$ are partitions of $[d]$ into e nonempty subsets, and $\underline{X} \neq \underline{Y}$. Then we have*

$$\gamma_{\underline{X}} \circ \phi_{\underline{Y}} = 0.$$

Proof. For some k, Y_k is not the union of X_j's. The image

$$\Phi_{\underline{Y}}(\Pi_1 \times \cdots \times \Pi_e)$$

consists of $\underline{Y}$-integral points. For any $x \in \mathbb{R}^{[d]}$, $x + \varepsilon y_{\underline{X}}$ is not $\underline{Y}$-integral for small $\varepsilon > 0$. It follows that

$$\gamma_{\underline{X}}(\phi_{\underline{Y}}(\mathbf{1}(\Pi_1 \times \cdots \times \Pi_e)))(x) = \gamma_{\underline{X}}(\mathbf{1}(\Phi_{\underline{Y}}(\Pi_1 \times \cdots \times \Pi_e)))(x) = 0$$

for all x. □

Theorem 3.8.3. *We have the following isomorphism*

$$\overline{\phi} : \bigoplus_{\substack{\underline{X}=(X_1,X_2,\ldots,X_e) \\ [d]=X_1 \sqcup X_2 \sqcup \cdots \sqcup X_e; X_1,\ldots,X_e \neq \emptyset}} \overline{P}_{*\mathrm{M}}(\underline{X}) \to \bigoplus_{r \in \mathbb{Z}} \overline{P}_{*\mathrm{M}}(d, r, e) \tag{3.8.1}$$

where $\overline{\phi} = \sum_{\underline{X}} \overline{\phi}_{\underline{X}}$.

Proof. We know that a $*$matroid base polytope of codimension e is a product of e $*$matroid base polytopes of codimension 1. This shows that $\overline{\phi}$ is surjective. It remains to show that $\overline{\phi}$ is injective.

Suppose that $\overline{\phi}(u) = 0$ where $u = \sum_{\underline{X}} u_{\underline{X}}$, and $u_{\underline{X}} \in \overline{P}_{*\mathrm{M}}(\underline{X})$ for all $\underline{X}$. We have $\gamma_{\underline{X}} \circ \phi_{\underline{Y}} = 0$ if $\underline{X} \neq \underline{Y}$ by Lemma 3.8.2. It follows that $\overline{\gamma}_{\underline{X}}(\overline{\phi}_{\underline{X}}(u_{\underline{X}})) = \overline{\gamma}_{\underline{X}}(\overline{\phi}(u)) = 0$. We can lift $u_{\underline{X}}$ to an element $\widetilde{u}_{\underline{X}} \in P_{\star\mathrm{M}}(\underline{X})$. Then we have that

$$\gamma_{\underline{X}}(\phi_{\underline{X}}(\widetilde{u}_{\underline{X}})) = \sum_i a_i \mathbf{1}(\Lambda_i)$$

where the Λ_i are $*$matroid polytopes of codimension $> e$. We have that $\mathbf{1}(\Lambda_i) \in \operatorname{im} \phi_{\underline{Y}'}$ for some partition $\underline{Y}'$ with more than e parts. Therefore $\mathbf{1}(\Lambda_i) \in \operatorname{im} \phi_{\underline{Y}}$ as well for any coarsening $\underline{Y}$ of $\underline{Y}'$ with e parts, and we may choose $\underline{Y}$ so that $\underline{Y} \neq \underline{X}$, so by Lemma 3.8.2, $\gamma_{\underline{X}}(\mathbf{1}(\Lambda_i)) = 0$ for all i. Therefore, we have

$$\gamma_{\underline{X}}(\phi_{\underline{X}}(\widetilde{u}_{\underline{X}})) = \gamma_{\underline{X}}(\gamma_{\underline{X}}(\phi_{\underline{X}}(\widetilde{u}_{\underline{X}}))) = \sum_i a_i \gamma_{\underline{X}}(\mathbf{1}(\Lambda_i)) = 0.$$

Note that $\gamma_{\underline{X}}$ induces a map $\gamma'_{\underline{X}} : P_{*\mathrm{M}}(\underline{X}) \to P_{*\mathrm{M}}(\underline{X})$ such that $\phi_{\underline{X}} \circ \gamma'_{\underline{X}} = \gamma_{\underline{X}} \circ \phi_{\underline{X}}$. We have that

$$\phi_{\underline{X}}(\widetilde{u}_{\underline{X}}) = (\mathrm{id} - \gamma_{\underline{X}})(\phi_{\underline{X}}(\widetilde{u}_{\underline{X}})) = \phi_{\underline{X}}((\mathrm{id} - \gamma'_{\underline{X}})(\widetilde{u}_{\underline{X}})).$$

Since $\phi_{\underline{X}}$ is injective, we have

$$\widetilde{u}_{\underline{X}} = (\mathrm{id} - \gamma'_{\underline{X}})(\widetilde{u}_{\underline{X}}).$$

So $\widetilde{u}_{\underline{X}}$ lies in the image of $\mathrm{id} - \gamma_{\underline{X}}$.

For $*$matroid polytopes $\Pi_1, \ldots, \Pi_e$ of codimension 1 in $\mathbb{R}^{|X_1|}, \ldots, \mathbb{R}^{|X_e|}$ respectively, we have

$$\gamma'_{\underline{X}}(\mathbf{1}(\Pi_1 \times \cdots \times \Pi_e))(x) = 1.$$

for any relative interior point x of $\Pi_1 \times \cdots \times \Pi_e$. It follows that

$$(\mathrm{id} - \gamma'_{\underline{X}})(\mathbf{1}(\Pi_1 \times \cdots \times \Pi_e)) = \sum_F a_F \mathbf{1}(F)$$

where F runs over the proper faces of $\Pi_1 \times \cdots \times \Pi_r$ and $a_F \in \mathbb{Z}$ for all F. Therefore, the composition

$$P_{*\mathrm{M}}(\underline{X}) \xrightarrow{\mathrm{id} - \gamma'_{\underline{X}}} P_{*\mathrm{M}}(\underline{X}) \longrightarrow \overline{P}_{*\mathrm{M}}(\underline{X}, e)$$

is equal to 0. Since $u_{\underline{X}}$ is the image of $\widetilde{u}_{\underline{X}} = (\mathrm{id} - \gamma'_{\underline{X}})(\widetilde{u}_{\underline{X}})$, we have that $u_{\underline{X}} = 0$. □

Let $p_{(\mathrm{P})\mathrm{M}}(d, r, e)$ be the rank of $\overline{P}_{(\mathrm{P})\mathrm{M}}(d, r, e)$, and $t_{(\mathrm{P})\mathrm{M}}(d, r) := p_{(\mathrm{P})\mathrm{M}}(d, r, 1)$ be the rank of $T_{(\mathrm{P})\mathrm{M}}(d, r)$.

Proof of Theorem 3.1.6(d). From Theorem 3.8.3 follows that

$$\exp\left(\sum_{d,r\geq 0} \frac{t_{\mathrm{PM}}(d,r)x^d y^r u}{d!}\right) = \sum_{e\geq 0} \frac{1}{e!}\Big(\sum_{d,r\geq 0} \frac{t_{\mathrm{PM}}(d,r)x^d y^r u}{d!}\Big)^e = \sum_{e,d,r\geq 0} \frac{p_{\mathrm{PM}}(d,r,e)}{d!} x^d y^r u^e$$

If we substitute $u = 1$, we get

$$\exp\left(\sum_{d,r\geq 0} \frac{t_{\mathrm{PM}}(d,r)x^d y^r}{d!}\right) = \sum_{e,d,r\geq 0} \frac{p_{\mathrm{PM}}(d,r,e)}{d!} x^d y^r = \frac{e^x(1-y)}{1-ye^x}.$$

It follows that

$$\sum_{d,r\geq 0} \frac{t_{\mathrm{PM}}(d,r)x^d y^r}{d!} = \log\big(\frac{e^x(1-y)}{1-ye^x}\big) = x + \log(1-y) - \log(1-ye^x) = x + \sum_{r\geq 1} \frac{(e^{rx}-1)y^r}{r}.$$

Comparing the coefficients of $x^d y^r$ gives

$$t_{\mathrm{PM}}(d,r) = \begin{cases} r^{d-1} & \text{if } d \geq 1; \\ 0 & \text{otherwise.} \end{cases}$$

(Recall that $0^0 = 1$.) □

We also have

$$\sum_{d,r\geq 0} \frac{p_{\mathrm{PM}}(d,r,e)t^d s^r u^e}{d!} = \exp\left(\log\left(\frac{e^t(1-s)}{1-se^t}\right)u\right) = \left(\frac{e^t(1-s)}{1-se^t}\right)^u.$$

Proof of Theorem 3.1.6(c). The proof is similar to the proof of part (d). We have

$$\sum_{d,r\geq 0} \frac{t_{\mathrm{M}}(d,r)x^{d-r}y^r}{d!} = \log\left(\sum_{d,r,e} \frac{p_{\mathrm{M}}(d,r,e)x^{d-r}y^r}{d!}\right) = \log\left(\frac{x-y}{xe^{-x}-ye^{-y}}\right), \tag{3.8.2}$$

and

$$\sum_{d,r,z\geq 0} \frac{p_{\mathrm{M}}(d,r,e)x^{d-r}y^r z^e}{d!} = \left(\frac{x-y}{xe^{-x}-ye^{-y}}\right)^z. \quad \square$$

A table for the values $t_{(\mathrm{P})\mathrm{M}}(d,r)$ can be found in Appendix 3.B.

If $d \geq 1$, let $\mathfrak{t}_{\mathrm{PM}}(d,r)$ be the set of all pairs $(\underline{X},\underline{r}) \in \mathfrak{p}_{\mathrm{PM}}(d,r)$ such that $r_1 > 0$, and $d \notin X_{k-1}$, where k is the length of $\underline{X}$. Similarly, if $d \geq 2$, let $\mathfrak{t}_{\mathrm{M}}(d,r)$ be the set of all pairs $(\underline{X},\underline{r}) \in \mathfrak{t}_{\mathrm{M}}(d,r)$ such that $r_1 > 0$, $|X_{k-1}| - r_{k-1} < d - r$, and $d \notin X_{k-1}$.

Lemma 3.8.4. *We have $|\mathfrak{t}_{(\mathrm{P})\mathrm{M}}(d,r)| = t_{(\mathrm{P})\mathrm{M}}(d,r)$ whenever the former is defined.*

Proof. For polymatroids. We revisit the bijection $f : \mathfrak{p}_{\mathrm{PM}}(d,r) \to \mathfrak{a}(d,r)$ defined in the proof of Theorem 3.1.5(d). It is easy to see that $\underline{a} \in f(\mathfrak{t}_{\mathrm{PM}}(d,r))$ if and only if $a_d = r_k = r$ and no a_i equals 0. Accordingly such an $\underline{a}$ has the form $(a_1,\ldots,a_{d-1},r)$ with a_i freely chosen from $\{1,\ldots,r\}$ for each $i = 1,\ldots,d-1$, so $|f(\mathfrak{t}_{\mathrm{PM}}(d,r))| = r^{d-1}$.

For matroids. We proceed by means of generating functions. We begin by invoking the exponential formula: the coefficient of $x^{d-r}y^r$ of the generating function

$$\exp\left(\sum_{d=0}^{\infty}\sum_{r=0}^{d} \frac{|\mathfrak{t}_{\mathrm{M}}(d,r)|}{d!}x^{d-r}y^r\right)$$

enumerates the ways to choose a partition $[d] = Z_1 \cup \cdots \cup Z_l$ and a composition $r = s_1 + \cdots + s_l$ and an element $(\underline{X}_i,\underline{r}_i)$ of $\mathfrak{t}_{\mathrm{M}}(|Z_i|,s_i)$ for each $i = 1,\ldots,l$. Let us denote by $\mathfrak{q}(d,r)$ the set of tuples $([d],r,(\underline{X}^{(1)},\underline{r}^{(1)}),\ldots,(\underline{X}^{(l)},\underline{r}^{(l)}))$.

We describe a bijection between $\mathfrak{q}(d,r)$ and $\mathfrak{p}_{\mathrm{M}}^{\mathrm{sym}}(d,r)$. Roughly, given $(\underline{X},\underline{r}) \in \mathfrak{p}_{\mathrm{M}}^{\mathrm{sym}}(d,r)$, we break it into pieces, breaking after X_i whenever $X_i \setminus X_{i-1}$ contains the largest remaining element of $[d] \setminus X_{i-1}$. More formally, given $(\underline{X},\underline{r}) \in \mathfrak{p}_{\mathrm{M}}^{\mathrm{sym}}(d,r)$, for each $j \geq 1$ let $Z_j = X_{i_j} \setminus X_{i_j-1}$ (taking $i_0 = 0$) where i_j is minimal such that X_{i_j} contains the maximum element of $[d] \setminus X_{i_{j-1}}$, and let $s_j = r_{i_j} - r_{i_{j-1}}$. This definition eventually fails, in that we cannot find a maximum element when $X_{i_{j-1}} = X_k = [d]$, so we stop there and let l be such that $i_l = k$.

For $j = 1, \ldots, l$, let $f_j : Z_j \to [|Z_j|]$ be the unique order-preserving map, and define the chain and list of integers $(\underline{X}^{(j)}, \underline{r}^{(j)})$ by

$$\begin{aligned} X_i^{(j)} &= f_j(X_{i_{j-1}+i} \setminus X_{i_{j-1}}), && (i = 1, \ldots, i_j - i_{j-1}) \\ r_i^{(j)} &= r_{i_{j-1}+i} - r_{i_{j-1}}. && (i = 1, \ldots, i_j - i_{i-1}) \end{aligned}$$

We have that $(\underline{X}^{(j)}, \underline{r}^{(j)}) \in \mathfrak{t}_{\mathrm{M}}(|Z_j|, s_j)$: the crucial property that $d \notin X_{k-1}$ obtains by choice of i_j and monotonicity of f_j. This finishes defining the bijection. Its inverse is easily constructed.

From this bijection and (3.8.2) it follows that

$$\exp\left(\sum_{d=1}^{\infty}\sum_{r=0}^{d}\frac{|\mathfrak{t}_{\mathrm{M}}(d,r)|}{d!}x^{d-r}y^r\right) = 1 + \sum_{d\geq 1}\sum_{r}\frac{|\mathfrak{q}(d,r)|x^{d-r}y^r}{d!} = \\ = \sum_{d,r}\frac{p_{\mathrm{M}}(d,r)x^{d-r}y^r}{d!} = \exp\left(\sum_{d=1}^{\infty}\sum_{r=0}^{d}\frac{t_{\mathrm{M}}(d,r)}{d!}x^{d-r}y^r\right).$$

□

Lemma 3.8.5. *The classes of* $\mathbf{1}(R_{(\mathrm{P})\mathrm{M}}(\underline{X}, \underline{r}))$ *for* $(\underline{X}, \underline{r}) \in \mathfrak{t}_{(\mathrm{P})\mathrm{M}}(d, r)$ *are linearly independent in* $T_{(\mathrm{P})\mathrm{M}}(d, r)$.

Proof. Let $y = (-1, \ldots, -1, d-1)$. Let $\Pi_{(\mathrm{P})\mathrm{M}}$ be the set of points $x \in \Delta_{(\mathrm{P})\mathrm{M}}(d, r)$ such that $x + \varepsilon y \in \Delta_{(\mathrm{P})\mathrm{M}}(d, r)$ for sufficiently small $\varepsilon > 0$. Choose some $(\underline{X}, \underline{r}) \in \mathfrak{t}_{(\mathrm{P})\mathrm{M}}(d, r)$. If $x \in R_{(\mathrm{P})\mathrm{M}}(\underline{X}, \underline{r}) \cap \Pi_{(\mathrm{P})\mathrm{M}}$, and $\varepsilon > 0$ is sufficiently small, we have $x + \varepsilon y \in R_{(\mathrm{P})\mathrm{M}}(\underline{X}, \underline{r})$, since the defining inequalities of $R_{(\mathrm{P})\mathrm{M}}(\underline{X}, \underline{r})$ involve only the variables $x_1, \ldots, x_{d-1}$. It follows that for $x \in \Pi_{(\mathrm{P})\mathrm{M}}$ we have

$$\mathbf{1}(R_{(\mathrm{P})\mathrm{M}}(\underline{X}, \underline{r}))(x) = \gamma_y(\mathbf{1}(R_{(\mathrm{P})\mathrm{M}}(\underline{X}, \underline{r})))(x).$$

We will write $\{[d]\}$ for the partition $[d] = \{1\} \cup \{2\} \cup \cdots \cup \{d\}$. Observe that y is $\{[d]\}$-balanced, so that for any point x, $x + \varepsilon y$ is $\{[d]\}$-regular for sufficiently small $\varepsilon > 0$.

Suppose the sum

$$S = \sum_{(\underline{X}, \underline{r}) \in \mathfrak{t}_{(\mathrm{P})\mathrm{M}}(d,r)} a(\underline{X}, \underline{r})\mathbf{1}(R_{(\mathrm{P})\mathrm{M}}(\underline{X}, \underline{r}))$$

vanishes in $T_{(\mathrm{P})\mathrm{M}}(d, r)$, i.e. is contained in $P_{(\mathrm{P})\mathrm{M}}(d, r, 2)$. Then the support of S contains no $\{[d]\}$-regular points. So for any $x \in \Pi_{(\mathrm{P})\mathrm{M}}$ we have $S(x) = \gamma_y(S)(x) = 0$.

We specialize now to the matroid case. If $r = d$, then $T_{\mathrm{M}}(d, r) = 0$ and the result is trivial. Otherwise let H be the hyperplane $\{x_d = 0\}$; we will examine the situation on restriction to H. Identifying H with $\mathbb{R}^{d-1}$ in the obvious fashion, we have $\Delta_{\mathrm{M}}(d, r) \cap H =$

$\Delta_{\mathrm{M}}(d-1,r)$, $\Pi_{\mathrm{M}} \cap H = \{x \in \Delta_{\mathrm{M}}(d-1,r) : x_i \neq 0 \text{ for all } i\}$. For any $(\underline{X},\underline{r}) \in \mathfrak{t}_{\mathrm{M}}(d,r)$, $R_{\mathrm{M}}(\underline{X},\underline{r}) \cap H = R_{\mathrm{M}}(\underline{X}',\underline{r}')$ where, supposing $\underline{X}$ has length k,

$$\underline{X}'' : \emptyset \subset X_1 \subset \cdots \subset X_{k-1} \subset X_k \setminus \{d\} = [d-1]$$

and $(\underline{X}',\underline{r}')$ is obtained from $(\underline{X}'',\underline{r})$ by dropping redundant entries as in the proof of Theorem 3.5.4.

Suppose $T \in P_{\mathrm{M}}(d-1,r)$ is supported on $\{x_i = 0\}$. By Theorem 3.5.4 we have a unique expression

$$T = \sum_{(\underline{X},\underline{r}) \in \mathfrak{p}_{\mathrm{M}}(d-1,r)} b(\underline{X},\underline{r}) \mathbf{1}(R_{\mathrm{M}}(\underline{X},\underline{r})).$$

But we also have

$$T = T|_{\{x_i=0\}} = \sum_{(\underline{X},\underline{r}) \in \mathfrak{p}_{\mathrm{M}}(d-1,r)} b(\underline{X},\underline{r}) \mathbf{1}(R_{\mathrm{M}}(\underline{X},\underline{r}) \cap \{x_i = 0\})$$

in which each $\mathbf{1}(R_{\mathrm{M}}(\underline{X},\underline{r}) \cap \{x_i = 0\})$ is either zero or another $\mathbf{1}(R_{\mathrm{M}}(\underline{X}',\underline{r}'))$, so that by uniqueness $b(\underline{X},\underline{r}) = 0$ when $R_{\mathrm{M}}(\underline{X},\underline{r}) \not\subseteq \{x_i = 0\}$.

The restriction $S|_H$ is supported on

$$\Delta_{\mathrm{M}}(d-1,r) \cap \left(\bigcup_{i=1}^{d-1} \{x_i = 0\} \right),$$

so it is a linear combination of those $\mathbf{1}(R_{\mathrm{M}}(\underline{X},\underline{r}))$ supported on some $\{x_i = 0\}$, i.e. those for which $r_1 = 0$. On the other hand,

$$S|_H = \sum_{(\underline{X},\underline{r}) \in \mathfrak{t}_{\mathrm{M}}(d,r)} a(\underline{X},\underline{r}) \mathbf{1}(R_{\mathrm{M}}(\underline{X},\underline{r}) \cap H)$$

in which each $R_{\mathrm{M}}(\underline{X},\underline{r}) \cap H$ is another matroid polytope $R_{\mathrm{M}}(\underline{X}',\underline{r})$ with $r_1 > 0$ (and X' only differing from X by dropping the d in the kth place). Note that $(\underline{X},\underline{r}) \in \mathfrak{t}_{\mathrm{M}}(d,r)$ is completely determined by $R_{\mathrm{M}}(\underline{X},\underline{r}) \cap H$. Therefore, by Theorem 3.5.4, $a(\underline{X},\underline{r}) = 0$ for all $(\underline{X},\underline{r}) \in \mathfrak{t}_{\mathrm{M}}(d,r)$.

The polymatroid case is similar, but in place of the hyperplane H we use all the hyperplanes $H_i = \{x_d = i\}$ for $i = 0, \ldots, r-1$.

Note that $\Delta_{\mathrm{PM}}(d,r) \cap H_i = \Delta_{\mathrm{PM}}(d, r-i)$. For $(\underline{X},\underline{r}) \in \mathfrak{t}_{\mathrm{PM}}(d,r)$, supposing $\underline{X}$ has length k,

$$R_{\mathrm{PM}}(\underline{X},\underline{r}) \cap H_i = \begin{cases} R_{\mathrm{M}}(\underline{X}',\underline{r}') & r_{k-1} \geq r-i \\ \emptyset & \text{otherwise} \end{cases}$$

where again

$$\underline{X}'' : \emptyset \subset X_1 \subset \cdots \subset X_{k-1} \subset X_k \setminus \{d\} = [d-1]$$

and

$$r'' = (r_1, r_2, \ldots, r_{k-1}, r_k - i).$$

and $(\underline{X}', \underline{r}')$ is obtained from $(\underline{X}'', \underline{r}'')$ by dropping redundant entries as in the proof of Theorem 3.5.4. Although $(\underline{X}, \underline{r}) \in \mathfrak{t}_{\mathrm{PM}}(d, r)$ is not completely determined by $S\mid_{H_0}$, the arguments in the matroid case still show that $S\mid_{H_0} = 0$, and $a(\underline{X}, \underline{r}) = 0$ for all $(\underline{X}, \underline{r})$ for which $X_{k-1} \neq [d-1]$. Restricting to H_{r-1} shows that $a(\underline{X}, \underline{r}) = 0$ for all $(\underline{X}, \underline{r})$ for which $X_{k-1} = [d-1]$ and $r_k = 1$. Proceeding by induction on i, we restrict S to H_{r-i} and see that $a(\underline{X}, \underline{r}) = 0$ for all $(\underline{X}, \underline{r})$ for which $X_{k-1} = [d-1]$ and $r_{k-1} = i$. □

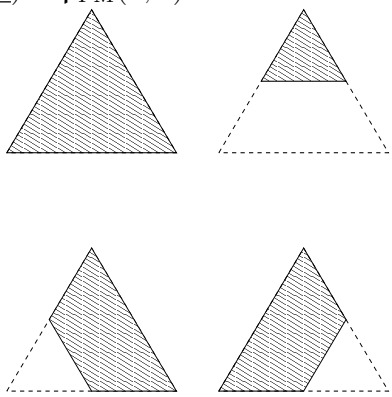
The following is an immediate consequence of Lemmas 3.8.4 and 3.8.5.

Theorem 3.8.6. *The group $T_{(\mathrm{P})\mathrm{M}}(d, r)$ is freely generated by all $\mathbf{1}(R_{(\mathrm{P})\mathrm{M}}(\underline{X}, \underline{r}))$ with $(\underline{X}, \underline{r}) \in \mathfrak{t}_{(\mathrm{P})\mathrm{M}}(d, r)$.*

Example 3.8.7. Consider again Example 3.6.7. The set $t_{\mathrm{PM}}(3, 2)$ consists of the following elements:

$\underline{X}: \ \{1,2,3\}$ $\underline{r} = \ (2)$	$\underline{X}: \ \{1,2\} \subset \{1,2,3\}$ $\underline{r} = \ (1,2)$
$\underline{X}: \ \{1\} \subset \{1,2,3\}$ $\underline{r} = \ (1,2)$	$\underline{X}: \ \{2\} \subset \{1,2,3\}$ $\underline{r} = \ (1,2)$

The polytopes $R_{\mathrm{PM}}(\underline{X}, \underline{r})$, $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{PM}}(3, 2)$ are

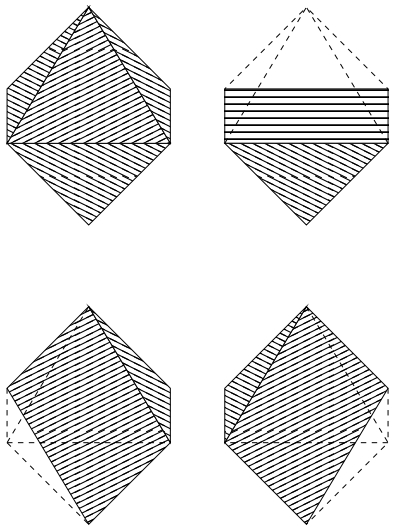


♢

Example 3.8.8. Consider again Example 3.6.8. The set $t_{\mathrm{M}}(4, 2)$ consists of the following elements:

$\underline{X}: \ \{1,2,3,4\}$ $\underline{r} = \ (2)$	$\underline{X}: \ \{1,2\} \subset \{1,2,3,4\}$ $\underline{r} = \ (1,2)$
$\underline{X}: \ \{1,3\} \subset \{1,2,3,4\}$ $\underline{r} = \ (1,2)$	$\underline{X}: \ \{2,3\} \subset \{1,2,3,4\}$ $\underline{r} = \ (1,2)$

The polytopes $R_{\mathrm{M}}(\underline{X}, \underline{r})$, $(\underline{X}, \underline{r}) \in \mathfrak{p}_{\mathrm{M}}(4, 2)$ are



♢

3.9 Additive invariants: the groups $T^{\mathrm{sym}}_{\mathrm{M}}$, $T^{\mathrm{sym}}_{\mathrm{PM}}$, $T^{\mathrm{sym}}_{\mathrm{MM}}$

The algebra $P^{\mathrm{sym}}_{\star M}$ of indicator functions of $*$-matroid polyhedra mod symmetries has a natural filtration:

$$\cdots \subseteq P^{\mathrm{sym}}_{\star \mathrm{M}}(d,r,2) \subseteq P^{\mathrm{sym}}_{\star \mathrm{M}}(d,r,1) \subseteq P^{\mathrm{sym}}_{\star M}(d,r,0) = P^{\mathrm{sym}}_{\star \mathrm{M}}(d,r).$$

Here $P^{\mathrm{sym}}_{\star \mathrm{M}}(d,r,e)$ is spanned by the indicator functions of all $*$matroid base polytopes of rank r and dimension $d-e$. Define $\overline{P}^{\mathrm{sym}}_{\star \mathrm{M}}(d,r,e) = P^{\mathrm{sym}}_{* \mathrm{M}}(d,r,e)/P^{\mathrm{sym}}_{* \mathrm{M}}(d,r,e+1)$. Let $\overline{P}^{\mathrm{sym}}_{\star \mathrm{M}} = \bigoplus_{d,r,e} \overline{P}^{\mathrm{sym}}_{\star \mathrm{M}}(d,r,e)$ be the associated graded algebra.

Define $T^{\mathrm{sym}}_{\star \mathrm{M}} = \bigoplus_{d,r} \overline{P}^{\mathrm{sym}}_{\star \mathrm{M}}(d,r,1)$. The following corollary follows from Theorem 3.8.3.

Theorem 3.9.1. *The algebra $\overline{P}^{\mathrm{sym}}_{\star \mathrm{M}}$ is the free symmetric algebra $S(T^{\mathrm{sym}}_{\star \mathrm{M}})$ on $T^{\mathrm{sym}}_{\star \mathrm{M}}$, and there exists an isomorphism*

$$S^e(T^{\mathrm{sym}}_{\star \mathrm{M}}) \cong \bigoplus_{d,r} \overline{P}^{\mathrm{sym}}_{\star \mathrm{M}}(d,r,e). \tag{3.9.1}$$

Proof. If we sum the isomorphism (3.8.1) in Theorem 3.8.3 over all d, we get an isomorphism

$$\bigoplus_{d,\underline{X}} \overline{P}(\underline{X}) \to \bigoplus_{d,r} \overline{P}_{\star M}(d,r,e)$$

where the sum on the left-hand side is over all d and all partitions $\underline{X}$ of $[d]$ into e nonempty subsets. If we divide out the symmetries on both sides, we get the isomorphism (3.9.1). □

Corollary 3.9.2. *The algebra $P^{\mathrm{sym}}_{\star \mathrm{M}}$ is a polynomial ring over $\mathbb{Z}$.*

Proof. Consider the surjective map

$$\bigoplus_{d,r} P^{\mathrm{sym}}_{\star \mathrm{M}}(d,r,1) \to \bigoplus_{d,r} \overline{P}^{\mathrm{sym}}_{\star \mathrm{M}}(d,r,1) = T^{\mathrm{sym}}_{\star \mathrm{M}}.$$

Suppose that G is a set of $\mathbb{Z}$-module generators of $T^{\mathrm{sym}}_{\star \mathrm{M}}$. Each element of G can be lifted to $\bigoplus_{d,r} P^{\mathrm{sym}}_{\star M}(d,r,e)$. Let $\widetilde{G}$ be the set of all lifts. Since G generates $\overline{P}^{\mathrm{sym}}_{\star M}$ by Theorem 3.9.1, $\widetilde{G}$ generates $P^{\mathrm{sym}}_{\star M}$ over $\mathbb{Z}$. Since G is an algebraically independent set, so is $\widetilde{G}$. So $P^{\mathrm{sym}}_{\star \mathrm{M}}$ is a polynomial ring over $\mathbb{Z}$, generated by $\widetilde{G}$. □

Proof of Theorem 3.1.6(a),(b). We prove the stated formulas after taking the reciprocal of both sides. Let $p^{\mathrm{sym}}_{\star \mathrm{M}}(d,r,e)$ be the rank of $\overline{P}^{\mathrm{sym}}_{\star \mathrm{M}}(d,r,e)$. Define $t^{\mathrm{sym}}_{\star \mathrm{M}}(d,r) := p_{\star \mathrm{M}}(d,r,1)$ as the rank of $T^{\mathrm{sym}}_{\star M}(d,r)$. From the matroid case of Theorem 3.9.1 follows that

$$\prod (1-x^r y^{d-r})^{-t^{\mathrm{sym}}_{\mathrm{M}}(d,r)} = \frac{1}{1-x-y}$$

and

$$\prod (1-ux^r y^{d-r})^{-t^{\mathrm{sym}}_{\mathrm{M}}(d,r)} = \sum_{d,r} p^{\mathrm{sym}}_{\mathrm{M}}(d,r,e) u^e x^r y^{d-r}$$

From the polymatroid case follows that

$$\prod (1-x^d y^r)^{-t^{\mathrm{sym}}_{\mathrm{PM}}(d,r)} = \frac{1-y}{1-x-y},$$

and

$$\prod (1-zx^d y^r)^{-t^{\mathrm{sym}}_{\mathrm{PM}}(d,r)} = \sum_{d,r} p^{\mathrm{sym}}_{\mathrm{PM}}(d,r,e) z^e x^d y^r.$$

□

3.10 Invariants as elements in free algebras

Let

$$(P^{\mathrm{sym}}_{*\mathrm{M}})^{\#} := \bigoplus_{d,r} P^{\mathrm{sym}}_{*\mathrm{M}}(d,r)^{\vee}$$

be the *graded* dual of $P^{\mathrm{sym}}_{*\mathrm{M}}$.

Proof of Theorem 3.1.7(a),(b). A basis of $(P^{\mathrm{sym}}_{\mathrm{PM}})^{\#} \otimes_{\mathbb{Z}} \mathbb{Q}$ is given by all u_α where α runs over all sequences of nonnegative integers, and a basis of $(P^{\mathrm{sym}}_{\mathrm{M}})^{\#} \otimes_{\mathbb{Z}} \mathbb{Q}$ is given by all u_α where

α is a sequence of 0s and 1s (see Corollaries 3.6.5 and 3.6.6). The multiplication in $(P^{\mathrm{sym}}_{*\mathrm{M}})^{\#}$ is given by

$$u_\alpha \cdot u_\beta = \binom{d+e}{d} u_{\alpha\beta},$$

where α has length d and β has length e. It follows that $(P^{\mathrm{sym}}_{\mathrm{PM}})^{\#} \otimes_{\mathbb{Z}} \mathbb{Q}$ is the free associative algebra $\mathbb{Q}\langle u_0, u_1, u_2, \dots \rangle$ generated by $u_0, u_1, u_2, \dots$ and $(P^{\mathrm{sym}}_{\mathrm{M}})^{\#} \otimes_{\mathbb{Z}} \mathbb{Q}$ is the free associative algebra $\mathbb{Q}\langle u_0, u_1\rangle$ (the binomial coefficients make no difference). The ordinary dual, $(P^{\mathrm{sym}}_{*\mathrm{M}})^{\vee}$ is a completion of the graded dual $(P^{\mathrm{sym}}_{*\mathrm{M}})^{\#}$. We get that $(P^{\mathrm{sym}}_{\mathrm{PM}})^{\vee} \otimes_{\mathbb{Z}} \mathbb{Q}$ is equal to $\mathbb{Q}\langle\langle u_0, u_1, u_2, \dots \rangle\rangle$ and $(P^{\mathrm{sym}}_{\mathrm{M}})^{\vee} \otimes_{\mathbb{Z}} \mathbb{Q}$ is equal to $\mathbb{Q}\langle\langle u_0, u_1\rangle\rangle$. □

Let $\mathfrak{m}_{\star\mathrm{M}} = \bigoplus_{d,r} P^{\mathrm{sym}}_{*\mathrm{M}}(d, r, 1)$. Then we have $\mathfrak{m}^2_{\star\mathrm{M}} = \bigoplus_{d,r} P^{\mathrm{sym}}_{*\mathrm{M}}(d, r, 2)$ and $T^{\mathrm{sym}}_{*\mathrm{M}} = \mathfrak{m}_{*\mathrm{M}}/\mathfrak{m}^2_{*\mathrm{M}}$.

The graded dual $\mathfrak{m}^{\#}_{\star\mathrm{M}}$ can be identified with

$$(P^{\mathrm{sym}}_{\star\mathrm{M}})^{\#}/P^{\mathrm{sym}}_{*\mathrm{M}}(0,0) \cong \bigoplus_{d=1}^{\infty} \bigoplus_{r} P^{\mathrm{sym}}_{\star\mathrm{M}}(d, r)^{\vee}.$$

So $\mathfrak{m}^{\#}_{\mathrm{PM}} \otimes_{\mathbb{Z}} \mathbb{Q}$ will be identified with the ideal $(u_0, u_1, \dots)$ of $\mathbb{Q}\langle u_0, u_1, \dots \rangle$ and $\mathfrak{m}^{\#}_{\mathrm{M}} \otimes_{\mathbb{Z}} \mathbb{Q}$ will be identified with the ideal (u_0, u_1) of $\mathbb{Q}\langle u_0, u_1\rangle$. The graded dual $(T^{\mathrm{sym}}_{\mathrm{PM}})^{\#} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a subalgebra (without 1) of the ideal $(u_0, u_1, \dots)$, and $(T^{\mathrm{sym}}_{\mathrm{PM}})^{\#} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a subalgebra of (u_0, u_1).

Lemma 3.10.1.

a. $u_0, u_1 \in (T^{\mathrm{sym}}_{\mathrm{M}})^{\vee} \otimes_{\mathbb{Z}} \mathbb{Q}$, *and* $u_i \in (T^{\mathrm{sym}}_{\mathrm{PM}})^{\vee} \otimes_{\mathbb{Z}} \mathbb{Q}$ *for all* i;

b. If $f, g \in (T^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}})^{\vee} \otimes_{\mathbb{Z}} \mathbb{Q}$, *then* $[f, g] = fg - gf \in (T^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}})^{\vee} \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof. Part (a) is clear. Suppose that $f, g \in (T^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}})^{\vee}$. Suppose that $a, b \in \mathfrak{m}_{\mathrm{PM}}$. We can write $\Delta(a) = a \otimes 1 + 1 \otimes a + a'$ and $\Delta(b) = b \otimes 1 + 1 \otimes b + b'$ where $a', b' \in \mathfrak{m}_{\mathrm{PM}} \otimes \mathfrak{m}_{\mathrm{PM}}$. Note that $a'(b \otimes 1), a'(1 \otimes b), a'b', b'(a \otimes 1), b'(1 \otimes a)$ lie in $\mathfrak{m}^2_{\mathrm{PM}} \otimes \mathfrak{m}_{\mathrm{PM}}$ or $\mathfrak{m}_{\mathrm{PM}} \otimes \mathfrak{m}^2_{\mathrm{PM}}$. It follows that

$$\begin{aligned} fg(ab) = (f \otimes g)((a \otimes 1 + 1 \otimes a)(b \otimes 1 + 1 \otimes b)) = \\ = f \otimes g(ab \otimes 1 + a \otimes b + b \otimes a + 1 \otimes ab) = f(a)g(b) + f(b)g(a). \end{aligned}$$

Similarly $gf(ab) = f(a)g(b) + f(b)g(a)$. We conclude that $[f, g](ab) = 0$. □

Proof of Theorem 3.1.7(c),(d). From Lemma 3.10.1 follows that $(T^{\mathrm{sym}}_{\mathrm{PM}})^{\#} \otimes_{\mathbb{Z}} \mathbb{Q}$ contains the free Lie algebra $Q\{u_0, u_1, u_2, \dots\}$ generated by $u_0, u_1, \dots,$ and $(T^{\mathrm{sym}}_{\mathrm{M}})^{\#} \otimes_{\mathbb{Z}} \mathbb{Q}$ contains $\mathbb{Q}\{u_0, u_1\}$. By the Poincaré-Birkhoff-Witt theorem, the graded Hilbert series of

$(P^{\mathrm{sym}}_{\mathrm{PM}})^{\#} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}\langle u_0, u_1, \dots \rangle$ is equal to the graded Hilbert series of the symmetric algebra on $\mathbb{Q}\{u_0, u_1, \dots\}$. On the other hand, the Hilbert series of $P^{\mathrm{sym}}_{\mathrm{PM}} \otimes \mathbb{Z}\mathbb{Q}$ is equal to the Hilbert series on the symmetric algebra on $T^{\mathrm{sym}}_{\mathrm{PM}} \otimes_{\mathbb{Z}} \mathbb{Q}$. So $(T^{\mathrm{sym}}_{\mathrm{PM}})^{\#} \otimes_Z \mathbb{Q}$ and $\mathbb{Q}\{u_0, u_1, \dots\}$ have the same graded Hilbert series, and must therefore be equal. If we take the completion, we get $(T^{\mathrm{sym}}_{\mathrm{PM}})^{\vee} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}\{\{u_0, u_1, \dots\}\}$. The proof for matroids is similar and $(T^{\mathrm{sym}}_{\mathrm{M}})^{\vee} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}\{\{u_0, u_1\}\}$. □

One can choose a basis in the free Lie algebra. We will use the Lyndon basis. A word (in some alphabet A with a total ordering) is a *Lyndon word* if it is strictly smaller than any cyclic permutation of w with respect to the lexicographic ordering. In particular, Lyndon words are aperiodic. If $\alpha \in \mathbb{N}$, we define $b(\alpha) := u_\alpha$. If $\alpha = \alpha_1\alpha_2 \cdots \alpha_d$ is a Lyndon word of length $d > 1$, we define $b(\alpha) = [b(u_\beta), b(u_\gamma)]$ where γ is a Lyndon word of maximal length for which $\alpha = \beta\gamma$ and β is a nontrivial word. The Lyndon basis of $\mathbb{Q}\{u_0, u_1\}$ (respectively $\mathbb{Q}\{u_0, u_1, \dots\}$) is the set of all $b(\alpha)$ where α is a word in $\{0, 1\}$ (respectively $\mathbb{N}$). For details, see [74]. Define $\mathfrak{t}^{\mathrm{sym}}_{\mathrm{M}}(d, r)$ (respectively $\mathfrak{t}^{\mathrm{sym}}_{\mathrm{PM}}(d, r)$) as the set of all Lyndon words α in the alphabet $\{0, 1\}$ (respectively $\mathbb{N}$) of length d with $|\alpha| = d$. The following theorem follows.

Theorem 3.10.2. *The space $(T^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}})^{\vee}(d, r) \otimes_{\mathbb{Z}} \mathbb{Q}$ of $\mathbb{Q}$-valuative additive invariants for (poly)-matroids on $[d]$ of rank r has the basis given by all $b(\alpha)$ with $\alpha \in \mathfrak{p}^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d, r)$.*

Example 3.10.3. For $d = 6$, $r = 3$ we have

$$\mathfrak{t}^{\mathrm{sym}}_{\mathrm{M}}(6, 3) = \{000111, 001011, 001101\}$$

and

$$\mathfrak{t}^{\mathrm{sym}}_{\mathrm{PM}}(6, 3) = \{000003, 000012, 000021, 000102, 000111, 000201, 001002, 001011, 001101\}.$$

♢

Proposition 3.10.4. *The Hopf algebra $P^{\mathrm{sym}}_{\mathrm{PM}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to the ring QSym of quasi-symmetric functions over $\mathbb{Q}$.*

Proof. If we set $u_i = p_{i+1}$ then the associative algebra $P^{\mathrm{sym}}_{\mathrm{PM}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to $NSym = \mathbb{Q}\langle p_1, p_2, \dots \rangle$. The ring *NSym* has a Hopf algebra structure with $\Delta(p_i) = p_i \otimes 1 + 1 \otimes p_i$ (see [29, §7.2]). The reader may verify that

$$\Delta(u_i) = \sum u_i \otimes 1 + 1 \otimes u_i.$$

This shows that the isomorphism is a Hopf-algebra isomorphism. It follows that $P^{\mathrm{sym}}_{\mathrm{PM}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to *QSym*, the Hopf-dual of *NSym*. □

If we identify $P_{\mathrm{PM}}^{\mathrm{sym}} \otimes_{\mathbb{Z}} \mathbb{Q}$ with $QSym$, then $\mathcal{G}$ is equal to $\psi_{\mathrm{PM}}^{\mathrm{sym}}$.

If a multiplicative invariant is also valuative, then there exists a group homomorphism $\widehat{f} : P_{*\mathrm{M}}^{\mathrm{sym}} \to A$ such that $f = \widehat{f} \circ \psi_{*\mathrm{M}}^{\mathrm{sym}}$. Since $\psi_{*\mathrm{M}}^{\mathrm{sym}}$ is onto, $\widehat{f}$ is a ring homomorphism as well. So there is a bijection between valuative, multiplicative invariants with values in A, and ring homomorphisms $\widehat{f} : P_{\star\mathrm{M}}^{\mathrm{sym}} \to A$. By Corollary 3.9.2, the ring $P_{\star\mathrm{M}}^{\mathrm{sym}}$ is a polynomial ring, so ring homomorphisms $P_{\star\mathrm{M}}^{\mathrm{sym}} \to A$ are in bijection with set maps to A from a set of generators $\widetilde{G}$ of $P_{\star\mathrm{M}}^{\mathrm{sym}}$. One such set is a lift of a basis of $\mathfrak{m}_{\star\mathrm{M}}/\mathfrak{m}_{\star\mathrm{M}}^2$. The next corollary follows.

Corollary 3.10.5. *The set of valuative, multiplicative invariants on the set of $*$matroids with values in A is isomorphic to* $\mathrm{Hom}_{\mathbb{Z}}(\mathfrak{m}_{*M}/\mathfrak{m}_{*M}^2, A)$.

3.A Equivalence of the weak and strong valuative properties

In this appendix we will prove that the weak valuative property and the strong valuative property are equivalent.

For a megamatroid polyhedron Π, let $\mathrm{vert}(\Pi)$ be the vertex set of the polyhedron. Let $W_{\mathrm{MM}}(d, r)$ be the subgroup of $Z_{\mathrm{MM}}(d, r)$ generated by all $m_{\mathrm{val}}(\Pi; \Pi_1, \ldots, \Pi_k)$ where $\Pi = \Pi_1 \cup \cdots \cup \Pi_k$ is a megamatroid polyhedron decomposition. Define $W_{\mathrm{MM}}(d, r, V)$ as the subgroup generated by all the $m_{\mathrm{val}}(\Pi; \Pi_1, \ldots, \Pi_k)$ where $\mathrm{vert}(\Pi) \subseteq V$.

A megamatroid $\mathrm{rk} : 2^{[d]} \to \mathbb{Z} \cup \infty$ is called *bounded from above* if $\mathrm{rk}([i]) < \infty$ for $i = 1, 2, \ldots, d$. The group $W_{\mathrm{MM}}^{+}(d, r)$ is the subgroup of $Z_{\mathrm{MM}}(d, r)$ generated by all $m_{\mathrm{val}}(\Pi; \Pi_1, \ldots, \Pi_k)$ where Π is bounded from above, and $W_{\mathrm{MM}}^{+}(d, r, V)$ is the subgroup of $Z_{\mathrm{MM}}(d, r)$ generated by all $m_{\mathrm{val}}(\Pi; \Pi_1, \ldots, \Pi_k)$ where Π is bounded from above and $\mathrm{vert}(\Pi) \subseteq V$.

Lemma 3.A.1. *If* rk *is a megamatroid bounded from above, then there exist megamatroids* $\mathrm{rk}_1, \ldots, \mathrm{rk}_k$ *which are bounded from above and integers* $a_1, \ldots, a_k$ *such that*

$$\langle \mathrm{rk} \rangle - \sum_{i=1}^{k} a_i \langle \mathrm{rk}_i \rangle \in W_{\mathrm{MM}}^{+}(d, r, \mathrm{vert}(\Pi))$$

and $\mathrm{vert}(\mathrm{Poly}(\mathrm{rk}_i))$ *consists of a single vertex of* $\Pi := \mathrm{Poly}(\mathrm{rk})$ *for all* i.

This lemma follows from the Lawrence-Varchenko polar decomposition of Poly(rk) [58, 89]. For explicitness we give a proof.

Proof. Let T be the group generated by $W_{\mathrm{MM}}^{+}(d, r, \mathrm{vert}(\Pi))$ and all megamatroid polyhedra Γ which are bounded from above, and whose vertex set consists of a single element of $\mathrm{vert}(\Pi)$. We prove the lemma by induction on $|\mathrm{vert}(\mathrm{Poly}(\mathrm{rk}))|$. If $|\mathrm{vert}(\Pi)| = 1$ then the result is

clear. Otherwise, we can find vertices v and w of Π such that $v - w$ is parallel to $e_i - e_j$ for some i, j with $i > j$. Consider the half-line $L = \mathbb{R}_{\geq 0}(e_i - e_j)$ where $\mathbb{R}_{\geq 0}$ is the set of nonnegative real numbers. Let $\Pi + L$ be the Minkowski sum. Let us call a facet F of Π a *shadow facet* if $(F + L) \cap \Pi = F$. Suppose that $F_1, \ldots, F_j$ are the shadow facets of Π.

We have a megamatroid polyhedron decomposition

$$\Pi + L = \Pi \cup (F_1 + L) \cup \cdots \cup (F_j + L).$$

Note that $\Pi + L, F_1 + L, \ldots, F_j + L$ are bounded from above. The set $\mathrm{vert}(\Pi + L)$ is a proper subset of $\mathrm{vert}(\Pi)$ because it cannot contain both v and w. Also $\mathrm{vert}(F_i + L)$ is contained in $\mathrm{vert}(F_i)$ for all i, and is therefore a proper subset of $\mathrm{vert}(\Pi)$ for all shadow facets F. The element

$$\langle \mathrm{rk} \rangle + m_{\mathrm{val}}(\Pi + L; \Pi, F_1 + L, \ldots, F_j + L)$$

is an integral combination of terms $\langle \mathrm{rk}' \rangle$ where $\mathrm{Poly}(\mathrm{rk}')$ is a face of $\Pi + L$ or a face of $F_i + L$ for some i. In particular, for each such term $\langle \mathrm{rk}' \rangle$, the polyhedron $\mathrm{Poly}(\mathrm{rk}')$ is bounded from above, and $\mathrm{vert}(\mathrm{Poly}(\mathrm{rk}'))$ is a proper subset of $\mathrm{vert}(\mathrm{Poly}(\mathrm{rk}))$. Hence by induction

$$\langle \mathrm{rk} \rangle + m_{\mathrm{val}}(\Pi + L; \Pi, F_1 + L, \ldots, F_j + L) \in T.$$

Now it follows that $\langle \mathrm{rk} \rangle \in T$. □

Proposition 3.A.2. *Suppose that* $\mathrm{rk}_1, \ldots, \mathrm{rk}_k$ *are megamatroids which are bounded from above and* $a_1, \ldots, a_k$ *are integers such that*

$$\sum_{i=1}^{k} a_i \mathbf{1}(\mathrm{Poly}(\mathrm{rk}_i)) = 0.$$

Then we have

$$\sum_{i=1}^{k} a_i \langle \mathrm{rk}_i \rangle \in W^{+}_{\mathrm{MM}}(d, r, V)$$

where $V = \bigcup_{i=1}^{k} \mathrm{vert}(\mathrm{Poly}(\mathrm{rk}_i))$.

Proof. First, assume that $\mathrm{Poly}(\mathrm{rk}_i)$ has only one vertex for all i. We prove the proposition by induction on d, the case $d = 1$ being clear. We will also use induction on k, the case $k = 0$ being obvious.

For vectors $y = (y_1, \ldots, y_d)$ and $z = (z_1, \ldots, z_d)$, we say that $y > z$ in the lexicographic ordering if there exists an i such that $y_j = z_j$ for $j = 1, 2, \ldots, i - 1$ and $y_i > z_i$. If rk is a megamatroid bounded from above, and $\mathrm{Poly}(\mathrm{rk})$ has only one vertex v, then v is the largest element of $\mathrm{Poly}(\mathrm{rk})$ with respect to the lexicographic ordering.

Assume $V = \{v_1, \ldots, v_m\}$, where $v_1 > v_2 > \cdots > v_m$ in the lexicographical ordering. Assume that $\mathrm{Poly}(\mathrm{rk}_1), \ldots, \mathrm{Poly}(\mathrm{rk}_n)$ are the only megamatroids among

$\mathrm{Poly}(\mathrm{rk}_1), \ldots, \mathrm{Poly}(\mathrm{rk}_k)$ which have v_1 as a vertex. Because v_1 is largest in lexicographic ordering, v_1 does not lie in any of the polyhedra $\mathrm{Poly}(\mathrm{rk}_{n+1}), \ldots, \mathrm{Poly}(\mathrm{rk}_k)$. Because these polyhedra are closed, there exists an open neighborhood U of v_1 such that $U \cap \mathrm{Poly}(\mathrm{rk}_j) = \emptyset$ for $j = n+1, \ldots, k$. If we restrict to U, we see that

$$\sum_{i=1}^{k} a_i \mathbf{1}(\mathrm{Poly}(\mathrm{rk}_i) \cap U) = \sum_{i=1}^{n} a_i \mathbf{1}(\mathrm{Poly}(\mathrm{rk}_i) \cap U) = 0$$

Since $\mathrm{Poly}(\mathrm{rk}_1), \ldots, \mathrm{Poly}(\mathrm{rk}_n)$ are cones with vertex v_1, we have

$$\sum_{i=1}^{n} a_i \mathbf{1}(\mathrm{Poly}(\mathrm{rk}_i)) = 0.$$

and

$$\sum_{i=n+1}^{k} a_i \mathbf{1}(\mathrm{Poly}(\mathrm{rk}_i)) = 0.$$

If $n < k$, then by the induction on k, we know that

$$\sum_{i=1}^{n} a_i \langle \mathrm{Poly}(\mathrm{rk})_i \rangle \in W^+_{\mathrm{MM}}(d, r, V)$$

and

$$\sum_{i=n+1}^{k} a_i \langle \mathrm{Poly}(\mathrm{rk}_i) \rangle \in W^+_{\mathrm{MM}}(d, r, V),$$

hence

$$\sum_{i=1}^{k} a_i \langle \mathrm{Poly}(\mathrm{rk}_i) \rangle \in W^+_{\mathrm{MM}}(d, r, V).$$

Assume that $n = k$, i.e., $\mathrm{Poly}(\mathrm{rk}_1), \ldots, \mathrm{Poly}(\mathrm{rk}_k)$ all have vertex v_1. After translation by $-v_1$, we may assume that $r = 0$, and $v_1 = 0$. Now $\mathrm{Poly}(\mathrm{rk}_1), \ldots, \mathrm{Poly}(\mathrm{rk}_k)$ are all contained in the halfspace defined by $y_d \geq 0$ inside the hyperplane $y_1 + \cdots + y_d = 0$.

Define

$$\rho : \{y \in \mathbb{R}^{d-1} \mid y_1 + \cdots + y_{d-1} = -1\} \to \{y \in \mathbb{R}^d \mid y_1 + \cdots + y_d = 0\}$$

by $\rho(y_1, \ldots, y_{d-1}) = (y_1, \ldots, y_{d-1}, 1)$. Assume that $\rho^{-1}(\mathrm{Poly}(\mathrm{rk}_i)) \neq \emptyset$ for $i = 1, 2, \ldots, t$ and $\rho^{-1}(\mathrm{Poly}(\mathrm{rk}_i)) = \emptyset$ for $i = t+1, \ldots, k$. For $i = 1, 2, \ldots, t$, define megamatroids $\mathrm{rk}'_i : 2^{[d-1]} \to \mathbb{Z} \cup \{\infty\}$ such that $\mathrm{Poly}(\mathrm{rk}'_i) = \rho^{-1}(\mathrm{Poly}(\mathrm{rk}_i))$. We have

$$\sum_{i=1}^{t} a_i \mathbf{1}(\mathrm{Poly}(\mathrm{rk}'_i)) = \sum_{i=1}^{n} a_i \mathbf{1}(\mathrm{Poly}(\mathrm{rk}_i)) \circ \rho = 0.$$

Note that $\mathrm{Poly}(\mathrm{rk}_i')$ is bounded from above and $\mathrm{vert}(\mathrm{Poly}(\mathrm{rk}_i')) \subseteq \{-e_1, \ldots, -e_{d-1}\}$ for $i = 1, 2, \ldots, t$. By induction on d we have

$$\sum_{i=1}^{t} a_i \langle \mathrm{rk}_i' \rangle \in W^+_{\mathrm{MM}}(d-1, -1, \{-e_1, -e_2, \ldots, -e_{d-1}\}). \tag{3.A.1}$$

If Γ is a megamatroid polyhedron inside $y_1 + \cdots + y_{d-1} = -1$ which is bounded from above, and $\mathrm{vert}(\Gamma) \subseteq \{-e_1, \ldots, -e_{d-1}\}$, then define $C(\Gamma)$ as the closure of $\mathbb{R}_{\geq 0}\rho(\Gamma)$. Note that $C(\Gamma)$ is also a megamatroid polyhedron. Define

$$\gamma : Z_{\mathrm{MM}}(d, -1, \{-e_2, \ldots, -e_d\}) \to Z_{\mathrm{MM}}(d, 0, \{0\})$$

by $\gamma(\langle \mathrm{rk} \rangle) = \langle \widehat{\mathrm{rk}} \rangle$, where $\widehat{\mathrm{rk}}$ is given by $\mathrm{Poly}(\widehat{\mathrm{rk}}) = C(\mathrm{Poly}(\mathrm{rk}))$.

If

$$\mathrm{Poly}(\mathrm{rk}') = \mathrm{Poly}(\mathrm{rk}_1') \cup \cdots \cup \mathrm{Poly}(\mathrm{rk}_s')$$

is a megamatroid decomposition inside $\{y \in \mathbb{R}^d \mid y_1 + \cdots + y_{d-1} = -1\}$, then

$$C(\mathrm{Poly}(\mathrm{rk}')) = C(\mathrm{Poly}(\mathrm{rk}_1')) \cup \cdots \cup C(\mathrm{Poly}(\mathrm{rk}_s'))$$

is also a megamatroid decomposition inside $y_1 + \cdots + y_d = 0$.

So γ maps $W^+_{\mathrm{MM}}(d, -1, \{-e_1, \ldots, -e_{d-1}\})$ to $W^+_{\mathrm{MM}}(d, 0, \{0\})$.

Applying γ to (3.A.1) we get

$$\gamma\Big(\sum_{i=1}^{t} a_i \langle \mathrm{rk}_i' \rangle\Big) = \sum_{i=1}^{t} a_i \langle \mathrm{rk}_i \rangle \in W^+_{\mathrm{MM}}(d, 0, \{0\}).$$

From this follows that $\sum_{i=1}^{t} a_i \mathbf{1}(\mathrm{Poly}(\mathrm{rk}_i)) = 0$. Since $\sum_{i=1}^{k} a_i \mathbf{1}(\mathrm{Poly}(\mathrm{rk}_i)) = 0$, we have that $\sum_{i=t+1}^{k} a_i \mathbf{1}(\mathrm{Poly}(\mathrm{rk}_i)) = 0$. Since $\mathrm{Poly}(\mathrm{rk}_i)$ is contained in the hyperplane defined by $y_d = 0$ for $i = t+1, \ldots, k$, we can again use induction on d to show that

$$\sum_{i=t+1}^{k} a_i \langle \mathrm{rk}_i \rangle \in W^+_{\mathrm{MM}}(d, r, \{0\}).$$

We conclude that

$$\sum_{i=1}^{k} a_i \langle \mathrm{rk}_i \rangle = \sum_{i=1}^{t} a_i \langle \mathrm{rk}_i \rangle + \sum_{i=t+1}^{k} a_i \langle \mathrm{rk}_i \rangle \in W^+_{\mathrm{MM}}(d, r, \{0\}).$$

Assume now we are in the case where $\mathrm{rk}_1, \ldots, \mathrm{rk}_k$ are arbitrary. By Lemma 3.A.1, we can find megamatroids $\mathrm{rk}_{i,j}$ bounded from above with only one vertex which is contained in the set V, and integers $c_{i,j}$ such that

$$\langle \mathrm{rk}_i \rangle - \sum_j c_{i,j} \langle \mathrm{rk}_{i,j} \rangle \in W^+_{\mathrm{MM}}(d, r, V)$$

It follows that $\sum_{i=1}^{k} a_i c_{i,j} \mathbf{1}(\mathrm{Poly}(\mathrm{rk}_{i,j})) = 0$. From the special case considered above, we obtain

$$\sum_{i=1}^{k} a_i \langle \mathrm{rk}_i \rangle = \sum_{i=1}^{k} a_i \sum_j c_{i,j} \langle \mathrm{rk}_{i,j} \rangle \in W^+_{\mathrm{MM}}(d, r, V).$$

□

Proof of Theorem 3.3.5. It suffices to show that the kernel of $\mathbf{1}_{\mathrm{MM}}$ is contained in $W_{\mathrm{MM}}(d, r)$. Suppose that

$$\mathbf{1}_{\mathrm{MM}}\Big(\sum_{i=1}^{k} a_i \langle \mathrm{rk}_i \rangle \Big) = \sum_{i=1}^{k} a_i \mathbf{1}(\mathrm{Poly}(\mathrm{rk}_i)) = 0.$$

Let $\mathrm{sgn} : \mathbb{R} \to \{-1, 0, 1\}$ be the signum function. For a vector $\gamma = (\gamma_1, \ldots, \gamma_d) \in \{-1, 0, 1\}^d$ and a megamatroid polyhedron Π, define

$$\Pi^\gamma = \{(y_1, \ldots, y_d) \in \Pi \mid \forall i \, (\mathrm{sgn}\, y_i = \gamma_i \text{ or } y_i = 0)\}.$$

For every j we have a megamatroid polyhedron decomposition

$$\Pi_j = \bigcup_{\gamma \in \{-1,1\}^d ; \Pi_j^\gamma \neq \emptyset} \Pi_j^\gamma \tag{3.A.2}$$

where γ runs over $\{-1, 1\}^d$. Intersections of the polyhedra Π_i^γ, $\gamma \in \{-1, 1\}$ are of the form Π_i^γ where $\gamma \in \{-1, 0, 1\}^d$. If $\Pi_i^\gamma \neq \emptyset$ define rk_i^γ such that $\mathrm{Poly}(\mathrm{rk}_i^\gamma) = \Pi_i^\gamma$. From (3.A.2) it follows that

$$m_{\mathrm{val}}(\Pi_j; \{\Pi_j^\gamma\}_{\gamma \in \{-1,1\}^d}) = \langle \mathrm{rk}_j \rangle - \sum_{\gamma \in \{-1,0,1\}^d ; \Pi_i^\gamma \neq \emptyset} b^\gamma \langle \mathrm{rk}_i^\gamma \rangle \in W_{\mathrm{MM}}(d, r) \tag{3.A.3}$$

where the coefficients $b^\gamma \in \mathbb{Z}$ only depend on γ. (One can show that $b_\gamma = (-1)^{z(\gamma)}$ where $z(\gamma)$ is the number of zeroes in γ, but we will not need this.)

For every γ we have

$$\sum_{i : \Pi_i^\gamma \neq \emptyset} a_i \mathbf{1}(\Pi_i^\gamma) = 0$$

For a given γ, we may assume after permuting the coordinates that $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_d$. It then follows that Π_i^γ is bounded from above for all i. By Proposition 3.A.2, we have

$$\sum_i a_i \langle \mathrm{rk}_i^\gamma \rangle \in W_{\mathrm{MM}}(d, r)$$

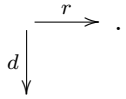
for all γ. By (3.A.3) we get

$$\sum_{i=1}^{k} a_i \langle \mathrm{rk}_i \rangle \in W_{\mathrm{MM}}(d, r).$$

□

3.B Tables

Below are the tables for the values of $p_{\mathrm{PM}}(d,r)$, $p_{\mathrm{M}}(d,r)$, $p_{\mathrm{PM}}^{\mathrm{sym}}(d,r)$, $p_{\mathrm{M}}^{\mathrm{sym}}(d,r)$, $t_{\mathrm{PM}}(d,r)$, $t_{\mathrm{M}}(d,r)$, $t_{\mathrm{PM}}^{\mathrm{sym}}(d,r)$, $t_{\mathrm{M}}^{\mathrm{sym}}(d,r)$ for $d \leq 6$ and $r \leq 6$. Rows correspond to values of d and columns correspond to values of r:



p_{PM}	0	1	2	3	4	5	6
0	1						
1	1	1	1	1	1	1	1
2	1	3	5	7	9	11	13
3	1	7	19	37	61	91	127
4	1	15	65	175	369	671	1105
5	1	31	211	781	2101	4651	9031
6	1	63	665	3367	11529	31031	70993

p_{M}	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	3	1				
3	1	7	7	1			
4	1	15	33	15	1		
5	1	31	131	131	31	1	
6	1	63	473	883	473	63	1

$p_{\mathrm{PM}}^{\mathrm{sym}}$	0	1	2	3	4	5	6
0	1						
1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7
3	1	3	6	10	15	21	28
4	1	4	10	20	35	56	84
5	1	5	15	35	70	126	210
6	1	6	21	56	126	252	462

$p_{\mathrm{M}}^{\mathrm{sym}}$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

t_{PM}	0	1	2	3	4	5	6
0							
1	1	1	1	1	1	1	1
2		1	2	3	4	5	6
3		1	4	9	16	25	36
4		1	8	27	64	125	216
5		1	16	81	256	625	1296
6		1	32	343	1024	3125	7776

t_{M}	0	1	2	3	4	5	6
0							
1	1	1					
2		1					
3		1	1				
4		1	4	1			
5		1	11	11	1		
6		1	26	66	26	1	

$t_{\mathrm{PM}}^{\mathrm{sym}}$	0	1	2	3	4	5	6
0							
1	1	1	1	1	1	1	1
2		1	1	2	2	3	3
3		1	2	3	5	7	9
4		1	2	5	8	14	20
5		1	3	7	14	25	42
6		1	3	9	20	42	75

$t_{\mathrm{M}}^{\mathrm{sym}}$	0	1	2	3	4	5	6
0							
1	1	1					
2		1					
3		1	1				
4		1	1	1			
5		1	2	2	1		
6		1	2	3	2	1	

The tables for $p_{\mathrm{M}}, t_{\mathrm{M}}, t_{\mathrm{PM}}^{\mathrm{sym}}, t_{\mathrm{M}}^{\mathrm{sym}}$ can be computed recursively using the equations for the generating functions in Theorems 3.1.5 and 3.1.6. The values for $p_{\mathrm{PM}}, p_{\mathrm{PM}}^{\mathrm{sym}}, p_{\mathrm{M}}^{\mathrm{sym}}, t_{\mathrm{PM}}$ are trivial to compute, but are included here for comparison. The tables of $p_{\mathrm{PM}}^{\mathrm{sym}}$ and $p_{\mathrm{M}}^{\mathrm{sym}}$ are of course related to Pascal's triangle. The table for $p_{\mathrm{M}}^{\mathrm{sym}}$ appears in Sloane's *Online Encyclopedia of Integer Sequences* [81] as sequence **A046802**. These numbers also appear in [80]. We have $t_{\mathrm{M}}(d, r) = E(d-1, r-1)$ for $d, r \geq 1$, where the $E(d, r)$ are the Eulerian numbers. See the *Handbook of Integer Sequences* [81], sequences **A008292** and **A123125**.

The sequences $t^{\mathrm{sym}}_{\mathrm{PM}}$ and $t^{\mathrm{sym}}_{\mathrm{M}}$ are related to sequences **A059966**, **A001037**, and the sequence **A051168** denoted by $T(h,k)$ in [81]. We have $t^{\mathrm{sym}}_{\mathrm{PM}}(d,r) = T(d-1,r)$ for $d \geq 1$ and $r \geq 0$, and $t^{\mathrm{sym}}_{\mathrm{M}}(d,r) = T(d-r-1,r)$ if $0 \leq r < d$.

3.C Index of notations used in this chapter

A subscript $_{\mathrm{MM}}$ or $_{\mathrm{PM}}$ or $_{\mathrm{M}}$ on a notation refers to the variant relating respectively to megamatroids or polymatroids or matroids. The subscript $_{*\mathrm{M}}$ stands in for any of $_{\mathrm{MM}}$ or $_{\mathrm{PM}}$ or $_{\mathrm{M}}$, while $_{(\mathrm{P})\mathrm{M}}$ stands in for either of $_{\mathrm{PM}}$ or $_{\mathrm{M}}$.

Notations below with a dagger may have the parenthesis (d,r) omitted, in which case they refer to direct sums over all d and r. These are introduced on page 61.

$V^\vee$	dual space of V, 34
$V^\#$	graded dual space of V, 74
$\mathbf{1}(\Pi)$	indicator function of a set Π, 33, 44
$\mathbf{1}_{*\mathrm{M}}$	the map $\mathbf{1}_{\mathrm{MM}} : Z_{\mathrm{MM}}(d,r) \to P_{\mathrm{MM}}(d,r)$, $\mathbf{1}_{*\mathrm{M}}(\langle \mathrm{rk} \rangle) = \mathbf{1}(\mathrm{Poly}(\mathrm{rk}))$, 44
$\mathbf{1}^\circ_{\mathrm{MM}}$	the map $\mathbf{1}_{\mathrm{MM}} : Z_{\mathrm{MM}}(d,r) \to P_{\mathrm{MM}}(d,r)$, $\mathbf{1}_{\mathrm{MM}}(\langle \mathrm{rk} \rangle) = \mathbf{1}(\mathrm{Poly}(\mathrm{rk})^\circ)$, 45
$A_{*\mathrm{M}}(d,r)$ †	the $\mathbb{Z}$-module generated by all $\mathbf{1}(R_{*\mathrm{M}}(\underline{X},\underline{r}))$ with $(\underline{X},\underline{r}) \in \mathfrak{a}_{*\mathrm{M}}(d,r)$, 57
$\mathfrak{a}_{*\mathrm{M}}(d,r)$	index set, 57
$B_{*\mathrm{M}}(d,r)$	the group generated by all $\mathbf{1}(\mathrm{Poly}(\mathrm{rk})) - \mathbf{1}(\mathrm{Poly}(\mathrm{rk} \circ \sigma))$, 56
E	the map $E(\langle \mathrm{rk} \rangle) = \sum_F \langle \mathrm{rk}_F \rangle$, F ranging over faces of rk, 43
face(Π)	the set of faces of a polyhedron Π, 43
$\mathcal{F}$	Billera-Jia-Reiner quasi-symmetric function, 33
$\mathcal{G}$	polymatroid invariant, 33
$\ell(\underline{X})$	length of a chain $\underline{X}$, 46
lhull(F)	linear hull of F, 39
$\mathfrak{m}_{*\mathrm{M}}$	$\bigoplus_{d,r} P^{\mathrm{sym}}_{*\mathrm{M}}(d,r,1)$, 75
$P_{*\mathrm{M}}(d,r)$ †	the $\mathbb{Z}$-module on indicator functions $\mathbf{1}(\mathrm{Poly}(\mathrm{rk}))$, 33, 44
$P_{*\mathrm{M}}(d,r,e)$	filtration of $P_{*\mathrm{M}}$, 65
$\overline{P}_{*\mathrm{M}}(d,r,e)$	associated graded of $P_{*\mathrm{M}}$, 65
$P^{\mathrm{sym}}_{*\mathrm{M}}(d,r)$ †	P/B, the symmetrized version of $P_{*\mathrm{M}}$, 34, 56
$P^{\mathrm{sym}}_{*\mathrm{M}}(d,r,e)$	filtration of P^{sym}_{rm*M}, 73
$\overline{P}^{\mathrm{sym}}_{*\mathrm{M}}(d,r,e)$	associated graded of $P^{\mathrm{sym}}_{*\mathrm{M}}$, 73
$p_{(\mathrm{P})\mathrm{M}}(d,r)$	rank of $P_{(\mathrm{P})\mathrm{M}}(d,r)$, the number of independent valuative functions, 34
$p^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)$	rank of $P^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r)$, the number of independent valuative invariants, 34
$p_{(\mathrm{P})\mathrm{M}}(d,r,e)$	rank of $P_{(\mathrm{P})\mathrm{M}}(d,r,e)$, 68
$p^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r,e)$	rank of $P^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d,r,e)$, 74
$\mathfrak{p}_{*\mathrm{M}}(d,r)$	index set for a basis of $P_{*\mathrm{M}}(d,r)$, 51
$\mathfrak{p}^{\mathrm{sym}}_{*\mathrm{M}}(d,r)$	index set for a basis of $P^{\mathrm{sym}}_{*\mathrm{M}}(d,r)$, 57

$\mathrm{Poly}(\mathrm{rk})$	base polytope of a megamatroid, 37
$R_{*\mathrm{M}}(\underline{X}, \underline{r})$	a (mega-, poly-)matroid whose polytope is a cone, 46
rk_Π	rank function of a polytope Π, 39
$s_{\underline{X},\underline{r}}$	the indicator function for the chain $\underline{X}$ having ranks $\underline{r}$, 50
$s^{\mathrm{sym}}_{\underline{X},\underline{r}}$	the average of $s_{\underline{X},\underline{r}}$ under the symmetric group action, 57
S	antipode $\mathcal{H} \to \mathcal{H}$ in a Hopf algebra, 63
$S_{*\mathrm{M}}(d, r)$	set of (mega-, poly-)matroids, 33, 42
$S^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d, r)$	isomorphism classes of (poly)matroids, 33
$\mathcal{T}$	Tutte polynomial, 33
$T_{*\mathrm{M}}(d, r)$ †	$\overline{P}_{*\mathrm{M}}(d, r, 1)$, 65
$T^{\mathrm{sym}}_{*\mathrm{M}}(d, r)$ †	$\overline{P}^{\mathrm{sym}}_{*\mathrm{M}}(d, r, 1)$, 73
$t_{(\mathrm{P})\mathrm{M}}(d, r)$	rank of $T_{(\mathrm{P})\mathrm{M}}(d, r)$, number of independent additive functions, 68
$t^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d, r)$	rank of $T^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d, r)$, number of independent additive invariants, 74
$\mathfrak{t}_{*\mathrm{M}}(d, r)$	index set for a basis of $T_{*\mathrm{M}}(d, r)$, 69
$\mathfrak{t}^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d, r)$	index set for a basis in $(T^{\mathrm{sym}}_{(\mathrm{P})\mathrm{M}}(d, r))^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$, 76
$\{U_\alpha\}$	basis of the ring of quasisymmetric functions, 33
$\{u_\alpha\}$	dual basis of $\{U_\alpha\}$, basis of $\mathbb{Q}$-valued invariants, 58
$\mathrm{vert}(\Pi)$	set of vertices of a polyhedron Π, 77
$W_{\mathrm{MM}}(d, r)$	subgroup of $Z_{\mathrm{MM}}(d, r)$ generated by all $m_{\mathrm{val}}(\Pi, \dots)$s, 77
$W_{\mathrm{MM}}(d, r, V)$	ditto, Π having all vertices in V, 77
$W_{\mathrm{MM}}(\dots)^+$	ditto, Π bounded from above, 77
$Y_{*\mathrm{M}}(d, r)$	the group generated by all $\langle \mathrm{rk} \rangle - \langle \mathrm{rk} \circ \sigma \rangle$, 55
$Z_{*\mathrm{M}}(d, r)$ †	the $\mathbb{Z}$-module on (mega-, poly-)matroids, 33, 42
$Z^{\mathrm{sym}}_{*\mathrm{M}}(d, r)$ †	Z/Y, the symmetrized version of Z, 33, 55
$\Delta_M(d, r)$	hypersimplex defined by $y_1 + \cdots + y_d = r$, $0 \leq y_i \leq 1$, 38
$\Delta_{PM}(d, r)$	simplex defined by $y_1 + \cdots + y_d = r$, $y_i \geq 0$, 38
Δ	comultiplication $\mathcal{H} \to \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ for a Hopf algebra $\mathcal{H}$, 61
η	unit in a Hopf algebra, 61
∇	multiplication $\mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ in a Hopf algebra, 61
ϵ	counit in a Hopf algebra, 62
Π°	relative interior of a polyhedron Π, 45
π_{MM}	the quotient map $Z_{\mathrm{MM}}(d, r) \to Z^{\mathrm{sym}}_{\mathrm{MM}}(d, r)$, 56
ρ_{MM}	the quotient map $P_{\mathrm{MM}}(d, r) \to P^{\mathrm{sym}}_{\mathrm{MM}}(d, r)$, 56

Chapter 4

The geometry of the Tutte polynomial

This chapter is joint work with David Speyer. It is on the arXiv with identifier 1004.2403, under the title *K-classes of matroids and equivariant localization*. (This version incorporates some minor changes, largely for consistency with other chapters.)

4.1 Introduction

Let H_1, H_2, ..., H_n be a collection of hyperplanes through the origin in $\mathbb{C}^d$. The study of such hyperplane arrangements is a major field of research, resting on the border between algebraic geometry and combinatorics. There are two natural objects associated to a hyperplane arrangement. We will describe both of these constructions in detail in Section 4.3.

The first is the *matroid* of the hyperplane arrangement, which can be thought of as encoding the combinatorial structure of the arrangement.

The second, which captures the geometric structure of the arrangement, is a point in the Grassmannian $G(d, n)$. There is ambiguity in the choice of this point; it is only determined up to the action of an n-dimensional torus on $G(d, n)$. So more precisely, to any hyperplane arrangement, we associate an orbit Y in $G(d, n)$ for this torus action. It is technically more convenient to work with the closure of this orbit. In [83], Speyer suggested that the K-class of this orbit could give rise to useful invariants of matroids, thus exploiting the geometric structure to study the combinatorial one. In this chapter, we continue that project.

One of our results is a formula for the Tutte polynomial, the most famous of matroid invariants, in terms of the K-class of Y. In addition, we rewrite all of the K-theoretic definitions in terms of moment graphs, something which was begun in the appendix of [83]. This makes our theory purely combinatorial and in principle completely computable. Many results which were shown for realizable matroids in [83] are now extended to all matroids.

We state our two main results. The necessary K-theoretic definitions will be given in the following section. Given integers $0 < d_1 < \cdots < d_s < n$, let $\mathcal{F}\ell(d_1, \ldots, d_s; n)$ be the partial

flag manifold of flags of dimensions $(d_1, \ldots, d_s)$. For instance, $\mathcal{F}\ell(d; n) = G(d, n)$. Note that $\mathcal{F}\ell(1, n-1; n)$ embeds as a hypersurface of bidegree $(1, 1)$ in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$, regarded as the space of pairs (`line`, `hyperplane`) in n-space.

We will be particularly concerned with the maps in diagram (4.1.1):

$$\begin{array}{ccccc} & & \mathcal{F}\ell(1, d, n-1; n) & & \\ & \swarrow^{\pi_d} & & \searrow & \\ G(d, n) & & & & \mathcal{F}\ell(1, n-1; n) \\ & & \swarrow_{\pi_{1(n-1)}} & & \downarrow \\ & & & & \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \end{array} \tag{4.1.1}$$

Here the maps $\mathcal{F}\ell(1, d, n-1; n) \to \mathcal{F}\ell(1, n-1; n)$ and $\mathcal{F}\ell(1, d, n-1; n) \to G(d, n)$ are given by respectively forgetting the d-plane and forgetting the line and hyperplane. The map $\pi_{1(n-1)}$ is defined by the composition $\mathcal{F}\ell(1, d, n-1; n) \to \mathcal{F}\ell(1, n-1; n) \to \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$.

Let T be the torus $(\mathbb{C}^*)^n$, which acts on the spaces in (4.1.1) in an obvious way (explicitly, T acts on $\mathbb{A}^n$ by $t \cdot x = (t_1^{-1}x_1, \ldots, t_n^{-1}x_n)$, and flags are taken to consist of subspaces of $\mathbb{A}^n$; see Convention 4.2.2). Let x be a point of $G(d, n)$, M the corresponding matroid, and $\overline{Tx}$ the closure of the T orbit through x. Let Y be the class of the structure sheaf of $\overline{Tx}$ in $K^0(G(d, n))$. Write $K^0(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}) = \mathbb{Q}[\alpha, \beta]/(\alpha^n, \beta^n)$, where α and β are the classes of the structure sheaves of hyperplanes.

We can now explain the geometric origin of the Tutte polynomial.

Theorem 4.7.1. *With the above notations,*

$$(\pi_{1(n-1)})_* \pi_d^* (Y \cdot [\mathcal{O}(1)]) = t_M(\alpha, \beta)$$

where t_M is the Tutte polynomial.

As usual, the sheaf $\mathcal{O}(1)$ on $G(d, n)$ is a generator of $\mathrm{Pic}(G(d, n)) \cong \mathbb{Z}$; it is the pullback of $\mathcal{O}(1)$ on $\mathbb{P}^N$ via the Plücker embedding. Also $\mathcal{O}(1) = \bigwedge^d S^\vee$, where S is the universal sub-bundle of Section 4.5.

The constant term of t_M is zero; this corresponds to the fact that $\pi_{1(n-1)}$ is not surjective onto $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ but, rather, has image lying in $\mathcal{F}\ell(1, n-1; n)$. The linear term of Tutte, $\beta(M)(\alpha + \beta)$, corresponds to the fact that the map $\pi_d^{-1}(\overline{Tx}) \to \mathcal{F}\ell(1, n-1; n)$ is finite of degree $\beta(M)$.

Theorems 4.8.1, 4.8.5. *Also with the above notations,*

$$(\pi_{1(n-1)})_* \pi_d^* (Y) = h_M(\alpha + \beta - \alpha\beta)$$

where h_M is Speyer's matroid invariant from [83].

Our results can be pleasingly presented in terms of $\alpha - 1$ and $\beta - 1$. For instance, in Theorem 4.8.1, h_M is a polynomial in $1 - (\alpha + \beta - \alpha\beta) = (\alpha - 1)(\beta - 1)$, and Theorem 4.7.1 obtains the rank generating function of M in the variables $\alpha - 1$, $\beta - 1$. In other words, we might take as a generating set for $K^0(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$ not the structure sheaves of linear spaces $\{\alpha^p \beta^q\}$, but the line bundles $\mathcal{O}(-p, -q) = \mathcal{O}(-1, 0)^p \ \mathcal{O}(0, -1)^q$, where $\mathcal{O}(-1, 0)$ is the pullback of $\mathcal{O}(-1)$ on $\mathbb{P}^{n-1}$ via projection to the first factor, and $\mathcal{O}(0, -1)$ is the analogue for the second factor. We have a short exact sequence

$$0 \to \mathcal{O}(-1, 0) \to \mathcal{O} \to \mathcal{O}_H \to 0, \tag{4.1.2}$$

where H is the divisor $\mathbb{P}^{n-2} \times \mathbb{P}^{n-1} \subseteq \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$, so $[\mathcal{O}(-1, 0)] = 1 - \alpha$, and similarly $[\mathcal{O}(0, -1)] = 1 - \beta$.

This chapter begins by introducing the limited subset of K-theory which we need, with particular attention to the method of *equivariant localization*. Many of our proofs, including those of Theorems 4.7.1 and Theorems 4.8.5 above, rely heavily on equivariant K-theory, even though they are theorems about ordinary K-theory. The end of Section 4.2 describes the K-theory of the Grassmannian from the equivariant perspective, and Section 4.3 describes the K-theory classes associated to matroids.

In equivariant K-theory, working with α and β presents a difficulty: the characters which appear when handling hyperplane classes equivariantly depend on the choice of hyperplane, and there is no canonical way to make this choice. Working with $\alpha - 1$ and $\beta - 1$ avoids this difficulty. Thus we get some results of independent interest in equivariant K-theory in terms of those classes, for instance Theorem 4.7.2.

Any function on matroids arising from $K^0(G(d, n))$ is a *valuation*. This is the subject of Section 4.4, where we show that the converse doesn't hold by exhibiting a valuative matroid invariant not arising from $K^0(G(d, n))$.

Section 4.5 proves Lemma 4.5.1, the core lemma which we use to push and pull K-classes in diagram (4.1.1). In conjunction with equivariant localization, our computations are reduced to manipulating sums of Hilbert series of certain infinite-dimensional T-representations, which we may regard as rational functions. We control these rational functions by expanding them as Laurent series with various domains of convergence. We collect a number of results on this subject in Section 4.6.

Sections 4.7 and Section 4.8 are the proofs of the theorems above. Finally, Section 4.9 takes results from [83], concerning the behavior of h_M under duality, direct sum and two-sum, and extends them to nonrealizable matroids.

4.1.1 Notation

We write $[n]$ for $\{1, 2, \ldots, n\}$. For any set S, we write $\binom{S}{k}$ for the set of k-element subsets of S and 2^S for the set of all subsets of S. Disjoint union is denoted by $\sqcup$. The use of the

notation $I \setminus J$ does not imply that J is contained in I. In addition to the notations $\mathbb{P}$, $G(d,n)$ and $\mathcal{F}\ell$ introduced above, we will often write $\mathbb{A}^n$ for affine space.

4.2 Background on K-theory

If X is any algebraic variety, then $K_0(X)$ denotes the free abelian group generated by isomorphism classes of coherent sheaves on X, subject to the relation $[A] + [C] = [B]$ whenever there is a short exact sequence $0 \to A \to B \to C \to 0$. The subspace generated by the classes of vector bundles is denoted $K^0(X)$. If X is smooth, as all the spaces we deal with will be, the inclusion $K^0(X) \hookrightarrow K_0(X)$ is an equality. (See [71, Proposition 2.1] for this fact, and its equivariant generalization.)

We put a ring structure on $K^0(X)$, generated by the relations $[E][F] = [E \otimes F]$ for any vector bundles E and F on X. The group $K_0(X)$ is a module for $K^0(X)$, with multiplication given by $[E][F] = [E \otimes F]$ where E is a vector bundle and F a coherent sheaf.

For any map $f : X \to Y$, there is a pull back map $f^* : K^0(Y) \to K^0(X)$ given by $f^*[E] = [f^*E]$. This is a ring homomorphism. If $f : X \to Y$ is a proper map, there is also a pushforward map $f_* : K_0(X) \to K_0(Y)$ given by

$$f_*[E] = \sum (-1)^i [R^i f_* E].$$

Here $R^i f_*$ are the right derived functors of the pushforward f_* of sheaves (see [21, paragraph 5.2.13]). These two maps are related by the projection formula, which asserts that f_* is a homomorphism of $K^0(Y)$-modules, where $K^0(X)$ has the module structure induced by f^*. In other words, for $E \in K^0(Y)$ and $F \in K_0(X)$, we have

$$f_*\big((f^*[E])[F]\big) = [E] f_*[F]. \tag{4.2.1}$$

We always have a map from X to a point. We denote the pushforward along this map by $\int$, or by $\int_X$ when necessary. (There are many analogies between K^0 and H^*. In cohomology, the pushforward from an oriented compact manifold to a point is often denoted by $\int$, because it is given by integration in the de Rham formulation of cohomology. We use the same symbol here by analogy.) Notice that $K_0(\mathtt{pt}) = K^0(\mathtt{pt}) = \mathbb{Z}$, and $\int [E]$ is the holomorphic Euler characteristic of the sheaf E.

If T is a torus acting on X, then we can form the analogous constructions using T-equivariant vector bundles and sheaves, denoted $K^0_T(X)$ and $K_0^T(X)$. The analogous properties hold of these. Writing $\mathrm{Char}(T)$ for the lattice of characters, $\mathrm{Hom}(T, \mathbb{C}^*)$, we have $K_0^T(\mathtt{pt}) = K^0_T(\mathtt{pt}) = \mathbb{Z}[\mathrm{Char}(T)]$. Explicitly, a T-equivariant sheaf on $\mathtt{pt}$ is simply a vector space with a T-action, and the corresponding element of $\mathbb{Z}[\mathrm{Char}(T)]$ is the character. Making a choice of coordinates, we will often take $\mathbb{Z}[\mathrm{Char}(T)] = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$.

We write $[E]^T$ for the class of the sheaf E in $K_T^0(X)$. We also write $\int^T$ for the pushforward to a point in equivariant cohomology.

We pause to discuss Hilbert series and sign conventions. If V is a finite dimensional representation of T, the Hilbert series of V is the sum

$$\mathrm{Hilb}(V) := \sum_{\chi \in \mathrm{Char}(T)} \dim \mathrm{Hom}(\chi, V) \cdot \chi$$

in $\mathbb{Z}[\mathrm{Char}(T)]$. If V isn't finite dimensional, but $\mathrm{Hom}(\chi, V)$ is for every character χ, then we can still consider this as a formal sum.

Here is one example of particular interest: let W be a finite dimensional representation of T with character $\sum \chi_i$. Suppose that all of the χ_i lie in an open half space in $\mathrm{Char}(T) \otimes \mathbb{R}$; if this condition holds, we say that W is *contracting*. Then the Hilbert series of $\mathrm{Sym}(W)$, defined as a formal power series, represents the rational function $1/(1-\chi_1)\cdots(1-\chi_r)$. If M is a finitely generated $\mathrm{Sym}(W)$ module, then the Hilbert series of M will likewise represent an element of $\mathrm{Frac}(\mathbb{Z}[\mathrm{Char}(T)])$ [68, Theorem 8.20].

Remark 4.2.1. If W is not contracting, then $\mathrm{Hom}(\chi, \mathrm{Sym}(W))$ will usually be infinite dimensional. It is still possible to define Hilbert series in this situation, see [68, Section 8.4], but we will not need this.

Sign conventions when working with group actions are potentially confusing. We now spell our choices out.

Convention 4.2.2. Suppose that a group G acts on a ring A. The group G then acts on $\mathrm{Spec}\, A$ by $g(a) = (g^{-1})^* a$. This definition is necessary in order to make sure that both actions are *left* actions. Although we will only consider actions of abelian groups, for which left and right actions are the same, we still follow this convention. This means that, if V is a vector space on which T acts by characters α_1, α_2, $\ldots$, α_r, then the coordinate ring of V is $\mathrm{Sym}(V^*)$ and has Hilbert series $1/\prod(1-\alpha_i^{-1})$. Now, let W be another T-representation, with characters β_1, β_2, $\ldots$, β_s. Consider $W \times V$ as a trivial vector bundle over V. The corresponding $\mathrm{Sym}(V^*)$ module is $W \otimes \mathrm{Sym}(V^*)$, and has Hilbert series $(\sum \beta_j)/\prod(1-\alpha_i^{-1})$. So one cannot simply memorize a rule like "always invert characters" or "never invert characters".

When we work out examples, we will need to specify how T acts on various partial flag varieties. Our convention is that T acts on $\mathbb{A}^n$ by the characters $t_1^{-1}, \ldots, t_n^{-1}$. Grassmannians, and other partial flag varieties, are flags of *subspaces*, not quotient spaces, and T acts on them by acting on the subobjects of $\mathbb{A}^n$. The advantage of this convention is that, for any ample line bundle L on $\mathcal{F}\ell(n)$, the pushforward $\int^T [L]$ will be composed of positive powers of the t_i.

Example 4.2.3. Let L be the d-plane $\mathrm{Span}(e_1, e_2, \ldots, e_d)$ and M be the $(n-d)$-plane $\mathrm{Span}(e_{d+1}, \ldots, e_n)$. Let $W \subset G(d,n)$ consist of those linear spaces which can be written as the graph of a linear map $L \to M$. This is an open neighborhood of L, sometimes called the

big Schubert cell. The cell W is a vector space of dimension $d(n-d)$, naturally identified with $\mathrm{Hom}(L, M)$. The torus T acts on the vector space W in the way induced from its action on $G(d,n)$, with characters $t_i t_j^{-1}$, for $1 \leq i \leq d$ and $d+1 \leq j \leq n$. So T acts on the coordinate ring of W with characters $t_i^{-1} t_j$, for i and j as above. $\diamondsuit$

4.2.1 Localization

The results in this section are well known to experts, but it seems difficult to find a reference that records them all in one place. We have attempted to do so; we have made no attempt to find the original sources for these results. The reader may want to compare our presentation to the description of equivariant cohomology in [54].

In this chapter, we will be only concerned with $K_0^T(X)$ for extremely nice spaces X. In fact, the only spaces we will need in the chapter are partial flag manifolds and products thereof. All of these spaces are *equivariantly formal* spaces, meaning that their K-theory can be described using the method of *equivariant localization*, which we now explain.

We will gradually add niceness hypotheses on X as we need them.

Condition 4.2.4. Let X be a smooth projective variety with an action of a torus T.

Writing X^T for the subvariety of T-fixed points, we have a restriction map

$$K_T^0(X) \to K_T^0(X^T) \cong K^0(X^T) \otimes K_T^0(\mathtt{pt}).$$

Suppose we have:

Condition 4.2.5. X has finitely many T-fixed points.

Theorem 4.2.6 ([71, Theorem 3.2], see also [53, Theorem A.4] and [90, Corollary 5.11])**.** *In the presence of Condition 4.2.4, the restriction map $K_T^0(X) \to K_T^0(X^T)$ is an injection. If we have Conditions 4.2.4 and 4.2.5, then $K_T^0(X^T)$ is simply the ring of functions from X^T to $K_T^0(\mathtt{pt})$.*

For example, if $X = G(d,n)$ and T is the standard n-dimensional torus, then X^T is $\binom{n}{d}$ distinct points, one for each d-dimensional coordinate plane in $\mathbb{C}^n$.

Let x be a fixed point of the torus action on X, so we have a restriction map $K_T^0(X) \to K_T^0(x) \cong K_T^0(\mathtt{pt})$. It is important to understand how this map is explicitly computed. For $\xi \in K_T^0(X)$, we write $\xi(x)$ for the image of ξ in $K_T^0(x)$.

We adopt a simplifying definition, which will hold in all of our examples: We say that X is *contracting at x* if there is a T-equivariant neighborhood of x which is isomorphic to $\mathbb{A}^N$ with T acting by a contracting linear representation. We will call the action of T on X *contracting* if it is contracting at every T-fixed point.

Let X be contracting at x. Let U be a T-equivariant neighborhood of x isomorphic to a contracting T-representation, and let $\chi_1, \ldots, \chi_N$ be the characters by which T acts on U. Let E be a T-equivariant coherent sheaf on U, corresponding to a graded, finitely generated $\mathcal{O}(U)$-module M. Then the Hilbert series of M lies in $\mathrm{Frac}(\mathbb{Z}[\mathrm{Char}(T)])$; it is a rational function of the form $k(E)/\prod(1-\chi_i^{-1})$ for some polynomial $k(E)$ in $\mathbb{Z}[\mathrm{Char}(T)]$.

Theorem 4.2.7. *If U is an open neighborhood of x as above then $K_T^0(U) \cong K_T^0(\mathtt{pt})$. With the above notations, $[E]^T(x) = k(E)$.*

Proof sketch. The restriction map $K_T^0(X) \to K_T^0(x)$ factors through $K_T^0(U)$, so it is enough to show that $[E]^T|_U$ is $k(E)$.

Let M be the $\mathcal{O}(U)$-module coresponding to U, and abbreviate $\mathcal{O}(U)$ to S. Then M has a finite T-graded resolution by free S-modules as in [68, Chapter 8], say

$$0 \to \bigoplus_{i=1}^{b_N} S[\chi_{iN}^{-1}] \to \cdots \to \bigoplus_{i=1}^{b_1} S[\chi_{i1}^{-1}] \to \bigoplus_{i=1}^{b_0} S[\chi_{i0}^{-1}] \to M \to 0.$$

Because we write our grading group multiplicatively, we write $S[\chi^{-1}]$ where $S[-\chi]$ might appear more familiar. This notation will only arise within this proof.

The sheafification of $S[\chi^{-1}]$, by definition, has class χ in $K_T^0(U)$. So

$$[E]^T = \sum_{j=1}^{N} (-1)^i \sum_{i=1}^{b_j} \chi_{ij}. \tag{4.2.2}$$

As the reader can easily check, or read in [68, Proposition 8.23], the sum in (4.2.2) is $k(E)$. □

Corollary 4.2.8. *If E is a vector bundle on U, and T acts on the fiber over x with character $\sum \eta_i$, then $[E]^T(x) = \sum \eta_i$.*

Remark 4.2.9. The positivity assumption is needed only for convenience. In general, let x be a smooth variety with T-action, x a fixed point of X, and let E be an equivariant coherent sheaf on X. Then $\mathcal{O}_x$ is a regular local ring, and E_x a finitely generated $\mathcal{O}_x$ module. Passing to the associated graded ring and module, gr E_x is a finitely generated, T-equivariant (gr $\mathcal{O}_x$)-module, and gr $\mathcal{O}_x$ is a polynomial ring. If the T-action on the tangent space at x is contracting, then we can define $[E]^T(x)$ using the Hilbert series of gr E_x; if not, we can use the trick of [68, Section 8.4] to define $k(\text{gr } E_x)$ and, hence, $[E]^T(x)$. But we will not need either of these ideas.

We have now described, given a T-equivariant sheaf E in $K_0^T(X)$, how to describe it as a function from X^T to $K_T^0(\mathtt{pt})$. It will also be worthwhile to know, given a function from X^T to $K_T^0(\mathtt{pt})$, when it is in $K_T^0(X)$. For this, we need

Condition 4.2.10. There are finitely many 1-dimensional T-orbits in X, each of which has closure isomorphic to $\mathbb{P}^1$.

A T-invariant subvariety of X isomorphic to $\mathbb{P}^1$ necessarily contains just two T-fixed points.

Theorem 4.2.11 ([90, Corollary 5.12], see also [53, Corollary A.5])**.** *Assume conditions 4.2.4, 4.2.5 and 4.2.10. Let f be a function from X^T to $K_T^0(\mathtt{pt})$. Then f is of the form $\xi(\cdot)$ for some $\xi \in K_T^0(X)$ if and only if the following condition holds: For every one dimensional orbit, on which T acts by character χ and for which x and y are the T-fixed points in the closure of the orbit, we have*

$$f(x) \equiv f(y) \mod 1 - \chi.$$

We cannot conclude that ξ is itself the class $[E]^T$ of a T-equivariant sheaf E, for reasons of positivity. For example, $\xi(x) = -1$ does not describe the class of a sheaf.

Let's see what this theorem means for the Grassmannian $G(d, n)$. Here $K_T^0(\mathtt{pt})$ is the ring of Laurent polynomials $\mathbb{Q}[t_1^{\pm}, t_2^{\pm}, \ldots, t_n^{\pm}]$. The fixed points $G(d, n)^T$ are the linear spaces of the form $\mathrm{Span}(e_i)_{i \in I}$ for $I \in \binom{[n]}{d}$. We will write this point as x_I for $I \in \binom{[n]}{d}$. So an element of $K_T^0(G(d, n))$ is a function $f : \binom{[n]}{d} \to K_T^0(\mathtt{pt})$ obeying certain conditions. What are those conditions? Each one-dimensional torus orbit joins x_I to x_J where $I = S \sqcup \{i\}$ and $J = S \sqcup \{j\}$ for some S in $\binom{[n]}{d-1}$. Thus an element of $K_T^0(G(d, n))$ is a function $f : \binom{[n]}{d} \to K_T^0(\mathtt{pt})$ such that

$$f(S \sqcup \{i\}) \equiv f(S \sqcup \{j\}) \mod 1 - t_i/t_j \tag{4.2.3}$$

for all $S \in \binom{[n]}{d-1}$ and $i, j \in [n] \backslash S$. In this case we have an even more concrete interpretation of this ring. The graph formed by the 0- and 1-dimensional orbits in $G(d, n)$ form the 1-skeleton of a polytope in the character lattice of T, namely the hypersimplex $\mathrm{conv}\{\prod_{i \in I} t_I : I \in \binom{[n]}{d}\}$. An edge in direction χ corresponds to a 1-dimensional orbit acted on by χ. Then $K_T^0(G(d, n))$ is the ring of splines, i.e. continuous piecewise polynomial functions, on the normal fan of the hypersimplex; condition (4.2.3) asserts continuity along the codimension 1 cones.

We now describe how to compute tensor products, pushforwards and pullbacks in the localization description. The first two are simple. Tensor product corresponds to multiplication. That is to say,

$$\left([E]^T [F]^T\right)(x) = [E]^T(x) \cdot [F]^T(x) \tag{4.2.4}$$

for $x \in X^T$. Pullback corresponds to pullback. That is to say, if X and Y are equivariantly formal spaces, and $\pi : X \to Y$ a T-equivariant map, then

$$\left(\pi^* [E]^T\right)(x) = [E]^T(\pi(x)) \tag{4.2.5}$$

for $x \in X^T$ and $[E]^T \in K_0^T(X)$. The proofs are simply to note that pullback to X^T and Y^T is compatible with pullback and with multiplication in the appropriate ways.

The formula for pushforward is somewhat more complex. Let X and Y be contracting at every fixed point and $\pi : X \to Y$ a T-equivariant map. For $x \in X^T$, let $\chi_1(x)$, $\chi_2(x)$, ..., $\chi_r(x)$ be the characters of T acting on a neighborhood of x; for $y \in Y^T$, define $\eta_1(y)$, ..., $\eta_s(y)$ similarly. Then we have the formula

$$\frac{(\pi_*[E]^T)(y)}{(1-\eta_1^{-1}(y))\cdots(1-\eta_s^{-1}(y))} = \sum_{x \in X^T,\ \pi(x)=y} \frac{[E]^T(x)}{(1-\chi_1^{-1}(x))\cdots(1-\chi_r^{-1}(x))}. \tag{4.2.6}$$

See [21, Theorem 5.11.7].

It is often more convenient to state this equation in terms of multi-graded Hilbert series. If $\mathrm{Hilb}(E_x)$ is the multi-graded Hilbert series of the stalk E_x, then equation (4.2.6) reads:

$$\mathrm{Hilb}(\pi_*(E)_y) = \sum_{x \in X^T,\ \pi(x)=y} \mathrm{Hilb}(E_x) \tag{4.2.7}$$

It is also important to note how this formula simplifies in the case of the pushforward to a point. In that case, we get

$$\int_X^T [E]^T = \sum_{x \in X^T} \mathrm{Hilb}(E_x) \tag{4.2.8}$$

This special case is more prominent in the literature than the general result (4.2.6); see for example [71, Section 4] for some classical applications.

Finally, we describe the relation between ordinary and T-equivariant K-theories. There is a map from equivariant K-theory to ordinary K-theory by forgetting the T-action. In particular, the map $K_T^0(\mathtt{pt}) \to K^0(\mathtt{pt}) = \mathbb{Z}$ just sends every character of T to 1. In this way, $\mathbb{Z}$ becomes a $K_T^0(\mathtt{pt})$-module. Thus, for any space X with a T-action, we get a map $K_T^0(X) \otimes_{K_T^0(\mathtt{pt})} \mathbb{Z} \to K^0(X)$. All we will need is that this map exists, but the reader might be interested to know the stronger result:

Theorem 4.2.12 ([66, Theorem 4.3])**.** *Assuming Condition 4.2.4, the map*

$$K_T^0(X) \otimes_{K_T^0(\mathtt{pt})} \mathbb{Z} \to K^0(X)$$

is an isomorphism.

4.3 Matroids and Grassmannians

Let E be a finite set (the *ground set*), which we will usually take to be $[n]$. For $I \subseteq \mathsf{E}$, we write e_I for the vector $\sum_{i \in I} e_i$ in $\mathbb{Z}^{\mathsf{E}}$.

Let M be a collection of d-element subsets of E. Let $\mathrm{Poly}(M)$ be the convex hull of the vectors e_I, as I runs through M. The collection M is called a matroid if it obeys any of a number of equivalent conditions. Our favorite is due to Gelfand, Goresky, MacPherson and Serganova:

Theorem 4.3.1 ([37, Theorem 4.1]). *M is a matroid if and only if M is nonempty and every edge of* $\mathrm{Poly}(M)$ *is in the direction $e_i - e_j$ for some i and $j \in \mathsf{E}$.*

See [23] for motivation and [70] for more standard definitions. We now explain the connection between matroids and Grassmannians. We assume basic familiarity with Grassmannians and their Plücker embedding. See [68, Chapter 14] for background. Given a point x in $G(d,n)$, the set of I for which the Plücker coordinate $p_I(x)$ is nonzero forms a matroid, which we denote $\mathrm{Mat}(x)$. (A matroid of this form is called *realizable*.) Let T be the torus $(\mathbb{C}^*)^n$, which acts on $G(d,n)$ in the obvious way, so that $p_I(tx) = t^{e_I} p_I(x)$ for $t \in T$. Clearly, $\mathrm{Mat}(tx) = \mathrm{Mat}(x)$ for any $t \in T$.

Remark 4.3.2. We pause to explain the connection to hyperplane arrangements, although this will only be needed for motivation. Let H_1, H_2, ..., H_n be a collection of hyperplanes through the origin in $\mathbb{C}^d$. Let v_i be a normal vector to H_i. Then the row span of the $d \times n$ matrix $(v_1\ v_2\ \cdots\ v_n)$ is a point in $G(d,n)$. This point is determined by the hyperplane arrangement, up to the action of T. Thus, it is reasonable to study hyperplane arrangements by studying T-invariant properties of x. In particular, $\mathrm{Mat}(x)$ is an invariant of the hyperplane arrangement. It follows easily from the definitions that $\{i_1, \ldots, i_d\}$ is in $\mathrm{Mat}(x)$ if and only if the hyperplanes H_{i_1}, ..., H_{i_d} are transverse.

We now discuss how we will bring K-theory into the picture. Consider the torus orbit closure $\overline{Tx}$. The orbit Tx is a translate (by x) of the image of the monomial map given by the set of characters $\{t^{-e_I} : p_I(x) \neq 0\}$. Essentially[1] by definition, $\overline{Tx}$ is the toric variety associated to the polytope $\mathrm{Poly}(\mathrm{Mat}(x))$ (see [22, Section 5]). In the appendix to [83], Speyer checked that the class of the structure sheaf of $\overline{Tx}$ in $K_T^0(G(d,n))$ depends only on $\mathrm{Mat}(x)$, and gave a natural way to define a class $y(M)$ in $K_T^0(G(d,n))$ for any matroid M of rank d on $[n]$, nonrealizable matroids included.

We review this construction here. For a polyhedron P and a point $v \in P$, define $\mathrm{Cone}_v(P)$ to be the positive real span of all vectors of the form $u - v$, with $u \in P$; if v is not in P, define $\mathrm{Cone}_v(P) = \emptyset$. Let $M \subseteq \binom{[n]}{d}$ be a matroid. We will abbreviate $\mathrm{Cone}_{e_I}(\mathrm{Poly}(M))$ by $\mathrm{Cone}_I(M)$. For a pointed rational polyhedron C in $\mathbb{R}^n$, define $\mathrm{Hilb}(C)$ to be the Hilbert series

$$\mathrm{Hilb}(C) := \sum_{a \in C \cap \mathbb{Z}^n} t^a.$$

[1]We say essentially for two reasons. First, Cox describes the toric variety associated to a polytope P as a the Zariski closure of the image of $t \mapsto (t^p)_{p \in P \cap \mathbb{Z}^n}$. We would rather describe it as the Zariski clsoure of the image of $t \mapsto (t^{-p})_{p \in P}$. These are the same subvariety of $G(d,n)$, and the same class in K-theory, but our convention makes the obvious torus action on the toric variety match the restriction of the torus action on $G(d,n)$. The reader may wish to check that our conventions are compatible with Example 4.2.3.

Second, there is a potential issue regarding normality here. According to most references, the toric variety associated to $\mathrm{Poly}(\mathrm{Mat}(x))$ is the normalization of $\overline{Tx}$. See the discussion in [22, Section 5]. However, this issue does not arise for us because $\overline{Tx}$ is normal and, in fact, projectively normal; see [93].

This is a rational function with denominator dividing $\prod_{i\in I}\prod_{j\notin I}(1-t_i^{-1}t_j)$ [84, Theorem 4.6.11]. We define the class $y(M)$ in $K_T^0(G(d,n))$ by

$$y(M)(x_I) := \mathrm{Hilb}(\mathrm{Cone}_I(M))\prod_{i\in I}\prod_{j\notin I}(1-t_i^{-1}t_j),$$

Note that $\mathrm{Hilb}(\mathrm{Cone}_I(M)) = 0$ for $I \notin M$.

To motivate this definition, suppose M is of the form $\mathrm{Mat}(x)$ for some $x \in G(d,n)$. For I in M, the toric variety $\overline{Tx}$ is isomorphic near x_I to $\operatorname{Spec}\mathbb{C}[\mathrm{Cone}_I(M)\cap\mathbb{Z}^n]$. In particular, the Hilbert series of the structure sheaf of $\overline{Tx}$ near x_I is $\mathrm{Hilb}(\mathrm{Cone}_I(M))$. So in this situation $y(M)$ is exactly the T-equivariant class of the structure sheaf of $\overline{Tx}$.

We now prove the following fact, which was stated without proof in [83] as Proposition A.6.

Proposition 4.3.3. *Whether or not M is realizable, the function $y(M)$ from $G(d,n)^T$ to $K_T^0(\mathtt{pt})$ defines a class in $K_T^0(G(d,n))$.*

This follows from a more general polyhedral result.

Lemma 4.3.4. *Let P be a lattice polytope in $\mathbb{R}^n$ and let u and v be vertices of P connected by an edge of P. Let e be the minimal lattice vector along the edge pointing from u to v, with $v = u + ke$. Then $\mathrm{Hilb}(\mathrm{Cone}_u(P)) + \mathrm{Hilb}(\mathrm{Cone}_v(P))$ is a rational function whose denominator is **not** divisible by $1-t^e$.*

It is not too hard to give a direct proof of this result, but we cheat and use Brion's formula.

Proof. Note that the truth of the claim is preserved under dilating the polytope by some positive integer N, since this does not effect the cones at the vertices.

Since u and v are joined by an edge, we can find a hyperplane H such that u and v lie on one side of H, and the other vertices of P lie on the other. Perturbing H, we may assume that it is not parallel to e, and that the defining equation of H has rational coefficients. Let H^+ be the closed half space bounded by H, containing u and v. Then $H^+\cap P$ is a bounded polytope and, after dilation, we may assume that it is a lattice polytope.

By Brion's formula ([10], [17]) applied to give the Ehrhart polynomial at 0,

$$\sum_{v\in\mathrm{Vert}(P\cap H^+)}\mathrm{Hilb}(\mathrm{Cone}_w(H^+\cap P)) = 1.$$

The terms coming from vertices w other than u and v have denominators not divisible by $(1-t^e)$. So $\mathrm{Hilb}(\mathrm{Cone}_u(P\cap H^+)) + \mathrm{Hilb}(\mathrm{Cone}_v(P\cap H^+))$, which is $\mathrm{Hilb}(\mathrm{Cone}_u(P)) + \mathrm{Hilb}(\mathrm{Cone}_v(P))$, also has denominator not divisible by $1-t^e$. □

Proof of Proposition 4.3.3. We must check the conditions of Theorem 4.2.11. If x_I and x_J are two fixed points of $G(d,n)$, joined by a one dimensional orbit, then we must have $I = S \cup \{i\}$ and $J = S \cup \{j\}$ for some $S \in \binom{[n]}{k-1}$ with $i, j \in [n] \setminus S$. We must check that $y(M)(x_I) \equiv y(M)(x_J) \mod 1 - t_i^{-1}t_j$. Abbreviate $\prod_{a \in I} \prod_{b \in [n] \setminus I} (1 - t_a^{-1} t_b)$ to d_I, and define d_J similarly. Observe that $d_I \equiv d_J \equiv 0 \mod 1 - t_i^{-1}t_j$ and $d_I \equiv -d_J \mod (1 - t_i t_j^{-1})^2$.

If I and J are not in M, then $y(M)(x_I) = y(M)(x_J) = 0$.

Suppose that $I \in M$ and $J \notin M$. Since $\mathrm{Hilb}(\mathrm{Cone}_I(M))$ has no edge in direction $e_i - e_j$, the denominator of $\mathrm{Hilb}(\mathrm{Cone}_I(M))$ is not divisible by $1 - t_i^{-1}t_j$. So $y(M)(x_I) = d_I \,\mathrm{Hilb}(\mathrm{Cone}_I(M))$ is divisible by $1 - t_i^{-1}t_j$, as required.

If I and J are in M, then we apply Lemma 4.3.4 to conclude that the denominator of $\mathrm{Hilb}(\mathrm{Cone}_I(M)) + \mathrm{Hilb}(\mathrm{Cone}_J(M))$ is not divisible by $1 - t_i^{-1}t_j$. Also, the denominator of $\mathrm{Hilb}(\mathrm{Cone}_J(M))$ is only divisble by $1 - t_i^{-1}t_j$ once.

Writing

$$y(M)_I - y(M)_J = d_I \,\mathrm{Hilb}(\mathrm{Cone}_I(M)) - d_J \,\mathrm{Hilb}(\mathrm{Cone}_J(M)) = \\ d_I \left(\mathrm{Hilb}(\mathrm{Cone}_I(M)) + \mathrm{Hilb}(\mathrm{Cone}_J(M))\right) - (d_I + d_J) \,\mathrm{Hilb}(\mathrm{Cone}_J(M)),$$

we see that each term on the righthand side is divisible by $1 - t_i^{-1}t_j$. □

Although we will not need this fact, it follows from the Basis Exchange theorem [72, Lemma 1.2.2] that the semigroup $\mathrm{Cone}_I(M) \cap \mathbb{Z}^n$ is generated by the first nonzero lattice point on each edge of $\mathrm{Cone}_I(M)$, i.e. by the vectors $e_j - e_i$, where (i,j) ranges over the pairs $i \in I$, $j \notin I$ such that $I \cup \{j\} \setminus \{i\}$ is in M.

Example 4.3.5. We work through these definitions for the case of a matroid in $G(2,4)$, namely

$$M = \{13, 14, 23, 24, 34\}.$$

This M is realizable, arising as $\mathrm{Mat}(x)$ when for instance x is the rowspan of $\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, with Plücker coordinates $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}) = (0, 1, 1, 1, 1, -1)$. For $t \in T$ the Plücker coordinates of tx are

$$(0, t_1t_3, t_1t_4, t_2t_3, t_2t_4, -t_3t_4).$$

Every point of $G(2,4)$ with $p_{12} = 0$ and the other Plücker coordinates nonzero can be written in this form, so $\overline{Tx}$ has defining equation $p_{12} = 0$ in $G(2,4)$. (That is, $\overline{Tx}$ is the Schubert divisor in $G(2,4)$, of which x is an interior point. Compare Remark 4.4.2, in which notation $M = SM(13)$ and the Schubert divisor is Ω_{13}.)

Computing $y(M)$ entails finding the Hilbert functions $\mathrm{Hilb}(\mathrm{Cone}_I(M))$ for each $I \in M$. The cone $\mathrm{Cone}_{13}(M)$ is a unimodular simplicial cone with ray generators $e_2 - e_1$, $e_4 - e_1$, and $e_4 - e_3$, so we have

$$\mathrm{Hilb}(\mathrm{Cone}_{13}(M)) = \frac{1}{(1 - t_1^{-1}t_2)(1 - t_1^{-1}t_4)(1 - t_3^{-1}t_4)}.$$

We can do similarly for the other bases 14, 23 and 24. At $I = 34$, the cone $\mathrm{Cone}_{34}(M)$ is the cone over a square with ray directions $e_1 - e_3$, $e_1 - e_4$, $e_2 - e_3$, and $e_2 - e_4$. By summing over a triangulation of this cone we find that

$$\mathrm{Hilb}(\mathrm{Cone}_{34}(M)) = \frac{1 - t_1 t_2 t_3^{-1} t_4^{-1}}{(1 - t_1 t_3^{-1})(1 - t_1 t_4^{-1})(1 - t_2 t_3^{-1})(1 - t_2 t_4^{-1})}.$$

Accordingly, $y(M)$ is sent under the localization map of Theorem 4.2.6 to

$$(0, 1 - t_2 t_3^{-1}, 1 - t_2 t_4^{-1}, 1 - t_1 t_3^{-1}, 1 - t_1 t_4^{-1}, 1 - t_1 t_2 t_3^{-1} t_4^{-1})$$

again ordering the coordinates lexicographically. We see that this satisfies the congruences in Theorem 4.2.11. ♢

4.4 Valuations

A subdivision of a polyhedron P is a polyhedral complex Σ with $|\Sigma| = P$. We use the names $P_1, \ldots, P_k$ for the facets of a typical subdivision Σ of P, and for $J \subseteq [k]$ nonempty we write $P_J = \bigcap_{j \in J} P_j$, which is a face of Σ. We also put $P_\emptyset = P$. Let $\mathcal{P}$ be a set of polyhedra in a vector space V, and A an abelian group. We say that a function $f : \mathcal{P} \to A$ is a *valuation* (or is *valuative*) if, for any subdivision such that $P_J \in \mathcal{P}$ for all $J \subseteq [k]$, we have

$$\sum_{J \subseteq [k]} (-1)^{|J|} f(P_J) = 0.$$

For example, one valuation of fundamental importance to the theory is the function $\mathbf{1}(\cdot)$ mapping each polytope P to its characteristic function. Namely, $\mathbf{1}(P)$ is the function $V \to \mathbb{Z}$ which takes the value 1 on P and 0 on $V \setminus P$.

We will be concerned in this chapter with the case $\mathcal{P} = \{\mathrm{Poly}(M) : M \text{ a matroid}\}$, and we will identify functions on $\mathcal{P}$ with the corresponding functions on matroids themselves. Many important functions of matroids, including the Tutte polynomial, are valuations.

We now summarize some results of Chapter 3. A function of matroid polytopes is a valuation if and only if it factors through $\mathbf{1}$. Therefore, the group of matroid polytope valuations valued in A is $\mathrm{Hom}(\mathcal{I}, A)$, where $\mathcal{I}$ is the $\mathbb{Z}$-module of functions $V \to \mathbb{Z}$ generated by indicator functions of matroid polytopes. (In Chapter 3, $\mathcal{I}$ was called P_{M}.) We are also interested in valuative matroid invariants, those valuations which take equal values on isomorphic matroids. For M a matroid on the ground set E and $\sigma \in \mathfrak{S}_{\mathsf{E}}$ a permutation, let $\sigma \cdot M$ be the matroid $\{\{\sigma(i_1), \ldots, \sigma(i_d)\} : \{i_1, \ldots, i_d\} \in \mathsf{E}\}$. This action of $\mathfrak{S}_n$ induces an action of $\mathfrak{S}_n$ on $\mathcal{I}$. We write $\mathcal{I}/\mathfrak{S}_n$ for the quotient of $\mathcal{I}$ by the subgroup generated by elements of the form $\sigma(M) - M$, with $\sigma \in \mathfrak{S}_n$ and $M \in \mathcal{I}$. The group of valuative invariants valued in A is $\mathrm{Hom}(\mathcal{I}, A)^{\mathfrak{S}_n} = \mathrm{Hom}(\mathcal{I}/\mathfrak{S}_n, A)$.

Given $I = \{i_1, \ldots, i_d\} \in \binom{[n]}{d}$ with $i_1 < \cdots < i_d$, the *Schubert matroid* $\mathrm{SM}(I)$ is the matroid consisting of all sets $\{j_1, \ldots, j_d\} \in \binom{[n]}{d}$, $j_1 < \ldots < j_d$ such that $j_k \geq i_k$ for each $k \in [d]$. In the notation of Chapter 3, $SM(I)$ is the matroid $R_{\mathrm{M}}(\underline{X}, \underline{r})$ where $X_i = [i]$ and $r_i = |I \cap [i]|$ for $1 \leq i \leq n$.

Theorem 4.4.1, which was Theorems 5.4 and 6.3 of Chapter 3, provides explicit bases for $\mathcal{I}$ and $\mathcal{I}/\mathfrak{S}_n$. The dual bases are bases for the groups of valuative matroid functions and invariants, respectively.

Theorem 4.4.1. *For $I \in \binom{[n]}{d}$, let $\rho(I) \subseteq \mathfrak{S}_n$ consist of one representative of each coset of the stabilizer $(\mathfrak{S}_n)_{\mathrm{SM}(I)}$.*

(a) *The set $\{\mathrm{Poly}(\sigma \cdot \mathrm{SM}(I)) : I \in \binom{[n]}{d}, \sigma \in \rho(I)\}$ is a basis for $\mathcal{I}$.*

(b) *The set $\{\mathrm{Poly}(\mathrm{SM}(I)) : I \in \binom{[n]}{d}\}$ is a basis for $\mathcal{I}/\mathfrak{S}_n$.*

Remark 4.4.2. We caution the reader that $y(\mathrm{SM}(I))$ is *not* in general the class of the structure sheaf of the Schubert variety Ω_I. Letting $I = \{i_1, \ldots, i_d\}$, $i_1 < \cdots < i_d$, this is the Schubert variety consisting of d-planes x whose Plücker coordinate $p_J(x)$ is zero for all $J = \{j_1, \ldots, j_d\}$, $j_1 < \cdots < j_d$, such that $j_k < i_k$ for some k. The two varieties differ in that Ω_I is the closure of the set of all points $x \in G(d, n)$ with $\mathrm{Mat}(x) = \mathrm{SM}(I)$, while $y(M)$ is the class of the closure of the torus orbit through a single point x with $\mathrm{Mat}(x) = \mathrm{SM}(I)$. Once Ω_I is large enough to have multiple torus orbits in its interior, there appears to be no relation between $y(\mathrm{SM}(I))$ and $[\mathcal{O}_{\Omega_I}]^T$.

We now discuss how valuations arise from K-theory. Let Σ be a subdivision of matroid polytopes, with facets $P_1, \ldots, P_k$, and let $P_J = \mathrm{Poly}(M_J)$. Then we have a linear relation of K-theory classes

$$\sum_{J \subseteq [k]} (-1)^{|J|} y(M_J) = 0. \tag{4.4.1}$$

That is,

Proposition 4.4.3. *The function y is a valuation of matroids valued in $K_T^0(G(d, n))$.*

Proof. Let $I \in \binom{[n]}{d}$. We will check that $\sum_{J \subseteq [k]} (-1)^{|J|} y(M_J)(x_I) = 0$. The nonempty cones among the $\mathrm{Cone}_I(M_j)$, $j = 1, \ldots, k$, are the facets of a polyhedral subdivision, and $\mathrm{Cone}_I(M_J) = \bigcap_{j \in J} \mathrm{Cone}_I(M_j)$. Then the proposition holds since taking the Hilbert series of a cone is a valuation. □

As a corollary, for any linear map $f : K_T^0(G(d, n)) \to A$, the composition $f \circ y$ is a valuation as well. In particular, all of the following are matroid valuations: the product of $y(M)$ with a fixed class $[E]^T \in K_T^0(G(d, n))$; any pushforward of such a product; and the non-equivariant version of any of these obtained by sending all characters of T to 1.

Note that $\mathfrak{S}_n$ acts trivially on $K^0(G(d,n))$, so $M \mapsto y(M)$ is a matroid invariant, and so is $M \mapsto \int y(M)[E]$ for any $E \in K^0(G(d,n))$. On the other hand, $\mathfrak{S}_n$ acts nontrivially on $K^0_T(G(d,n))$, so valuations built from equivariant K-theory need not be matroid invariants.

As the reader can see from Theorem 4.4.1, $\mathcal{I}/\mathfrak{S}_n$ is free of rank $\binom{n}{d}$. The group $K^0(G(d,n))$ is also free of rank $\binom{n}{d}$. This gives rise to the hope that every valuative matroid invariant might factor through $M \mapsto y(M)$, i.e. that every matroid valuation might come from K-theory. This hope is quite false. We give a conceptual explanation for why it is wrong, followed by a counterexample.

The reason this should be expected to be false is that no torus orbit closure can have dimension greater than that of T, namely $n-1$. Therefore, $\int y(M)[E]$ vanishes whenever E is supported in codimension n or greater. This imposes nontrivial linear constraints on $y(M)$, so the classes $y(M)$ span a proper subspace of $K^0(G(d,n))$.

Example 4.4.4. We exhibit an explicit non-K-theoretic valuative matroid invariant. Up to isomorphism, there are 7 matroids of rank 2 on [4]. Six of them are Schubert matroids $\mathrm{SM}(I)$; the last is $M_0 := \{13, 23, 14, 24\}$. The unique linear relation in $\mathcal{I}/\mathfrak{S}_4$ is

$$[M_0] = 2[\mathrm{SM}(13)] - [\mathrm{SM}(12)],$$

corresponding to the unique matroid polytope subdivision of these matroids, an octahedron cut into two square pyramids along a square. However, in $K^0(G(2,4))$, we have the additional relation

$$y(M_0) = y(\mathrm{SM}(14)) + y(\mathrm{SM}(23)) - y(\mathrm{SM}(24)).$$

The reader can verify this relation by using equivariant localization to express $y(M_0)$ as a $K^0_T(\mathtt{pt})$-linear combination of the $y(SM(I))$, and then applying Theorem 4.2.12.

Consider the matroid invariant given by $z(\mathrm{SM}(12)) = z(\mathrm{SM}(13)) = z(M_0) = 1$ and $z(\mathrm{SM}(I)) = 0$ for all other I (extended to be $\mathfrak{S}_4$-invariant in the unique way).The reader may prefer the following description: $z(M)$ is 1 if $\mathrm{Poly}(M)$ contains $(1/2, 1/2, 1/2, 1/2)$ and 0 otherwise. Then z is valuative, but does not extend to a linear function on $K^0(G(d,n))$. $\diamondsuit$

4.5 A fundamental computation

Let $[E]$ be a class in $K^0(G(d,n))$. Recall from Section 4.1 the maps $\pi_d : \mathcal{F}\ell(1,d,n-1;n) \to G(d,n)$ and $\pi_{1(n-1)} : \mathcal{F}\ell(1,d,n-1;n) \to \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$, and the notations α and β for the hyperplane classes in $K^0(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$.

Over $G(d,n)$, we have the tautological exact sequence

$$0 \to S \to \mathbb{C}^n \to Q \to 0. \tag{4.5.1}$$

Here S and Q, the *tautological sub-* and *quotient bundles*, are the natural bundles such that over each point of $G(d,n)$, the fiber of S is the corresponding d-dimensional vector space, and that of Q is the $(n-d)$-dimensional quotient.

The point of this section is the following computation:

Lemma 4.5.1. *Given $[E] \in K^0(G(d,n))$, define a formal polynomial in u and v by*

$$R(u,v) := \int_{G(d,n)} [E] \sum [\textstyle\bigwedge^p S][\bigwedge^q (Q^\vee)] u^p v^q.$$

Then

$$(\pi_{1(n-1)})_* \pi_d^* [E] = R(\alpha - 1, \beta - 1).$$

Remark 4.5.2. Lemma 4.5.1 is an equality in non-equivariant K-theory. In equivariant K-theory, we may only speak of the class of a hyperplane if it is a coordinate hyperplane, and then the class depends on which coordinate hyperplane it is. We do not have an equivariant generalization of Lemma 4.5.1.

For the purposes of this section we will write $\kappa = [\mathcal{O}(1,0)]$ and $\lambda = [\mathcal{O}(0,1)]$. Recall that $\kappa^{-1} = 1 - \alpha$ and $\lambda^{-1} = 1 - \beta$, by exact sequence (4.1.2). For k, $\ell \geq 0$, we have $(\pi_d)_*(\pi_{1(n-1)})^*(\kappa^k \lambda^\ell) = [\mathrm{Sym}^k Q \otimes \mathrm{Sym}^\ell S^\vee]$.

From the sequence (4.5.1), we have $[S] + [Q] = n$. Similarly, we have a filtration $0 \subseteq \bigwedge^k S = F_n \subseteq F_{n-1} \subseteq \cdots \subseteq F_0 = \bigwedge^k \mathbb{C}^n$, where F_i is spanned by wedges i of whose terms lie in S. Its successive quotients are $\bigwedge^k S$, $\bigwedge^{k-1} S \otimes Q$, $\ldots$, $\bigwedge^k Q$, giving the relation $\sum_{i=0}^k [\bigwedge^i S][\bigwedge^{k-i} Q] = \binom{n}{k}$. We can encode all of these relations as a formal power series in u, with coefficients in $K^0(G(d,n))$:

$$\left(\sum_p [\textstyle\bigwedge^p (S)] u^p \right) \left(\sum_\ell [\bigwedge^\ell (Q)] u^\ell \right) = (1+u)^n$$

Also, from the exactness of the Koszul complex [34, appendix A2.6.1],

$$\left(\sum_k (-1)^k [\mathrm{Sym}^k (Q)] u^k \right) \left(\sum_\ell [\textstyle\bigwedge^\ell (Q)] u^\ell \right) = 1.$$

So

$$\sum [\textstyle\bigwedge^p (S)] u^p = (1+u)^n \left(\sum (-1)^k [\mathrm{Sym}^k (Q)] u^k \right).$$

The right hand side is

$$\left((\pi_d)_* \pi_{1(n-1)}^* \sum (-1)^k \kappa^k u^k \right) (1+u)^n = (1+u)^n (\pi_d)_* \pi_{1(n-1)}^* \left(\frac{1}{1+u\kappa} \right).$$

Similarly,

$$\sum [\textstyle\bigwedge^q (Q)^\vee] v^q = (1+v)^n (\pi_d)_* \pi_{1(n-1)}^* \left(\frac{1}{1+v\lambda} \right).$$

So,

$$R(u,v) = (1+u)^n(1+v)^n \int_{G(d,n)} [E](\pi_d)_*\pi^*_{1(n-1)} \left(\frac{1}{(1+u\kappa)(1+v\lambda)} \right).$$

By the projection formula (equation 4.2.1),

$$R(u,v) = (1+u)^n(1+v)^n \int_{\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}} \left((\pi_{1(n-1)})_*\pi^*_d[E] \right) \frac{1}{(1+u\kappa)(1+v\lambda)}.$$

Since $\kappa = (1-\alpha)^{-1}$ and $\lambda = (1-\beta)^{-1}$, we get

$$R(u,v) = \int \left((\pi_{1(n-1)})_*\pi^*_d[E] \right) \frac{(1+u)^n(1+v)^n}{(1+u(1-\alpha)^{-1})(1+v(1-\beta)^{-1})}.$$

The quantity multiplying $(\pi_{1(n-1)})_*\pi^*_d[E]$ can be expanded as a geometric series

$$\sum (1-\alpha)(1-\beta)\alpha^k\beta^\ell(1+u)^{n-1-k}(1+v)^{n-1-\ell}.$$

The sum is finite because $\alpha^n = \beta^n = 0$.

Let $((\pi_{1(n-1)})_*\pi^*_d[E]) = \sum T_{ij}\alpha^i\beta^j$. Now, $\int_{\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}} \alpha^i\beta^j$ is 1 if i and j are both less than n, and zero otherwise. So

$$\int \alpha^i\beta^j(1-\alpha)(1-\beta)\alpha^k\beta^\ell = \begin{cases} 1 & \text{if } i = n-1-k \text{ and } j = n-\ell-1 \\ 0 & \text{otherwise} \end{cases}.$$

We deduce that

$$R(u,v) = \sum T_{ij}(1+u)^i(1+v)^j \text{ and } R(u-1,v-1) = \sum T_{ij}u^iv^j.$$

Looking at the definition of the T_{ij}, we have deduced Lemma 4.5.1.

4.6 Flipping cones

Let f be a rational function in $\mathbb{Q}(z_1, z_2, \ldots, z_n)$. It is possible that many different Laurent power series represent f on different domains of convergence. In this section, we will study this phenomenon. Everything in this section is surely known to experts, but we could not find references for much of it. ([43] treats a case, but doesn't contain the lemmas we need). We recommend [9] as a general introduction to generating functions for lattice points in cones. The results here can be thought of as generalizations of the relationships between the

lattice point enumeration formulas of Brianchon-Gram, Brion and Lawrence-Varchenko. We recommend [10] as an introduction to these formulas.

Let $\mathcal{P}_n$ be the vector space of real-valued functions on $\mathbb{Z}^n$ which are linear combinations of the characteristic functions of finitely many lattice polytopes. (A polytope need not be bounded.) We denote the characteristic function of the polytope P by $\mathbf{1}(P)$. If P is a pointed polytope, then the sum $\sum_{e \in P} z^e$ converges somewhere, and the value it converges to is a rational function in $\mathbb{R}(z_1, \ldots, z_n)$ which we denote $\mathrm{Hilb}(P)$.

It is a theorem of Lawrence [59], and later Khovanski-Pukhlikov [51], that $\mathbf{1}(P) \mapsto \mathrm{Hilb}(P)$ extends to a linear map $\mathrm{Hilb} : \mathcal{P} \to \mathbb{Q}(z_1, \ldots, z_n)$. If P is a polytope with nontrivial lineality space then $\mathrm{Hilb}(\mathbf{1}(P)) = 0$.

Let $\boldsymbol{\zeta} := (\zeta_1, \zeta_2, \ldots, \zeta_n)$ be a basis for $(\mathbb{R}^n)^*$, which for expediency we identify with $\mathbb{R}^n$ via the standard inner product. Define an order $<_{\boldsymbol{\zeta}}$ on $\mathbb{Q}^n$ by $x <_{\boldsymbol{\zeta}} y$ if, for some index i, we have $\langle \zeta_1, x \rangle = \langle \zeta_1, y \rangle$, $\langle \zeta_2, x \rangle = \langle \zeta_2, y \rangle$, $\ldots$, $\langle \zeta_{i-1}, x \rangle = \langle \zeta_{i-1}, y \rangle$ and $\langle \zeta_i, x \rangle < \langle \zeta_i, y \rangle$.

Remark 4.6.1. Note that, if the components of ζ_1 are linearly independent over $\mathbb{Q}$, we can disregard the later components of $\boldsymbol{\zeta}$. For any finite collection of vectors in $\mathbb{Q}^n$, we can find ζ_1' with such linearly independent components so that $<_{\boldsymbol{\zeta}}$ and $<_{\zeta_1'}$ agree on this collection. We could use this trick to reduce to the case of a single vector in all of our applications, but the freedom to use vectors with integer entries will be convenient.

We'll say that a polytope P is $\boldsymbol{\zeta}$*-pointed* if, for every $a \in \mathbb{R}^n$, the intersection $P \cap \{e : e <_{\boldsymbol{\zeta}} a\}$ is bounded. We'll say that an element in $\mathcal{P}_n$ is $\boldsymbol{\zeta}$-pointed if it is supported on a finite union of $\boldsymbol{\zeta}$-pointed polytopes. Let $\mathcal{P}_n^{\boldsymbol{\zeta}}$ be the vector space of $\boldsymbol{\zeta}$-pointed elements in $\mathcal{P}_n$.

Lemma 4.6.2. *The vector space $\mathcal{P}_n$ is spanned by the classes of simplicial cones.*

Proof. Let P be any polytope. By the Brianchon-Gram formula ([15, 40], see also [79] for a modern exposition), $[P]$ is a linear combination of classes of cones. We can triangulate those cones into simplicial cones. □

Lemma 4.6.3. *The restriction of* Hilb *to $\mathcal{P}_n^{\boldsymbol{\zeta}}$ is injective.*

Proof of Lemma 4.6.3. Suppose, for the sake of contradiction, that $\mathrm{Hilb}(b) = 0$ for some nonzero $b \in \mathcal{P}_n^{\boldsymbol{\zeta}}$. Note that $\mathcal{P}_n$ is a $\mathbb{Q}[t_1, \ldots, t_n]$ module with the multiplication $t_i * \mathbf{1}(P) = \mathbf{1}(P + e_i)$. For any simplicial cone C, there is a nonzero polynomial $q(t) \in \mathbb{Q}[t_1, \ldots, t_n]$ such that $q * \mathbf{1}(C)$ has finite support. We can take $q(t) = \prod (1 - t^e)$ where the product is over the minimal lattice vectors on the rays of C [84, Theorem 4.6.11]. So, by Lemma 4.6.2, we can find a nonzero $q \in \mathbb{Q}[t_1, \ldots, t_n]$ such that $q * b$ is finitely supported.

Now, Hilb is clearly $\mathbb{Q}[t_1, \ldots, t_n]$-linear. So $\mathrm{Hilb}(q * b) = q \cdot \mathrm{Hilb}(b) = 0$. But $q * b$ is finitely supported, so $q * b = 0$.

Thus far, we have not used that b is $\boldsymbol{\zeta}$-pointed. We use this now. Let e be the $\boldsymbol{\zeta}$-minimal element of $\mathbb{Z}^n$ for which $b(e) \neq 0$. Also, let d be the $\boldsymbol{\zeta}$-minimal exponent for which t^d occurs in q. Then the coefficient of $d + e$ in $q * b$ is nonzero, a contradiction. □

We will usually use the above lemma in the following, obviously equivalent, form:

Corollary 4.6.4. *Suppose that we have functions f_1, f_2, ..., f_r, g_1, g_2, ..., g_s in $\mathcal{P}_n^{\boldsymbol{\zeta}}$ and scalars a_1, ..., a_r, b_1, ..., b_s such that $\sum a_i \operatorname{Hilb}(f_i) = \sum b_j \operatorname{Hilb}(g_j)$ in $\mathbb{Q}(z_1, \ldots, z_n)$. Let e be any lattice point in $\mathbb{Z}^n$. Then $\sum a_i f_i(e) = \sum b_j g_j(e)$ in $\mathcal{P}_n^{\boldsymbol{\zeta}}$.*

Let C be a simplicial cone with vertex w, spanned by rays v_1, v_2, ..., v_r. Reorder the v_i so that $v_i <_{\boldsymbol{\zeta}} 0$ for $1 \leq i \leq \ell$ and $v_i >_{\boldsymbol{\zeta}} 0$ for $\ell + 1 \leq i \leq r$. Define the set $C^{\boldsymbol{\zeta}}$ to be

$$C^{\boldsymbol{\zeta}} = \{w + \sum_{i=1}^{r} a_i v_i : \ a_i < 0 \text{ for } 1 \leq i \leq \ell \text{ and } a_i \geq 0 \text{ for } \ell + 1 \leq i \leq n\}$$

and define

$$\mathbf{1}(C)^{\boldsymbol{\zeta}} = (-1)^{\ell} \mathbf{1}(C^{\boldsymbol{\zeta}}).$$

Note that $C^{\boldsymbol{\zeta}}$ is $\boldsymbol{\zeta}$-pointed.

Lemma 4.6.5. *With the above notation,*

$$\operatorname{Hilb}(\mathbf{1}(C)) = \operatorname{Hilb}(\mathbf{1}(C)^{\boldsymbol{\zeta}}).$$

An example of Lemma 4.6.5 is that $\sum_{i \geq 0} z^i$ and $-\sum_{i<0} z^i$ both converge to $1/(1-z)$, on different domains. This shows that Corollary 4.6.4 is quite false if the f_i and g_i are taken to be in $\mathcal{P}_n$ rather than $\mathcal{P}_n^{\boldsymbol{\zeta}}$, taking f_1 and g_1 to be these two series and $a_1 = b_1 = 1$.

Proof. For I a subset of $\{1, 2, \ldots, \ell\}$, set

$$C_I := \{w + \sum_{i=1}^{r} a_i v_i : a_i \geq 0 \text{ for } i \notin I, \ a_i \in \mathbb{R} \text{ for } i \in I\}.$$

So $C_{\emptyset} = C$. Then

$$\mathbf{1}(C)^{\boldsymbol{\zeta}} = \sum_{I \subset [\ell]} (-1)^{|I|} \mathbf{1}(C_I).$$

Applying Hilb to both sides of the equation, all the terms drop out except

$$\operatorname{Hilb}(\mathbf{1}(C)^{\boldsymbol{\zeta}}) = \operatorname{Hilb}(\mathbf{1}(C_{\emptyset})) = \operatorname{Hilb}(\mathbf{1}(C)).$$

□

The following lemma, in the case that ζ_1 has linearly independent components over $\mathbb{Q}$, is the main result of [43].

Lemma 4.6.6. *Let $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n)$ be as above. For every $f \in \mathcal{P}_n$, there is a unique $f^{\boldsymbol{\zeta}} \in \mathcal{P}_n^{\boldsymbol{\zeta}}$ such that $\operatorname{Hilb}(f) = \operatorname{Hilb}(f^{\boldsymbol{\zeta}})$. The map $f \mapsto f^{\boldsymbol{\zeta}}$ is linear.*

By Lemma 4.6.5, this notation f^ζ is consistent with the earlier notation $\mathbf{1}(C)^\zeta$.

Proof. We get uniqueness from Lemma 4.6.3. It is clearly enough to prove existence in the case $f = \mathbf{1}(P)$ for some polytope P. By Lemma 4.6.2, it is enough to show $\mathbf{1}(D)^\zeta$ exists for D a simplicial cone. This is Lemma 4.6.5.

Finally, we must establish linearity. Let f and $g \in \mathcal{P}_n$ and let a and b be scalars. Then

$$\operatorname{Hilb}((af+bg)^\zeta) = \operatorname{Hilb}(af+bg) = a \operatorname{Hilb}(f) + b \operatorname{Hilb}(g) = \\ a \operatorname{Hilb}(f^\zeta) + b \operatorname{Hilb}(g^\zeta) = \operatorname{Hilb}(a(f^\zeta) + b(g^\zeta)).$$

By the uniqueness, we must have $(af+bg)^\zeta = a(f^\zeta) + b(g^\zeta)$. □

Remark 4.6.7. We warn the reader that, when C is not simplicial, $\mathbf{1}(C)^\zeta$ need not be of the form $\pm\mathbf{1}(C')$. For example, let C be the span of $(0,0,1)$, $(1,0,1)$, $(0,1,1)$ and $(1,1,1)$. Choose ζ_1 to be negative on $(0,0,1)$, $(1,0,1)$ and positive on $(0,1,1)$ and $(1,1,1)$. Then $\mathbf{1}(C)^\zeta = \mathbf{1}(U) - \mathbf{1}(V)$ where

$$U = \{a(1,0,0) + b(0,0,-1) + c(0,-1,-1) : a \geq 0,\ b,c > 0\}$$

and

$$V = \{a(1,0,0) + b(1,0,1) + c(1,1,1) : a > 0, b, c \geq 0\}.$$

Lemma 4.6.8. *Let C be a pointed cone with vertex at w. Then $\mathbf{1}(C)^\zeta$ is contained in the half space $\{x : \langle \zeta_1, x\rangle \geq \langle \zeta_1, w\rangle\}$. Furthermore, if C is not contained in $\{x : \langle \zeta_1, x\rangle \geq \langle \zeta_1, w\rangle\}$, then $\mathbf{1}(C)^\zeta$ is in the open half space $\{x : \langle \zeta, x\rangle > \langle \zeta_1, w\rangle\}$.*

Proof. For simplicial cones, this follows from the explicit description of $\mathbf{1}(C)^\zeta$ in Lemma 4.6.5. Since any cone can be triangulated, the statement about the closed half space follows immediately from linearity and the simplicial case.

If C is not contained in $\{x : \langle \zeta_1, x\rangle \geq \langle \zeta_1, w\rangle\}$ then there is some ray of C in direction v with $\langle \zeta_1, v\rangle < 0$. Choose a triangulation of C in which every interior face uses the ray v. For example, we can triangulate the faces of C which do not contain v, then cone that triangulation from v. (This is called a *pulling triangulation*.)

Letting $\mathcal{F}$ be the set of interior cones of this triangulation, we have

$$\mathbf{1}(C) = \sum_{F \in \mathcal{F}} (-1)^{\dim C - \dim F} \mathbf{1}(F)$$

and

$$\mathbf{1}(C)^\zeta = \sum_{F \in \mathcal{F}} (-1)^{\dim C - \dim F} \mathbf{1}(F)^\zeta.$$

By the simplicial computation, each summand on the right is supported on the required open half space. □

Corollary 4.6.9. *Let C_i be a finite sequence of pointed cones in $\mathbb{R}^n$, with the vertex of C_i at w_i. Let a_i be a finite sequence of scalars. Suppose that we know $\sum a_i \operatorname{Hilb}(C_i)$ is a Laurent polynomial. Then the Newton polytope of this polynomial is contained in the convex hull of the w_i.*

Proof. Let P be the Newton polytope in question and let $\sum_{e \in P} f(e) z^e$ be the polynomial. Extend f to $\mathbb{Z}^n$ by $f(e) = 0$ for $e \notin P$. Since P is a bounded polytope, f is $\boldsymbol{\zeta}$-pointed for every $\boldsymbol{\zeta}$ and, thus, $f^{\boldsymbol{\zeta}} = f$ for every $\boldsymbol{\zeta}$.

Let e be a lattice point which is not contained in the convex hull of the w_i. By the Farkas lemma [95, Proposition 1.10], there is some ζ_1 such that $\langle \zeta_1, e \rangle < \langle \zeta_1, w_i \rangle$ for all i. Complete ζ_1 to a basis $\boldsymbol{\zeta}$ of $\mathbb{R}^n$. For this ζ, Lemma 4.6.8 shows that f^{ζ} does not contain e. But, as noted above, $f^{\zeta} = f$. So $f(e) = 0$. We have shown that $f(e) = 0$ whenever e is not in the convex hull of the w_i, which is the required claim. □

4.7 Proof of the formula for the Tutte polynomial

Let M be a rank d matroid on the ground set $[n]$, and let ρ_M be the rank function of M. The rank generating function of M is

$$r_M(u, v) := \sum_{S \subset [n]} u^{d - \rho_M(S)} v^{|S| - \rho_M(S)}.$$

The Tutte polynomial is defined by $t_M(z, w) = r_M(z-1, w-1)$. See [19] for background on the Tutte polynomial, including several alternate definitions.

We continue to use the notations π_d, $\pi_{1(n-1)}$, α and β from section 4.1, and the notation $K_T^0(\mathtt{pt})$ for $K^0(\mathtt{pt}) = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$.

The aim of this section is to prove:

Theorem 4.7.1. *We have*

$$(\pi_{1(n-1)})_* \pi_d^* \left(y(M) \cdot [\mathcal{O}(1)] \right) = t_M(\alpha, \beta).$$

By Lemma 4.5.1, it is enough to show instead that

$$\int y(M) \cdot [\mathcal{O}(1)] \cdot \sum_{p=0}^{d} \sum_{q=0}^{n-d} [\textstyle\bigwedge^p S] \; [\textstyle\bigwedge^q (Q^\vee)] u^p v^q = r_M(u, v).$$

In fact, we will show something stronger.

Theorem 4.7.2. *In equivariant K-theory, we have*

$$\int^T \sum_{p=0}^{d} \sum_{q=0}^{n-d} y(M)\ [\mathcal{O}(1)]^T\ [\textstyle\bigwedge^p S]^T\ [\bigwedge^q (Q^\vee)]^T\ u^p v^q = \sum_{S \subset [n]} t^{e_S} u^{d-\rho_M(S)} v^{|S|-\rho_M(S)}.$$

That is, the integral (4.7.2) is a generating function in $K_T^0(\mathtt{pt})[u,v]$ recording the subsets of $[n]$ which $r_M(u,v)$ enumerates.

Proof. As defined earlier, let $e_S = \sum_{i \in S} e_i$. We now begin computing the left hand side of (4.7.2), using localization. Let $I \in \binom{[n]}{d}$ and abbreviate $[n] \backslash I$ by J. Because the localization of a vector bundle at x_I is the character of its stalk there, we have

$$\begin{aligned}\left([\mathcal{O}(1)]^T [\textstyle\bigwedge^p S]^T [\bigwedge^q (Q^\vee)]^T\right)(I) &= (t_{i_1} \cdots t_{i_d}) E_p(t_i^{-1})_{i \in I} E_q(t_j)_{j \in J} \\ &= E_{d-p}(t_i)_{i \in I} E_q(t_j)_{j \notin I}\end{aligned}$$

where E_k is the k-th elementary symmetric function. Summing over p and q and expanding, we get

$$\sum_{p=0}^{d} \sum_{q=0}^{n-d} \left([\mathcal{O}(1)]^T [\textstyle\bigwedge^p S]^T [\bigwedge^q (Q^\vee)]^T\right)(I)\ u^p v^q = \sum_{P \subseteq I} \sum_{Q \subseteq J} t^{e_P + e_Q} u^{d-|P|} v^{|Q|}.$$

So we want to compute

$$\sum_{I \in M} \mathrm{Hilb}(\mathrm{Cone}_I(M)) \sum_{P \subseteq I} \sum_{Q \subseteq J} t^{e_P + e_Q} u^{d-|P|} v^{|Q|}. \tag{4.7.1}$$

The reader may want to consult Example 4.7.4 at this time.

Although by its looks this sum is a rational function in the t_i, it is a class in $K_T^0(\mathtt{pt})$ and is therefore a Laurent polynomial. By Corollary 4.6.9, all the exponents appearing with nonzero coefficient in this polynomial must be in the convex hull of the set of all exponents which can be written as $e_P + e_Q$, for P and Q as above. Since P and Q are disjoint, all of these exponents are in the cube $\{0,1\}^n$, so the polynomial (4.7.1) must be supported on monomials of the form t^{e_S}. Fix a subset S of $[n]$. Our goal is now to compute the coefficient of t^{e_S} in (4.7.1).

Choose $\zeta_1 \in \mathbb{R}^n$ such that the components of ζ_1 are linearly independent over $\mathbb{Q}$, the component $(\zeta_1)_i$ is negative for $i \in S$ and $(\zeta_1)_i$ is positive for $i \notin S$. Clearly, on the cube $\{0,1\}^n$, the minimum value of ζ_1 occurs at e_S. Complete ζ_1 to a basis $\boldsymbol{\zeta}$ of $\mathbb{R}^n$. Note that ζ_1 assumes distinct values on the 2^n points of the unit cube. Then (4.7.1) is equal to

$$\sum_{I \in M} \mathrm{Hilb}(\mathbf{1}(\mathrm{Cone}_I(M))^{\boldsymbol{\zeta}}) \sum_{P \subseteq I} \sum_{Q \subseteq J} t^{e_{P \cup Q}} u^{d-|P|} v^{|Q|}. \tag{4.7.2}$$

By Corollary 4.6.4 we can compute the coefficient of t^{e_S} in this polynomial by adding up the coefficients of t^{e_S} in each term.

We therefore consider the coefficient of t^{e_S} in $t^{e_{P\cup Q}} \operatorname{Hilb}(\mathbf{1}(\operatorname{Cone}_I(M)^\varsigma))$. The function $e_{P\cup Q} + \mathbf{1}(\operatorname{Cone}_I(M))^\varsigma$ is supported on a cone whose tip is at $e_{P\cup Q}$, and which is contained in the half space $\{x : \langle \zeta_1, x\rangle \geq \langle \zeta_1, e_{P\cup Q}\rangle\}$. Since $e_{P\cup Q}$ is in the unit cube $\{0,1\}^n$, our choice of ζ_1 implies that $\langle \zeta_1, e_S\rangle \leq \langle \zeta_1, e_{P\cup Q}\rangle$. So $t^{e_{P\cup Q}} \operatorname{Hilb}(\mathbf{1}(\operatorname{Cone}_I(M))^\varsigma)$ contains a t^{e_S} term only if $S = P \cup Q$. Even if $S = P \cup Q$, by Lemma 4.6.8, the coefficient of t^{e_S} is nonzero only if $\operatorname{Cone}_I(M)$ is in the half space where ζ_1 is nonnegative. This occurs if and only if $\zeta_1(e_I) \leq \zeta_1(e_{I'})$ for every $I' \in M$. In short, the coefficient of t^{e_S} receives nonzero contributions from those triples (I, P, Q) such that

1. The function ζ_1, on $\operatorname{Poly}(M)$, is minimized at e_I.
2. $P \subseteq I$ and $Q \subseteq [n] \setminus I$.
3. $S = P \cup Q$.

The contribution from such a triple is $u^{d-|P|}v^{|Q|}$.

Because ζ_1 takes distinct values on $\{0,1\}^n$, there is only one basis of M at which ζ_1 is minimized. Call this basis I_0. Moreover, there is only one way to write S as $P \cup Q$ with $P \subseteq I_0$ and $Q \subseteq [n] \setminus I_0$; we must take $P = S \cap I_0$ and $Q = S \cap ([n] \setminus I_0)$. So the coefficient of t^{e_S} is $u^{d-|S\cap I_0|}v^{|S\cap([n]\setminus I_0)|}$.

From the way we chose ζ_1, we see that I_0 is an element of M with maximal intersection with S. In other words, $|S \cap I_0| = \rho_M(S)$. From the description in the previous paragraph, the coefficient of t^{e_S} is $u^{d-\rho_M(S)}v^{|S|-\rho_M(S)}$. Summing over S, we have equation (4.7.2), and Theorems 4.7.1 and 4.7.2 are proved. □

Question 4.7.3. Is there an equivariant version of Lemma 4.5.1 which provides a generating function in $K_T^0(\mathtt{pt})[u,v]$ for the bases of given activity, parallel to Theorem 4.7.2 for the rank generating function?

Example 4.7.4. We compute the sum in (4.7.1) for the matroid from Example 4.3.5. We can shorten our expressions slightly by defining

$$\begin{aligned} s_I &:= \sum_{P\subseteq I}\sum_{Q\subseteq J} t^{e_P+e_Q}u^{d-|P|}v^{|Q|} = \prod_{i\in I}(u+t_i)\prod_{j\in J}(1+vt_j) \\ h_I &:= \prod_{i\in I}\prod_{j\in J}(1-t_i^{-1}t_j)^{-1}. \end{aligned}$$

where, recall, $J = [n] \setminus I$. Then (4.7.1) is

$$\begin{aligned} s_{13}h_{13}(1-t_2t_3^{-1}) + s_{14}h_{14}(1-t_2t_4^{-1}) + s_{23}h_{23}(1-t_1t_3^{-1}) \\ + s_{24}h_{24}(1-t_2t_4^{-1}) + s_{34}h_{34}(1-t_1t_2t_3^{-1}t_4^{-1}), \end{aligned}$$

which is

$$(t_1t_3 + t_2t_3 + t_1t_4 + t_2t_4 + t_3t_4) + (t_1 + t_2 + t_3 + t_4) \cdot u \\ + (t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4) \cdot v + u^2 + (t_1t_2) \cdot uv + (t_1t_2t_3t_4) \cdot v^2$$

Specializing the t_i to 1 gives the rank-generating function

$$5 + 4u + 4v + u^2 + uv + v^2.$$

Setting $u = z - 1$ and $v = w - 1$ gives the Tutte polynomial

$$w + z + w^2 + wz + z^2.$$

$\diamondsuit$

4.8 Proof of the formula for Speyer's h

In this section, we discuss the relation between localization methods and the matroid invariant h_M discovered by Speyer. Our first aim is to prove Theorem 4.8.1 below, defining a polynomial H_M. We will then discuss the relation of H_M to h_M.

Theorem 4.8.1. *Let M be a rank d matroid on $[n]$ without loops or coloops. Let the maps π_d and $\pi_{1(n-1)}$ and the classes α and β be as in Section 4.1. Then there exists a polynomial $H_M \in \mathbb{Z}[s]$ such that*

$$(\pi_{1(n-1)})_* \pi_d^* y(M) = H_M(\alpha + \beta - \alpha\beta).$$

Because $(\alpha+\beta-\alpha\beta)^n = 0$, there is more than one polynomial which obeys this condition. We make H_M unique by defining it to have degree $< n$.

The heart of our proof is the following lemma:

Lemma 4.8.2. *In the setup of Theorem 4.8.1, $\int^T y(M)[\bigwedge^p S]^T[\bigwedge^q (Q^\vee)]^T \in \mathbb{Z}$ for any p and q, and equals 0 when $p \neq q$.*

Proof of Theorem 4.8.1 from Lemma 4.8.2. Suppose that $(\pi_{1(n-1)})_* \pi_d^* (y(M)) = F(\alpha, \beta)$. To say that F is a polynomial in $\alpha + \beta - \alpha\beta$ is the same as to say that it is a polynomial in $1 - \alpha + \beta - \alpha\beta = (\alpha - 1)(\beta - 1)$. So, by Lemma 4.5.1, it is equivalent to show that $\int y(M) \sum [\bigwedge^p S][\bigwedge^q (Q^\vee)] u^p v^q$ is a polynomial in uv. By Lemma 4.8.2, the coefficient of $u^p v^q$ is zero whenever $p \neq q$, so this sum is a polynomial in uv. $\square$

As in the proof of Theorem 4.7.1, the proof of Lemma 4.8.2 will be by equivariant localization.

Proof. Fix p and q. For any $I \in \binom{[n]}{d}$, we have

$$[\textstyle\bigwedge^p S]^T [\bigwedge^q (Q^\vee)]^T(I) = E_p(t_i^{-1})_{i \in I} E_q(t_j)_{j \in [n] \setminus I}$$

where E_k is the k-th elementary symmetric function. So

$$\int y(M) [\textstyle\bigwedge^p S]^T [\bigwedge^q (Q^\vee)]^T = \sum_{I \in M} \mathrm{Hilb}(\mathrm{Cone}_I(M)) \sum_{P \in \binom{I}{p}} t^{-e_P} \sum_{Q \in \binom{[n] \setminus I}{q}} t^{e_Q}. \tag{4.8.1}$$

The reader may wish to consult example 4.8.3 at this time.

By Corollary 4.6.9, t^a may only appear with nonzero coefficient if a is in the convex hull of $\{e_P - e_Q\}$ where P and Q are as above. In particular, every coordinate of a must be -1, 0 or 1. We will now establish that, in fact, every coordinate must be zero.

Consider any index i in $[n]$. Let $\zeta_1 = e_i$ and complete ζ_1 to a basis $\boldsymbol{\zeta}$ of $\mathbb{R}^n$. We abbreviate the half space $\{x : x_i \geq 0\}$ by H, and $\{x : x_i > 0\}$ by H_+.

The sum in (4.8.1) is equal to

$$\sum_{I \in M} \mathrm{Hilb}(\mathbf{1}(\mathrm{Cone}_I(M))^{\boldsymbol{\zeta}}) \sum_{P \in \binom{I}{p}} \sum_{Q \in \binom{[n] \setminus I}{q}} t^{e_Q - e_P}. \tag{4.8.2}$$

By Corollary 4.6.4, it is legitimate to extract the coefficient of a particular term.

Suppose that $i \notin I$. Then i cannot be in P, so the i-th coordinate in $e_Q - e_P$ is nonnegative. Also, by Lemma 4.6.8, $\mathbf{1}(\mathrm{Cone}_I(M))^{\boldsymbol{\zeta}}$ is supported in H. So such $t^{e_Q - e_P} \mathrm{Hilb}(\mathbf{1}(\mathrm{Cone}_I(M))^{\boldsymbol{\zeta}})$ cannot contribute any monomial of the form t^a with $a_i < 0$.

Now, suppose that $i \in I$. Since i is not a coloop of M, the cone $\mathrm{Cone}_I(M)$ has a ray with negative i-th coordinate. So, by Lemma 4.6.8, $\mathbf{1}(\mathrm{Cone}_I(M))^{\boldsymbol{\zeta}}$ lies in the open halfplane H_+. In particular, if $\mathbf{1}(\mathrm{Cone}_I(M))^{\boldsymbol{\zeta}}(a)$ is nonzero for some lattice point a then $a_i \geq 1$. So, again, $t^{e_Q - e_P} \mathrm{Hilb}(\mathbf{1}(\mathrm{Cone}_I(M))^{\boldsymbol{\zeta}})$ cannot contribute any monomial of the form t^a with $a_i < 0$.

A very similar argument shows that no monomial with any positive exponent can occur in (4.8.2). So the only monomial in (4.8.2) is t^0, i.e. (4.8.2) is in $\mathbb{Z}$. Additionally, (4.8.2) is homogenous of degree $q-p$, which is nonzero if $p \neq q$. So we deduce that in that case (4.8.2) is zero, as desired. □

Example 4.8.3. We compute H_M for the matroid M from Example 4.3.5. For brevity, we write

$$\begin{aligned} s'_I &:= \sum_{P \subseteq I} \sum_{Q \subseteq J} t^{-e_P + e_Q} u^{|P|} v^{|Q|} = \prod_{i \in I} (1 + u t_i^{-1}) \prod_{j \in J} (1 + v t_j) \\ h_I &:= \prod_{i \in I} \prod_{j \in J} (1 - t_i^{-1} t_j)^{-1}. \end{aligned}$$

We must compute

$$s'_{13}h_{13}(1-t_2t_3^{-1}) + s'_{14}h_{14}(1-t_2t_4^{-1}) + s'_{23}h_{23}(1-t_1t_3^{-1}) \\ + s'_{24}h_{24}(1-t_2t_4^{-1}) + s'_{34}h_{34}(1-t_1t_2t_3^{-1}t_4^{-1}). \quad (4.8.3)$$

The reader may enjoy typing (4.8.3) into a computer algebra system and watching it simplify to $1-uv$. So $H_M = 1-(1-\alpha)(1-\beta) = \alpha+\beta-\alpha\beta$ and $h_M(s) = s$. ♢

We now show that the polynomial H_M is equal to the polynomial h_M from Speyer's work [83].

Remark 4.8.4. In [83], two closely related polynomials are introduced, $h_M(s)$ and $g_M(s)$. These obey $g_M(s) = (-1)^c h_M(-s)$, where c is the number of connected components of M. As discussed in [83, Section 3], g_M behaves more nicely in combinatorial formulas; its coefficients are positive and formulas involving g_M have fewer signs. However, h_M is more directly related to algebraic geometry. The fact that h_M arises more directly in this chapter is another indication of this.

We review some relevant notation. Let i be an index between 1 and d. Choose a line ℓ in n-space and an $n-i$ plane M containing ℓ. Let $\Omega_i \subset G(d,n)$ be the Schubert cell of those d-planes L such that $\ell \subset L$ and $L+M$ is contained in a hyperplane. If $i > d$, we define Ω_i to be Ω_d. Then $h_M(s)$ was defined by

$$\frac{h_M(s)}{1-s} = \sum_{i=1}^{\infty} \int_{G(d,n)} y(M)[\mathcal{O}_{\Omega_i}]s^i.$$

In other words, the coefficient of s^i in $h_M(s)$ is

$$\int_{G(d,n)} y(M)\left([\mathcal{O}_{\Omega_i}] - [\mathcal{O}_{\Omega_{i-1}}]\right).$$

Theorem 4.8.5. *With the above definitions, we have $H_M(s) = h_M(s)$.*

Proof. We will show that the coefficient of s^i in both cases is the same. Notice that the coefficient of s^i in $H_M(s)$ will also be the coefficient of β^i in $H_M(\alpha+\beta-\alpha\beta)$. As we computed in the proof of Lemma 4.5.1, the dual basis to $\alpha^i\beta^j$ is $\alpha^{d-1-i}\beta^{n-d-1-j}(1-\alpha)(1-\beta)$. In particular, the coefficient of β^i in $(\pi_{1(n-1)})_*\pi_d^* y(M)$ is $\int \left((\pi_{1(n-1)})_*\pi_d^* y(M)\right)\alpha^{n-1}\beta^{n-1-i}(1-\beta)$.

Now, α^{n-1} intersects the hypersurface $\mathcal{F}\ell(1,n-1;n)$ in the set of pairs $(\texttt{line}, \texttt{hyperplane})$ where $\texttt{line}$ has a given value ℓ. Intersecting further with β^{n-i-1} imposes in addition the condition that $\texttt{hyperplane}$ contain a certain generic $n-i-1$ plane N. But, since the hyperplane is already forced to contain ℓ, it is equivalent to say that the hyperplane contains the $n-i$ plane $N+\ell$. In short, $\alpha^{n-1}\beta^{n-i-1} \cap \mathcal{F}\ell(1,n-1;n)$ is represented by the

Schubert variety of pairs (l, H) where l is a given line ℓ and H contains a given $n-i$ plane M containing ℓ.

Now, the pushforward of the structure sheaf of a variety is always the structure sheaf of a Schubert variety. In the case at hand, $(\pi_{1(n-1)})_*\pi_d^*\alpha^{n-1}\beta^{n-i-1}$ is the class of the Schubert variety of d-planes L such that $\ell \subset L$ and $L + M$ is contained in a hyperplane. This is to say, $(\pi_d)_*\pi_{1(n-1)}^*\alpha^{n-1}\beta^{n-i-1} = [\mathcal{O}_{\Omega_i}]$. Using (4.2.1), we see that the coefficient of s^i in $H_M(s)$ is

$$\int_{\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}} \left((\pi_{1(n-1)})_*\pi_d^* y(M)\right) \alpha^{n-1}\beta^{n-1-i}(1-\beta) = \int_{G(d,n)} y(M)\left([\mathcal{O}_{\Omega_i}] - [\mathcal{O}_{\Omega_{i-1}}]\right),$$

as desired. □

4.9 Geometric interpretations of matroid operations

In [83], a number of facts about the behavior of h_M under standard matroid operations were proved geometrically. In this section we re-establish these using our algebraic tools of localization and Lemma 4.5.1. Following the established pattern, our proofs will be equivariant. We first introduce slightly more general polynomials for which our results hold

Define $F_M^m(u, v)$ to be the unique polynomial, of degree $\leq n$ in u and v, such that

$$F_M^m(\mathcal{O}(1,0), \mathcal{O}(0,1)) = (\pi_{1(n-1)})_*\pi_d^*\left([\mathcal{O}(m)]y(M)\right). \tag{4.9.1}$$

We also define an equivariant generalization of this by

$$F_M^{m,T}(u, v) := \int y(M)[\mathcal{O}(m)]^T \sum_{p,q} [\textstyle\bigwedge^p S]^T [\bigwedge^q (Q^\vee)]^T u^p v^q$$

In the previous sections, we checked that $F_M^{0,T}(u, v) = H_M(1 - uv) = h_M(1 - uv)$, that $F_M^{1,T}(u, v)$ and $F_M^1(u, v)$ are the weighted and unweighted rank generating functions, and that $F_M^1(u - 1, v - 1)$ is the Tutte polynomial. The entire collection of $F_M^{m,T}$ can be seen as a generalization of the Ehrhart polynomial of Poly(M). Specifically, $F_M^m(0, 0) = \#(m \cdot \text{Poly}(M) \cap \mathbb{Z}^n)$ for $m \geq 0$.

Write M^* for the matroid dual to M.

Proposition 4.9.1. *We have* $F_M^m(u, v) = F_{M^*}^m(v, u) \in \mathbb{Z}[u, v]$.

Proof. Equivariantly, we will show that $F_M^{m,T}(t)(u, v) = t^{me_{[n]}} F_{M^*}^{m,T}(t^{-1})(v, u)$. (Here the symbol $F^{m,T}(t^{-1})$ means that we are to take the coefficients of $F^{m,T}$, which are in $\mathbb{Z}[\text{Char}(T)]$, and apply the linear map which inverts each character of T.)

We must show that for any p and q,

$$\left(\int^T y(M)[\mathcal{O}(m)]^T[\textstyle\bigwedge^p S]^T[\bigwedge^q(Q^\vee)]^T\right)(t) \\ = t^{me_{[n]}}\left(\int^T y(M^*)[\mathcal{O}(m)]^T[\textstyle\bigwedge^q S]^T[\bigwedge^p(Q^\vee)]^T\right)(t^{-1}). \quad (4.9.2)$$

By localization, the left side is

$$\sum_{I\in M}\mathrm{Hilb}(\mathrm{Cone}_I(M))(t)\ t^{me_I}\sum_{P\in\binom{I}{p}}\sum_{Q\in\binom{[n]\setminus I}{q}} t^{e_Q-e_P}.$$

The polytope $\mathrm{Poly}(M^*)$ is the image of $\mathrm{Poly}(M)$ under the reflection $x\mapsto e_{[n]}-x$. Therefore $\mathrm{Hilb}(\mathrm{Cone}_I(M))(t)=\mathrm{Hilb}(\mathrm{Cone}_{[n]\setminus I}(M^*))(t^{-1})$. So the left side of (4.9.2), reindexing the sum by $J=[n]\setminus I$, is

$$\sum_{J\in M^*}\mathrm{Hilb}(\mathrm{Cone}_J(M^*))(t^{-1})\ t^{me_{[n]\setminus J}}\sum_{P\in\binom{[n]\setminus J}{p}}\sum_{Q\in\binom{J}{q}} t^{e_Q-e_P} \\ = t^{me_{[n]}}\sum_{J\in M^*}\mathrm{Hilb}(\mathrm{Cone}_J(M^*))(t^{-1})\ t^{-e_J}\sum_{Q\in\binom{J}{q}}\sum_{P\in\binom{[n]\setminus J}{p}} t^{-e_P+e_Q}$$

which is the right side of (4.9.2). □

Given matroids M and M', we denote their direct sum by $M\oplus M'$.

Proposition 4.9.2. *We have $F^m_M F^m_{M'}=F^m_{M\oplus M'}$.*

Proof. Localization gives

$$F^m_M=\sum_{I\in M}\mathrm{Hilb}(\mathrm{Cone}_I(M))\ t^{me_I}\sum_{P\subseteq I}\sum_{Q\subseteq \mathsf{E}\setminus I} t^{e_Q-e_P}u^{|P|}v^{|Q|} \quad (4.9.3)$$

and analogous expansions for M' and $M\oplus M'$. Since $\mathrm{Poly}(M\oplus M')=\mathrm{Poly}(M)\times\mathrm{Poly}(M')$, we have

$$\mathrm{Hilb}(\mathrm{Cone}_I(M))\,\mathrm{Hilb}(\mathrm{Cone}_{I'}(M'))=\mathrm{Hilb}(\mathrm{Cone}_{I\cup I'}(M\oplus M')).$$

The proposition follows immediately by multiplying out expansions like (4.9.3). □

For $k=1,2$, let M_k be a matroid on the ground set E_k and let $i_k\in\mathsf{E}_k$. Consider the larger ground set $\mathsf{E}=\mathsf{E}_1\sqcup\mathsf{E}_2\setminus\{i_1,i_2\}\cup\{i\}$, where i should be regarded as the identification of i_1 and i_2. There are three standard matroid operations one can perform in this setting.

In the next definitions, I_1 and I_2 range over elements of M_1 and M_2 respectively. The series connection M_{ser} of M_1 and M_2 is the matroid

$$\begin{aligned}&\{I_1 \sqcup I_2 : |(I_1 \sqcup I_2) \cap \{i_1, i_2\}| = 0\}\\ \cup\, &\{(I_1 \sqcup I_2) \setminus \{i_1, i_2\} \cup \{i\} : |(I_1 \sqcup I_2) \cap \{i_1, i_2\}| = 1\}\end{aligned}$$

on E; their parallel connection M_{par} is the matroid

$$\begin{aligned}&\{(I_1 \sqcup I_2) \setminus \{i_1, i_2\} : |(I_1 \sqcup I_2) \cap \{i_1, i_2\}| = 1\}\\ \cup\, &\{(I_1 \sqcup I_2) \setminus \{i_1, i_2\} \cup \{i\} : |(I_1 \sqcup I_2) \cap \{i_1, i_2\}| = 2\}\end{aligned}$$

on E; and their two-sum $M_{2\text{sum}}$ is the matroid

$$\{(I_1 \sqcup I_2) \setminus \{i_1, i_2\} : |(I_1 \sqcup I_2) \cap \{i_1, i_2\}| = 1\}$$

on $\mathsf{E} \setminus \{i\}$. For the reader with little intuition for these matroids, they can be realized as vector arrangements (the dual realization to hyperplane arrangements) as follows. For $i = 1, 2$, let $A_i = \{v_1^i, \ldots, v_{n_i}^i\}$ be an arrangement of vectors in the vector space V_i realizing M_i. Our realizations will be variations on the realization $(A_1, 0) \cup (0, A_2) \subseteq V_1 \oplus V_2$ of $M_1 \oplus M_2$: for M_{ser}, replace $(v_{i_1}, 0)$ and $(0, v_{i_2})$ by (v_{i_1}, v_{i_2}); for M_{par}, project to the quotient $V_1 \oplus V_2/(v_{i_1}, -v_{i_2})$ and identify the images of $(v_{i_1}, 0)$ and $(0, v_{i_2})$ as a single point; and for $M_{2\text{sum}}$, take the same projection but remove those two images.

The next property has the nicest form for the particular case of F_M^0, on account of Lemma 4.8.2.

Theorem 4.9.3. *We have*

$$F^m_{M_1 \oplus M_2} = (1+v)F^m_{M_{\text{ser}}} + (1+u)F^m_{M_{\text{par}}} - (1+v)(1+u)F^m_{M_{2\text{sum}}}.$$

In particular, $F^0_{M_{2\text{sum}}} = F^0_{M_{\text{ser}}} = F^0_{M_{\text{par}}} = F^0_{M_1 \oplus M_2}/(1-uv)$.

The series, respectively parallel, extension of a matroid M_1 along i_1 is its series, respectively parallel, connection to the uniform matroid $U_{1,2}$. Two-sum with $U_{1,2}$ leaves M_1 unchanged. If $M_2 = U_{1,2}$ then we have

$$F^0_{M_{\text{ser}}} = \frac{F^0_{M_1 \oplus U_{1,2}}}{1-uv} = \frac{F^0_{M_1} F^0_{U_{1,2}}}{1-uv} = \frac{F^0_{M_1}(1-uv)}{1-uv} = F^0_{M_1}$$

using Proposition 4.9.2, and an analogue holds for the parallel extension. This implies one of the most characteristic combinatorial properties of h from [83].

Corollary 4.9.4. *The values of h_M, H_M and F_M^0 are unchanged by series and parallel extensions.*

Proof of Theorem 4.9.3. Let M_k have rank d_k, $d = d_1+d_2$, and $n = |\mathsf{E}_1|+|\mathsf{E}_2|$. Let $T = (\mathbb{C}^*)^n$ be the torus acting on $G(d,n)$. Our aim is to relate $y(M_1 \oplus M_2) \in K_T^0(G(d,n))$ to $y(M_{\text{ser}})$, $y(M_{\text{par}})$, and $y(M_{2\text{sum}})$. Localization renders the problem one of relating cones at vertices of certain polytopes. Define

$$\begin{aligned} P_{\text{ser}} &= \text{Poly}(M_1 \oplus M_2) \cap \{x_{i_1} + x_{i_2} \le 1\} \\ P_{\text{par}} &= \text{Poly}(M_1 \oplus M_2) \cap \{x_{i_1} + x_{i_2} \ge 1\} \\ P_{2\text{sum}} &= \text{Poly}(M_1 \oplus M_2) \cap \{x_{i_1} + x_{i_2} = 1\} \end{aligned}$$

Then

$$\mathbf{1}(\text{Poly}(M_1 \oplus M_2)) = \mathbf{1}(P_{\text{ser}}) + \mathbf{1}(P_{\text{par}}) - \mathbf{1}(P_{2\text{sum}}).$$

(If P_{ser} and P_{par} have the same dimension as $\text{Poly}(M_1 \oplus M_2)$ they will be the facets of a subdivision, with $P_{2\text{sum}}$ the unique other interior face.) This implies that, for $I \in \binom{n}{d}$,

$$\text{Hilb}(\text{Cone}_{e_I}(M_1 \oplus M_2)) = \text{Hilb}(\text{Cone}_{e_I}(P_{\text{ser}})) + \text{Hilb}(\text{Cone}_{e_I}(P_{\text{par}})) - \text{Hilb}(\text{Cone}_{e_I}(P_{2-\text{sum}})). \tag{4.9.4}$$

We'll use L to denote one of the symbols ser, par, 2sum. Let $p : \mathbb{R}^{\mathsf{E}_1 \sqcup \mathsf{E}_2} \to \mathbb{R}^{\mathsf{E}}$ be the linear projection with $p(e_{i_1}) = p(e_{i_2}) = e_i$ and $p(e_j) = e_j$ for $j \neq i_1, i_2$, and let $\iota : \mathbb{R}^{\mathsf{E}\setminus\{i\}} \to \mathbb{R}^{\mathsf{E}}$ be the inclusion into the i-th coordinate hyperplane. Then

$$\begin{aligned} p(P_{\text{ser}}) &= \text{Poly}(M_{\text{ser}}) \\ p(P_{\text{par}}) &= \text{Poly}(M_{\text{par}}) + e_i \\ p(P_{2\text{sum}}) &= \iota(\text{Poly}(M_{2\text{sum}})) + e_i \end{aligned}$$

where $+e_i$ denotes a translation.

The polytope $\text{Poly}(M_1 \oplus M_2)$ lies in the hyperplane $\{\sum_{j\in\mathsf{E}_1} x_j = d_1\}$, which intersects $\ker p$ transversely, so p is an isomorphism on the polytopes P_L. In particular for any $I \in M_1 \oplus M_2$ we have $\text{Cone}_{p(e_I)}(p(P_L)) = p(\text{Cone}_{e_I}(P_L))$. Also, if u is a lattice point then $p(u)$ is. Define $r : K_T^0(\mathtt{pt}) \to K_{T'}^0(\mathtt{pt})$ to be the restriction from characters of T to characters of its codimension 1 subtorus

$$T' = \{(t_j)_{j\in\mathsf{E}_1\sqcup\mathsf{E}_2} \in T : t_{i_1} = t_{i_2}\},$$

so that $t^{p(e_I)} = r(t^{e_I})$. We write t_i for the common restriction of t_{i_1} and t_{i_2} to T'. We will also occassionally need a notation for the torus T'' which is the projection of T' under forgetting the i-th coordinate.

Let A be the subring of $\text{Frac}\, K_T^0(\mathtt{pt})$ consisting of rational functions whose denominator is not divisible by $t_{i_1} - t_{i_2}$. The map r extends to a map $r : A \to \text{Frac}\, K_{T'}^0(\mathtt{pt})$. Because P_L is in the hyperplane $\sum_{j\in\mathsf{E}_1} x_j = d_1$, the edges of P_L do not point in direction $e_{i_1} - e_{i_2}$, so

$\mathrm{Hilb}(\mathrm{Cone}_{e_I}(P_L))$ is in A and we have

$$\begin{aligned}\mathrm{Hilb}(\mathrm{Cone}_{p(e_I)}(M_L)) &= \mathrm{Hilb}(\mathrm{Cone}_{p(e_I)}(p(P_L)))\\ &= \mathrm{Hilb}(p(\mathrm{Cone}_{e_I}(P_L)))\\ &= r(\mathrm{Hilb}(\mathrm{Cone}_{e_I}(P_L)))\,. \end{aligned} \tag{4.9.5}$$

We now embark on the computation of $F^m_{M_1\oplus M_2}$ by equivariant localization. We have

$$F^{m,T}_{M_1\oplus M_2}(u,v) = \sum_{I\in M_1\oplus M_2} \mathrm{Hilb}(\mathrm{Cone}_I(M_1\oplus M_2))\, t^{me_I} \sum_{P\subseteq I}\sum_{Q\subseteq \mathsf{E}_1\sqcup\mathsf{E}_2\setminus I} t^{e_Q-e_P}u^{|P|}v^{|Q|}.$$

Expanding as dictated by (4.9.4), this is

$$\begin{aligned}F^{m,T}_{M_1\oplus M_2}(u,v) = \sum_{I\in M_1\oplus M_2} \Big(&\mathrm{Hilb}(\mathrm{Cone}_{e_I}(P_{\mathrm{ser}})) + \mathrm{Hilb}(\mathrm{Cone}_{e_I}(P_{\mathrm{par}}))\\ &- \mathrm{Hilb}(\mathrm{Cone}_{e_I}(P_{2\mathrm{sum}}))\Big)\cdot t^{me_I}\sum_{P\subseteq I}\sum_{Q\subseteq \mathsf{E}_1\sqcup\mathsf{E}_2\setminus I} t^{e_Q-e_P}u^{|P|}v^{|Q|}\end{aligned} \tag{4.9.6}$$

We will eventually be applying the map $K^0_T(\mathtt{pt})\to K^0(\mathtt{pt})=\mathbb{Z}$ replacing all characters by 1 to get a nonequivariant result. This map factors through r. As explained above, all of the terms in equation (4.9.6) lie in the ring A, so we may apply r to both sides.

We take the three terms inside the large parentheses in (4.9.6) individually. The three are similar, and we will only work through the first, involving P_{ser}, in detail. Temporarily denote this subsum Σ_{ser}, i.e.

$$\Sigma_{\mathrm{ser}} = \sum_{I\in M_1\oplus M_2} \mathrm{Hilb}(\mathrm{Cone}_{e_I}(P_{\mathrm{ser}}))\, t^{me_I} \sum_{P\subseteq I}\sum_{Q\subseteq \mathsf{E}_1\sqcup\mathsf{E}_2\setminus I} t^{e_Q-e_P}u^{|P|}v^{|Q|}.$$

By (4.9.5) and the definition of r we have

$$r(\Sigma_{\mathrm{ser}}) = \sum_{I\in M_1\oplus M_2} \mathrm{Hilb}(\mathrm{Cone}_{p(e_I)}(M_{\mathrm{ser}}))\, t^{p(me_I)} \sum_{P\subseteq I}\sum_{Q\subseteq \mathsf{E}_1\sqcup\mathsf{E}_2\setminus I} t^{p(e_Q-e_P)}u^{|P|}v^{|Q|}.$$

For any $I\in M_1\oplus M_2$ such that $p(e_I)\in \mathrm{Poly}(M_{\mathrm{ser}})$, not both i_1 and i_2 are in I, so $p(e_I)=e_J$ for some $I'\subseteq\mathsf{E}$, and we have

$$\sum_{P\subseteq I}\sum_{Q\subseteq \mathsf{E}_1\sqcup\mathsf{E}_2\setminus I} t^{p(e_Q-e_P)}u^{|P|}v^{|Q|} = (1+vt_i)\sum_{P\subseteq I'}\sum_{Q\subseteq \mathsf{E}\setminus I'} t^{e_Q-e_P}u^{|P|}v^{|Q|}$$

where the factor $(1+vt_i)$ comes from dropping one of i_1 and i_2 not contained in I from the sum over Q. Therefore

$$\begin{aligned} r(\Sigma_{\text{ser}}) &= (1+vt_i) \sum_{I' \in M_{\text{ser}}} \text{Hilb}(\text{Cone}_{I'}(M_{\text{ser}}))\, t^{me_{I'}} \sum_{P \subseteq I'} \sum_{Q \subseteq \mathsf{E} \setminus I'} t^{e_Q - e_P} u^{|P|} v^{|Q|} \\ &= (1+vt_i) F^m_{M_{\text{ser}}}(u,v). \end{aligned}$$

A similar argument for each of the other two summands in (4.9.6) yields

$$\begin{aligned} &r\left(F^{m,T}_{M_1 \oplus M_2}(u,v)\right) = \\ &\quad (1+vt_i) F^{m,T'}_{M_{\text{ser}}}(u,v) + (1+ut_i^{-1}) F^{m,T'}_{M_{\text{par}}}(u,v) - (1+vt_i)(1+ut_i^{-1}) F^{m,T''}_{M_{2\text{sum}}}(u,v). \quad (4.9.7) \end{aligned}$$

In the last term, we are implictly using the injection $K_0^{T''}(\mathtt{pt}) \hookrightarrow K_0^{T'}(\mathtt{pt})$ coming from the projection $T \to T''$.

On passing to non-equivariant K-theory, this becomes the first assertion of the theorem. For the second, Lemma 4.8.2 says that H_M is a polynomial in uv for any matroid M. Thus, putting $m = 0$ in (4.9.7), the terms on the right containing an unmatched v must cancel, implying $F^0_{M_{\text{ser}}} = F^0_{M_{2\text{sum}}}$. The same goes for the terms containing an unmatched u, implying $F^0_{M_{\text{par}}} = F^0_{M_{2\text{sum}}}$. Making these substitutions and simplifying, (4.9.7) becomes the second assertion of the theorem. $\square$

Chapter 5

Tropical cycles and Chow polytopes

5.1 Introduction

Several well understood classes of tropical variety are known to correspond to certain regular subdivisions of polytopes, in a way that provides a bijection of combinatorial types.

1. *Hypersurfaces* in $\mathbb{P}^{n-1}$ are set-theoretically cut out by principal prime ideals. If the base field has trivial valuation, then $\operatorname{Trop}\mathbf{V}(f)$ is[1] the fan of all cones of positive codimension in the normal fan to its Newton polytope $\operatorname{Newt}(f)$. In the case of general valuation, the valuations of coefficients in f induce a regular subdivision of $\operatorname{Newt}(f)$, and $\operatorname{Trop}\mathbf{V}(f)$ consists of the non-full-dimensional faces in the normal complex (in the sense of Section 5.2.2).

2. *Linear spaces* in $\mathbb{P}^{n-1} = \mathbb{P}(\mathbb{K}^n)$ are cut out by ideals generated by linear forms. To a linear space $\mathbf{X}$ of dimension $n-d-1$ is associated a matroid $M(\mathbf{X})$, whose bases are the sets $I \in \binom{[n]}{d}$ such that the projection of $\mathbf{X}$ to the coordinate subspace $\mathbb{K}\{e_i : i \notin I\}$ has full rank. If the base field has trivial valuation, then $\operatorname{Trop}\mathbf{X}$ is a subfan (the *Bergman fan* [7]) of the normal fan to the matroid polytope

$$\operatorname{Poly}(M(\mathbf{X})) = \operatorname{conv}\{\textstyle\sum_{j\in J} e_j : J \text{ is a basis of } M(\mathbf{X})\} \tag{5.1.1}$$

of $M(\mathbf{X})$. In the case of general valuations, the valuations of the Plücker coordinates induce a regular subdivision of $\operatorname{Poly}(M(\mathbf{X}))$ into matroid polytopes as discussed in Chapter 3, and $\operatorname{Trop}\mathbf{X}$ consists of appropriate faces of the normal complex.

[1]Throughout this chapter we use boldface for classical algebro-geometric objects (except those with standard symbols in blackboard bold or roman, which we preserve), and plain italic for tropical ones.

3. *Zero-dimensional tropical varieties* are simply point configurations. A zero-dimensional tropical variety X is associated to an arrangement $\mathcal{H}$ of *upside-down* tropical hyperplanes with cone points at the points of X: for instance, the tropical convex hull of the points of X is a union of closed regions in the polyhedral complex determined by $\mathcal{H}$. The arrangement $\mathcal{H}$ is dual to a fine mixed subdivision of a simplex, and X consists of the faces dual to little simplices in the normal complex of this subdivision.

The polytopes in this list are special cases of the *Chow polytope*, associated to any cycle $\mathbf{X}$ on $\mathbb{P}^{n-1}$ as the weight polytope of the point representing $\mathbf{X}$ in the *Chow variety*, the parameter space of cycles. This suggests that there should be a general description of tropical varieties as somehow dual to associated Chow polytope subdivisions.

This chapter's main theorem, Theorem 5.4.1, provides a simple tropical formula for this Chow polytope subdivision in terms of the tropical variety $\operatorname{Trop}\mathbf{X}$, making use of *stable* Minkowski sums of tropical cycles introduced in Section 5.3. There is however no general map in the reverse direction, from Chow polytope subdivision to tropical variety, and in Section 5.6 we present an example of two distinct tropical varieties with the same Chow polytope. In Section 5.5 we use this machinery to at last give a proof of the fact that tropical linear spaces are exactly tropical varieties of degree 1.

5.2 Setup

5.2.0 Polyhedral notations and conventions

For Π a polyhedron in a real vector space V and $u : V \to \mathbb{R}$ a linear functional, $\operatorname{face}_u \Pi$ is the face of Π on which u is minimised, if such a face exists. For Π, P polyhedra, $\Pi + \mathrm{P}$ is the Minkowski sum $\{\pi + \rho : \pi \in \Pi, \rho \in \mathrm{P}\}$, and we write $-\mathrm{P} = \{-\rho : \rho \in \mathrm{P}\}$ and $\Pi - \mathrm{P} = \Pi + (-\mathrm{P})$.

5.2.1 Tropical cycles

Let $N_{\mathbb{R}}$ be a real vector space containing a distinguished full-dimensional lattice N, so that $N_{\mathbb{R}} = N \otimes \mathbb{R}$. For a polyhedron $\sigma \subseteq N_{\mathbb{R}}$, let $\operatorname{lin}\sigma$ be the translate of the affine hull of σ to the origin. We say that σ is *rational* if $N_\sigma := N \cap \operatorname{lin}\sigma$ is a lattice of rank $\dim\sigma$.

The fundamental tropical objects we will be concerned with are abstract *tropical cycles* in $N_{\mathbb{R}}$. See [3, Section 5] for a careful exposition of tropical cycles. Loosely, a tropical cycle X of dimension k consists of the data of a rational polyhedral complex Σ pure of dimension k, and for each facet σ of Σ an integer multiplicity m_σ satisfying a *balancing condition* at codimension 1 faces, modulo identifications which ensure that the precise choice of polyhedral complex structure, among those with a given support, is unimportant. A *tropical variety* is an effective tropical cycle, one in which all multiplicities m_σ are nonnegative.

We write Z_k for the additive group of tropical cycles in $N_{\mathbb{R}}$ of dimension k. We also write $Z = \bigoplus_k Z_k$, and use upper indices for codimension, $Z^k = Z_{\dim N_{\mathbb{R}} - k}$. If Σ is a polyhedral complex, then by $Z(\Sigma)$ (and variants with superscript or subscript) we denote the group of tropical cycles X (of appropriate dimension) which can be given some polyhedral complex structure with underlying polyhedral complex Σ. Our notations Z and Z^k are compatible with [3], but we use $Z(\Sigma)$ differently (in [3] it refers merely to cycles contained as sets in Σ, a weaker condition).

If a tropical cycle X can be given a polyhedral complex structure which is a fan over the origin, we call it a *fan cycle*. We prefer this word "fan", as essentially in [36], over "constant-coefficient", for brevity and for not suggesting tropicalisation; and over the "affine" of [3], since *tropical affine space* should refer to a particular partial compactification of $N_{\mathbb{R}}$. We use notations based on the symbol Z^{fan} for groups of tropical fan cycles.

In a few instances it will be technically convenient to work with objects which are like tropical cycles except that the balancing condition is not required. We call these *unbalanced cycles* and use notations based on the symbol Z^{unbal}. That is, Z^{unbal} simply denotes the free Abelian group on the cones of Δ. If $\sigma \subseteq N_{\mathbb{R}}$ is a k-dimensional polyhedron, we write $[\sigma]$ for the unbalanced cycle σ bearing multiplicity 1, and observe the convention $[\emptyset] = 0$. Then every tropical cycle can be written as an integer combination of various $[\sigma]$.

It is a central fact of tropical intersection theory that Z^{fan} is a graded ring, with multiplication given by (*stable*) *tropical intersection*, which we introduce next, and grading given by codimension. The invocation of these notions in the toric context [35, Section 4] prefigured the tropical approach:

Theorem 5.2.1 (Fulton–Sturmfels)**.** *Given a complete fan Σ, $Z^{\mathrm{fan}}(\Sigma)$ is the Chow cohomology ring of the toric variety associated to Σ.*

Given two rational polyhedra σ and τ, we define a multiplicity $\mu_{\sigma,\tau}$ arising from the lattice geometry, namely the index

$$\mu_{\sigma,\tau} = [N_{\sigma+\tau} : N_\sigma + N_\tau].$$

We define two variations where we require, respectively, transverse intersection and linear independence:

$$\mu^{\bullet}_{\sigma,\tau} = \begin{cases} \mu_{\sigma,\tau} & \text{if } \operatorname{codim}(\sigma \cap \tau) = \operatorname{codim}\sigma + \operatorname{codim}\tau \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu^{\boxplus}_{\sigma,\tau} = \begin{cases} \mu_{\sigma,\tau} & \text{if } \dim(\sigma + \tau) = \dim\sigma + \dim\tau \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively, $\mu^{\boxplus}_{\sigma,\tau}$ is the absolute value of the determinant of a block matrix consisting of a block whose rows generate N_σ as a $\mathbb{Z}$-module above a block whose rows generate N_τ, in

coordinates providing a basis for any $(\dim\sigma + \dim\tau)$-dimensional lattice containing $N_{\sigma+\tau}$. Likewise $\mu^\bullet_{\sigma,\tau}$ can be computed from generating sets for the dual lattices.

If σ and τ are polytopes in $N_\mathbb{R}$ which are either disjoint or intersect transversely in the relative interior of each, their stable tropical intersection is

$$[\sigma]\cdot[\tau] = \mu^\bullet_{\sigma,\tau}[\sigma\cap\tau]. \tag{5.2.1}$$

If $X = \sum_\sigma m_\sigma[\sigma]$ and $Y = \sum_\tau n_\tau[\tau]$ are unbalanced cycles such that every pair of facets σ of X and τ of Y satisfy this condition, then their stable tropical intersection is obtained by linear extension,

$$X\cdot Y = \sum_{\sigma,\tau} m_\sigma n_\tau\cdot\mu^\bullet_{\sigma,\tau}[\sigma\cap\tau]. \tag{5.2.2}$$

If X and Y are tropical cycles, so is $X\cdot Y$ (see [3]). For a point $v\in N_\mathbb{R}$, let $[v]\boxplus Y$ denote the translation of Y by v; this is a special case of a notation we introduce in Section 5.3. If X and Y are rational tropical cycles with no restrictions, then for generic small displacements $v\in N_\mathbb{R}$ the faces of X and $[v]\boxplus Y$ intersect suitably for equation (5.2.2) to be applied. In fact the facets of the intersection $X\cdot([v]\boxplus Y)$ vary continuously with v, in a way that can be continuously extended to all v. This is essentially the *fan displacement rule* of [35], which ensures that $X\cdot Y$ is always well-defined.

Definition 5.2.2. Given two tropical cycles X, Y, their (*stable*) *tropical intersection* is

$$X\cdot Y = \lim_{v\to 0} X\cdot([v]\boxplus Y).$$

We introduce a few more operations on cycles. Firstly, there is a cross product defined in the expected fashion. Temporarily write $Z(V)$ for the ring of tropical cycles defined in the vector space V. Let $(N_i)_\mathbb{R}$, $i=1,2$, be two real vector spaces. Then there is a well-defined bilinear cross product map

$$\times : Z^{\mathrm{unbal}}((N_1)_\mathbb{R})\otimes Z^{\mathrm{unbal}}((N_2)_\mathbb{R}) \to Z^{\mathrm{unbal}}((N_1\oplus N_2)_\mathbb{R})$$

linearly extending $[\sigma]\times[\tau] = [\sigma\times\tau]$, and the exterior product of tropical cycles is a tropical cycle.

Let $h : N\to N'$ be a linear map of lattices, inducing a map of real vector spaces which we will also denote $h : N_\mathbb{R}\to N'_\mathbb{R}$ (an elementary case of a tropical morphism). Cycles can be pushed forward and pulled back along h. These are special cases of notions defined in tropical intersection theory even in ambient tropical varieties other than $\mathbb{R}^n$ (in the general case, one can push forward general cycles but only pull back complete intersections of Cartier divisors [3]).

Given a cycle $Y = \sum_\sigma m_\sigma[\sigma]$ on $N'_\mathbb{R}$, its pullback is defined in [2] as follows. This is shown in [35, Proposition 2.7] to agree with the pullback on Chow rings of toric varieties.

$$h^*(Y) = \sum_{\sigma\ :\ \sigma \text{ meets } \mathrm{im}\, h \text{ transversely}} m_\sigma[N_{h^{-1}(\sigma)} : h^{-1}(N'_\sigma)][h^{-1}(\sigma)]$$

The pushforward is defined in [36] in the tropical context, and is shown to coincide with the cohomological pushforward in [49, Lemma 4.1]. If $X = \sum_\sigma m_\sigma[\sigma]$ is a cycle on $N_\mathbb{R}$, its pushforward is

$$h_*(X) = \sum_{\sigma \,:\, h|_\sigma \text{ injective}} m_\sigma[N'_{h(\sigma)} : h(N_\sigma)][h(\sigma)].$$

In these two displays, the conditions on σ in the sum are equivalent to $h^{-1}(\sigma)$ or $h(\sigma)$, respectively, having the expected dimension. Pushforwards and pullbacks of tropical cycles are tropical cycles.

5.2.2 Normal complexes

Write $M = N^\vee$, $M_\mathbb{R} = N_\mathbb{R}{}^\vee$ for the dual lattice and real vector space. Let $\pi : M_\mathbb{R} \times \mathbb{R} \to M_\mathbb{R}$ be the projections to the first factor. A polytope $\Pi \subseteq \mathbb{M}_\mathbb{R} \times \mathbb{R}$ induces a regular subdivision Σ of $\pi(\Pi)$. Our convention will be that regular subdivisions are determined by lower faces: so the faces of Σ are the projections $\pi(\text{face}_{(u,1)}\,\Pi)$. We will also write $\text{face}_u\,\Sigma$ to refer to this last face. In general, we will not consider regular subdivisions Σ by themselves but will also want to retain the data of Π. More precisely, what is necessary is to have a well-defined normal complex; for this we need only Σ together with the data of the heights of the vertices of Π visible from underneath, equivalently the lower faces of Π. (When we refer to "vertex heights" we shall always mean only the lower vertices.)

Definition 5.2.3. The (*inner*) *normal complex* $\mathcal{N}(\Sigma, \Pi)$ to the regular subdivision Σ induced by Π is the polyhedral subdivision of $N_\mathbb{R}$ with a face

$$\text{normal}(F) = \{u \in N_\mathbb{R} : W \subseteq \text{face}_{(u,1)}(\Pi)\}$$

for each face $F = \text{conv}(\pi(W))$ of Σ.

We will allow ourselves to write $\mathcal{N}^1(\Sigma)$ for $\mathcal{N}^1(\Sigma, \Pi)$ when Π is clear from context. If Π is contained in $M_\mathbb{R} \times \{0\}$, which we identify with $M_\mathbb{R}$, then $\mathcal{N}(\Sigma, \Pi)$ is the *normal fan* of Π.

We give multiplicities to the faces of the skeleton $\mathcal{N}^e(\Sigma, \Pi)$ of $\mathcal{N}(\Sigma, \Pi)$ so as to make it a cycle, which we also denote $\mathcal{N}^e(\Sigma, \Pi)$. To each face $\text{normal}(F) \in \mathcal{N}(\Sigma, \Pi)$ of codimension e, we associate the multiplicity $m_{\text{normal}(F)} = \text{vol}\, F$ where vol is the normalised lattice volume, i.e. the Euclidean volume on $\text{lin}\, F$ rescaled so that any simplex whose edges incident to one vertex form a basis for N_F has volume 1. In fact $\mathcal{N}^e(\Sigma, \Pi)$ is a tropical cycle. In codimension 1 a converse holds as well.

Theorem 5.2.4.

(a) For any rational regular subdivision Σ in $M_\mathbb{R}$ induced by a polytope Π in $M_\mathbb{R} \times \mathbb{R}$, the skeleton $\mathcal{N}^e(\Sigma, \Pi)$ is a tropical variety.

(b) For any tropical variety $X \in Z^1(N_\mathbb{R})$, there exists a rational polytope Π in $M_\mathbb{R} \times \mathbb{R}$ and induced regular subdivision Σ, unique up to translation and adding a constant to the vertex heights, such that $X = \mathcal{N}^1(\Sigma, \Pi)$.

Part (a) in the case of fans, i.e. $\Pi \subseteq M_\mathbb{R} \times \{0\}$, is a foundational result in the polyhedral algebra [64, Section 11]. The statement for general tropical varieties follows since the normal complex of Σ is just the slice through the normal fan of Π at height 1, and this slicing preserves the balancing condition. Part (b) is also standard, and is a consequence of ray-shooting algorithms, the codimension 1 case of Theorem 5.2.11.

5.2.3 Chow polytopes

See [47], [38, ch. 4] and [28] for fuller treatments of Chow polytopes and Chow forms, the first for the toric background, the second in the context of elimination theory, and the last especially from a computational standpoint.

Let $\mathbb{K}$ be an algebraically closed field. Let $(\mathbb{K}^*)^n$ be an algebraic torus acting via a linear representation on a vector space V, or equivalently on its projectivisation $\mathbb{P}(V)$. Suppose that the action of $(\mathbb{K}^*)^n$ is *diagonalisable*, i.e. V can be decomposed as a direct sum $V = \bigoplus V_i$ where $(\mathbb{K}^*)^n$ acts on each V_i by a character or *weight* $\chi_i : (\mathbb{K}^*)^n \to \mathbb{K}^*$. A character χ_i corresponds to a point w_i in the character lattice of $(\mathbb{K}^*)^n$, via $\chi_i(t) = t^{w_i}$. We shall always assume V is finite-dimensional, except in a few instances where we explicitly waive this assumption for technical convenience. If V is finite-dimensional, the action of $(\mathbb{K}^*)^n$ is necessarily diagonalisable.

Definition 5.2.5. Given a point $v \in V$ of the form $v = \sum_{k \in K} v_{i_k}$ with each $v_{i_k} \in V_{i_k}$ nonzero, the *weight polytope* of v is $\operatorname{conv}\{w_k : k \in K\}$.

If $\mathbf{X} \subseteq \mathbb{P}(V)$ is a $(\mathbb{K}^*)^n$-equivariant subvariety, this defines the weight polytope of a point $x \in \mathbf{X}$.

The *Chow variety* $\operatorname{Gr}(d, n, r)$ of $\mathbb{P}^{n-1}$, introduced by Chow and van der Waerden in 1937 [20], is the parameter space of effective cycles of dimension $d-1$ and degree r in $\mathbb{P}^{n-1}$. When we invoke homogeneous coordinates on $\mathbb{P}^{n-1}$ we will name them $x_1, \ldots, x_n$.

Example 5.2.6.

1. The variety $\operatorname{Gr}(n-1, n, r)$ parametrising degree r cycles of codimension 1 is $\mathbb{P}(\mathbb{K}[x_1, \ldots, x_n]_r) \cong \mathbb{P}^{\binom{r+n-1}{r}-1}$. An irreducible cycle is represented by its defining polynomial.

2. The variety $\operatorname{Gr}(d, n, 1)$ parametrises degree 1 effective cycles, which must be irreducible and are therefore linear spaces. So $\operatorname{Gr}(d, n, 1)$ is simply the Grassmannian $\operatorname{Gr}(d, n)$, motivating the notation. ♢

The Chow variety $\mathrm{Gr}(d,n,r)$ is projective. Indeed, we can present the coordinate ring of $\mathrm{Gr}(n-d,n)$ in terms of (primal) *Plücker coordinates*, which we write as brackets:

$$\mathbb{K}[\mathrm{Gr}(n-d,n)] = \mathbb{K}\Big[\,[J] : J \in \binom{[n]}{n-d}\,\Big]/(\text{Plücker relations}).$$

For our purposes the precise form of the Plücker relations will be unimportant. Then $\mathrm{Gr}(d,n,r)$ has a classical embedding into the space $\mathbb{P}(\mathbb{K}[\mathrm{Gr}(n-d,n)]_r)$ of homogeneous degree r polynomials on $\mathrm{Gr}(n-d,n)$ up to scalars, given by the *Chow form* [20]. For $\mathbf{X}$ irreducible, the Chow form $R_{\mathbf{X}}$ of $\mathbf{X}$ can be constructed as the defining polynomial of the locus of linear subspaces of $\mathbb{P}^{n-1}$ of dimension $n-d-1$ which intersect $\mathbf{X}$. (There is a single defining polynomial since $\mathrm{Pic}(\mathrm{Gr}(n-d,n)) = \mathbb{Z}$.)

The natural componentwise action $(\mathbb{K}^*)^n \curvearrowright \mathbb{K}^n$ induces an action $(\mathbb{K}^*)^n \curvearrowright S^*(\bigwedge^{n-d}\mathbb{K}^n)$. The ring $\mathbb{K}[\mathrm{Gr}(n-d,n)]$ is a quotient of this symmetric algebra by the ideal of Plücker relations. This ideal is homogeneous in the weight grading, so the quotient inherits an $(\mathbb{K}^*)^n$-action. The Chow variety is an $(\mathbb{K}^*)^n$-equivariant subvariety of $\mathbb{P}(\mathbb{K}[\mathrm{Gr}(n-d,n)]_r)$, so we also get an action $(\mathbb{K}^*)^n \curvearrowright \mathrm{Gr}(n-d,n)$. The weight spaces of $\mathbb{K}[\mathrm{Gr}(n-d,n)]$ under the $(\mathbb{K}^*)^n$-action are spanned by monomials in the brackets $[J]$. The weight of a bracket monomial $\prod_i [J_i]^{m_i}$ is $\sum_i m_i e_{J_i}$.

Definition 5.2.7. If $\mathbf{X}$ is a cycle on $\mathbb{P}^{n-1}$ represented by the point x of $\mathrm{Gr}(d,n,r)$, the *Chow polytope* $\mathrm{Chow}(\mathbf{X})$ of $\mathbf{X}$ is the weight polytope of x.

Example 5.2.8.

1. The Chow form of a hypersurface $\mathbf{V}(f)$ is simply its defining polynomial f with the variables x_k replaced by brackets $[k]$, so that the Chow polytope $\mathrm{Chow}(\mathbf{V}(f))$ is the Newton polytope $\mathrm{Newt}(f)$.

2. The Chow form of a $(d-1)$-dimensional linear space $\mathbf{X}$ is a linear form in the brackets, $\sum_J p_J[J]$, where the p_J are the *dual* Plücker coordinates of $\mathbf{X}$ for $J \in \binom{[n]}{n-d}$. Accordingly $\mathrm{Chow}(\mathbf{X})$ is the polytope $\mathrm{Poly}(M(\mathbf{X})^*)$ of the dual matroid. Note that this is simply the image of $\mathrm{Poly}(M(\mathbf{X}))$ under a reflection.

3. For $\mathbf{X} = \mathbf{X}_A$ an embedded *toric variety* in $\mathbb{P}^{n-1}$, the Chow polytope $\mathrm{Chow}(\mathbf{X})$ is the *secondary polytope* associated to the vector configuration A [38, Chapter 8.3]. $\diamondsuit$

From a tropical perspective, the preceding setup has all pertained to the constant-coefficient case. Suppose now that the field $\mathbb{K}$ has a nontrivial valuation $\nu : \mathbb{K}^* \to \mathbb{Q}$, with residue field $\Bbbk \hookrightarrow \mathbb{K}$. For instance we might take $\mathbb{K} = \Bbbk\{\!\{t\}\!\}$ the field of Puiseux series over an algebraically closed field $\Bbbk$, with the valuation $\nu : \mathbb{K}^* \to \mathbb{Q}$ by least degree of t. Let $\mathbf{X}$ be a cycle on $\mathbb{P}^{n-1}$ with Chow form $R_{\mathbf{X}} \in \mathbb{K}[\mathrm{Gr}(n-d,n)]$. Let $\tau_1, \ldots, \tau_m \in \mathbb{K}$ be the coefficients

of $R[X]$, so that $R[X]$ is defined over the subfield $\Bbbk[\tau_1^{\pm 1}, \dots, \tau_n^{\pm 1}] \subseteq \mathbb{K}$. The restriction of ν to this subfield is a discrete valuation, so we may assume that all the $\nu(\tau_i)$ are integers.

The torus $(\Bbbk^*)^n$ acts on $\Bbbk[\mathrm{Gr}(n-d,n)]$ just as before, and therefore acts on $\Bbbk[\mathrm{Gr}(n-d,n)][\tau_1^{\pm 1}, \dots, \tau_n^{\pm 1}]$. Let $(\Bbbk^*)^n \times \Bbbk^* \curvearrowright \Bbbk[\tau^{\pm 1}][\mathrm{Gr}(n-d,n)]$ where the right factor acts on Laurent monomials in $\tau_1, \dots, \tau_n$, with τ^e having weight $\sum_{i=1}^m \nu(\tau_i) e_i$. Let Π be the weight polytope of the Chow form $R_{\mathbf{X}}$ with respect to this action.

Definition 5.2.9. The *Chow subdivision* of a cycle $\mathbf{X}$ on $\mathbb{P}^{n-1}$ over $(\mathbb{K}, \nu)$ is the regular subdivision $\mathrm{Chow}_\nu(\mathbf{X})$ induced by Π.

The Chow subdivision is the non-constant-coefficient analogue of the Chow polytope, generalising the polytope subdivision of the opening examples. It appears as the *secondary subdivision* in Definition 5.5 of [48], but nothing is done with the definition in that work, and we believe this chapter is the first study to investigate it in any detail. Observe that $\mathrm{Chow}_\nu(\mathbf{X})$ is a subdivision of $\mathrm{Chow}(\mathbf{X})$, and if ν is the trivial valuation, $\mathrm{Chow}_\nu(\mathbf{X})$ is $\mathrm{Chow}(\mathbf{X})$ unsubdivided. By $\mathcal{N}(\mathrm{Chow}_\nu(\mathbf{X}))$ we will always mean $\mathcal{N}(\mathrm{Chow}_\nu(\mathbf{X}), \Pi)$.

If $(u, v) : \Bbbk^* \to (\Bbbk^*)^n \times \Bbbk^*$ is a one-parameter subgroup which as an element of $N \times \mathbb{Z}$ has negative last coordinate, then $\mathrm{face}_u \mathrm{Chow}_\nu(\mathbf{X}) = \mathrm{face}_{(u,v)} \Pi$ is bounded. We observe that a bounded face $F = \mathrm{face}_u \mathrm{Chow}_\nu(\mathbf{X})$ of $\mathrm{Chow}_\nu(\mathbf{X})$ is the weight polytope of the toric degeneration $\lim_{t \to 0} u(t) \cdot \mathbf{X}$. This follows from an unbounded generalisation of Proposition 1.3 of [47], which describes the toric degenerations of a point in terms of the faces of its weight polytope.

Example 5.2.10. Perhaps the simplest varieties not among our opening examples are conic curves in $\mathbb{P}^3$. Let $\mathbb{K} = \mathbb{C}\{\!\{t\}\!\}$, and let $\mathbf{X} \subseteq \mathbb{P}^3$ be the conic defined by the ideal

$$(tx - y + z - t^3 w, yz + tz^2 + t^2 yw - zw + (t^3 - t^7)w^2)$$

where $(x : y : z : w)$ are coordinates on $\mathbb{P}^3$. The Chow form of $\mathbf{X}$ can be computed by the algorithm of [28, Section 3.1]. It is

$$\begin{aligned}(2t^7 + t^6 + t^5 - t^3)[zw][yw] + (t^7 + t^5 - t^3)[yw]^2 + (2t^4 + t^3 + t^2 - 1)[zw][yz] + (-t^3 + t^2 - 1)[yw][yz] \\ + (-t - 1)[yz]^2 + (2t^8 - t^4)[zw][xw] + (2t^8 + t^6 - 2t^4)[yw][xw] + (t^9 - t^5)[xw]^2 + (2t^5 - t)[zw][xz] \\ + (-t^4 + t^3 - 2t)[yw][xz] + (-2t^2 - t)[yz][xz] + (-t^2)[xw][xz] + (-t^3)[xz]^2 + (-t^4 - 2t^3 + t)[zw][xy] \\ + (-t^3)[yw][xy] + t[yz][xy] + (-t^4)[xw][xy] + t^2[xz][xy].\end{aligned}$$

The Chow subdivision $\mathrm{Chow}_\nu(\mathbf{X})$ is the regular subdivision induced by the valuations of these coefficients. It is a 3-polytope subdivided into 5 pieces, depicted in Figure 5.1. The polytope $\mathrm{Chow}(\mathbf{X})$ of which it is a subdivision is an octahedron with two opposite corners truncated (it is not the whole octahedron, which is the generic Chow polytope for conics in $\mathbb{P}^3$). ♢

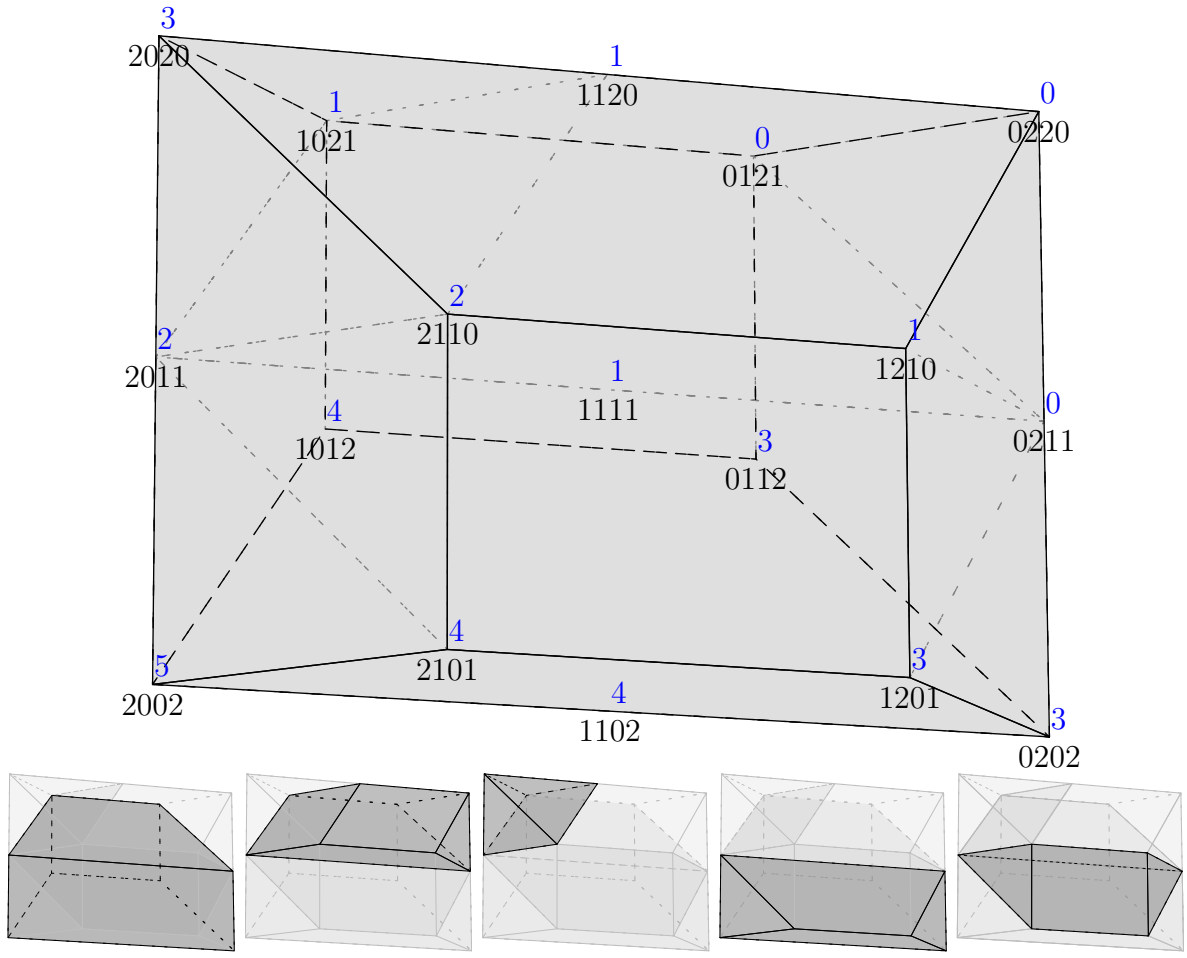


Figure 5.1: The Chow subdivision of Example 5.2.10. Top: coordinates of points (black) and lifting heights (blue). Bottom: the pieces.

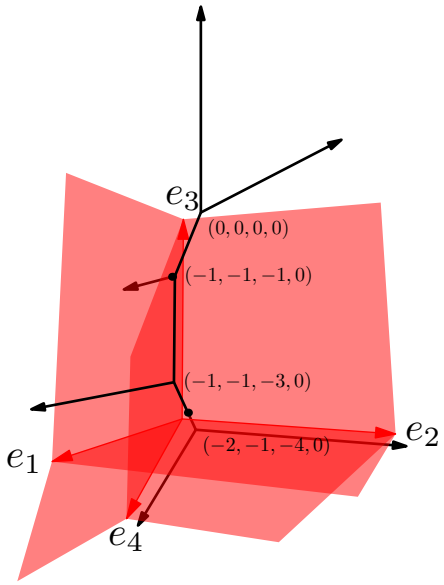


Figure 5.2: Identifying a vertex of a Chow subdivision by Theorem 5.2.11. Coordinates of vertices of the curve are given in $\mathbb{R}^4/(1,1,1,1)$.

Theorem 2.2 of [31] provides a procedure that determines the polytope $\mathrm{Chow}(\mathbf{X})$ given a fan tropical variety $X = \mathrm{Trop}\,\mathbf{X}$. That procedure is the constant-coefficient case of the next theorem, Theorem 5.2.11, which can be interpred as justifying our definition of the Chow subdivision. Theorem 5.2.11 determines $\mathrm{Chow}_\nu(\mathbf{X})$ for $X = \mathrm{Trop}\,\mathbf{X}$ not necessarily a fan, by identifying the regions of the complement of $\mathcal{N}^1(\mathrm{Chow}_\nu(\mathbf{X}))$ and the vertex of $\mathrm{Chow}_\nu(\mathbf{X})$ each of these regions is dual to.

Theorem 5.2.11. *Let $u \in N_\mathbb{R}$ be a linear functional such that $\mathrm{face}_u \mathrm{Chow}_\nu(\mathbf{X})$ is a vertex of $\mathrm{Chow}_\nu(\mathbf{X})$. Then*

$$\mathrm{in}_u R_\mathbf{X} = \prod_{J \in \binom{[n]}{n-d}} [J]^{\deg([u+\mathbb{R}_{\geq 0}\{e_j : j \in J\}]\cdot X)},$$

i.e.

$$\mathrm{vertex}_u \mathrm{Chow}_\nu(\mathbf{X}) = \sum_{J \in \binom{[n]}{n-d}} \deg([u + \mathbb{R}_{\geq 0}\{e_j : j \in J\}] \cdot X) e_J. \tag{5.2.3}$$

The condition that $\mathrm{face}_u \mathrm{Chow}_\nu(\mathbf{X})$ be a vertex is the genericity condition necessary for the set-theoretic intersection $(u + \mathbb{R}_{\geq 0}\{e_j : j \in J\}) \cap X$ to be a finite set of points.

The constant-coefficient case of Theorem 5.2.11 is known as *ray-shooting*, and the general case as *orthant-shooting*, since the positions of the vertices of $\mathrm{Chow}_\nu(\mathbf{X})$ are read off from intersection numbers of X and orthants C_J shot from the point u.

Example 5.2.12. Let $\mathbf{X}$ be the conic curve of Example 5.2.10. The black curve in Figure 5.2 is $X = \operatorname{Trop}\mathbf{X}$. Arbitrarily choosing the cone point of the red tropical plane to be $u \in N_{\mathbb{R}}$, we see that there are two intersection points among the various $[u + C_J] \cdot X$, the two points marked as black dots. Each has multiplicity 1, and they occur one each for $J = \{1, 3\}$ and $J = \{2, 3\}$. Accordingly $e_{\{1,3\}} + e_{\{2,3\}} = (1, 1, 2, 0)$ is the corresponding vertex of $\operatorname{Chow}_\nu(X)$ (compare Figure 5.1). ♢

Theorem 5.2.11 is proved in the literature, in a few pieces. The second assertion, orthant-shooting in the narrow sense, for arbitrary valued fields is Theorem 10.1 of [48]. The first assertion, describing initial forms in the Chow form, is essentially Theorem 2.6 of [47]. This is stated in the trivial valuation case but of course extends to arbitrary valuations with our machinery of regular subdivisions in one dimension higher. The connection of that result with orthant shooting is as outlined in Section 5.4 of [87].

5.3 Minkowski sums of cycles

Let N be any lattice. For a tropical cycle $X = \sum m_\sigma[\sigma]$, we let $X^{\text{refl}} = \sum m_\sigma[-\sigma]$ denote its reflection about the origin. (This is the pushforward or pullback of X along the linear isomorphism $x \mapsto -x$.)

Given two polyhedra $\sigma, \tau \subseteq N_{\mathbb{R}}$, define the (*stable*) *Minkowski sum*

$$[\sigma] \boxplus [\tau] = \mu^{\boxplus}_{\sigma,\tau}[\sigma + \tau]. \tag{5.3.1}$$

Compare (5.2.1). If X and Y are cycles in $N_{\mathbb{R}}$, then we can write their intersection and Minkowski sum in terms of their exterior product $X \times Y \in N_{\mathbb{R}} \times N_{\mathbb{R}}$. We have an exact sequence

$$0 \to N_{\mathbb{R}} \xrightarrow{\iota} N_{\mathbb{R}} \times N_{\mathbb{R}} \xrightarrow{\phi} N_{\mathbb{R}} \to 0$$

of vector spaces where ι is the inclusion along the diagonal and ϕ is subtraction, $(x, y) \mapsto x - y$. It is then routine to check from the definitions that

$$\begin{aligned} X \cdot Y &= \iota^*(X \times Y) \\ X \boxplus Y^{\text{refl}} &= \phi_*(X \times Y) \end{aligned} \tag{5.3.2}$$

Since pullback is well-defined and takes tropical cycles to tropical cycles, it follows immediately that there is a well-defined bilinear map $\boxplus : Z^{\text{unbal}} \otimes Z^{\text{unbal}} \to Z^{\text{unbal}}$ extending (5.3.1), restricting to a bilinear map $\boxplus : Z \otimes Z \to Z$.

A notion of Minkowski sum for tropical varieties arose in [27] as the tropicalisation of the *Hadamard product* for classical varieties. The Minkowski sum of two tropical varieties in that paper's sense can have dimension less than the expected dimension. By contrast our bilinear operation $\boxplus$ should be regarded as a *stable* Minkowski sum for tropical cycles. It is

additive in dimension, i.e. $Z_d \boxplus Z_{d'} \subseteq Z_{d+d'}$, just as stable tropical intersection is additive in codimension. The next lemma further relates intersection and Minkowski sum.

The balancing condition implies that for any tropical cycle X in $N_{\mathbb{R}}$ of dimension $\dim N_{\mathbb{R}}$, $X(u)$ is constant for any $u \in N_{\mathbb{R}}$ for which it's defined. We shall denote this constant $\deg X$. Similarly, if $\dim X = 0$, then X is a finite sum of points with multiplicities, and we will let $\deg X$ be the sum of these multiplicities. These are both special cases of Definition 5.3.2, to come.

Lemma 5.3.1. *Let X and Y be tropical cycles on $N_{\mathbb{R}}$, of complementary dimensions. Then*

$$\deg(X \cdot Y) = \deg(X \boxplus Y^{\mathrm{refl}}).$$

Proof. Let $u \in N_{\mathbb{R}}$ be generic. Let $\Sigma(X)$ and $\Sigma(Y)$ be polyhedral complex structures on X and Y. The multiplicity of $X \boxplus Y^{\mathrm{refl}}$ at a point $u \in N_{\mathbb{R}}$ is

$$(X \boxplus Y^{\mathrm{refl}})(u) = \sum_{\sigma,\tau} \mu^{\boxplus}_{\sigma,\tau}, \tag{5.3.3}$$

summing over only those $\sigma \in \Sigma(X)$ and $\tau \in \Sigma(Y)^{\mathrm{refl}}$ with $u \in \sigma + \tau$, i.e. with $(\{u\} - \tau) \cap \sigma$ nonempty. These $\{u\} - \tau$ are the cones of $\Sigma(Y')$, where $Y' = [u] \boxplus Y$. Then by (5.2.2), $\deg(X \cdot Y')$ is given by the very same expression (5.3.3) except with $\mu^{\bullet}$ in place of $\mu^{\boxplus}$; and by the fan displacement rule preceding Definition 5.2.2, $\deg(X \cdot Y) = \deg(X \cdot Y')$. But $\mu^{\bullet}_{\sigma,\tau} = \mu^{\boxplus}_{\sigma,\tau}$ when σ and τ are of complementary dimensions. □

Let $\mathcal{L}$ be the fan of the ambient toric variety $\mathbb{P}^n$, which is the normal fan in N to the standard simplex $\mathrm{conv}\{e_i\}$. The ray generators of $\mathcal{L}$ are $e_i \in N$, and every proper subset of the rays span a face, which is simplicial. For $J \subsetneq [n]$ let $C_J = \mathbb{R}_{\geq 0}\{e_j : j \in J\}$ be the face of $\mathcal{L}$ indexed by J. Let $\mathcal{L}_k$ be the dimension k skeleton of $\mathcal{L}$ with multiplicities 1, that is, the canonical k-dimensional tropical fan linear space.

Definition 5.3.2 ([3, Definition 9.13])**.** The *degree* of a tropical cycle $X \in Z^e(N_{\mathbb{R}})$ is $\deg X := \deg(X \cdot \mathcal{L}_e)$.

The symbol deg appearing on the right side is the special case defined just above for cycles of dimension 0. It is a consequence of the fan displacement rule that $\deg X = \deg(X \cdot ([v] \boxplus \mathcal{L}_e))$ for any $v \in N_{\mathbb{R}}$.

Lemma 5.3.3. *Let $X \in Z^e$. Then*

$$\deg(X \boxplus {\mathcal{L}_{e-1}}^{\mathrm{refl}}) = e \deg X.$$

Proof. By Lemma 5.3.1 we have

$$\begin{aligned}
\deg(X \boxplus {\mathcal{L}_{e-1}}^{\mathrm{refl}}) &= \deg((X \boxplus {\mathcal{L}_{e-1}}^{\mathrm{refl}}) \cdot \mathcal{L}_1) \\
&= \deg(X \boxplus {\mathcal{L}_{e-1}}^{\mathrm{refl}} \boxplus {\mathcal{L}_1}^{\mathrm{refl}}) \\
&= \deg(({\mathcal{L}_{e-1}}^{\mathrm{refl}} \boxplus {\mathcal{L}_1}^{\mathrm{refl}}) \cdot X^{\mathrm{refl}}) \\
&= \deg((\mathcal{L}_{e-1} \boxplus \mathcal{L}_1) \cdot X) \\
&= \deg((e\mathcal{L}_e) \cdot X) \\
&= e \deg(X \cdot \mathcal{L}_e) \\
&= e \deg X.
\end{aligned}$$

□

Remark 5.3.4. The classical projection formula of intersection theory is valid tropically [3, Proposition 7.7], and has an analogue for $\boxplus$. For a linear map of lattices $h : N \to N'$ and cycles $X \in Z(N_{\mathbb{R}})$ and $Y \in Z(N'_{\mathbb{R}})$, we have

$$\begin{aligned}
h_*(X \cdot h^*(Y)) &= h_*(X) \cdot Y, \\
X \boxplus h^*(Y) &= h^*(h_*(X) \boxplus Y).
\end{aligned}$$

The facts in this section, as well as the duality given by polarisation in the algebra of cones which exchanges intersection and Minkowski sum, are all suggestive of the existence of a duality between tropical stable intersection and stable Minkowski sum. However, we have not uncovered a better statement of such a duality than equations (5.3.2).

5.4 From tropical variety to Chow polytope

Henceforth $d \leq n$ will be a fixed integer, and $\mathbf{X}$ will be a $(d-1)$-dimensional subvariety of $\mathbb{P}^{n-1}$. We will tropicalise $\mathbf{X}$ with respect to the torus $(\mathbb{K}^*)^n/\mathbb{K}^* \subseteq \mathbb{P}^n$, where $\mathbb{K}$ embeds diagonally, so that $X := \operatorname{Trop} \mathbf{X}$ is a tropical fan in $N_{\mathbb{R}} = \mathbb{R}^n/(1,\ldots,1)$. Let $N = \mathbb{Z}^n/(1,\ldots,1)$ be the lattice of integer points within $N_{\mathbb{R}}$, and let $M_{\mathbb{R}} = (1,\ldots,1)^{\perp} = (N_{\mathbb{R}})^{\vee}$.

As explained in [47], the torus $(\mathbb{K}^*)^n$ acts on the Hilbert scheme $\operatorname{Hilb}(\mathbb{P}^{n-1})$ in the fashion induced from its action on $\mathbb{P}^{n-1}$, and the map $\operatorname{Hilb}(\mathbb{P}^{n-1}) \to \operatorname{Gr}(d,n,r)$ sending each ideal to the corresponding cycle is $(\mathbb{K}^*)^n$-equivariant. This implies that deformations in $\operatorname{Hilb}(\mathbb{P}^{n-1})$ determine those in $\operatorname{Gr}(d,n,r)$: if $u, u' \in N_{\mathbb{R}}$ are such that $\operatorname{in}_u \mathcal{I}(\mathbf{X}) = \operatorname{in}_{u'} \mathcal{I}(\mathbf{X})$, where $\mathcal{I}$ denotes the defining ideal, then also $\operatorname{in}_u R_{\mathbf{X}} = \operatorname{in}_{u'} R_{\mathbf{X}}$. Accordingly each initial ideal of $\mathcal{I}(\mathbf{X})$ determines a face of $\operatorname{Chow}(\mathbf{X})$, so that the Gröbner fan of $\mathbf{X}$ is a refinement of the normal fan of $\operatorname{Chow}(\mathbf{X})$.

The standard construction of the tropical variety X via initial ideals [75, Theorem 2.6] shows that X is a subfan of the Gröbner fan. But in fact X is a subfan of the coarser fan $\mathcal{N}(\operatorname{Chow}(\mathbf{X}))$, since the normal cone of a face $\operatorname{face}_u \operatorname{Chow}(\mathbf{X})$ appears in X if and only if $\mathbf{X}$ meets the maximal torus $(\mathbb{K}^*)^n/\mathbb{K}^* \subseteq \mathbb{P}^{n-1}$, and whether this happens is determined

by the cycle associated to $\mathbf{X}$. The analogue of this holds in the non-fan case as well. This reflects the principle that the information encoded in the Hilbert scheme but not in the Chow variety pertains essentially to nonreduced structure, while tropical varieties have no notion of embedded components and only multiplicities standing in for full-dimensional non-reduced structure.

The machinery of Section 5.3 allows us to give a lean combinatorial characterisation of the Chow subdivision in terms of Theorem 5.2.11.

Main theorem 5.4.1. *We have*

$$\mathcal{N}^1(\mathrm{Chow}_\nu(\mathbf{X})) = X \boxplus \mathcal{L}_{n-d-1}{}^{\mathrm{refl}}.$$

To reiterate: *Let* $\mathbf{X}$ *be a* $(d-1)$*-cycle in* $\mathbb{P}^{n-1}$*, and let* $X = \operatorname{Trop}\mathbf{X}$*. Then the codimension 1 part of the normal subdivision to the Chow subdivision of* $\mathbf{X}$ *is the stable Minkowski sum of* X *and the reflected linear space* $\mathcal{L}_{n-d-1}{}^{\mathrm{refl}}$*.* By Theorem 5.2.4(b), this uniquely determines $\mathrm{Chow}_\nu(\mathbf{X})$ in terms of X, up to translation and adding a constant to the vertex heights.

Theorem 5.4.1 should be taken as providing the extension of the notion of Chow polytope (via its normal fan) to tropical varieties.

Definition 5.4.2. Let the *Chow map ch* be the map taking a tropical cycle X of dimension d to its (*tropical*) *Chow hypersurface*, the cycle $ch(X) = X \boxplus \mathcal{L}_{n-d-1}{}^{\mathrm{refl}}$.

The dimension of $ch(X)$ is $(d-1)+(n-d-1) = n-2$, so its codimension is 1. Indeed ch is a linear map $Z_{d-1} \to Z^1$.

In the classical world, the most natural object to be called the "Chow hypersurface" of a $(d-1)$-cycle $\mathbf{X}$ in $\mathbb{P}^{n-1}$ is the hypersurface in $\mathrm{Gr}(n-d,n)$ defined by the Chow form $R_{\mathbf{X}}$. Our tropical Chow hypersurface $ch(X)$ however lives in the tropical torus $(\mathbb{K}^*)^n/\mathbb{K}$, as does X, and not in a tropical Grassmannian. One can of course associate to $\mathbf{X}$ a hypersurface Y in $\operatorname{Trop}\mathrm{Gr}(n-d,n)$, namely the tropicalisation of the ideal generated by $R_{\mathbf{X}}$ and the Plücker relations. The torus action $(\mathbb{K}^*)^n/\mathbb{K}^* \curvearrowright \mathrm{Gr}(n-d,n)$ tropicalises to an action of $N_\mathbb{R}$ on $\operatorname{Trop}\mathrm{Gr}(n-d,n)$ by translation, i.e. an $(n-1)$-dimensional lineality space. Denote by $N_\mathbb{R}+0$ the orbit of the origin in $\operatorname{Trop}\mathrm{Gr}(n-d,n)$; this is the parameter space for tropical linear spaces in $N_\mathbb{R}$ that are translates of $\mathcal{L}_{n-d-1}$. Then we have $ch(X) = Y \cap (N_\mathbb{R}+0)$.

Lemma 5.3.3 is also seen to be about Chow hypersurfaces, in which context it says

$$\deg ch(X) = \operatorname{codim} X \deg X.$$

This should be compared to the fact that the Chow form of a cycle $\mathbf{X}$ in $\mathrm{Gr}(d,n,r)$ is of degree $r = \deg\mathbf{X}$ in $\mathbb{K}[\mathrm{Gr}(n-d,n)]$, and this ring is generated by brackets in $n-d = \operatorname{codim}\mathbf{X}$ letters.

Proof of Theorem 5.4.1. Given a regular subdivision T of lattice polytopes in M induced by Π, its support function $V_T : u \mapsto \mathrm{face}_u T$ is a piecewise linear function whose domains of linearity are $\mathcal{N}^0(T,\Pi)$. We can view V_T as an element of $(Z^{\mathrm{unbal}})^0 \otimes M$.

We take a linear map $\delta : (Z^{\mathrm{unbal}})^0 \otimes M \to (Z^{\mathrm{unbal}})^1$ such that $\delta(V_T) = \mathcal{N}^1(T,\Pi) \in Z^1$ for any regular subdivision T. The restriction of δ to the linear span of all support functions is a canonical map δ', which has been constructed as the map from Cartier divisors supported on $\mathcal{N}(T,\Pi)$ to Weil divisors on $\mathcal{N}(T,\Pi)$ in the framework of [3], or as the map from piecewise polynomials to Minkowski weights given by equivariant localisation in [50]. Roughly, $\delta'(V)$ is the codimension 1 tropical cycle whose multiplicity at a facet τ records the difference of the values taken by V on either side of τ. We can take δ as any linear map extending δ' such that $\delta(V)$ still only depends on V locally; our only purpose in making this extension is to allow formal manipulations using unbalanced cycles.

Let $V = V_{\mathrm{Chow}_\nu(\mathbf{X})}$, and write $X = \sum_{\sigma\in\Sigma} m_\sigma[\sigma]$. Expanding (5.2.3) in terms of this sum, the value of V at $u \in N_{\mathbb{R}}$ is

$$\sum_{\sigma\in\Sigma} m_\sigma \sum_{J\in\binom{[n]}{n-d}} \deg([\sigma]\cdot[u+C_J])e_J.$$

The intersection $[\sigma]\cdot[u+C_J]$ is zero if $u \notin \sigma - C_J$, and if $u \in \sigma - C_J$ it is one point with multiplicity $\mu^\bullet_{\sigma,C_J}$. So

$$V = \sum_{\sigma\in\Sigma} m_\sigma \sum_{J\in\binom{[n]}{n-d}} \mu^\bullet_{\sigma,C_J}[\sigma - C_J]\otimes e_J.$$

Let V_σ be the inner sum here, so that $V = \sum_{\sigma\in\Sigma} m_\sigma V_\sigma$. Then

$$\delta(V_\sigma) = \sum_{J\in\binom{[n]}{n-d}} \mu^\bullet_{\sigma,C_J} \sum_{\substack{\tau \text{ a facet}\\ \text{of } \sigma - C_J}} \delta([\tau]\otimes e_J).$$

Here, if τ is a facet of form $\sigma' - C_J$ for σ' a facet of σ, then $e_J \in \mathbb{R}\tau$ so $\delta([\tau]\otimes e_J) = 0$ and the τ term vanishes. Otherwise τ has the form $\sigma - C_{J'}$ where $J' = J\setminus\{j\}$ for some $j \in J$. Regrouping the sum by J' gives

$$\delta(V_\sigma) = \sum_{J'\in\binom{[n]}{n-d-1}} \left(\sum_{j\in[n]\setminus J'} \mu^\bullet_{\sigma,C_{J'\cup\{j\}}} \delta([\sigma - C_{J'}]\otimes e_j) \right) \tag{5.4.1}$$

where again we have omitted the terms $\delta([\sigma - C_{J'}] \otimes e_{J'}) = 0$. Now, if $j \notin J'$ then

$$\begin{aligned}
\mu^{\bullet}_{\sigma, C_{J'\cup\{j\}}} &= \mu_{\sigma, C_{J'\cup\{j\}}} \\
&= [N_{\sigma + C_{J'\cup\{j\}}} : N_\sigma + N_{C_{J'\cup\{j\}}}] \\
&= [N : N_\sigma + N_{C_{J'}} + \mathbb{Z}e_j] \\
&= [N : N_{\sigma + C_{J'}} + \mathbb{Z}e_j][N_{\sigma + C_{J'}} + \mathbb{Z}e_j : N_\sigma + N_{C_{J'}} + \mathbb{Z}e_i] \\
&= [N : N_{\sigma + C_{J'}} + \mathbb{Z}e_j][N_{\sigma + C_{J'}} : N_\sigma + N_{C_{J'}}] \\
&= \langle e_j, p\rangle \mu^{\boxplus}_{\sigma, C_{J'}}
\end{aligned}$$

where p is the first nonzero lattice point in the appropriate direction on a line in $M_{\mathbb{R}}$ normal to $\sigma + C_{J'}$. Then the components of p are the minors of a matrix of lattice generators for $\sigma + C_{J'}$ by Cramer's rule, and the last equality is a row expansion of the determinant computing $\mu^{\boxplus}_{\sigma, C_{J'}}$. If $j \in J'$ then $\mu^{\bullet}_{\sigma, C_{J'\cup\{j\}}} = 0 = \langle e_j, p\rangle \mu^{\boxplus}_{\sigma, C_{J'}}$ also. So it's innocuous to let the inner sum in (5.4.1) run over all $j \in [n]$, and we get

$$\begin{aligned}
\delta(V_\sigma) &= \sum_{J' \in \binom{[n]}{n-d-1}} \left(\sum_{j \in [n]} \mu^{\boxplus}_{\sigma, C_{J'}} \langle e_j, p\rangle \delta([\sigma - C_{J'}] \otimes e_j) \right) \\
&= \sum_{J' \in \binom{[n]}{n-d-1}} \mu^{\boxplus}_{\sigma, C_{J'}} \delta([\sigma - C_{J'}] \otimes p) \\
&= \sum_{J' \in \binom{[n]}{n-d-1}} \mu^{\boxplus}_{\sigma, C_{J'}} [\sigma - C_{J'}] \\
&= ([\sigma] \boxplus \mathcal{L}_{n-d-1}{}^{\mathrm{refl}}).
\end{aligned}$$

We conclude that

$$\mathcal{N}^1(\mathrm{Chow}_\nu(X)) = \delta(V) = \sum_\sigma m_\sigma([\sigma] \boxplus \mathcal{L}_{n-d-1}{}^{\mathrm{refl}}) = X \boxplus \mathcal{L}_{n-d-1}{}^{\mathrm{refl}}. \quad \square$$

5.5 Linear spaces

A *matroid subdivision* (of rank r) is a regular subdivision of a matroid polytope (of rank r) all of whose facets are matroid polytopes, i.e. polytopes of the form $\mathrm{Poly}(M)$ defined in (5.1.1). The *hypersimplex* $\Delta(r, n)$ is the polytope $\mathrm{conv}\{e_J : J \in \binom{[n]}{r}\}$. The vertices of a rank r matroid polytope are a subset of those of $\Delta(r, n)$. We have the following polytopal characterisation of matroid polytopes due to Gelfand, Goresky, MacPherson, and Serganova.

Theorem 5.5.1 ([37])**.** *A polytope $\Pi \subseteq \mathbb{R}^n$ is a matroid polytope if and only if $\Pi \subseteq [0,1]^n$ and each edge of Π is a parallel translate of $e_i - e_j$ for some i, j.*

Definition 5.5.2. Given a regular matroid subdivision Σ, its *Bergman complex* $\mathcal{B}(\Sigma)$ and *co-Bergman complex* $\mathcal{B}^*(\Sigma)$ are subcomplexes of $\mathcal{N}(\Sigma)$. The face of $\mathcal{N}(\Sigma)$ normal to $F \in \Sigma$

- is a face of $\mathcal{B}(\Sigma)$ if and only if F is the polytope of a loop-free matroid;
- is a face of $\mathcal{B}^*(\Sigma)$ if and only if F is the polytope of a coloop-free matroid.

We make $\mathcal{B}(\Sigma)$ and $\mathcal{B}^*(\Sigma)$ into tropical varieties by giving each facet multiplicity 1.

The *Bergman fan*, the fan case of the Bergman complex, was introduced in [7] (where an object named the "Bergman complex" different to ours also appears). Bergman complexes are much used in tropical geometry, on account of the following standard definition, appearing for instance in [82].

Definition 5.5.3. A *tropical linear space* is the Bergman complex of a regular matroid subdivision.

In the context of Chow polytopes it is the co-Bergman complex rather than the Bergman complex that arises naturally, on account of the duality mentioned in Example 5.2.8(2). Observe that the co-Bergman complex of a matroid subdivision is a reflection of the Bergman complex of the dual matroid subdivision; in particular any Bergman complex is a co-Bergman complex and vice versa.

Since there is a good notion of tropical degree (Definition 5.3.2), the following alternative definition seems natural.

Definition 5.5.4. A *tropical linear space* is a tropical variety of degree 1.

Theorem 5.5.5. *Definitions 5.5.3 and 5.5.4 are equivalent.*

The equivalence in Theorem 5.5.5 was noted by Mikhalkin, Sturmfels, and Ziegler and recorded in [45], but no proof was provided. One implication, that Bergman complexes of matroids have degree 1, follows from Proposition 3.1 of [82], which implies that the tropical stable intersection of a $(d-1)$-dimensional Bergman complex of a matroid subdivision with $\mathcal{L}_{n-d}$ (the Bergman complex of a uniform matroid) is a 0-dimensional Bergman complex, i.e. a point with multiplicity 1. Thus it remains to prove that degree 1 tropical varieties are (co-)Bergman complexes. In fact, let $X \subseteq N_{\mathbb{R}}$ be a degree 1 tropical variety of dimension $d-1$. We will show

1. The regular subdivision Σ such that $ch(X) = \mathcal{N}^1(\Sigma)$ is dual to a matroid subdivision of rank $n-d$.
2. We have $X = \mathcal{B}^*(\Sigma)$.

Tropical varieties have an analogue of Bézout's theorem. See for instance Theorem 9.16 of [3], which however only proves equality under genericity assumptions, not the inequality below. We will only need the theorem in the case that the varieties being intersected have degree 1.

Theorem 5.5.6 (Tropical Bézout's theorem)**.** *Let X and Y be tropical varieties of complementary dimensions. We have $\deg(X \cdot Y) \leq \deg X \deg Y$, and equality is attained if X and Y are of sufficiently generic combinatorial type.*

Lemma 5.5.7. *If a tropical variety X of degree 1 contains a ray in direction $-e_i$ for $i \in [n]$, then $-e_i$ is contained in the lineality space of X.*

Proof. Consider the set

$$Y = \{u \in N_{\mathbb{R}} : u - ae_i \in X \text{ for } a \gg 0\}.$$

By assumption on X, Y is nonempty. This Y is the underlying set of a polyhedral complex; make it into a cycle by giving each facet multiplicity 1. In fact, Y is a tropical variety, as any face τ of Y corresponds to a face σ of X such that $\tau = \sigma + \mathbb{R}e_i$, and so Y inherits balancing from X. Also $\dim Y = \dim X =: d - 1$. Since Y is effective, some translate and therefore any translate of $\mathcal{L}_{n-d-1}$ intersects Y stably in at least one point.

Suppose X had a facet σ whose linear span didn't contain $-e_i$. Then there is some translate $[u] \boxplus \mathcal{L}_{n-d-1}$ which intersects $\operatorname{relint}\sigma$, with the intersection lying on a face $u + C_J$ of $[u] \boxplus \mathcal{L}_{n-d-1}$ with $i \in J$. Given this translate, any other translate $[u - ae_i] \boxplus \mathcal{L}_{n-d-1}$ with $a \geq 0$ will intersect X transversely in the same point of $\operatorname{relint}\sigma$. For a sufficiently large, one of the points of $Y \cdot ([u - ae_i] \boxplus \mathcal{L}_{n-d-1})$ lies in X, providing a second intersection point of X and $[u - ae_i] \boxplus \mathcal{L}_{n-d-1}$. By Bézout's theorem this contradicts the assumption that $\deg X = 1$. □

Proof of Theorem 5.5.5. To (1). Suppose $l \subseteq N_{\mathbb{R}}$ is a classical line in any direction e_J, $J \subseteq [n]$. By Lemma 5.3.1 and Theorem 5.5.6 we have

$$\deg(ch(X) \cdot [l]) = \deg((X \boxplus \mathcal{L}_{(n-d-1)}{}^{\text{refl}}) \cdot [l]) = \deg((\mathcal{L}_{(n-d-1)} \boxplus [l]) \cdot X) \leq 1 \qquad (5.5.1)$$

because $\mathcal{L}_{(n-d-1)} \boxplus [l]$ is a degree 1 tropical variety. Since intersection multiplicities are positive, if l intersects a facet σ of $ch(X)$ then the multiplicity of the intersection is $\mu^{\bullet}_{\sigma,l} = 1$.

Let σ be a facet of $ch(X)$, and l a line in direction e_J intersecting it. Then $\mu^{\bullet}_{\sigma,l} = \langle m, e_J \rangle$ where $m \in M_{\mathbb{R}}$ is the difference of the endpoints of the edge of Σ dual to σ. Then m is the product of a primitive normal vector to σ and the multiplicity m_σ. The positive components of m cannot have sum $k \geq 2$, or else, for a suitable choice of J, we would achieve $\mu^{\bullet}_{\sigma,l} = \langle m, e_J \rangle = k$. Since m is nonzero and normal to $(1, \ldots, 1)$ we must have $m = e_i - e_j$ for some $i \neq j \in [n]$. It follows that each edge of Σ is a parallel translate of some $e_i - e_j$.

Furthermore, let $l \subseteq N_{\mathbb{R}}$ be a line in direction e_i, for $i \in [n]$. The vertices of Σ attained as $\mathrm{face}_u \Sigma$ for some $u \in l$ are in bijection with the connected components of the complement of $ch(X)$. So there are at most two of these vertices, and if there are two, say m_0 and m_1, we have $\langle m_1 - m_0, e_i \rangle = 1$. But among the vertices $\mathrm{face}_u \Sigma$ for $u \in l$ are vertices m minimising and maximising the pairing $\langle m, e_i \rangle$. Therefore, the projection of Σ to the ith coordinate axis has length either 0 or 1.

For the remainder of the proof we fix a particular translation representative of Σ, namely the one whose projection onto the ith coordinate axis is either the point $\{0\}$ or the interval $[0,1]$ for each $i \in [n]$. For this particular Σ, Theorem 5.5.1 implies that Σ is a matroid subdivision.

Let r be the rank of the matroid subdivision Σ. Let e_J be one vertex of Σ, so that $|J| \in \binom{[n]}{r}$, and let u be a linear form with $\mathrm{face}_u \Sigma = e_J$. Then, for any $i \in [n] \setminus J$ and any $a > 0$, we have $\mathrm{face}_{u+ae_i} \Sigma = e_J$, since $e_J \in \mathrm{face}_{e_i} \Sigma$. On the other hand, for any $i \in J$ and sufficiently large $a \gg 0$, we have $\mathrm{face}_{u+ae_i} \Sigma \not\ni e_J$, and indeed $\mathrm{face}_{u+ae_i} \Sigma$ will contain some vertex $e_{J'}$ with $i \notin J'$, whose existence is assured by our choice of translation representative for Σ. It follows that a ray $[u] \boxplus [R_{\geq 0}\{e_i\}]$ of $[u] \boxplus \mathcal{L}_1$ intersects $ch(X)$ if and only if $i \in J$. Each intersection must have multiplicity 1, so

$$\deg(ch(X)) = \deg(ch(X) \cap ([u] \boxplus \mathcal{L}_1)) = |J| = r.$$

But by Proposition 5.3.3 we have that $\deg(ch(X)) = n - d$, so $r = n - d$ as claimed.

To (2). Fix some polyhedral complex structure on X. Given any $u \in N_{\mathbb{R}}$ in the support of $ch(X)$, its multiplicity is $ch(X)(u) = 1$, and therefore by positivity there is a unique choice of a facet τ of X and $J \in \binom{[n]}{n-d-1}$ such that $u \in X - C_J$. Write $J = J(u)$. On the other hand, Σ has a canonical coarsest possible polyhedral complex structure, on account of being a normal complex. We claim that $J(u)$ is constant for u in the relative interior of each facet σ of Σ, and thus we can write $J(\sigma) := J(u)$. Suppose not. Consider the common boundary ρ of two adjacent regions σ_1, σ_2 of σ on which $J(u)$ is constant. Suppose $\sigma_1 \subseteq \tau - C_{J_1}$. We have $\rho \subseteq \tau - C_K$ for $K \in \binom{[n]}{n-d-2}$. There is a facet of Σ of form $\sigma_j \subseteq \tau - C_{K \cup k}$ incident to ρ for each $k \in [n] \setminus K$ such that e_k is not contained in the affine hull of τ. Since $\dim \tau = d - 1$, and any d of the e_k are independent in $N_{\mathbb{R}}$, there exist at most $d - 1$ indices $k \in [n]$ such that e_k is not contained in the affine hull of τ, and hence at least

$$|[n] \setminus K| - (d - 1) = 3$$

indices $k \in [n]$ yielding facets of Σ. In particular σ_1 and σ_2 cannot be the only $(d-1)$-dimensional regions in Σ incident to ρ, and this implies σ cannot be a facet of Σ, contradiction.

Now, every facet σ of $ch(X)$ is normal to an edge of Σ, say $E_\sigma = \mathrm{conv}\{e_K + e_j, e_K + e_k\}$ for $K \in \binom{[n]}{n-d-1}$. Since $\Sigma \subseteq \Delta(n-d, n)$, σ must contain a translate of the normal cone to E_σ in $\mathcal{N}^1(\Delta(n-d, n))$, namely

$$\mathrm{normal}(E_\sigma) = \{u \in N_{\mathbb{R}} : u_j = u_k, u_i \leq u_j \text{ for } i \in K, u_i \geq u_j \text{ for } i \notin K \cup \{j, k\}\}.$$

In particular σ contains exactly $n-d-1$ rays in directions $-e_i$, those with $i \in K$.

Let R be the set of directions $-e_1, \ldots, -e_n$. Suppose for the moment that X contains no lineality space in any direction $-e_i$. We have that $\sigma \subseteq X \boxplus [-C_{J(\sigma)}]$. By Lemma 5.5.7, X contains no rays in directions in R, so we must have that $J(\sigma) = K$ and $-C_{J(\sigma)}$ contains a ray in direction $-e_i$ for all $i \in K$. Now consider any face ρ of σ containing no rays in directions in R. Then we claim $\rho \in X$. If this weren't so, then there would be another face σ' parallel to σ and with $J(\sigma) = J(\sigma')$. But the edge E_σ is determined by $J(\sigma) = K$ and the normal direction to σ, so $E_\sigma = E_{\sigma'}$, implying $\sigma = \sigma'$. On the other hand, the relative interior of any face of σ containing a ray in direction R is disjoint from X, since if u is a point in such a face there exists $v \in -C_{J(\sigma)} \setminus \{0\}$ such that $u - v \in X$. So X consists exactly of the faces of $ch(X)$ containing no ray in a direction in R.

If X has a lineality space containing those $-e_j$ with $j \in J$, then let X' be the pullback of X along a linear projection with kernel $\operatorname{span}\{-e_j : j \in J\}$. Then we can repeat the last argument using X', and we get that X consists exactly of the faces of $ch(X)$ containing no ray in a direction in $R \setminus \{-e_j : j \in J\}$.

Now, a face normal(F) of $\mathcal{N}(\Sigma)$ contains a ray in direction $-e_i$ if and only if the linear functional $\langle m, -e_i \rangle$ is constant on $m \in F$ and equal to its maximum for $m \in \Sigma$. The projection of F to the ith coordinate axis is either $\{0\}$, $\{1\}$, or $[0,1]$, so normal(F) contains a ray in direction $-e_i$ if and only if the projection of F is $\{1\}$, or the projection of F and of Σ are both $\{0\}$. Projections taking Σ to $\{0\}$ correspond to lineality directions in X, so we have that X consists exactly of the faces of $ch(X)$ which don't project to $\{1\}$ along any coordinate axis. These are exactly the coloop-free faces. □

5.6 The kernel of the Chow map

In this section we will show that the Chow map $ch : Z_{d-1} \to Z^1$ has a nontrivial kernel. This implies that there exist distinct tropical varieties with the same Chow polytope: Y and $X + Y$ will be a pair of such varieties for any nonzero $X \in \ker ch$, choosing Y to be any effective tropical cycle such that $X + Y$ is also effective (for instance, let Y be a sum of classical linear spaces containing the facets of X that have negative multiplicity). Thus Chow subdivisions do not lie in a combinatorial bijection with general tropical varieties, as was the case for our opening examples.

There are a few special cases in which ch is injective. In the case $d = n - 1$ of hypersurfaces, ch is the identity. In the case $d = 1$, in which X is a point set with multiplicity, $ch(X)$ is a sum of reflected tropical hyperplanes with multiplicity, from which X is easily recoverable. Furthermore, Conjecture 5.6.2 below would imply restrictions on the rays in any one-dimensional tropical fan cycle in $\ker ch$, and one can check that no cycle with these restrictions lies in $\ker ch$.

Example 5.6.1 provides an explicit tropical fan cycle in $\ker ch$ in the least case, $(d, n) =$

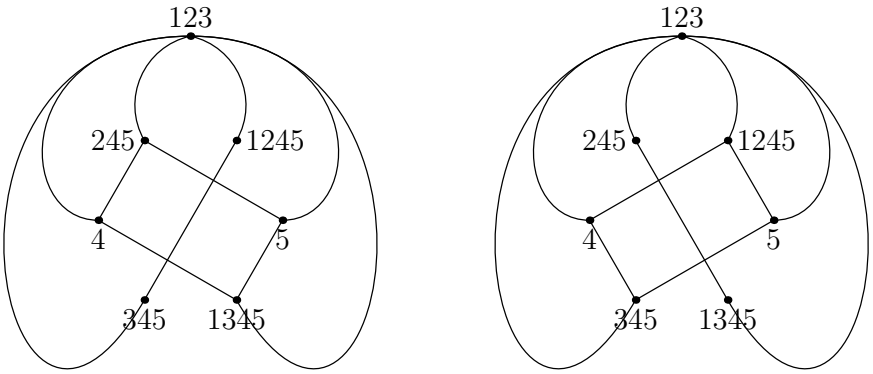


Figure 5.3: Two tropical surfaces with the same Chow hypersurface. A point labelled $j_1 \cdots j_k$ represents the ray $\mathbb{R}_{\geq 0} e_{\{j_1,\ldots,j_k\}}$.

$(3,5)$, not among those just mentioned. First we introduce the fan on which the example depends, which seems to be of critical importance to the behaviour of $\ker ch$ in general.

Let $\mathcal{A}_n \subseteq \mathbb{R}^{n-1}$ be the fan in $N_{\mathbb{R}}$ consisting of the cones $\mathbb{R}_{\geq 0}\{e_{J_1}, \ldots, e_{J_i}\}$ for all chains of subsets

$$\emptyset \subsetneq J_1 \subsetneq \cdots \subsetneq J_i \subsetneq [n].$$

This fan $\mathcal{A}_n$ makes many appearances in combinatorics. It is the normal fan of the permutahedron, and by Theorem 5.5.1 also the common refinement of all normal fans of matroid polytopes. Its face poset is the order poset of the boolean lattice. Moreover, its codimension 1 skeleton is supported on the union of the hyperplanes $\{\{x_i = x_j\} : i \neq j \in [n]\}$ of the *type A reflection arrangement*, i.e. the *braid arrangement*.

As in Section 5.2.1, the ring $Z^{\text{fan}}(\mathcal{A}_n)$ is the Chow cohomology ring of the toric variety associated to Σ. This toric variety is the closure of the torus orbit of a generic point in the complete flag variety (which, to say it differently, is $\mathbb{P}^{n-1}$ blown up along all the coordinate subspaces). The cohomology of this variety has been studied by Stembridge [85]. We have that $\dim Z^{\text{fan}}(\mathcal{A}_n) = n!$, and $\dim(Z^{\text{fan}})^k(\mathcal{A}_n)$ is the *Eulerian number* $E(n,k)$, the number of permutations of $[n]$ with k descents.

For any cone $\sigma = \mathbb{R}_{\geq 0}\{e_{J_1}, \ldots, e_{J_d}\}$ of $\mathcal{A}_n$, and any orthant ${\sigma_{J'}}^{\text{refl}} = \mathbb{R}_{\geq 0}\{-e_j : j \in J'\}$, the Minkowski sum $\sigma + {\sigma_{J'}}^{\text{refl}}$ is again a union of cones of $\mathcal{A}_n$. Therefore $ch(Z_d^{\text{fan}}(\mathcal{A}_n)) \subseteq (Z^{\text{fan}})^1(\mathcal{A}_n)$ always, and we find nontrivial elements of $\ker ch$ whenever the dimension of $Z_d^{\text{fan}}(\mathcal{A}_n)$ exceeds that of $(Z^{\text{fan}})^1(\mathcal{A}_n)$, i.e. when $E(n, n-d) > E(n,1)$, equivalently when $2 < d < n-1$.

Example 5.6.1. For $(d,n) = (3,5)$, we have $E(5,5-3) = 66 > 26 = E(5,1)$, and the kernel of ch restricted to $Z_2^{\text{fan}}(\mathcal{A}_5)$ is 40-dimensional. Two tropical varieties in $Z_2^{\text{fan}}(\mathcal{A}_5)$ within $N_{\mathbb{R}} = \mathbb{R}^4$ with equal Chow hypersurfaces are depicted in Figure 5.3. As one often does, we have dropped one dimension in the drawing by actually drawing the intersections of these

2-dimensional tropical fans with a sphere centered at the origin in $\mathbb{R}^4$, which are graphs in $\mathbb{R}^3$. The difference of these varieties is an actual element of $\ker ch$, involving the six labelled rays other than 123, which form an octahedron. $\diamondsuit$

The property of $\mathcal{A}_n$ that this example exploits appears to be essentially unique: this is part (a) of the next conjecture. This property, together with experimentation with fan varieties of low degree in low ambient dimension, also suggests part (b).

Conjecture 5.6.2.

(a) Let Σ be a complete fan such that the stable Minkowski sum of any cone of Σ and any ray $\mathbb{R}_{\geq 0}(-e_i)$ is a sum of cones of Σ. Then $\mathcal{A}_n$ is a refinement of Σ.

(b) The kernel of the restriction of ch to fan varieties is generated by elements of $Z^{\text{fan}}(\mathcal{A}_n)$.

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