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Construction of Weak Mirror Pairs by Deformations

A Dissertation submitted in partial satisfaction
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by

Brian John Rolle

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To my wife, Michelle.

ABSTRACT OF THE DISSERTATION

Construction of Weak Mirror Pairs by Deformations

by

Brian John Rolle

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2011
Dr. Yat Sun Poon, Chairperson

The central idea in weak mirror symmetry is relating a complex manifold and a symplectic manifold by comparing their induced differential Gerstenhaber algebras (DGAs). If they are quasi-isomorphic, we say the complex and symplectic manifold form a weak mirror pair. Complex and symplectic manifold live in a larger category of generalized complex manifolds. We can use the deformation theory of generalized complex geometry to deform some complex manifolds into symplectic manifolds. It is then natural to ask when the undeformed object and the deformed one form a weak mirror pair. This thesis provides a sufficient condition for when a complex manifold can be deformed to form a weak mirror pair. We also use this condition to determine when complex symplectic algebras can be deformed to provide a weak mirror pair.

Contents

Chapter 1

Introduction

Homological mirror symmetry came about as tool in string theory. It relates field theories on two Calabi-Yau manifolds by matching the symplectic structure on one with the complex structure on the other. This led mathematicians to examine this pairing in general, relating symplectic manifold to complex manifolds by a quasi-isomorphism. This is known as weak mirror symmetry, as developed by Merkulov in [13]. Weak mirror symmetry has been studied in many different cases. Most famously, there is the SYZ conjecture, relating mirror symmetry to T-duality in [15]. Using this idea, Cleyton, Lauret and Poon studied weak mirror symmetry on Lie algebras in [3]. Cleyton, Ovando and Poon also studied mirror symmetry by way of T-duality for complex symplectic algebras in [4]. Additionally, Jian Zhou examined weak mirror symmetry for various algebras in [16] and with Cao in [2].

Traditionally, symplectic manifolds and complex manifolds have been viewed as unrelated objects, even though they have a few similarities. However Hitchin introduced the notion of a generalized complex structures in [8], and his student Gualtieri developed them further in his thesis [6]. These structure contain all complex manifolds and all symplectic manifolds as examples. Thus, they provided a frame work for relating these two objects.

Generalized complex structures have a deformation theory, and so there is a simple way to turn some complex manifolds into symplectic manifolds. The central result of this thesis is a classification of when these deformations will yield weak mirror pairs, Theorem 17.

In chapter 2, we present background information on generalized complex structures, differential Gerstenhaber algebras, and define the notion of a weak mirror pair. Lie bi-algebroids and the Courant bracket are used extensively. Next, in chapter 3, we examine certain deformations of generalized complex structures. We see which deformations can turn a complex structure into a symplectic structure. We also develop a chain of isomorphisms that, if it exists, will make these complex and symplectic structures into weak mirror pairs. Only one of the links in this chain can fail. Theorem 16 give the sufficient conditions for this link to hold, the existence of what we call a compatible pair. Then in chapter 4, we examine when a certain class of examples, complex symplectic algebras, have compatible pairs. These algebras have been studied by Andrada in [1] and Cleyton, Ovando and Poon in [4]. Thus we apply the work of the previous chapter to say when complex symplectic algebras can be deformed to yield a mirror pair in Theorem 29, and give these pairs an explicit formulation. We also look at a few low dimensional examples. In chapter 5 we lay some preliminary ground work for future work on principal tori bundles. Lastly, in chapter 6 we present some preliminary ideas in how the framework developed here could be used on other classes of examples.

Chapter 2

Preliminary Material

2.1 Generalized Complex Structures

The following section draws heavily on the material in [6].

Let M be a manifold of real dimension $2n$, with tangent bundle TM and cotangent bundle T^*M. On $(TM \oplus T^*M)_{\mathbb{C}}$ we define the following symmetric inner product:

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\eta(X) + \xi(Y)) \tag{2.1}$$

where X and Y are vector fields and ξ and η are one-forms. The signature of this operation is $(2n, 2n)$. We also define the Courant bracket, first present by Courant in [5]:

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(\eta(X) - \xi(Y)), \tag{2.2}$$

where $[X, Y]$ is the usual Lie bracket of the vector fields X and Y and $\mathcal{L}_X\eta$ is the Lie derivative of η with respect to X.

We define a *generalized complex structure* on M in two different ways.

Let $L < (TM \oplus T^*M)_{\mathbb{C}}$ be a sub-bundle of the tangent plus cotangent bundle. We say that L is isotropic if $\langle l_1, l_2 \rangle = 0$ for all $l_1, l_2 \in L$. L is maximally isotropic if it is isotropic and not properly contained in any isotropic sub-bundle. If $L \cap \overline{L} = \{0\}$

and L is maximally isotropic, then L is an almost generalized complex structure. The integrability condition is that $C^\infty(L)$ is closed under the Courant bracket. Therefore if $[l_1, l_2] \in C^\infty(L)$ for all $l_1, l_2 \in C^\infty(L)$, then L is a generalized complex structure.

We also define generalized complex structures in a different way. Let $J : TM \oplus T^*M \to TM \oplus T^*M$. If J is a smooth bundle map with $J \circ J = -Id$ and $J^* = -J$, then J is an *almost generalized complex structure*. We say J is an *generalized complex structure* is its Nijenhuis tensor vanishes for all smooth sections of $TM \oplus T^*M$. That is, for any $l_1, l_2 \in C^\infty(TM \oplus T^*M)$, we have

$$N_J(l_1, l_2) = [l_1, l_2] - [Jl_1, Jl_2] + J([Jl_1, l_2] + [l_1, Jl_2]) = 0 \tag{2.3}$$

We can view J as being made up of several parts based on the fact that is exchanges some vectors for one-forms, while mapping other vectors to vectors.

$$J = \begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix}$$

where $A : TM \to TM$ and $A^* : T^*M \to T^*M$ is the adjoint of A defined by $(A^*\eta)(X) = \eta(AX)$ for a vector field X and a one-form η. Also $B : TM \to T^*M$ and $\beta : T^*M \to TM$, with $B^* = -B$ and $\beta^* = -\beta$. This means we can view B as a two-form.

These two definitions are equivalent. Given a J as above, we notice it has eigenvalues $\pm i$. Set L to be the $+i$ eigenspace of J in $(TM \oplus T^*M)_{\mathbb{C}}$. Then L will be totally real and maximally isotropic. The condition given by (2.3) is equivalent to the closure of $C^\infty(L)$ under the Courant bracket. This is analogous to the case in complex geometry with an almost complex structure and the bundle $T^{1,0}$.

Any complex manifold and any symplectic manifold are also examples of generalized complex manifolds. Let M a complex manifold with complex structure $\mathcal{J}$. Then the generalized complex structure is given by

$$J_{\mathcal{J}} = \begin{pmatrix} \mathcal{J} & 0 \\ 0 & -\mathcal{J}^* \end{pmatrix},$$

with $L_{\mathcal{J}} = T^{1,0} \oplus T^{*(0,1)}$. The integrability of this generalized complex structure is equivalent to the integrability of the complex structure $\mathcal{J}$.

Let M be a symplectic manifold with symplectic form ω. We can view the two-form ω as a map $\omega : TM \to T^*M$, by $\omega(X)(Y) = \omega(X,Y)$. Then the generalized complex structure is given by

$$J_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix},$$

with $L_\omega = \{X - i\omega(X) : X \in TM_{\mathbb{C}}\}$. In this case the integrability conditions for L_ω is equivalent to $d\omega = 0$.

We can view generalized complex structures as a bridge between complex and symplectic structures. Let $\rho : (TM \oplus T^*M)_{\mathbb{C}} \to TM_{\mathbb{C}}$ by $\rho(X + \xi) = X$. We call this map ρ the anchor map. Note that at a point $p \in M$, $\rho(L_p) \subset T_pM_{\mathbb{C}}$ is a subspace of $T_pM_{\mathbb{C}}$. We define the type of L at a point p to be the complex dimension of $T_pM_{\mathbb{C}}$ minus the complex dimension of $\rho(L)$ at p. The type may change from point to point over a manifold. If the manifold has real dimension $2n$, the type of a generalized complex structure on it is an integer between 0 and n. As we can see from the examples above, a complex manifold of real dimension $2n$ has type n, and a symplectic manifold has type 0.

The type of a generalized complex structure can change from point to point. However, it can only change by an even number. Further, each point has a neighborhood where the type does not increase. A point that has a neighborhood where the type is constant is called *regular*.

As an example, let $M = \mathbb{C}^2$ with coordinates z_1, z_2. The coordinate tangent vectors will be denoted by ∂_{z_i}, and the one-forms by dz_i. Define
$L = \text{span}\{z_1\partial_{z_1} - dz_2, z_1\partial_{z_2} + dz_1, \partial_{\bar{z}_1}, \partial_{\bar{z}_2}\}$. Then L is maximally isotropic. Also L is involutive as the Courant bracket on these elements is trivial. If $z_1 \neq 0$, then $\rho(L) = \text{span}\{\partial_{z_1}, \partial_{z_2}, \partial_{\bar{z}_1}, \partial_{\bar{z}_2}\} = TM_{\mathbb{C}}$ and so when $z_1 \neq 0$, L has type 0. On the plane $z_1 = 0$, $\rho(L) = \text{span}\{\partial_{\bar{z}_1}, \partial_{\bar{z}_2}\} = T^{0,1}$, and so L has type 2. In fact, when $z_1 = 0$ this is the conjugate of the standard complex structure. However, when $z_1 \neq 0$, even though the

generalized complex structure is type 0, it does not arise directly from a symplectic structure. So we see here that the type jumps by 2 on the hyperplane $z_1 = 0$. The regular points are the ones with $z_1 \neq 0$.

Let B be a two-form on M. We define the B *field transformation* as a map $e^B : (TM \oplus T^*M)_{\mathbb{C}} \to (TM \oplus T^*M)_{\mathbb{C}}$ by $e^B(X+\xi) = X + \xi + B(X)$. Note that $e^B([X+\xi, Y+\eta]) = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(\eta(X) - \xi(Y)) + B([X,Y])$, and

$$\begin{aligned}[e^B(X+\xi), e^B(Y+\eta)] &= [X+\xi+B(X), Y+\eta+B(Y)] \\ &= [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \tfrac{1}{2}d(\eta(X)-\xi(Y)) \\ &+\mathcal{L}_X B(Y) - \mathcal{L}_Y B(X) - \tfrac{1}{2}d(B(Y,X) - B(X,Y)).\end{aligned}$$

These will be equal if

$$B([X,Y]) = \mathcal{L}_X B(Y) - \mathcal{L}_Y B(X) - \frac{1}{2}d(B(Y,X) - B(X,Y)). \tag{2.4}$$

However the above equation is equivalent to the closure of B. Therefore, if L is a generalized complex structure and B is closed, then $e^B([l_1, l_2]) = [e^B(l_1), e^B(l_2)]$ for any $l_1, l_2 \in C^\infty(L)$. This means that $e^B L = \{X+\xi+B(X) : X+\xi \in L\}$ is also a generalized complex structure. Further, $e^B(L)$ and L have the same type at each point in M, since $\rho(e^B(X+\xi)) = \rho(X+\xi+B(X)) = X = \rho(X+\xi)$.

In complex geometry we have the Newlander-Nirenburg theorem that describes the local structure of complex manifolds, and in symplectic geometry we have Darboux's theorem for describing the local structure of symplectic manifolds. We have an analogous theorem for generalized complex structures.

Theorem 1 (Generalized Darboux Theorem) *Let M be a manifold with real dimension $2n$. Let $p \in M$ be a regular point of type k. Then there is a neighborhood U of p and a closed two-form B such that $e^B(L|_U) \cong \mathbb{C}^k \times (\mathbb{R}^{2n-2k}, \omega_k)$ where $\mathbb{R}^{2n-2k}$ has coordinates $\{x_1, y_1, \ldots, x_{n-k}, y_{n-k}\}$ and $\omega_k = dx_1 \wedge dy_1 + \cdots + dx_{n-k} \wedge dy_{n-k}$ and $\mathbb{C}^k$ has the standard complex structure.*

This theorem only describes the local structure. However if the generalized complex structure has type 0, we have a global result.

Proposition 2 *Let L be a generalized complex structure on a manifold M. If L has type 0, then L is the B-field transform of a symplectic generalized complex structure.*

Proof. If L has type 0, then for all $X \in TM_{\mathbb{C}}$, there is a unique $\xi \in T^*M_{\mathbb{C}}$ such that $X + \xi \in L$. We prove uniqueness by assuming $X + \xi^1, X + \xi^2 \in L$. Then $(X + \xi^1) - (X + \xi^2) = (\xi^1 - \xi^2) \in L$. L is isotropic, so for any $Y + \eta \in L$, $0 = 2\langle Y + \eta, \xi^1 - \xi^2\rangle = (\xi^1 - \xi^2)(Y)$, which implies $\xi^1 = \xi^2$.

Then $L = \{X + \xi_X : X \in TM_{\mathbb{C}}\}$, where $\xi_X \in T^*M_{\mathbb{C}}$ depends on X. We define $\epsilon : TM_{\mathbb{C}} \to T^*M_{\mathbb{C}}$ by $\epsilon(X) = \xi_X$. Note that if X is a vector field on M, then ξ_X is a one-form on M, and so ϵ is smooth.

If $X, Y \in TM_{\mathbb{C}}$, then there exist $\xi_X, \xi_Y \in T^*M_{\mathbb{C}}$ such that $X + \xi_X, Y + \xi_Y \in L$. Since $(X + \xi_X) + (Y + \xi_Y) = (X + Y) + (\xi_X + \xi_Y)$, $\epsilon(X + Y) = \xi_X + \xi_Y$, and so ϵ is linear.

Since L is isotropic, $0 = 2\langle X + \xi_X, Y + \xi_Y\rangle = \xi_Y(X) + \xi_X(Y)$
$= \epsilon(Y, X) + \epsilon(X, Y)$, so ϵ is skew. Therefore ϵ is a complex two-form on M.

As above with the B-field in equation (2.4), the condition that ϵ is closed is equivalent to L being integrable, which is equivalent to $C^\infty(L)$ being closed under the Courant bracket.

We set $\mathrm{Re}(\epsilon) = B$ and $\mathrm{Im}(\epsilon) = -\omega$, so $\epsilon = B - i\omega$, where B and ω are real two-forms. Each is closed, since ϵ is closed.

Lastly we show ω is non-degenerate. For any real vector $X \in TM$, $X + B(X) - i\omega(X) \in L$ and $X + B(X) + i\omega(X) \in \overline{L}$. If $\omega(X) = 0$, then $X + B(X) \in L \cap \overline{L} = \{0\}$, and so $X = 0$. So ω is a symplectic form.

Therefore $L = \{X - i\omega(X) + B(X) : X \in TM_{\mathbb{C}}\} = e^B L_\omega$. ■

2.2 Lie Bialgebroids

The theory of Lie bialgebriods is developed in [12]. Let $L < (TM \oplus T^*M)_{\mathbb{C}}$ be an isotropic sub-bundle. Let $\rho : L \to TM_{\mathbb{C}}$ be the *anchor map* defined above. If X is a vector and ξ is a one-form with $X + \xi \in L$, then $\rho(X + \xi) = X$.

Let $[-,-] : C^\infty(L) \times C^\infty(L) \to C^\infty(L)$ be a bilinear map over $\mathbb{R}$. The pair $(L, [-,-])$ is a *Lie algebroid* if the following conditions hold for all $l_1, l_2, l_3 \in C^\infty(L)$ and smooth functions f.

$$[l_1, l_2] = -[l_2, l_1]$$

$$[l_1, fl_2] = f[l_1, l_2] + (\rho(l_1)f)l_2$$

$$\rho([l_1, l_2]) = [\rho(l_1), \rho(l_2)]$$

$$[l_1, [l_2, l_3]] + [l_2, [l_3, l_1]] + [l_3, [l_1, l_2]] = 0$$

The last equation is the Jacobi identity. Note that if L is a generalized complex structure and the bracket is the Courant bracket, then these conditions will be satisfied, even though the Courant bracket does not usually satisfy the Jacobi identity on $(TM \oplus T^*M)_{\mathbb{C}}$. We call any subbundle that is maximally isotropic with respect to (2.1) whose smooth sections are closed under the Courant bracket a *Dirac structure.*

Let L^* be the dual of L. We define the *differential* $\delta : C^\infty(\bigwedge^n L^*) \to C^\infty(\bigwedge^{n+1} L^*)$ as follows. If $l_1, \ldots, l_{n+1} \in C^\infty(L)$ and $\sigma \in C^\infty(\bigwedge^n L^*)$, then

$$\begin{aligned} &\delta\sigma(l_1, \ldots, l_{n+1}) \\ &= \sum_{r=1}^{n+1} (-1)^{r+1} \rho(l_r) \left(\sigma(l_1, \ldots, \widehat{l_r}, \ldots, l_{n+1}) \right) \\ &+ \sum_{r<s} (-1)^{r+s} \sigma([l_r, l_s], l_1, \ldots, \widehat{l_r}, \ldots, \widehat{l_s}, \ldots, l_{n+1}). \end{aligned} \tag{2.5}$$

where $\widehat{l_r}$ denotes that l_r is omitted.

We also define the Lie derivative $\mathcal{L}$. If $l \in C^\infty(L)$, then $\mathcal{L}_l : C^\infty(\bigwedge^n L^*) \to C^\infty(\bigwedge^n L^*)$. If $\sigma \in C^\infty(\bigwedge^n L^*)$ and $l_1, \ldots, l_n \in C^\infty(L)$, then

$$(\mathcal{L}_l\sigma)(l_1, \ldots, l_n) = \rho(l)\left(\sigma(l_1, \ldots, l_n)\right) - \sum_{r=1}^{n} \sigma\left(l_1, \ldots, [l, l_r], \ldots, l_n\right).$$

Next we define the interior product ι. If $l \in C^\infty(L)$, then $\iota_l : C^\infty(\bigwedge^{n+1} L^*) \to C^\infty(\bigwedge^n L^*)$. If $\sigma \in C^\infty\left(\bigwedge^{n+1} L^*\right)$ and $l_1, \ldots, l_n \in C^\infty(L)$, then

$$(\iota_l\sigma)(l_1, \ldots, l_n) = \sigma(l, l_1, \ldots, l_n).$$

The above operations satisfy the following relationships for all smooth functions f and smooth sections l, l_i of L.

Proposition 3 *Basic formula for Lie algebroid calculus.*

1. $\iota_l(f\sigma) = f\iota_l\sigma$;
2. $\iota_{fl}\sigma = f\iota_l(\sigma)$;
3. $\mathcal{L}_l(f\sigma) = f\mathcal{L}_l\sigma + (\rho(l)f)\,\sigma$;
4. $\mathcal{L}_{[l_1,l_2]} = \mathcal{L}_{l_1} \circ \mathcal{L}_{l_2} - \mathcal{L}_{l_2} \circ \mathcal{L}_{l_1}$;
5. $\mathcal{L}_l = \iota_l \circ \delta + \delta \circ \iota_l$;
6. $\mathcal{L}_l \circ \delta = \delta \circ \mathcal{L}_l$;
7. $\mathcal{L}_{l_1} \circ \iota_{l_2} - \iota_{l_2} \circ \mathcal{L}_{l_1} = \iota_{[l_1,l_2]}$;
8. $\left(\mathcal{L}_{fl}\sigma\right)(l_1, \ldots, l_n) = f\left(\mathcal{L}_l\sigma\right)(l_1, \ldots, l_n) - \sum_{r=1}^{n}(-1)^r\left(\rho(l_r)f\right)(\iota_l\sigma)(l_1, \ldots, \widehat{l_r}, \ldots, l_n)$.

Let L and K be two Dirac structures with $L \oplus K = (TM \oplus T^*M)_{\mathbb{C}}$. Since $K \cap L = \{0\}$, we can identify K with L^* by $k(l) = 2\langle l, k\rangle$. Then L is identified with K^* as well. With this we can define a differential for K by

$$(\delta_K l)(k_1, k_2) = 2\big(\rho(k_1)\langle l, k_2\rangle - \rho(k_2)\langle l, k_1\rangle - \langle l, [k_1, k_2]\rangle\big). \tag{2.6}$$

Note that $\delta_K : C^\infty(\bigwedge^n K^*) \to C^\infty(\bigwedge^{n+1} K^*)$, so δ_K acts on L. We are most interested in the case where $K = \overline{L}$. Then we use ∂ to denote $\delta_{\overline{L}}$ and $\overline{\partial}$ to denote δ_L. In this case, when $l \in L$ and $\bar{l} \in \overline{L}$, we define,

$$\bar{l}(l) = 2\langle l, \bar{l}\rangle = l(\bar{l}) \tag{2.7}$$

We will frequently use the following computational lemma in later sections.

Lemma 4 *Let $l \in C^\infty(L)$, $\Lambda \in C^\infty(\bigwedge^2 L)$ and $\bar{l}_1, \bar{l}_2 \in C^\infty(\overline{L})$, then*

$$[\Lambda, l](\bar{l}_1, \bar{l}_2) = -\rho(l)(\bar{l}_1(\Lambda(\bar{l}_2))) - \bar{l}_1([\Lambda(\bar{l}_2), l]) + \bar{l}_2([\Lambda(\bar{l}_1), l]). \tag{2.8}$$

Proof.

By the linearity of $[-,-]$ it is sufficient to prove the claim when $\Lambda = l_1 \wedge l_2$. We have that $(l_1 \wedge l_2)(\bar{l}) = ((l_1(\bar{l}))l_2 - (l_2(\bar{l}))l_1)$. Then the left side of equation (2.8) is

$$[l_1 \wedge l_2, l](\bar{l}_1, \bar{l}_2) = (l_1 \wedge [l_2, l])(\bar{l}_1, \bar{l}_2) - (l_2 \wedge [l_1, l])(\bar{l}_1, \bar{l}_2)$$
$$= l_1(\bar{l}_1)[l_2, l](\bar{l}_2) - l_1(\bar{l}_2)[l_2, l](\bar{l}_1) - l_2(\bar{l}_1)[l_1, l](\bar{l}_2) + l_2(\bar{l}_2)[l_1, l](\bar{l}_1)$$

The right side of equation (2.8) is

$$\begin{aligned}
& -\rho(l)(\bar{l}_1((l_1 \wedge l_2)(\bar{l}_2))) - \bar{l}_1([(l_1 \wedge l_2)(\bar{l}_2), l]) + \bar{l}_2([(l_1 \wedge l_2)(\bar{l}_1), l]) \\
= \; & -\rho(l)((l_1(\bar{l}_2))(\bar{l}_1(l_2))) + \rho(l)((l_2(\bar{l}_2))(\bar{l}_1(l_1))) - \bar{l}_1([(l_1(\bar{l}_2))l_2, l]) + \bar{l}_1([(l_2(\bar{l}_2))l_1, l]) \\
+ \; & \bar{l}_2([(l_1(\bar{l}_1))l_2, l]) - \bar{l}_2([(l_2(\bar{l}_1))l_1, l]) \\
= \; & -\rho(l)(l_1(\bar{l}_2))\bar{l}_1(l_2) - l_1(\bar{l}_2)\rho(l)(\bar{l}_1(l_2)) + \rho(l)(l_2(\bar{l}_2))\bar{l}_1(l_1) + l_2(\bar{l}_2)\rho(l)(\bar{l}_1(l_1)) \\
- \; & \bar{l}_1((l_1(\bar{l}_2))[l_2, l]) + \bar{l}_1(\rho(l)(l_1(\bar{l}_2))l_2) + \bar{l}_1((l_2(\bar{l}_2))[l_1, l]) - \bar{l}_1(\rho(l)(l_2(\bar{l}_2))l_1) \\
+ \; & \bar{l}_2((l_1(\bar{l}_1))[l_2, l]) - \bar{l}_2(\rho(l)(l_1(\bar{l}_1))l_2) - \bar{l}_2((l_2(\bar{l}_1))[l_1, l]) + \bar{l}_2(\rho(l)(l_2(\bar{l}_1))l_1) \\
= \; & -\bar{l}_1((l_1(\bar{l}_2))[l_2, l]) + \bar{l}_1((l_2(\bar{l}_2))[l_1, l]) + \bar{l}_2((l_1(\bar{l}_1))[l_2, l]) - \bar{l}_2((l_2(\bar{l}_1))[l_1, l]).
\end{aligned}$$

So the left side and the right side are equal. ■

2.3 Differential Gerstenhaber Algebras

The definitions in this section are also given by Poon in [14] for the Schouten bracket. $\left(\bigwedge L, [\quad,\quad], \wedge, \overline{\partial}\right)$ is a differential Gerstenhaber algebra if the following axioms hold for all $l_1 \in C^\infty(\bigwedge^a L)$, $l_2 \in C^\infty(\bigwedge^b L)$ and $l_3 \in C^\infty(\bigwedge^c L)$.

$$[L^i, L^j] \subset L^{i+j-1} \tag{2.9}$$

$$[l_1, l_2] = (-1)^{ab+a+b}[l_2, l_1] \tag{2.10}$$

$$[l_1, [l_2, l_3]] = [[l_1, l_2], l_3] - (-1)^{ab+a+b}[l_2, [l_1, l_3]] \tag{2.11}$$

$$L^i \wedge L^j \subset L^{i+j} \tag{2.12}$$

$$l_1 \wedge l_2 = (-1)^{ab} l_2 \wedge l_1 \tag{2.13}$$

$$[l_1 \wedge l_2, l_3] = l_1 \wedge [l_2, l_3] + (-1)^{ab} l_2 \wedge [l_1, l_3] \tag{2.14}$$

$$\overline{\partial} L^i \subset L^{i+1} \tag{2.15}$$

$$\overline{\partial} \circ \overline{\partial} = 0 \tag{2.16}$$

$$\overline{\partial}[l_1, l_2] = [\overline{\partial} l_1, l_2] - (-1)^a [l_1, \overline{\partial} l_2] \tag{2.17}$$

$$\overline{\partial}(l_1 \wedge l_2) = (\overline{\partial} l_1) \wedge l_2 + (-1)^a l_1 \wedge (\overline{\partial} l_2) \tag{2.18}$$

If instead only (2.12), (2.13), (2.15), (2.16) and (2.18), we say $\left(\bigwedge L, \wedge, \overline{\partial}\right)$ is a differential exterior algebra. If only (2.9), (2.10), (2.11), (2.12), (2.13) and (2.14) hold, we have a Gerstenhaber algebra. We include some explicit forms for the differential in the complex and symplectic case.

Proposition 5 *If M is a complex manifold, let $L = T^{1,0} \oplus T^{*(0,1)}$. If $\{X_1, \ldots, X_n\}$ is a local frame for $T^{1,0}$ and $\{\xi^1, \ldots, \xi^n\}$ is the dual local frame for $T^{*(1,0)}$, then the differential $\overline{\partial} : L \to \bigwedge^2 L$ can be expressed as*

$$\overline{\partial} Y = \textstyle\sum_{j=1}^{n} [Y, \overline{X}_j]^{1,0} \wedge \overline{\xi}^j \tag{2.19}$$

$$\overline{\partial} \overline{\eta} = (d\overline{\eta})^{0,2}. \tag{2.20}$$

Proof. Equation (2.6) shows that $\overline{\partial}Y(\overline{Z}_1, \overline{Z}_2) = 0$ and $\overline{\partial}Y(\alpha_1, \alpha_2) = 0$ for any $\overline{Z}_1, \overline{Z}_2 \in C^\infty(T^{0,1})$ and $\alpha_1, \alpha_2 \in C^\infty(T^{*(1,0)})$. So $\overline{\partial}Y \in C^\infty(T^{1,0} \wedge T^{*(0,1)})$ and we contract with a $(1,0)$ form β and a $(0,1)$ vector $\overline{Z}$.

$$\begin{aligned}
\left(\textstyle\sum_{j=1}^n [Y, \overline{X}_j]^{1,0} \wedge \overline{\xi}^j\right)(\beta, \overline{Z}) &= \textstyle\sum_{j=1}^n \beta([Y, \overline{X}_j]^{1,0})\overline{\xi}^j(\overline{Z}) \\
= \textstyle\sum_{j=1}^n \beta(\overline{\xi}^j(\overline{Z})[Y, \overline{X}_j]) &= \textstyle\sum_{j=1}^n \beta([Y, \overline{\xi}^j(\overline{Z})\overline{X}_j] - (Y(\overline{\xi}^j(\overline{Z})))\overline{X}_j) \\
= \textstyle\sum_{j=1}^n \beta([Y, \overline{\xi}^j(\overline{Z})\overline{X}_j]) &= \beta([Y, \textstyle\sum_{j=1}^n \overline{\xi}^j(\overline{Z})\overline{X}_j]) = \beta([Y, \overline{Z}]) \\
= -\overline{Z}(\beta(Y)) + [\overline{Z}, \beta](Y) &= 2(\rho(\beta)\langle Y, \overline{Z}\rangle - \rho(\overline{Z})\langle Y, \beta\rangle - \langle Y, [\beta, \overline{Z}]\rangle) = \overline{\partial}Y(\beta, \overline{Z})
\end{aligned}$$

This proves equation (2.19). Equation (2.6) shows that $\overline{\partial}\overline{\eta}(\overline{Z}, \alpha_1) = 0$ and $\overline{\partial}Y(\alpha_1, \alpha_2) = 0$ for any $\overline{Z} \in C^\infty(T^{0,1})$ and $\alpha_1, \alpha_2 \in C^\infty(T^{*(1,0)})$. So $\overline{\partial}\overline{\eta} \in C^\infty(T^{*(0,1)} \wedge T^{*(0,1)})$ and we contract with $\overline{Z}_1, \overline{Z}_2 \in C^\infty(T^{0,1})$.

$$\begin{aligned}
(d\overline{\eta})^{0,2}(\overline{Z}_1, \overline{Z}_2) &= d\overline{\eta}(\overline{Z}_1, \overline{Z}_2) = \overline{Z}_1\overline{\eta}(\overline{Z}_2) - \overline{Z}_2\overline{\eta}(\overline{Z}_1) - \overline{\eta}([\overline{Z}_1, \overline{Z}_2]) \\
&= 2(\rho(\overline{Z}_1)\langle\overline{\eta}, \overline{Z}_2\rangle - \rho(\overline{Z}_2)\langle\overline{\eta}, \overline{Z}_1\rangle - \langle\overline{\eta}, [\overline{Z}_1, \overline{Z}_2]\rangle) = \overline{\partial}\overline{\eta}(\overline{Z}_1, \overline{Z}_2)
\end{aligned}$$

This proves equation (2.20). ∎

Proposition 6 *If M is a symplectic manifold with symplectic form ω, let $L = \{X - i\omega(X) : X \in TM\}$. Then the differential $\overline{\partial} : L \to \bigwedge^2 L$ can be expressed as*

$$\overline{\partial}(X - i\omega(X)) = -2id(\omega(X)).$$

Proof. Let $Y_1 + i\omega(Y_1), Y_2 + i\omega(Y_2) \in C^\infty(\overline{L})$.

$$\overline{\partial}(X - i\omega(X))(Y_1 + i\omega(Y_1), Y_2 + i\omega(Y_2))$$

$$= 2(\rho(Y_1 + i\omega(Y_1))\langle X - i\omega(X), Y_2 + i\omega(Y_2)$$

$$-\rho(Y_2 + i\omega(Y_2))\langle X - i\omega(X), Y_1 + i\omega(Y_1)\rangle$$

$$-\langle X - i\omega(X), [Y_1 + i\omega(Y_1), Y_2 + i\omega(Y_2)]\rangle)$$

$$= Y_1(-i\omega(X, Y_2) + i\omega(Y_2, X)) - Y_2(-i\omega(X, Y_1) + i\omega(Y_1, X))$$

$$-2\langle X - i\omega(X), [Y_1, Y_2] + i\mathcal{L}_{Y_1}\omega(Y_2) - i\mathcal{L}_{Y_2}\omega(Y_1) - \tfrac{1}{2}d(i\omega(Y_2, Y_1) - i\omega(Y_1, Y_2))\rangle$$

$$= -2iY_1(\omega(X, Y_2)) + 2iY_2(i\omega(X, Y_1)) + i\omega(X, [Y_1, Y_2])$$

$$-i(\mathcal{L}_{Y_1}\omega(Y_2))X + i(\mathcal{L}_{Y_2}\omega(Y_1))X - iX\omega(Y_1, Y_2)$$

$$= -2iY_1(\omega(X, Y_2)) + 2iY_2(i\omega(X, Y_1)) + 2i\omega(X, [Y_1, Y_2])$$

$$= -2id(\omega(X))(Y_1, Y_2) = -2id(\omega(X))(Y_1 + i\omega(Y_1), Y_2 + i\omega(Y_2))$$

Where $\omega(X, [Y_1, Y_2]) = -(\mathcal{L}_{Y_1}\omega(Y_2))X + (\mathcal{L}_{Y_2}\omega(Y_1))X - X\omega(Y_1, Y_2)$ since ω is closed. ■

2.4 Weak Mirror Pairs

Introductory material in weak mirror symmetry can be found in [14]. Given a differential Gerstenhaber algebra $\left(\bigwedge L, [\ \ ,\ \ ], \wedge, \overline{\partial}\right)$, we can define $\mathcal{Z}^n = \{A \in \bigwedge^n L : \overline{\partial} A = 0\}$ and $\mathcal{B}^n = \{A \in \bigwedge^n L : A = \overline{\partial} B, B \in \bigwedge L\}$. The axioms for a DGA imply that $\bigoplus_n \mathcal{Z}^n$ and $\bigoplus_n \mathcal{B}^n$ are Gerstenhaber algebras under the $[-,-]$ and $\wedge$ operations. Since $\overline{\partial} \circ \overline{\partial} = 0$, we also have that $\mathcal{B}^n \subset \mathcal{Z}^n$.

Let $\mathcal{H}^n = \mathcal{Z}^n / \mathcal{B}^n = \ker \overline{\partial} / \operatorname{Im} \overline{\partial}$. Let $\mathcal{H} = \bigoplus_n \mathcal{H}^n$. Then $(\mathcal{H}, [-,-], \wedge)$ is the *Gerstenhaber algebra* of $\left(\bigwedge L, [\ \ ,\ \ ], \wedge, \overline{\partial}\right)$ with the inherited $[-,-]$ and $\wedge$ operations.

Let L_1 and L_2 be generalized complex structures with associated Gerstenhaber algebras $(\mathcal{H}_1, [-,-], \wedge)$ and $(\mathcal{H}_2, [-,-], \wedge)$, respectively. Let $\Phi : \left(\bigwedge L_1, [\ \ ,\ \ ], \wedge, \overline{\partial}_1\right) \to \left(\bigwedge L_2, [\ \ ,\ \ ], \wedge, \overline{\partial}_2\right)$ be a DGA homomorphism. Then there is an induces homomorphism $\tilde{\Phi} : (\mathcal{H}_1, [-,-], \wedge) \to (\mathcal{H}_2, [-,-], \wedge)$. If $\tilde{\Phi}$ is an isomorphism, we say Φ is a *quasi-isomorphism* and that the DGA's are *quasi-isomorphic*. In the case where L_1 is a complex structure and L_2 is a symplectic structure, we say $\left(\bigwedge L_1, [\ \ ,\ \ ], \wedge, \overline{\partial}_1\right)$ and $\left(\bigwedge L_2, [\ \ ,\ \ ], \wedge, \overline{\partial}_2\right)$ form a *weak mirror pair*.

Chapter 3

Weak Mirror Pairs by Deformations

In this chapter, we outline how we can construct weak mirror pairs by deforming generalized complex structures. First we will examine deformations of generalized complex structures, and see how they can change type. Then we will examine the DGA's of these deformed structures, and examine isomorphisms between the deformed and undeformed structures and present a theorem stating when they will be isomorphic.

3.1 Deformation of Generalized Complex Structures

Let L be a generalized complex structure on a manifold M. Recall that its conjugate $\overline{L}$ is also its dual. Let $\overline{\Gamma} \in C^\infty(\bigwedge^2 \overline{L})$. Then $\overline{\Gamma} : L \to \overline{L}$. We will use $\overline{\Gamma}$ to deform L to $L_{\overline{\Gamma}} = \{l + \overline{\Gamma}(l) : l \in L\}$.

We need to examine whether $L_{\overline{\Gamma}}$ is even a generalized complex structure. First we will see if it is isotropic. Let $l_1, l_2 \in L$. Then $l_1 + \overline{\Gamma}(l_1), l_2 + \overline{\Gamma}(l_2) \in L_{\overline{\Gamma}}$, and

$$\langle l_1 + \overline{\Gamma}(l_1), l_2 + \overline{\Gamma}(l_2)\rangle = \langle l_1, l_2\rangle + \langle l_1, \overline{\Gamma}(l_2)\rangle + \langle \overline{\Gamma}(l_1), l_2\rangle + \langle \overline{\Gamma}(l_1), \overline{\Gamma}(l_2)\rangle$$
$$= \langle l_1, \overline{\Gamma}(l_2)\rangle + \langle \overline{\Gamma}(l_1), l_2\rangle = \tfrac{1}{2}(l_1(\overline{\Gamma}(l_2)) + \overline{\Gamma}(l_1)(l_2)) = \tfrac{1}{2}(\overline{\Gamma}(l_2, l_1) + \overline{\Gamma}(l_1, l_2)) = 0,$$

where we use equation (2.7) and the fact that $\overline{\Gamma}$ is skew, as well as the fact that L and $\overline{L}$ are isotropic. So $L_{\overline{\Gamma}}$ is isotropic.

The maximality of $L_{\overline{\Gamma}}$ follow from the maximality of L. Suppose $L_{\overline{\Gamma}}$ is a proper subset of an isotropic bundle K. Then L is a proper subset $K_{-\overline{\Gamma}}$ and $K_{-\overline{\Gamma}}$ is still isotropic. This contradicts the maximality of L. Therefore $L_{\overline{\Gamma}}$ is maximally isotropic.

We also need $L_{\overline{\Gamma}} \cap \overline{L_{\overline{\Gamma}}} = 0$. Note that $\overline{L_{\overline{\Gamma}}} = \overline{\{l + \overline{\Gamma}(l) : l \in L\}}$
$= \{\bar{l} + \Gamma(\bar{l}) : \bar{l} \in \overline{L}\} = \overline{L}_\Gamma$. Suppose $l_1, l_2 \in L$ are non-zero and $l_1 + \overline{\Gamma}(l_1) = \bar{l}_2 + \Gamma(l_2) \in \overline{L}_\Gamma$. Then we have that $l_1 = \Gamma(\bar{l}_2)$ and $\bar{l}_2 = \overline{\Gamma}(l_1)$, or $\Gamma(\overline{\Gamma}(l_1)) = l_1$, or $\Gamma\overline{\Gamma}$ has a non-trivial fixed point. This gives us our first non-trivial condition for $L_{\overline{\Gamma}}$ to be a generalized complex structure.

Proposition 7 *Let $\Gamma : \overline{L} \to L$ and its conjugate $\overline{\Gamma} : L \to \overline{L}$, then $\Gamma\overline{\Gamma} : L \to L$. If $\Gamma\overline{\Gamma}$ has no non-trivial fixed points, then $L_{\overline{\Gamma}} \cap \overline{L_{\overline{\Gamma}}} = 0$.*

If $\Gamma\overline{\Gamma} = 0$, then it will have no non-trivial fixed points. In practice, this will always be satisfied when we are trying to build weak mirror pairs, due to our choice of Γ.

The last condition we have to satisfy is that $L_{\overline{\Gamma}}$ is involutive, or
$[C^\infty(L_{\overline{\Gamma}}), C^\infty(L_{\overline{\Gamma}})] \subset C^\infty(L_{\overline{\Gamma}})$. For this we refer to Theorem 6.1 in [12], where 3.1 is Maurer-Cartan equation.

Theorem 8 *Let L be a generalized complex structure. Then $L_{\overline{\Gamma}}$ is involutive if and only if*

$$\partial\overline{\Gamma} + \frac{1}{2}[\overline{\Gamma}, \overline{\Gamma}] = 0 \tag{3.1}$$

We prove a special case of this theorem in Proposition (19).

Now we have can state the following theorem.

Theorem 9 *Let L be a generalized complex structure and $\overline{\Gamma} \in C^\infty(\bigwedge^2 \overline{L})$. If $\Gamma\overline{\Gamma} = 0$ and $\partial\overline{\Gamma} + \frac{1}{2}[\overline{\Gamma}, \overline{\Gamma}] = 0$, then $L_{\overline{\Gamma}} = \{l + \overline{\Gamma}(l) : l \in L\}$ is a generalized complex structure.*

The theorem stated above is for arbitrary deformations of arbitrary generalized complex structures. However, our goal is to build weak mirror pairs. To do this, we will start with a complex manifold M of real dimension $2n$. Our generalized complex structure will be $L = T^{1,0} \oplus T^{*(0,1)}$, which has type n. Our goal is to deform to a symplectic structure, which will have type 0. This leads to some restrictions on our choice of $\overline{\Gamma}$ as well. Since $\overline{L} = T^{0,1} \oplus T^{*(1,0)}$, we can view $\overline{\Gamma} : T^{1,0} \oplus T^{*(0,1)} \to T^{0,1} \oplus T^{*(1,0)}$ as $\overline{\Gamma} = \overline{\Gamma}_1 + \overline{\Gamma}_2 + \overline{\Gamma}_3 + \overline{\Gamma}_4$, with

$$\overline{\Gamma}_1 : T^{1,0} \to T^{0,1}$$

$$\overline{\Gamma}_2 : T^{(1,0)} \to T^{*(1,0)}$$

$$\overline{\Gamma}_3 : T^{*(0,1)} \to T^{0,1}$$

$$\overline{\Gamma}_4 : T^{*(0,1)} \to T^{*(1,0)}$$

In order to decrease the type of L, we need to add in vectors that are in $T^{0,1}$. This means only using $\overline{\Gamma}_1$ and $\overline{\Gamma}_3$. So we can set $\overline{\Gamma}_2 = \overline{\Gamma}_4 = 0$. As for $\overline{\Gamma}_1$ we note that $\rho(L) = T^{1,0}$ and $\rho(L_{\overline{\Gamma}_1}) = \{Z + \overline{\Gamma}_1(Z) | Z \in T^{1,0}\}$. Since these spaces have the same dimension, L and $L_{\overline{\Gamma}_1}$ have the same type. So we also set $\overline{\Gamma}_1 = 0$.

However, $L_{\overline{\Gamma}_3} = T^{1,0} \oplus \{\overline{\xi} + \Gamma(\overline{\xi}) | \overline{\xi} \in T^{*(0,1)}\}$, and so $\rho(L_{\overline{\Gamma}_3})$ $= T^{1,0} \oplus \{\overline{\Gamma}(\overline{\xi}) | \overline{\xi} \in T^{*(0,1)}\}$. If $\Gamma(\overline{\xi})$ is non-zero, then it is a (0,1) vector, so $\overline{\Gamma}_3$ can decrease type. Also, $\Gamma_3 : T^{*(1,0)} \to T^{1,0}$, meaning that $\Gamma_3 \overline{\Gamma}_3 = 0$ trivially. So $\overline{\Gamma}_3$ is the only part of $\overline{\Gamma}$ we want. We shall change notation and set $\overline{\Gamma} = \overline{\Gamma}_3 = \overline{\Lambda}$ where $\overline{\Lambda} \in C^\infty(T^{0,1} \wedge T^{0,1})$.

We can now update theorem 8. Before we do though, note that $\partial \overline{\Lambda} \in C^\infty(T^{0,1} \wedge T^{0,1} \wedge T^{1,0})$ when we use the ∂ for a complex structure defined in the Proposition 5, and $[\overline{\Lambda}, \overline{\Lambda}] \in C^\infty(T^{0,1} \wedge T^{0,1} \wedge T^{0,1})$. Therefore $\partial \overline{\Lambda} + \frac{1}{2}[\overline{\Lambda}, \overline{\Lambda}] = 0$ if and only if $\partial \overline{\Lambda} = 0$ and $[\overline{\Lambda}, \overline{\Lambda}] = 0$. So theorem 8 is now as follows.

Theorem 10 *Let M be a complex manifold and $L = T^{1,0} \oplus T^{*(0,1)}$ and let $\overline{\Lambda} \in C^\infty(T^{0,1} \wedge T^{0,1})$. If $\partial \overline{\Lambda} = 0$ and $[\overline{\Lambda}, \overline{\Lambda}] = 0$, then $L_{\overline{\Lambda}} = \{X + \overline{\xi} + \overline{\Lambda}(\overline{\xi}) : X \in T^{1,0}, \overline{\xi} \in T^{*(0,1)}\}$ is a generalized complex structure.*

Now we examine how the $\overline{\Lambda}$ changes the type of L. Note that $\rho(L_{\overline{\Lambda}}) = T^{1,0} \oplus \{\overline{\Lambda}(\overline{\xi}) | \overline{\xi} \in T^{*(0,1)}\}$. Since $\overline{\Lambda} : T^{*(0,1)} \to T^{0,1}$ is skew, it has even rank. So the complex dimension of $\{\overline{\Lambda}(\overline{\xi}) | \overline{\xi} \in T^{*(0,1)}\}$ is even. This means that the $\overline{\Lambda}$ changes the type of L by an even number.

If we want to deform from complex to symplectic, we need the complex dimension of our complex manifold to be even. Also, we need $\overline{\Lambda} : T^{*(0,1)} \to T^{0,1}$ to be non-degenerate. Under these conditions we have the following theorem.

Theorem 11 *Let M be a complex manifold of real dimension $4n$ and let $\overline{\Lambda} \in C^\infty(T^{0,1} \wedge T^{0,1})$ be non-degenerate as a map $\overline{\Lambda} : T^{*(0,1)} \to T^{0,1}$. Then $L = T^{1,0} \oplus T^{*(0,1)}$ has type $2n$. If $\partial\overline{\Lambda} = 0$ and $[\overline{\Lambda}, \overline{\Lambda}] = 0$, then $L_{\overline{\Lambda}} = \{X + \overline{\xi} + \overline{\Lambda}(\overline{\xi}) : X \in T^{1,0}, \overline{\xi} \in T^{*(0,1)}\}$ is a generalized complex structure of type 0.*

This is almost what we want. Every type 0 generalized complex structure is the B-field transformation of a symplectic structure, as proved in Proposition 2. Using this, we now have a theorem describing when we can deform from a complex structure to a symplectic structure.

Theorem 12 *Let M be a complex manifold of real dimension $4n$ and let $\overline{\Lambda} \in C^\infty(T^{0,1} \wedge T^{0,1})$ be non-degenerate as a map $\overline{\Lambda} : T^{*(0,1)} \to T^{0,1}$. Let $L = T^{1,0} \oplus T^{*(0,1)}$. If $\partial\overline{\Lambda} = 0$ and $[\overline{\Lambda}, \overline{\Lambda}] = 0$, then there is a closed two-form B on M such that $e^B L_{\overline{\Lambda}}$ is a symplectic structure.*

3.2 B-field Isomorphism

Now, we show that the DGA of a generalized complex structure L is isomorphic to the DGA of any B-field transformation $e^B L$ by a closed two-form B.

Define the map $e^B : L \to e^B L$ by $e^B(X+\xi) = X + \xi + B(X)$ where X is a vector field and ξ is a one-form. We extend this map to $e^B : \bigwedge L \to \bigwedge L$ by $e^B(l_1 \wedge \cdots \wedge l_n) = (e^B l_1) \wedge \cdots \wedge (e^B l_n)$. By the discussion of B-field transformation in section (2.1), we know that $e^B([l_1, l_2]) = [e^B l_1, e^B l_2]$ if and only if B is closed.

Note that $(e^B L)^* = e^B \overline{L}$, since B is a real form. In order for $(\bigwedge L, [-,-], \wedge, \overline{\delta})$ to be isomorphic to $(\bigwedge e^B L, [-,-], \wedge, \overline{\delta}_B)$, we need the following diagram to commute.

$$\begin{array}{ccc} \wedge^n L & \xrightarrow{\overline{\delta}} & \wedge^{n+1} L \\ e^B \downarrow & & \downarrow e^B \\ \wedge^n e^B L & \xrightarrow{\overline{\delta}_B} & \wedge^{n+1} e^B L \end{array} .$$

This means we need $e^B \overline{\delta} = \overline{\delta}_B e^B$. The right-hand side is

$$\begin{aligned}
&\overline{\delta}(X+\xi+B(X))(\overline{Y}_1+\overline{\eta}^1+B(\overline{Y}_1), \overline{Y}_2+\overline{\eta}^2+B(\overline{Y}_2)) \\
&= 2(\overline{Y}_1\langle X+\xi+B(X), \overline{Y}_2+\overline{\eta}^2+B(\overline{Y}_2)\rangle \\
&-\overline{Y}_2\langle X+\xi+B(X), \overline{Y}_1+\overline{\eta}^1+B(\overline{Y}_1)\rangle \\
&-\langle X+\xi+B(X), [\overline{Y}_1+\overline{\eta}^1+B(\overline{Y}_1), \overline{Y}_2+\overline{\eta}^2+B(\overline{Y}_2)]\rangle) \\
&= 2(\overline{Y}_1\langle X+\xi, \overline{Y}_2+\overline{\eta}^2\rangle + \overline{Y}_1(B(\overline{Y}_2, X)+B(X, \overline{Y}_2)) \\
&-\overline{Y}_2\langle X+\xi, \overline{Y}_1+\overline{\eta}^1\rangle - \overline{Y}_2(B(\overline{Y}_1, X)+B(X, \overline{Y}_1)) \\
&-\langle X+\xi+B(X), [\overline{Y}_1+\overline{\eta}^1, \overline{Y}_2+\overline{\eta}^2]+B([Y_1, Y_2])\rangle) \\
&= 2(\overline{Y}_1\langle X+\xi, \overline{Y}_2+\overline{\eta}^2\rangle - \overline{Y}_2\langle X+\xi, \overline{Y}_1+\overline{\eta}^1\rangle - \langle X+\xi, [\overline{Y}_1+\overline{\eta}^1, \overline{Y}_2+\overline{\eta}^2]\rangle \\
&-(B([\overline{Y}_1, \overline{Y}_2], X)+B(X, [\overline{Y}_1, \overline{Y}_2]))) \\
&= \overline{\delta}(X+\xi)(\overline{Y}_1+\overline{\eta}^1, \overline{Y}_2+\overline{\eta}^2).
\end{aligned}$$

We have repeatedly used the fact that B is skew, and $e^B([l_1, l_2]) = [e^B l_1, e^B l_2]$. So we see that $\overline{\delta}(e^B l)(e^B \bar{l}_1, e^B \bar{l}_2) = \overline{\delta} l(\bar{l}_1, \bar{l}_2)$. Also note that

$$(e^B l)(e^B \overline{k}) = 2\langle X + \xi + B(X), \overline{Y} + \overline{\eta} + B(\overline{Y})\rangle$$
$$= 2\langle X + \xi, \overline{Y} + \overline{\eta}\rangle + B(\overline{Y}, X) + B(X, \overline{Y}) = l(\overline{k}),$$

and so,

$$(e^B(l_1 \wedge l_2))(e^B \overline{k}_1, e^B \overline{k}_2) = ((e^B l_1) \wedge (e^B l_2))(e^B \overline{k}_1, e^B \overline{k}_2)$$
$$= ((e^B l_1)(e^B \overline{k}_1))((e^B l_2)(e^B \overline{k}_2)) - ((e^B l_1)(e^B \overline{k}_2))((e^B l_2)(e^B \overline{k}_1))$$
$$= l_1(\overline{k}_1) l_2(\overline{k}_2) - l_1(\overline{k}_2) l_2(\overline{k}_1) = (l_1 \wedge l_2)(\overline{k}_1, \overline{k}_2).$$

Since e^B is linear and $\overline{\delta} l \in C^\infty(\bigwedge^2 L)$, we have $(e^B(\overline{\delta} l))(e^B \bar{l}_1, e^B \bar{l}_2) = \delta l(\bar{l}_1, \bar{l}_2)$ $= (\overline{\delta}_B(e^B l))(e^B \bar{l}_1, e^B \bar{l}_2)$, and the diagram above commutes. Since we can extend this to higher powers by induction, we have that $(\bigwedge L, [-,-], \wedge, \overline{\delta})$ and $(\bigwedge e^B L, [-,-], \wedge, \overline{\delta}_B)$ are isomorphic.

3.3 Deformed Lie Bialgebroids

As we saw in section 2.2, the pair $(L, [-,-], \wedge, \overline{\partial})$ and $(\overline{L}, [-,-], \wedge, \partial)$ form a Lie bialgebroid for any generalized complex structure L. We can also form a Lie bialgebroid out of $L_{\overline{\Gamma}}$ and $\overline{L}_{\Gamma}$ when these are generalized complex structures, that is when $\overline{\partial}\Gamma + \frac{1}{2}[\Gamma, \Gamma] = 0$. The differentials will be given by equation (2.5).

Our goal in this section is to build an isomorphism from the DGA defined by $L_{\overline{\Gamma}}$ with its natural differential to one defined by L with a different differential. This will let us change the problem from one about deformed spaces to one about deformed differentials.

Let $\Gamma \in C^{\infty}(\bigwedge^2 L)$. Define $A_{\Gamma} : L \oplus \overline{L} \to L \oplus \overline{L}$ by $A_{\Gamma}(l) = l + \overline{\Gamma}(l)$ for $l \in L$ and $A_{\Gamma}(\bar{l}) = \bar{l} + \Gamma(\bar{l})$ for $\bar{l} \in \overline{L}$. Then $A_{\Gamma}(L) = L_{\overline{\Gamma}}$ and $A_{\Gamma}(\overline{L}) = \overline{L}_{\Gamma}$. Let $\overline{\delta}$ be the differential for $L_{\overline{\Gamma}}$ defined by formula (2.5) using $(L_{\overline{\Gamma}})^* = \overline{L}_{\Gamma}$.

Theorem 13 *If* $\overline{\partial}\Gamma + \frac{1}{2}[\Gamma, \Gamma] = 0$*, then* $A_{\Gamma} : (\bigwedge^* L, [-,-], \wedge, \overline{\partial}_{\Gamma}) \to (\bigwedge^* L_{\overline{\Gamma}}, [-,-], \wedge, \overline{\delta})$ *is an isomorphism, where*

$$\overline{\partial}_{\Gamma} l = \overline{\partial} l + [\Gamma, l].$$

Proof. We are extending A_{Γ} to the wedge product by $A_{\Gamma}(l_1 \wedge l_2) = (A_{\Gamma} l_1) \wedge (A_{\Gamma} l_2)$.

For the bracket we need $[l_1 + \overline{\Gamma}(l_1), l_2 + \overline{\Gamma}(l_2)] = l_3 + \overline{\Gamma}(l_3)$ for some $l_3 \in L$. This is a equivalent to $\overline{\partial}\Gamma + \frac{1}{2}[\Gamma, \Gamma] = 0$, as in Theorem (8).

In order for $(\bigwedge^* L, [-,-], \wedge, \overline{\partial}_{\Gamma})$ to be a differential Gerstenhaber algebra, we need to show that $\overline{\partial}_{\Gamma}$ conditions (2.16), (2.17) and (2.18). That is

$$\overline{\partial}_{\Gamma} \circ \overline{\partial}_{\Gamma} = 0,$$

$$\overline{\partial}_{\Gamma}[l_1, l_2] = [\overline{\partial}_{\Gamma} l_1, l_2] - (-1)^a [l_1, \overline{\partial}_{\Gamma} l_2],$$

$$\overline{\partial}_{\Gamma}(l_1 \wedge l_2) = (\overline{\partial}_{\Gamma} l_1) \wedge l_2 + (-1)^a l_1 \wedge (\overline{\partial}_{\Gamma} l_2),$$

where $l_1 \in C^{\infty}(\bigwedge^a L)$. To prove that $\overline{\partial}_{\Gamma} \circ \overline{\partial}_{\Gamma} = 0$, we first note that $[[\Gamma, \Gamma], l] = 2[\Gamma, [\Gamma, l]]$

by the Jacobi identity, equation (2.11). So

$$\overline{\partial}_\Gamma \overline{\partial}_\Gamma l = \overline{\partial}_\Gamma(\overline{\partial} l + [\Gamma, l]) = \overline{\partial}\overline{\partial} l + \overline{\partial}[\Gamma, l] + [\Gamma, \overline{\partial} l] + [\Gamma, [\Gamma, l]]$$
$$= [\overline{\partial}\Gamma, l] - [\Gamma, \overline{\partial} l] + [\Gamma, \overline{\partial} l] + \tfrac{1}{2}[[\Gamma, \Gamma], l] = [\overline{\partial}\Gamma + \tfrac{1}{2}[\Gamma, \Gamma], l].$$

Therefore $\overline{\partial}_\Gamma \overline{\partial}_\Gamma l = 0$ if $\overline{\partial}\Gamma + \frac{1}{2}[\Gamma, \Gamma] = 0$. The other two conditions are proved by straightforward computation.

The last thing we need to prove is that $\overline{\delta} \circ A_\Gamma = A_\Gamma \circ \overline{\partial}_\Gamma$, or that the diagram below commutes.

$$\begin{array}{ccc} \bigwedge^n L & \xrightarrow{\overline{\partial}_\Gamma} & \bigwedge^{n+1} L \\ A_\Gamma \downarrow & & \downarrow A_\Gamma \\ \bigwedge^n L_\Gamma & \xrightarrow{\overline{\delta}} & \bigwedge^{n+1} L_\Gamma \end{array} .$$

This is true by theorem 6.1 in [12]. A special case of this is proved in Proposition 20. Therefore A_Γ is an isomorphism of differential Gerstenhaber algebras. ∎

3.4 Construction of Weak Mirror Pairs

On a complex manifold M with real dimension $4n$, let $L = T^{1,0} \oplus T^{*(0,1)}$, let $\overline{\Lambda} \in C^\infty(T^{0,1} \wedge T^{0,1})$ be non-degenerate. Then $L_{\overline{\Lambda}}$ is a generalized complex structure of type 0. We want to find conditions that will guarantee that the DGA's defined by L and $L_{\overline{\Lambda}}$ are isomorphic. Since every isomorphism is a quasi-ismorphism, we will have a weak mirror pair.

Our goal in this section is to build an isomorphism from $(\bigwedge L, [-,-], \wedge, \overline{\partial}_\Lambda)$ to $(\bigwedge L, [-,-], \wedge, \overline{\partial})$. Since $(\bigwedge L, [-,-], \wedge, \overline{\partial}_\Lambda)$ is isomorphic to $(\bigwedge L_{\overline{\Lambda}}, [-,-], \wedge, \overline{\delta})$, we will have an isomorphism from the complex $(\bigwedge L, [-,-], \wedge, \overline{\partial})$ to $(\bigwedge L_{\overline{\Lambda}}, [-,-], \wedge, \overline{\delta})$, which will be the B-field transform of a symplectic structure if $\overline{\Lambda}$ satisfies the conditions in Theorem 12. We will now derive conditions for $(\bigwedge L, [-,-], \wedge, \overline{\partial}_\Lambda)$ and $(\bigwedge L, [-,-], \wedge, \overline{\partial})$ to be isomorphic. We will assume $\overline{\partial}\Lambda = 0$ and $[\Lambda, \Lambda] = 0$, where $\Lambda \in C^\infty(T^{1,0} \wedge T^{1,0})$ so that $L_{\overline{\Lambda}}$ is a generalized complex structure and $\overline{\partial}_\Lambda$ is a differential, and that A_Λ is an isomorphism of DGA's as in Theorem 13. Let $\Phi : L \to L$ be an isomorphism. We will extend Φ to the wedge product by

$$\Phi(l_1 \wedge l_2) = \Phi(l_1) \wedge \Phi(l_2). \tag{3.2}$$

We assume Φ is a DGA isomorphism, then

$$[\Phi(l_1), \Phi(l_2)] = \Phi([l_1, l_2]). \tag{3.3}$$

Also, the following diagram will commute.

$$\begin{array}{ccc} \wedge^n L & \xrightarrow{\overline{\partial}_\Lambda} & \wedge^{n+1} L \\ \Phi \downarrow & & \downarrow \Phi \\ \wedge^n L & \xrightarrow{\overline{\partial}} & \wedge^{n+1} L \end{array} .$$

This means that $\overline{\partial}_\Lambda = \Phi^{-1}\overline{\partial}\Phi$, or that for any $A \in \bigwedge^* L$,

$$\overline{\partial}A + [\Lambda, A] = \Phi^{-1}(\overline{\partial}(\Phi(A))). \tag{3.4}$$

To better understand this equation, we will consider a family of deformations, $t\Lambda$. Then, for all t, $t\Lambda$ satisfies $[t\Lambda, t\Lambda] = t^2[\Lambda, \Lambda] = 0$ and $\overline{\partial}(t\Lambda) = t\overline{\partial}\Lambda = 0$. Therefore $t\Lambda$ defines an integrable deformation for all t. For each t, there will be a different Φ_t for the above equation. Equation(3.4) becomes,

$$\overline{\partial}A + [t\Lambda, A] = \Phi_t^{-1}(\overline{\partial}(\Phi_t(A))). \tag{3.5}$$

When $t = 0$, the deformation is trivial and $\overline{\partial}_{0\Lambda} = \overline{\partial}$, so $\Phi_0 = Id$. When we differentiate equation (3.5) with respect to t, we get,

$$[\Lambda, A] = \frac{d\Phi_t^{-1}}{dt}(\overline{\partial}(\Phi_t(A))) + \Phi_t^{-1}\left(\overline{\partial}\left(\frac{d\Phi_t}{dt}(A)\right)\right). \tag{3.6}$$

We define $\phi := \left.\frac{d\Phi_t}{dt}\right|_{t=0}$, then $\left.\frac{d\Phi_t^{-1}}{dt}\right|_{t=0} = -\phi$. Evaluating equation (3.6) at $t = 0$ yields

$$[\Lambda, A] = -\phi(\overline{\partial}(A)) + \overline{\partial}(\phi(A)). \tag{3.7}$$

Differentiating equation (3.2) yields,

$$\frac{d\Phi_t}{dt}(l_1 \wedge l_2) = \frac{d}{dt}(\Phi_t(l_1) \wedge \Phi_t(l_2)) = \left(\frac{d\Phi_t}{dt}(l_1)\right) \wedge \Phi_t(l_2) + \Phi_t(l_1) \wedge \left(\frac{d\Phi_t}{dt}(l_2)\right).$$

Evaluating this at $t = 0$ yields,

$$\phi(l_1 \wedge l_2) = (\phi(l_1)) \wedge l_2 + l_1 \wedge (\phi(l_2)). \tag{3.8}$$

A similar computation on equation (3.3) yields

$$\phi([l_1, l_2]) = [\phi(l_1), l_2] + [l_1, \phi(l_2)] \tag{3.9}$$

Obviously ϕ depends on the choice of Λ, and so we make the following definition.

Definition 14 *Let $L = T^{1,0} \oplus T^{*(0,1)}$ and let $\Lambda \in C^\infty(T^{1,0} \wedge T^{1,0})$ and $\phi : L \to L$ be a vector bundle homomorphism. We call Λ and ϕ a* compatible pair *if the following* *conditions hold for all $l, l_1, l_2 \in C^\infty(L)$:*

$$\overline{\partial}\Lambda = 0 \qquad \text{and} \qquad [\Lambda, \Lambda] = 0 \tag{3.10}$$

$$\phi(l_1 \wedge l_2) = (\phi(l_1)) \wedge l_2 + l_1 \wedge (\phi(l_2)) \tag{3.11}$$

$$\phi([l_1, l_2]) = [\phi(l_1), l_2] + [l_1, \phi(l_2)] \tag{3.12}$$

$$\overline{\partial}(\phi(l)) - \phi(\overline{\partial}(l)) = [\Lambda, l] \tag{3.13}$$

Theorem 15 *Let $\Phi : (\bigwedge L, [-,-], \wedge, \overline{\partial}_\Lambda) \to (\bigwedge L, [-,-], \wedge, \overline{\partial})$ be a DGA isomorphism with $\overline{\partial}\Lambda = 0$ and $[\Lambda, \Lambda] = 0$. Then there exists a $\phi : L \to L$ such that ϕ and Λ are a compatible pair.*

Our goal now is to prove the converse of this theorem. If Λ and ϕ are a compatible pair, we would like to be able to find an isomorphism Φ. Before we do that, we will investigate ϕ, much in the same way we looked at $\overline{\Gamma}$, and reduced it to a $(0, 2)$ field.

Since $\phi : T^{1,0} \oplus T^{*(0,1)} \to T^{1,0} \oplus T^{*(0,1)}$, we have $\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4$, where

$$\phi_1 : T^{1,0} \to T^{1,0}$$

$$\phi_2 : T^{1,0} \to T^{*(0,1)}$$

$$\phi_3 : T^{*(0,1)} \to T^{1,0}$$

$$\phi_4 : T^{*(0,1)} \to T^{*(0,1)}.$$

We now check which of these maps works with equation (3.13). Since $l \in C^\infty(T^{1,0} \oplus T^{*(0,1)})$, we can examine when $l = X \in C^\infty(T^{1,0})$ and $l = \overline{\xi} \in C^\infty(T^{*(0,1)})$. We note that $[\Lambda, X] \in C^\infty(\bigwedge^2 T^{1,0})$ and $[\Lambda, \overline{\xi}] \in C^\infty(T^{1,0} \wedge T^{*(0,1)})$. This means, in order for ϕ and Λ to be compatible, we need $\overline{\partial}(\phi(X)) - \phi(\overline{\partial}(X)) \in C^\infty(\bigwedge^2 T^{1,0})$ and $\overline{\partial}(\phi(\overline{\xi})) - \phi(\overline{\partial}(\overline{\xi})) \in C^\infty(T^{1,0} \wedge T^{*(0,1)})$.

Also, $\overline{\partial}X \in C^\infty(T^{1,0} \wedge T^{*(0,1)})$ and $\overline{\partial}\overline{\xi} \in C^\infty(\bigwedge^2 T^{*(0,1)})$. So, using equation (3.11),

$$\begin{aligned}
&\overline{\partial}\phi_1(X) - \phi_1(\overline{\partial}X) \in C^\infty(T^{1,0} \wedge T^{*(0,1)}) \\
&\overline{\partial}\phi_1(\overline{\xi}) - \phi_1(\overline{\partial}\overline{\xi}) = 0 \\
&\overline{\partial}\phi_2(X) - \phi_2(\overline{\partial}X) \in C^\infty(T^{*(0,1)} \wedge T^{*(0,1)}) \\
&\overline{\partial}\phi_2(\overline{\xi}) - \phi_2(\overline{\partial}\overline{\xi}) = 0 \\
&\overline{\partial}\phi_3(X) - \phi_3(\overline{\partial}X) \in C^\infty(T^{1,0} \wedge T^{1,0}) \\
&\overline{\partial}\phi_3(\overline{\xi}) - \phi_3(\overline{\partial}\overline{\xi}) \in C^\infty(T^{1,0} \wedge T^{*(0,1)}) \\
&\overline{\partial}\phi_4(X) - \phi_4(\overline{\partial}X) \in C^\infty(T^{1,0} \wedge T^{(1,0)}) \\
&\overline{\partial}\phi_4(\overline{\xi}) - \phi_4(\overline{\partial}\overline{\xi}) = C^\infty(T^{*(0,1)} \wedge T^{*(0,1)}).
\end{aligned}$$

So for ϕ and Λ to be compatible, we need $\phi = \phi_3$, or $\phi : T^{*(0,1)} \to T^{1,0}$. With this, we can now prove the converse to theorem (15).

Theorem 16 *Let $L = T^{1,0} \oplus T^{*(0,1)}$ and $\Lambda \in C^\infty(\bigwedge^2 T^{1,0})$ and let $\phi : T^{*(0,1)} \to T^{1,0}$ be a vector bundle homomorphism. Let $\Phi : L \to L$ be defined by $\Phi = 1+\phi$. If Λ and ϕ are a compatible pair, then $\Phi : (\bigwedge L, [-,-], \wedge, \overline{\partial}_\Lambda) \to (\bigwedge L, [-,-], \wedge, \overline{\partial})$ is DGA isomorphism.*

Proof. For Φ to be a DGA isomorphism, we must show that, for all $l, l_1, l_2 \in C^\infty(L)$,

$$\Phi(l_1 \wedge l_2) = \Phi(l_1) \wedge \Phi(l_2) \tag{3.14}$$

$$\Phi([l_1, l_2]) = [\Phi(l_1), \Phi(l_2)] \tag{3.15}$$

$$\Phi(\overline{\partial}_\Lambda l) = \overline{\partial}(\Phi(l)) \tag{3.16}$$

Since $\phi : T^{*(0,1)} \to T^{1,0}$, $\phi \circ \phi = 0$. When we use equation (3.11), the left side of (3.14) is,

$$\Phi(l_1 \wedge l_2) = l_1 \wedge l_2 + \phi(l_1 \wedge l_2) = l_1 \wedge l_2 + (\phi(l_1)) \wedge l_2 + l_1 \wedge (\phi(l_2)).$$

The right side is,

$$\Phi(l_1)\wedge\Phi(l_2)=(l_1+\phi(l_1))\wedge(l_2+\phi(l_2)) \;=\; l_1\wedge l_2+(\phi(l_1))\wedge l_2+l_1\wedge(\phi(l_2)) + (\phi(l_1))\wedge(\phi(l_2)).$$

So for these to be equal, we need $(\phi(l_1))\wedge(\phi(l_2))=0$. However, by equation (3.11), we have

$$\phi((\phi(l_1))\wedge l_2)=(\phi(\phi(l_1)))\wedge(\phi(l_2))+(\phi(l_1))\wedge(\phi(l_2))=(\phi(l_1))\wedge(\phi(l_2)),$$

and

$$\phi(l_1\wedge(\phi(l_2)))=(\phi(l_1))\wedge(\phi(l_2))+(\phi(l_1))\wedge(\phi(\phi(l_2)))=(\phi(l_1))\wedge(\phi(l_2)),$$

so

$$2((\phi(l_1))\wedge(\phi(l_2)))=\phi((\phi(l_1))\wedge l_2)+\phi(l_1\wedge(\phi(l_2)))$$
$$=\phi((\phi(l_1))\wedge l_2+l_1\wedge(\phi(l_2)))=\phi(\phi(l_1\wedge l_2))=0.$$

Therefore, $\Phi(l_1\wedge l_2)=\Phi(l_1)\wedge\Phi(l_2)$. A similar computation using equation (3.12) shows that $[\phi(l_1),\phi(l_2)]=0$, and so

$$\Phi([l_1,l_2])=[l_1,l_2]+\phi([l_1,l_2])=[l_1,l_2]+[\phi(l_1),l_2]+[l_1,\phi(l_2)]$$
$$=[l_1,l_2]+[\phi(l_1),l_2]+[l_1,\phi(l_2)]+[\phi(l_1),\phi(l_2)]$$
$$=[l_1+\phi(l_1),l_2+\phi(l_2)]=[\Phi(l_1),\Phi(l_2)]$$

Now we examine equation (3.16). Using equation (3.13), we see

$$\Phi(\overline{\partial}_\Lambda l)=\Phi(\overline{\partial}l+[\Lambda,l])=\overline{\partial}l+[\Lambda,l]+\phi(\overline{\partial}l)+\phi([\Lambda,l])$$
$$=\overline{\partial}l+\overline{\partial}\phi(l)+\phi([\Lambda,l])=\overline{\partial}(\Phi(l))+\phi([\Lambda,l]).$$

So for equation (3.16) to hold, we need $\phi([\Lambda, l]) = 0$. Evaluating ϕ on equation (3.13) yields $\phi([\Lambda, l]) = \phi(\overline{\partial}(\phi l)) - \phi(\phi(\overline{\partial} l)) = \phi(\overline{\partial}(\phi l))$. Replacing l with $\phi(l)$ in equation yields

$$[\Lambda, \phi(l)] = \overline{\partial}(\phi(\phi(l))) - \phi(\overline{\partial}(\phi(l))) = -\phi(\overline{\partial}(\phi(l))).$$

Together, these computations show that

$$\phi([\Lambda, l]) = -[\Lambda, \phi(l)]. \tag{3.17}$$

Now, let v, w be (1,0) vector fields. Then $\phi(v) = 0 = \phi(w)$ and,

$$\begin{aligned}
&\phi([v \wedge w, l]) = \phi(v \wedge [w, l] - w \wedge [v, l]) \\
&= \phi(v) \wedge [w, l] + v \wedge \phi([w, l]) - \phi(w) \wedge [v, l] - w \wedge \phi([v, l]) \\
&= v \wedge [\phi(w), l] + v \wedge [w, \phi(l)] - w \wedge [\phi(v), l] - w \wedge [v, \phi(l)] \\
&= v \wedge [w, \phi(l)] - w \wedge [v, \phi(l)] = [v \wedge w, \phi(l)].
\end{aligned}$$

By the linearity of $[-, l]$, we have,

$$\phi([\Lambda, l]) = [\Lambda, \phi(l)]. \tag{3.18}$$

Therefore, by equations (3.17) and (3.18), $\phi([\Lambda, l]) = 0$, and equation (3.16) holds.

While equations (3.14), (3.15) and (3.16) are only in degree one they hold in higher degrees as well when ϕ is extended so that $\phi(l_1 \wedge l_2 \wedge \cdots l_n) = \phi(l_1) \wedge l_2 \wedge \cdots l_n + l_1 \wedge \phi(l_2) \wedge \cdots l_n + \cdots + l_1 \wedge l_2 \wedge \cdots \phi(l_n)$. Therefore Φ is a DGA homomorphism.

Lastly, we look at the kernel and image of $\Phi = 1 + \phi$. If $X \in T^{1,0}$ and $\overline{\xi} \in T^{*(0,1)}$, then $\Phi(X + \overline{\xi}) = X + \overline{\xi} + \phi(X) + \phi(\overline{\xi}) = X + \phi(\overline{\xi}) + \overline{\xi}$, where $\phi(\overline{\xi}) \in T^{1,0}$. If $X + \phi(\overline{\xi}) + \overline{\xi} = 0$, then $\overline{\xi} = 0$, which means $\phi(\overline{\xi}) = 0$, and so $X = 0$. So $\ker \Phi = \{0\}$. Also, if $X \in T^{1,0}$, then $\Phi(X) = X + \phi(X) = X$. If $\overline{\eta} \in T^{*(0,1)}$, let $Y = \phi(\overline{\eta})$. Then $\Phi(-Y + \overline{\eta}) = -Y + \overline{\eta} - \phi(Y) + \phi(\overline{\eta}) = -Y + \overline{\eta} + Y = \overline{\eta}$. So Φ is surjective. Therefore, Φ is an isomorphism. ■

3.5 Main Theorem

Our original goal was to build weak mirror pairs, and we are now in a position to do that. Let $L = T^{1,0} \oplus T^{*(0,1)}$ be a generalized complex structure defined on a complex manifold M with real dimension $4n$. Then the type of L is $2n$. Let $\overline{\Lambda} \in C^\infty(\bigwedge^2 T^{0,1})$. If $\partial\overline{\Lambda} = 0$ and $[\overline{\Lambda}, \overline{\Lambda}] = 0$, then $L_{\overline{\Lambda}}$ is a generalized complex structure. If $\overline{\Lambda} : T^{*(0,1)} \to T^{0,1}$ is non-degenerate, then the $L_{\overline{\Lambda}}$ has type 0. So, by Proposition 2 there will be a closed two-form B so that $e^B L_{\overline{\Lambda}}$ is symplectic.

If there exists ϕ compatible with Λ, then $\Phi = 1+\phi$ will be a DGA isomorphism, and we will have the following chain of isomorphisms.

$$(\bigwedge L, [-,-], \wedge, \overline{\partial})$$

$$\Phi^{-1} \downarrow$$

$$(\bigwedge L, [-,-], \wedge, \overline{\partial}_\Lambda)$$

$$A_\Lambda \downarrow$$

$$(\bigwedge L_{\overline{\Lambda}}, [-,-], \wedge, \overline{\delta})$$

$$e^B \downarrow$$

$$(\bigwedge e^B L_{\overline{\Lambda}}, [-,-], \wedge, \overline{\delta}_B)$$

Therefore $(\bigwedge L, [-,-], \wedge, \overline{\partial})$ and $(\bigwedge e^B L_{\overline{\Lambda}}, [-,-], \wedge, \overline{\delta}_B)$ define a weak mirror pair, since every isomorphism is a quasi-isomorphism. The only isomorphism in this chain that is non-trivial is Φ, and we can express the existence of this chain in terms of the existence of Φ. It should be noted that $\Phi^{-1} = 1 - \phi$, since $(1+\phi) \circ (1-\phi) = 1 \circ 1 + 1 \circ (-\phi) + \phi \circ 1 + \phi \circ (-\phi) = 1 - \phi + \phi - \phi \circ \phi = 1$.

Theorem 17 *Let $L = T^{1,0} \oplus T^{*(0,1)}$ be a generalized complex structure defined on a complex manifold M of real dimension $4n$. Let $\Lambda \in C^\infty(\bigwedge^2 T^{1,0})$ be non-degenerate as a map $\Lambda : T^{*(1,0)} \to T^{1,0}$. Let $\phi : T^{*(0,1)} \to T^{1,0}$ be a vector bundle isomorphism. If Λ and ϕ satisfy the following conditions for all $l, l_1, l_2 \in C^\infty(L)$,*

$$\overline{\partial}\Lambda = 0 \qquad \text{and} \qquad [\Lambda, \Lambda] = 0$$

$$\phi(l_1 \wedge l_2) = (\phi(l_1)) \wedge l_2 + l_1 \wedge (\phi(l_2))$$

$$\phi([l_1, l_2]) = [\phi(l_1), l_2] + [l_1, \phi(l_2)]$$

$$\overline{\partial}(\phi(l)) - \phi(\overline{\partial}(l)) = [\Lambda, l],$$

then there exists a closed two-form B such that $(\bigwedge L, [-,-], \wedge, \overline{\partial})$ and $(\bigwedge e^B L_{\overline{\Lambda}}, [-,-], \wedge, \overline{\delta}_B)$ are a weak mirror pair.

This reduces the problem of finding weak mirror pairs to the problem of finding compatible pairs. In general, the existence of Λ has studied been studied by Hitchin [9] and Gualtieri [7]. Such Λ are examples of holomorphic Poisson bi-vector fields.

In practice below, we will start with a canonical choice of Λ and solve for a compatible ϕ. The nature of this ϕ, as well the B and symplectic form, will hopefully be significant to the problem in question.

3.6 Additional Calculations

In this section we show that the Maurer-Cartan equation implies that our deformation above are integrable, and that the deformed differential is a differential. We restrict our attention to the cases that are most important to this thesis. In this section $L = T^{1,0} \oplus T^{*(0,1)}$ is generalized complex structure defined by an integrable complex structure, and $\Lambda \in C^{\infty}(T^{1,0} \wedge T^{1,0})$ with $\overline{\partial}\Lambda = 0$ and $[\Lambda, \Lambda] = 0$. First we prove a useful computational result.

Lemma 18

$$[\overline{\Lambda}, \overline{\Lambda}](\overline{\xi}^1, \overline{\xi}^2, \overline{\xi}^3) = 2\overline{\xi}^1([\overline{\Lambda}(\overline{\xi}^2), \overline{\Lambda}(\overline{\xi}^3)]) - 2(\overline{\Lambda}(\overline{\xi}^1)(\overline{\Lambda}(\overline{\xi}^2, \overline{\xi}^3))) + c.p.$$

where the cyclic permutations are in $\overline{\xi}^1, \overline{\xi}^2, \overline{\xi}^3$.

Proof. Look at

$$\begin{aligned}
&[\overline{\Lambda}, \overline{X}_1 \wedge \overline{X}_2](\overline{\xi}^1, \overline{\xi}^2, \overline{\xi}^3) \\
&= -\overline{X}_1 \wedge [\overline{\Lambda}, \overline{X}_2](\overline{\xi}^1, \overline{\xi}^2, \overline{\xi}^3) + \overline{X}_2 \wedge [\overline{\Lambda}, \overline{X}_1](\overline{\xi}^1, \overline{\xi}^2, \overline{\xi}^3) \\
&= -\overline{\xi}^1(\overline{X}_1)[\overline{\Lambda}, \overline{X}_2](\overline{\xi}^2, \overline{\xi}^3) + \xi^1(\overline{X}_2)[\overline{\Lambda}, \overline{X}_1](\overline{\xi}^2, \overline{\xi}^3) + c.p. \\
&= \overline{\xi}^1(\overline{X}_1)(\overline{X}_2(\overline{\xi}^2(\overline{\Lambda}(\overline{\xi}^3))) + \overline{\xi}^2([\overline{\Lambda}(\overline{\xi}^3), \overline{X}_2]) - \overline{\xi}^3([\overline{\Lambda}(\overline{\xi}^2), \overline{X}_2])) \\
&-\overline{\xi}^1(\overline{X}_2)(\overline{X}_1(\overline{\xi}^2(\overline{\Lambda}(\overline{\xi}^3))) - \overline{\xi}^2([\overline{\Lambda}(\overline{\xi}^3), \overline{X}_1]) + \overline{\xi}^3([\overline{\Lambda}(\overline{\xi}^2), \overline{X}_1])) + c.p. \\
&= (\overline{\xi}^1(\overline{X}_1)\overline{X}_2 - \overline{\xi}^1(\overline{X}_2)\overline{X}_1)(\overline{\xi}^2(\overline{\Lambda}(\overline{\xi}^3))) \\
&+\overline{\xi}^2(\overline{\xi}^1(\overline{X}_1)[\overline{\Lambda}(\overline{\xi}^3), \overline{X}_2] - \overline{\xi}^1(\overline{X}_2)[\overline{\Lambda}(\overline{\xi}^3), \overline{X}_1]) \\
&-\overline{\xi}^3(\overline{\xi}^1(\overline{X}_1)[\overline{\Lambda}(\overline{\xi}^2), \overline{X}_2] - \overline{\xi}^1(\overline{X}_2)[\overline{\Lambda}(\overline{\xi}^2), \overline{X}_1]) + c.p. \\
&= ((\overline{X}_1 \wedge \overline{X}_2)(\overline{\xi}^1))(\overline{\xi}^2(\overline{\Lambda}(\overline{\xi}^3))) \\
&+\overline{\xi}^2([\overline{\Lambda}(\overline{\xi}^3), \overline{\xi}^1(\overline{X}_1)\overline{X}_2] - [\overline{\Lambda}(\overline{\xi}^3), \overline{\xi}^1(\overline{X}_2)\overline{X}_1]) \\
&-\overline{\xi}^2((\overline{\Lambda}(\overline{\xi}^3)(\overline{\xi}^1(\overline{X}_1)))\overline{X}_2 - ((\overline{\Lambda}(\overline{\xi}^3))(\overline{\xi}^1(\overline{X}_2))\overline{X}_1)) \\
&-\overline{\xi}^3([\overline{\Lambda}(\overline{\xi}^2), \overline{\xi}^1(\overline{X}_1)\overline{X}_2] - [\overline{\Lambda}(\overline{\xi}^2), \overline{\xi}^1(\overline{X}_2)\overline{X}_1]) \\
&+\overline{\xi}^3((\overline{\Lambda}(\overline{\xi}^2)(\overline{\xi}^1(\overline{X}_1)))\overline{X}_2 - ((\overline{\Lambda}(\overline{\xi}^2))(\overline{\xi}^1(\overline{X}_2))\overline{X}_1)) + c.p.
\end{aligned}$$

$$= ((\overline{X}_1 \wedge \overline{X}_2)(\overline{\xi}^1))(\overline{\xi}^2(\overline{\Lambda}(\overline{\xi}^3)))$$

$$+\overline{\xi}^2([\overline{\Lambda}(\overline{\xi}^3), (\overline{X}_1 \wedge \overline{X}_2)(\overline{\xi}^1)]) - \overline{\xi}^3([\overline{\Lambda}(\overline{\xi}^2), (\overline{X}_1 \wedge \overline{X}_2)(\overline{\xi}^1)])$$

$$-(\overline{\Lambda}(\overline{\xi}^3)(\overline{\xi}^1(\overline{X}_1)))(\overline{\xi}^2(\overline{X}_2)) + (\overline{\Lambda}(\overline{\xi}^3)(\overline{\xi}^1(\overline{X}_2)))(\overline{\xi}^2(\overline{X}_1))$$

$$+(\overline{\Lambda}(\overline{\xi}^2)(\overline{\xi}^1(\overline{X}_1)))(\overline{\xi}^3(\overline{X}_2)) - (\overline{\Lambda}(\overline{\xi}^2)(\overline{\xi}^1(\overline{X}_2)))(\overline{\xi}^3(\overline{X}_1)) + c.p.$$

Focusing on last two lines shows,

$$-(\overline{\Lambda}(\overline{\xi}^3)(\overline{\xi}^1(\overline{X}_1)))(\overline{\xi}^2(\overline{X}_2)) + (\overline{\Lambda}(\overline{\xi}^3)(\overline{\xi}^1(\overline{X}_2)))(\overline{\xi}^2(\overline{X}_1))$$

$$+(\overline{\Lambda}(\overline{\xi}^2)(\overline{\xi}^1(\overline{X}_1)))(\overline{\xi}^3(\overline{X}_2)) - (\overline{\Lambda}(\overline{\xi}^2)(\overline{\xi}^1(\overline{X}_2)))(\overline{\xi}^3(\overline{X}_1)) + c.p.$$

$$= -(\overline{\Lambda}(\overline{\xi}^3)(\overline{\xi}^1(\overline{X}_1)))(\overline{\xi}^2(\overline{X}_2)) + (\overline{\Lambda}(\overline{\xi}^3)(\overline{\xi}^1(\overline{X}_2)))(\overline{\xi}^2(\overline{X}_1))$$

$$+(\overline{\Lambda}(\overline{\xi}^2)(\overline{\xi}^1(\overline{X}_1)))(\overline{\xi}^3(\overline{X}_2)) - (\overline{\Lambda}(\overline{\xi}^2)(\overline{\xi}^1(\overline{X}_2)))(\overline{\xi}^3(\overline{X}_1))$$

$$-(\overline{\Lambda}(\overline{\xi}^1)(\overline{\xi}^2(\overline{X}_1)))(\overline{\xi}^3(\overline{X}_2)) + (\overline{\Lambda}(\overline{\xi}^1)(\overline{\xi}^2(\overline{X}_2)))(\overline{\xi}^3(\overline{X}_1))$$

$$+(\overline{\Lambda}(\overline{\xi}^3)(\overline{\xi}^2(\overline{X}_1)))(\overline{\xi}^1(\overline{X}_2)) - (\overline{\Lambda}(\overline{\xi}^3)(\overline{\xi}^2(\overline{X}_2)))(\overline{\xi}^1(\overline{X}_1))$$

$$-(\overline{\Lambda}(\overline{\xi}^2)(\overline{\xi}^3(\overline{X}_1)))(\overline{\xi}^1(\overline{X}_2)) + (\overline{\Lambda}(\overline{\xi}^2)(\overline{\xi}^3(\overline{X}_2)))(\overline{\xi}^1(\overline{X}_1))$$

$$+(\overline{\Lambda}(\overline{\xi}^1)(\overline{\xi}^3(\overline{X}_1)))(\overline{\xi}^2(\overline{X}_2)) - (\overline{\Lambda}(\overline{\xi}^1)(\overline{\xi}^3(\overline{X}_2)))(\overline{\xi}^2(\overline{X}_1))$$

$$= -\overline{\Lambda}(\overline{\xi}^1)((\overline{X}_1 \wedge \overline{X}_2)(\overline{\xi}^2, \overline{\xi}^3)) - \overline{\Lambda}(\overline{\xi}^2)((\overline{X}_1 \wedge \overline{X}_2)(\overline{\xi}^3, \overline{\xi}^1))$$

$$-\overline{\Lambda}(\overline{\xi}^3)((\overline{X}_1 \wedge \overline{X}_2)(\overline{\xi}^1, \overline{\xi}^2)) = -\overline{\Lambda}(\overline{\xi}^1)((\overline{X}_1 \wedge \overline{X}_2)(\overline{\xi}^2, \overline{\xi}^3)) + c.p.$$

Therefore,

$$[\overline{\Lambda}, \overline{X}_1 \wedge \overline{X}_2](\overline{\xi}^1, \overline{\xi}^2, \overline{\xi}^3) = ((\overline{X}_1 \wedge \overline{X}_2)(\overline{\xi}^1))(\overline{\xi}^2(\overline{\Lambda}(\overline{\xi}^3)))$$

$$+\overline{\xi}^2([\overline{\Lambda}(\overline{\xi}^3), (\overline{X}_1 \wedge \overline{X}_2)(\overline{\xi}^1)]) - \overline{\xi}^3([\overline{\Lambda}(\overline{\xi}^2), (\overline{X}_1 \wedge \overline{X}_2)(\overline{\xi}^1)])$$

$$-\overline{\Lambda}(\overline{\xi}^1)((\overline{X}_1 \wedge \overline{X}_2)(\overline{\xi}^2, \overline{\xi}^3)) + c.p.$$

and

$$[\overline{\Lambda},\overline{\Lambda}](\overline{\xi}^1,\overline{\xi}^2,\overline{\xi}^3) = \overline{\Lambda}(\overline{\xi}^1)(\overline{\Lambda}(\overline{\xi}^3,\overline{\xi}^2)) + \overline{\xi}^2([\overline{\Lambda}(\overline{\xi}^3),\overline{\Lambda}(\overline{\xi}^1)])$$
$$-\overline{\xi}^3([\overline{\Lambda}(\overline{\xi}^2),\overline{\Lambda}(\overline{\xi}^1)]) - \overline{\Lambda}(\overline{\xi}^1)(\overline{\Lambda}(\overline{\xi}^2,\overline{\xi}^3)) + c.p.$$
$$= -2\overline{\Lambda}(\overline{\xi}^1)(\overline{\Lambda}(\overline{\xi}^2,\overline{\xi}^3)) + \overline{\xi}^2([\overline{\Lambda}(\overline{\xi}^3),\overline{\Lambda}(\overline{\xi}^1)]) - \overline{\xi}^3([\overline{\Lambda}(\overline{\xi}^2),\overline{\Lambda}(\overline{\xi}^1)])$$
$$-2\overline{\Lambda}(\overline{\xi}^2)(\overline{\Lambda}(\overline{\xi}^3,\overline{\xi}^1)) + \overline{\xi}^3([\overline{\Lambda}(\overline{\xi}^1),\overline{\Lambda}(\overline{\xi}^2)]) - \overline{\xi}^1([\overline{\Lambda}(\overline{\xi}^3),\overline{\Lambda}(\overline{\xi}^2)])$$
$$-2\overline{\Lambda}(\overline{\xi}^3)(\overline{\Lambda}(\overline{\xi}^1,\overline{\xi}^2)) + \overline{\xi}^1([\overline{\Lambda}(\overline{\xi}^2),\overline{\Lambda}(\overline{\xi}^3)]) - \overline{\xi}^2([\overline{\Lambda}(\overline{\xi}^1),\overline{\Lambda}(\overline{\xi}^3)])$$
$$= 2\overline{\xi}^1([\overline{\Lambda}(\overline{\xi}^2),\overline{\Lambda}(\overline{\xi}^3)]) - 2\overline{\Lambda}(\overline{\xi}^1)(\overline{\Lambda}(\overline{\xi}^2,\overline{\xi}^3)) + c.p.$$

■

Also, we have the following two formulas,

$$\overline{\partial}\Lambda(\overline{z},\xi^1,\xi^2) = \overline{z}(\Lambda(\xi^1,\xi^2)) - \Lambda([\overline{z},\xi^1],\xi^2) + \Lambda([\overline{z},\xi^2],\xi^1)$$
$$\partial\overline{\Lambda}(z,\overline{\xi}^1,\overline{\xi}^2) = z(\overline{\Lambda}(\overline{\xi}^1,\overline{\xi}^2)) - \overline{\Lambda}([z,\overline{\xi}^1],\overline{\xi}^2) + \overline{\Lambda}([z,\overline{\xi}^2],\overline{\xi}^1)$$
$$= -z(\overline{\Lambda}(\overline{\xi}^1,\overline{\xi}^2)) - \overline{\xi}^1([z,\overline{\Lambda\xi}^2]) + \overline{\xi}^2([z,\overline{\Lambda\xi}^1])$$

Proposition 19 *If $L = T^{1,0} \oplus T^{*(0,1)}$ and $\partial\overline{\Lambda} = 0$ and $[\overline{\Lambda},\overline{\Lambda}] = 0$, then $C^\infty(L_{\overline{\Lambda}})$ is closed under the Courant bracket.*

Proof. Since $L_{\overline{\Lambda}}$ is maximally isotropic, we need to show that $\langle[l_1+\overline{\Lambda}l_1, l_2+\overline{\Lambda}l_2], l_3+\overline{\Lambda}l_3\rangle = 0$ for all $l_1,l_2,l_3 \in C^\infty(L)$. We break this condition in to all possible cases where $l_i = z_i$ or $l_i = \overline{\xi}^i$. We list the non-trivial cases below.

Case:$(l_1 = z_1, l_2 = \overline{\xi}^2, l_3 = \overline{\xi}^3)$

$$2\langle[z_1,\overline{\xi}^2+\overline{\Lambda\xi}^2],\overline{\xi}^3+\overline{\Lambda\xi}^3\rangle = \overline{\xi}^3([z_1,\overline{\Lambda\xi}^2]) + [z_1,\overline{\xi}^2](\overline{\Lambda\xi}^3)$$
$$= z_1(\overline{\xi}^2(\overline{\Lambda\xi}^3)) - \overline{\xi}^2([z_1,\overline{\Lambda\xi}^3]) + \overline{\xi}^3([z_1,\overline{\Lambda\xi}^2])$$
$$= -z_1(\overline{\Lambda}(\overline{\xi}^2,\overline{\xi}^3)) - \overline{\xi}^2([z_1,\overline{\Lambda\xi}^3]) + \overline{\xi}^3([z_1,\overline{\Lambda\xi}^2])$$
$$= \partial\overline{\Lambda}(z_1,\overline{\xi}^2,\overline{\xi}^3) = 0.$$

Case:$(l_1 = \overline{\xi}^1, l_2 = \overline{\xi}^2, l_3 = \overline{\xi}^3)$

$$
\begin{aligned}
&2\langle[\overline{\xi}^1 + \overline{\Lambda\xi}^1, \overline{\xi}^2 + \overline{\Lambda\xi}^2], \overline{\xi}^3 + \overline{\Lambda\xi}^3\rangle \\
&= \overline{\xi}^3([\overline{\Lambda\xi}^1, \overline{\Lambda\xi}^2]) + [\overline{\Lambda\xi}^1, \overline{\xi}^2](\overline{\Lambda\xi}^3) - [\overline{\Lambda\xi}^2, \overline{\xi}^1](\overline{\Lambda\xi}^3) \\
&= \overline{\xi}^3([\overline{\Lambda\xi}^1, \overline{\Lambda\xi}^2]) + \overline{\Lambda\xi}^1(\overline{\xi}^2(\overline{\Lambda\xi}^3)) - \overline{\xi}^2([\overline{\Lambda\xi}^1, \overline{\Lambda\xi}^3]) - \tfrac{1}{2}\overline{\Lambda\xi}^3(\overline{\xi}^2(\overline{\Lambda\xi}^1)) \\
&\quad -\overline{\Lambda\xi}^2(\overline{\xi}^1(\overline{\Lambda\xi}^3)) + \overline{\xi}^1([\overline{\Lambda\xi}^2, \overline{\Lambda\xi}^3]) + \tfrac{1}{2}\overline{\Lambda\xi}^3(\overline{\xi}^1(\overline{\Lambda\xi}^2)) \\
&= \overline{\xi}^3([\overline{\Lambda\xi}^1, \overline{\Lambda\xi}^2]) - \overline{\Lambda\xi}^1(\overline{\Lambda}(\overline{\xi}^2, \overline{\xi}^3)) + \overline{\xi}^2([\overline{\Lambda\xi}^3, \overline{\Lambda\xi}^1]) \\
&\quad -\overline{\Lambda\xi}^3(\overline{\Lambda}(\overline{\xi}^1, \overline{\xi}^2)) - \overline{\Lambda\xi}^2(\overline{\Lambda}(\overline{\xi}^3, \overline{\xi}^1)) + \overline{\xi}^1([\overline{\Lambda\xi}^2, \overline{\Lambda\xi}^3]) \\
&= \tfrac{1}{2}[\overline{\Lambda}, \overline{\Lambda}](\overline{\xi}^1, \overline{\xi}^2, \overline{\xi}^3) = 0.
\end{aligned}
$$

All other cases are trivially satisfied, so $L_{\overline{\Lambda}}$ is involutive under the Courant bracket. ■

Proposition 20 *If $L = T^{1,0} \oplus T^{*(0,1)}$ and $\overline{\partial}_\Lambda = \overline{\partial} + [\Lambda, -]$ for $\Lambda \in C^\infty(T^{2,0})$, then the diagram below commutes,*

$$
\begin{array}{ccc}
\bigwedge^n L & \xrightarrow{\overline{\partial}_\Lambda} & \bigwedge^{n+1} L \\
A_\Lambda \downarrow & & \downarrow A_\Lambda \\
\bigwedge^n L_{\overline{\Lambda}} & \xrightarrow{\overline{\delta}} & \bigwedge^{n+1} L_{\overline{\Lambda}}
\end{array} . \tag{3.19}
$$

where $\overline{\delta}$ is defined by equation (2.5).

Proof. We prove this by induction. When $n = 1$, then let $l \in C^\infty(L)$. Note that

$$
\begin{aligned}
&A_\Lambda \overline{\partial}_\Lambda l(\bar{l}_1 + \Lambda\bar{l}_1, \bar{l}_2 + \Lambda\bar{l}_2) \\
&= (\overline{\partial} l + [\Lambda, l] + \overline{\Lambda}\overline{\partial} l + \overline{\Lambda}[\Lambda, l])(\bar{l}_1 + \Lambda\bar{l}_1, \bar{l}_2 + \Lambda\bar{l}_2) \\
&= \overline{\partial} l(\bar{l}_1, \bar{l}_2) + [\Lambda, l](\bar{l}_1, \bar{l}_2) + \overline{\Lambda}\overline{\partial} l(\Lambda\bar{l}_1, \Lambda\bar{l}_2) + \overline{\Lambda}[\Lambda, l](\Lambda\bar{l}_1, \Lambda\bar{l}_2)
\end{aligned}
$$

and

$$
\begin{aligned}
&\overline{\delta} A_\Lambda l(\bar{l}_1 + \Lambda\bar{l}_1, \bar{l}_2 + \Lambda\bar{l}_2) \\
&= \overline{\delta}(l + \overline{\Lambda} l)(\bar{l}_1 + \Lambda\bar{l}_1, \bar{l}_2 + \Lambda\bar{l}_2)
\end{aligned}
$$

Since $[\Lambda, z] \in T^{(2,0)}$ and $[\Lambda, \xi] \in T^{1,0} \wedge T^{*(0,1)}$, $\overline{\Lambda}[\Lambda, l] = 0$ for all $l \in T^{1,0} \oplus T^{*(0,1)}$. So we need to show that,

$$\overline{\partial} l(\bar{l}_1, \bar{l}_2) + [\Lambda, l](\bar{l}_1, \bar{l}_2) + \overline{\Lambda \partial} l(\Lambda \bar{l}_1, \Lambda \bar{l}_2) = \overline{\delta}(l + \overline{\Lambda} l)(\bar{l}_1 + \Lambda \bar{l}_1, \bar{l}_2 + \Lambda \bar{l}_2) \tag{3.20}$$

for all $l \in C^\infty(L)$ and $\bar{l}_1, \bar{l}_2 \in C^\infty(\overline{L})$. We examine the cases where $l = z$ or $l = \overline{\xi}$, and $\bar{l}_i = \overline{z}_i$ or $\bar{l}_i = \xi^i$. Also note that $\Lambda z = 0 = \overline{\Lambda} \overline{z}_i$ and $\overline{\Lambda \partial} l(\bar{l}_1 + \Lambda \bar{l}_1, \bar{l}_2 + \Lambda \bar{l}_2) = 0$, due to type considerations.

Case: $(l = z, \bar{l}_1 = \overline{z}_1, \bar{l}_2 = \overline{z}_2)$

$$A_\Lambda \overline{\partial}_\Lambda(z)(\overline{z}_1, \overline{z}_2) = \overline{\partial} z(\overline{z}_1, \overline{z}_2) + [\Lambda, z](\overline{z}_1, \overline{z}_2) = 0$$

$$\overline{\delta} A_\Lambda z(\overline{z}_1, \overline{z}_2) = \overline{\delta} z(\overline{z}_1, \overline{z}_2) = 2\big(\overline{z}_1 \langle z, \overline{z}_2 \rangle - \overline{z}_2 \langle z, \overline{z}_1 \rangle - \langle z, [\overline{z}_1, \overline{z}_2] \rangle\big) = 0$$

So $A_\Lambda \overline{\partial}_\Lambda(z)(\overline{z}_1, \overline{z}_2) = \overline{\delta} A_\Lambda z(\overline{z}_1, \overline{z}_2)$.

Case: $(l = z, \bar{l}_1 = \overline{z}_1, \bar{l}_2 = \xi^2)$

$$\begin{aligned} A_\Lambda \overline{\partial}_\Lambda(z)(\overline{z}_1, \xi^2 + \Lambda \xi^2) &= \overline{\partial} z(\overline{z}_1, \xi^2) + [\Lambda, z](\overline{z}_1, \xi^2) = \overline{\partial} z(\overline{z}_1, \xi^2) \\ &= \overline{z}_1 \xi^2(z) - [\overline{z}_1, \xi^2](z) \end{aligned}$$

$$\begin{aligned} \overline{\delta} A_\Lambda z(\overline{z}_1, \xi^2 + \Lambda \xi^2) &= \overline{\delta} z(\overline{z}_1, \xi^2 + \Lambda \xi^2) \\ &= \overline{z}_1 \xi^2(z) - [\overline{z}_1, \xi^2](z) \end{aligned}$$

So $A_\Lambda \overline{\partial}_\Lambda(z)(\overline{z}_1, \xi^2 + \Lambda \xi^2) = \overline{\delta} A_\Lambda z(\overline{z}_1, \xi^2 + \Lambda \xi^2)$.

Case: $(l = z, \bar{l}_1 = \xi^1, \bar{l}_2 = \xi^2)$

$$\begin{aligned} A_\Lambda \overline{\partial}_\Lambda(z)(\xi^1 + \Lambda \xi^1, \xi^2 + \Lambda \xi^2) &= \overline{\partial} z(\xi^1, \xi^2) + [\Lambda, z](\xi^1, \xi^2) \\ &= z(\Lambda(\xi^1, \xi^2)) - \xi^1([\Lambda(\xi^2), z]) + \xi^2([\Lambda(\xi^1), z]) \end{aligned}$$

$$\overline{\delta} A_\Lambda z(\xi^1 + \Lambda\xi^1, \xi^2 + \Lambda\xi^2) = \overline{\delta} z(\xi^1 + \Lambda\xi^1, \xi^2 + \Lambda\xi^2)$$

$$= \Lambda\xi^1(\xi^2(z)) - \Lambda\xi^2(\xi^1(z)) - [\xi^1 + \Lambda\xi^1, \xi^2 + \Lambda\xi^2]z$$

$$= \Lambda\xi^1(\xi^2(z)) - \Lambda\xi^2(\xi^1(z)) - \Lambda\xi^1(\xi^2(z)) + \xi^2([\Lambda\xi^1, z])$$

$$+\Lambda\xi^2(\xi^1(z)) - \xi^1([\Lambda\xi^2, z]) + z\Lambda(\xi^1, \xi^2)$$

$$= \xi^2([\Lambda\xi^1, z]) - \xi^1([\Lambda\xi^2, z]) + z\Lambda(\xi^1, \xi^2)$$

So $A_\Lambda \overline{\partial}_\Lambda(z)(\xi^1 + \Lambda\xi^1, \xi^2 + \Lambda\xi^2) = \overline{\delta} A_\Lambda z(\xi^1 + \Lambda\xi^1, \xi^2 + \Lambda\xi^2)$.

Case: $(l = \overline{\xi}, \overline{l}_1 = \overline{z}_1, \overline{l}_2 = \xi^2)$

$$A_\Lambda \overline{\partial}_\Lambda(\overline{\xi})(\overline{z}_1, \xi^2 + \Lambda\xi^2) = \overline{\partial\xi}(\overline{z}_1, \xi^2) + [\Lambda, \overline{\xi}](\overline{z}_1, \xi^2)$$

$$= -[\Lambda\xi^2, \overline{\xi}](\overline{z}_1) = -\Lambda\xi^2(\overline{\xi}(\overline{z}_1)) + \overline{\xi}([\Lambda\xi^2, \overline{z}_1])$$

$$\overline{\delta} A_\Lambda \overline{\xi}(\overline{z}_1, \xi^2 + \Lambda\xi^2) = \overline{\delta}(\overline{\xi} + \overline{\Lambda\xi})(\overline{z}_1, \xi^2 + \Lambda\xi^2)$$

$$= -\Lambda\xi^2(\overline{\xi}(\overline{z}_1)) - \overline{\xi}([\overline{z}_1, \Lambda\xi^2]) - [\overline{z}_1, \xi^2](\overline{\Lambda\xi})$$

$$= -\Lambda\xi^2(\overline{\xi}(\overline{z}_1)) - \overline{\xi}([\overline{z}_1, \Lambda\xi^2])$$

So $A_\Lambda \overline{\partial}_\Lambda(\overline{\xi})(\overline{z}_1, \xi^2 + \Lambda\xi^2) = \overline{\delta} A_\Lambda \overline{\xi}(\overline{z}_1, \xi^2 + \Lambda\xi^2)$.

Case: $(l = \overline{\xi}, \overline{l}_1 = \xi^1, \overline{l}_2 = \xi^2)$

$$A_\Lambda \overline{\partial}_\Lambda(\overline{\xi})(\xi^1 + \Lambda\xi^1, \xi^2 + \Lambda\xi^2)$$

$$= \overline{\partial\xi}(\xi^1, \xi^2) + [\Lambda, \overline{\xi}](\xi^1, \xi^2) + \overline{\Lambda\partial\xi}(\Lambda\xi^1, \Lambda\xi^2) = 0$$

$$\overline{\delta} A_\Lambda \overline{\xi}(\xi^1 + \Lambda\xi^1, \xi^2 + \Lambda\xi^2) = \overline{\delta}(\overline{\xi} + \overline{\Lambda\xi})(\xi^1 + \Lambda\xi^1, \xi^2 + \Lambda\xi^2)$$

$$= \Lambda\xi^1(\overline{\xi}(\Lambda\xi^2) + \xi^2(\overline{\Lambda\xi})) - \Lambda\xi^1(\overline{\xi}(\Lambda\xi^2) + \xi^2(\overline{\Lambda\xi}))$$

$$-2\langle\overline{\xi} + \overline{\Lambda\xi}, [\xi^1 + \Lambda\xi^1, \xi^2 + \Lambda\xi^2]\rangle = 0$$

So $A_\Lambda \overline{\partial}_\Lambda(\overline{\xi})(\xi^1 + \Lambda\xi^1, \xi^2 + \Lambda\xi^2) = \overline{\delta} A_\Lambda \overline{\xi}(\xi^1 + \Lambda\xi^1, \xi^2 + \Lambda\xi^2)$.

So we have $A_\Lambda\overline{\partial}_\Lambda(l) = \overline{\delta}A_\Lambda l$ for all $l \in L = T^{1,0} \oplus T^{*(0,1)}$. This is the base step for our induction. Assuming $A_\Lambda\overline{\partial}_\Lambda(l_1 \wedge \cdots \wedge l_n) = \overline{\delta}A_\Lambda(l_1 \wedge \cdots \wedge l_n)$ for $l_i \in L$, then,

$$
\begin{gathered}
A_\Lambda\overline{\partial}_\Lambda(l_0 \wedge l_1 \wedge \cdots \wedge l_n) \\
= A_\Lambda((\overline{\partial}_\Lambda l_0) \wedge l_1 \wedge \cdots \wedge l_n - l_0 \wedge \overline{\partial}_\Lambda(l_1 \wedge \cdots \wedge l_n)) \\
= (A_\Lambda\overline{\partial}_\Lambda l_0) \wedge A_\Lambda(l_1 \wedge \cdots \wedge l_n) - (A_\Lambda l_0) \wedge (A_\Lambda\overline{\partial}_\Lambda)(l_1 \wedge \cdots \wedge l_n) \\
= (\overline{\delta}A_\Lambda l_0) \wedge A_\Lambda(l_1 \wedge \cdots \wedge l_n) - (A_\Lambda l_0) \wedge (\overline{\delta}A_\Lambda)(l_1 \wedge \cdots \wedge l_n) \\
= \overline{\delta}((A_\Lambda l_0) \wedge A_\Lambda(l_1 \wedge \cdots \wedge l_n)) = \overline{\delta}A_\Lambda(l_0 \wedge l_1 \wedge \cdots \wedge l_n)
\end{gathered}
$$

Therefore diagram (3.19) commutes. ■

Chapter 4

Complex Symplectic Algebras

In this chapter, we apply the Theorem 17 to a specific class of examples, complex symplectic algebras. We find a compatible pair and outline the construction of weak mirror pairs. We also will look at four dimensional examples of complex symplectic algebras. These algebra were studied by Andrada in [1] and Cleyton, Ovando and Poon in [4].

4.1 Definition

Let $\mathfrak{g}$ be a real Lie algebra, and let ω be a symplectic form on $\mathfrak{g}$. Let V be the underlying vector space of $\mathfrak{g}$. Let $\gamma : \mathfrak{g} \to \mathrm{End}(V)$ satisfy the following properties,

$$\text{Torsion-free: } \gamma(x)y - \gamma(y)x = [x, y],$$

$$\text{Symplectic: } \omega(\gamma(x)y, z) + \omega(y, \gamma(x)z) = 0,$$

$$\text{Flat: } \gamma([x, y]) = \gamma(x)\gamma(y) - \gamma(y)\gamma(x).$$

Then γ is a symplectic representation on $\mathfrak{g}$. Also, we can view γ as a flat, symplectic invarient connection on G, a Lie group with Lie algebra $\mathfrak{g}$. We use this representation to define semi-direct product $\mathfrak{h} = \mathfrak{g} \ltimes V$, with Lie bracket defined by

$$[(x, 0), (y, 0)] = ([x, y], 0) \text{ for all } x, y \in \mathfrak{g},$$

$$[(x, 0), (0, v)] = (0, \gamma(x)v) \text{ for all } x \in \mathfrak{g}, v \in V.$$

On $\mathfrak{h}$, we can define a complex structure J defined by

$$J(x,u) = (-u,x).$$

This structure is integrable for any $\mathfrak{g}$ by the torsion-free condition, and so $\mathfrak{h}$ is a complex algebra. With this complex structure, the (1,0) vectors are $\mathfrak{h}^{1,0} = \{(x,-ix) : x \in \mathfrak{g}\}$ and the (1,0) forms are $\mathfrak{h}^{*(1,0)} = \{(\beta, i\beta) : \beta \in \mathfrak{g}^*\}$. Also, define

$$E(x,u) = (u,x).$$

Then $E \circ E = Id$, so E is a product structure. It is also integrable since γ is torsion-free. Also, $J \circ E = -E \circ J$ and so $\mathfrak{h}$ is hypersymplectic under (J,E), as studied by Hitchin in [8] and Andrada in [1].

Next we define the following two-forms on $\mathfrak{h}$,

$$\Omega_1((x,u),(y,v)) := -\omega(x,v) - \omega(u,y),$$
$$\Omega_2((x,u),(y,v)) := \omega(x,y) - \omega(u,v),$$
$$\Omega_3((x,u),(y,v)) := \omega(x,y) + \omega(u,v).$$

Since ω is closed and non-degenerate on $\mathfrak{g}$, for $i = 1,2,3$, $\Omega_i : \mathfrak{h} \to \mathfrak{h}^*$ are each non-degenerate and closed, each Ω_i is a symplectic form on $\mathfrak{h}$. Let $\Omega_c = \Omega_1 + i\Omega_2$. Then Ω_c is a closed (2,0) form. Also, Ω_3 is a closed (1,1) form. As maps,

$$\Omega_c : \mathfrak{h}^{1,0} \to \mathfrak{h}^{*(1,0)} \quad \text{and} \quad \Omega_3 : \mathfrak{h}^{1,0} \to \mathfrak{h}^{*(0,1)}$$

are non-degenerate. Then J and Ω_c form a complex symplectic structure on $\mathfrak{h}$.

We introduce one new form as well. Let $\mathfrak{h} = \mathfrak{g} \ltimes V$ be a complex symplectic algebra. Let g be a non-degenerate, symmetric bilinear form on V, the underlying vector space of $\mathfrak{g}$. We define the following non-degenerate (1,1) form.

$$\Omega_4((x,u),(y,v)) := g(x,v) - g(y,u).$$

Lemma 21 Ω_4 *is closed if and only if*

$g(\gamma(x)y, w) - g(\gamma(y)x, w) - g(x, \gamma(y)w) + g(y, \gamma(x)w) = 0$ *for all* $x, y, w \in \mathfrak{g}$.

Proof.

$$
\begin{gathered}
d\Omega_4((x,u),(y,v),(z,w)) \\
= -\Omega_4([(x,u),(y,v)],(z,w)) - \Omega_4([(y,v),(z,w)],(x,u)) - \Omega_4([(z,w),(x,u)],(y,v)) \\
= -\Omega_4(([x,y], \gamma(x)v - \gamma(y)u),(z,w)) - \Omega_4(([y,z], \gamma(y)w - \gamma(z)v),(x,u)) \\
-\Omega_4(([z,x], \gamma(z)u - \gamma(x)w),(y,v)) \\
= -g([x,y],w) + g(z, \gamma(x)v) - g(z, \gamma(y)u) \\
-g([y,z],u) + g(x, \gamma(y)w) - g(x, \gamma(z)v) \\
-g([z,x],v) + g(y, \gamma(z)u) - g(y, \gamma(x)w) \\
= -g(\gamma(x)y, w) + g(\gamma(y)x, w) + g(x, \gamma(y)w) - g(y, \gamma(x)w) \\
-g(\gamma(z)x, v) + g(\gamma(x)z, v) + g(z, \gamma(x)v) - g(x, \gamma(z)v) \\
-g(\gamma(y)z, u) + g(\gamma(z)y, u) + g(y, \gamma(z)u) - g(z, \gamma(y)u)
\end{gathered}
$$

where $[x,y] = \gamma(x)y - \gamma(y)x$. Since the last three lines are cyclic permutations of (x,u), (y,v) and (z,w), if one of these lines is 0, all three equal 0 and therefore $d\Omega_4 = 0$. Conversely, if $d\Omega_4 = 0$, set $z = u = v = 0$. Then the last two lines equal 0, and so $-g(\gamma(x)y, w) + g(\gamma(y)x, w) + g(x, \gamma(y)w) - g(y, \gamma(x)w) = 0$. ∎

We now prove a few computational lemmas, which will be used in solving for ϕ.

Lemma 22 $\Omega_3(a, -ia) = \Omega_4(-ig^{-1}\omega(a), -g^{-1}\omega(a))$ *and*
$\Omega_3(a, ia) = \Omega_4(ig^{-1}\omega(a), -g^{-1}\omega(a))$.

Proof. $\Omega_3((a,-ia),(n,in)) = \omega(a,n) + \omega(-ia, in) = 2\omega(a,n)$ and
$\Omega_4((-ig^{-1}\omega(a), -g^{-1}\omega(a)),(n,in)) = g(-ig^{-1}\omega(a), in) - g(-g^{-1}\omega(a), n)$
$= 2g(g^{-1}\omega(a), n) = 2\omega(a,n)$ for all (0,1) vectors (n, in). So
$\Omega_3(a,-ia) = \Omega_4(-ig^{-1}\omega(a), -g^{-1}\omega(a))$, and the second equality in the lemma is the conjugate of the first. ∎

Lemma 23 $\Omega_c(a, -ia) = \Omega_4(-2g^{-1}\omega(a), -2ig^{-1}\omega(a))$.

Proof. $\Omega_c((a, -ia), (n, -in)) = -\omega(a, -in) - \omega(-ia, n) + i(\omega(a, n) - \omega(-ia, -in)) = 4i\omega(a, n)$ and also $\Omega_4((-2g^{-1}\omega(a), -2ig^{-1}\omega(a)), (n, -in))$
$= g(-2g^{-1}\omega(a), -in) - g(-2ig^{-1}\omega(a), n) = 4ig(g^{-1}\omega(a), n) = 4i\omega(a, n)$ for all (1,0) vectors $(n, -in)$. So $\Omega_c(a, -ia) = \Omega_4(-2g^{-1}\omega(a), -2ig^{-1}\omega(a))$. ■

Lemma 24 $[(a, -ia), (b, ib)]^{(1,0)} = (-\gamma(b)(a), i\gamma(b)(a))$ *and*
$[(a, -ia), (b, ib)]^{(0,1)} = (\gamma(a)(b), i\gamma(a)(b))$

Proof.

$$[(a, -ia), (b, ib)] = ([a, b], \gamma(a)(ib) - \gamma(b)(-ia))$$
$$= (\gamma(a)(b) - \gamma(b)(a), i\gamma(a)(b) + i\gamma(b)(a))$$
$$= (\gamma(a)(b), i\gamma(a)(b)) + (-\gamma(b)(a), i\gamma(b)(a))$$

where $(-\gamma(b)(a), i\gamma(b)(a))$ is a (1,0) vector and $(\gamma(a)(b), i\gamma(a)(b))$ is a (0,1) vector. ■

4.2 Choice of Λ

Our goal is to build a compatible pair on complex symplectic $\mathfrak{h}$, using Ω_c, Ω_3 and Ω_4. Since $\mathfrak{g}$ is symplectic, it will have even real dimension $2n$. V will also have dimension $2n$ and so $\mathfrak{h}$ will have dimension $4n$.

We need $\Lambda : \mathfrak{h}^{*(1,0)} \to \mathfrak{h}^{1,0}$ as part of our compatible pair. We have $\Omega_c : \mathfrak{h}^{1,0} \to \mathfrak{h}^{*(1,0)}$ as a non-degenerate map from above. So we set $\Lambda = \Omega_c^{-1}$.

Proposition 25 *The tensor Λ satisfies the following conditions.*

1. *For any $\alpha, \beta \in \mathfrak{h}^{*(1,0)}$, $\Lambda(\alpha, \beta) = -\Omega_c(\Omega_c^{-1}(\alpha), \Omega_c^{-1}(\beta))$.*
2. *$\Lambda \in \wedge^2 \mathfrak{h}^{1,0}$.*
3. *$[\Lambda, \Lambda] = 0$.*
4. *$\overline{\partial}\Lambda = 0$.*

Proof. For $A, B \in \mathfrak{h}^{1,0}$ we set $\alpha = \Omega_c(A)$ and $\beta = \Omega_c(B)$. Then for part *1* we see that $\Lambda(\alpha, \beta) = \Lambda(\alpha)(\beta) = \iota_A(\beta) = \iota_A \Omega_c(B) = \Omega_c(B, A)$ $= \Omega_c(\Omega_c^{-1}(\beta), \Omega_c^{-1}(\alpha)) = -\Omega_c(\Omega_c^{-1}(\alpha), \Omega_c^{-1}(\beta))$. For part *2*, note that Λ is antisymmetric by the antisymmetry of Ω_c, and since $\Lambda(\overline{\alpha}) = 0$, $\Lambda \in \wedge^2 \mathfrak{h}^{1,0}$.

For part *3*, first note that for any $V_1, V_2 \in \mathfrak{h}^{1,0}$

$$[\Lambda, V_1 \wedge V_2](\alpha, \beta, \gamma) = ([\Lambda, V_1] \wedge V_2 - [\Lambda, V_2] \wedge V_1)(\alpha, \beta, \gamma)$$

$$= [\Lambda, V_1](\alpha, \beta)\gamma(V_2) + [\Lambda, V_1](\beta, \gamma)\alpha(V_2) + [\Lambda, V_1](\gamma, \alpha)\beta(V_2)$$

$$-[\Lambda, V_2](\alpha, \beta)\gamma(V_1) - [\Lambda, V_2](\beta, \gamma)\alpha(V_1) + [\Lambda, V_2](\gamma, \alpha)\beta(V_1)$$

which by Lemma 2.8

$$\begin{aligned}
&= -\alpha([\Lambda(\beta), V_1])\gamma(V_2) + \beta([\Lambda(\alpha), V_1])\gamma(V_2) - \beta([\Lambda(\gamma), V_1])\alpha(V_2) \\
&+\gamma([\Lambda(\beta), V_1])\alpha(V_2) - \gamma([\Lambda(\alpha), V_1])\beta(V_2) + \alpha([\Lambda(\gamma), V_1])\beta(V_2) \\
&+\alpha([\Lambda(\beta), V_2])\gamma(V_1) - \beta([\Lambda(\alpha), V_2])\gamma(V_1) + \beta([\Lambda(\gamma), V_2])\alpha(V_1) \\
&-\gamma([\Lambda(\beta), V_2])\alpha(V_1) + \gamma([\Lambda(\alpha), V_2])\beta(V_1) - \alpha([\Lambda(\gamma), V_2])\beta(V_1) \\
&= (\mathcal{L}_{\Lambda(\beta)}\alpha)(V_1)\gamma(V_2) - (\mathcal{L}_{\Lambda(\alpha)}\beta)(V_1)\gamma(V_2) + (\mathcal{L}_{\Lambda(\gamma)}\beta)(V_1)\alpha(V_2) \\
&-(\mathcal{L}_{\Lambda(\beta)}\gamma)(V_1)\alpha(V_2) + (\mathcal{L}_{\Lambda(\alpha)}\gamma)(V_1)\beta(V_2) - (\mathcal{L}_{\Lambda(\gamma)}\alpha)(V_1)\beta(V_2) \\
&-(\mathcal{L}_{\Lambda(\beta)}\alpha)(V_2)\gamma(V_1) + (\mathcal{L}_{\Lambda(\alpha)}\beta)(V_2)\gamma(V_1) - (\mathcal{L}_{\Lambda(\gamma)}\beta)(V_2)\alpha(V_1) \\
&+(\mathcal{L}_{\Lambda(\beta)}\gamma)(V_2)\alpha(V_1) - (\mathcal{L}_{\Lambda(\alpha)}\gamma)(V_2)\beta(V_1) + (\mathcal{L}_{\Lambda(\gamma)}\alpha)(V_2)\beta(V_1) \\
&= (V_1 \wedge V_2)(\mathcal{L}_{\Lambda(\beta)}\alpha, \gamma) - (V_1 \wedge V_2)(\mathcal{L}_{\Lambda(\alpha)}\beta, \gamma) + (V_1 \wedge V_2)(\mathcal{L}_{\Lambda(\gamma)}\beta, \alpha) \\
&-(V_1 \wedge V_2)(\mathcal{L}_{\Lambda(\beta)}\gamma, \alpha) + (V_1 \wedge V_2)(\mathcal{L}_{\Lambda(\alpha)}\gamma, \beta) - (V_1 \wedge V_2)(\mathcal{L}_{\Lambda(\gamma)}\alpha, \beta)
\end{aligned}$$

Where we use the fact that

$$\alpha([\Lambda(\beta), V_1]) = \mathcal{L}_{\Lambda(\beta)}(\alpha(V_1)) - (\mathcal{L}_{\Lambda(\beta)}\alpha)(V_1) = -(\mathcal{L}_{\Lambda(\beta)}\alpha)(V_1) \tag{4.1}$$

By the linearity of the bracket, we see that,

$$\begin{aligned}
[\Lambda, \Lambda](\alpha, \beta, \gamma) &= \Lambda(\mathcal{L}_{\Lambda(\beta)}\alpha, \gamma) - \Lambda(\mathcal{L}_{\Lambda(\alpha)}\beta, \gamma) + \Lambda(\mathcal{L}_{\Lambda(\gamma)}\beta, \alpha) \\
&-\Lambda(\mathcal{L}_{\Lambda(\beta)}\gamma, \alpha) + \Lambda(\mathcal{L}_{\Lambda(\alpha)}\gamma, \beta) - \Lambda(\mathcal{L}_{\Lambda(\gamma)}\alpha, \beta) \\
&= -\Omega_c(\Omega_c^{-1}(\mathcal{L}_{\Lambda(\beta)}\alpha), \Omega_c^{-1}(\gamma)) + \Omega_c(\Omega_c^{-1}(\mathcal{L}_{\Lambda(\alpha)}\beta), \Omega_c^{-1}(\gamma)) \\
&-\Omega_c(\Omega_c^{-1}(\mathcal{L}_{\Lambda(\gamma)}\beta), \Omega_c^{-1}(\alpha)) + \Omega_c(\Omega_c^{-1}(\mathcal{L}_{\Lambda(\beta)}\gamma), \Omega_c^{-1}(\alpha)) \\
&-\Omega_c(\Omega_c^{-1}(\mathcal{L}_{\Lambda(\alpha)}\gamma), \Omega_c^{-1}(\beta)) + \Omega_c(\Omega_c^{-1}(\mathcal{L}_{\Lambda(\gamma)}\alpha), \Omega_c^{-1}(\beta)) \\
&= \Omega_c(\Omega_c^{-1}(\mathcal{L}_{\Lambda(\alpha)}\beta - \mathcal{L}_{\Lambda(\beta)}\alpha), \Omega_c^{-1}(\gamma)) + \Omega_c(\Omega_c^{-1}(\mathcal{L}_{\Lambda(\beta)}\gamma - \mathcal{L}_{\Lambda(\gamma)}\beta), \Omega_c^{-1}(\alpha)) \\
&+\Omega_c(\Omega_c^{-1}(\mathcal{L}_{\Lambda(\gamma)}\alpha - \mathcal{L}_{\Lambda(\alpha)}\gamma), \Omega_c^{-1}(\beta))
\end{aligned}$$

By (4.1),

$$(\mathcal{L}_A\beta - \mathcal{L}_B\alpha)C = \alpha([\Lambda(\beta), C]) - \beta([\Lambda(\alpha), C]) = \Omega_c(A, [B, C]) - \Omega_c(B, [A, C])$$
$$= -\Omega_c([B, C], A) - \Omega_c([C, A], B) = \Omega_c([A, B], C) = \Omega_c([\Lambda(\alpha), \Lambda(\beta)], C)$$

since $d\Omega_c(A, B, C) = 0$. Therefore,

$$\Omega_c^{-1}(\mathcal{L}_{\Lambda(\alpha)}\beta - \mathcal{L}_{\Lambda(\beta)}\alpha) = [\Lambda(\alpha), \Lambda(\beta)] \tag{4.2}$$

and so,

$$[\Lambda, \Lambda](\alpha, \beta, \gamma)$$
$$= \Omega_c([\Lambda(\alpha), \Lambda(\beta)], \Lambda(\gamma)) + \Omega_c([\Lambda(\beta), \Lambda(\gamma)], \Lambda(\alpha)) + \Omega_c([\Lambda(\gamma), \Lambda(\alpha)], \Lambda(\beta))$$
$$= d\Omega_c(\Lambda(\alpha), \Lambda(\beta), \Lambda(\gamma)) = 0$$

For part *4*, first we note that $[\overline{X}, \alpha]Y = -\alpha([\overline{X}, Y]) = -\Omega_c(A, [\overline{X}, Y])$ $= -\Omega_c([Y, A], \overline{X}) - \Omega_c([A, \overline{X}], Y) = \Omega_c([\overline{X}, A], Y)$ where $\Omega_c([Y, A], \overline{X}) = 0$ since Ω_c is a (2,0) form. So $\Lambda([\overline{X}, \alpha]) = [\overline{X}, \Lambda(\alpha)]$.

Also since $\Lambda \in \mathfrak{h}^{1,0} \wedge \mathfrak{h}^{1,0}$, $\overline{\partial}\Lambda \in \mathfrak{h}^{*0,1} \wedge \mathfrak{h}^{1,0} \wedge \mathfrak{h}^{1,0}$ so we consider

$$\overline{\partial}\Lambda(\overline{X}, \alpha, \beta) = -\Lambda([\overline{X}, \alpha], \beta) - \Lambda([\alpha, \beta], \overline{X}) - \Lambda([\beta, \overline{X}], \alpha)$$
$$= \Lambda([\overline{X}, \beta], \alpha) - \Lambda([\overline{X}, \alpha], \beta) = \beta([\Lambda(\alpha), \overline{X}]) - \alpha([\Lambda(\beta), \overline{X}])$$
$$= (\mathcal{L}_{\Lambda(\alpha)}\beta - \mathcal{L}_{\Lambda(\beta)}\alpha)(\overline{X}) = \Omega_c([\Lambda(\alpha), \Lambda(\beta)], \overline{X}) = 0.$$

■

So Λ is a valid choice for our compatible pair. Let $L = \mathfrak{h}^{1,0} \oplus \mathfrak{h}^{*(0,1)}$. Then the type of the generalized complex structure L is $2n$. Also, $L_{\overline{\Lambda}}$ is a generalized complex structure, since $\partial\overline{\Lambda} + \frac{1}{2}[\overline{\Lambda}, \overline{\Lambda}] = 0$. Since Ω_c is non-degenerate, Λ is non-degenerate. So the type of $L_{\overline{\Lambda}}$ is 0. Λ gives us the kind of deformation we want. Now our goal is to find a compatible ϕ and use Theorem 17.

4.3 Choice of ϕ

Given g as above with $d\Omega_4 = 0$, let $\phi = \lambda\Omega_3^{-1} + \mu\Omega_4^{-1}$. We seek λ and μ such that,

$$\phi([l_1, l_1]) = [\phi l_1, l_2] + [l_1, \phi l_2] \quad \text{and} \tag{4.3}$$

$$\overline{\partial}\phi l - \phi\overline{\partial} l = [\Lambda, l], \tag{4.4}$$

for all $l, l_1, l_2 \in L = \mathfrak{h}^{1,0} \oplus \mathfrak{h}^{*(0,1)}$. The first equation is linear, so we can set $\phi = c\Omega_3^{-1} + \Omega_4^{-1}$ where $c = \frac{\lambda}{\mu}$, if $\mu \neq 0$.

If $l_1, l_2 \in \mathfrak{h}^{1,0}$, equation (4.3) is trivially satisfied.

If $l_1 = A = (a, -ia) \in \mathfrak{h}^{(1,0)}$ and $l_2 = \overline{\beta} \in \mathfrak{h}^{*(0,1)}$, equation (4.3) reduces to $\phi([A, \overline{\beta}]) = [A, \phi\overline{\beta}]$. If $\overline{\beta} = \Omega_3(b, -ib)$, then $\overline{\beta} = \Omega_4(-ig^{-1}\omega(b), -g^{-1}\omega(b))$, by lemma (22). Since $\Omega_3^{-1}([A, \overline{\beta}])$, $\Omega_4^{-1}([A, \overline{\beta}])$, $[A, \Omega_3^{-1}\overline{\beta}]$ and $[A, \Omega_4^{-1}\overline{\beta}]$ are all (1,0) vectors, we contract with a (1,0) form $\eta = \Omega_3(n, in) = \Omega_4(ig^{-1}\omega(n), -g^{-1}\omega(n))$. Then equation (4.3) becomes,

$$\begin{aligned}
0 &= \eta(\phi([A, \overline{\beta}]) - [A, \phi\overline{\beta}]) \\
&= c\eta(\Omega_3^{-1}([A, \overline{\beta}])) + \eta(\Omega_4^{-1}([A, \overline{\beta}])) - c\eta([A, \Omega_3^{-1}\overline{\beta}]) - \eta([A, \Omega_4^{-1}\overline{\beta}]) \\
&= c\Omega_3((n, in), \Omega_3^{-1}([A, \overline{\beta}])) + \Omega_4((ig^{-1}\omega(n), -g^{-1}\omega(n)), \Omega_4^{-1}([A, \overline{\beta}])) \\
&\quad - c\Omega_3((n, in), [(a, -ia), (b, -ib)]) \\
&\quad - \Omega_4((ig^{-1}\omega(n), -g^{-1}\omega(n)), [(a, -ia), (-ig^{-1}\omega(b), -g^{-1}\omega(b))]) \\
&= -c[A, \overline{\beta}](n, in) - [A, \overline{\beta}](ig^{-1}\omega(n), -g^{-1}\omega(n)) - c\Omega_3((n, in), ([a, b], -i[a, b])) \\
&\quad - \Omega_4((ig^{-1}\omega(n), -g^{-1}\omega(n)), (-i[a, g^{-1}\omega(b)], -[a, g^{-1}\omega(b)]))
\end{aligned}$$

$$= c\overline{\beta}((\gamma(a)n, i\gamma(a)n)) + \overline{\beta}((i\gamma(a)g^{-1}\omega(n), -\gamma(a)g^{-1}\omega(n)))$$

$$-c\Omega_3((n, in), ([a,b], -i[a,b]))$$

$$-\Omega_4((ig^{-1}\omega(n), -g^{-1}\omega(n)), (-i[a, g^{-1}\omega(b)], -[a, g^{-1}\omega(b)]))$$

$$= c\Omega_3((b, -ib), (\gamma(a)n, i\gamma(a)n))$$

$$+\Omega_4((-ig^{-1}\omega(b), -g^{-1}\omega(b)), (i\gamma(a)g^{-1}\omega(n), -\gamma(a)g^{-1}\omega(n)))$$

$$-c\Omega_3((n, in), ([a,b], -i[a,b]))$$

$$-\Omega_4((ig^{-1}\omega(n), -g^{-1}\omega(n)), (-i[a, g^{-1}\omega(b)], -[a, g^{-1}\omega(b)]))$$

Using the definitions for Ω_3 and Ω_4,

$$0 = 2c\omega(b, \gamma(a)n) + 2ig(g^{-1}\omega(b), \gamma(a)g^{-1}\omega(n))$$

$$-2c\omega(n, [a,b]) + 2ig(g^{-1}\omega(n), [a, g^{-1}\omega(b)])$$

$$= 2c(-\omega(\gamma(a)b, n) + \omega(\gamma(a)b - \gamma(b)a, n)) + 2ig(g^{-1}\omega(b), \gamma(a)g^{-1}\omega(n))$$

$$+2ig(g^{-1}\omega(n), \gamma(a)g^{-1}\omega(b)) - 2ig(g^{-1}\omega(n), \gamma(g^{-1}\omega(b))a)$$

$$= -2c\omega(\gamma(b)a, n) + 2ig(a, \gamma(g^{-1}\omega(b))g^{-1}\omega(n))$$

where the last equality is by Lemma 21. Then by equation (4.1),

$$0 = -2c\omega(\gamma(b)n, a) + 2ig(\gamma(g^{-1}\omega(b))g^{-1}\omega(n), a).$$

Since this must hold for and $a, b, n \in \mathfrak{g}$,

$$cg^{-1}\omega(\gamma(b)n) = i\gamma(g^{-1}\omega(b))(g^{-1}\omega(n)) \tag{4.5}$$

If $l_1 = \overline{\alpha} \in \mathfrak{h}^{*(0,1)}$ and $l_2 = \overline{\beta} \in \mathfrak{h}^{*(0,1)}$, equation (4.3) is $0 = [\phi\overline{\alpha}, \overline{\beta}] + [\overline{\alpha}, \phi\overline{\beta}]$. If $\overline{\alpha} = \Omega_3(a, -ia)$ and $\overline{\beta} = \Omega_3(b, -ib)$, then by Lemma (22) $\overline{\alpha} = \Omega_4(-ig^{-1}\omega(a), -g^{-1}\omega(a))$ and $\overline{\beta} = \Omega_4(-ig^{-1}\omega(b), -g^{-1}\omega(b))$. Since $[\phi\overline{\alpha}, \overline{\beta}]$and $[\overline{\alpha}, \phi\overline{\beta}]$ are (0,1) forms, we contract with a (0,1) vector (n, in).

$$0 = c[\Omega_3^{-1}\overline{\alpha}, \overline{\beta}](n, in) + [\Omega_4^{-1}\overline{\alpha}, \overline{\beta}](n, in) - c[\Omega_3^{-1}\overline{\beta}, \overline{\alpha}](n, in) - [\Omega_4^{-1}\overline{\beta}, \overline{\alpha}](n, in)$$

$$= c[(a, -ia), \overline{\beta}](n, in) + [(-ig^{-1}\omega(a), -g^{-1}\omega(a)), \overline{\beta}](n, in)$$

$$-c[(b, -ib), \overline{\alpha}](n, in) - [(-ig^{-1}\omega(b), -g^{-1}\omega(b)), \overline{\alpha}](n, in)$$

$$= -c\overline{\beta}([(a, -ia), (n, in)]^{(0,1)}) - \overline{\beta}([(-ig^{-1}\omega(a), -g^{-1}\omega(a)), (n, in)]^{(0,1)})$$

$$+c\overline{\alpha}([(b, -ib), (n, in)]^{(0,1)}) + \overline{\alpha}([(-ig^{-1}\omega(b), -g^{-1}\omega(b)), (n, in)]^{(0,1)})$$

$$= -c\Omega_3((b, -ib), (\gamma(a)n, i\gamma(a)n)) + c\Omega_3((a, -ia), (\gamma(b)n, i\gamma(b)n))$$

$$-\Omega_4((-ig^{-1}\omega(b), -g^{-1}\omega(b)), (-i\gamma(g^{-1}\omega(a))n, \gamma(g^{-1}\omega(a)n)))$$

$$+\Omega_4((-ig^{-1}\omega(a), -g^{-1}\omega(a)), (-i\gamma(g^{-1}\omega(b))n, \gamma(g^{-1}\omega(b))n))$$

Using the definition for Ω_3 and Ω_4, along with Lemma 21 and equation (4.1) yields,

$$0 = -2c(\omega(b, \gamma(a)n) - \omega(a, \gamma(b)n))$$

$$+2i(g(g^{-1}\omega(b), \gamma(g^{-1}\omega(a))n) - g(g^{-1}\omega(a), \gamma(g^{-1}\omega(b))n))$$

$$= 2c(\omega(\gamma(a)b, n) - \omega(\gamma(b)a, n))$$

$$+2i(g(\gamma(g^{-1}\omega(b))(g^{-1}\omega(a)), n) - g(\gamma(g^{-1}\omega(a))(g^{-1}\omega(b)), n))$$

$$= 2c\omega([a, b], n) - 2ig([g^{-1}\omega(a), g^{-1}\omega(b)], n)$$

Since this must hold for all $a, b, n \in \mathfrak{g}$,

$$cg^{-1}\omega([a, b]) = i[g^{-1}\omega(a), g^{-1}\omega(b)] \tag{4.6}$$

However, this is implied by the previous condition, equation (4.5), so we have the following proposition.

Proposition 26 *If g is an invariant non-degenerate symmetric form on $\mathfrak{g}$ and $\Omega_4((x,u),(y,v)) = g(x,v) - g(y,u)$ is closed, and if there is a $c \in \mathbb{C}$ such that $cg^{-1}\omega(\gamma(a)b) = i\gamma(g^{-1}\omega(a))(g^{-1}\omega(b))$ for all $a, b \in \mathfrak{g}$, then $\phi = c\mu\Omega_3^{-1} + \mu\Omega_4^{-1}$ satisfies $\phi([l_1, l_2]) = [\phi(l_1), l_2] + [l_1, \phi(l_2)]$ for all $l_1, l_2 \in \mathfrak{h}^{(1,0)} \oplus \mathfrak{h}^{*(0,1)}$ for any $\mu \in \mathbb{C}$.*

Now we seek conditions on $\phi = c\mu\Omega_3^{-1} + \mu\Omega_4^{-1}$ so that $\overline{\partial}\phi l - \phi\overline{\partial} l = [\Lambda, l]$ where $l \in \mathfrak{h}^{(1,0)} \oplus \mathfrak{h}^{*(0,1)}$. With this ϕ equation (4.4) becomes,

$$\overline{\partial}\phi l - \phi\overline{\partial} l = [\Lambda, l]$$

$$c\mu\overline{\partial}\Omega_3^{-1}(l) + \mu\overline{\partial}\Omega_4^{-1}(l) - c\mu\Omega_3^{-1}\overline{\partial}(l) - \mu\Omega_4^{-1}\overline{\partial}(l) = [\Lambda, l]$$

$$c\mu(\overline{\partial}\Omega_3^{-1}(l) - \Omega_3^{-1}\overline{\partial}(l)) + \mu(\overline{\partial}\Omega_4^{-1}(l) - \Omega_4^{-1}\overline{\partial}(l)) = [\Lambda, l]$$

When $l = z \in \mathfrak{h}^{1,0}$, $\overline{\partial}\Omega_3^{-1}(z) = \overline{\partial}\Omega_4^{-1}(z) = 0$. The other parts of the equation are $(2,0)$ bivectors, so we contract with $\alpha, \beta \in \mathfrak{h}^{*(1,0)}$, with $\alpha = \Omega_c(a, -ia) = \Omega_4(-2g^{-1}\omega(a), -2ig^{-1}\omega(a))$ and $\beta = \Omega_c(b, -ib)$ $= \Omega_4(-2g^{-1}\omega(b), -2ig^{-1}\omega(b))$. Equation (4.4) reduces to,

$$c\mu(-\Omega_3^{-1}\overline{\partial}(z)) + \mu(-\Omega_4^{-1}\overline{\partial}(z))(\alpha, \beta) = [\Lambda, z](\alpha, \beta) \tag{4.7}$$

Then, by Lemma 13, the right hand side of (4.7) is

$$[\Lambda, z](\alpha, \beta) = -\alpha([\Lambda(\beta), z]) + \beta([\Lambda(\alpha), z]) = -\alpha([B, z]) + \beta([A, z])$$

$$= -\Omega_c(A, [B, z]) + \Omega_c(B, [A, z]) = \Omega_c(z, [A, B])$$

since $d\Omega_c(A, B, z) = 0$. Letting $A = (a, -ia)$ and $B = (b, -ib)$ and $z = (x, -ix)$, we get

$$[\Lambda, z](\alpha, \beta) = \Omega_c(z, [A, B]) = \Omega_c((x, -ix), [(a, -ia), (b, -ib)])$$

$$= \Omega_c((x, -ix), [a, b] - i\gamma(a)b + i\gamma(b)a) = \Omega_c((x, -ix), ([a, b], -i[a, b]))$$

$$= \Omega_1((x, -ix), ([a, b], -i[a, b])) + i\Omega_2((x, -ix), ([a, b], -i[a, b]))$$

$$= -\omega(x, -i[a, b]) - \omega(-ix, [a, b]) + i\omega(x, [a, b]) - i\omega(-ix, -i[a, b])$$

$$= 4i\omega(x, [a, b])$$

Therefore,

$$[\Lambda, z](\alpha, \beta) = 4i\omega(x, [a, b]) \tag{4.8}$$

For each part of left hand side of equation (4.7), we use the differential formula (2.19) with basis elements $\overline{\alpha}^j = (e^j, -ie^j)$ and $\overline{z}_j = \frac{1}{2}(e_j, ie_j)$, with basis elements $e_j \in \mathfrak{g}$, $e^j \in \mathfrak{g}^*$ and $e^i(e_j) = \delta^i_j$. The the first term in the left hand side of equation (4.7) is,

$$-\Omega_3^{-1}\overline{\partial} z(\alpha, \beta) = \sum [z, \overline{z}_j]^{1,0} \wedge \Omega_3^{-1}(\overline{\alpha}^j)(\beta, \alpha)$$

$$= \sum(\beta([z, \overline{z}_j])\alpha(\Omega_3^{-1}(\overline{\alpha}^j)) - \alpha([z, \overline{z}_j])\beta(\Omega_3^{-1}(\overline{\alpha}^j)))$$

$$= \sum(\beta([z, \overline{z}_j])\Omega_3(\Omega_3^{-1}(\alpha), \Omega_3^{-1}(\overline{\alpha}^j)) - \alpha([z, \overline{z}_j])\Omega_3(\Omega_3^{-1}(\beta), \Omega_3^{-1}(\overline{\alpha}^j)))$$

$$= \sum(\beta([(x, -ix), \tfrac{1}{2}(e_j, ie_j)])(-4ie^j(a)) - \alpha([(x, -ix), \tfrac{1}{2}(e_j, ie_j)])(-4ie^j(b)))$$

$$= -2i\sum(\beta([(x, -ix), (e^j(a)e_j, ie^j(a)e_j)]) - \alpha([(x, -ix), (e^j(b)e_j, ie^j(b)e_j)]))$$

$$= -2i(\beta([(x, -ix), (a, ia)]) - \alpha([(x, -ix), (b, ib)]))$$

$$= -2i(\beta([x, a], i\gamma(x)a + i\gamma(a)x) - \alpha([x, b], i\gamma(x)b + i\gamma(b)x))$$

$$= -2i(\Omega_c((b, -ib), ([x, a], i\gamma(x)a + i\gamma(a)x)) - \Omega_c((a, -ia), ([x, b], i\gamma(x)b + i\gamma(b)x)))$$

$$= -2i(-\omega(b, i\gamma(x)a + i\gamma(a)x) - \omega(-ib, [x,a]) + i\omega(b, [x,a])$$

$$-i\omega(-ib, i\gamma(x)a + i\gamma(a)x) + \omega(a, i\gamma(x)b + i\gamma(b)x)$$

$$+\omega(-ia, [x,b]) - i\omega(a, [x,b]) + i\omega(-ia, i\gamma(x)b + i\gamma(b)x))$$

$$= -4(\omega(b, \gamma(x)a + \gamma(a)x) - \omega(b, [x,a]) - \omega(a, \gamma(x)b + \gamma(b)x) + \omega(a, [x,b]))$$

$$= -8(\omega(b, \gamma(a)x) - \omega(a, \gamma(b)x))$$

$$= 8(\omega(\gamma(a)b, x) - \omega(\gamma(b)a, x)) = 8(\omega(\gamma(a)b - \gamma(b)a, x)) = 8\omega([a,b], x)$$

After noting that $\omega(b, \gamma(a)x) = -\omega(\gamma(a)b, x)$, since γ is symplectic. So,

$$-\Omega_3^{-1}\overline{\partial}z(\alpha, \beta) = -8\omega(x, [a,b]) \tag{4.9}$$

The second term in the left hand side of equation (4.7) is,

$$-\Omega_4^{-1}\overline{\partial}(z)(\alpha,\beta) = -\Omega_4^{-1}\sum([z,\overline{z}_j] \wedge \overline{\alpha}^j)(\alpha,\beta) = -\sum([z,\overline{z}_j] \wedge \Omega_4^{-1}(\overline{\alpha}^j))(\alpha,\beta)$$

$$= -\sum(\alpha([z,\overline{z}_j])\beta(\Omega_4^{-1}(\overline{\alpha}^j)) - \beta([z,\overline{z}_j])\alpha((\Omega_4^{-1}(\overline{\alpha}^j))))$$

$$= \sum(\alpha([z,\overline{z}_j]^{(1,0)})\Omega_4(\Omega_4^{-1}(\overline{\alpha}^j), (-2g^{-1}\omega(b), -2ig^{-1}\omega(b)))$$

$$-\beta([z,\overline{z}_j]^{(1,0)})\Omega_4(\Omega_4^{-1}(\overline{\alpha}^j), (-2g^{-1}\omega(a), -2ig^{-1}\omega(a))))$$

$$= \sum(\alpha(\tfrac{1}{2}(-\gamma(e_j)x, i\gamma(e_j)x))e^j(-4g^{-1}\omega(b))$$

$$-\beta(\tfrac{1}{2}(-\gamma(e_j)x, i\gamma(e_j)x))e^j(-4g^{-1}\omega(a)))$$

$$= -2(\alpha((-\gamma(g^{-1}\omega(b))x, i\gamma(g^{-1}\omega(b))x)) - \beta((-\gamma(g^{-1}\omega(a))x, i\gamma(g^{-1}\omega(a))x)))$$

$$= 2(\Omega_4((-2g^{-1}\omega(a), -2ig^{-1}\omega(a)), (\gamma(g^{-1}\omega(b))x, -i\gamma(g^{-1}\omega(b))x))$$

$$-\Omega_4((-2g^{-1}\omega(b), -2ig^{-1}\omega(b)), (\gamma(g^{-1}\omega(a))x, -i\gamma(g^{-1}\omega(a))x)))$$

$$= -4(-2ig(g^{-1}\omega(a), \gamma(g^{-1}\omega(b))x) + 2ig(g^{-1}\omega(b), \gamma(g^{-1}\omega(a))x))$$

Now, using Lemma 21 and equation (4.5), we see this equals

$$= 8i(g(\gamma(g^{-1}\omega(a))(g^{-1}\omega(b)), x) - g(\gamma(g^{-1}\omega(b))(g^{-1}\omega(a)), x))$$

$$= 8ig([g^{-1}\omega(a), g^{-1}\omega(b)], x) = 8cg(g^{-1}\omega([a,b], x)) = -8c\omega(x, [a,b])$$

This means,

$$-\Omega_4^{-1}\overline{\partial}(z)(\alpha,\beta) = -8c\omega(x,[a,b]) \tag{4.10}$$

So, using equations (4.8), (4.9) and (4.10), equation (4.7) becomes,

$$c\mu(-\Omega_3^{-1}\overline{\partial}(z)(\alpha,\beta)) + \mu(-\Omega_4^{-1}\overline{\partial}(z)(\alpha,\beta)) = [\Lambda, z](\alpha,\beta)$$
$$-16c\mu\omega(x,[a,b]) = 4i\omega(x,[a,b])$$
$$-4c\mu\omega(x,[a,b]) = i\omega(x,[a,b])$$

This will be satisfied when $c\mu = \lambda = -\frac{i}{4}$.

When $l = \overline{\alpha} \in \mathfrak{h}^{*(0,1)}$, equation (4.4) becomes

$$c\mu(\overline{\partial}\Omega_3^{-1}(\overline{\alpha}) - \Omega_3^{-1}\overline{\partial}(\overline{\alpha})) + \mu(\overline{\partial}\Omega_4^{-1}(\overline{\alpha}) - \Omega_4^{-1}\overline{\partial}(\overline{\alpha})) = [\Lambda, \overline{\alpha}]. \tag{4.11}$$

Each part is in $\mathfrak{h}^{1,0} \wedge \mathfrak{h}^{*(0,1)}$, so we evaluate on a (1,0) form β and a (0,1) vector $\overline{z}$. By Lemma 4,

$$[\Lambda, \overline{\alpha}](\beta, \overline{z}) = -\beta([\Lambda(\overline{z}), \overline{\alpha}]) + \overline{z}([\Lambda(\beta), \overline{\alpha}])$$
$$= \overline{z}([\Lambda(\beta), \overline{\alpha}]) = d\overline{\alpha}(\Lambda(\beta), \overline{z}) = -\overline{\alpha}([\Lambda(\beta), \overline{z}])$$

We set $\overline{\alpha} = \Omega_3(a, -ia) = (\omega(a), -i\omega(a)) = \Omega_4(-ig^{-1}\omega(a), -g^{-1}\omega(a))$, $\beta = \Omega_c(b, -ib)$ $= 2i(\omega(b), i\omega(b))$ so $\Lambda(\beta) = \Omega_c^{-1}(\beta) = (b, -ib)$ and $\Omega_3^{-1}\beta = 2i(b, ib) = 2i\bar{B}$, where $\bar{B} = (b, ib)$. Then $\beta = \Omega_4(-2g^{-1}\omega(b), -2ig^{-1}\omega(b))$. Also we set $\overline{z} = (x, ix)$. So right hand side of equation (4.11) becomes,

$$[\Lambda, \overline{\alpha}](\beta, \overline{z}) = -\overline{\alpha}([\Lambda(\beta), \overline{z}]) = -(\omega(a), -i\omega(a))([(b, -ib), (x, ix)])$$
$$= -(\omega(a), -i\omega(a))([b, x], i\gamma(b)x + i\gamma(x)b) = -2\omega(a, \gamma(b)x)$$

This gives,

$$[\Lambda, \overline{\alpha}](\beta, \overline{z}) = -2\omega(a, \gamma(b)x) \tag{4.12}$$

The left hand side of equation (4.11) has four parts. We will examine each of them.

$$\overline{\partial}\Omega_3^{-1}(\overline{\alpha})(\beta,\overline{z}) = -\Omega_3^{-1}(\overline{\alpha})([\beta,\overline{z}]) = -\Omega_3^{-1}(\overline{\alpha})([\beta,\overline{z}]) = d\beta((x,ix),(a,-ia))$$
$$= -\beta([(x,ix),(a,-ia)]) = -2i(\omega(b),i\omega(b))([x,a],-i\gamma(x)a - i\gamma(a)x)$$
$$= -2i(\omega(b,[x,a]) + i\omega(b,-i\gamma(x)a - i\gamma(a)x)) = -4i\omega(b,\gamma(x)a)$$

So we have,

$$\overline{\partial}\Omega_3^{-1}(\overline{\alpha})(\beta,\overline{z}) = -4i\omega(b,\gamma(x)a) \tag{4.13}$$

For $\overline{\partial}\Omega_4^{-1}(\overline{\alpha})$, we have,

$$\overline{\partial}\Omega_4^{-1}(\overline{\alpha})(\beta,\overline{z}) = -[\beta,\overline{z}](\Omega_4^{-1}(\overline{\alpha})) = \beta([\Omega_4^{-1}(\overline{\alpha}),\overline{z}]^{(1,0)})$$
$$= \beta([(-ig^{-1}\omega(a),-g^{-1}\omega(a)),(x,ix)]^{(1,0)})$$
$$= \Omega_4((-2g^{-1}\omega(b),-2ig^{-1}\omega(b)),(i\gamma(x)(g^{-1}\omega(a)),\gamma(x)(g^{-1}\omega(a))))$$
$$= -4g(g^{-1}\omega(b),\gamma(x)(g^{-1}\omega(a))).$$

So we have,

$$\overline{\partial}\Omega_4^{-1}(\overline{\alpha})(\beta,\overline{z}) = -4g(g^{-1}\omega(b),\gamma(x)(g^{-1}\omega(a))). \tag{4.14}$$

For the other two parts, we note that $\overline{\partial}\overline{\alpha}$ is a (0,2) form, so it is a sum of terms of the form $\overline{\theta}_1 \wedge \overline{\theta}_2$ where $\overline{\theta}_1$ and $\overline{\theta}_2$ are (0,1) forms. As before, for $i = 3,4$, we extend $\Omega_i^{-1} : \bigwedge^2 L \to \bigwedge^2 L$ by $\Omega_i^{-1}(l_1 \wedge l_2) = (\Omega_i^{-1}(l_1)) \wedge l_2 + l_1 \wedge (\Omega_i^{-1}(l_2))$. Since Ω_3^{-1} and Ω_4^{-1} are linear, we examine $-\Omega_i^{-1}(\overline{\theta}_1 \wedge \overline{\theta}_2)(\beta,\overline{z})$.

$$-\Omega_3^{-1}(\overline{\theta}_1 \wedge \overline{\theta}_2)(\beta,\overline{z}) = (-\Omega_3^{-1}(\overline{\theta}_1) \wedge \overline{\theta}_2 - \overline{\theta}_1 \wedge \Omega_3^{-1}(\overline{\theta}_2))(\beta,\overline{z})$$
$$= -\beta(\Omega_3^{-1}(\overline{\theta}_1))\overline{\theta}_2(\overline{z}) + \overline{\theta}_1(\overline{z})\beta(\Omega_3^{-1}(\overline{\theta}_2)) = -\beta(\Omega_3^{-1}(\overline{\theta}_1))\overline{\theta}_2(\overline{z}) + \beta(\Omega_3^{-1}(\overline{\theta}_2))\overline{\theta}_1(\overline{z})$$
$$= -\Omega_3(2i\bar{B},\Omega_3^{-1}(\overline{\theta}_1))\overline{\theta}_2(\overline{z}) + \Omega_3(2i\bar{B},\Omega_3^{-1}(\overline{\theta}_2))\overline{\theta}_1(\overline{z})$$
$$= (\overline{\theta}_1)(2iB)\overline{\theta}_2(\overline{z}) - (\overline{\theta}_2)(2iB)\overline{\theta}_1(\overline{z}) = (\overline{\theta}_1 \wedge \overline{\theta}_2)(2i\bar{B},\overline{z}) = (\overline{\theta}_1 \wedge \overline{\theta}_2)(\Omega_3^{-1}\beta,\overline{z}).$$

So,

$$-\Omega_3^{-1}(\overline{\partial}\overline{\alpha})(\beta,\overline{z}) = (\overline{\partial}\overline{\alpha})(\Omega_3^{-1}\beta,\overline{z}). \tag{4.15}$$

Also,

$$-\Omega_4^{-1}(\overline{\theta}_1 \wedge \overline{\theta}_2)(\beta,\overline{z}) = (-\Omega_4^{-1}(\overline{\theta}_1) \wedge \overline{\theta}_2 - \overline{\theta}_1 \wedge \Omega_4^{-1}(\overline{\theta}_2))(\beta,\overline{z})$$
$$= -\beta(\Omega_4^{-1}(\overline{\theta}_1))\overline{\theta}_2(\overline{z}) + \overline{\theta}_1(\overline{z})\beta(\Omega_4^{-1}(\overline{\theta}_2)) = -\beta(\Omega_4^{-1}(\overline{\theta}_1))\overline{\theta}_2(\overline{z}) + \beta(\Omega_4^{-1}(\overline{\theta}_2))\overline{\theta}_1(\overline{z})$$
$$= -\Omega_4(-2g^{-1}\omega(b), -2ig^{-1}\omega(b))(\Omega_4^{-1}(\overline{\theta}_1))\overline{\theta}_2(\overline{z})$$
$$+\Omega_4(-2g^{-1}\omega(b), -2ig^{-1}\omega(b))(\Omega_4^{-1}(\overline{\theta}_2))\overline{\theta}_1(\overline{z})$$
$$= \overline{\theta}_1((-2g^{-1}\omega(b), -2ig^{-1}\omega(b)))\overline{\theta}_2(\overline{z}) - \overline{\theta}_2((-2g^{-1}\omega(b), -2ig^{-1}\omega(b)))\overline{\theta}_1(\overline{z})$$
$$= (\overline{\theta}_1 \wedge \overline{\theta}_2)(\Omega_4^{-1}\beta,\overline{z}).$$

So,

$$-\Omega_4^{-1}(\overline{\partial}\overline{\alpha})(\beta,\overline{z}) = (\overline{\partial}\overline{\alpha})(\Omega_4^{-1}\beta,\overline{z}). \tag{4.16}$$

Using equation (4.15) shows,

$$-\Omega_3^{-1}(\overline{\partial}\overline{\alpha})(\beta,\overline{z}) = (\overline{\partial}\overline{\alpha})(\Omega_3^{-1}\beta,\overline{z}) = -\overline{\alpha}([\Omega_3^{-1}\beta,\overline{z}]) = -\overline{\alpha}([2i(b,ib),(x,ix)])$$
$$= -2i(\omega(a), -i\omega(a))([b,x], i[b,x]) = -4i\omega(a,[b,x]).$$

Yielding,

$$-\Omega_3^{-1}(\overline{\partial}\overline{\alpha})(\beta,\overline{z}) = -4i\omega(a,[b,x]). \tag{4.17}$$

Also, using equation (4.16) yields,

$$-\Omega_4^{-1}\overline{\partial}(\overline{\alpha})(\beta,\overline{z}) = \overline{\partial}\overline{\alpha}(\Omega_4^{-1}\beta,\overline{z}) = \overline{\partial}\overline{\alpha}((-2g^{-1}\omega(b), -2ig^{-1}\omega(b)),(x,ix))$$
$$= -\overline{\alpha}([(-2g^{-1}\omega(b), -2ig^{-1}\omega(b)),(x,ix)])$$
$$= 2\Omega_4((-ig^{-1}\omega(a), -g^{-1}\omega(a)),([g^{-1}\omega(b),x], i[g^{-1}\omega(b),x]))$$
$$= 4g(g^{-1}\omega(a),[g^{-1}\omega(b),x])$$

So,

$$-\Omega_4^{-1}\overline{\partial}(\overline{\alpha})(\beta,\overline{z}) = 4g(g^{-1}\omega(a),[g^{-1}\omega(b),x]). \tag{4.18}$$

Using equations (4.13) and (4.17) yields,

$$\overline{\partial}\Omega_3^{-1}(\overline{\alpha})(\beta,\overline{z}) - \Omega_3^{-1}\overline{\partial}(\overline{\alpha})(\beta,\overline{z}) = -4i\omega(b,\gamma(x)a) - 4i\omega(a,[b,x])$$
$$= -4i\omega(b,\gamma(x)a) - 4i\omega(a,\gamma(b)x) + 4i\omega(a,\gamma(x)b)$$

Using equations (4.14) and (4.18), as well as lemma (21) (with $y = g^{-1}\omega(b)$, $w = g^{-1}\omega(a)$ and $x = x$), and equation (4.5) yields,

$$\overline{\partial}\Omega_4^{-1}(\overline{\alpha})(\beta,\overline{z}) - \Omega_4^{-1}\overline{\partial}(\overline{\alpha})(\beta,\overline{z})$$
$$= -4g(g^{-1}\omega(b),\gamma(x)(g^{-1}\omega(a))) + 4g(g^{-1}\omega(a),[g^{-1}\omega(b),x])$$
$$= -4(g(g^{-1}\omega(b),\gamma(x)(g^{-1}\omega(a))) - g(g^{-1}\omega(a),\gamma(g^{-1}\omega(b))x)$$
$$+g(g^{-1}\omega(a),\gamma(x)g^{-1}\omega(b)))$$
$$= -4g(x,\gamma(g^{-1}\omega(b))g^{-1}\omega(a)) = -4g(x,-icg^{-1}\omega(\gamma(b)a))$$
$$= 4ic(g(g^{-1}\omega(\gamma(b)a)))(x) = 4ic\omega(\gamma(b)a,x)$$

Combining these results with with equation (4.12), as well as the property that γ is symplectic, in equation (4.11) yields,

$$\overline{\partial}\phi\overline{\alpha}(\beta,\overline{z}) - \phi\overline{\partial}\overline{\alpha}(\beta,\overline{z}) = [\Lambda,\overline{\alpha}](\beta,\overline{z})$$
$$c\mu(\overline{\partial}\Omega_3^{-1}(\overline{\alpha}) - \Omega_3^{-1}\overline{\partial}(\overline{\alpha}))(\beta,\overline{z}) + \mu(\overline{\partial}\Omega_4^{-1}(\overline{\alpha}) - \Omega_4^{-1}\overline{\partial}(\overline{\alpha}))(\beta,\overline{z}) = [\Lambda,\overline{\alpha}](\beta,\overline{z})$$
$$-4ic\mu\omega(b,\gamma(x)a) - 4ic\mu\omega(a,\gamma(b)x) + 4ic\mu\omega(a,\gamma(x)b) + 4ic\mu\omega(\gamma(b)a,x)$$
$$= -2\omega(a,\gamma(b)x)$$
$$-4ic\mu\omega(a,\gamma(x)b) - 4ic\mu\omega(a,\gamma(b)x) + 4ic\mu\omega(a,\gamma(x)b) - 4ic\mu\omega(a,\gamma(b)x)$$
$$= -2\omega(a,\gamma(b)x)$$
$$4ic\mu\omega(a,\gamma(b)x) = \omega(a,\gamma(b)x)$$

This will be satisfied when $c\mu = \lambda = -\frac{i}{4}$, just as in the case where $l = z$. Combining this with Proposition 26 gives the following proposition.

Proposition 27 *If g is an invariant, non-degenerate, symmetric bi-linear form on $\mathfrak{g}$ and*

$\Omega_4((x,u),(y,v)) = g(x,v) - g(y,u)$ is closed, and if there is a $c \in \mathbb{C}$ such that
$cg^{-1}\omega(\gamma(a)b) = i\gamma(g^{-1}\omega(a))(g^{-1}\omega(b))$ for all $a, b \in \mathfrak{g}$, then
$\phi = -\frac{i}{4}\Omega_3^{-1} - \frac{i}{4c}\Omega_4^{-1}$ satisfies $\phi([l_1, l_2]) = [\phi(l_1), l_2] + [l_1, \phi(l_2)]$ and
$\overline{\partial}\phi(l) - \phi\overline{\partial}(l) = [\Lambda, l]$ for all $l, l_1, l_2 \in \mathfrak{h}^{(1,0)} \oplus \mathfrak{h}^{*(0,1)}$.

4.4 Weak Mirror Pair

Let $\phi = -\frac{i}{4}\Omega_3^{-1} - \frac{i}{4c}\Omega_4^{-1}$ for some $c \in \mathbb{C}$. If we define the convention $\Omega_i^{-1}(l_1 \wedge l_2) = \Omega_i^{-1}(l_1) \wedge l_2 + l_1 \wedge \Omega_i^{-1}(l_2)$, then $\phi(l_1 \wedge l_2) = \phi(l_1) \wedge l_2 + l_1 \wedge \phi(l_2)$. Combining propositions (25) and (27) yields the following proposition.

Proposition 28 *If g is a symmetric bi-linear form on $\mathfrak{g}$ and $\Omega_4((x,u),(y,v)) = g(x,v) - g(y,u)$ is closed, and if there is a $c \in \mathbb{C}$ such that $cg^{-1}\omega(\gamma(a)b) = i\gamma(g^{-1}\omega(a))(g^{-1}\omega(b))$ for all $a, b \in \mathfrak{g}$, then $\Lambda = \Omega_c^{-1}$ and $\phi = -\frac{i}{4}\Omega_3^{-1} - \frac{i}{4c}\Omega_4^{-1}$ form a compatible pair.*

Since we have a compatible pair, we should have a weak mirror pair. Now we examine what that pair is. If $L = \mathfrak{h}^{1,0} \oplus \mathfrak{h}^{*(0,1)}$, then $L_{\overline{\Lambda}} = \mathfrak{h}^{1,0} \oplus \{\overline{\xi} + \overline{\Lambda}(\overline{\xi}) | \overline{\xi} \in \mathfrak{h}^{*(0,1)}\}$, where $\overline{\Lambda} = \overline{\Omega}_c^{-1}$. Since $L_{\overline{\Lambda}}$ is a generalized complex structure of type 0, $L_{\overline{\Lambda}} = e^B L_\sigma$ for some closed two-form B and symplectic form σ. $L_\sigma = \{X - i\sigma(X) | X \in \mathfrak{h}\}$, so $e^B L_\sigma = \{X - i\sigma(X) + B(X) | X \in \mathfrak{h}\}$. For an arbitrary $X \in \mathfrak{h}^{1,0}$ and $\overline{Y} \in \mathfrak{h}^{0,1}$, the corresponding element in $e^B L_\sigma$ is $X + \overline{Y} - i\sigma(X) - i\sigma(\overline{Y}) + B(X) + B(\overline{Y})$. The corresponding elements in $L_{\overline{\Lambda}}$ is $X + \overline{\Omega}_c(\overline{Y}) + \overline{\Lambda}(\overline{\Omega}_c(\overline{Y})) = X + \overline{Y} + \overline{\Omega}_c(\overline{Y})$. Since $\overline{\Omega}$ is a (0,2) form, $\overline{\Omega}_c(\overline{Y}) = \overline{\Omega}_c(X + \overline{Y}) = \Omega_1(X) + \Omega_1(\overline{Y}) - i\Omega_2(X) - i\Omega_2(\overline{Y})$. So,

$$X + \overline{Y} - i\sigma(X) - i\sigma(\overline{Y}) + B(X) + B(\overline{Y}) = X + \overline{Y} + \Omega_1(X) + \Omega_1(\overline{Y}) - i\Omega_2(X) - i\Omega_2(\overline{Y}).$$

Therefore, $\sigma = \Omega_2$ and $B = \Omega_1$, and we have the following theorem.

Theorem 29 *Let $(\mathfrak{g}, \omega)$ be a symplectic Lie algebra and let $\mathfrak{h} = \mathfrak{g} \ltimes V$ be a complex symplectic algebra with Ω_1 and Ω_2 defined as above. If g is a symmetric bi-linear form on $\mathfrak{g}$ and $\Omega_4((x,u),(y,v)) = g(x,v) - g(y,u)$ is closed, and if there is a $c \in \mathbb{C}$ such that $cg^{-1}\omega(\gamma(a)b) = i\gamma(g^{-1}\omega(a))(g^{-1}\omega(b))$ for all $a, b \in \mathfrak{g}$, then the DGA's of $L = \mathfrak{h}^{1,0} \oplus \mathfrak{h}^{*(0,1)}$ and $e^{\Omega_1} L_{\Omega_2}$ form a weak mirror pair.*

4.5 Examples

There are 3 non trivial examples of 4 dimensional complex symplectic algebras on $\mathfrak{g} \ltimes V$, where $\mathfrak{g} = \langle e_1, e_2 \rangle$ and $V = \langle v_1, v_2 \rangle$. These are given by Clayton, Ovando and Poon in [4], based off of Andrada's work in [1]. In these examples $\Omega_1 = -e^1 \wedge v^2 + e^2 \wedge v^1 = \frac{1}{2}(z^1 \wedge z^2 - \overline{z}^1 \wedge \overline{z}^2)$ and $\Omega_2 = e^1 \wedge e^2 - v^1 \wedge v^2 = \frac{1}{2}(z^1 \wedge z^2 + \overline{z}^1 \wedge \overline{z}^2)$ where $z_1 = \frac{1}{2}(e_1 - iv_1)$ and $z_2 = \frac{1}{2}(e_2 - iv_2)$ are the (1,0) vectors and $z^1 = e^1 + iv^1$ and $z^2 = e^2 + iv^2$ are the (1,0) forms. This means that $\Omega_c = iz^1 \wedge z^2$ and so $\Lambda = iz_1 \wedge z_2$. Also $\Omega_3 = e^1 \wedge e^2 + v^1 \wedge v^2 = \frac{1}{2}(z^1 \wedge \overline{z}^2 + \overline{z}^1 \wedge z^2)$ and so $\Omega_3^{-1} = 2(\overline{z}_2 \wedge z_1 + z_2 \wedge \overline{z}_1)$. $\Omega_3^{-1}(\overline{z}^1) = -2z_2$ and $\Omega_3^{-1}(\overline{z}^2) = 2z_1$.

In these examples we choose g such that $g(e_i, e_j) = 1$ is $i \neq j$ and $g(e_i, e_i) = 0$. So $g^{-1}\omega(e_1) = e_1$ and $g^{-1}\omega(e_2) = -e_2$. With this g, $\Omega_4 = e^1 \wedge v^2 - v^1 \wedge e^2$ $= \frac{i}{2}(z^1 \wedge \overline{z}^2 - \overline{z}^1 \wedge z^2)$ and so $\Omega_4^{-1} = 2i(z_2 \wedge \overline{z}_1 - \overline{z}_2 \wedge z_1)$. Therefore $\Omega_4^{-1}(z_2) = 0 = \Omega_4^{-1}(z_2)$, $\Omega_4^{-1}(\overline{z}^1) = -2iz_2$ and $\Omega_4^{-1}(\overline{z}^2) = -2iz_1$

Example 1: The structure equations for the first algebra are:

$$[e_1, v_1] = v_2$$

When translated into complex coordinates, this yields,

$$[z_1, \overline{z}^2] = -\tfrac{1}{2}\overline{z}^1, \qquad \overline{\partial} z_1 = \tfrac{1}{2}\overline{z}^1 \wedge z_2$$

We can see that $\overline{\partial}\Lambda = 0$ and $[\Lambda, \Lambda] = 0$. We set $\phi = \lambda\Omega_3^{-1} + \mu\Omega_4^{-1}$, where $\mu = \frac{\lambda}{c} = -\frac{i}{4c}$. The representation is γ is given by:

$$\gamma(e_1)v_1 = v_2$$

We check if there is a $c \in \mathbb{C}$ such that $cg^{-1}\omega(\gamma(a)b) = i\gamma(g^{-1}\omega(a))(g^{-1}\omega(b))$. When $a = e_1, b = v_1$:

$$cg^{-1}\omega(\gamma(e_1)v_1) = cg^{-1}\omega(v_2) = -cv_2$$
$$= i\gamma(g^{-1}\omega(e_1))(g^{-1}\omega(v_1)) = i\gamma(e_1)(v_1) = iv_2$$

When $a = e_1, b = v_2$:

$$cg^{-1}\omega(\gamma(e_1)v_2) = cg^{-1}\omega(0) = c0$$
$$= i\gamma(g^{-1}\omega(e_1))(g^{-1}\omega(v_2)) = i\gamma(e_1)(v_2) = i0$$

When $a = e_2, b = v_1$:

$$cg^{-1}\omega(\gamma(e_2)v_1) = cg^{-1}\omega(0) = c0$$
$$= i\gamma(g^{-1}\omega(e_2))(g^{-1}\omega(v_1)) = i\gamma(-e_2)(v_1) = i0$$

When $a = e_2, b = v_2$:

$$cg^{-1}\omega(\gamma(e_2)v_2) = cg^{-1}\omega(0) = c0$$
$$= i\gamma(g^{-1}\omega(e_2))(g^{-1}\omega(v_2)) = i\gamma(-e_2)(-v_2) = i0$$

So all of these equations are satisfied when $c = -i$. Therefore $\mu = \frac{1}{4}$ and $\phi = \lambda\Omega_3^{-1} + \mu\Omega_4^{-1} = -\frac{i}{4}2(\overline{z}_2 \wedge z_1 + z_2 \wedge \overline{z}_1) + \frac{1}{4}2i(z_2 \wedge \overline{z}_1 - \overline{z}_2 \wedge z_1) = -i\overline{z}_2 \wedge z_1$. Λ and this ϕ are a compatible pair for this algebra.

Example 2: The structure equations for the second algebra are:

$$[e_1, e_2] = e_2, \qquad [e_1, v_1] = -v_1, \qquad [e_1, v_2] = v_2$$

When translated into complex coordinates, this yields,

$$[z_1, z_2] = \tfrac{1}{2}z_2, \qquad [z_1, \overline{z}^1] = \tfrac{1}{2}\overline{z}^1, \qquad [z_1, \overline{z}^2] = -\tfrac{1}{2}\overline{z}^2$$
$$\overline{\partial} z_1 = -\tfrac{1}{2}\overline{z}^1 \wedge z_1, \qquad \overline{\partial} z_2 = \tfrac{1}{2}\overline{z}^1 \wedge z_2, \qquad \overline{\partial}\overline{z}^2 = -\tfrac{1}{2}\overline{z}^1 \wedge \overline{z}^2$$

We can see that $\overline{\partial}\Lambda = i\overline{\partial} z_1 \wedge z_2 = i(\overline{\partial} z_1) \wedge z_2 - iz_1 \wedge (\overline{\partial} z_2)$ $= -\frac{i}{2}\overline{z}^1 \wedge z_1 \wedge z_2 - \frac{i}{2} z_1 \wedge \overline{z}^1 \wedge z_2 = 0$ and $[\Lambda, \Lambda] = [iz_1 \wedge z_2, iz_1 \wedge z_2] = 2z_1 \wedge z_2 \wedge [z_1, z_2] = 0$. We set $\phi = \lambda\Omega_3^{-1} + \mu\Omega_4^{-1}$, where $\mu = \frac{\lambda}{c} = -\frac{i}{4c}$. The representation is γ is given by:

$$\gamma(e_1)v_1 = -v_1, \quad \gamma(e_1)v_2 = v_2$$

We check if there is a $c \in \mathbb{C}$ such that $cg^{-1}\omega(\gamma(a)b) = i\gamma(g^{-1}\omega(a))(g^{-1}\omega(b))$. When $a = e_1, b = v_1$:

$$cg^{-1}\omega(\gamma(e_1)v_1) = cg^{-1}\omega(-v_1) = -cv_1$$
$$= i\gamma(g^{-1}\omega(e_1))(g^{-1}\omega(v_1)) = i\gamma(e_1)(v_1) = -iv_1$$

When $a = e_1, b = v_2$:

$$cg^{-1}\omega(\gamma(e_1)v_2) = cg^{-1}\omega(v_2) = -cv_2$$
$$= i\gamma(g^{-1}\omega(e_1))(g^{-1}\omega(v_2)) = i\gamma(e_1)(-v_2) = -iv_2$$

When $a = e_2, b = v_1$:

$$cg^{-1}\omega(\gamma(e_2)v_1) = cg^{-1}\omega(0) = c0$$
$$= i\gamma(g^{-1}\omega(e_2))(g^{-1}\omega(v_1)) = i\gamma(-e_2)(v_1) = i0$$

When $a = e_2, b = v_2$:

$$cg^{-1}\omega(\gamma(e_2)v_2) = cg^{-1}\omega(0) = c0$$
$$= i\gamma(g^{-1}\omega(e_2))(g^{-1}\omega(v_2)) = i\gamma(-e_2)(-v_2) = i0$$

So all of these equations are satisfied when $c = i$. Therefore, $\mu = -\frac{1}{4}$ and $\phi = \lambda\Omega_3^{-1} + \mu\Omega_4^{-1} = -\frac{i}{4}2(\overline{z}_2 \wedge z_1 + z_2 \wedge \overline{z}_1) - \frac{1}{4}2i(z_2 \wedge \overline{z}_1 - \overline{z}_2 \wedge z_1) = -iz_2 \wedge \overline{z}_1$. So Λ and this ϕ are a compatible pair for this algebra.

Example 3: The structure equations for the third algebra are:

$$[e_1, e_2] = e_2, \qquad [e_1, v_1] = -\frac{1}{2}v_1, \qquad [e_1, v_2] = \frac{1}{2}v_2, \qquad [e_2, v_1] = -\frac{1}{2}v_2$$

When translated into complex coordinates, this yields,

$$[z_1, z_2] = \tfrac{1}{2}z_2, \qquad [z_1, \overline{z}^1] = \tfrac{1}{4}\overline{z}^1, \qquad [z_1, \overline{z}^2] = -\tfrac{1}{4}\overline{z}^2, \qquad [z_2, \overline{z}^2] = \tfrac{1}{4}\overline{z}^1$$

$$\overline{\partial} z_1 = -\tfrac{1}{4}\overline{z}^1 \wedge z_1 - \tfrac{1}{4}\overline{z}^2 \wedge z_2, \qquad \overline{\partial} z_2 = \tfrac{1}{4}\overline{z}^1 \wedge z_2, \qquad \overline{\partial}\overline{z}^2 = -\tfrac{1}{2}\overline{z}^1 \wedge \overline{z}^2$$

We can see that $\overline{\partial}\Lambda = i\overline{\partial} z_1 \wedge z_2 = i(\overline{\partial} z_1) \wedge z_2 - iz_1 \wedge (\overline{\partial} z_2)$
$= -\frac{i}{4}(\overline{z}_1 \wedge z_1 + \overline{z}^2 \wedge z_2) \wedge z_2 - \frac{i}{4} z_1 \wedge \overline{z}^1 \wedge z_2 = 0$ and $[\Lambda, \Lambda] = [iz_1 \wedge z_2, iz_1 \wedge z_2]$
$= -2z_1 \wedge z_2 \wedge [z_1, z_2] = 0$. We set $\phi = \lambda\Omega_3^{-1} + \mu\Omega_4^{-1}$, where $\mu = \frac{\lambda}{c} = -\frac{i}{4c}$. The representation is given by:

$$\gamma(e_1)v_1 = -\tfrac{1}{2}v_1, \quad \gamma(e_1)v_2 = \tfrac{1}{2}v_2, \quad \gamma(e_2)v_1 = -\tfrac{1}{2}v_2$$

We check if there is a $c \in \mathbb{C}$ such that $cg^{-1}\omega(\gamma(a)b) = i\gamma(g^{-1}\omega(a))(g^{-1}\omega(b))$. When $a = e_1, b = v_1$:

$$cg^{-1}\omega(\gamma(e_1)v_1) = cg^{-1}\omega(-\tfrac{1}{2}v_1) = -c\tfrac{1}{2}v_1$$
$$= i\gamma(g^{-1}\omega(e_1))(g^{-1}\omega(v_1)) = i\gamma(e_1)(v_1) = -i\tfrac{1}{2}v_1$$

When $a = e_1, b = v_2$:

$$cg^{-1}\omega(\gamma(e_1)v_2) = cg^{-1}\omega(\tfrac{1}{2}v_2) = -c\tfrac{1}{2}v_2$$
$$= i\gamma(g^{-1}\omega(e_1))(g^{-1}\omega(v_2)) = i\gamma(e_1)(-v_2) = -i\tfrac{1}{2}v_2$$

When $a = e_2, b = v_1$:

$$cg^{-1}\omega(\gamma(e_2)v_1) = cg^{-1}\omega(-\tfrac{1}{2}v_2) = c\tfrac{1}{2}v_2$$
$$= i\gamma(g^{-1}\omega(e_2))(g^{-1}\omega(v_1)) = i\gamma(-e_2)(v_1) = i\tfrac{1}{2}v_2$$

When $a = e_2, b = v_2$:

$$cg^{-1}\omega(\gamma(e_2)v_2) = cg^{-1}\omega(0) = c0$$
$$= i\gamma(g^{-1}\omega(e_2))(g^{-1}\omega(v_2)) = i\gamma(-e_2)(-v_2) = i0$$

So all of these equations are satisfied when $c = i$. Therefore, $\mu = -\frac{1}{4}$ and $\phi = \lambda\Omega_3^{-1} + \mu\Omega_4^{-1} = -\frac{i}{4}2(\overline{z}_2 \wedge z_1 + z_2 \wedge \overline{z}_1) - \frac{1}{4}2i(z_2 \wedge \overline{z}_1 - \overline{z}_2 \wedge z_1) = -iz_2 \wedge \overline{z}_1$.

However Λ and ϕ are not a compatible pair for this algebra, because Ω_4 is not closed, by Lemma 21. We can see this, as $g(\gamma(e_2)e_1, e_1) - g(\gamma(e_1)e_2, e_1) - g(e_2, \gamma(e_1)e_1) + g(e_1, \gamma(e_2)e_1) = -1 \neq 0$. Due to this $\phi([l_1, l_1]) \neq [\phi(l_1), l_2] + [l_1, \phi(l_2)]$ and so $\Phi([l_1, l_2]) \neq [\Phi(l_1), \Phi(l_2)]$.

We can show that there is no ϕ compatible with $\Lambda = iz_1 \wedge z_2$. If there was such a ϕ, then $\Phi = 1 + \phi$ would have to be of the following form:

$$\Phi(z_1) = z_1$$
$$\Phi(z_2) = z_2$$
$$\Phi(\overline{z}^1) = \overline{z}^1 + az_1 + bz_2$$
$$\Phi(\overline{z}^2) = \overline{z}^2 + cz_1 + dz_2.$$

Using the structure equations and the formula $\Phi([l_1, l_2]) = [\Phi(l_1), \Phi(l_2)]$ to solve for a, b, c and d shows the following:

$$\Phi([z_1, \overline{z}^1]) = \frac{1}{4}\Phi(\overline{z}^1) = \frac{1}{4}(\overline{z}^1 + az_1 + bz_2)$$
$$[\Phi(z_1), \Phi(\overline{z}^1)] = [z_1, \overline{z}^1 + az_1 + bz_2] = \frac{1}{4}\overline{z}^1 + \frac{b}{2}z_2$$
$$\Phi([z_1, \overline{z}^2]) = -\frac{1}{4}\Phi(\overline{z}^2) = -\frac{1}{4}(\overline{z}^2 + cz_1 + dz_2)$$
$$[\Phi(z_1), \Phi(\overline{z}^2)] = [z_1, \overline{z}^2 + cz_1 + dz_2] = -\frac{1}{4}\overline{z}^2 + \frac{d}{2}z_2$$

The first two equations imply that $a = 0 = b$ and the second two equations show that $c = 0 = d$. So our only choice is $\Phi = 1$ the identity. So the condition $\Phi(\overline{\partial}_\Lambda l) = \overline{\partial}(\Phi(l))$

becomes $\overline{\partial} l = \overline{\partial}_\Lambda l = \overline{\partial} l + [\Lambda, l]$, or $[\Lambda, l] = 0$ for all $l \in L$. However, $[\Lambda, \overline{z}^1] = [iz_1 \wedge z_2, \overline{z}^1] = -\frac{i}{4} z_2 \wedge \overline{z}^1 \neq 0$. So there is no isomorphism between the DGA's of L and $L_{\overline{\Lambda}}$.

Chapter 5

Principal Torus Bundles

In this chapter, we examine how the integrability of a generalized complex structure on a principal T^{2n} bundle effects the connection on that bundle. This is preliminary work in a effort to apply theorem 17 to principal bundles. The primary reference for this chapter is Kobayashi and Nomizu [10].

Let P be a principle fiber bundle with even dimensional base manifold M and $\pi : P \to M$. The fibers will be T^{2n}, the torus with real dimension $2n$. We assume a connection on P, which defines horizontal vector fields and horizontal forms. If $X \in C^\infty(TP)$, then $X = V + H$, where V is a vertical vector field and H is a horizontal vector field. Also, if $\xi \in C^\infty(T^*P)$, then $\xi = \nu + h$, where ν is a vertical form and h is a horizontal form. Therefore $TP = \mathcal{V} \oplus \mathcal{H}$ and $T^*P = \mathcal{V}^* \oplus \mathcal{H}^*$, where $C^\infty(\mathcal{V})$ are the vertical vector fields, $C^\infty(\mathcal{V}^*)$ are the vertical one-forms, $C^\infty(\mathcal{H})$ are the horizontal vector fields and $C^\infty(\mathcal{H}^*)$ are the horizontal one-forms.

If V_1 and V_2 are vertical vector fields, then $[V_1, V_2]$ is also vertical. However, if H_1 and H_2 are horizontal vector fields, $[H_1, H_2]$ is not required to be horizontal. The vertical portion of $[H_1, H_2]$ depends on the curvature of the connection.

T^{2n} acts freely and transitively on P. Every vertical vector at a point $p \in P$ corresponds to an element in $\mathfrak{g}$, the Lie algebra of the fibers. As in [10], we call a vertical vector field V *fundamental* if $V(p)$ corresponds to the same element of $\mathfrak{g}$ for all p. We will extend the definition of fundamental to include more than just vertical vector fields. A horizontal vector field will be *fundamental* if it is invariant under the group action on P. Then there is a one to one correspondence between fundamental horizontal field and lifts of vector fields on the base manifold. A vertical form ν will be *fundamental* if, whenever V is a fundamental, vertical vector field, $\nu(V)$ is constant on P. A horizontal form h will be *fundamental* if, whenever H is a fundamental, horizontal vector field, $h(H)$ is constant on each fiber. Note that every fundamental object is invariant under the group action, and every horizontal object that is invariant under the group action is fundamental, but there are vertical objects that are invariant under the group action that are not fundamental.

For any fundamental vertical field V and fundamental horizontal field H, $[V, H] = 0$. As such, $[V, h] = 0$ and $[H, \nu] = 0$ where h is a fundamental horizontal form and ν is a fundamental vertical form.

Any fundamental vertical field corresponds to an element of the Lie algebra for the fiber. Therefore $[V_1, V_2] = 0$, since the the Lie algebra for T^{2n} is abelian. This means $[V, \nu] = 0$ as well.

Let $J : TP \oplus T^*P \to TP \oplus T^*P$ be a generalized complex structure. Then, for all $l_1, l_2 \in TP \oplus T^*P$, the generalized Nijenhuis tensor vanishes. That is,

$$N_J(l_1, l_2) = [Jl_1, Jl_2] - [l_1, l_2] - J[Jl_1, l_2] - J[l_1, Jl_2] = 0. \tag{5.1}$$

Let $l_1, l_2 \in C^\infty(TP \oplus T^*P)$. Then at a point $p \in P$ there exist $\tilde{l}_1, \tilde{l}_2 \in C^\infty(TP \oplus T^*P)$ such that $\tilde{l}_1, \tilde{l}_2$ are fundamental and $\tilde{l}_i(p) = l_i(p)$. Since N_J is a tensor, $N_J(\tilde{l}_1, \tilde{l}_2)(p)$ $= N_J(l_1, l_2)(p)$. As such, we will only consider fundamental objects in this chapter.

We will make two further assumptions, so that J respects the fibration. First, that $J(\mathcal{V} \oplus \mathcal{V}^*) \subset \mathcal{V} \oplus \mathcal{V}^*$ and $J(\mathcal{H} \oplus \mathcal{H}^*) \subset \mathcal{H} \oplus \mathcal{H}^*$. Then $J = J_H + J_V$, where J_H vanishes on vertical objects and J_V vanishes on horizontal objects. J_H and J_V are almost generalized complex structures, whose integrability needs to be checked.

Second, we assume that if $X + \xi$ is fundamental, then $J(X + \xi)$ is fundamental as well. Since every horizontal object is the lift an object on the base manifolds, J_H is the lift of an almost generalized complex structure on the base manifold. We can write J_H in matrix form as

$$J_{\mathcal{H}} = \begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix} = \begin{pmatrix} \pi_*^{-1} & 0 \\ 0 & \pi^* \end{pmatrix} \begin{pmatrix} \widetilde{A} & \widetilde{\beta} \\ \widetilde{B} & -\widetilde{A}^* \end{pmatrix} \begin{pmatrix} \pi_* & 0 \\ 0 & (\pi^*)^{-1} \end{pmatrix}$$

where the middle matrix acts on $TM \oplus T^*M$, π_* is the pushforward of vectors and π^* is the pullback of forms. If we only consider horizontal objects, π_* is well-defined. As such, we will use

$$J_{\mathcal{H}} = \begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix}$$

on the total space and on the base manifold.

For everything below, H_i will be a fundamental horizontal vector, V_i will be a fundamental vertical vector, h_i will be a fundamental horizontal form, and ν_i will be a fundamental vertical form. We will now see what conditions equation (5.1) imposes on J_V, J_H and the connection on P. We will examine $N_J(l_1, l_2) = 0$, letting l_i vary between horizontal and vertical vectors and forms.

Since $[V + \nu, H + h] = 0$ for fundamental objects, $N_J(l_1, l_2) = 0$ is trivial when $l_1 \in \mathcal{V} \oplus \mathcal{V}^*$ and $l_2 \in \mathcal{H} \oplus \mathcal{H}^*$.

Since $[V_1 + \nu_1, V_2 + \nu_2] = 0$ for fundamental objects, N_{J_V} vanishes on each fiber. So we have the following proposition.

Proposition 30 *J_V is integrable on each fiber.*

This now leaves only the horizontal conditions. Let $l_i = H_i + h_i$ be fundamental horizontal objects. Let $[l_1, l_2]_V$ be the vertical part of $[l_1, l_2]$ and $[l_1, l_2]_H$ be the horizontal part of $[l_1, l_2]$. Then

$$\begin{aligned} 0 = N_J(l_1, l_2) &= [Jl_1, Jl_2] - [l_1, l_2] - J([Jl_1, l_2] + [l_1, Jl_2]) \\ &= [J_H l_1, J_H l_2]_H - [l_1, l_2]_H - J_H([J_H l_1, l_2] + [l_1, J_H l_2]) \\ &\quad [J_H l_1, J_H l_2]_V - [l_1, l_2]_V - J_V([J_H l_1, l_2] + [l_1, J_H l_2]). \end{aligned}$$

Requiring the horizontal part to be 0 yields,

$$[J_H l_1, J_H l_2]_H - [l_1, l_2]_H - J_H([J_H l_1, l_2]_H + [l_1, J_H l_2]_H) = 0. \tag{5.2}$$

Requiring the vertical part to be 0 yields,

$$[J_H l_1, J_H l_2]_V - [l_1, l_2]_V - J_V([J_H l_1, l_2]_V + [l_1, J_H l_2]_V) = 0. \tag{5.3}$$

Equation (5.2) gives the following proposition.

Proposition 31 *J_H is integrable on the base manifold M.*

Lastly, we seek to understand equation (5.3). Since we have a connection, we have a connection form $\theta : TP \to \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra for the fibers. For any basis $\{e_i\}$ of $\mathfrak{g}$, we have $\theta = \sum \theta_i e_i$, where θ_i are fundamental vertical one-forms on P. If V is a vertical vector field and H is a horizontal vector field, then at a point $p \in P$, $\theta(V + H)(p)$ is the element of $\mathfrak{g}$ corresponding to $\tilde{V}$, where $\tilde{V}$ is fundamental and vertical and $V(p) = \tilde{V}(p)$. Then Ω, the curvature of θ is,

$$\Omega(X, Y) = d\theta(X, Y) + \frac{1}{2}[\theta(X), \theta(Y)] \tag{5.4}$$

For $l_1, l_2 \in C^\infty(TP \oplus T^*P)$, we extend Ω as $\Omega(l_1, l_2) = \Omega(\rho(l_1), \rho(l_2))$ where $\rho : TP \oplus T^*P \to TP$ is the anchor map. We make the following definition.

Definition 32 *Ω is* type (1,1) with respect to J *if* $\Omega(l_1 - iJ(l_1), l_2 - iJ(l_2)) = 0$ *for all* $l_1, l_2 \in C^\infty(TP \oplus T^*P)$.

When J comes from a complex structure, this definition is equivalent to Ω being type (1,1) in the classical sense.

If l_j is horizontal, then $J_H(l_j)$ is horizontal and $\theta(l_j - iJ_H(l_j)) = 0$. If Ω is type (1,1), then

$$\begin{aligned}
0 &= \Omega(l_1 - iJ_H(l_1), l_2 - iJ_H(l_2)) \\
&= d\theta(l_1 - iJ_H(l_1), l_2 - iJ_H(l_2)) \\
&= -\theta([l_1 - iJ_H(l_1), l_2 - iJ_H(l_2)]_V) \\
&= -\theta([l_1, l_2]_V) + \theta([J_H l_1, J_H l_2]_V) \\
&+ i(\theta([l_1, J_H(l_2)]_V) + \theta([J_H(l_1), l_2]_V)) \\
&= \theta([J_H l_1, J_H l_2]_V - [l_1, l_2]_V) \\
&+ i(\theta([l_1, J_H(l_2)]_V + [J_H(l_1), l_2]_V))
\end{aligned}$$

We note that θ is defined by $\{\theta_i\}$, and so $\theta_i([l_1, J_H(l_2)] + [J_H(l_1), l_2]) = 0$. Since $\{\theta_i\}$ is a basis for fundamental vertical forms, if ν is a fundamental vertical form then $\nu([l_1, J_H(l_2)] + [J_H(l_1), l_2]) = 0$. Equivalently, $\nu([l_1, l_2] - [J_H(l_1), J(l_2)]) = 0$. Since Ω is a tensor, we only consider fundamental objects.

Proposition 33 *Ω is type (1,1) if and only if $\nu([l_1, J_H(l_2)] + [J_H(l_1), l_2]) = 0$ for all fundamental vertical forms ν and fundamental horizontal objects l_j.*

We now examine equation (5.3) in several different cases. In each case, we will examine equation (5.3) for each combination of horizontal vectors H_i and horizontal forms h_i. As a reminder, the Courant bracket of two one-forms is always 0.

First we assume that J_V is of complex type. Then $J(\mathcal{V}) = \mathcal{V}$ and $J(\mathcal{V}^*) = \mathcal{V}^*$.

If J_H is complex, so that $B = 0$ and $\beta = 0$ in the matrix form of J, then we have the following conditions.

$$[AH_1, AH_2]_V - [H_1, H_2]_V - J_V([AH_1, H_2]_V + [H_1, AH_2]_V) = 0$$
$$-[AH_1, A^*h_2]_V - [H_1, h_2]_V - J_V([AH_1, h_2]_V - [H_1, A^*h_2]_V) = 0$$

In this case, A is an integrable complex structure. Contracting the second condition with a fundamental vertical vector V and recalling from above that $[H, V] = 0$ show that the second condition is always satisfied. However, it is interesting to note that when we contract with a vertical vector V, the second equation reduces to $h_2(A([AH_1, V] + [H_1, J_V V] - [AH_1, JV] + [H_1, V])) = 0$. The first condition is satisfied when Ω is type (1,1).

If J_H is symplectic instead, so that $A = 0$ and $A^* = 0$, then we have the following conditions.

$$-[H_1, H_2]_V - J_V([BH_1, H_2] + [H_1, BH_2]) = 0$$
$$[BH_1, \beta h_2] - [H_1, h_2]_V - J_V([H_1, \beta h_2]_V) = 0$$
$$[\beta h_1, \beta h_2] - J_V([\beta h_1, h_2] + [h_1, \beta h_2]) = 0$$

These terms are all equivalent, after substitution. The vector part of each condition is $[H_1, H_2]_V = 0$, which implies that $\Omega = 0$, and the connection is flat. Since J_H is integrable, B is viewed as the lift of a symplectic form on the base manifold. The form part of the above conditions is satisfied since B is closed.

Now we assume J_V is symplectic, so that $J(\mathcal{V}) = \mathcal{V}^*$ and $J(\mathcal{V}^*) = \mathcal{V}$. The conditions will look similar to the ones above, but the reduce nicely.

If J_H is complex, we have the following conditions.

$$[AH_1, AH_2]_V - [H_1, H_2]_V - J_V([AH_1, H_2]_V + [H_1, AH_2]_V) = 0$$
$$-[AH_1, A^* h_2]_V - [H_1, h_2]_V - J_V([AH_1, h_2]_V - [H_1, A^* h_2]_V) = 0$$

The vector part of the first equation is $[AH_1, AH_2]_V - [H_1, H_2] = 0$ and the form part is $J_V([AH_1, H_2]_V + [H_1, AH_2]_V) = 0$, which implies $[AH_1, H_2]_V + [H_1, AH_2]_V = 0$. These conditions are equivalent to Ω being type (1,1).

Lastly, if J_H is symplectic, we have,

$$-[H_1, H_2]_V - J_V([BH_1, H_2] + [H_1, BH_2]) = 0$$
$$[BH_1, \beta h_2]_V - [H_1, h_2]_V - J_V([H_1, \beta h_2]_V) = 0$$
$$[\beta h_1, \beta h_2]_V - J_V([\beta h_1, h_2]_V + [h_1, \beta h_2]_V) = 0$$

The symplectic form is given by $B(H) = \omega(H, -)$ and
$0 = d\omega(H_1, H_2, V) = B([H_1, V])H_2 - B([H_2, V])H_1 = ([BH_1, H_2] + [H_1, BH_2])(V)$, so we get that $[H_1, H_2]_V = 0$ and the connection is flat.

This tells us about the structure of some basic types of generalized complex structures on principal tori bundles which we summarize below.

Theorem 34 *Let P be a principal tori bundle. Let J be a generalized complex structure on P that preserves horizontal and vertical objects and is invariant under the T^{2n} action. Then $J = J_H + J_V$, where J_H is the lift of a generalized complex structure on the base manifold and J_V is a generalized complex structure on the fibers.*
If J_H is symplectic and J_V is complex or symplectic, the connection is flat.
If J_H is complex and J_V is complex, the connection is type (1,1).
If J_H and J_V are complex, we have a condition on the curvature that is satisfied if the curvature is type (1,1).

Chapter 6

Future Agenda

In this chapter we outline some possible applications of the results of this thesis.

Starting with a complex manifold, Theorem 17 gives the sufficient conditions for there to be a deformation to a symplectic manifold with an isomorphic DGA. The condition is the existence of a compatible pair. In the work on complex symplectic algebras, one part of this pair was defined canonically, our Λ. We then solved for ϕ. This seems likely to be the best approach, and so a starting point for future work is to ask the following:

Which complex manifolds (of even complex dimension) have a non-degenerate (2,0) vector field Λ satisfying $\overline{\partial}\Lambda = 0$ and $[\Lambda, \Lambda] = 0$?

Any such manifold is a good candidate for the machinery of this thesis, and there are a lot of them. If we drop the non-degeneracy condition, such Λ are called holomorphic Poisson fields and have been studied by Hitchin in [9], Gaultieri in [7] and Laurent-Gengoux, Stienon and Xu in [11]. As we saw in Proposition 25, if a manifold has a closed, non-degenerate two-form Ω, then $\Lambda = \Omega^{-1}$ will satisfy $\overline{\partial}\Lambda = 0$ and $[\Lambda, \Lambda] = 0$.

In particular, any principal T^{2n} fiber bundle over a manifold M of real dimension $2n$ where the complex structure is totally real with respect to the connection for the fibration could have a non-degenerate (2,0) field. Whether or not such a complex structure exists would depend on the base manifold M. This is somewhat more general that the work we did on complex symplectic manifolds.

Also a generalized complex structure on a principal tori bundle could respect the fibration, so that it decomposes into a structure on the fibers and one that projects to the base manifold. We have done some preliminary work on generalized complex structures on principal tori bundles and what restrictions they place on the connection and its curvature in chapter 5. Deformations of these structures would be interesting as well.

Some K3 surfaces, Calabi-Yau manifolds and hyper-Kahler manifolds are holomorphic Poisson. While Calabi-Yau 3-folds are of most interest to physicists, the methods of this thesis cannot work directly on them, as they have complex dimension 3. However, maybe when viewed as part of something else, particularly as the base manifold of a bundle, some other method might be found. In all these cases, while there may be obvious choices for Λ, finding ϕ is where the challenge will be.

Most work on weak mirror symmetry involves a concept called T-duality as in [15]. This is how Clayton, Ovando and Poon studied mirror symmetry in [4]. T-duality is generally applied larger class of examples, as it does not require the weak mirror pair to have the same base manifold. In the cases where T-duality does yield the same base manifold, it would be interesting to understand how those weak mirror pairs relate to mirror pairs created by deformations.

Finally, most of the results in this thesis are specific to deformations of complex structure to symplectic structures. It is still interesting to look at deformations of arbitrary generalized complex structures, and see when the DGA's are preserved. Some preliminary work has been done with Yat Sun Poon and Daniele Grandini at UCR.

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