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### Author

Yu, Yifeng

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# Maximal and minimal solutions of an Aronsson equation: $L^\infty$ variational problems versus the game theory

Yifeng Yu

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**Abstract** The Dirichlet problem

$$\begin{cases} \Delta_\infty u - |Du|^2 = 0 & \text{on } \Omega \subset \mathbb{R}^n \\ u|_{\partial\Omega} = g \end{cases}$$

might have many solutions, where  $\Delta_\infty u = \sum_{1 \leq i, j \leq n} u_{x_i} u_{x_j} u_{x_i x_j}$ . In this paper, we prove that the maximal solution is the unique absolute minimizer for  $H(p, z) = \frac{1}{2}|p|^2 - z$  from calculus of variations in  $L^\infty$  and the minimal solution is the continuum value function from the “tug-of-war” game. We will also characterize graphs of solutions which are neither an absolute minimizer nor a value function. A remaining interesting question is how to interpret those intermediate solutions. Most of our approaches are based on an idea of Barles–Busca (Commun Partial Differ Equ 26(11–12):2323–2337, 2001).

**Mathematics Subject Classification (2000)** Primary 35J70; Secondary 35D99 · 91A05

## 1 Introduction

Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$ . Peres et al. [14] introduced a two-person differential game called “tug-of-war”. Starting from a point  $x \in \Omega$ , at each step with fixed length, two players toss a fair coin to determine the order of move. One player tries to maximize the payoff function and the other wants to minimize it. The game will stop if one of them reaches the boundary of  $\Omega$ . In this paper, let us assume that the running payoff function is a constant  $-\tau$  and the terminal payoff function is  $g \in W^{1,\infty}(\Omega)$ . Owing to [14], as the step size tends

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Y. Yu (✉)

Department of Mathematics, University of California at Irvine, Irvine, USA  
e-mail: yyu1@math.uci.edu

to zero, value functions of the game will converge to the unique viscosity solution of the equation

$$\begin{cases} \frac{\Delta_\infty u}{|Du|^2} = \tau & \text{on } \Omega \\ u|_{\partial\Omega} = g. \end{cases} \tag{1.1}$$

Following the terminology in [14], we call a viscosity solution of Eq. (1.1) a *continuum value function of the “tug-of-war” game*. See the *User’s Guide* Crandall–Ishii–Lions [8] for definitions of viscosity solutions of general nonlinear elliptic equations. Here we should be careful about the operator  $\frac{\Delta_\infty u}{|Du|^2}$  when  $|Du|$  vanishes. According to the definition in [14], if a  $C^2$  test function  $\phi$  touches  $u$  at  $x \in \Omega$  from above (or below) and  $D\phi(x) = 0$ , we require that  $\max_{\{|p|=1\}} p \cdot D^2\phi(x) \cdot p \geq \tau$  (or  $\min_{\{|p|=1\}} p \cdot D^2\phi(x) \cdot p \leq \tau$ ). When  $n = 1$ ,

$$\max_{\{|p|=1\}} p \cdot \phi''(x) \cdot p = \min_{\{|p|=1\}} p \cdot \phi''(x) \cdot p = \phi''(x).$$

Hence Eq. (1.1) is just  $u'' = \tau$ .

Multiplying  $|Du|^2$  on both side, we derive that the value function is also a viscosity solution of the equation

$$\Delta_\infty u - \tau|Du|^2 = 0. \tag{1.2}$$

However, except when  $\tau = 0$ , solutions of Eq. (1.2) might not be solutions of Eq. (1.1). Here is a simple example.

*Example 1*  $u_1 = 0$  and  $u_2 = \frac{1}{2}x^2 - \frac{1}{2}$  are both smooth solutions of

$$\begin{cases} (u')^2 u'' - |u'|^2 = 0 & \text{on } (-1, 1) \\ u(-1) = u(1) = 0. \end{cases}$$

But  $u_1 = 0$  is not a solution of  $u'' = \frac{(u')^2 u''}{(u')^2} = 1$ .

According to [14], Eq. (1.1) admits a unique solution. But the above example suggests that Eq. (1.2) might have multiple solutions with prescribed boundary value. Equation (1.2) is a so called Aronsson equation associated to  $H(p, z) = \frac{1}{2}|p|^2 - \tau z$ . For general  $H = H(p, z, x) \in C^1(\mathbb{R}^n \times \mathbb{R} \times \Omega)$ , the correspondent Aronsson equation is

$$A_H(u) = H_p(x, u, Du) \cdot D_x(H(x, u, Du)) = 0 \quad \text{in } \Omega.$$

Here  $H_p$  is the partial derivative of  $H$  with respect to  $p$  and  $D_x$  represents the derivative with respect to  $x$  of  $H(x, u(x), Du(x))$ . Aronsson equations are Euler–Lagrangian equations for “calculus of variations in  $L^\infty$ ” which were initiated by Aronsson in 1960s ([1–4]). Here is the general definition of minimizers of such highly nonconventional variational problems. For  $H = H(p, z, x) \in C(\mathbb{R}^n \times \mathbb{R} \times \Omega)$ , we say that  $u \in W_{loc}^{1,\infty}(\Omega)$  is an *absolute minimizer for  $H$  in  $\Omega$*  if for any open set  $V \subset \bar{V} \subset \Omega$  and  $v \in W^{1,\infty}(V)$ ,

$$u|_{\partial V} = v|_{\partial V}$$

implies that

$$\text{esssup}_V H(Du, u, x) \leq \text{esssup}_V H(Dv, v, x).$$

Crandall proved in [7] (see also Barron–Jensen–Wang [6]) that if  $H \in C^2$  and is quasiconvex in  $p$ , then an absolute minimizer for  $H = H(p, z, x)$  in  $\Omega$  is a viscosity solution of the

Aronsson equation

$$A_H(u) = 0 \text{ in } \Omega.$$

A function  $f$  is *quasiconvex* if the set  $\{f < t\}$  are convex for all  $t \in \mathbb{R}$ .

Let us focus on  $H = \frac{1}{2}|p|^2 - \tau z$ . Then any absolute minimizer for  $H$  is a viscosity solution of Eq. (1.2) in  $\Omega$ . However, except when  $\tau = 0$ , the converse might not be true. In Example I,  $u_2 = \frac{1}{2}x^2 - \frac{1}{2}$  is not an absolute minimizer. In fact,  $u_2|_{\partial\Omega} = 0$ , but

$$\frac{1}{2} = \text{esssup}_{(-1,1)} \left( \frac{1}{2}(u_2')^2 - u_2 \right) > 0.$$

Hence two natural questions arise.

- (1) Is an absolute minimizer for  $H$  unique with prescribed boundary value?
- (2) If uniqueness holds, what are the positions of the continuum value function from the game theory and the absolute minimize among all viscosity solutions of Eq. (1.2)?

When  $\tau = 0$ , Eq. (1.2) is the famous infinity Laplacian equation. Jensen proved in [11] that Dirichlet problem of the infinity Laplacian equation has a unique solution. Hence the continuum value function and the absolute minimizer coincide in this case. So let us look at  $\tau \neq 0$ . By properly scaling and changing signs, we may assume that  $\tau = 1$ . The following is our main result.

**Theorem 1.1** *Suppose that  $g \in W^{1,\infty}(\Omega)$ . Then there exists a unique absolute minimizer for  $H = \frac{1}{2}|p|^2 - z$  in  $\Omega$  with boundary value  $g$ . The absolute minimizer is the maximal viscosity solution of*

$$\begin{cases} \Delta_\infty u - |Du|^2 = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

*Moreover, the continuum value function from the game theory is the minimal viscosity solution of above equation.*

**Remark 1.2** In Example I,  $u_1 = 0$  is the absolute minimizer and  $u_2 = \frac{1}{2}x^2 - \frac{1}{2}$  is the continuum value function. Also, it is easy to deduce from Theorem 1.1 that for general  $\tau > 0$  ( $\tau < 0$ ), the absolute minimizer is the maximal (minimal) solution and the continuum value function is the minimal (maximal) solution.

In Theorem 1.1, the uniqueness of an absolute minimizer follows immediately after we prove that an absolute minimizer is the maximal solution. There were various results on uniqueness of absolute minimizers from  $L^\infty$ -variational problems. See for instance Crandall–Gunnarsson–Wang [9], Jensen [11], Jensen–Wang–Yu [12], Juutinen [13], Barles–Busca [5], etc. However all those results depend on uniqueness of solutions of Dirichlet problems for correspondent Aronsson equations, which, as suggested by Example I, might not hold in our case. To prove that an absolute minimizer is the maximal solution, we first use an idea from [5] to reduce inhomogeneous boundary conditions to homogeneous boundary conditions. Then, by combine use of the PDE (1.3) and the definition of absolute minimizers, we prove that if an absolute minimizer vanishes on the boundary, then it must be zero.

We want to point out that the existence of absolute minimizers does not follow directly from the usual  $L^p$  approximation introduced by Aronsson (see [6]) since  $H = \frac{1}{2}|p|^2 - z$  is not bounded from below. What we do is to introduce an auxiliary  $\hat{H} \geq 0$  and show that absolute minimizers for  $\hat{H}$  are also absolute minimizers for  $H = \frac{1}{2}|p|^2 - z$ . Our approach

relies on the fact that any solution of Eq. (1.3) is bounded from above by its maximum value on  $\partial\Omega$ . This is because a viscosity solution of Eq. (1.3) is a viscosity subsolution of the infinity Laplacian equation.

**Outline of our paper** In Sect. 2, we will prove Theorem 1.1. In Sect. 3, we give a characterization of solutions of Eq. (1.3) which are neither the absolute minimizer nor the continuum value function. A remaining interesting question is how to interpret those solutions.

**Notations** We denote  $B_r(x_0)$  as an open ball centered at  $x_0$  with radius  $r$ . For  $\delta > 0$ , we write

$$\Omega_\delta = \{x \in \Omega \mid d(x, \partial\Omega) > \delta\}.$$

If  $V$  is a subset of  $\mathbb{R}^n$ ,  $\partial V$  denotes its boundary and  $\bar{V}$  the closure. Moreover, if  $f$  is a semiconvex function, i.e.  $f(x) + C|x|^2$  is convex for some  $C > 0$ , we denote  $D^-f(x_0)$  as the subdifferentials of  $f$  at  $x_0$ . That is

$$D^-f(x_0) = \{p \in \mathbb{R}^n \mid f(x) \geq f(x_0) + p \cdot (x - x_0) - o(|x - x_0|)\},$$

where  $o(|x - x_0|)$  means that  $\lim_{x \rightarrow x_0} \frac{o(|x - x_0|)}{|x - x_0|} = 0$ .

*Remark 1.3* From now on,  $H = \frac{1}{2}|p|^2 - z$ . Moreover, we use ‘‘absolute minimizer(s)’’ as an abbreviation for ‘‘absolute minimizer(s) for  $H$  in  $\Omega$ ’’ unless we specify the functional.

## 2 Proofs

We first use an idea from [5] to prove a key lemma.

**Lemma 2.1** *Suppose that  $u \in C(\bar{\Omega})$  is a semiconvex viscosity subsolution of equation*

$$\Delta_\infty u - |Du|^2 = 0 \quad \text{in } \Omega \tag{2.1}$$

*and  $v \in C(\bar{\Omega})$  is a viscosity solution of the above equation. Assume that*

$$\max_{\bar{\Omega}}(u - v) > \max_{\partial\Omega}(u - v).$$

*If  $u(x_0) - v(x_0) = \max_{\bar{\Omega}}(u - v)$  for some  $x_0 \in \Omega$ , then there exists  $r_0 > 0$  such that*

$$u(x) = u(x_0) \quad \text{for } x \in B_{r_0}(x_0).$$

*Proof* For  $\delta > 0$  and  $h \in B_\delta(0)$ , denote  $M_\delta(h) = \max_{\bar{\Omega}_\delta}(u(x + h) - v(x))$ . It is clear that  $M_\delta(h)$  is a semiconvex function of  $h$ . Since the maximum value of  $u - v$  is not attained  $\partial\Omega$ , there should exist  $\delta_1 > 0$  such that for all  $h \in B_{\delta_1}(0)$ ,

$$\{x \in \bar{\Omega}_{\delta_1} \mid u(x + h) - v(x) = M_{\delta_1}(h)\} \subset \Omega_{2\delta_1}. \tag{2.2}$$

Now I claim that

$$0 \in D^-M_{\delta_1}(h) \quad \text{for all } h \in B_{\delta_1}(0). \tag{2.3}$$

In fact, fix  $h$  and let us denote

$$w_{\epsilon,h}(x, y) = (1 + \epsilon)u(x + h) - v(y) - \frac{1}{2\epsilon}|x - y|^2.$$

Suppose that  $(\bar{x}, \bar{y}) \in \{ (x, y) \in \bar{\Omega}_{\delta_1} \times \bar{\Omega}_{\delta_1} \mid w_{\epsilon,h}(\bar{x}, \bar{y}) = \max_{x,y \in \bar{\Omega}_{\delta_1}} w_{\epsilon,h} \}$ . Owing to (2.2), when  $\epsilon$  is small enough, we have that  $(\bar{x}, \bar{y}) \in \Omega_{\delta_1} \times \Omega_{\delta_1}$ . According to [8], there exist  $X$  and  $Y$  such that

- (1)  $(\frac{\bar{x}-\bar{y}}{\epsilon}, X) \in \bar{J}_{\Omega_{\delta_1}}^{2,+}[(1+\epsilon)u(\bar{x}+h)], (\frac{\bar{x}-\bar{y}}{\epsilon}, Y) \in \bar{J}_{\Omega_{\delta_1}}^{2,-}v(\bar{y}),$
- (2)  $-\frac{3}{\epsilon}I_n \leq X \leq Y \leq \frac{3}{\epsilon}I_n.$

Here  $X, Y, \bar{x}$  and  $\bar{y}$  all depend on  $\epsilon$ . See [8] for definitions of  $\bar{J}_V^{2,+}$  and  $\bar{J}_V^{2,-}$ . Owing to Eq. (2.1), we have that

$$\frac{\bar{x}-\bar{y}}{\epsilon} \cdot X \cdot \frac{\bar{x}-\bar{y}}{\epsilon} \geq (1+\epsilon)|\frac{\bar{x}-\bar{y}}{\epsilon}|^2$$

and

$$\frac{\bar{x}-\bar{y}}{\epsilon} \cdot Y \cdot \frac{\bar{x}-\bar{y}}{\epsilon} \leq |\frac{\bar{x}-\bar{y}}{\epsilon}|^2.$$

Due to (2) above, we must have that

$$\frac{\bar{x}-\bar{y}}{\epsilon} = 0.$$

Since  $u$  is semiconvex,  $u$  is differentiable at  $\bar{x}+h$  and

$$D^-u(\bar{x}+h) = \{0\}.$$

Passing to a subsequence if necessary, we may assume that

$$\lim_{\epsilon \rightarrow 0} \bar{x} = \lim_{\epsilon \rightarrow 0} \bar{y} = z_0.$$

It is clear that  $z_0 \in \{\bar{x} \in \bar{\Omega}_{\delta_1} \mid u(x+h) - v(x) = M_{\delta_1}(h)\}$ . Since  $u(\cdot+h)$  is semiconvex, the set  $D^-u(x)$  is upper-semicontinuous. Therefore

$$0 \in D^-u(z_0+h).$$

Hence

$$u(z_0+\hat{h}) \geq u(z_0+h) - o(|\hat{h}-h|).$$

Therefore

$$M_{\delta_1}(\hat{h}) \geq u(z_0+\hat{h}) - v(z_0) \geq u(z_0+h) - v(z_0) - o(|\hat{h}-h|) = M_{\delta_1}(h) - o(|\hat{h}-h|).$$

So

$$0 \in D^-M_{\delta_1}(h).$$

Hence our claim holds. Therefore

$$M_{\delta_1}(h) = M_{\delta_1}(0) \text{ for } |h| \leq \delta_1.$$

Accordingly,

$$u(x_0+h) - v(x_0) \leq M_{\delta_1}(h) = M_{\delta_1}(0) = u(x_0) - v(x_0).$$

This implies that

$$u(x_0+h) \leq u(x_0) \text{ for } |h| \leq \delta_1.$$

Since  $u$  is a viscosity subsolution of Eq. (2.1),  $u$  is a viscosity subsolution of the infinity Laplacian equation

$$\Delta_\infty u = 0.$$

According to the well known differential Harnack inequality (see Lemma 2.5 in CEG [10] for instance), we must have that

$$u(x_0 + h) = u(x_0) \quad \text{for } |h| \leq \delta_1. \quad (2.4)$$

□

The following lemma says that the graph of an absolute minimizer can not contain wells. Its proof is a combine use of the PDE and the definition of absolute minimizers.

**Lemma 2.2** *Suppose that  $V$  is a bounded open set in  $\mathbb{R}^n$ . Assume that  $w$  is an absolute minimizer for  $H$  on  $V$  and*

$$w = c \quad \text{on } \partial V.$$

*Then*

$$w \equiv c \quad \text{in } V.$$

*Proof* Since  $w - c$  is also an absolute minimizer, we may assume that  $c = 0$ . Since  $w$  is an absolute minimizer, it is a viscosity solution of Eq. (2.1). So it is a viscosity subsolution of the infinity Laplacian equation

$$\Delta_\infty w = 0.$$

Owing to the *maximum principle* for the infinity Laplacian equation, we have that

$$w \leq 0 \quad \text{in } V.$$

Since  $w$  is an absolute minimizer and vanishes on the boundary, according to the definition of absolute minimizers,

$$\text{esssup}_{x \in V} (|Dw|^2 - w) \leq 0.$$

So

$$w \geq 0 \quad \text{in } V.$$

Therefore,

$$w \equiv 0.$$

□

Next lemma says that graphs of continuum value functions can not contain flat pieces.

**Lemma 2.3** *Suppose that  $u$  is a viscosity subsolution of Eq. (1.1) with  $\tau = 1$ . Then there does not exist a nonempty open set  $V$  such that*

$$u \equiv \text{constant} \quad \text{in } V.$$

*Proof* We argue by contradiction. Suppose that there exists such  $V$ . Choose a point  $x_0 \in V$ . Then the quadratic polynomial

$$P(x) = u(x_0) + \frac{1}{4}|x - x_0|^2$$

touches  $u$  at  $x_0$  from the above in  $V$ . Since  $DP(x_0) = 0$ , owing to the definition of viscosity subsolutions of Eq. (2.1), we should have that

$$\max_{\{|\xi|=1\}} \xi \cdot D^2P(x_0) \cdot \xi \geq 1.$$

However,  $\max_{\{|\xi|=1\}} \xi \cdot D^2P(x_0) \cdot \xi = \frac{1}{2}$ . This is a contradiction. Therefore our lemma holds.  $\square$

*Proof of Theorem 1.1* Step I: (Existence of an absolute minimizer). We may assume that  $g \leq 0$ . Now let us consider a new Hamiltonian

$$\hat{H}(p, z) = \frac{1}{2}|p|^2 - z^-,$$

where  $z^- = \min\{z, 0\}$ . Clearly,  $\hat{H} \geq \frac{1}{2}|p|^2$ . So, the existence of an absolute minimizer for  $\hat{H}$  with boundary value  $g$  follows from the usual  $L^p$  approximation. See for instance [6]. Suppose that  $w$  is an absolute minimizer for  $\hat{H}$  with boundary value  $g \leq 0$ . I want to show that  $w$  is also an absolute minimizer for  $H$ . In fact, assume that  $V$  is an open subset of  $\Omega$  and  $f \in W^{1,\infty}(V)$  such that

$$f = w \quad \text{on } \partial V.$$

We need to prove that

$$\text{esssup}_V \left( \frac{1}{2}|Dw|^2 - w \right) \leq \text{esssup}_V \left( \frac{1}{2}|Df|^2 - f \right). \tag{2.5}$$

First I claim that  $w \leq 0$ . We argue by contradiction. If not, since  $w|_{\partial\Omega} \leq 0$ , then there exists an open subset  $U \subset \bar{U} \subset \Omega$  such that

$$w > 0 \quad \text{in } U$$

and

$$w = 0 \quad \text{on } \partial U.$$

Since  $w$  is an absolute minimizer for  $\hat{H}$ ,

$$\text{esssup}_U \hat{H}(Dw, w) \leq \text{esssup}_U \hat{H}(0, 0) = 0.$$

Since  $\hat{H} \geq \frac{1}{2}|p|^2$ , we get that

$$|Dw| = 0 \quad \text{a.e. in } U.$$

Accordingly,  $w \equiv 0$  in  $U$ . This is a contradiction. Therefore our claim holds, i.e,  $w \leq 0$  in  $\Omega$ . Hence

$$f \leq 0 \quad \text{on } \partial V.$$

Therefore,

$$\text{esssup}_V \left( \frac{1}{2}|Df|^2 - f \right) \geq 0.$$

Hence

$$\text{esssup}_V \left( \frac{1}{2}|Df|^2 - f \right) \geq \text{esssup}_V \left( \frac{1}{2}|Df^-|^2 - f^- \right). \tag{2.6}$$



Note that

$$\frac{1}{2}|Df^-|^2 - f^- = \hat{H}(Df^-, f^-).$$

Since  $w$  is an absolute minimizer for  $\hat{H}$  and  $w = f = f^-$  on  $\partial V$ , we have that

$$\text{esssup}_V \hat{H}(Dw, w) \leq \text{esssup}_V \hat{H}(Df^-, f^-) = \text{esssup}_V \left( \frac{1}{2}|Df^-|^2 - f^- \right). \tag{2.7}$$

Since  $w \leq 0$ ,  $w = w^-$ . Hence

$$\frac{1}{2}|Dw|^2 - w = \hat{H}(Dw, w) \quad \text{a.e in } \Omega. \tag{2.8}$$

Combining (2.6)–(2.8), (2.5) holds. So  $w$  is indeed an absolute minimizer for  $H = \frac{1}{2}|p|^2 - z$ .

Step II: Next we show that an absolute minimizer is the maximal viscosity solution of Eq. (1.3). Assume that  $w$  is an absolute minimizer and  $u$  is an arbitrary viscosity subsolution of Eq. (1.3). Our goal is to prove that

$$w \geq u \quad \text{in } \bar{\Omega}. \tag{2.9}$$

By considering super-convolution of  $u$  and routine modifications, we may assume that  $u$  is semiconvex. If (2.9) does not hold, there must exist  $x_0 \in \Omega$  such that

$$u(x_0) - w(x_0) = \max_{\Omega} (u - w) > 0.$$

According to Lemma 2.1, there exists  $r > 0$  such that  $\bar{B}_r(x_0) \subset \Omega$  and

$$u \equiv u(x_0) \quad \text{in } B_r(x_0).$$

We say that  $\mathcal{O}$  is an *admissible open subset of  $\Omega$*  if  $x_0 \in \mathcal{O}$  and

$$u \equiv u(x_0) \quad \text{in } \mathcal{O}.$$

Denote

$$V = \cup \{ \mathcal{O} \text{ is an admissible open subset of } \Omega \} \mathcal{O}.$$

Note that  $V$  is not empty since  $B_r(x_0) \subset V$ . I claim that for  $y \in \partial V$ ,

$$w(y) > w(x_0).$$

Owing to the choice of  $x_0$ , it is clear that  $w(y) \geq w(x_0)$ . If  $y \in \partial\Omega$ , it is easy to see that  $w(y) = u(y) = u(x_0) > w(x_0)$ . If  $y \in \Omega$  and  $w(y) = w(x_0)$ , then

$$u(y) - w(y) = u(x_0) - w(x_0) = \max_{\Omega} (u - w) > 0.$$

By Lemma 2.1, there exists  $r' > 0$  such that

$$u \equiv u(y) = u(x_0) \quad B_{r'}(y).$$

Hence  $B_{r'}(y) \cup V$  is an admissible open subset of  $\Omega$ . By the definition of  $V$ , we have that  $B_{r'}(y) \subset V$ . This contradicts to  $y \in \partial V$ . Hence our claim holds. Accordingly, there must exist a  $\delta > 0$  and an open subset  $V' \subset \bar{V}' \subset V$  such that  $x_0 \in V'$ ,

$$w(x) < w(x_0) + \delta \quad \text{in } V' \tag{2.10}$$

and

$$w(x) = w(x_0) + \delta \quad \text{on } \partial V'.$$

Hence by Lemma 2.2,  $w \equiv w(x_0) + \delta$  in  $V'$ . This contradicts to (2.10). Therefore (2.9) holds.

Step III: Finally, we need to show that the continuum value function from the ‘‘tug-of-war’’ game is the minimal viscosity solution of equation. Suppose  $u$  is a viscosity subsolution of Eq. (1.1) and  $v$  is an arbitrary viscosity solution of Eq. (1.3). We need to show that

$$v \geq u \quad \text{in } \Omega. \tag{2.11}$$

By super-convolution and routine modifications, we may assume that  $u$  is semiconvex. We argue by contradiction. If (2.11) is not true, owing to Lemma 2.1, there must exist a nonempty open subset  $V$  of  $\Omega$  such that

$$u \equiv c \quad \text{in } V.$$

This is impossible according to Lemma 2.3. Hence (2.11) holds. □

The following theorem provides an alternative way to see why the continuum value function is the minimal solution.

**Theorem 2.4** *Any viscosity solution  $u$  of Eq. (1.3) is a viscosity supersolution of Eq. (1.1) with  $\tau = 1$ .*

*Proof* We argue by contradiction. If not, then there exists  $x_0 \in \Omega$  and  $\phi \in C^2(\Omega)$  such that

$$\phi(x) - u(x) < \phi(x_0) - u(x_0) = 0 \quad \text{for } x \in \Omega \setminus \{x_0\} \tag{2.12}$$

and

$$\min_{\{|p|=1\}} p \cdot D^2\phi(x_0) \cdot p > 1.$$

Hence the least eigenvalue of the Hessian matrix  $D^2\phi(x_0)$  must be larger than 1. Therefore

$$D^2\phi(x_0) > I_n, \tag{2.13}$$

where  $I_n$  is the  $n \times n$  identity matrix. Since  $u$  is a viscosity solution of (1.3), we have that

$$D\phi(x_0) \cdot D^2\phi(x_0) \cdot D\phi(x_0) \leq |D\phi(x_0)|^2.$$

By (2.13),  $D\phi(x_0) = 0$ . Also owing to (2.13), there exists  $\delta > 0$  such that

$$D\phi(x) \neq 0 \quad \text{for } x \in B_\delta(x_0) \setminus \{x_0\}. \tag{2.14}$$

For  $h \in B_r(0)$ , we choose  $x_h \in \Omega_r$  such that

$$\phi(x_h + h) - u(x_h) = \max_{\Omega_r} (\phi(x + h) - u(x)).$$

According to (2.12), it is easy to see that when  $r$  is small,  $x_h$  will be close to  $x_0$ . Hence when  $r$  is small enough, we have that

$$D^2\phi(x_h + h) > I_n. \tag{2.15}$$

Since  $u$  is a viscosity solution of (1.3), we have that

$$D\phi(x_h + h) \cdot D^2\phi(x_h + h) \cdot D\phi(x_h + h) \leq |D\phi(x_h + h)|^2.$$

According to (2.15),

$$D\phi(x_h + h) = 0.$$

By (2.14), when  $r$  is sufficiently small, we must have that

$$x_h + h = x_0.$$

Hence due to the choice of  $x_h$ ,

$$\phi(x_0) - u(x_0 - h) \geq \phi(x_0 + h) - u(x_0).$$

So

$$\phi(x_0 + h) + \phi(x_0 - h) \leq \phi(x_0 + h) + u(x_0 - h) \leq 2\phi(x_0).$$

This contradicts to (2.13) when  $h$  is small. Hence our claim holds. □

### 3 Other solutions of equation (1.3)

In this section, we will give a characterization of graphes of *intermediate solutions*, i.e. those solutions between the absolute minimizer and the continuum value function. Before stating the theorem, we define some terminologies.

We say that the graph of a function  $f \in C(\bar{\Omega})$  has a *well* if there exists a open set  $V \subset \Omega$  such that

$$\min_{\bar{V}} f < \min_{\partial V} f.$$

We say that the graph of  $f \in C(\bar{\Omega})$  has a *flat piece* if  $f$  is constant in some open subset of  $\Omega$ .

**Theorem 3.1** *Suppose that  $u$  is a viscosity solution of Eq. (1.3). Then*

- (i)  $u$  is not the absolute minimizer if and only its graph has wells.
- (ii)  $u$  is not the value function if and only if its graph has flat pieces.

*Especially,  $u$  is an intermediate solution if and only if its graph has both wells and flat pieces.*

*Proof* (i) Note that in Step II of the proof of Theorem 1.1, we only use the fact that the graph of an absolute minimizer has no well. Hence (i) holds.

(ii) The sufficiency part of (ii) is Lemma 2.3. Hence we only need to prove the necessity part. Assume that  $v$  is the viscosity solution of Eq. (1.1) with  $\tau = 1$ . Suppose that  $u \neq v$ . We are going to show that there exists a open set  $U \subset \Omega$  such that  $u$  is constant in  $U$ . Since  $u \neq v$ , we have that

$$\max_{\Omega} (u - v) > 0.$$

Hence there must exist  $\delta > 0$  such that for  $h \in B_{\delta}(0)$

$$\{x \in \bar{\Omega}_{\delta} \mid u(x + h) - v(x) = \max_{\bar{\Omega}_{\delta}} (u(\cdot + h) - v)\} \subset \Omega_{3\delta}.$$

Now fix  $\delta$ . For  $\epsilon > 0$ , we denote  $u_{\epsilon}$  as the super-convolution of  $u$ , i.e.,

$$u_{\epsilon}(x) = \max_{y \in \bar{\Omega}} \left( u(y) - \frac{1}{\epsilon} |x - y|^2 \right).$$

It is clear that when  $\epsilon$  is small enough,  $u_\epsilon$  is a viscosity subsolution of Eq. (1.3) in  $\Omega_{\frac{\delta}{2}}$  and for  $h \in B_{\frac{\delta}{4}}(0)$

$$\{x \in \bar{\Omega}_\delta \mid u_\epsilon(x+h) - v(x) = \max_{\Omega_\delta}(u_\epsilon(\cdot+h) - v)\} \subset \Omega_{2\delta}.$$

Note that  $u_\epsilon$  is semiconvex. Choose  $x_\epsilon \in \Omega_\delta$  such that

$$u_\epsilon(x_\epsilon) - v(x_\epsilon) = \max_{\Omega_\delta}(u_\epsilon - v).$$

Owing to (2.4),

$$u_\epsilon \equiv u_\epsilon(x_\epsilon) \quad \text{in } B_{\frac{\delta}{4}}(x_\epsilon).$$

Passing to a subsequence if necessary, we may assume that

$$\lim_{\epsilon \rightarrow 0} x_\epsilon = x_0 \in \bar{\Omega}_\delta.$$

Then

$$u \equiv u(x_0) \quad \text{in } B_{\frac{\delta}{4}}(x_0).$$

□

*Remark 3.2* Equation in Example 1 actually possesses infinitely many intermediate solutions. This motivates us to ask the following two questions which we will investigate in the future.

- Q1 Is it true that there are infinitely many intermediate solutions if the absolute minimizer and the value function do not coincide?  
 Q2 How to interpret those intermediate solutions?

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