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A BIJECTIVE PROOF OF KOHNERT'S RULE FOR SCHUBERT POLYNOMIALS

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Abstract. Kohnert proposed a formula for Schubert polynomials as the generating polynomial for certain unit cell diagrams obtained from the diagram of a permutation. Billey, Jockusch and Stanley proved a formula for Schubert polynomials as the generating polynomial for compatible sequences of reduced words. In this paper, we give an explicit bijection between these two models, thereby proving Kohnert's rule for Schubert polynomials.

Keywords. Schubert polynomials, Kohnert's rule, Kohnert diagrams, reduced words

Mathematics Subject Classifications. 05A05, 05A19, 14N10, 14N15

Schubert polynomials, introduced by Lascoux and Schützenberger in 1982 [6], are polynomial representatives of Schubert classes for the cohomology of the flag manifold whose structure constants precisely give the Schubert cell decomposition for the corresponding product of Schubert classes. They can be defined in terms of ***divided difference operators*** ∂_i acting on polynomials $f \in \mathbb{Z}[x_1, x_2, \ldots]$ by

$$\partial_i(f) = \frac{f - s_i \cdot f}{x_i - x_{i+1}}, \tag{1}$$

where s_i exchanges x_i and x_{i+1}. Let $\mathrm{Red}(w)$ denote the set of ***reduced words*** for a permutation w, sequences $\rho = (\rho_\ell, \ldots, \rho_1)$ such that $s_{\rho_\ell} \cdots s_{\rho_1} = w$ has ℓ inversions. The ∂_i satisfy the braid relations, so for $\rho \in \mathrm{Red}(w)$ we may define

$$\partial_w \;=\; \partial_{\rho_\ell} \cdots \partial_{\rho_1}. \tag{2}$$

Definition 1 ([6]). The ***Schubert polynomial*** $\mathfrak{S}_w$ for $w \in \mathcal{S}_n$ is

$$\mathfrak{S}_w = \partial_{w^{-1}w_0}\left(x_1^{n-1} x_2^{n-2} \cdots x_{n-1}\right), \tag{3}$$

where $w_0 = n \cdots 21$ is the longest permutation of $\mathcal{S}_n$ of length $\binom{n}{2}$.

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Schubert polynomials are monomial positive, and so one naturally seeks a positive combinatorial expression in terms of monomials. Kohnert proposed the first such formula in 1990 using combinatorics of ***diagrams***, finite collections of points, called cells, in the first quadrant of $\mathbb{Z} \times \mathbb{Z}$. Kohnert's elegant combinatorial model [5] uses the following operation.

Definition 2 ([5])**.** A ***Kohnert move*** on a diagram selects the rightmost cell of a given row and moves the cell down within its column to the first available position below, if it exists, jumping over other cells in its way as needed.

Let $\mathrm{KD}(w)$ denote the set of ***Kohnert diagrams for*** w, defined as those diagrams obtainable from the ***Rothe diagram*** $\mathbb{D}(w)$ via Kohnert moves, where

$$\mathbb{D}(w) = \{(i, w_j) \mid i < j \text{ and } w_i > w_j\}. \tag{4}$$

For example, Fig. 1 shows the Rothe diagram for $w = 152869347$ on the left, and all other diagrams are obtained from Kohnert moves on the shaded cells.

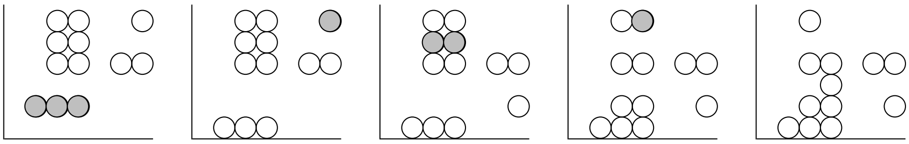

Figure 1: Several Kohnert diagrams for 152869347.

Definition 3. The ***Kohnert polynomial*** for a permutation w is

$$\mathfrak{K}_w = \sum_{T \in \mathrm{KD}(w)} x_1^{\mathbf{wt}(T)_1} \cdots x_n^{\mathbf{wt}(T)_n}. \tag{5}$$

where the ***weight of*** T satisfies $\mathbf{wt}(T)_i$ equals the number of cells of T in row i.

Kohnert asserted $\mathfrak{S}_w = \mathfrak{K}_w$. Winkel [7, 8] gives two proofs, though neither the original nor revised proof is universally accepted. Bergeron [3] proposed and proved a similar but more complicated diagram model for Schubert polynomials. Independently, Billey, Jockusch, and Stanley [4] proved a simple formula for Schubert polynomials in terms of compatible sequences for reduced words. For $\rho \in \mathrm{Red}(w)$, a word α is ***ρ-compatible*** if $\alpha_j \leqslant \rho_j$, $\alpha_{j+1} \geqslant \alpha_j$, and $\alpha_{j+1} > \alpha_j$ whenever $\rho_{j+1} > \rho_j$. In this case, we call (ρ, α) a ***compatible pair***.

$$\begin{array}{ccccccccccccccccccccccc}
6 & & 4 & & 5 & & 7 & & 8 & & 6 & & 4 & & 5 & & 7 & & 2 & & 3 & & 4 \\
\rotatebox{-90}{$\geqslant$} & & \rotatebox{-90}{$\geqslant$} & & \rotatebox{-90}{$\geqslant$} & & \rotatebox{-90}{$\geqslant$} & & \rotatebox{-90}{$\geqslant$} & & \rotatebox{-90}{$\geqslant$} & & \rotatebox{-90}{$\geqslant$} & & \rotatebox{-90}{$\geqslant$} & & \rotatebox{-90}{$\geqslant$} & & \rotatebox{-90}{$\geqslant$} & & \rotatebox{-90}{$\geqslant$} & & \rotatebox{-90}{$\geqslant$} \\
\alpha_{12} & > & \alpha_{11} & \geqslant & \alpha_{10} & \geqslant & \alpha_9 & \geqslant & \alpha_8 & > & \alpha_7 & > & \alpha_6 & \geqslant & \alpha_5 & \geqslant & \alpha_4 & > & \alpha_3 & \geqslant & \alpha_2 & \geqslant & \alpha_1
\end{array}$$

Figure 2: Required inequalities for ρ-compatible words.

For example, Fig. 2 shows the inequalities for $\rho = (6, 4, 5, 7, 8, 6, 4, 5, 7, 2, 3, 4)$, giving two solutions $(\alpha_{12}, 4, 4, 4, 4, 3, 2, 2, 2, 1, 1, 1)$ with $\alpha_{12} = 5, 6$. By [4, Theorem 1.1], we have:

Theorem 4 ([4])**.** *The Schubert polynomial $\mathfrak{S}_w$ is given by*

$$\mathfrak{S}_w = \sum_{\substack{\rho \in R(w) \\ \alpha\ \rho-\text{compatible}}} x_{\alpha_1} \cdots x_{\alpha_\ell}. \tag{6}$$

We prove (5) and (6) agree term by term by giving weight-preserving bijections

$$\{(\rho, \alpha) \mid \rho \in \mathrm{Red}(w),\ \alpha\ \rho - \text{compatible}\} \overset{\mathbb{D}}{\underset{\mathbb{W}}{\rightleftarrows}} \mathrm{KD}(w).$$

These recursively defined maps are easily implemented, proving the following.

Theorem 5 (Kohnert's rule)**.** *Given a permutation w, we have*

$$\mathfrak{K}_w = \mathfrak{S}_w. \tag{7}$$

As motivation, consider the canonical word associated with $\mathbb{D}(w)$ [1, Def. 2.3].

Definition 6 ([1])**.** A word π is ***super-Yamanouchi*** if factors into increasing words $(\pi^{(k)}| \cdots |\pi^{(1)})$ with each $\pi^{(i)}$ an interval and $\min(\pi^{(i+1)}) > \min(\pi^{(i)})$.

By [1, Prop. 2.4], each permutation w has a unique super-Yamanouchi reduced word which we denote by π_w. By [1, Prop. 3.2], π_w is obtained by filling cells in row r of $\mathbb{D}(w)$ with entries $r, r+1, \ldots$ from left to right. By [1, Prop. 3.3], column c of $\mathbb{D}(w)$ so filled has entries $c, c+1, \ldots$ from bottom to top. For example, the super-Yamanouchi reduced word for the permutation 152869347 is $\pi_w = (6, 7, 8 \mid 5, 6 \mid 4, 5, 6, 7 \mid 2, 3, 4)$. Fig. 3 shows $\mathbb{D}(w)$ labeled by π_w. Notice the rows of letters of π_w correspond to values of the maximal compatible sequence.

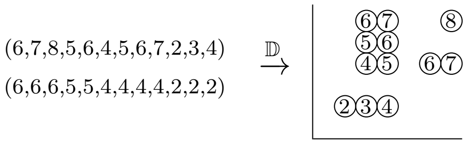

Figure 3: The super-Yamanouchi word and Rothe diagram for 152869347.

We use super-Yamanouchi words to relate Rothe diagrams for permutations related in Bruhat order, thus motivating the action of permutations on diagrams.

Lemma 7. *Let $\pi = \pi_w$. If $i = 1$ or $(\pi_w)_i > (\pi_w)_{i-1}$, then $\widehat{\pi} = \pi_\ell \cdots \pi_{i+1}\pi_{i-1} \cdots \pi_1$ is super-Yamanouchi. Moreover, if π_i lies in column c of $\mathbb{D}(w)$, then no lower row of $\mathbb{D}(w)$ has its rightmost cell in column c if and only if $\widehat{\pi} = \pi_{\widehat{w}}$ where $\widehat{w} = ws_c$, and in this case $\mathbb{D}(\widehat{w})$ is $\mathbb{D}(w) \setminus \{(\pi_w)_i\}$ with columns $c, c+1$ permuted.*

Proof. If $i = 1$, then $\widehat{\pi}$ must still factor into intervals, with the last now one letter shorter (possibly deleted), but the smallest of each interval is unchanged, and so $\widehat{\pi}$ is super-Yamanouchi. If $i = \ell$, then $\widehat{\pi}$ again factors into intervals, where now we have deleted the leftmost, so $\widehat{\pi}$ remains super-Yamanouchi. If $\ell > i > 1$, then either $\pi_{i+1} = \pi_i + 1$, and so we have removed an interval, or $\pi_{i+1} = \pi_i - 1$ lies in the same interval as π_i, so the smallest letter in that interval (and hence π_{i+1} as well) must be larger than π_{i-1}, thus ensuring the super-Yamanouchi criteria are met.

Since $\pi_i > \pi_{i-1}$ (or $i = 1$), π_i must lie at the end of its row, say r, in $\mathbb{D}(w)$. Since rows and columns of $\mathbb{D}(w)$ are intervals, if the nearest occupied row below r is shorter than row r, then all letters in that row are less than $\pi_i - 1$, and so π_i commutes with them. If the row is strictly longer, then it contains π_{i-1} followed by π_i, and so π_i may braid through this row and become $\pi_i - 1$, at which point it commutes with any remaining letters in the row, all of which are at least $\pi_i + 1$. The same logic now continues down, until at last we have transformed π into the word $\widehat{\pi}c$, where c is the last value through which π_i braided, which by [1, Prop. 3.3] is the column index in which it sits. Thus $\widehat{\pi} = \pi_{ws_c}$.

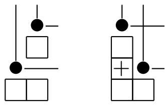

Figure 4: Going up in Bruhat order on Rothe diagrams.

Since $\ell(ws_c) = \ell(w) - 1$, we have $w_c > w_{c+1}$. In the Rothe diagrams, cells in rows $r < w_{c+1}$ are in either both or neither columns $c, c+1$, and no cells lie in either column weakly above row w_c. In rows $w_{c+1} < r < w_c$, $\mathbb{D}(ws_c)$ has no cells in column c while in $\mathbb{D}(w)$ has no cells in column $c+1$, and any cells in this range move between the two columns with $\mathbb{D}(w)$ having the additional cell in row w_{c+1}, column c corresponding to the inversion $w_c > w_{c+1}$; see Fig. 4. □

Lemma 7 relates the Rothe diagrams for w and ws_c. To relate their Kohnert diagrams, we use a special case of *rectification* [2, Def. 4.2.4] of diagrams.

Definition 8 ([2])**.** Let T be a diagram and $c \geqslant 1$ a column index. Iteratively ***pair*** any unpaired cell in column $c+1$ with an unpaired cell in column c weakly above it whenever all cells in columns $c, c+1$ in rows strictly between are already paired. Set $\mathfrak{s}_c\mathbf{T}$ to be the diagram obtained by moving all unpaired cells in column $c+1$ to column c, maintaining their rows; see Fig. 5.

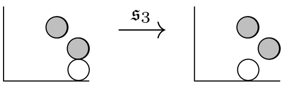

Figure 5: An example of the diagram map $\mathfrak{s}_c$ with paired cells shaded.

From the definition, if a cell x in column $c+1$ is unpaired, then there is no cell immediately to its left in column c. Thus the map $\mathfrak{s}_c$ is well-defined on diagrams. By [2, Lemma 6.3.5], this map induces a bijection on Kohnert diagrams of D and $\mathfrak{s}_c D$ whenever D is ***southwest***, meaning any two cells positioned with one strictly northwest of the other there exists a cell at their southwest corner, and column c is a subset of column $c+1$. Rothe diagrams are easily seen to be southwest and, moreover, every pair of adjacent columns of $\mathbb{D}(w)$ is ordered by inclusion.

Lemma 9 ([2]). *For w a permutation and $c \geqslant 1$, the map $\mathfrak{s}_c$ is a weight-preserving bijection between* $\mathrm{KD}(\mathbb{D}(w))$ *and* $\mathrm{KD}(\mathfrak{s}_c\mathbb{D}(w))$.

Definition 10. For a compatible pair (ρ, α), the ***Kohnert diagram for*** (ρ, α) is

$$\mathbb{D}(\rho, \alpha) = x \cup \mathfrak{s}_{\rho_1}\mathbb{D}((\rho_\ell, \ldots, \rho_2), (\alpha_\ell, \ldots, \alpha_2)), \tag{8}$$

where x is the cell in row α_1, column ρ_1 and $\mathbb{D}(\varnothing, \varnothing) = \varnothing$.

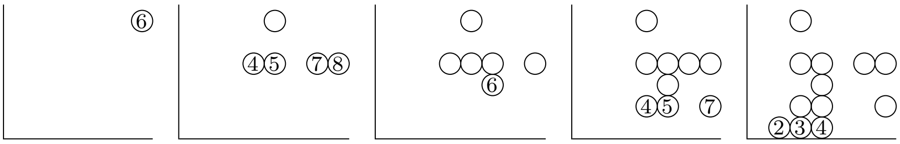

Figure 6: The Kohnert diagram for the compatible pair $\rho = (6,4,5,7,8,6,4,5,7,2,3,4)$, $\alpha = (6,4,4,4,4,3,2,2,2,1,1,1)$.

Fig. 6 shows the Kohnert diagram for $(6 \mid 4,5,7,8 \mid 6 \mid 4,5,7 \mid 2,3,4)$ with compatible word $(6,4,4,4,4,3,2,2,2,1,1,1)$. We begin with a cell in row 6, column 6; then we apply $\mathfrak{s}_8\mathfrak{s}_7\mathfrak{s}_5\mathfrak{s}_4$ and add cells in row 4, columns $4,5,7,8$; then we apply $\mathfrak{s}_6$ and add a cell in row 3, column 6; then we apply $\mathfrak{s}_7\mathfrak{s}_5\mathfrak{s}_4$ and add cells in row 2, columns $4,5,7$; finally we apply $\mathfrak{s}_4\mathfrak{s}_3\mathfrak{s}_2$ and add cells in row 1, columns $2,3,4$.

The map from compatible pairs to diagrams is easily reversed as follows.

Definition 11. For T a nonempty diagram, the ***reduced word for*** T is

$$\mathbb{W}(T) = \mathbb{W}(\mathfrak{s}_c(T \setminus x))\ c \tag{9}$$

where x is the lowest then rightmost cell of T, and c is its column index.

To show $\mathbb{D}$ is well-defined, we wish to match letters of $\rho \in \mathrm{Red}(w)$ with cells of $\mathbb{D}(w)$. As letters of π_w naturally pair with cells of $\mathbb{D}(w)$, we use [1, Def 2.7] to define a matching from letters of π_w (and hence cells of $\mathbb{D}(w)$) to letters of $\rho \in \mathrm{Red}(w)$.

Definition 12 ([1]). For $\rho, \pi \in \mathrm{Red}(w)$ with π super-Yamanouchi, the ***matching of*** ρ ***to*** π is constructed as follows: for i from $\ell(w)$ to 1, set $k = \pi_i$ and for j from $\ell(w)$ to 1 if ρ_j is already paired then decrement j to $j-1$; otherwise if $\rho_j = k$ then pair ρ_j and π_i; otherwise if $\rho_j = k-1$ then decrement k to $k-1$ and j to $j-1$; otherwise decrement j to $j-1$.

This algorithm is well-defined [1, Thm. 2.8] and can be used to compute the minimum number of Coxeter moves needed to obtain ρ from π. For example, see Fig. 7.

To ensure Lemma 7 applies when removing letters of ρ, we have the following.

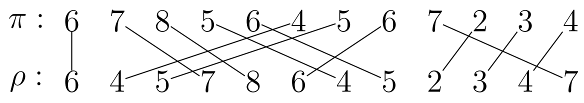

Figure 7: The matching of $\rho \in \mathrm{Red}(w)$ to π_w.

Lemma 13. *For $\rho \in \mathrm{Red}(w)$ with π super-Yamanouchi, suppose π_i matches to ρ_{p_i}. Then $\rho_{p_i} \leqslant \pi_i$, and if $\pi_{i+1} < \pi_i$, then $p_{i+1} > p_i$.*

Proof. The algorithm attempts to match π_i with a letter of ρ equal to π_i but can decrement the target value, proving $\rho_{j_1} \leqslant \pi_i$. Suppose $\pi_{i+1} < \pi_i$, so that $\pi_i = \pi_{i+1} + 1$ by Definition 6. We prove the following by induction on the number of Coxeter relations needed to obtain ρ from π:

(i) if $\rho_{p_{i+1}} < \rho_{p_i}$, then $\rho_{p_i} = \rho_{p_{i+1}} + 1$;

(ii) if $\rho_{p_{i+1}} \geqslant \rho_{p_i}$, then there exist letters $\rho_{p_{i+1}} + 1 > \cdots > \rho_{p_i} + 1$ lying strictly between them in this order.

For π, (i) is trivial and (ii) is vacuous, proving the base case. Assuming the claims for ρ, suppose σ is obtained from ρ by a single relation, and consider the matching of π to ρ then to σ by this relation. If neither π_{i+1} nor π_i matches to the letters affect by the relation, then the claims holds for σ. Thus we assume at least one of π_{i+1}, π_i matches to an affected letter of ρ.

Case (commutation): Suppose σ is obtained by a commutation relation from ρ, say swapping adjacent letters j, k with $|j-k| > 1$ and j to the left of k. If only one of π_{i+1}, π_i matches to the commuting letters, then the relative positions and values of the images of π_{i+1}, π_i remain the same in σ as in ρ, proving the claims still holds. Thus assume π_{i+1}, π_i match to j, k, respectively as shown on the left side of Fig. 8. If $j > k$, then ρ contradicts to (ii); if $j = k$, then ρ is not reduced; and if $j < k$, then $|j - k| > 1$ contradicts (i). Thus these cases cannot occur.

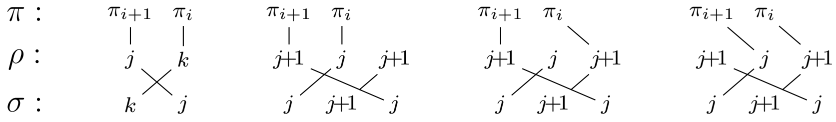

Figure 8: The four potential cases of π matching to ρ with one additional Coxeter relation.

Case (Yang–Baxter): Suppose σ is obtained by a commutation relation from ρ, say braiding $j+1, j, j+1$ to $j, j+1, j$. If only one of π_{i+1}, π_i matches to one of the braiding letters, then the relative positions of the images of π_{i+1}, π_i remain the same in σ as in ρ, proving the first claim. If π_{i+1} matches to an unaffected letter, then either π_i matches to the same value with the same set of larger values between them, or π_i matches to j with $j+1$ immediately to its left, proving the claims for σ. Similarly, if π_i matches to an unaffected letter, then either π_{i+1} matches to the same value or to j, again proving the claims for σ. Finally, we have left the three cases where both π_{i+1}, π_i match to affected letters of ρ; see Fig. 8. The middle two cases contradict claim (ii), and so cannot occur, and the rightmost case maintains relative positions and values for the images in σ, proving the claims. ☐

Theorem 14. *For* (ρ, α) *a compatible pair with* $\rho \in \mathrm{Red}(w)$*,* $\mathbb{D}(\rho, \alpha)$ *is a well-defined Kohnert diagram for* $\mathbb{D}(w)$ *and*

$$x_{\alpha_1} \cdots x_{\alpha_n} = x_1^{\mathbf{wt}(\mathbb{D}(\rho,\alpha))_1} \cdots x_n^{\mathbf{wt}(\mathbb{D}(\rho,\alpha))_n}. \tag{10}$$

Proof. If $\ell(w) = 1$, then $\mathrm{Red}(w) = \{(r)\}$. Here $\mathbb{D}(w)$ is the diagram with a single cell in row r, column r. The diagram $\mathbb{D}((r), \alpha)$ is a single cell in row α_1, column r where $\alpha = (\alpha_1)$. Any compatible word $\alpha = (\alpha_1)$ has $\alpha_1 \leqslant r$, and so $\mathbb{D}((r), \alpha)$ is obtained by exactly $r - \alpha_1 \geqslant 0$ Kohnert moves on $\mathbb{D}(w)$. Thus, $\mathbb{D}((r), \alpha) \in \mathrm{KD}(w)$, so we proceed by induction on $\ell(w)$.

Suppose $\rho \in \mathrm{Red}(w)$ with $\ell = \ell(w) > 1$, and take α any ρ-compatible word. Let $\widehat{\rho} = (\rho_\ell, \ldots, \rho_2)$, $\widehat{\alpha} = (\alpha_\ell, \ldots, \alpha_2)$, and $\widehat{w} = ws_{\rho_1}$. Then $\widehat{\rho} \in \mathrm{Red}(\widehat{w})$ has compatible word $\widehat{\alpha}$ and $\ell(\widehat{w}) = \ell(w) - 1$. By induction, $\mathbb{D}(\widehat{\rho}, \widehat{\alpha}) \in \mathrm{KD}(\widehat{w})$ and (10) holds for $\widehat{\rho}, \widehat{\alpha}$. Since every pair of adjacent columns of $\mathbb{D}(\widehat{w})$ is ordered by containment, $\mathfrak{s}_{\rho_1}\mathbb{D}(\widehat{\rho}, \widehat{\alpha})$ is a well-defined Kohnert diagram for $\mathfrak{s}_{\rho_1}\mathbb{D}(\widehat{w})$. The ρ-compatible conditions state $\alpha_{i+1} \geqslant \alpha_i$ with $\alpha_{i+1} > \alpha_i$ whenever $\rho_{i+1} > \rho_i$. Thus if $\rho_2 > \rho_1$, then the new cell x is placed into a strictly lower row, and if $\rho_2 < \rho_1$, then x is placed into a strictly rightward column. Therefore $\mathbb{D}(\rho, \alpha)$ is well-defined and (10) holds for $\mathbb{D}(\rho, \alpha)$. It remains only to show $\mathbb{D}(\rho, \alpha) \in \mathrm{KD}(w)$.

Suppose ρ_1 matches to some letter π_r for $\pi = \pi_w$ super-Yamanouchi. By Lemma 13, either $r = 1$ or $\pi_r > \pi_{r-1}$, and either way π_r lies at the end of its row in $\mathbb{D}(w)$. Thus by Lemma 7, $\widehat{\pi} = \pi_\ell \cdots \pi_{i+1}\pi_{i-1} \cdots \pi_1$ is super-Yamanouchi for $\widehat{w}$ and $\mathbb{D}(\widehat{w}) = \mathfrak{s}_{\rho_1}(\mathbb{D}(w) \setminus \{(\pi_w)_i\})$. Thus we can insert a cell in row π_r, column $\rho_1 + 1$ of $\mathbb{D}(\widehat{w})$ that we push, via Kohnert moves, down to row α_1, and then perform the remaining Kohnert moves to obtain $\mathfrak{s}_{\rho_1}\mathbb{D}(\widehat{\rho}, \widehat{\alpha})$ union a cell in row α_1, column ρ_1. Therefore $\mathbb{D}(\rho, \alpha)$ is indeed a Kohnert diagram for w. ☐

To show $\mathbb{W}$ is well-defined, we give the analog of Lemma 13 for Kohnert diagrams.

Lemma 15. *Let* $T \in \mathrm{KD}(w)$ *and suppose* x *is the lowest then rightmost cell of* T*. If* x *lies in column* c*, then some row of* $\mathbb{D}(w)$ *has its rightmost cell in column* c*.*

Proof. Let $r_1 < \cdots < r_k$ be the rows of $\mathbb{D}(w)$ with a cell in column c. If some row r_i has its rightmost cell in column c, then the lemma is proved. Otherwise, we may take $c' > c$ the smallest column index such that some row r_i has a cell in column c'. We claim all rows r_i have a cell in column c'. If not, then let i be the smallest row index such that row r_i has no cell in column c'. We may describe $\mathbb{D}(w)$ by placing bullets in row i, column w_i and killing all cells above and to the right, as illustrated in Fig. 4. If $i = 1$, then since r_1 has a cell in column $c < c'$ and no cell in column c', there is a bullet in column c' below row r_1 ensuring no row $r > r_1$ can have a cell in column c', contradicting the choice of c'. If $i > 1$, then there is a bullet column c' in some row r with $r_{i-1} > r \geqslant r_i$. Since both rows r_{i-1}, r_i have a cell in column c, the bullet in column c lies above row $r_i > r$, and so row r must have a cell in column c as well, in which case $r = r_i$. However, by choice of c' and r_i, row r_i must have a cell strictly to the right of c', a contradiction. Therefore all rows r_i have cells in both columns $c < c'$, and no cells between these. To arrive at a contradiction, we claim for any $S \in \mathrm{KD}(w)$ and any $r \geqslant 1$, we have

$$\#\{x \in S \mid \mathrm{row}(x) \leqslant r,\ \mathrm{col}(x) = c\} \leqslant \#\{x \in S \mid \mathrm{row}(x) \leqslant r,\ \mathrm{col}(x) = c'\}. \tag{11}$$

As the row of x in T violates this inequality, we have a contradiction, thereby proving the lemma. Suppose $S \in \mathrm{KD}(w)$ satisfies (11), but applying a Kohnert move to a cell y in S results in a violation. Since y moved down, we must have y in column c, say going from row s down to the nearest empty position in row $r < s$ causing (11) to fail for row r. Since (11) holds for S, we must have equality at row r for S. Since pushing y is a Kohnert move, every row above r and weakly below s has a cell in column c. For (11) to hold for S for rows up to and including s, the same must be true for column c' as well. However, this means there is a cell in row s, column c', a contradiction to pushing y being a Kohnert move. Therefore Kohnert moves preserve (11). ☐

Theorem 16. *For T a Kohnert diagram for $\mathbb{D}(w)$, $\mathbb{W}(T)$ is a well-defined reduced word for w and $\alpha(T) = (n^{\mathbf{wt}(T)_n}, \ldots, 1^{\mathbf{wt}(T)_1})$ is $\mathbb{W}(T)$-compatible.*

Proof. If $\ell(w) = 1$, then $\mathrm{Red}(w) = \{(r)\}$ and $\mathbb{D}(w)$ has a single cell in row r, column r. Every Kohnert diagram T has a single cell in column r in some row $\alpha_1 \leqslant r$. Hence $\mathbb{W}(T) = (r) \in \mathrm{Red}(w)$ and $\alpha(T) = (\alpha_1)$. Thus we may proceed by induction on $\ell(w)$.

Suppose $T \in \mathrm{KD}(w)$ with $\ell = \ell(w) > 1$, and let x be the lowest then rightmost cell of T, say in column c. By Lemma 15, some row of $\mathbb{D}(w)$ ends in column c. Thus there exists an index r such that for $\pi = \pi_w$ super-Yamanouchi, π_r lies at the end of its row and in column c and no lower row has its rightmost cell in column c. By Lemma 7, deleting this letter from π yields the super-Yamanouchi word for $\widehat{w} = w s_{\rho_c}$ and $\mathbb{D}(\widehat{w}) = \mathfrak{s}_c\,(\mathbb{D}(w) \setminus \{\pi_r\})$. Thus $T \backslash x \in \mathrm{KD}(\mathbb{D}(w) \backslash \{\pi_r\})$. By Lemma 9, we have $\mathfrak{s}_c(T \setminus x) \in \mathrm{KD}(\mathfrak{s}_c\,(\mathbb{D}(w) \setminus \{\pi_r\}) = \mathrm{KD}(\widehat{w})$. Since $\ell(\widehat{w}) = \ell(w) - 1$, by induction $\mathbb{W}(\mathfrak{s}_c(T \setminus x)) \in \mathrm{Red}(\widehat{w})$, and so $\mathbb{W}(T) = \mathbb{W}(\mathfrak{s}_c(T \setminus x))c \in \mathrm{Red}(w)$.

From (4), the lowest cell of $\mathbb{D}(w)$ in column c is at most the smallest index i such that $w_i > c$. Thus the lowest cell in column c lies weakly below row c since otherwise the first c letters of w must be strictly less than c, an impossibility. Extrapolating, the kth lowest cell in column c must lie weakly below row $c+k-1$ for all c, k. Kohnert moves push cells down within their columns, so the kth lowest cell in column c of $T \in \mathrm{KD}(w)$ must also lie weakly below row $c+k-1$. Thus for x the lowest then rightmost cell of T and c its column index, $\alpha(T)_1$ is the row of x, and so $\alpha(T)_1 \leqslant c$. Removing x ensures $\mathfrak{s}_c \cdot (T \backslash x)$ also has the property that the kth lowest cell in column c' lies weakly below row $c' + k$ for all c', and we may thus proceed as before. At each step, the recorded letter is at least as large as the row index of the removed cell, showing $\alpha(T)_i \leqslant \mathbb{W}(T)_i$. Moreover, if $\mathbb{W}(T)_{i+1} > \mathbb{W}(T)_i$, then the corresponding removed cells x_{i+1}, x_i must have x_i weakly left of x_{i+1}. Since we remove the lowest then rightmost cell, this ensures x_{i+1} lies strictly above x_i, and so $\alpha(T)_{i+1} > \alpha(T)_i$. Thus $\alpha(T)$ is $\mathbb{W}(T)$-compatible. ☐

The maps $\mathbb{D}$ and $\mathbb{W}$ are clearly inverse to one another, and so Theorems 14 and 16 give a bijective proof of Kohnert's rule for Schubert polynomials.

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