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Logarithmic Sobolev Inequalities for Gaussian Convolutions of Compactly Supported Measures

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by

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The dissertation of David Sawyer Zimmermann is approved, and it is acceptable in quality and form for publication on microfilm:

Chair

University of California, San Diego

2015

DEDICATION

To my parents, Steve and Jan.

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Chapters 2,3,4, and 5 are, in part, a reprint of material from three articles. The first: D. Zimmermann. Logarithmic Sobolev inequalities for mollified compactly supported measures. *J. Funct. Anal.*, 265:1064–1083, 2013. (See [33].) The second: D. Zimmermann. Bounds for logarithmic Sobolev constants for Gaussian convolutions. Submitted for publication in Annales de l'Institute Henri Poincaré. (See [31].) The third: D. Zimmermann. Elementary proof of logarithmic Sobolev inequalities for Gaussian convolutions on $\mathbb{R}$. Submitted for publication in Annales Mathématiques Blaise Pascal. (See [32].) The dissertation author was the author for this material.

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ABSTRACT OF THE DISSERTATION

Logarithmic Sobolev Inequalities for Gaussian Convolutions of Compactly Supported Measures

by

David Sawyer Zimmermann
Doctor of Philosophy in Mathematics

University of California San Diego, 2015

Professor Todd Kemp, Chair

We give a brief exposition of logarithmic Sobolev Inequalities (LSIs) for probability measures on $\mathbb{R}^n$, as well as some known sufficient conditions on such measures for a LSI to hold. We show that the convolution of a compactly supported probability measure on $\mathbb{R}^n$ with a Gaussian measure satisfies a LSI, and look at some examples. We conclude with an application of this result by showing that the empirical law of eigenvalues of an $n \times n$ symmetric random matrix converges weakly to its mean as $n \to \infty$.

Chapter 1

Introduction

1.1 Background

A probability measure μ on $\mathbb{R}^n$ (or more generally, a Riemannian manifold) is said to satisfy a logarithmic Sobolev inequality (LSI) with constant $c \in \mathbb{R}$ if

$$\mathrm{Ent}_\mu(f^2) \leq 2c\, \mathscr{E}(f,f)$$

for all locally Lipschitz functions $f : \mathbb{R}^n \to \mathbb{R}_+$ for which both sides of the inequality are finite, where Ent_μ, called the entropy functional, is defined as

$$\mathrm{Ent}_\mu(f) \coloneqq \int f \log \frac{f}{\int f \; d\mu} \, d\mu$$

and $\mathscr{E}(f,f)$, the energy of f, is defined as

$$\mathscr{E}(f,f) \coloneqq \int |\nabla f|^2 d\mu,$$

with $|\nabla f|$ defined as

$$|\nabla f|(x) \coloneqq \limsup_{y \to x} \frac{|f(x) - f(y)|}{|x-y|}$$

so that $|\nabla f|$ is defined everywhere and coincides with the usual notion of gradient where f is differentiable. The smallest c for which a LSI with constant c holds is called the optimal log-Sobolev constant for μ.

LSIs show up as an important tool in many areas of mathematics, such as geometry [1, 2, 7, 11, 12, 13, 21], probability [9, 14, 18], and optimal transport [22, 24], as well as statistical physics [28, 29, 30]. A 2003 paper of Ledoux [23] uses LSI (in its equivalent form *hypercontractivity*, see [15]) to determine tail bounds for the largest eigenvalue of a large symmetric random matrix with Gaussian entries. Another important application of LSI in probability is the *Herbst inequality* (see [16], p.301, Ex. 3.4):

Theorem 1.1.1. (Herbst). *Let μ be a probability measure on $\mathbb{R}^n$ satisfying a LSI with constant c, and let $F : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz. Then for all $\lambda \in \mathbb{R}$,*

$$\mu\left\{\left|F - \int F\, d\mu\right| \geq \lambda\right\} \leq 2\exp\left(-\frac{\lambda^2}{2c||F||^2_{\text{Lip}}}\right).$$

Because of the widespread utility of LSI, it is of great interest to know which measures satisfy a LSI, and for those that do, what the optimal log-Sobolev constants are. The prototypical example of a measure that satisfies a LSI is the standard Gaussian measure on $\mathbb{R}^n$, which Gross proved satisfies a LSI with constant 1 in his early seminal work [15] in the field. There are many known sufficient conditions on μ in order for μ to satisfy a LSI (for example, [4, 5, 8, 20, 25]), as well as some known necessary conditions (for example, Theorem 1.1.1 above implies that μ must have sub-Gaussian tails if it satisfies a LSI). The next two sufficient conditions for μ to satisfy a LSI will be used in later chapter chapters. The first of these two is due to Cattiaux, Guillin, and Wu (see [10, Thm. 1.2]):

Theorem 1.1.2 (Cattiaux, Guillin, Wu)**.** *Let μ be a probability measure on $\mathbb{R}^n$ with $d\mu(x) = e^{-V(x)}dx$ for some $V \in C^2(\mathbb{R}^n)$. Suppose the following:*

1. *There exists a constant $K \leq 0$ such that* $\text{Hess}(V) \geq KI$.

2. *There exists a $W \in C^2(\mathbb{R}^n)$ with $W \geq 1$ and constants $b, c > 0$ such that*

$$\Delta W(x) - \langle \nabla V, \nabla W\rangle(x) \leq (b - c|x|^2)W(x)$$

for all $x \in \mathbb{R}^n$.

Then μ satisfies a LSI.

On the real line, Bobkov and Götze gave the following necessary *and* sufficient condition (see [6, p.25, Thm 5.3]):

Theorem 1.1.3 (Bobkov, Götze)**.** *Let μ be a Borel probability measure on $\mathbb{R}$ with distribution function $F(x) = \mu((-\infty, x])$. Let p be the density of the absolutely continuous part of μ with respect to Lebesgue measure, and let m be a median of μ. Let*

$$D_0 = \sup_{x<m} \left(F(x) \cdot \log \frac{1}{F(x)} \cdot \int_x^m \frac{1}{p(t)} dt \right),$$
$$D_1 = \sup_{x>m} \left((1 - F(x)) \cdot \log \frac{1}{1 - F(x)} \cdot \int_m^x \frac{1}{p(t)} dt \right),$$

defining D_0 and D_1 to be zero if $\mu((-\infty, m)) = 0$ or $\mu((m, \infty)) = 0$, respectively, and using the convention $0 \cdot \infty = 0$. Then the optimal log Sobolev constant c for μ satisfies $\frac{1}{150}(D_0 + D_1) \le c \le 468(D_0 + D_1)$. In particular, μ satisfies a LSI if and only if D_0 and D_1 are finite.

From this, one can glean some sufficient conditions for a LSI to hold. For example, if μ is supported in the interval $[a, b]$, and the absolutely continuous part of μ has a density whose reciprocal is in $L^1([a, b])$, then μ satisfies a LSI with constant bounded by an absolute constant times $||1/p||_1$. This is seen by the following rough estimate for D_1 (the estimate for D_0 is similar):

$$D_1 = \sup_{x>m} \left((1 - F(x)) \cdot \log \frac{1}{1 - F(x)} \cdot \int_m^x \frac{1}{p(t)} dt \right) \le \sup_{0<u<1} \left(u \cdot \log \frac{1}{u} \right) \cdot \int_a^b \frac{1}{p(t)} dt$$
$$= \frac{1}{e} ||1/p||_1.$$

We further remark that it is not necessary for $1/p$ to be L^1 for μ to satisfy a LSI. For example, on $[0, 1]$, let $d\mu(t) = (\alpha + 1)t^\alpha dt$ for any $\alpha \ge 1$. Then one can explicitly compute the integrals defining D_0 and D_1 to check that μ satisfies a LSI.

Surprisingly absent in the literature is the idea of approximation of arbitrary measures by measures that satisfy a LSI; this will be the focus of this dissertation. We will approximate by using convolution with Gaussian measures. (Since the time of publication of [33] by the present author, other work has been done. See the recent paper [25] for statements about convolutions involving more general classes of probability measures on $\mathbb{R}^n$ than what we investigate here.)

Convolution of an arbitrary measure with a Gaussian does not necessarily yield a LSI; for example, consider the exponential distribution on $\mathbb{R}$: $d\mu(t) = \exp(-t)\, dt$,

$t \geq 0$. The right tail is not sub-Gaussian; therefore by the Herbst inequality (Theorem 1.1.1 above), μ does not satisfy a LSI. If we convolve μ with the standard Gaussian measure, then the right tail of the convolved measure has density p given by

$$\begin{aligned} p(x) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) \exp(-(x-y)) \cdot \mathbb{1}_{\{y>0\}}(x-y)dy \\ &= \exp\left(-x+\frac{1}{2}\right) \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y+1)^2}{2}\right) dy \\ &\geq \frac{1}{2} \exp\left(-x+\frac{1}{2}\right), \qquad \text{for} \quad x \geq -1. \end{aligned}$$

Thus the convolved measure still has an exponential, hence not sub-Gaussian, right tail and therefore does not satisfy a LSI either. So this approximation scheme does not work in general. However, if we restrict our attention to compactly supported measures, then convolution will yield a LSI; this is stated precisely in the next section.

1.2 Results

In this section, we state all of the main results. We first introduce some conventions: we will denote by γ_δ the centered Gaussian of variance δ; i.e., $d\gamma_\delta(x) = (2\pi\delta)^{-n/2} \exp(-\frac{|x|^2}{2\delta})dx$. Given a measure μ, we will sometimes write μ_δ as shorthand for $\mu * \gamma_\delta$, the convolution of μ with γ_δ. Since γ_δ has a smooth density, μ_δ also has a smooth density, which we denote p_δ or sometimes just p (except in Section 2.3, where we use p to denote the Gaussian density, and q to denote to convolved density). In general, any symbol decorated with a δ will denote some object associated to $\mu * \gamma_\delta$.

We remark that some of the theorems stated below are subsumed by other theorems stated below. For the sake of exposition, we include all of their statements (and proofs, in the subsequent chapters).

The first three theorems below concern measures on the real line.

Theorem 1.2.1. *Let μ be a compactly supported probability measure on $\mathbb{R}$. Let γ_δ be the centered Gaussian with variance $\delta > 0$. Then $\mu * \gamma_\delta$ satisfies a LSI with constant c for some $c = c(\delta)$.*

Theorem 1.2.2. *Let μ be a probability measure on $\mathbb{R}$ whose support is contained in an interval of length $2R$, and let γ_δ be the centered Gaussian of variance $\delta > 0$.*

Then for some absolute constants K_i, the optimal log-Sobolev constant $c(\delta)$ for $\mu * \gamma_\delta$ satisfies

$$c(\delta) \leq K_1 \frac{\delta^{3/2} R}{4R^2 + \delta} \exp\left(\frac{2R^2}{\delta}\right) + K_2\,(\sqrt{\delta} + 2R)^2.$$

In particular, if $\delta \leq R^2$, then

$$c(\delta) \leq K_3 \frac{\delta^{3/2}}{R} \exp\left(\frac{2R^2}{\delta}\right).$$

The K_i can be taken in the above inequalities to be $K_1 = 6905, K_2 = 4989, K_3 = 7803$.

Theorem 1.2.3. *Let μ be a probability measure on $\mathbb{R}$ whose support is contained in an interval of length $2R$, and let γ_δ be the centered Gaussian of variance $\delta > 0$. Then the optimal log-Sobolev constant $c(\delta)$ for $\mu * \gamma_\delta$ satisfies*

$$c(\delta) \leq \max\left(\delta \exp\left(\frac{4R^2}{\delta} + \frac{4R}{\sqrt{\delta}} + \frac{1}{4}\right), \delta \exp\left(\frac{12R^2}{\delta}\right)\right).$$

In particular, if $\delta \leq 3R^2$, we have

$$c(\delta) \leq \delta \exp\left(\frac{12R^2}{\delta}\right).$$

We remark here that while Theorem 1.2.3 is quantitatively no better than Theorem 1.2.2 (for small δ), its novelty lies in its proof, which does not rely heavily on any sophisticated machinery or deep theorems.

The next two theorems are statements about measures on $\mathbb{R}^n$.

Theorem 1.2.4. *Let μ be a probability measure on $\mathbb{R}^n$ whose support is contained in a ball of radius R. Then for all $\delta > 2R^2 n$, the optimal log-Sobolev constant $c(\delta)$ for $\mu * \gamma_\delta$ satisfies*

$$c(\delta) \leq \frac{\delta^2}{\delta - 2R^2 n}.$$

Theorem 1.2.5. *Let μ be a probability measure on $\mathbb{R}^n$ whose support is contained in a ball of radius R, and let γ_δ be the centered Gaussian of variance δ with $0 < \delta \leq R^2$, i.e., $d\gamma_\delta(x) = (2\pi\delta)^{-n/2} \exp(-\frac{|x|^2}{2\delta})dx$. Then for some absolute constant K, the optimal log-Sobolev constant $c(\delta)$ for $\mu * \gamma_\delta$ satisfies*

$$c(\delta) \leq K\, R^2 \exp\left(20n + \frac{5R^2}{\delta}\right).$$

K can be taken above to be 289.

The bounds stated in Theorems 1.2.3, 1.2.2, and 1.2.5 are bounded by an exponential in R^2/δ. We show in Example 4.0.6 that one cannot do better than exponential in R^2/δ for small δ.

Our last main result is an application of Theorem 1.2.5. We prove, under weaker hypotheses than classically stated, the universality theorem in random matrix theory that the empirical law of eigenvalues of an $n \times n$ real symmetric random matrix converges weakly to its mean in probability as $n \to \infty$. (Further exposition is given in Chapter 5.) Before we state that theorem, we state some terminology. Given a set S, we say $\Pi = \{P_1, P_2, \ldots, P_m\}$ is a partition of S if the P_k are disjoint non-empty subsets of S whose union equals S. Also, recall that a family $\{f_\alpha\}_{\alpha \in A}$ of random variables is said to be uniformly integrable if for every $\epsilon > 0$ there exists a $C \geq 0$ such that

$$\mathbb{E}\left(f_\alpha \cdot \mathbb{1}_{\{|f_\alpha| > C\}}\right) < \epsilon$$

for all $\alpha \in A$.

Theorem 1.2.6. *For each natural number n, let Y_n be an $n \times n$ random real symmetric matrix, and let $X_n = \frac{1}{\sqrt{n}} Y_n$. Suppose the following:*

1. *The family*

$$\left\{[\mathring{Y}_n]_{ij}^2\right\}_{n \in \mathbb{N}, 1 \leq i,j \leq n}$$

is uniformly integrable, where for a random variable Z, $\mathring{Z} \coloneqq Z - \mathbb{E}(Z)$.

2. *For each n, there exists d_n and a partition $\Pi = \{P_1, P_2, \ldots, P_m\}$ of $\{[Y_n]_{ij}\}_{1 \leq i \leq j \leq n}$ such that:*

   (a) *For each $1 \leq k \leq m$, $|P_k| \leq d_n$.*

   (b) *For each $1 \leq k \leq m$, every entry in P_k is independent of $\bigcup_{l \neq k} P_l$.*

   (c) *As $n \to \infty$,*

$$\frac{d_n}{\log n} \to 0.$$

Then the empirical law of eigenvalues μ_{X_n} of X_n converges weakly to its mean in probability.

The remainder of this dissertation is outlined as follows:

In Chapter 2, we present the proofs of our 1-dimensional results: Theorems 1.2.1, 1.2.2, and 1.2.3.

In Chapter 3, we present the proofs of our n-dimensional results: Theorems 1.2.4 and 1.2.5.

In Chapter 4, we look at LSIs for specific examples. We consider compactly supported measures on $\mathbb{R}$ with densities bounded above and below, as well as measures with disconnected support; in particular, 2-point measures.

In Chapter 5, we give a brief exposition of Theorem 1.2.6, and present the proof of Theorem 1.2.6.

Chapter 2

The 1-dimensional case

2.1 Proof of Theorem 1.2.1

The main tool for proving Theorem 1.2.1 is Theorem 1.1.3. The key idea is the fact that we can describe the tail behavior of the convolution of a compactly supported measure with a Gaussian.

Proof of Theorem 1.2.1. Suppose $\operatorname{supp}(\mu) \subseteq [a, b]$. We will apply Theorem 1.1.3 to the probability measure $\mu * \gamma_\delta$. We will show D_0 and D_1, as defined in Theorem 1.1.3, are finite; at the moment we consider D_0. Since γ_δ has a smooth density, $\mu * \gamma_\delta$ has a smooth density p. Note that p is nonzero everywhere since γ_δ has strictly positive density. We therefore want to show

$$D_0 = \sup_{x<m} \left(\int_{-\infty}^{x} p(t)dt \cdot \log \frac{1}{\int_{-\infty}^{x} p(t)dt} \cdot \int_{x}^{m} \frac{1}{p(t)} dt \right)$$

is finite. Since the above expression is continuous in x for all $x \in \mathbb{R}$, it is bounded on every compact interval. We therefore only need to show that

$$\limsup_{x\to-\infty} \left(\int_{-\infty}^{x} p(t)dt \cdot \log \frac{1}{\int_{-\infty}^{x} p(t)dt} \cdot \int_{x}^{m} \frac{1}{p(t)} dt \right)$$

is finite. We will do this by giving asymptotics for $\int_{-\infty}^{x} p(t)dt$ and $\int_{x}^{m} \frac{1}{p(t)}dt$.

Lemma 2.1.1.

$$\lim_{x\to-\infty} \frac{\delta p'(x)}{-xp(x)} = 1.$$

Proof. By definition of p,

$$p(x) = \int \frac{1}{\sqrt{2\pi\delta}} \exp\left(\frac{-(x-t)^2}{2\delta}\right) d\mu(t),$$

so

$$\frac{p(x+h)-p(x)}{h} = \int \frac{1}{\sqrt{2\pi\delta}} \frac{1}{h} \left(\exp\left(\frac{-(x+h-t)^2}{2\delta}\right) - \exp\left(\frac{-(x-t)^2}{2\delta}\right)\right) d\mu(t).$$

Since, by the Mean Value Theorem, the integrand in the above equation is dominated uniformly in h by $\max_{t\in\mathbb{R}} \frac{-t}{\delta\sqrt{2\pi\delta}} \exp(\frac{-t^2}{2\delta}) < \infty$, we can let $h \to 0$ and apply the Dominated Convergence Theorem to differentiate under the integral and get

$$p'(x) = \int \frac{-1}{\delta\sqrt{2\pi\delta}}(x-t) \exp\left(\frac{-(x-t)^2}{2\delta}\right) d\mu(t).$$

Then

$$\begin{aligned}
\frac{\delta p'(x)}{-xp(x)} &= \frac{\delta \int \frac{-1}{\delta\sqrt{2\pi\delta}}(x-t)\exp\left(\frac{-(x-t)^2}{2\delta}\right) d\mu(t)}{-x\int \frac{1}{\sqrt{2\pi\delta}} \exp\left(\frac{-(x-t)^2}{2\delta}\right) d\mu(t)} \\
&= \frac{\int (x-t)\exp\left(\frac{-(x-t)^2}{2\delta}\right) d\mu(t)}{x\int \exp\left(\frac{-(x-t)^2}{2\delta}\right) d\mu(t)} \\
&= 1 - \frac{\int t\exp\left(\frac{-(x-t)^2}{2\delta}\right) d\mu(t)}{x\int \exp\left(\frac{-(x-t)^2}{2\delta}\right) d\mu(t)}.
\end{aligned}$$

But

$$\begin{aligned}
\left|\frac{\int t\exp\left(\frac{-(x-t)^2}{2\delta}\right) d\mu(t)}{x\int \exp\left(\frac{-(x-t)^2}{2\delta}\right) d\mu(t)}\right| &\le \frac{\int |t|\exp\left(\frac{-(x-t)^2}{2\delta}\right) d\mu(t)}{|x|\int \exp\left(\frac{-(x-t)^2}{2\delta}\right) d\mu(t)} \\
&\le \frac{\max(|a|,|b|)\int \exp\left(\frac{-(x-t)^2}{2\delta}\right) d\mu(t)}{|x|\int \exp\left(\frac{-(x-t)^2}{2\delta}\right) d\mu(t)} \\
&= \frac{\max(|a|,|b|)}{|x|} \\
&\to 0 \quad \text{as} \quad x \to -\infty,
\end{aligned}$$

so $\frac{\delta p'(x)}{-xp(x)} \to 1$ as $x \to -\infty$.

□

The next two lemmas give asymptotics for $\int_{-\infty}^{x} p(t)dt$ and $\int_{x}^{m} \frac{1}{p(t)}dt$. We will say $f(x) \sim g(x)$ if $\frac{f(x)}{g(x)} \to 1$ as $x \to -\infty$.

Lemma 2.1.2.

$$\int_{-\infty}^{x} p(t)dt \sim -\frac{\delta}{x}p(x).$$

Proof. Observe that both $\int_{-\infty}^{x} p(t)dt$ and $-\frac{\delta}{x}p(x)$ tend to 0 as $x \to -\infty$ and apply L'Hôpital's Rule and Lemma 2.1.1:

$$\begin{aligned}
\lim_{x\to-\infty} \frac{\int_{-\infty}^{x} p(t)dt}{-\frac{\delta}{x}p(x)} &= \lim_{x\to-\infty} \frac{p(x)}{\frac{\delta}{x^2}p(x) - \frac{\delta}{x}p'(x)} \\
&= \lim_{x\to-\infty} \frac{1}{\frac{\delta}{x^2} + \frac{\delta p'(x)}{-xp(x)}} \\
&= 1.
\end{aligned}$$

□

Lemma 2.1.3.

$$\int_{x}^{m} \frac{1}{p(t)}dt \sim -\frac{\delta}{xp(x)}.$$

Observe that this claim shows that the above integral asymptotically does not depend on m. Since p is continuous and nonzero on $\mathbb{R}$ and $m \in [a, b]$, $\int_{x}^{m} \frac{1}{p(t)}dt$ is finite for each x; and since $\frac{1}{p(t)}$ blows up as $x \to -\infty$, any dependence on m of $\int_{x}^{m} \frac{1}{p(t)}dt$ is diminished as $x \to -\infty$.

Proof. For $x \leq a$,

$$\begin{aligned}
p(x) &= \int \frac{1}{\sqrt{2\pi\delta}} \exp\left(\frac{-(x-t)^2}{2\delta}\right) d\mu(t) \\
&\leq \int \frac{1}{\sqrt{2\pi\delta}} \exp\left(\frac{-(x-a)^2}{2\delta}\right) d\mu(t) \\
&= \frac{1}{\sqrt{2\pi\delta}} \exp\left(\frac{-(x-a)^2}{2\delta}\right),
\end{aligned}$$

so that

$$\frac{1}{p(x)} \geq \sqrt{2\pi\delta} \cdot \exp\left(\frac{(x-a)^2}{2\delta}\right).$$

So $-\frac{\delta}{xp(x)}$ tends to $+\infty$ as $x \to -\infty$ and we can again use L'Hôpital's Rule and

Lemma 2.1.1:

$$\lim_{x\to-\infty}\frac{\int_x^m \frac{1}{p(t)}dt}{-\frac{\delta}{xp(x)}} = \lim_{x\to-\infty}\frac{-\frac{1}{p(x)}}{\frac{\delta}{(xp(x))^2}(p(x)+xp'(x))}$$
$$= \lim_{x\to-\infty}\frac{1}{\frac{-\delta}{x^2}+\frac{\delta p'(x)}{-xp(x)}}$$
$$= 1.$$

□

Before we proceed, we need the following fact about asymptotics of logs: if $f(x) \sim g(x)$ and $g(x) \to \infty$ as $x \to -\infty$, then $\log f(x) \sim \log g(x)$. This follows by observing that

$$\frac{\log f(x)}{\log g(x)} = 1 + \frac{\log\left(\frac{f(x)}{g(x)}\right)}{\log g(x)}$$

and letting $x \to -\infty$.

Proposition 2.1.4. *D_0 and D_1 are finite.*

Proof. We first consider D_0. By the observations made at the beginning of this section, it suffices to show that

$$\limsup_{x\to-\infty}\left(\int_{-\infty}^x p(t)dt \cdot \log\frac{1}{\int_{-\infty}^x p(t)dt}\cdot\int_x^m \frac{1}{p(t)}dt\right)$$

is finite. By Lemmas 2.1.2 and 2.1.3,

$$\limsup_{x\to-\infty}\left(\int_{-\infty}^x p(t)dt \cdot \log\frac{1}{\int_{-\infty}^x p(t)dt}\cdot\int_x^m \frac{1}{p(t)}dt\right)$$
$$=\limsup_{x\to-\infty} -\frac{\delta}{x}p(x)\cdot\log\left(-\frac{x}{\delta}\frac{1}{p(x)}\right)\cdot\left(-\frac{\delta}{xp(x)}\right)$$
$$=\limsup_{x\to-\infty}\frac{\delta^2}{x^2}\left(\log\left(\frac{-x}{\delta}\right)-\log p(x)\right)$$
$$=\limsup_{x\to-\infty}\frac{\delta^2}{x^2}\left(-\log p(x)\right).$$

We just now need to show that $\limsup_{x\to-\infty} \frac{\delta^2}{x^2}(-\log p(x)) < \infty$. But for $x \le a$,

$$\begin{aligned} p(x) &= \int \frac{1}{\sqrt{2\pi\delta}} \exp\left(\frac{-(x-t)^2}{2\delta}\right) d\mu(t) \\ &\ge \int \frac{1}{\sqrt{2\pi\delta}} \exp\left(\frac{-(x-b)^2}{2\delta}\right) d\mu(t) \\ &= \frac{1}{\sqrt{2\pi\delta}} \exp\left(\frac{-(x-b)^2}{2\delta}\right) \end{aligned}$$

so that

$$\begin{aligned} &\limsup_{x\to-\infty} \frac{\delta^2}{x^2}(-\log p(x)) \\ \le &\limsup_{x\to-\infty} -\frac{\delta^2}{x^2} \log\left(\frac{1}{\sqrt{2\pi\delta}} \exp\left(\frac{-(x-b)^2}{2\delta}\right)\right) \\ = &\limsup_{x\to-\infty} -\frac{\delta^2}{x^2}\left(\log\left(\frac{1}{\sqrt{2\pi\delta}}\right) + \frac{-(x-b)^2}{2\delta}\right) \\ = &\frac{\delta}{2} < \infty. \end{aligned}$$

Therefore $D_0 < \infty$.

The proof that $D_1 < \infty$ is practically identical, the relevant ingredients being the following:

$$\begin{aligned} &1 - F(x) = \int_x^\infty p(t)dt, \\ &\lim_{x\to+\infty} \frac{\delta p'(x)}{-xp(x)} = 1, \\ &\int_x^\infty p(t)dt \sim \frac{\delta}{x}p(x) \quad \text{as} \quad x \to +\infty, \quad \text{and} \\ &\int_m^x \frac{1}{p(t)}dt \sim \frac{\delta}{xp(x)} \quad \text{as} \quad x \to +\infty. \end{aligned}$$

Details are omitted. □

Theorem 2 now immediately follows from Proposition 2.1.4.

□

2.2 Proof of Theorem 1.2.2

The approach to proving Theorem 1.2.2 uses Theorem 1.1.3 as was done in the previous section, but by carefully bounding D_0 and D_1 in Theorem 1.1.3, both on a bounded neighborhood of supp(μ) and outside of that neighborhood, we can construct explicit upper bounds for the optimal log-Sobolev constant for $\gamma * \mu$.

Proof of Theorem 1.2.2. Fix μ with support contained in an interval of length $2R$. Since satisfaction of a LSI is translation invariant, we can assume that the support of μ is contained in the interval $[-R, R]$. Throughout, we will let m_δ denote the median of μ_δ, and $D_0(\delta)$ and $D_1(\delta)$ be as defined in Theorem 1.1.3, as applied to the measure μ_δ. We therefore want to bound

$$D_0(\delta) = \sup_{x<m_\delta} \left(\int_{-\infty}^{x} p_\delta(t)dt \cdot \log \frac{1}{\int_{-\infty}^{x} p_\delta(t)dt} \cdot \int_{x}^{m_\delta} \frac{1}{p_\delta(t)} dt \right)$$

and

$$D_1(\delta) = \sup_{x>m_\delta} \left(\int_{x}^{\infty} p_\delta(t)dt \cdot \log \frac{1}{\int_{x}^{\infty} p_\delta(t)dt} \cdot \int_{m_\delta}^{x} \frac{1}{p_\delta(t)} dt \right).$$

As in the previous section, note that

$$p_\delta(t)dt = \int_{-R}^{R} \frac{1}{\sqrt{2\pi\delta}} \exp\left(-\frac{(t-s)^2}{2\delta}\right) d\mu(s)$$

and

$$p'_\delta(t)dt = \int_{-R}^{R} \frac{1}{\sqrt{2\pi\delta}} \cdot \frac{s-t}{\delta} \exp\left(-\frac{(t-s)^2}{2\delta}\right) d\mu(s).$$

Lemma 2.2.1. *For $x \geq R$,*

$$\int_{x}^{\infty} p_\delta(t)dt \leq \frac{4}{3} \cdot \frac{\delta}{x-R+\sqrt{\delta}}\, p_\delta(x).$$

Proof. We have

$$\begin{aligned}
&\frac{4}{3} \cdot \frac{\delta}{x-R+\sqrt{\delta}}\, p_\delta(x) - \int_{x}^{\infty} p_\delta(t)dt \\
&= -\frac{4}{3}\delta \int_{x}^{\infty} \frac{d}{dt}\left(\frac{p_\delta(t)}{t-R+\sqrt{\delta}}\right) dt - \int_{x}^{\infty} p_\delta(t)dt \\
&= \int_{x}^{\infty} \left(-\frac{4}{3}\delta \cdot \frac{p'_\delta(t)(t-R+\sqrt{\delta}) - p_\delta(t)}{(t-R+\sqrt{\delta})^2} - p_\delta(t)\right) dt.
\end{aligned}$$

Writing out the integral expressions for p, p' and simplifying, we get that the above expression is equal to

$$
\begin{aligned}
&\int_x^\infty \frac{1}{(t-R+\sqrt{\delta})^2}\frac{1}{\sqrt{2\pi\delta}} \\
&\cdot \int_{-R}^R \left(\frac{4}{3}\delta + \frac{4}{3}(t-R+\sqrt{\delta})(t-s) - (t-R+\sqrt{\delta})^2\right)\exp\left(-\frac{(t-s)^2}{2\delta}\right) d\mu(s)dt \\
\geq &\int_x^\infty \frac{1}{(t-R+\sqrt{\delta})^2}\frac{1}{\sqrt{2\pi\delta}} \\
&\int_{-R}^R \left(\frac{4}{3}\delta + \frac{4}{3}(t-R+\sqrt{\delta})(t-R) - (t-R+\sqrt{\delta})^2\right)\exp\left(-\frac{(t-s)^2}{2\delta}\right) d\mu(s)dt \\
&\text{since } t \geq R \text{ and } s \leq R \\
= &\int_x^\infty \frac{1}{(t-R+\sqrt{\delta})^2}\frac{1}{\sqrt{2\pi\delta}}\int_{-R}^R \frac{1}{3}(t-R-\sqrt{\delta})^2 \exp\left(-\frac{(t-s)^2}{2\delta}\right) d\mu(s)dt \\
\geq &\, 0,
\end{aligned}
$$

as desired. □

The next lemma is an elementary calculation; we omit the details.

Lemma 2.2.2. *For $x \geq 0$,*

$$\int_x^\infty \exp\left(-\frac{u^2}{2}\right) du \geq \frac{1}{x+1}\exp\left(-\frac{x^2}{2}\right)$$

and

$$\int_0^x \exp\left(\frac{u^2}{2}\right) du \leq \frac{2x}{x^2+1}\exp\left(\frac{x^2}{2}\right).$$

Lemma 2.2.3. *For $x \geq R$,*

$$\int_x^\infty p_\delta(t)dt \geq \frac{\sqrt{\delta}}{\sqrt{2\pi}(x+R+\sqrt{\delta})}\exp\left(-\frac{(x+R)^2}{2\delta}\right).$$

Proof. We have

$$
\begin{aligned}
\int_x^\infty p_\delta(t)dt &= \int_x^\infty \int_{-R}^R \frac{1}{\sqrt{2\pi\delta}}\exp\left(-\frac{(t-s)^2}{2\delta}\right) d\mu(s)dt \\
&\geq \int_x^\infty \int_{-R}^R \frac{1}{\sqrt{2\pi\delta}}\exp\left(-\frac{(t+R)^2}{2\delta}\right) d\mu(s)dt \\
&= \int_x^\infty \frac{1}{\sqrt{2\pi\delta}}\exp\left(-\frac{(t+R)^2}{2\delta}\right) dt,
\end{aligned}
$$

the above inequality being because ($t \geq R$ and $-R \leq s \leq R$) $\Rightarrow -(t-s)^2 \geq -(t+R)^2$. Letting $u = (t+R)/\sqrt{\delta}$ in the last integral above, we get

$$\begin{aligned}\int_x^\infty p_\delta(t)dt \geq & \frac{1}{\sqrt{2\pi}} \int_{(x+R)/\sqrt{\delta}}^\infty \exp\left(-\frac{u^2}{2}\right) du \\ \geq & \frac{1}{\sqrt{2\pi}} \frac{1}{(x+R)/\sqrt{\delta}+1} \exp\left(-\frac{(x+R)^2}{2\delta}\right) \\ & \text{by Lemma 2.2.2} \\ = & \frac{\sqrt{\delta}}{\sqrt{2\pi}(x+R+\sqrt{\delta})} \exp\left(-\frac{(x+R)^2}{2\delta}\right),\end{aligned}$$

as desired. □

Lemma 2.2.4. *For $x \geq R$,*

$$\int_R^x \frac{1}{p_\delta(t)} dt \leq \frac{2\delta(x-R)}{((x-R)^2+\delta)p_\delta(x)}.$$

Proof. We have

$$\frac{2\delta(x-R)}{((x-R)^2+\delta)p_\delta(x)} - \int_R^x \frac{1}{p_\delta(t)} dt = \int_R^x \left(\frac{d}{dt}\left(\frac{2\delta(t-R)}{((t-R)^2+\delta)p_\delta(t)} \right) - \frac{1}{p_\delta(t)} \right) dt.$$

Letting $u = (t-R)/\sqrt{\delta}$ and writing out the integral expressions for p, p' and simplifying, we get that the above expression is equal to

$$\begin{aligned}& \int_0^{(x-R)/\sqrt{\delta}} \frac{1}{\sqrt{2\pi}(u^2+1)^2 p_\delta(\sqrt{\delta}u+R)^2} \\ & \cdot \int_{-R}^R \left(-u^4 - 4u^2 + 1 + 2(u^3+u)\left(u + \frac{R-s}{\sqrt{\delta}}\right)\right) \exp\left(-\frac{(u+R-s)^2}{2\delta}\right) d\mu(s) du \\ \geq & \int_0^{(x-R)/\sqrt{\delta}} \frac{1}{\sqrt{2\pi}(u^2+1)^2 p_\delta(\sqrt{\delta}u+R)^2} \\ & \cdot \int_{-R}^R \left(-u^4 - 4u^2 + 1 + 2(u^3+u)\cdot u\right) \exp\left(-\frac{(u+R-s)^2}{2\delta}\right) d\mu(s) du \\ & \text{since } u \geq 0 \text{ and } s \leq R \\ = & \int_0^{(x-R)/\sqrt{\delta}} \frac{1}{\sqrt{2\pi}(u^2+1)^2 p_\delta(\sqrt{\delta}u+R)^2} \\ & \cdot \int_{-R}^R (u^2-1)^2 \exp\left(-\frac{(u+R-s)^2}{2\delta}\right) d\mu(s) du \\ \geq & \, 0,\end{aligned}$$

as desired. □

Lemma 2.2.5.

$$D_1(\delta) \leq \frac{8\sqrt{2\pi}}{e} \cdot \frac{\delta^{3/2}R}{4R^2+\delta} \exp\left(\frac{2R^2}{\delta}\right) + \frac{2}{3}(2\pi+1)(1+\sqrt{2})(\sqrt{\delta}+2R)^2$$

and

$$D_0(\delta) \leq \frac{8\sqrt{2\pi}}{e} \cdot \frac{\delta^{3/2}R}{4R^2+\delta} \exp\left(\frac{2R^2}{\delta}\right) + \frac{2}{3}(2\pi+1)(1+\sqrt{2})(\sqrt{\delta}+2R)^2.$$

Proof. We only present the proof for the bound on $D_1(\delta)$; the proof for the bound on $D_0(\delta)$ involves analogous lemmas and identical reasoning, and is therefore omitted.

By definition of $D_1(\delta)$,

$$\begin{aligned}
D_1(\delta) &= \sup_{x>m_\delta} \int_x^\infty p_\delta(t)dt \cdot \log \frac{1}{\int_x^\infty p_\delta(t)dt} \cdot \int_{m_\delta}^x \frac{1}{p_\delta(t)}dt \\
&= \max\Bigg(\sup_{m_\delta < x \leq R} \int_x^\infty p_\delta(t)dt \cdot \log \frac{1}{\int_x^\infty p_\delta(t)dt} \cdot \int_{m_\delta}^x \frac{1}{p_\delta(t)}dt, \\
&\qquad \sup_{x \geq R} \int_x^\infty p_\delta(t)dt \cdot \log \frac{1}{\int_x^\infty p_\delta(t)dt} \cdot \int_{m_\delta}^x \frac{1}{p_\delta(t)}dt \Bigg) \\
&\leq \max\Bigg(\sup_{m_\delta < x \leq R} \int_x^\infty p_\delta(t)dt \cdot \log \frac{1}{\int_x^\infty p_\delta(t)dt} \cdot \int_{m_\delta}^x \frac{1}{p_\delta(t)}dt, \\
&\qquad \sup_{x \geq R} \int_x^\infty p_\delta(t)dt \cdot \log \frac{1}{\int_x^\infty p_\delta(t)dt} \cdot \int_{m_\delta}^R \frac{1}{p_\delta(t)}dt \\
&\qquad + \sup_{x \geq R} \int_x^\infty p_\delta(t)dt \cdot \log \frac{1}{\int_x^\infty p_\delta(t)dt} \cdot \int_R^x \frac{1}{p_\delta(t)}dt \Bigg) \\
&= \max(A, B+C),
\end{aligned}$$

where

$$\begin{aligned}
A &:= \sup_{m_\delta < x \leq R} \int_x^\infty p_\delta(t)dt \cdot \log \frac{1}{\int_x^\infty p_\delta(t)dt} \cdot \int_{m_\delta}^x \frac{1}{p_\delta(t)}dt, \\
B &:= \sup_{x \geq R} \int_x^\infty p_\delta(t)dt \cdot \log \frac{1}{\int_x^\infty p_\delta(t)dt} \cdot \int_{m_\delta}^R \frac{1}{p_\delta(t)}dt, \\
C &:= \sup_{x \geq R} \int_x^\infty p_\delta(t)dt \cdot \log \frac{1}{\int_x^\infty p_\delta(t)dt} \cdot \int_R^x \frac{1}{p_\delta(t)}dt.
\end{aligned}$$

By Lemmas 2.2.1,2.2.3, and 2.2.4,

$$\begin{aligned} C \leq & \sup_{x \geq R} \frac{4}{3} \cdot \frac{\delta}{x - R + \sqrt{\delta}}\ p_\delta(x) \cdot \log\left(\frac{\sqrt{2\pi}}{\sqrt{\delta}}(x + R + \sqrt{\delta}) \exp\left(\frac{(x+R)^2}{2\delta}\right)\right) \\ & \cdot \frac{2\delta(x - R)}{((x-R)^2 + \delta)p_\delta(x)} \\ = & \sup_{u \geq 0} \frac{8}{3} \frac{\delta u}{(u+1)(u^2+1)} \cdot \left[\frac{1}{2}\log\left(\frac{2\pi}{\delta}(\sqrt{\delta}(u+1) + 2R)^2\right) + \frac{(\sqrt{\delta}u + 2R)^2}{2\delta}\right] \\ & \text{where } u = \frac{x - R}{\sqrt{\delta}}. \end{aligned}$$

Since $\log y \leq y/e$, we get

$$\begin{aligned} C \leq & \sup_{u \geq 0} \frac{8}{3} \frac{\delta u}{(u+1)(u^2+1)} \cdot \left[\frac{1}{2e} \cdot \frac{2\pi}{\delta}(\sqrt{\delta}(u+1) + 2R)^2 + \frac{(\sqrt{\delta}(u+1) + 2R)^2}{2\delta}\right] \\ = & \sup_{u \geq 0} \frac{4}{3e}(2\pi + e) \cdot \frac{(\sqrt{\delta}(u+1) + 2R)^2}{u+1} \cdot \frac{u}{u^2+1}. \end{aligned}$$

Using $u/(u^2+1) \leq (1+\sqrt{2})/2(u+1)$ and simplifying, we finally get

$$C \leq \sup_{u \geq 0} \frac{2}{3e}(2\pi + e)(1 + \sqrt{2})\left(\sqrt{\delta} + \frac{2R}{u+1}\right)^2 = \frac{2}{3e}(2\pi + e)(1+\sqrt{2})(\sqrt{\delta} + 2R)^2.$$

We now bound A. We have

$$\begin{aligned} A \leq & \sup_{m_\delta < x \leq R}\left(\int_x^\infty p_\delta(t)dt \cdot \log\frac{1}{\int_x^\infty p_\delta(t)dt}\right) \cdot \sup_{m_\delta < x \leq R} \int_{m_\delta}^x \frac{1}{p_\delta(t)}dt \\ \leq & \sup_{0 < u < 1}\left(u \log\frac{1}{u}\right) \cdot \int_{m_\delta}^R \frac{1}{p_\delta(t)}dt \\ \leq & \frac{1}{e}\int_{-R}^R \frac{1}{p_\delta(t)}dt. \end{aligned}$$

But

$$\begin{aligned} p_\delta(t) = & \frac{1}{\sqrt{2\pi\delta}}\int_{-R}^R \exp\left(-\frac{(t-s)^2}{2\delta}\right)d\mu(s) \\ \geq & \frac{1}{\sqrt{2\pi\delta}}\int_{-R}^R \exp\left(-\frac{(|t|+R)^2}{2\delta}\right)d\mu(s) \\ & \text{since } -R \leq s \leq R \Rightarrow -(t-s)^2 \geq -(|t|+R)^2 \\ = & \frac{1}{\sqrt{2\pi\delta}}\exp\left(-\frac{(|t|+R)^2}{2\delta}\right), \end{aligned}$$

so

$$A \le \frac{1}{e}\int_{-R}^{R}\sqrt{2\pi\delta}\exp\left(\frac{(|t|+R)^2}{2\delta}\right)dt = \frac{2}{e}\int_{0}^{R}\sqrt{2\pi\delta}\exp\left(\frac{(t+R)^2}{2\delta}\right)dt$$
$$\le \frac{2}{e}\int_{-R}^{R}\sqrt{2\pi\delta}\exp\left(\frac{(t+R)^2}{2\delta}\right)dt.$$

Letting $u = (t+R)/\sqrt{\delta}$ above and applying Lemma 2.2.2, we get

$$A \le \frac{2\sqrt{2\pi}\delta}{e}\int_{0}^{2R/\sqrt{\delta}}\exp\left(\frac{u^2}{2}\right)du \le \frac{2\sqrt{2\pi}\delta}{e}\cdot\frac{2\cdot 2R/\sqrt{\delta}}{4R^2/\delta+1}\exp\left(\frac{2R^2}{\delta}\right)$$
$$= \frac{8\sqrt{2\pi}}{e}\cdot\frac{\delta^{3/2}R}{4R^2+\delta}\exp\left(\frac{2R^2}{\delta}\right).$$

Similarly,

$$B \le \sup_{0<u<1}\left(u\log\frac{1}{u}\right)\cdot\int_{m_\delta}^{R}\frac{1}{p_\delta(t)}dt \le \frac{8\sqrt{2\pi}}{e}\cdot\frac{\delta^{3/2}R}{4R^2+\delta}\exp\left(\frac{2R^2}{\delta}\right).$$

So

$$D_1(\delta) \le \max\Bigg[\frac{8\sqrt{2\pi}}{e}\cdot\frac{\delta^{3/2}R}{4R^2+\delta}\exp\left(\frac{2R^2}{\delta}\right),$$
$$\frac{8\sqrt{2\pi}}{e}\cdot\frac{\delta^{3/2}R}{4R^2+\delta}\exp\left(\frac{2R^2}{\delta}\right) + \frac{2}{3e}(2\pi+e)(1+\sqrt{2})(\sqrt{\delta}+2R)^2\Bigg]$$
$$= \frac{8\sqrt{2\pi}}{e}\cdot\frac{\delta^{3/2}R}{4R^2+\delta}\exp\left(\frac{2R^2}{\delta}\right) + \frac{2}{3e}(2\pi+e)(1+\sqrt{2})(\sqrt{\delta}+2R)^2.$$

□

To bound $c(\delta)$ and conclude the proof of Theorem 1.2.2, we apply Theorem 1.1.3 and Lemma 2.2.5:

$$c(\delta) \le 468(D_0(\delta)+D_1(\delta))$$
$$\le 468\cdot 2\cdot\left(\frac{8\sqrt{2\pi}}{e}\cdot\frac{\delta^{3/2}R}{4R^2+\delta}\exp\left(\frac{2R^2}{\delta}\right) + \frac{2}{3e}(2\pi+e)(1+\sqrt{2})(\sqrt{\delta}+2R)^2\right)$$
$$\le 6905\cdot\frac{\delta^{3/2}R}{4R^2+\delta}\exp\left(\frac{2R^2}{\delta}\right) + 4989\cdot(\sqrt{\delta}+2R)^2.$$

In particular, suppose $\delta \le R^2$. Now by elementary calculus, $\frac{\delta^{3/2}}{R}\exp\left(\frac{2R^2}{\delta}\right)$ is decreasing in δ for $\delta \le R^2$, so

$$\frac{\delta^{3/2}}{R}\exp\left(\frac{2R^2}{\delta}\right) \ge \left[\frac{\delta^{3/2}}{R}\exp\left(\frac{2R^2}{\delta}\right)\right]_{\delta=R^2} = e^2R^2,$$

giving

$$R^2 \leq e^{-2} \cdot \frac{\delta^{3/2}}{R} \exp\left(\frac{2R^2}{\delta}\right).$$

Therefore

$$\begin{aligned} c(\delta) \leq & 468 \cdot 2 \cdot \left(\frac{8\sqrt{2\pi}}{e} \cdot \frac{\delta^{3/2} R}{4R^2 + \delta} \exp\left(\frac{2R^2}{\delta}\right) + \frac{2}{3e}(2\pi + e)(1+\sqrt{2})(\sqrt{\delta} + 2R)^2\right) \\ \leq & 936 \cdot \left(\frac{8\sqrt{2\pi}}{e} \cdot \frac{\delta^{3/2} R}{4R^2} \exp\left(\frac{2R^2}{\delta}\right) + \frac{2}{3e}(2\pi + e)(1+\sqrt{2})(3R)^2\right) \\ \leq & 936 \cdot \left(\frac{2\sqrt{2\pi}}{e} + \frac{6}{e}(2\pi + e)(1+\sqrt{2})e^{-2}\right) \frac{\delta^{3/2}}{R} \exp\left(\frac{2R^2}{\delta}\right) \\ \leq & 7803 \cdot \frac{\delta^{3/2}}{R} \exp\left(\frac{2R^2}{\delta}\right). \end{aligned}$$

This concludes the proof. □

2.3 Elementary proof of Theorem 1.2.3

The proof of Theorem 1.2.3 is based on two facts: first, the Gaussian measure γ_1 of unit variance satisfies a LSI with constant 1. Second, Lipshitz functions preserve LSIs. We give a precise statement of this second fact below.

Proposition 2.3.1. *Let μ be a measure on $\mathbb{R}^m$ that satisfies a LSI with constant c, and let $T : \mathbb{R}^m \to \mathbb{R}^n$ be Lipschitz. Then the push-forward measure $T_*\mu$ also satisfies a LSI with constant $c||T||_{\mathrm{Lip}}^2$.*

Proof. Let $g : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz. Then $g \circ T$ is locally Lipschitz, so by the LSI for μ,

$$\int (g \circ T)^2 \log \frac{(g \circ T)^2}{\int (g \circ T)^2 \, d\mu} \, d\mu \leq c \int |\nabla(g \circ T)|^2 d\mu. \tag{2.1}$$

But since T is Lipschitz,

$$|\nabla(g \circ T)| \leq (|\nabla g| \circ T)||T||_{\mathrm{Lip}}.$$

So by a change of variables, (2.1) simply becomes

$$\int g^2 \log \frac{g^2}{\int g^2 \, dT_*\mu} \, dT_*\mu \leq c||T||_{\mathrm{Lip}}^2 \int |\nabla g|^2 dT_*\mu,$$

as desired. □

We now prove Theorem 1.2.3.

Proof of Theorem 1.2.3. In light of Proposition 2.3.1, we will establish the theorem by showing that $\mu * \gamma_\delta$ is the push-forward of γ_1 under a Lipschitz map. By translation invariance of LSI, we can assume that $\mathrm{supp}(\mu) \subseteq [-R, R]$. We will also first assume that $\delta = 1$ (the general case will be handled at the end of the proof by a scaling argument).

Let F and G be the cumulative distribution functions of γ_1 and $\mu * \gamma_1$, i.e.,

$$F(x) = \int_{-\infty}^{x} p(t)\,dt, \qquad G(x) = \int_{-\infty}^{x} q(t)\,dt,$$

where

$$p(t) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) \qquad \text{and} \qquad q(t) = \int_{-R}^{R} p(t-s)\,d\mu(s).$$

Notice that q is smooth and strictly positive, so that $G^{-1} \circ F$ is well-defined and smooth. It is readily seen that $(G^{-1} \circ F)_*(\gamma_1) = \mu * \gamma_1$, so to establish the theorem we simply need to bound the derivative of $G^{-1} \circ F$.

Now

$$(G^{-1} \circ F)'(x) = \frac{1}{G'((G^{-1} \circ F)(x))} \cdot F'(x) = \frac{p(x)}{q((G^{-1} \circ F)(x))}.$$

We will bound the above derivative in cases – when $x \geq 2R$, when $-2R \leq x \leq 2R$, and when $x \leq -2R$.

We first consider the case $x \geq 2R$. Define

$$\Lambda(x) = \int_{-R}^{R} e^{xs} d\mu(s), \qquad K(x) = \frac{\log \Lambda(x) + R}{x}.$$

Note Λ and K are smooth for $x \neq 0$.

Lemma 2.3.2. *For $x \geq 2R$,*

$$\exp\left(-2R^2 - 2R - \frac{1}{8}\right) p(x) \leq q(x + K(x)) \leq e^{-R} p(x).$$

Proof. By definition of q, p, Λ, and K,

$$\begin{aligned} q(x+K(x)) &= \int_{-R}^{R} p(x+K(x)-s)\, d\mu(s) \\ &= p(x)\cdot e^{-xK(x)} \int_{-R}^{R} \exp\left(-\frac{(K(x)-s)^2}{2}\right)\cdot e^{xs}\, d\mu(s) \\ &= \frac{e^{-R}\, p(x)}{\Lambda(x)} \int_{-R}^{R} \exp\left(-\frac{(K(x)-s)^2}{2}\right)\cdot e^{xs}\, d\mu(s) \\ &\le \frac{e^{-R}\, p(x)}{\Lambda(x)} \int_{-R}^{R} e^{xs}\, d\mu(s) \\ &= e^{-R}\, p(x). \end{aligned}$$

To get the other inequality, first note that $e^{-Rx} \le \Lambda(x) \le e^{Rx}$. (These are just the maximum and minimum values in the integrand defining Λ.) This implies that $-R + R/x \le K(x) \le R + R/x$, so for $-R \le s \le R$ and $x \ge 2R$, we have

$$-2R - \frac{R}{x} \le -2R + \frac{R}{x} \le K(x) - s \le 2R + \frac{R}{x}$$

so that

$$\begin{aligned} \exp\left(-\frac{(K(x)-s)^2}{2}\right) \ge \exp\left(-\frac{(2R+R/x)^2}{2}\right) &\ge \exp\left(-\frac{(2R+R/(2R))^2}{2}\right) \\ &= \exp\left(-2R^2 - R - \frac{1}{8}\right). \end{aligned}$$

Therefore

$$\begin{aligned} q(x+K(x)) &= \frac{e^{-R}\, p(x)}{\Lambda(x)} \int_{-R}^{R} \exp\left(-\frac{(K(x)-s)^2}{2}\right)\cdot e^{xs}\, d\mu(s) \\ &\ge \exp\left(-2R^2 - 2R - \frac{1}{8}\right) p(x). \end{aligned}$$

□

Lemma 2.3.3. *$K'(x) \le R$ for $x \ge 2R$.*

Proof. Recall that $e^{-Rx} \le \Lambda(x)$. (Again, e^{-Rx} is the minimum value in the integrand defining Λ). We therefore have

$$\begin{aligned} K'(x) = \frac{\Lambda'(x)}{x\Lambda(x)} - \frac{\log \Lambda(x)}{x^2} - \frac{R}{x^2} &= \frac{\int_{-R}^{R} s\, e^{sx}\, d\mu(s)}{x\Lambda(x)} - \frac{\log\Lambda(x)}{x^2} - \frac{R}{x^2} \\ &\le \frac{R\int_{-R}^{R} e^{sx}\, d\mu(s)}{x\Lambda(x)} + \frac{Rx}{x^2} - \frac{R}{x^2} \\ &= \frac{2R}{x} - \frac{R}{x^2}. \end{aligned}$$

By elementary calculus, the above has a maximum value of R. □

Lemma 2.3.4. *For $x \geq 2R$,*

$$x - R \leq (G^{-1} \circ F)(x) \leq x + K(x).$$

Proof. Since G and G^{-1} are increasing, the lemma is equivalent to

$$G(x - R) \leq F(x) \leq G(x + K(x)).$$

The first inequality follows from the definition of G and the Fubini-Tonelli Theorem:

$$\begin{aligned} G(x-R) = \int_{-\infty}^{x-R} q(t)\,dt &= \int_{-\infty}^{x} \int_{-R}^{R} p(t-s)\,d\mu(s)\,dt \\ &= \int_{-R}^{R} \int_{-\infty}^{x-R} p(t-s)\,dt\,d\mu(s) \\ &= \int_{-R}^{R} \int_{-\infty}^{x-R+s} p(u)\,du\,d\mu(s) \\ &\quad \text{where } u = t - s \\ &\leq \int_{-R}^{R} \int_{-\infty}^{x} p(u)\,dt\,d\mu(s) \\ &= F(x). \end{aligned}$$

To establish the other inequality, we use Lemmas 2.3.2 and 2.3.3:

$$\begin{aligned} 1 - G(x + K(x)) = \int_{x+K(x)}^{\infty} q(t)\,dt &= \int_{x}^{\infty} q(u + K(u))(1 + K'(u))\,du \\ &\quad \text{where } t = u + K(u) \\ &\leq \int_{x}^{\infty} p(u) e^{-R}(1+R)\,du \\ &\quad \text{by Lemmas 2.3.2 and 2.3.3} \\ &\leq \int_{x}^{\infty} p(u)\,du \\ &\quad \text{since } e^{R} \geq 1 + R \\ &= 1 - F(x), \end{aligned}$$

so that $F(x) \leq G(x + K(x))$, as desired. □

We are almost ready to bound $(G^{-1} \circ F)'(x)$ for $x \geq 2R$. The last observation to make is that q is decreasing on $[R, \infty)$ since

$$q'(t) = \int_{-R}^{R} p'(t-s)\,d\mu(s) = \int_{-R}^{R} -(t-s)p(t-s)\,d\mu(s) \le 0 \qquad \text{for } t \ge R.$$

So for $x \ge 2R$ we have, by lemma 2.3.4,

$$q((G^{-1} \circ F)(x)) \ge q(x + K(x)).$$

Combining this with Lemma 2.3.2, we get

$$(G^{-1} \circ F)'(x) = \frac{p(x)}{q((G^{-1} \circ F)(x))} \le \frac{p(x)}{q(x + K(x))} \le \exp\left(2R^2 + 2R + \frac{1}{8}\right)$$

for $x \ge 2R$.

In the case where $-2R \le x \le 2R$, first note that for all x,

$$x - R \le (G^{-1} \circ F)(x) \le x + R;$$

the first inequality above was done in Lemma 2.3.4, and the second inequality is proven in the same way. So

$$\begin{aligned}\sup_{-2R \le x \le 2R} (G^{-1} \circ F)'(x) = \sup_{-2R \le x \le 2R} \frac{p(x)}{q((G^{-1} \circ F)(x))} &\le \sup_{\substack{-2R \le x \le 2R \\ -R \le y \le R}} \frac{p(x)}{q(x+y)} \\ &= \left(\inf_{\substack{-2R \le x \le 2R \\ -R \le y \le R}} \frac{q(x+y)}{p(x)} \right)^{-1}.\end{aligned}$$

For convenience, let $S = \{(x, y) : -2R \le x \le 2R, -R \le y \le R\}$. Now

$$\inf_{(x,y) \in S} \frac{q(x+y)}{p(x)} = \inf_{(x,y) \in S} \frac{1}{p(x)} \int_{-R}^{R} p(x + y - s)\,d\mu(s).$$

Since p has no local minima, the minimum value of the above integrand occurs at either $s = R$ or $s = -R$. Without loss of generality, we assume the minimum is achieved at $s = R$ (otherwise, we can replace (x, y) with $(-x, -y)$ by symmetry of S and p). So

$$\inf_{(x,y) \in S} \frac{q(x+y)}{p(x)} \ge \inf_{(x,y) \in S} \frac{1}{p(x)} \cdot p(x + y + R).$$

Elementary calculus shows that the above infimum is equal to e^{-6R^2} (achieved at $x = 2R, y = R$). Therefore

$$\sup_{-2R\le x\le 2R} (G^{-1} \circ F)'(x) \le \left(\inf_{(x,y)\in S} \frac{q(x+y)}{p(x)} \right)^{-1} \le e^{6R^2}.$$

The case $x \le -2R$ is dealt with in the same way as the case $x \ge 2R$, the analogous statements being:

$$\exp\left(-2R^2 - 2R - \frac{1}{8}\right) p(x) \le q(x + K(x)) \le e^{-R} p(x),$$

$$K'(x) \le R,$$

$$x + K(x) \le (G^{-1} \circ F)(x) \le x + R,$$

and q is increasing for $x \le -2R$. The upper bound for $(G^{-1} \circ F)'(x)$ obtained in this case is the same as the one in the case $x \ge 2R$.

We therefore have

$$||G^{-1} \circ F||_{\text{Lip}} \le \max\left(\exp\left(2R^2 + 2R + \frac{1}{8}\right), e^{6R^2}\right)$$

So by Proposition 2.3.1, $\mu * \gamma_1$ satisfies a LSI with constant $c(1)$ satisfying

$$c(1) \le ||G^{-1} \circ F||^2_{\text{Lip}} \le \max\left(\exp\left(4R^2 + 4R + \frac{1}{4}\right), e^{12R^2}\right).$$

This proves the theorem for the case $\delta = 1$.

To establish the theorem for a general $\delta > 0$, first observe that

$$\mu * \gamma_\delta = (h_{\sqrt{\delta}})_* \left(((h_{1/\sqrt{\delta}})_*\mu) * \gamma_1\right),$$

where h_λ denotes the scaling map with factor λ, i.e., $h_\lambda(x) = \lambda\, x$. Now $(h_{1/\sqrt{\delta}})_*\mu$ is supported in $[-R/\sqrt{\delta}, R/\sqrt{\delta}]$, so by the case $\delta = 1$ just proven, $((h_{1/\sqrt{\delta}})_*\mu) * \gamma_1$ satisfies a LSI with constant

$$\max\left(\exp\left(4(R/\sqrt{\delta})^2 + 4(R/\sqrt{\delta}) + \frac{1}{4}\right), e^{12(R/\sqrt{\delta})^2}\right).$$

Finally, since $||h_{\sqrt{\delta}}||^2_{\text{Lip}} = \delta$, we have by Proposition 2.3.1,

$$c(\delta) \le \max\left(\delta \exp\left(\frac{4R^2}{\delta} + \frac{4R}{\sqrt{\delta}} + \frac{1}{4}\right), \delta \exp\left(\frac{12R^2}{\delta}\right)\right).$$

In particular, when $\delta \leq 3R^2$ (in fact when $\delta \leq (160 - 64\sqrt{6})R^2 \approx 3.23R^2$), we have

$$\delta \exp\left(\frac{4R^2}{\delta} + \frac{4R}{\sqrt{\delta}} + \frac{1}{4}\right) \leq \delta \exp\left(\frac{12R^2}{\delta}\right)$$

so the above bound on $c(\delta)$ simplifies to

$$c(\delta) \leq \delta \exp\left(\frac{12R^2}{\delta}\right).$$

□

Chapter 2 is, in part, a reprint of material from three articles. The first: D. Zimmermann. Logarithmic Sobolev inequalities for mollified compactly supported measures. *J. Funct. Anal.*, 265:1064–1083, 2013. (See [33].) The second: D. Zimmermann. Bounds for logarithmic Sobolev constants for Gaussian convolutions. Submitted for publication in Annales de l'Institute Henri Poincaré. (See [31].) The third: D. Zimmermann. Elementary proof of logarithmic Sobolev inequalities for Gaussian convolutions on ℝ. Submitted for publication in Annales Mathématiques Blaise Pascal. (See [32].) The dissertation author was the author for this material.

Chapter 3

The n-dimensional case

3.1 Proof of Theorem 1.2.4

Theorem 1.2.4 is based on the following theorem due to Bakry, Émery and Ledoux:

Theorem 3.1.1. (Bakry, Émery, Ledoux). *Let ν be a probability measure on $\mathbb{R}^n$ with smooth, strictly positive density p. If there exists $c > 0$ such that $\mathrm{Hess}(-\log p)(x) - \frac{1}{c}I_n$ is positive semidefinite for all $x \in \mathbb{R}^n$, where I_n is the $n \times n$ identity matrix, then ν satisfies a LSI with constant c.*

We remark that the above theorem was stated by Bakry and Émery in [4] in a slightly different context from what is given here; it was stated in the above form by Ledoux; for a proof of Theorem 3.1.1, see [17, p. 55].

Proof of Theorem 1.2.4. Suppose $\delta > 2R^2n$. By translation invariance of LSI, we may suppose that the ball containing $\mathrm{supp}(\mu)$ is centered at 0. Then $\mu * \gamma_\delta$ has smooth, strictly positive density p given by

$$p(x) = \int (2\pi\delta)^{-n/2} \exp\left(\frac{-(x-y)^2}{2\delta}\right) d\mu(y) = \int d\nu_x(y),$$

where $d\nu_x(y) = (2\pi\delta)^{-n/2} \exp\left(\frac{-(x-y)^2}{2\delta}\right) d\mu(y)$. It is then straightforward to compute that, for $i \neq j$,

$$\partial_i p(x) = -\frac{1}{\delta}\left(x_i \int d\nu_x(y) - \int y_i \, d\nu_x(y)\right),$$
$$\partial_{ii} p(x) = -\frac{1}{\delta^2}\left(\delta \int d\nu_x(y) - x_i^2 \int d\nu_x(y) + 2x_i \int y_i \, d\nu_x(y) - \int y_i^2 \, d\nu_x(y)\right), \quad \text{and}$$
$$\partial_{ij} p(x) = \frac{1}{\delta^2}\left(x_i x_j \int d\nu_x(y) - x_i \int y_j \, d\nu_x(y) - x_j \int y_i \, d\nu_x(y) + \int y_i y_j \, d\nu_x(y)\right);$$

differentiation under the integral is justified by the Dominated Convergence Theorem since the integrands are smooth and have bounded partial derivatives of all orders.

We now show $\delta \cdot \mathrm{Hess}(-\log p)$ converges uniformly to the $n \times n$ identity matrix as $\delta \to \infty$. For $i \neq j$,

$$\begin{aligned}
&(\partial_i p \cdot \partial_j p - p \cdot \partial_{ij} p)(x) \\
=&\frac{1}{\delta^2}\left(x_i \int d\nu_x(y) - \int y_i \, d\nu_x(y)\right)\left(x_j \int d\nu_x(y) - \int y_j \, d\nu_x(y)\right) \\
-&\frac{1}{\delta^2}\int d\nu_x(y)\left(x_i x_j \int d\nu_x(y) - x_i \int y_j \, d\nu_x(y) - x_j \int y_i \, d\nu_x(y) + \int y_i y_j \, d\nu_x(y)\right) \\
=&\frac{1}{\delta^2}\left(\int y_i \, d\nu_x(y) \int y_j \, d\nu_x(y) - \int y_i y_j \, d\nu_x(y)\right),
\end{aligned}$$

so

$$\begin{aligned}
\partial_{ij}(-\log p(x)) &= \frac{\partial_i p(x) \partial_j p(x) - p(x) \partial_{ij} p(x)}{p(x)^2} \\
&= \frac{\int y_i \, d\nu_x(y) \int y_j \, d\nu_x(y) - \int y_i y_j \, d\nu_x(y)}{\delta^2 \left(\int d\nu_x(y)\right)^2}.
\end{aligned}$$

Thus

$$\begin{aligned}
|\delta \cdot \partial_{ij}(-\log p(x))| &\leq \frac{\int |y_i| \, d\nu_x(y) \int |y_j| \, d\nu_x(y) + \int |y_i||y_j| \, d\nu_x(y)}{\delta \left(\int d\nu_x(y)\right)^2} \\
&\leq \frac{R^2 \left(\int d\nu_x(y)\right)^2 + R^2 \left(\int d\nu_x(y)\right)^2}{\delta \left(\int d\nu_x(y)\right)^2} \\
&= \frac{2R^2}{\delta}.
\end{aligned}$$

We also compute

$$\begin{aligned}
\partial_{ii}(-\log p(x)) &= \frac{(\partial_i p(x))^2 - p(x) \partial_{ii} p(x)}{p(x)^2} \\
&= \frac{\left(\int y_i \, d\nu_x(y)\right)^2 + \delta \left(\int d\nu_x(y)\right)^2 - \int d\nu_x(y) \int y_i^2 \, d\nu_x(y)}{\delta^2 \left(\int d\nu_x(y)\right)^2},
\end{aligned}$$

so

$$\begin{aligned}
|\delta \cdot \partial_{ii}(-\log p(x)) - 1| &= \left| \frac{\left(\int y_i \, d\nu_x(y)\right)^2 - \int d\nu_x(y) \int y_i^2 \, d\nu_x(y)}{\delta \left(\int d\nu_x(y)\right)^2} \right| \\
&\leq \frac{\left(\int |y_i| \, d\nu_x(y)\right)^2 + \int d\nu_x(y) \int y_i^2 \, d\nu_x(y)}{\delta \left(\int d\nu_x(y)\right)^2} \\
&\leq \frac{R^2 \left(\int d\nu_x(y)\right)^2 + R^2 \left(\int d\nu_x(y)\right)^2}{\delta \left(\int d\nu_x(y)\right)^2} \\
&= \frac{2R^2}{\delta}.
\end{aligned}$$

So $\delta \cdot \text{Hess}(-\log p) = I_n + A(\delta)$, where $A(\delta)$ is an $n \times n$ real symmetric matrix whose entries are all uniformly bounded in absolute value by $2R^2/\delta$. We therefore have for all $\mathbf{v} \in \mathbb{R}^n$, $c \in \mathbb{R}$,

$$\begin{aligned}
\langle (\text{Hess}(-\log p) - \frac{1}{c} I_n)\mathbf{v}, \mathbf{v} \rangle &= \langle \frac{1}{\delta}(I_n + A(\delta))\mathbf{v}, \mathbf{v} \rangle - \frac{1}{c} ||\mathbf{v}||^2 \\
&= \left(\frac{1}{\delta} - \frac{1}{c} \right) ||\mathbf{v}||^2 + \frac{1}{\delta} \langle A(\delta)\mathbf{v}, \mathbf{v} \rangle \\
&\geq \left(\frac{1}{\delta} - \frac{1}{c} \right) ||\mathbf{v}||^2 - \frac{1}{\delta} |\langle A(\delta)\mathbf{v}, \mathbf{v} \rangle|
\end{aligned}$$

But by Cauchy-Schwarz, we have

$$\begin{aligned}
|\langle A(\delta)\mathbf{v}, \mathbf{v} \rangle|^2 &= |\sum_{i,j} A_{ij} v_i v_j|^2 \\
&\leq \sum_{i,j} |A_{ij}|^2 \cdot \sum_{i,j} |v_i v_j|^2 \\
&\leq \sum_{i,j} \left(\frac{2R^2}{\delta} \right)^2 \cdot \sum_i |v_i|^2 \cdot \sum_j |v_j|^2 \\
&= n^2 \left(\frac{2R^2}{\delta} \right)^2 ||\mathbf{v}||^2 \cdot ||\mathbf{v}||^2 \\
&= \left(\frac{2R^2 n}{\delta} ||\mathbf{v}||^2 \right)^2,
\end{aligned}$$

so for sufficiently large c,

$$\begin{aligned}
\langle (\text{Hess}(-\log p) - \frac{1}{c} I_n)\mathbf{v}, \mathbf{v} \rangle &\geq \left(\frac{1}{\delta} - \frac{1}{c} \right) ||\mathbf{v}||^2 - \frac{1}{\delta} |\langle A(\delta)\mathbf{v}, \mathbf{v} \rangle| \\
&\geq \left(\frac{1}{\delta} - \frac{1}{c} \right) ||\mathbf{v}||^2 - \frac{1}{\delta} \cdot \frac{2R^2 n}{\delta} ||\mathbf{v}||^2 \\
&= \frac{1}{\delta^2} \left(\delta - 2R^2 n - \frac{\delta^2}{c} \right) ||\mathbf{v}||^2 \\
&\geq 0
\end{aligned}$$

since $\delta > 2R^2n$. In particular, the above is satisfied for $c \geq \delta^2/(\delta - 2R^2n)$. So by Theorem 3.1.1, $\mu * \gamma_\delta$ satisfies a LSI with constant $\delta^2/(\delta - 2R^2n)$. □

3.2 Proof of Theorem 1.2.5

To prove Theorem 1.2.5, we use the Theorem 1.1.2, restated below in more detail than was given in the introduction.

Theorem 1.1.2. (Cattiaux, Guillin, Wu). *Let μ be a probability measure on $\mathbb{R}^n$ with $d\mu(x) = e^{-V(x)}dx$ for some $V \in C^2(\mathbb{R}^n)$. Suppose the following:*

1. *There exists a constant $K \leq 0$ such that* $\mathrm{Hess}(V) \geq KI$.
2. *There exists a $W \in C^2(\mathbb{R}^n)$ with $W \geq 1$ and constants $b, c > 0$ such that*

$$\Delta W(x) - \langle \nabla V, \nabla W\rangle(x) \leq (b - c|x|^2)W(x)$$

for all $x \in \mathbb{R}^n$.

Then μ satisfies a LSI.

In particular, let $r_0, b', \lambda > 0$ be such that

$$\Delta W(x) - \langle \nabla V, \nabla W\rangle(x) \leq -\lambda W(x) + b'\mathbb{1}_{B_{r_0}}$$

where B_{r_0} denotes the ball centered at 0 of radius r_0 (the existence of such r_0, b', λ is implied by Assumption 2). By [3, p.61, Thm. 1.4], μ satisfies a Poincaré inequality with constant C_P; that is, for every sufficiently smooth g with $\int g \, d\mu = 0$,

$$\int g^2 d\mu \leq C_P \int |\nabla g|^2 d\mu;$$

C_P can be taken to be $(1 + b'\kappa_{r_0})/\lambda$, where κ_{r_0} is the Poincaré constant of μ restricted to B_{r_0}. A bound for κ_{r_0} is

$$\kappa_{r_0} \leq Dr_0^2 \frac{\sup_{x \in B_{r_0}} p(x)}{\inf_{x \in B_{r_0}} p(x)},$$

where $p(x) = e^{-V(x)}$ and D is some absolute constant that can be taken to be $4/\pi^2$. Let

$$\begin{aligned} A =& \frac{2}{c}\left(\frac{1}{\epsilon} - \frac{K}{2}\right) + \epsilon \\ B =& \frac{2}{c}\left(\frac{1}{\epsilon} - \frac{K}{2}\right)\left(b + c\int |x|^2 d\mu(x)\right), \end{aligned}$$

where ϵ is an arbitrarily chosen parameter. Then μ satisfies a LSI with constant $A + (B+2)C_P$.

We remark that the statement of Theorem 1.1.2 is given in [10] in the more general context of Riemannian manifolds. Also, the constants given above are derived in [10] but not presented there; for our purposes we have collected those constants and presented them here.

With the above, we now prove Theorem 1.2.5, which we restate here for the reader's convenience.

Theorem 1.2.5. *Let μ be a probability measure on $\mathbb{R}^n$ whose support is contained in a ball of radius R, and let γ_δ be the centered Gaussian of variance δ with $0 < \delta \leq R^2$, i.e., $d\gamma_\delta(x) = (2\pi\delta)^{-n/2}\exp(-\frac{|x|^2}{2\delta})dx$. Then for some absolute constant K, the optimal log-Sobolev constant $c(\delta)$ for $\mu * \gamma_\delta$ satisfies*

$$c(\delta) \leq K\, R^2 \exp\left(20n + \frac{5R^2}{\delta}\right).$$

K can be taken above to be 289.

Proof. By translation invariance of LSI, we will assume that μ is supported in B_R. We will apply Theorem 1.1.2 to μ_δ and compute the appropriate bounds and expressions for K, W, b, c, r_0, b', λ, κ_{r_0}, C_P, $\int |x|^2 d\mu_\delta(x)$, A, and B.

To find K, b, and c, we follow the computations as done in [25, pp. 7-8]. Let $V(x) = \frac{x^2}{2\delta}$ and $V_\delta(x) = -\log(p_\delta(x))$, so

$$d\mu_\delta(x) = e^{-V_\delta(x)}dx = d(e^{-V} * \mu)(x).$$

Also let

$$d\mu_x(z) = \frac{1}{p_\delta(x)}e^{-V(x-z)}d\mu(z),$$

so μ_x is a probability measure for each $x \in \mathbb{R}^n$. Then for $X \in \mathbb{R}^n$ with $|X| = 1$,

$$
\begin{aligned}
&\mathrm{Hess}(V_\delta)(X,X)(x)\\
=&\left(\int_{B_R} \nabla_X V(x-z)d\mu_x(z)\right)^2\\
&-\int_{B_R}\left(|\nabla_X V(x-z)|^2-\mathrm{Hess}(V)(X,X)(x-z)\right)d\mu_x(z)\\
=&\frac{1}{\delta}-\left(\int_{B_R}|\nabla_X V(x-z)|^2 d\mu_x(z)-\left(\int_{B_R}\nabla_X V(x-z)d\mu_x(z)\right)^2\right)\\
&\text{since } \mathrm{Hess}(V)=\frac{1}{\delta}I.
\end{aligned}
$$

But for any C^1 function f,

$$
\begin{aligned}
\int_{B_R} f^2 d\mu_x(z)-\left(\int_{B_R} f\ d\mu_x(z)\right)^2 =&\frac{1}{2}\int_{B_R\times B_R}(f(z)-f(y))^2 d\mu_x(z)d\mu_x(y)\\
\le& 2R^2\sup|\nabla f|^2,
\end{aligned}
$$

so for $f=\nabla_X V$, we get

$$
\mathrm{Hess}(V_\delta)(X,X)(x)\ge\frac{1}{\delta}-2R^2\sup|\nabla(\nabla_X V)|^2=\frac{1}{\delta}-\frac{2R^2}{\delta^2}.
$$

So we take

$$
K=\frac{1}{\delta}-\frac{2R^2}{\delta^2}.
$$

Note $K\le 0$ since $\delta\le R^2$.

Let

$$
W(x)=\exp\left(\frac{|x|^2}{16\delta}\right).
$$

Then

$$
\begin{aligned}
\frac{\Delta W-\langle\nabla V_\delta,\nabla W\rangle}{W}(x) =&\frac{n}{8\delta}+\frac{|x|^2}{64\delta^2}-\frac{1}{16\delta}\int_{B_R}\langle x,\nabla V(x-z)\rangle d\mu_x(z)\\
=&\frac{n}{8\delta}+\frac{|x|^2}{64\delta^2}-\frac{1}{16\delta^2}\int_{B_R}\left(|x|^2-\langle x,z\rangle\right)d\mu_x(z)\\
\le&\frac{n}{8\delta}-\frac{3|x|^2}{64\delta^2}+\frac{1}{16\delta^2}\sup_{z\in B_R}\langle x,z\rangle\\
=&\frac{n}{8\delta}-\frac{3|x|^2}{64\delta^2}+\frac{1}{16\delta^2}R|x|.
\end{aligned}
$$

Using $|x| \le |x|^2/2R + R/2$ above, we get

$$\begin{aligned}\frac{\Delta W - \langle \nabla V_\delta, \nabla W\rangle}{W}(x) \le& \frac{n}{8\delta} - \frac{3|x|^2}{64\delta^2} + \frac{1}{16\delta^2} R\left(\frac{|x|^2}{2R} + \frac{R}{2}\right)\\ =& \frac{n}{8\delta} + \frac{R^2}{32\delta^2} - \frac{1}{64\delta^2}|x|^2,\end{aligned}$$

so we take

$$b = \frac{n}{8\delta} + \frac{R^2}{32\delta^2}, \qquad c = \frac{1}{64\delta^2}.$$

Now let

$$r_0 = \sqrt{16n\delta + 2R^2}, \qquad b' = \frac{1}{4\delta}\exp\left(n + \frac{R^2}{8\delta} - 1\right), \qquad \lambda = \frac{n}{8\delta}.$$

We claim that

$$b - c|x|^2 \le -\lambda + b' \exp\left(-\frac{|x|^2}{16\delta}\right)\mathbb{1}_{B_{r_0}}, \quad \text{i.e.,} \quad \frac{b + \lambda - c|x|^2}{b'}\exp\left(\frac{|x|^2}{16\delta}\right) \le \mathbb{1}_{B_{r_0}},$$

so that

$$\Delta W(x) - \langle \nabla V, \nabla W\rangle(x) \le -\lambda W(x) + b'\mathbb{1}_{B_{r_0}}.$$

We have

$$\begin{aligned}&\frac{b + \lambda - c|x|^2}{b'}\exp\left(\frac{|x|^2}{16\delta}\right)\\ =& 4\delta \exp\left(-n - \frac{R^2}{8\delta} + 1\right)\left(\frac{n}{8\delta} + \frac{R^2}{32\delta^2} + \frac{n}{8\delta} - \frac{|x|^2}{64\delta^2}\right)\exp\left(\frac{|x|^2}{16\delta}\right)\\ =& \left(n + \frac{R^2}{8\delta} - \frac{|x|^2}{16\delta}\right)\exp\left(-\left(n + \frac{R^2}{8\delta} - \frac{|x|^2}{16\delta}\right) + 1\right).\end{aligned}$$

For $|x| \ge r_0$, the above expression is nonpositive, and for $|x| \le r_0$, the above expression is of the form ue^{-u+1}, which has a maximum value of 1, as desired.

Now we estimate κ_{r_0} by estimating $\sup_{x\in B_{r_0}} p_\delta(x)$ and $\inf_{x\in B_{r_0}} p_\delta(x)$. For $x \in B_{r_0}$, we have

$$p_\delta(x) = \int_{B_R} (2\pi\delta)^{-n/2}\exp\left(-\frac{|x-y|^2}{2\delta}\right)d\mu(y) \le \int_{B_R}(2\pi\delta)^{-n/2}d\mu(y) = (2\pi\delta)^{-n/2}$$

and

$$\begin{aligned}p_\delta(x) =& \int_{B_R} (2\pi\delta)^{-n/2}\exp\left(-\frac{|x-y|^2}{2\delta}\right)d\mu(y)\\ \ge& \int_{B_R} (2\pi\delta)^{-n/2}\exp\left(-\frac{(r_0+R)^2}{2\delta}\right)d\mu(y)\\ =& (2\pi\delta)^{-n/2}\exp\left(-\frac{(r_0+R)^2}{2\delta}\right),\end{aligned}$$

so

$$\kappa_{r_0} \leq Dr_0^2 \frac{\sup_{x\in B_{r_0}} p(x)}{\inf_{x\in B_{r_0}} p(x)} \leq Dr_0^2 \exp\left(\frac{(r_0+R)^2}{2\delta}\right).$$

We then take

$$\begin{aligned}
C_P =& \frac{1+b'\kappa_{r_0}}{\lambda} \\
\leq& \frac{8\delta}{n}\left(1+\frac{1}{4\delta}\exp\left(n+\frac{R^2}{8\delta}-1\right)\cdot Dr_0^2\exp\left(\frac{(r_0+R)^2}{2\delta}\right)\right) \\
=& \frac{8\delta}{n}+\frac{D}{e}\left(32\delta+\frac{4R^2}{n}\right)\exp\left(n+\frac{R^2}{8\delta}+\frac{(\sqrt{16n\delta+2R^2}+R)^2}{2\delta}\right).
\end{aligned}$$

Using $\sqrt{a}+\sqrt{b}\leq\sqrt{2(a+b)}$ and the assumptions $\delta\leq R^2$ and $n\geq 1$ above, we get

$$\begin{aligned}
C_P \leq& \frac{8R^2}{1}+\frac{D}{e}\left(32R^2+\frac{4R^2}{1}\right)\exp\left(n+\frac{R^2}{8\delta}+\frac{\sqrt{2(16n\delta+2R^2+R^2)}^2}{2\delta}\right) \\
=& 8R^2+\frac{36D}{e}R^2\exp\left(17n+\frac{25R^2}{8\delta}\right) \\
\leq& \left(8+\frac{36D}{e}\right)R^2\exp\left(17n+\frac{25R^2}{8\delta}\right).
\end{aligned}$$

Next, we estimate $\int |x|^2 d\mu_\delta(x)$:

$$\begin{aligned}
\int_{\mathbb{R}^n} |x|^2 d\mu_\delta(x) =& \int_{\mathbb{R}^n}\int_{B_R} |x|^2 (2\pi\delta)^{-n/2}\exp\left(-\frac{|x-y|^2}{2\delta}\right)d\mu(y)dx \\
=& (2\pi\delta)^{-n/2}\int_{B_R}\int_{\mathbb{R}^n}|x+y|^2\exp\left(-\frac{|x|^2}{2\delta}\right)dx\,d\mu(y) \\
&\text{by replacing } x\to x+y \\
=& (2\pi\delta)^{-n/2}\int_{B_R}\int_{\mathbb{R}^n}(|x|^2+|y|^2)\exp\left(-\frac{|x|^2}{2\delta}\right)dx\,d\mu(y) \\
&+(2\pi\delta)^{-n/2}\int_{B_R}\int_{\mathbb{R}^n}2\langle x,y\rangle\exp\left(-\frac{|x|^2}{2\delta}\right)dx\,d\mu(y).
\end{aligned}$$

The second integral in the last expression above equals 0 since the integrand is an odd function of x. So

$$\begin{aligned}\int_{\mathbb{R}^n} |x|^2 d\mu_\delta(x) =&(2\pi\delta)^{-n/2} \int_{B_R} \int_{\mathbb{R}^n} (|x|^2 + |y|^2) \exp\left(-\frac{|x|^2}{2\delta}\right) dx\, d\mu(y) \\ \leq&(2\pi\delta)^{-n/2} \int_{\mathbb{R}^n} \int_{B_R} (|x|^2 + R^2) \exp\left(-\frac{|x|^2}{2\delta}\right) d\mu(y) dx \\ =&(2\pi\delta)^{-n/2} \int_{\mathbb{R}^n} (|x|^2 + R^2) \exp\left(-\frac{|x|^2}{2\delta}\right) dx \\ =&n\delta + R^2,\end{aligned}$$

the last integral computed using polar coordinates.

To get expressions for A, B, we choose $\epsilon = 16\delta$; then A, B satisfy

$$\begin{aligned}A = \frac{2}{c}\left(\frac{1}{\epsilon} - \frac{K}{2}\right) + \epsilon =&128\delta^2 \left(\frac{1}{16\delta} - \left(\frac{1}{2\delta} - \frac{R^2}{\delta^2}\right)\right) + 16\delta \\ =&128R^2 - 40\delta \leq 128R^2\end{aligned}$$

and

$$\begin{aligned}B =&\frac{2}{c}\left(\frac{1}{\epsilon} - \frac{K}{2}\right)\left(b + c\int |x|^2 d\mu_\delta(x)\right) \\ \leq&128\delta^2 \left(\frac{1}{16\delta} - \left(\frac{1}{2\delta} - \frac{R^2}{\delta^2}\right)\right)\left(\frac{n}{8\delta} + \frac{R^2}{32\delta^2} + \frac{1}{64\delta^2}\left(n\delta + R^2\right)\right) \\ =&\frac{18nR^2}{\delta} + \frac{6R^4}{\delta^2} - \frac{63n}{8} - \frac{21R^2}{8} \\ \leq&\frac{18nR^2}{\delta} + \frac{6R^4}{\delta^2} - 2.\end{aligned}$$

Putting everything together, we get that the optimal log-Sobolev constant $c(\delta)$ for μ_δ satisfies

$$\begin{aligned}c(\delta) \leq&A + (B+2)C_P \\ \leq&128R^2 + \left(\frac{18nR^2}{\delta} + \frac{6R^4}{\delta^2} - 2 + 2\right)\left(8 + \frac{36D}{e}\right) R^2 \exp\left(17n + \frac{25R^2}{8\delta}\right) \\ =&128R^2 + 12 \cdot \frac{R^2}{2\delta}\left(3n + \frac{R^2}{\delta}\right)\left(8 + \frac{36D}{e}\right) R^2 \exp\left(17n + \frac{25R^2}{8\delta}\right).\end{aligned}$$

Applying $u \leq e^u$ to two of the terms in the expression above, we get

$$\begin{aligned}
c(\delta) \leq & 128R^2 + 12\exp\left(\frac{R^2}{2\delta}\right)\exp\left(3n + \frac{R^2}{\delta}\right)\left(8 + \frac{36D}{e}\right)R^2\exp\left(17n + \frac{25R^2}{8\delta}\right) \\
= & 128R^2 + \left(96 + \frac{432D}{e}\right)R^2\exp\left(20n + \frac{37R^2}{8\delta}\right) \\
\leq & \left(128 + 96 + \frac{432D}{e}\right)R^2\exp\left(20n + \frac{5R^2}{\delta}\right) \\
\leq & 289R^2\exp\left(20n + \frac{5R^2}{\delta}\right).
\end{aligned}$$

This concludes the proof of Theorem 1.2.5. □

We conjecture that the optimal upper bound for $c(\delta)$ is independent of n; see Example 4.0.6 and the remark following that example.

Chapter 3 is, in part, a reprint of material from two articles. The first: D. Zimmermann. Logarithmic Sobolev inequalities for mollified compactly supported measures. *J. Funct. Anal.*, 265:1064–1083, 2013. (See [33].) The second: D. Zimmermann. Bounds for logarithmic Sobolev constants for Gaussian convolutions. Submitted for publication in Annales de l'Institute Henri Poincaré. (See [31].) The dissertation author was the author for this material.

Chapter 4

Examples

In this section, we first examine $c(\delta)$ for a compactly supported measure on $\mathbb{R}$ with density bounded above and below by positive constants, and show that such a measure itself satisfies a LSI. We then show that measures with disconnected support cannot satisfy a LSI, and then give tight (up to absolute constants) upper and lower bounds on $c(\delta)$ for the symmetric 2-point measure on $\mathbb{R}$; we demonstrate this same (n-independent) lower bound on $c(\delta)$ for the symmetric 2-point measure on $\mathbb{R}^n$.

Example 4.0.1. On $\mathbb{R}$, let μ be a probability measure on $[-R, R]$ whose absolutely continuous part has a density that is bounded below by some constant $a > 0$. Then there are absolute constants K_i such that for $0 < \delta \leq R^2$,

$$c(\delta) \leq K_1 \frac{R}{a} + K_2 \, \delta + K_3 \; \delta \, \log\left(\frac{1}{a^2 \delta}\right).$$

The K_i can be taken above to be $K_1 = 2067, K_2 = 9016, K_3 = 1248$.

Proof. Defining A, B, C as done in the proof of Lemma 2.2.5, we have

$$A \leq \sup_{0<u<1} \left(u \log \frac{1}{u}\right) \cdot \int_{m_\delta}^{R} \frac{1}{p_\delta(t)} dt \leq \frac{1}{e} \int_{-R}^{R} \frac{1}{p_\delta(t)} dt.$$

Now

$$p_\delta(t) \geq \int_{-R}^{R} \frac{1}{\sqrt{2\pi\delta}} \exp\left(-\frac{(t-s)^2}{2\delta}\right) \cdot a \, ds = a \int_{(t-R)/\sqrt{\delta}}^{(t+R)/\sqrt{\delta}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du$$

$$\text{where } u = \frac{t-s}{\sqrt{\delta}}.$$

But for $0 \le t \le R$ we have $[0, R/\sqrt{\delta}] \subseteq [(t-R)/\sqrt{\delta}, (t+R)/\sqrt{\delta}]$, and for $-R \le t \le 0$ we have $[-R\sqrt{\delta}, 0] \subseteq [(t-R)/\sqrt{\delta}, (t+R)/\sqrt{\delta}]$. So by symmetry of the above integrand we have for $\delta \le R^2$,

$$p_\delta(t) \ge a \int_0^{R/\sqrt{\delta}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du \ge a \int_0^1 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du \ge \frac{a}{3}$$

for $-R \le t \le R$. So

$$A \le \frac{1}{e} \int_{-R}^{R} \frac{3}{a}\, dt = \frac{6R}{e\,a}.$$

Similarly, we have

$$B \le \frac{6R}{e\,a}.$$

To estimate C, we note that

$$\begin{aligned}
\int_x^\infty p_\delta(t) dt \ge& \int_x^\infty \int_{-R}^{R} \frac{1}{\sqrt{2\pi\delta}} \exp\left(-\frac{(t-s)^2}{2\delta}\right) \cdot a\, ds\, dt \\
\ge& \frac{a}{\sqrt{2\pi\delta}} \int_x^\infty \int_{R-\sqrt{\delta}}^{R} \exp\left(-\frac{(t-s)^2}{2\delta}\right) ds\, dt \\
& \text{since } \delta \le R^2 \\
\ge& \frac{a}{\sqrt{2\pi\delta}} \int_x^\infty \int_{R-\sqrt{\delta}}^{R} \exp\left(-\frac{(t-R+\sqrt{\delta})^2}{2\delta}\right) ds\, dt \\
& \text{since } R-\sqrt{\delta} \le s \le t \Rightarrow -(t-s)^2 \ge -(t-R+\sqrt{\delta})^2 \\
=& \frac{a}{\sqrt{2\pi\delta}} \int_x^\infty \sqrt{\delta} \exp\left(-\frac{(t-R+\sqrt{\delta})^2}{2\delta}\right) dt.
\end{aligned}$$

Letting $u = (x - R + \sqrt{\delta})/\sqrt{\delta}$ in the last integral above, we get that for $x \ge R$,

$$\begin{aligned}
\int_x^\infty p_\delta(t) dt \ge& \frac{a\sqrt{\delta}}{\sqrt{2\pi}} \int_{(x-R+\sqrt{\delta})/\sqrt{\delta}}^\infty \exp\left(-\frac{u^2}{2}\right) du \\
\ge& \frac{a\sqrt{\delta}}{\sqrt{2\pi}} \cdot \frac{\sqrt{\delta}}{x-R+2\sqrt{\delta}} \exp\left(-\frac{(x-R+\sqrt{\delta})^2}{2\delta}\right) \\
& \text{by Lemma 2.2.2} \\
\ge& \frac{a\,\delta}{2\sqrt{2\pi}(x-R+\sqrt{\delta})} \exp\left(-\frac{(x-R+\sqrt{\delta})^2}{2\delta}\right).
\end{aligned}$$

Then we have (by reusing Lemmas 2.2.1 and 2.2.4),

$$\begin{aligned} C \leq & \sup_{x\geq R} \frac{4}{3} \cdot \frac{\delta}{x-R+\sqrt{\delta}}\, p_\delta(x) \cdot \log\left(\frac{2\sqrt{2\pi}}{a\,\delta}(x-R+\sqrt{\delta}) \exp\left(\frac{(x-R+\sqrt{\delta})^2}{2\delta}\right)\right) \\ & \cdot \frac{2\delta(x-R)}{((x-R)^2+\delta)p_\delta(x)} \\ = & \sup_{u\geq 0} \frac{8}{3} \frac{\delta u}{(u+1)(u^2+1)} \cdot \left[\log\left(\frac{2\sqrt{2\pi}}{a\sqrt{\delta}}\right) + \frac{1}{2}\log\left((u+1)^2\right) + \frac{(u+1)^2}{2}\right] \\ & \text{where } u = \frac{x-R}{\sqrt{\delta}}. \end{aligned}$$

Using $\log y \leq y$ above and simplifying, we get

$$\begin{aligned} C \leq & \sup_{u\geq 0} \frac{8}{3}\delta \, \log\left(\frac{2\sqrt{2\pi}}{a\sqrt{\delta}}\right) \frac{u}{(u+1)(u^2+1)} + \sup_{u\geq 0} \frac{8}{3}\delta \, \frac{u(u+1)}{u^2+1} \\ \leq & \frac{8}{3}\delta \, \log\left(\frac{2\sqrt{2\pi}}{a\sqrt{\delta}}\right) + \frac{16}{3}\delta \\ = & \left(\frac{8}{3}\log\left(2\sqrt{2\pi}\right) + \frac{16}{3}\right)\delta + \frac{4}{3}\delta \, \log\left(\frac{1}{a^2\delta}\right). \end{aligned}$$

Proceeding as done in Section 2.2, we therefore have

$$D_1(\delta) \leq B + C \leq \frac{6R}{e\,a} + \left(\frac{8}{3}\log\left(2\sqrt{2\pi}\right) + \frac{16}{3}\right)\delta + \frac{4}{3}\delta \, \log\left(\frac{1}{a^2\delta}\right).$$

So

$$\begin{aligned} c(\delta) \leq & 468 \cdot 2 \cdot (B+C) \\ \leq & 936 \cdot \left(\frac{6R}{e\,a} + \left(\frac{8}{3}\log\left(2\sqrt{2\pi}\right) + \frac{16}{3}\right)\delta + \frac{4}{3}\delta \, \log\left(\frac{1}{a^2\delta}\right)\right) \\ \leq & 2067\,\frac{R}{a} + 9016\,\delta + 1248\;\delta\,\log\left(\frac{1}{a^2\delta}\right). \end{aligned}$$

□

The next example requires a quick lemma, whose simple proof is left to the reader:

Lemma 4.0.2. *Let f be a bounded continuous function, and for every n, let f_n be a continuous, bounded function and let μ_n, μ be probability measures on $\mathbb{R}$. If $f_n \to f$ uniformly and $\mu_n \to \mu$ weakly, then $\int f_n \; d\mu_n \to \int f \; d\mu$.*

Example 4.0.3. On $\mathbb{R}$, let μ be a probability measure on $[-R, R]$ that has a density p such that $0 < a \leq p \leq b$ for some constants $a, b > 0$. Then for some absolute constant K, we have μ satisfies a LSI with constant c where c satisfies

$$c \leq K\,\frac{R}{a}.$$

K can be taken above to be 2067.

We remark that the claim in Example 4.0.3 above follows from Theorem 1.1.3; here, we will prove the above claim by using approximating convolutions and showing that the log Sobolev constants behave well under weak convergence.

Proof. Let $f \in H^1([-R, R])$, the Sobolev space on $[-R, R]$. Extend f to a continuous bounded function $\tilde{f}$ on $\mathbb{R}$ by defining $\tilde{f}(x) = \tilde{f}(-R)$ for $x \leq -R$ and $\tilde{f}(x) = \tilde{f}(R)$ for $x \geq R$. Take any positive sequence δ_n converging to 0. Then μ_{δ_n} satisfies a LSI with some constant c_n. For brevity, denote μ_{δ_n} by μ_n.

If $\int f^2\, d\mu = 0$, then $f = 0$ μ-almost everywhere and LSI for μ clearly holds. Otherwise, $\int f^2\, d\mu > 0$ so that $\int \tilde{f}^2\, d\mu_n > 0$ for each n. We have that

$$\int \tilde{f}^2 \log \frac{\tilde{f}^2}{\int \tilde{f}^2 d\mu_n} d\mu_n \leq c_n \int (\tilde{f}')^2 d\mu_n$$

for each n, so

$$\liminf_{n\to\infty} \int \tilde{f}^2 \log \frac{\tilde{f}^2}{\int \tilde{f}^2 d\mu_n} d\mu_n \leq \liminf_{n\to\infty}\; c_n \int (\tilde{f}')^2 d\mu_n. \tag{4.1}$$

To simplify the left hand side of (4.1), note first that the function

$$\tilde{f}^2 \log \frac{\tilde{f}^2}{\int \tilde{f}^2 d\mu}$$

is bounded and continuous since $\tilde{f}$ is. We also have

$$\tilde{f}^2 \log \frac{\tilde{f}^2}{\int \tilde{f}^2 d\mu_n} \to \tilde{f}^2 \log \frac{\tilde{f}^2}{\int \tilde{f}^2 d\mu}$$

uniformly as $n \to \infty$ since

$$\left|\tilde{f}(x)^2 \log \frac{\tilde{f}(x)^2}{\int \tilde{f}^2 d\mu_n} - \tilde{f}(x)^2 \log \frac{\tilde{f}(x)^2}{\int \tilde{f}^2 d\mu}\right| \leq \sup_{x\in\mathbb{R}} \left|\tilde{f}(x)^2\right| \left|\log \frac{\int \tilde{f}^2 d\mu}{\int \tilde{f}^2 d\mu_n}\right| \to 0.$$

So by Lemma 4.0.2,

$$\int f^2 \log \frac{f^2}{\int f^2 d\mu_n} d\mu_n \to \int f^2 \log \frac{f^2}{\int f^2 d\mu} d\mu.$$

To simplify the right hand side of (4.1), note first that by Example 4.0.1,

$$\liminf_{n\to\infty} c_n \le K\, R^2.$$

Also, p_n, the density for μ_n, satisfies

$$\begin{aligned} p_n(t) = \int_{-R}^{R} \frac{1}{\sqrt{2\pi\delta}} \exp\left(-\frac{(t-s)^2}{2\delta}\right) \cdot p(s) ds &\le b \int_{-R}^{R} \frac{1}{\sqrt{2\pi\delta}} \exp\left(-\frac{(t-s)^2}{2\delta}\right) ds \\ &\le b, \end{aligned}$$

and

$$p_n(t) \to p(t)$$

as $n \to \infty$ for (Lebesgue) almost every $-R \le t \le R$. Finally, note $(\tilde{f}')^2 \in L^1(\mathbb{R})$ since $f' \in L^2([-R, R])$ and $\tilde{f}' = 0$ outside of $[-R, R]$. so by the Dominated Convergence Theorem (with dominating function $b \cdot (\tilde{f}')^2$), we have

$$\begin{aligned} \liminf_{n\to\infty} c_n \int (\tilde{f}')^2 d\mu_n \le K\, R^2 \lim_{n\to\infty} \int_{-R}^{R} (\tilde{f}'(t))^2 p_n(t) dt &= K\, R^2 \int_{-R}^{R} (\tilde{f}'(t))^2 p(t) dt \\ &= K\, R^2 \int (\tilde{f}')^2 d\mu. \end{aligned}$$

So (4.1) becomes

$$\int \tilde{f}^2 \log \frac{\tilde{f}^2}{\int \tilde{f}^2 d\mu} d\mu \le K\, R^2 \int (\tilde{f}')^2 d\mu.$$

But $\tilde{f} = f$ on $\text{supp}(\mu)$, so we have

$$\int f^2 \log \frac{f^2}{\int f^2 d\mu} d\mu \le K\, R^2 \int (f')^2 d\mu$$

as desired. □

Our last two examples involve convolutions of 2-point measures; we now briefly show that those 2-point measures themselves do not satisfy a LSI.

Example 4.0.4. On $\mathbb{R}^n$, let μ be a probability measure with disconnected support. Then μ does not satisfy a LSI.

Proof. Suppose $\operatorname{supp}(\mu) \subseteq U \cup V$ for some disjoint open sets U, V such that $\operatorname{supp}(\mu) \cap U \neq \emptyset$ and $\operatorname{supp}(\mu) \cap V \neq \emptyset$. In particular, since $\operatorname{supp}(\mu)$ is closed (and $\mathbb{R}^n$ is a metric space), we can take U and V to have disjoint closures. So we can find a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f \equiv 1$ on U and $f \equiv 0$ on V. Since U, V are open, we therefore have $\nabla f \equiv 0$ on $U \cup V$. Note also that $0 < \mu(U) < 1$ by definition of U and V. Then we can easily compute that $\int |\nabla f|^2 \, d\mu = 0$ but $\operatorname{Ent}_\mu(f^2) = \mu(U) \log(1/\mu(U)) > 0$, so a LSI cannot hold. □

We remark that if the μ in the above example is compactly supported, then μ_δ satisfies a LSI, and one can use the above f as a test function to show that the optimal log-Sobolev constant for μ_δ can be bounded below by $\exp(C/\delta)$ for sufficiently small δ and some constant C that depends on μ (see Example 4.0.6 for an illustration of this idea). Details are omitted.

Example 4.0.5. On $\mathbb{R}$, let $\mu = \frac{1}{2}(\delta_{-R} + \delta_R)$ (so the support of μ is contained in an interval of length $2R$). Then there are absolute constants K_i such that for $0 < \delta \leq R^2$,

$$K_1 \frac{\delta^{3/2}}{R} \exp\left(\frac{R^2}{2\delta}\right) \leq c(\delta) \leq K_2 \frac{\delta^{3/2}}{R} \exp\left(\frac{R^2}{2\delta}\right).$$

The K_i can be taken to be $K_1 = 1/11, K_2 = 117942$.

Proof. We first show the upper bound for $c(\delta)$. The density p_δ for μ_δ is given by

$$p_\delta(t) = \frac{1}{2\sqrt{2\pi\delta}} \left(\exp\left(-\frac{(t+R)^2}{2\delta}\right) + \exp\left(-\frac{(t-R)^2}{2\delta}\right)\right)$$

with median $m_\delta = 0$.

Defining A, B, C as done in the proof of Lemma 2.2.5, we have

$$A \leq \sup_{0<u<1} \left(u \log \frac{1}{u}\right) \cdot \int_{m_\delta}^R \frac{1}{p_\delta(t)} dt \leq \frac{1}{e} \int_0^R 2\sqrt{2\pi\delta} \exp\left(\frac{(t-R)^2}{2\delta}\right) dt$$

$$\text{since } p_\delta(t) \geq \frac{1}{2\sqrt{2\pi\delta}} \exp\left(-\frac{(t-R)^2}{2\delta}\right).$$

Letting $u = (R-t)/\sqrt{\delta}$ above, we get

$$A \leq \frac{2\sqrt{2\pi}}{e}\delta \int_0^{R/\sqrt{\delta}} \exp\left(\frac{u^2}{2}\right) du \leq \frac{2\sqrt{2\pi}}{e}\delta \cdot \frac{2\,R/\sqrt{\delta}}{(R/\sqrt{\delta})^2+1} \exp\left(\frac{R^2}{2\delta}\right)$$

by Lemma 2.2.2

$$= \frac{4\sqrt{2\pi}}{e}\delta^{3/2} \frac{R}{R^2+\delta} \exp\left(\frac{R^2}{2\delta}\right)$$

$$\leq \frac{4\sqrt{2\pi}}{e} \cdot \frac{\delta^{3/2}}{R} \exp\left(\frac{R^2}{2\delta}\right).$$

Similarly, we have

$$B \leq \frac{4\sqrt{2\pi}}{e} \cdot \frac{\delta^{3/2}}{R} \exp\left(\frac{R^2}{2\delta}\right).$$

Also, as done in Lemma 2.2.5,

$$C \leq \frac{2}{3}(2\pi+1)(1+\sqrt{2})(\sqrt{\delta}+2R)^2 \leq 6(2\pi+1)(1+\sqrt{2})R^2$$

for $\delta \leq R^2$.

Again proceeding as done in Section 2.2, we have

$$\begin{aligned} D_1(\delta) \leq& B + C \\ \leq& \frac{4\sqrt{2\pi}}{e} \cdot \frac{\delta^{3/2}}{R} \exp\left(\frac{R^2}{2\delta}\right) + 6(2\pi+1)(1+\sqrt{2})R^2. \end{aligned}$$

Now

$$R^2 \leq 3\sqrt{3}e^{-3/2} \cdot \frac{\delta^{3/2}}{R} \exp\left(\frac{R^2}{2\delta}\right),$$

which can be seen by using elementary calculus to minimize the right-hand side of the above inequality over δ. So

$$\begin{aligned} c(\delta) \leq& 468 \cdot 2 \cdot D_1(\delta) \\ \leq& 936 \cdot \left(\frac{4\sqrt{2\pi}}{e} \cdot \frac{\delta^{3/2}}{R} \exp\left(\frac{R^2}{2\delta}\right) + 6(2\pi+1)(1+\sqrt{2})R^2\right) \\ \leq& 936 \cdot \left(\frac{4\sqrt{2\pi}}{e} + 6(2\pi+1)(1+\sqrt{2}) \cdot 3\sqrt{3}e^{-3/2}\right) \frac{\delta^{3/2}}{R} \exp\left(\frac{R^2}{2\delta}\right) \\ \leq& 117942 \cdot \frac{\delta^{3/2}}{R} \exp\left(\frac{R^2}{2\delta}\right). \end{aligned}$$

The proof of the lower bound on $c(\delta)$ is done in Example 4.0.6 below. $\square$

Example 4.0.6. On $\mathbb{R}^n$, let $\mu = \frac{1}{2}\delta_{(R,0,\ldots,0)} + \frac{1}{2}\delta_{(-R,0,\ldots,0)}$ (so the support of μ is contained in a ball of radius R). Then there is some absolute constant K such that for $0 < \delta \le R^2$,

$$c(\delta) \ge K\,\frac{\delta^{3/2}}{R}\exp\left(\frac{R^2}{2\delta}\right).$$

K can be taken above to be $1/11$.

Proof. Given $0 < \delta \le R^2$, let f be the continuous piecewise linear function on $\mathbb{R}^n$ such that $f = 0$ on $\{x_1 \le 0\}$, $f = 1$ on $\{x_1 \ge \delta/R\}$, and $f(x) = \frac{R}{\delta}x_1$ on $\{0 \le x_1 \le \delta/R\}$. Then

$$c(\delta) = \sup_g \frac{\mathrm{Ent}_{\mu_\delta}(g^2)}{\mathscr{E}(g,g)} \ge \frac{\mathrm{Ent}_{\mu_\delta}(f^2)}{\mathscr{E}(f,f)}.$$

Now

$$\int f^2 d\mu_\delta \le \int_{\{x_1 \ge 0\}} d\mu_\delta = \frac{1}{2},$$

so

$$\begin{aligned}
\mathrm{Ent}_{\mu_\delta}(f^2) &= \int f^2 \log \frac{f^2}{\int f^2 d\mu_\delta} d\mu_\delta \\
&\ge \int f^2 \log(2f^2) d\mu_\delta \\
&= \int_{\{0 \le x_1 \le \delta/R\}} f^2 \log(2f^2) d\mu_\delta + \int_{\{x_1 \ge \delta/R\}} f^2 \log(2f^2) d\mu_\delta.
\end{aligned}$$

Since $u\log(2u) \ge -1/2e$ and $f^2\log(2f^2) = \log 2$ on $\{x_1 \ge \delta/R\}$, we have

$$\mathrm{Ent}_{\mu_\delta}(f^2) \ge -\frac{1}{2e}\mu_\delta(\{0 \le x_1 \le \delta/R\}) + \log 2 \cdot \mu_\delta(\{x_1 \ge \delta/R\}).$$

By definition of μ_δ,

$$\begin{aligned}
&\mu_\delta(\{0 \le x_1 \le \delta/R\}) \\
&= \int_{\{0 \le x_1 \le \delta/R\}} \frac{1}{2(2\pi\delta)^{n/2}} \left(\exp\left(-\frac{|x-(R,0,\ldots,0)|^2}{2\delta}\right) \right. \\
&\qquad\qquad \left. + \exp\left(-\frac{|x+(R,0,\ldots,0)|^2}{2\delta}\right)\right) dx \\
&\le \int_{\{0 \le x_1 \le \delta/R\}} \frac{1}{(2\pi\delta)^{n/2}} \exp\left(-\frac{|x-(R,0,\ldots,0)|^2}{2\delta}\right) dx \\
&\quad \text{since } \exp\left(-\frac{|x+(R,0,\ldots,0)|^2}{2\delta}\right) \le \exp\left(-\frac{|x-(R,0,\ldots,0)|^2}{2\delta}\right) \text{ for } x_1 \ge 0.
\end{aligned}$$

Integrating in the first component x_1 in the above integral, we get

$$\begin{aligned}
&\mu_\delta(\{0 \le x_1 \le \delta/R\}) \\
\le &\int_0^{\delta/R} \exp\left(-\frac{(x_1-R)^2}{2\delta}\right) dx_1 \cdot \int_{\mathbb{R}^{n-1}} \frac{1}{(2\pi\delta)^{n/2}} \exp\left(-\frac{x_2^2+\cdots+x_n^2}{2\delta}\right) dx_2 \ldots dx_n \\
= &\frac{1}{\sqrt{2\pi\delta}} \int_0^{\delta/R} \exp\left(-\frac{(x_1-R)^2}{2\delta}\right) dx_1,
\end{aligned}$$

the second integral in the first expression above being an $(n-1)$-dimensional Gaussian integral. Since the integrand in the last expression above is bounded by 1, we get for $\delta \le R^2$,

$$\mu_\delta(\{0 \le x_1 \le \delta/R\}) \le \frac{1}{\sqrt{2\pi\delta}} \cdot \frac{\delta}{R} \le \frac{1}{\sqrt{2\pi}}.$$

We also have

$$\begin{aligned}
&\mu_\delta(\{x_1 \ge \delta/R\}) \\
= &\int_{\{x_1 \ge \delta/R\}} \frac{1}{2(2\pi\delta)^{n/2}} \left(\exp\left(-\frac{|x-(R,0,\ldots,0)|^2}{2\delta}\right)\right. \\
&\qquad\qquad \left. + \exp\left(-\frac{|x+(R,0,\ldots,0)|^2}{2\delta}\right)\right) dx \\
\ge &\int_{\{x_1 \ge \delta/R\}} \frac{1}{2(2\pi\delta)^{n/2}} \exp\left(-\frac{|x-(R,0,\ldots,0)|^2}{2\delta}\right) dx \\
\ge &\int_{\{x_1 \ge R\}} \frac{1}{2(2\pi\delta)^{n/2}} \exp\left(-\frac{|x-(R,0,\ldots,0)|^2}{2\delta}\right) dx \\
&\text{since } \delta \le R^2.
\end{aligned}$$

Letting $u = x - (R, 0, \ldots, 0)$ above, we get

$$\mu_\delta(\{x_1 \ge \delta/R\}) \ge \int_{\{u_1 \ge 0\}} \frac{1}{2(2\pi\delta)^{n/2}} \exp\left(-\frac{|u|^2}{2\delta}\right) du = \frac{1}{4}.$$

So

$$\mathrm{Ent}_{\mu_\delta}(f^2) \ge -\frac{1}{2e}\mu_\delta(\{0 \le x_1 \le \delta/R\}) + \log 2 \cdot \mu_\delta(\{x_1 \ge \delta/R\}) \ge \frac{\log 2}{4} - \frac{1}{2e\sqrt{2\pi}}.$$

Also,

$$\mathscr{E}(f,f) = \int |\nabla f|^2 d\mu_\delta = \frac{R^2}{\delta^2} \cdot \mu_\delta(\{0 \le x_1 \le \delta/R\}).$$

But as we have just shown,

$$\begin{aligned}\mu_\delta(\{0 \le x_1 \le \delta/R\}) \le& \frac{1}{\sqrt{2\pi\delta}} \int_0^{\delta/R} \exp\left(-\frac{(t-R)^2}{2\delta}\right) dt \\ \le& \frac{1}{\sqrt{2\pi\delta}} \cdot \frac{\delta}{R} \exp\left(-\frac{(\delta/R-R)^2}{2\delta}\right) \\ & \text{since } t \le \delta/R \le R \Rightarrow -(t-R)^2 \le -(\delta/R-R)^2,\end{aligned}$$

so

$$\begin{aligned}\mathscr{E}(f,f) \le \frac{R^2}{\delta^2} \cdot \frac{1}{\sqrt{2\pi\delta}} \cdot \frac{\delta}{R} \exp\left(-\frac{(\delta/R-R)^2}{2\delta}\right) =& \frac{R}{\sqrt{2\pi}\delta^{3/2}} \exp\left(-\frac{\delta}{2R^2} + 1 - \frac{R^2}{2\delta}\right) \\ \le& \frac{eR}{\sqrt{2\pi}\delta^{3/2}} \exp\left(-\frac{R^2}{2\delta}\right).\end{aligned}$$

Therefore

$$\begin{aligned}c(\delta) \ge \frac{\mathrm{Ent}_{\mu_\delta}(f^2)}{\mathscr{E}(f,f)} \ge \frac{\frac{\log 2}{4} - \frac{1}{2e\sqrt{2\pi}}}{\frac{eR}{\sqrt{2\pi}\delta^{3/2}} \exp\left(-\frac{R^2}{2\delta}\right)} =& \frac{\sqrt{2\pi}}{e}\left(\frac{\log 2}{4} - \frac{1}{2e\sqrt{2\pi}}\right) \frac{\delta^{3/2}}{R} \exp\left(\frac{R^2}{2\delta}\right) \\ \ge& \frac{1}{11} \cdot \frac{\delta^{3/2}}{R} \exp\left(\frac{R^2}{2\delta}\right).\end{aligned}$$

□

We conjecture that the bound stated in Example 4.0.5 is, up to absolute constant, the best upper bound (uniform over all measures whose supports have a given radius) that could have been given in Theorems 1.2.2 and 1.2.5. In particular, the upper bound should be independent of dimension.

Chapter 4 is, in part, a reprint of material from the following article: D. Zimmermann. Bounds for logarithmic Sobolev constants for Gaussian convolutions. Submitted for publication in Annales de l'Institute Henri Poincaré. (See [31].) The dissertation author was the author for this material.

Chapter 5

An application to random matrices

In this section, we give an application of Theorem 1.2.5 to random matrix theory. For each natural number n, let Y_n be an $n \times n$ random real symmetric matrix whose upper triangular entries are independent, and let $X_n = \frac{1}{\sqrt{n}} Y_n$. By a classical result in random matrix theory due to Wigner [26, 27], if the entries of Y_n are identically distributed and the common distribution has finite second moment, then the empirical law of eigenvalues of X_n converges weakly in probability to its mean. That is: let $\lambda_1^n, \lambda_2^n, \ldots, \lambda_n^n$ be the (necessarily real) eigenvalues of X_n, and let

$$\mu_{X_n} = \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k^n}.$$

Then for all $\epsilon > 0$ and all Lipschitz $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{P}\left(\left| \int f \; d\mu_{X_n} - \mathbb{E}\left(\int f \; d\mu_{X_n} \right) \right| \geq \epsilon \right) \to 0$$

as $n \to \infty$. In particular, Wigner showed that if the common distribution has mean 0 and variance 1, then the empirical law of eigenvalues of X_n converges weakly to the semicircle distribution.

The original proof of this fact was combinatorial in nature; in 2008, Guionnet (see [17, p.70, Thm. 6.6]) proved this convergence using logarithmic Sobolev inequalities in the special case where the joint laws of entries of the X_n satisfy a LSI, using the following theorem:

Theorem 5.0.7. (Guionnet). *Let Y_n, X_n be as above. If the joint law of entries of*

Y_n satisfies a LSI with constant c, then for all $\epsilon > 0$ and all Lipschitz $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{P}\left(\left|\int f\ d\mu_{X_n} - \mathbb{E}\left(\int f\ d\mu_{X_n}\right)\right| \geq \epsilon\right) \leq 2\exp\left(\frac{-n^2\epsilon^2}{4c||f||_{\text{Lip}}^2}\right).$$

Given independence of the entries, LSIs for the distributions of the individual entries are related to LSIs for the joint laws of entries by the following product property of LSIs (see [15, p. 1074, Rk. 3.3]):

Theorem 5.0.8. (Segal's Lemma). *Let ν_1, ν_2 be probability measures on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ that each satisfy a LSI with constants c_1, c_2, respectively. Then the probability measure $\nu_1 \otimes \nu_2$ on $\mathbb{R}^{n_1+n_2}$ satisfies a LSI with constant $\max(c_1, c_2)$.*

The convergence proven by Wigner is in fact almost sure convergence; if the joint laws of entries of the Y_n also satisfy a LSI with constants that do not grow too large with n (for example, in the case where the entries are i.i.d. with common distribution satisfying a LSI), then one could also use Theorem 5.0.7 and the Borel-Cantelli lemma to deduce almost sure convergence. Using Theorem 1.2.5, we will show that, under certain integrability and independence assumptions, the empirical law of eigenvalues μ_{X_n} converges weakly in probability to its mean, *regardless* of whether or not the joint laws of entries satisfy a LSI. We first state a lemma from matrix theory (see [19, p.37, Thm. 1, and p.39, Rk. 2]):

Lemma 5.0.9. (Hoffman, Wielandt). *Let A, B be symmetric $n \times n$ matrices with eigenvalues $\lambda_1^A \leq \lambda_2^A \leq \ldots \leq \lambda_n^A$ and $\lambda_1^B \leq \lambda_2^B \leq \ldots \leq \lambda_n^B$. Then*

$$\sum_{i=1}^{n}(\lambda_i^A - \lambda_i^B)^2 \leq \text{Tr}[(A-B)^2].$$

We now prove Theorem 1.2.6, which we restate here for the reader's convenience.

Theorem 1.2.6. *For each natural number n, let Y_n be an $n \times n$ random real symmetric matrix, and let $X_n = \frac{1}{\sqrt{n}} Y_n$. Suppose the following:*

1. *The family*

$$\left\{[\mathring{Y}_n]_{ij}^2\right\}_{n\in\mathbb{N}, 1\leq i,j\leq n}$$

is uniformly integrable, where for a random variable Z, $\mathring{Z} := Z - \mathbb{E}(Z)$.

2. *For each n, there exists d_n and a partition $\Pi = \{P_1, P_2, \ldots, P_m\}$ of $\{[Y_n]_{ij}\}_{1\le i\le j\le n}$ such that:*

   (a) *For each $1 \le k \le m$, $|P_k| \le d_n$.*

   (b) *For each $1 \le k \le m$, every entry in P_k is independent of $\bigcup_{l\neq k} P_l$.*

   (c) *As $n \to \infty$,*
$$\frac{d_n}{\log n} \to 0.$$

Then the empirical law of eigenvalues μ_{X_n} of X_n converges weakly to its mean in probability.

We remark that Assumption (1) above is the analogue of the assumption of finite variance of the entries in the case where the entries are assumed to be i.i.d.; Assumption (2) roughly says that we can allow for small groups of dependence of the entries if these groups are independent of each other and each group has size less than $\log n$.

Proof. Suppose $\left\{[\mathring{Y}_n]_{ij}^2\right\}_{n\in\mathbb{N},1\le i,j\le n}$ is uniformly integrable. We will first suppose for every i, j, n that $\operatorname{ran}([Y_n]_{ij})$ is contained in an interval of length $2R$; we will remove this assumption later in the proof (on page 51).

Let $\epsilon > 0$, and let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz. For each n, let $\widetilde{Y}_n = Y_n + \sqrt{\delta} G_n$ and $\widetilde{X}_n = \frac{1}{\sqrt{n}} \widetilde{Y}_n$, where G_n is a random symmetric matrix whose upper triangular entries are independent (and independent of Y_n) and all have a Gaussian distribution with mean 0 and variance 1, and $\delta = \delta(n)$ is a positive real number that we will send to 0 as $n \to \infty$ (later in the proof).

Let ν be the joint law of entries of Y_n, and $\widetilde{\nu}$ be the joint law of entries of $\widetilde{Y}_n$. By Assumption (2b), ν can be expressed as a product $\nu = \nu_1 \otimes \nu_2 \otimes \cdots \otimes \nu_m$, where each ν_k is a probability measure on $\mathbb{R}^{|P_k|}$. By construction, $\widetilde{\nu} = \gamma_\delta * \nu$. Since γ_δ is itself a product of Gaussians, convolution by γ_δ distributes across product so we have $\widetilde{\nu} = (\gamma_\delta * \nu_1) \otimes (\gamma_\delta * \nu_2) \otimes \cdots \otimes (\gamma_\delta * \nu_m)$. (We suppress further notation, but each Gaussian here is now of the appropriate dimension.) By Assumption (2a), each ν_k is supported in a ball of radius at most $R\sqrt{|P_k|} \le R\sqrt{d_n}$. So by Theorems 1.2.5 and

5.0.8, $\widetilde{\nu}$ satisfies a LSI with constant

$$\begin{aligned} c(\delta) \leq & K(R\sqrt{d_n})^2 \exp\left(20d_n + \frac{5(R\sqrt{d_n})^2}{\delta}\right) \\ \leq & KR^2 \exp\left(21d_n + \frac{5R^2 d_n}{\delta}\right) \\ & \text{since } d_n \leq \exp(d_n). \end{aligned}$$

Now

$$\begin{aligned} & \mathbb{P}\left(\left|\int f \, d\mu_{X_n} - \mathbb{E}\left(\int f \, d\mu_{X_n}\right)\right| \geq \epsilon\right) \\ \leq & \mathbb{P}\left(\left|\int f \, d\mu_{X_n} - \int f \, d\mu_{\widetilde{X}_n}\right| \geq \frac{\epsilon}{3}\right) \\ + & \mathbb{P}\left(\left|\int f \, d\mu_{\widetilde{X}_n} - \mathbb{E}\left(\int f \, d\mu_{\widetilde{X}_n}\right)\right| \geq \frac{\epsilon}{3}\right) \\ + & \mathbb{P}\left(\left|\mathbb{E}\left(\int f \, d\mu_{\widetilde{X}_n}\right) - \mathbb{E}\left(\int f \, d\mu_{X_n}\right)\right| \geq \frac{\epsilon}{3}\right), \end{aligned} \tag{5.1}$$

where μ_{X_n} and $\mu_{\widetilde{X}_n}$ are the empirical laws of eigenvalues for X_n and $\widetilde{X}_n$. We will show that each of the three terms on the right hand side of (5.1) tend to 0 as $n \to \infty$.

Lemma 5.0.10.

$$\mathbb{P}\left(\left|\int f \, d\mu_{X_n} - \int f \, d\mu_{\widetilde{X}_n}\right| \geq \frac{\epsilon}{3}\right) \leq \frac{9||f||_{\text{Lip}}^2}{\epsilon^2} \, \delta.$$

Proof. Let $\lambda_1^n \leq \lambda_2^n \leq \ldots \leq \lambda_n^n$ and $\widetilde{\lambda}_1^n \leq \widetilde{\lambda}_2^n \leq \ldots \leq \widetilde{\lambda}_n^n$ be the eigenvalues of X_n and $\widetilde{X}_n$. Then by the Cauchy-Schwarz inequality and Lemma 5.0.9,

$$\begin{aligned} \left|\int f \, d\mu_{X_n} - \int f \, d\mu_{\widetilde{X}_n}\right| = \left|\frac{1}{n}\sum_{i=1}^n f(\lambda_i^n) - f(\widetilde{\lambda}_i^n)\right| & \leq \frac{1}{n}\sum_{i=1}^n ||f||_{\text{Lip}} \left|\lambda_i^n - \widetilde{\lambda}_i^n\right| \\ & \leq \frac{||f||_{\text{Lip}}}{\sqrt{n}} \left(\sum_{i=1}^n (\lambda_i^n - \widetilde{\lambda}_i^n)^2\right)^{1/2} \\ & \leq \frac{||f||_{\text{Lip}}}{\sqrt{n}} \left(\text{Tr}[(X_n - \widetilde{X}_n)^2]\right)^{1/2}. \end{aligned}$$

By Markov's inequality, we therefore have

$$\begin{aligned}
\mathbb{P}\left(\left|\int f\ d\mu_{X_n} - \int f\ d\mu_{\widetilde{X}_n}\right| \geq \frac{\epsilon}{3}\right) &\leq \mathbb{P}\left(\frac{||f||_{\text{Lip}}}{\sqrt{n}}\left(\text{Tr}[(X_n - \widetilde{X}_n)^2]\right)^{1/2} \geq \frac{\epsilon}{3}\right) \\
&= \mathbb{P}\left(\text{Tr}[(X_n - \widetilde{X}_n)^2] \geq \frac{\epsilon^2 n}{9||f||_{\text{Lip}}^2}\right) \\
&\leq \frac{9||f||_{\text{Lip}}^2}{\epsilon^2 n}\ \mathbb{E}\left(\text{Tr}[(X_n - \widetilde{X}_n)^2]\right) \\
&= \frac{9||f||_{\text{Lip}}^2}{\epsilon^2 n} \sum_{1 \leq i,j \leq n} \mathbb{E}\left(([X_n]_{ij} - [\widetilde{X}_n]_{ij})^2\right) \\
&= \frac{9||f||_{\text{Lip}}^2}{\epsilon^2 n} \sum_{1 \leq i,j \leq n} \mathbb{E}\left(\frac{\delta}{n}[G_n]_{ij}^2\right) \\
&= \frac{9||f||_{\text{Lip}}^2}{\epsilon^2}\ \delta.
\end{aligned}$$

□

Lemma 5.0.11.

$$\mathbb{P}\left(\left|\int f\ d\mu_{\widetilde{X}_n} - \mathbb{E}\left(\int f\ d\mu_{\widetilde{X}_n}\right)\right| \geq \frac{\epsilon}{3}\right) \leq 2\exp\left(\frac{-n^2\epsilon^2}{36c||f||_{\text{Lip}}^2}\right),$$

where $c = c(\delta)$ is the log Sobolev constant for $\widetilde{\nu}$.

Proof. This is immediate from Theorem 5.0.7. □

Lemma 5.0.12. *If $\delta(n) \to 0$ as $n \to \infty$, then*

$$\mathbb{P}\left(\left|\mathbb{E}\left(\int f\ d\mu_{\widetilde{X}_n}\right) - \mathbb{E}\left(\int f\ d\mu_{X_n}\right)\right| \geq \frac{\epsilon}{3}\right) = 0$$

for all n sufficiently large.

Proof. Note that the sequence

$$\left|\mathbb{E}\left(\int f\ d\mu_{\widetilde{X}_n}\right) - \mathbb{E}\left(\int f\ d\mu_{X_n}\right)\right|$$

is a sequence of real numbers, so the above probability will eventually be equal to 0 if $\left|\mathbb{E}\left(\int f\ d\mu_{\widetilde{X}_n}\right) - \mathbb{E}\left(\int f\ d\mu_{X_n}\right)\right|$ converges to 0 as $n \to \infty$. Doing similar estimates as in Lemma 5.0.10, we get

$$\begin{aligned}
\left|\mathbb{E}\left(\int f\ d\mu_{\widetilde{X}_n}\right) - \mathbb{E}\left(\int f\ d\mu_{X_n}\right)\right| &\leq \mathbb{E}\left(\left|\int f\ d\mu_{\widetilde{X}_n} - \int f\ d\mu_{X_n}\right|\right) \\
&\leq \mathbb{E}\left(\frac{||f||_{\text{Lip}}}{\sqrt{n}}\left(\text{Tr}[(X_n - \widetilde{X}_n)^2]\right)^{1/2}\right) \\
&\leq \frac{||f||_{\text{Lip}}}{\sqrt{n}}\left(\mathbb{E}\left(\text{Tr}[(X_n - \widetilde{X}_n)^2]\right)\right)^{1/2} \\
&= ||f||_{\text{Lip}}\sqrt{\delta(n)} \\
&\to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}$$

the third inequality above following from the Cauchy-Schwarz inequality applied to $\left(\text{Tr}[(X_n - \widetilde{X}_n)^2]\right)^{1/2}$ and the constant function 1. So
$\mathbb{P}\left(\left|\mathbb{E}\left(\int f\ d\mu_{\widetilde{X}_n}\right) - \mathbb{E}\left(\int f\ d\mu_{X_n}\right)\right| \geq \frac{\epsilon}{3}\right) = 0$ for all sufficiently large n. □

We now construct our $\delta = \delta(n)$ so that $\delta \to 0$ at the appropriate rate as $n \to \infty$. For each n sufficiently large, let

$$\delta(n) = \frac{5R^2 d_n}{\log \frac{n}{KR^2} - 21 d_n}.$$

Note $\delta(n) > 0$, and $\delta(n) \to 0$ by Assumption (2c). We have

$$\begin{aligned}
c(\delta(n)) \leq& KR^2 \exp\left(21 d_n + \frac{5R^2 d_n}{\delta(n)}\right) \\
=& n.
\end{aligned}$$

Applying Lemmas 5.0.10, 5.0.11, and 5.0.12, to (5.1), we get that for sufficiently large n,

$$\begin{aligned}
&\mathbb{P}\left(\left|\int f\ d\mu_{X_n} - \mathbb{E}\left(\int f\ d\mu_{X_n}\right)\right| \geq \epsilon\right) \\
\leq& \frac{9||f||_{\text{Lip}}^2}{\epsilon^2}\ \delta(n) + 2\exp\left(\frac{-n^2\epsilon^2}{36 c(\delta(n))||f||_{\text{Lip}}^2}\right) + 0 \\
\leq& \frac{9||f||_{\text{Lip}}^2}{\epsilon^2}\ \delta(n) + 2\exp\left(\frac{-n\epsilon^2}{36||f||_{\text{Lip}}^2}\right) \\
\to& 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

We therefore have weak convergence in probability.

To obtain convergence in the general case where the entries of the Y_n need not be bounded, we apply a standard "cutoff" argument. Let $\epsilon > 0$, and let $\eta > 0$. By uniform integrability, there exists some $C \geq 0$ such that

$$\mathbb{E}\left([\mathring{Y}_n]_{ij}^2 \cdot \mathbb{1}_{\{|[\mathring{Y}_n]_{ij}|>C\}}\right) < \min(1,\eta) \cdot \epsilon^2/(9||f||_{\text{Lip}}^2)$$

for all i, j, n. Let $\widehat{Y}_n$ be the matrix defined by

$$[\widehat{Y}_n]_{ij} = [Y_n]_{ij} - [\mathring{Y}_n]_{ij} \cdot \mathbb{1}_{\{|[\mathring{Y}_n]_{ij}|>C\}},$$

and $[\widehat{X}_n]_{ij} = \frac{1}{\sqrt{n}}[\widehat{Y}_n]_{ij}$. Note that for all n, i, j,

$$\begin{aligned}
[\widehat{Y}_n]_{ij} - \mathbb{E}([Y_n]_{ij}) =& [Y_n]_{ij} - [\mathring{Y}_n]_{ij} \cdot \mathbb{1}_{\{|[\mathring{Y}_n]_{ij}|>C\}} - \mathbb{E}([Y_n]_{ij}) \\
=& [\mathring{Y}_n]_{ij} - [\mathring{Y}_n]_{ij} \cdot \mathbb{1}_{\{|[\mathring{Y}_n]_{ij}|>C\}} \\
=& [\mathring{Y}_n]_{ij} \cdot \mathbb{1}_{\{|[\mathring{Y}_n]_{ij}|\leq C\}}
\end{aligned}$$

so

$$\operatorname{ran}\left([\widehat{Y}_n]_{ij}\right) \subseteq \left[\mathbb{E}([Y_n]_{ij}) - C, \mathbb{E}([Y_n]_{ij}) + C\right],$$

which is an interval of length $2C$. (We remark that it is not necessary to normalize $\widehat{Y}_n$ since no assumptions on the values of the mean or the variance of Y_n were used.) Then, similarly as before, we have

$$\begin{aligned}
&\mathbb{P}\left(\left|\int f\ d\mu_{X_n} - \mathbb{E}\left(\int f\ d\mu_{X_n}\right)\right| \geq \epsilon\right) \\
\leq& \mathbb{P}\left(\left|\int f\ d\mu_{X_n} - \int f\ d\mu_{\widehat{X}_n}\right| \geq \frac{\epsilon}{3}\right) \\
&+ \mathbb{P}\left(\left|\int f\ d\mu_{\widehat{X}_n} - \mathbb{E}\left(\int f\ d\mu_{\widehat{X}_n}\right)\right| \geq \frac{\epsilon}{3}\right) \\
&+ \mathbb{P}\left(\left|\mathbb{E}\left(\int f\ d\mu_{\widehat{X}_n}\right) - \mathbb{E}\left(\int f\ d\mu_{X_n}\right)\right| \geq \frac{\epsilon}{3}\right).
\end{aligned} \tag{5.2}$$

The first term on the right hand side of (5.2) is bounded using the same reasoning as done in the proof of Lemma 5.0.10:

$$\begin{aligned}
\mathbb{P}\left(\left|\int f\ d\mu_{X_n} - \int f\ d\mu_{\widehat{X}_n}\right| \geq \frac{\epsilon}{3}\right) &\leq \frac{9||f||_{\text{Lip}}^2}{\epsilon^2 n} \sum_{1\leq i,j\leq n} \frac{1}{n}\mathbb{E}\left(([Y_n]_{ij} - [\widehat{Y}_n]_{ij})^2\right) \\
&= \frac{9||f||_{\text{Lip}}^2}{\epsilon^2 n} \sum_{1\leq i,j\leq n} \frac{1}{n}\mathbb{E}\left([\mathring{Y}_n]_{ij}^2 \cdot \mathbb{1}_{\{|[\mathring{Y}_n]_{ij}|>C\}}\right) \\
&< \eta.
\end{aligned}$$

The second term on the right hand side of (5.2) goes to 0 as $n \to \infty$ by the case we just proved.

The third term is bounded as done in Lemma 5.0.12:

$$\begin{aligned}\left|\mathbb{E}\left(\int f\ d\mu_{\widehat{X}_n}\right) - \mathbb{E}\left(\int f\ d\mu_{X_n}\right)\right| &\leq \frac{||f||_{\mathrm{Lip}}}{\sqrt{n}}\left(\mathbb{E}\left(\mathrm{Tr}[(X_n - \widehat{X}_n)^2]\right)\right)^{1/2} \\ &= \frac{||f||_{\mathrm{Lip}}}{\sqrt{n}}\left(\sum_{1\leq i,j\leq n}\frac{1}{n}\mathbb{E}\left([\mathring{Y}_n]_{ij}^2 \cdot \mathbb{1}_{\{|[\mathring{Y}_n]_{ij}|>C\}}\right)\right)^{1/2} \\ &< \frac{\epsilon}{3},\end{aligned}$$

so $\mathbb{P}\left(\left|\mathbb{E}\left(\int f\ d\mu_{\widehat{X}_n}\right) - \mathbb{E}\left(\int f\ d\mu_{X_n}\right)\right| \geq \frac{\epsilon}{3}\right) = 0$. So

$$\limsup_{n\to\infty}\ \mathbb{P}\left(\left|\int f\ d\mu_{X_n} - \mathbb{E}\left(\int f\ d\mu_{X_n}\right)\right| \geq \epsilon\right) \leq \eta.$$

Since $\eta > 0$ was arbitrary, we have $\mathbb{P}\left(\left|\int f\ d\mu_{X_n} - \mathbb{E}\left(\int f\ d\mu_{X_n}\right)\right| \geq \epsilon\right) \to 0$ as $n \to \infty$, giving convergence in probability. □

Chapter 5 is, in part, a reprint of material from the following article: D. Zimmermann. Bounds for logarithmic Sobolev constants for Gaussian convolutions. Submitted for publication in Annales de l'Institute Henri Poincaré. (See [31].) The dissertation author was the author for this material.

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