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ASYMPTOTIC PERTURBATION OF DIFFERENTIAL EQUATIONS

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ABSTRACT

The eigenvalue problems given by

$$\frac{d^2u}{dx^2} + \{\lambda - q(x) - \varepsilon p(x)\} u = 0, \qquad 0 < x < \infty,$$

and

$$\nabla^2 u + \{\lambda - q(x, y, z) - \varepsilon p(x, y, z)\} u = 0,$$

in ordinary three-dimensional space, are considered. It is assumed that ϵ is a small real quantity. The expansions for the eigenvalues and eigenfunctions which are given by formal perturbation theory are justified as asymptotic series, valid for a finite number of terms as $\epsilon \to 0$. The approximations are established rigorously up to second order by placing certain restrictions on the function p.

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INTRODUCTION

We consider the eigenvalue problem given by the differential equation $\frac{d^2u}{dx^2} + \{\lambda - q(x)\}u = 0$

with $-\infty \leq a < x < b \leq +\infty$ and λ a complex number. The function q(x) is a real-valued continuous function in (a, b). In this paper we treat the so-called singular case; that is, the interval (a, b) is infinite, or q(x) is singular at one or both end points.

Now consider the above equation perturbed as follows,

$$\frac{d^2u}{dx^2} + \{\lambda - q(x) - \varepsilon p(x)\}u = 0$$

where ε is a small real quantity. The problem of perturbation theory is to calculate the eigenvalues and eigenfunctions of the perturbed problem in terms of the known eigenvalues and eigenfunctions of the unperturbed problem.

In this paper we consider the discrete part of the spectrum of the unperturbed problem and assume conditions on p(x) such that the corresponding perturbed spectrum is discrete.

If we take λ_n and $\phi_n(x)$ to be an unperturbed eigenvalue and an unperturbed eigenfunction, the formal perturbation procedure is to assume that the infinite series

$$\lambda_{\varepsilon n} = \lambda_n + \varepsilon \lambda_n^{(1)} + \varepsilon^2 \lambda_n^{(2)} + \cdots\cdots$$

$$\phi_{\varepsilon n} = \phi_n + \varepsilon \phi_n^{(1)} + \varepsilon^2 \phi_n^{(2)} + \cdots\cdots$$

give the perturbed eigenvalue and eigenfunction. The coefficients are computed by substituting the above series in the perturbed equation and equating like powers of ε .

The justification of this formal procedure in the case where the above series are convergent in the usual sense is called analytic perturbation theory, and has been given for linear operators in a normed linear space by Rellich (1, 2, 3, 4, 5), Sz. Nagy (1), Wolf (1), Kato (5, 7) and others. In particular for operators of the type we are considering, Rellich (4) and Kato (5) establish the following two independent conditions for convergent series in powers of ε . The first condition is

$$\left(\int_a^b \{p(x)f(x)\}^2 dx\right)^{1/2} \leq \alpha\left(\int_a^b \{f(x)\}^2 dx\right)^{1/2} + \beta\left(\int_a^b \{q(x)f(x) - f''(x)\}^2 dx\right)^{1/2}$$

for $\alpha \geq 0$, $\beta \geq 0$ and $f(x)$ such that f, Lf are in $L_2(a,b)$ where $Lf = qf - f''$. The second condition is

$$\left|\int_a^b p(x)\{f(x)\}^2 dx\right| \leq \alpha\int_a^b \{f(x)\}^2 dx + \beta\left|\int_a^b f(x)\{q(x)f(x) - f''(x)\}dx\right|.$$

In this paper we consider cases where the above conditions are not fulfilled yet the perturbation method is known to give useful results by taking a few terms of the perturbation series. Mathematically the series are considered as asymptotic series, valid for a finite number of terms as $\varepsilon \to 0$. This interpretation was first given by Titchmarsh (2, 3) for differential operators and by Kato (5, 6) and V. Kramer (1) for semibounded operators in an abstract Hilbert space. Kato (1) has generalized the latter to asymptotic perturbation for semibounded quadratic forms in Hilbert space.

In this paper we derive theorems giving asymptotic perturbation series for ordinary and partial differential operators of second order which are semibounded. We derive these theorems using the abstract theory of Kato (1).

In the first chapter we summarize the theory of semibounded, closed, quadratic forms and the asymptotic perturbation theory in Hilbert space.

In the second chapter the eigenvalue and eigenfunction approximations are established up to second order for singular ordinary differential operators by placing certain restrictions on the function p (x).

As a by-product of the abstract method used in the second chapter, approximations for solutions of a nonhomogeneous two-point boundary value problem are given.

In the third chapter the theory is extended to the equation

$$\nabla^2 u + \{\lambda - q(x, y, z) - \varepsilon p(x, y, z)\} u = 0$$

in ordinary three-dimensional space. The problem of degenerate eigenvalues occurs and is considered.

I wish to thank Professor Frantisek Wolf, who suggested the problem and gave me advice and encouragement throughout, and Professor Tosio Kato, whose teaching and guidance were invaluable.

I.

ASYMPTOTIC PERTURBATION THEORY IN HILBERT SPACE

1. Semibounded, closed quadratic forms

We base the asymptotic perturbation theory of ordinary and partial differential equations on the asymptotic perturbation theory of quadratic forms in Hilbert space which has been developed by Kato.[1] The theory developed by Kato is for semibounded, closed forms. The theory of semibounded quadratic forms was introduced by Friedrichs[2] and applied by Rellich and Friedrichs to differential operators[3] and by Rellich to analytic perturbation theory.[4] Kato[1] has developed the theory considerably and has applied it to asymptotic perturbation theory.

We introduce the following definitions:

Definition. Let $\mathfrak{D}$ be a dense linear subset of an abstract Hilbert space, $\mathfrak{H}$. A functional $J[u, v]$ defined for u, v in $\mathfrak{D}$ is called a Hermitian bilinear form if

i) $\overline{J[u, v]} = J[v, u]$,

ii) $J[u, v]$ is linear in u, that is,

$$J[\alpha u_1 + \beta u_2, v] = \alpha J[u_1, v] + \beta J[u_2, v].$$

$\mathfrak{D}$ is called the domain of J and denoted by $\mathfrak{D}[J]$

$J[u] \equiv J[u, u]$ is called a quadratic form.

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[1] Kato (1)

[2] Friedrichs (1)

[3] Friedrichs (1), (2); Rellich (6)

[4] Rellich (3)

Definition. A quadratic form J is said to be bounded from below if there is a real number γ such that $J[u] \geq \gamma \|u\|^2$ for all u in $\mathcal{D}[J]$ This is denoted $J \geq \gamma$. Such forms are called lower semibounded. Henceforth we shall consider only lower semibounded forms.

Definition. Let J_1, J_2 be two forms such that $\mathcal{D}[J_1] \subset \mathcal{D}[J_2]$ and $J_1[u,v] = J_2[u,v]$ for u, v in $\mathcal{D}[J_1]$. Then J_2 is called an extension of J_1 and J_1 a restriction of J_2. We denote this $J_2 \supset J_1$ or $J_1 \subset J_2$.

As an example let K be a linear symmetric operator in $\mathcal{H}$ and let $J[u,v] = (Ku, v) = (u, Kv)$ with $\mathcal{D}[J] = \mathcal{D}[K]$. J is clearly a Hermitian form and bounded from below if K is bounded from below. A symmetric operator is bounded from below if $(Ku, u) \geq \gamma(u, u)$ for all u in $\mathcal{D}[K]$ where $\gamma > -\infty$.

To continue with the necessary theory of quadratic forms we need a few more definitions. Consider a form J with $\mathcal{D}[J]$ as its domain and a sequence $\{u_n\}$ such that each u_n is in $\mathcal{D}[J]$.

Definition. The convergence $u_n \xrightarrow{J} u$ means $\|u_n - u\| \to 0$ as $n \to \infty$ and $J[u_n - u_m] \to 0$ as $n, m \to \infty$.

Definition. A form J is called closed if $u_n \xrightarrow{J} u$ implies u in $\mathcal{D}[J]$ and $J[u_n - u] \to 0$.

Definiton. A form J is said to be closable if it has a closed extension.

With these definitions we can state the following theorem.

Theorem A (Kato[5]). A necessary and sufficient condition that a form J be closable is that $u_n \xrightarrow{J} 0$ imply $J[u_n] \to 0$. Then J has a least closed extension, $\tilde{J}$, called the closure of J. $\tilde{J}$ has the same lower bound as J. $\mathcal{D}[\tilde{J}]$ is the set of u in $\mathcal{H}$ such that there is a sequence $\{u_n\}$ with $u_n \xrightarrow{J} u$ and $J[u,v] = \lim_{n\to\infty} J[u_n, v_n]$ where $v_n \xrightarrow{J} v$, arbitrary.

If we consider again the example $J[u,v] = (Ku, v)$ where K is lower semibounded we shall see that J is closable. Assume without loss of generality that $K \geq 0$. Let $u_n \xrightarrow{J} 0$. Then

$\|u_n\| \to 0$ and $(K(u_n - u_m), u_n - u_m) \longrightarrow 0$.

For $\varepsilon > 0$, there exists an N such that

$$\varepsilon > (K(u_n - u_m), u_n - u_m)$$
$$= (Ku_n, u_n) + (Ku_m, u_m) - 2\,\mathrm{Re}(Ku_n, u_m)$$

for $m, n > N$. Let $m \longrightarrow \infty$ for a fixed n. As $(Ku_m, u_m) \geq 0$ it follows that $(Ku_n, u_n) \leq \varepsilon$ for $n > N$. So $J[u_n] \longrightarrow 0$.

From now on we shall consider only semibounded, closed forms -- or, at any rate, closable forms. We shall proceed to state some of their more important properties.

Definition. A linear subset $\mathcal{D}'$ of $\mathcal{D}[J]$ of a closed form J is called a core of J if the restriction of J to $\mathcal{D}'$ has J as its closure.

Theorem B (Kato[6]). Let $J_1, \ldots\ldots, J_k$ be forms with dense intersection. If $J_1, \ldots\ldots, J_k$ are closed so is $J_1 + \ldots\ldots + J_k$. If $J_1, \ldots\ldots J_k$ are closable, so is $J_1 + \ldots\ldots\ldots + J_k$, and

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[5] Kato (1) Theorem 2.3

[6] Kato (1) Theorem 3.1

$$(J_1 + \cdots + J_k)^{\sim} \subset \tilde{J}_1 + \cdots + \tilde{J}_k .$$

Theorem C (Kato[7]). Let $J, \ldots\ldots J_k$ be closable forms derived from symmetric operators $K_1, \ldots\ldots, K_k$ where $\mathcal{D}[J_i] = \mathcal{D}[K_i]$ and $J_i[u,v] = (K_i u, v)$. If the operator $K_1 + \ldots\ldots + K_k$ is essentially self-adjoint,[8] then

$$(J_1 + \cdots + J_k)^{\sim} = \tilde{J}_1 + \cdots + \tilde{J}_k .$$

The main result concerning semibounded, closed forms is embodied in the following theorem due to Friedrichs.

Theorem D (Kato[9]). If J is a semibounded, closed form, there exists a self-adjoint operator, H, such that (i) $\mathcal{D}[H] \subset \mathcal{D}[J]$ and $J[u,v] \doteq (Hu, v)$ for every u in $\mathcal{D}[H]$ and v in $\mathcal{D}[J]$; (ii) $\mathcal{D}[H]$ is a core of J; (iii) if u in $\mathcal{D}[J]$, u^* in $\mathcal{H}$ and $J[u,v] = (u^*, v)$ for all v of a core of J then u in $\mathcal{D}[H]$, and $Hu = u^*$; (iv) H has the same lower bound as J. H is uniquely determined by (i).

H is called the self-adjoint operator associated with J.

An important consequence of this theorem is stated in the following theorem.

Theorem E (Kato[10]). Let J be a semibounded, closed form and let H be the associated self-adjoint operator. Let $\gamma \leq \gamma_J = \gamma_H$. Then $\mathcal{D}[J] = \mathcal{D}[(H-\gamma)^{\frac{1}{2}}]$ and $J[u,v] = ((H-\gamma)^{\frac{1}{2}} u, (H-\gamma)^{\frac{1}{2}} v) + \gamma(u,v)$ for every u, v in $\mathcal{D}[J]$.

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[7] Kato (1) Theorem (4.9)

[8] Stone (1) page 51

[9] Friedrichs (1), Kato (1) Theorem (4.1)

[10] Kato (1) Theorem 4.2

If we consider the example $J[u,v]=(Ku,v)$ with $\mathcal{D}[J]=\mathcal{D}[K]$ we have[11] that H is a self-adjoint extension of K with the same lower bound as K. $\mathcal{D}[K]$ is a core of $(H-\gamma)^{\frac{1}{2}}$ for any $\gamma < \gamma_k$. In this case H is called the Friedrichs extension of K.

2. Monotone sequences of forms

We now state some theorems and definitions concerning sequences of forms.

Definition. A form J_1 is said to be not smaller than a form J_2, $J_1 \geqslant J_2$, if $\mathcal{D}[J_1] \subset \mathcal{D}[J_2]$ and $J_1[u] \geq J_2[u]$

holds for every u in $\mathcal{D}[J_1]$.

Definition. A sequence $\{J_n\}$ of forms is said to be nondecreasing (nonincreasing) if $J_m \leq J_n$ $(J_m \geq J_n)$ for $m<n$.

Definition. A sequence $\{J_n\}$ of forms is said to be dominated or bounded from above (below) by a form J if $J_n \leq J$ $(J_n \geqslant J)$ for all n. A sequence $\{J_n\}$ is dominated from below if and only if it is uniformly bounded from below, i.e. $J_n \geq \gamma$ where γ is real and independent of n.

Theorem F (Kato[12]). Let $\{J_n\}$ be a nonincreasing sequence of forms uniformly bounded below. Then there is a greatest lower bound, J, of the sequence with the following properties:

(i) $$\mathcal{D}[J] = \bigcup_{n=1}^{\infty} \mathcal{D}[J_n],$$

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[11] Kato (1) Theorem 8.1

[12] Kato (1) Theorem 9.3

(ii) $J[u,v] = \lim_{n\to\infty} J_n[u,v]$,

(iii) $J \leq J_n$ for all n and $J \geq J'$

for any J' such that $J' \leq J_n$ for all n. J is uniquely determined and we write $J = \inf J_n$

We now consider nonincreasing sequences of closed forms and give the following two theorems.

Theorem G (Kato[13]). Let $\{J_n\}$ be a nonincreasing sequence of closed forms uniformly bounded from below by $J_n \geq \gamma_0$. Then there is a greatest lower bound J of the sequence with the following properties:

(i) J is closed and $J \leq \inf J_n$; $J \geq J^1$ for any J^1 such that $J^1 \leq J_n$ for all n;

(ii) Let H_n be the self-adjoint operators associated with J_n and let H be the self-adjoint operator associated with J. Then for any $\gamma < \gamma_0$

$$(H_n - \gamma)^{-1} \longrightarrow (H-\gamma)^{-1}$$

$$(H_n - \gamma)^{\frac{1}{2}} u \xrightarrow{w} (H-\gamma)^{\frac{1}{2}} u$$

for u in $\bigcup_n \mathcal{D}[J_n]$;

(iii) If in particular $\inf J_n$ is closable,

$$J = (\inf J_n)^{\sim}$$ and

$$(H_n - \gamma)^{\frac{1}{2}} u \longrightarrow (H-\gamma)^{\frac{1}{2}} u$$ for

u in $\bigcup_n \mathcal{D}[J_n]$ and $\gamma < \gamma_0$.

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[13] Kato (1) Theorem 10.2

J is uniquely determined by $\{J_n\}$ and we write

$$J = c - \inf J_n .$$

Note: For a sequence of linear operators $\{T_n\}$ the symbol $T_n \longrightarrow T$ means $\|T_n u - Tu\| \to 0$ as $n \to \infty$ for all u in $\bigcap_n \mathcal{D}[T_n]$, $T_n u \xrightarrow{w} Tu$ means $(T_n u, v) \to (Tu, v)$ for v in $\mathcal{H}$.

Theorem H (Kato[14]). In Theorem G let the spectrum of H consist of discrete eigenvalues $\mu_1 < \mu_2 < \mu_3 < \cdots$ with finite multiplicities $m_1, m_2, m_3, \ldots$ at least in its lower part. For each i and for sufficiently large n, there are exactly m eigenvalues of H_n in each neighborhood of u_i, and these m_i eigenvalues converge to μ_i for $n \to \infty$. The projection on the m_i-dimensional subspace determined by all the eigenvectors corresponding to these eigenvalues converges uniformly to the projection on the eigenspace of H corresponding to the eigenvalue μ_i.

3. Asymptotic perturbation series for $H_\varepsilon^{-1} u$ and $(H_\varepsilon^{-1} u, v)$; first-order approximation.

We consider now the asymptotic perturbation theory of semi-bounded closed forms. Consider a closed form $\tilde{J}_\varepsilon = (J + \varepsilon J'_\varepsilon)^\sim$, $\varepsilon > 0$ where $J \geq 1$, $J' \geq 0$. In the cases that we shall consider we have $\tilde{J}_\varepsilon = \tilde{J} + \varepsilon \tilde{J}'_\varepsilon$. However, the next three theorems hold if $\tilde{J}_\varepsilon \subset \tilde{J} + \varepsilon \tilde{J}'_\varepsilon$ and $\bigcup_{0<\varepsilon<\varepsilon_0} \mathcal{D}[\tilde{J}_\varepsilon]$ is a core of $\tilde{J}$.

We wish to study the spectral properties of the self-adjoint

[14] Kato (1) Theorem 10.3

operator H_ε associated with $\tilde{\mathfrak{J}}_\varepsilon$ in the sense of Theorem D. In the following theorems we get asymptotic expansions for the quantities $H_\varepsilon^{-1}u$ and $(H_\varepsilon^{-1}u, v)$.

Theorem 1.1 (Kato[15]). Let $\tilde{\mathfrak{J}} \geq 1$ be a closed form. For each ε, $0 < \varepsilon < \varepsilon_0$, let $\tilde{\mathfrak{J}}_\varepsilon' \geq 0$ be a closed form nonincreasing as $\varepsilon \to +0$, such that $\inf_{\varepsilon \to +0} \tilde{\mathfrak{J}}_\varepsilon'$ is closable. Let $\mathcal{D}[\tilde{\mathfrak{J}}] \cap \mathcal{D}[\tilde{\mathfrak{J}}_{\varepsilon_0}']$ be dense in $\mathcal{H}$ so that $\tilde{\mathfrak{J}} + \varepsilon \tilde{\mathfrak{J}}_\varepsilon'$ is a closed form for $0 < \varepsilon < \varepsilon_0$ nonincreasing as $\varepsilon \to +0$. Let $\tilde{\mathfrak{J}}_\varepsilon \subset \tilde{\mathfrak{J}} + \varepsilon \tilde{\mathfrak{J}}_\varepsilon'$ be closed and nonincreasing as $\varepsilon \to +0$. Let $\bigcup_{0<\varepsilon<\varepsilon_0} \mathcal{D}[\tilde{\mathfrak{J}}_\varepsilon]$ be a core of $\tilde{\mathfrak{J}}$. Let $\tilde{\mathfrak{J}}' = c - \inf_{\varepsilon \to +0} \tilde{\mathfrak{J}}_\varepsilon'$. Let H_ε, H, H_ε', H' be the self-adjoint operators belonging to $\tilde{\mathfrak{J}}_\varepsilon$, $\tilde{\mathfrak{J}}$, $\tilde{\mathfrak{J}}_\varepsilon'$, $\tilde{\mathfrak{J}}'$ respectively. Let $A_\varepsilon = H^{\frac{1}{2}} H_\varepsilon^{-\frac{1}{2}}$ and $B_\varepsilon' = H_\varepsilon'^{\frac{1}{2}} H_\varepsilon^{-\frac{1}{2}}$.

Then (i) H_ε^{-1}, H^{-1}, A_ε, B_ε' are bounded operators and

$$\|H_\varepsilon^{-1}\| \leq 1, \quad \|A_\varepsilon\| \leq 1, \quad \|B_\varepsilon'\| \leq \varepsilon^{-\frac{1}{2}},$$

(ii)
$$A_\varepsilon \to 1, \quad A_\varepsilon^* \to 1, \quad \varepsilon^{\frac{1}{2}} B_\varepsilon' \to 0, \quad \varepsilon^{\frac{1}{2}} B_\varepsilon'^* \to 0,$$
$$\text{for } \varepsilon \to +0;$$

(iii)
$$\tilde{\mathfrak{J}} = c - \inf_{\varepsilon \to +0} \tilde{\mathfrak{J}}_\varepsilon \quad ;$$

(iv)
$$H_\varepsilon^{-1} \to H^{-1}, \quad \varepsilon \to +0;$$

(v) for any u such that $H^{-1}u$ in $\mathcal{D}[\tilde{\mathfrak{J}}_{\varepsilon_1}]$,

$$H_\varepsilon^{-1}u = H^{-1}u - \varepsilon H_\varepsilon^{-\frac{1}{2}} B_\varepsilon'^* H_\varepsilon'^{\frac{1}{2}} H^{-1}u, \quad 0 < \varepsilon \leq \varepsilon_1,$$

$$H_\varepsilon^{-1}u = H^{-1}u + o(\varepsilon^{\frac{1}{2}}), \quad \varepsilon \to +0;$$

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15 Kato (1) Theorem 15.1

(vi) for u, v such that $H^{-1}u$, $H^{-1}v$ in $\mathcal{D}[\tilde{\mathfrak{J}}_{\varepsilon_1}]$,

$$(H_\varepsilon^{-1}u, v) = (H^{-1}u, v) - \varepsilon(H_\varepsilon'^{\frac{1}{2}} H^{-1}u, H_\varepsilon'^{\frac{1}{2}} H^{-1}v) + \varepsilon^2 (B_\varepsilon'^{*} H_\varepsilon'^{\frac{1}{2}} H^{-1}u, B_\varepsilon'^{*} H_\varepsilon'^{\frac{1}{2}} H^{-1}v), \quad 0 < \varepsilon \le \varepsilon_1,$$

$$(H_\varepsilon^{-1}u, v) = (H^{-1}u, v) - \varepsilon(H'^{\frac{1}{2}} H^{-1}u, H'^{\frac{1}{2}} H^{-1}v) + o(\varepsilon), \quad \varepsilon \to +0.$$

4. <u>Improved first order approximation</u>

We can improve the first order of approximation for the case $\tilde{\mathfrak{J}}_\varepsilon \subset \tilde{\mathfrak{J}} + \varepsilon \tilde{\mathfrak{J}}'$ by restricting the domains of $H^{-1}u$ and $H^{-1}v$ to $\mathcal{D}[\tilde{\mathfrak{J}}_{\varepsilon_1}] \cap \mathcal{D}[H'^{\alpha}]$, $\frac{1}{2} \le \alpha < 1$. We establish the following theorem.

<u>Theorem 1.2</u>. Let $\tilde{\mathfrak{J}} \ge 1$, $\tilde{\mathfrak{J}}' \ge 0$ be closed forms. Let $\mathcal{D}[\tilde{\mathfrak{J}}] \cap \mathcal{D}[\tilde{\mathfrak{J}}']$ be dense in $\mathcal{H}$. Let $\tilde{\mathfrak{J}}_\varepsilon \subset \tilde{\mathfrak{J}} + \varepsilon \tilde{\mathfrak{J}}'$ be a closed form, nonincreasing as $\varepsilon \to 0$. Let H_ε, H, H' be the self-adjoint operators belonging to $\tilde{\mathfrak{J}}_\varepsilon$, $\tilde{\mathfrak{J}}$, $\tilde{\mathfrak{J}}'$ respectively. Let $C_\varepsilon = H'^{\frac{1}{2}-\beta} H_\varepsilon^{-\frac{1}{2}+\beta}$ where β is a real number such that $0 \le \beta < \frac{1}{2}$. Then

(i) C_ε is a bounded operator and $\|C_\varepsilon\| = \|C_\varepsilon^*\| \le \varepsilon^{-\frac{1}{2}+\beta}$;

(ii) $\varepsilon^{\frac{1}{2}-\beta} C_\varepsilon^* \to 0$ as $\varepsilon \to +0$;

(iii) for any u such that $H^{-1}u$ is in $\mathcal{D}[\tilde{\mathfrak{J}}_{\varepsilon_1}] \cap \mathcal{D}[H'^{\alpha}]$, $0 < \varepsilon \le \varepsilon_1$, where $\alpha = \beta + \frac{1}{2}$, $\frac{1}{2} \le \alpha < 1$ we have

$$H_\varepsilon^{-1}u = H^{-1}u - \varepsilon H_\varepsilon^{-\frac{1}{2}-\beta} C_\varepsilon^* H'^{\alpha} H^{-1}u,$$

$$H_\varepsilon^{-1}u = H^{-1}u + o(\varepsilon^{\alpha}), \quad \varepsilon \to +0;$$

(iv) If u, v are such that $H^{-1}u$, $H^{-1}v$ are in $\mathcal{D}[\tilde{J}_\varepsilon] \cap \mathcal{D}[H^\alpha]$,

then $$(H_\varepsilon^{-1}u, v) = (H^{-1}u, v) - \varepsilon\,(H'^{\frac{1}{2}}H^{-1}u, H'^{\frac{1}{2}}H^{-1}v) + \varepsilon^2(H_\varepsilon^{-\beta}C_\varepsilon^* H'^\alpha H^{-1}u, H_\varepsilon^{-\beta}C_\varepsilon^* H'^\alpha H^{-1}v),$$

and

$$(H_\varepsilon^{-1}u, u) = (H^{-1}u, u) - \varepsilon\|H'^{\frac{1}{2}}H^{-1}u\|^2 + o(\varepsilon^{2\alpha}), \ \varepsilon \to +0.$$

Proof: (i) We first prove that C_ε is a bounded operator. We need to give the following definition. Let A, B be linear operators in $\mathcal{H}$. A is said to be metrically not smaller than B, A >>B, if $\mathcal{D}[A] \subset \mathcal{D}[B]$ and $\|Au\| \geq \|Bu\|$ for every u in $\mathcal{D}[A]$. We then need the following lemma.

Lemma 1.1. (Heinz (1) Satz 3; Kato (8) Theorem 2) Let A and B be self-adjoint operators and let $A \geq 0$, $B \geq 0$. If A >>B then $A^\nu >> B^\nu$ for $0 \leq \nu \leq 1$.

We have $\mathcal{D}[\tilde{J}_\varepsilon] \subset \mathcal{D}[\tilde{J}']$ and $\tilde{J}_\varepsilon[u] \geq \varepsilon\,\tilde{J}'[u]$ for u in $\mathcal{D}[\tilde{J}_\varepsilon]$. We also know from Theorem E that

$$\tilde{J}_\varepsilon[u] = \|H_\varepsilon^{\frac{1}{2}}u\|^2, \quad \mathcal{D}[\tilde{J}_\varepsilon] = \mathcal{D}[H_\varepsilon^{\frac{1}{2}}],$$

$$\tilde{J}'[u] = \|H'^{\frac{1}{2}}u\|^2, \quad \mathcal{D}[\tilde{J}'] = \mathcal{D}[H'^{\frac{1}{2}}],$$

consequently, we have $\varepsilon^{\frac{1}{2}}H'^{\frac{1}{2}} << H_\varepsilon^{\frac{1}{2}}$.

Now, using Lemma 1.1, we have

$$\varepsilon^{\frac{1}{2}-\beta}H'^{\frac{1}{2}-\beta} << H_\varepsilon^{\frac{1}{2}-\beta}.$$

Hence $$\|\varepsilon^{\frac{1}{2}-\beta}H'^{\frac{1}{2}-\beta}H_\varepsilon^{-\frac{1}{2}+\beta}\| \leq 1,$$

$$\|\varepsilon^{\frac{1}{2}-\beta}C_\varepsilon\| \leq 1.$$

So we have $\|C_\varepsilon\| \leq \varepsilon^{-\frac{1}{2}+\beta}$.

The adjoint $C_\varepsilon^* \supset H_\varepsilon^{-\frac{1}{2}+\beta} H'^{\frac{1}{2}-\beta}$ and $\|C_\varepsilon^*\| = \|C_\varepsilon\|$.

(ii) We wish now to prove

$$\varepsilon^{\frac{1}{2}-\beta} C_\varepsilon^* \longrightarrow 0 \quad , \quad \varepsilon \to +0 .$$

We have $C_\varepsilon^* u = H_\varepsilon^{-\frac{1}{2}+\beta} H'^{\frac{1}{2}-\beta} u$ for u in $\mathcal{D}[\tilde{J}']$, and

$$\| C_\varepsilon^* u \| = \| H_\varepsilon^{-\frac{1}{2}+\beta} H'^{\frac{1}{2}-\beta} u \| \leq \| H'^{\frac{1}{2}-\beta} u \| ,$$

because $H_\varepsilon \geq 1$ so $\| H_\varepsilon^{-\frac{1}{2}+\beta} \| \leq 1$. Hence $\| \varepsilon^{\frac{1}{2}-\beta} C_\varepsilon^* u \| = O(\varepsilon^{\frac{1}{2}-\beta})$.

Hence $\varepsilon^{\frac{1}{2}-\beta} C_\varepsilon^* u \longrightarrow 0$ as $\varepsilon \longrightarrow +0$ for u in $\mathcal{D}[\tilde{J}']$ which is a dense set in $\mathfrak{H}$. We have that $\varepsilon^{\frac{1}{2}-\beta} C_\varepsilon^*$ is uniformly bounded in ε, $\| \varepsilon^{\frac{1}{2}-\beta} C_\varepsilon^* \| \leq 1$ by (i) of the proof. We have then that $\| \varepsilon^{\frac{1}{2}-\beta} C_\varepsilon^* u \| \longrightarrow 0$ for u in dense set in $\mathfrak{H}$ and $\| \varepsilon^{\frac{1}{2}-\beta} C_\varepsilon^* u \| \leq 1$ independent of ε, so by Kato (1) page 86 we have $\| \varepsilon^{\frac{1}{2}-\beta} C_\varepsilon^* u \| \longrightarrow 0$ for all u in $\mathfrak{H}$, which is $\varepsilon^{\frac{1}{2}-\beta} C_\varepsilon^* \longrightarrow 0$, $\varepsilon \longrightarrow +0$.

(iii) From Theorem 1.1 (v) we have

$$H_\varepsilon^{-1} u = H^{-1} u - \varepsilon H_\varepsilon^{-\frac{1}{2}} B_\varepsilon'^{*} H'^{\frac{1}{2}} H^{-1} u \quad , \quad H^{-1} u \in \mathcal{D}[\tilde{J}_\varepsilon] .$$

Let

$$B'_\varepsilon = H'^{\frac{1}{2}} H_\varepsilon^{-\frac{1}{2}} = H'^{\beta} H'^{\frac{1}{2}-\beta} H_\varepsilon^{-\frac{1}{2}+\beta} H_\varepsilon^{-\beta}$$

$$= H'^{\beta} C_\varepsilon H_\varepsilon^{-\beta} .$$

Then we have $H_\varepsilon^{-\beta} C_\varepsilon^* H'^{\beta} \subset B_\varepsilon'^{*}$. Now if we assume $H^{-1}u$ in $\mathcal{D}[\tilde{J}_\varepsilon] \cap \mathcal{D}[H'^{\alpha}]$ that is that $H'^{\alpha} H^{-1} u$ exists, then

$$B_\varepsilon'^{*} H'^{\frac{1}{2}} H^{-1}u = H_\varepsilon^{-\beta} C_\varepsilon^{*} H'^{\beta+\frac{1}{2}} H^{-1}u$$
$$= H_\varepsilon^{-\beta} C_\varepsilon^{*} H'^{\alpha} H^{-1}u\,.$$

On substituting this expression into the previous expansion for $H_\varepsilon^{-1}u$ we get

$$H_\varepsilon^{-1}u = H^{-1}u - \varepsilon H_\varepsilon^{-\frac{1}{2}-\beta} C_\varepsilon^{*} H'^{\alpha} H^{-1}u\,.$$

We can let

$$\varepsilon H_\varepsilon^{-\frac{1}{2}-\beta} C_\varepsilon^{*} H'^{\alpha} H^{-1}u = \varepsilon^{\frac{1}{2}+\beta} H_\varepsilon^{-\frac{1}{2}-\beta} \varepsilon^{\frac{1}{2}-\beta} C_\varepsilon^{*} H'^{\alpha} H^{-1}u,$$

and since $\varepsilon^{\frac{1}{2}-\beta} C_\varepsilon^{*} \to 0$ as $\varepsilon \to +0$, we have

$$H_\varepsilon^{-\frac{1}{2}-\beta} \varepsilon^{\frac{1}{2}-\beta} C_\varepsilon^{*} H'^{\alpha} H^{-1}u \longrightarrow 0\,, \quad \varepsilon \to +0\,.$$

Consequently, the second term in the expansion is $o(\varepsilon^{\frac{1}{2}+\beta}) = o(\varepsilon^{\alpha})$.

Hence

$$H_\varepsilon^{-1}u = H^{-1}u + o(\varepsilon^{\alpha})\,, \quad \varepsilon \to +0.$$

(iv) We now derive the asymptotic expansion for $(H_\varepsilon^{-1}u, v)$ under the conditions stated in (iv). From (iii) we have, for u such that $H^{-1}u$ is in $\mathcal{D}[\tilde{J}_\varepsilon] \cap \mathcal{D}[H'^{\alpha}]$, and for v in $\mathcal{H}$, the following expansion

$$(H_\varepsilon^{-1}u, v) = (H^{-1}u, v) - \varepsilon\,(H_\varepsilon^{-\frac{1}{2}-\beta} C_\varepsilon^{*} H'^{\alpha} H^{-1}u, v)$$
$$= (H^{-1}u, v) - \varepsilon(H'^{\alpha} H^{-1}u,\; C_\varepsilon H_\varepsilon^{-\frac{1}{2}-\beta} v).$$

If we take v such that $H^{-1}v$ is in $\mathcal{D}[\tilde{J}_\varepsilon] \cap \mathcal{D}[H'^{\alpha}]$ then we have

$$C_\varepsilon H_\varepsilon^{-\frac{1}{2}-\beta} v = H'^{\frac{1}{2}-\beta} H_\varepsilon^{-\frac{1}{2}+\beta} H_\varepsilon^{-\frac{1}{2}-\beta} v$$
$$= H'^{\frac{1}{2}-\beta} H_\varepsilon^{-1} v$$
$$= H'^{\frac{1}{2}-\beta} H^{-1} v - \varepsilon H'^{\frac{1}{2}-\beta} H_\varepsilon^{-\frac{1}{2}-\beta} C_\varepsilon^{*} H'^{\alpha} H^{-1} v.$$

If we substitute the above expression back into the expansion for $(H_\varepsilon^{-1}u, v)$ we then have

$$
\begin{aligned}
(H_\varepsilon^{-1}u, v) &= (H^{-1}u, v) - \varepsilon(H'^{\alpha}H^{-1}u, H'^{\frac{1}{2}-\beta}H^{-1}v) \\
&\quad + \varepsilon^2(H'^{\alpha}H^{-1}u, H'^{\frac{1}{2}-\beta}H_\varepsilon^{-\frac{1}{2}-\beta}C_\varepsilon^{*}H'^{\alpha}H^{-1}v) \\
&= (H^{-1}u, v) - \varepsilon(H'^{\beta}H'^{\frac{1}{2}}H^{-1}u, H'^{-\beta}H'^{\frac{1}{2}}H^{-1}v) \\
&\quad + \varepsilon^2(H'^{\alpha}H^{-1}u, H'^{\frac{1}{2}-\beta}H_\varepsilon^{-\frac{1}{2}+\beta}H_\varepsilon^{-2\beta}C_\varepsilon^{*}H'^{\alpha}H^{-1}v) \\
&= (H^{-1}u, v) - \varepsilon(H'^{\frac{1}{2}}H^{-1}u, H'^{\frac{1}{2}}H^{-1}v) \\
&\quad + \varepsilon^2(H'^{\alpha}H^{-1}u, C_\varepsilon H_\varepsilon^{-\beta}H_\varepsilon^{-\beta}C_\varepsilon^{*}H'^{\alpha}H^{-1}v) \\
&= (H^{-1}u, v) - \varepsilon(H'^{\frac{1}{2}}H^{-1}u, H'^{\frac{1}{2}}H^{-1}v) \\
&\quad + \varepsilon^2(H_\varepsilon^{-\beta}C_\varepsilon^{*}H'^{\alpha}H^{-1}u, H_\varepsilon^{-\beta}C_\varepsilon^{*}H'^{\alpha}H^{-1}v).
\end{aligned}
$$

So we have established the first part of (iv). To prove the second part we write

$$
\begin{aligned}
(H_\varepsilon^{-1}u, u) &= (H^{-1}u, u) - \varepsilon\|H'^{\frac{1}{2}}H^{-1}u\|^2 \\
&\quad + \varepsilon^2\|H_\varepsilon^{-\beta}C_\varepsilon^{*}H'^{\alpha}H^{-1}u\|^2.
\end{aligned}
$$

We must show that

$$
\varepsilon^2\|H_\varepsilon^{-\beta}C_\varepsilon^{*}H'^{\alpha}H^{-1}u\|^2 = o(\varepsilon^{2\alpha}).
$$

That is, we need to show that

$$\lim_{\varepsilon\to+0} \frac{\varepsilon \| H_\varepsilon^{-\beta} C_\varepsilon^* H'^{\alpha} H^{-1} u \|}{\varepsilon^\alpha} = 0$$

We can write $\varepsilon^{1-\alpha} = \varepsilon^{\frac{1}{2}-\beta}$ then

$$\varepsilon^{1-\alpha} \| H_\varepsilon^{-\beta} C_\varepsilon^* H'^{\alpha} H^{-1} u \| = \| H_\varepsilon^{-\beta} \varepsilon^{\frac{1}{2}-\beta} C_\varepsilon^* H'^{\alpha} H^{-1} u \| ;$$

but $\varepsilon^{\frac{1}{2}-\beta} C_\varepsilon^* \longrightarrow 0$ as $\varepsilon \to +0$ by (ii), so

$$\varepsilon^{1-\alpha} \| H_\varepsilon^{-\beta} C_\varepsilon^* H'^{\alpha} H^{-1} u \| \longrightarrow 0 \qquad \text{as } \varepsilon \to +0.$$

We then have

$$(H_\varepsilon^{-1} u, u) = (H^{-1} u, u) - \varepsilon \| H'^{\frac{1}{2}} H^{-1} u \|^2 + o(\varepsilon^{2\alpha}),$$

which completes the proof of Theorem 1.2.

5. Second-order approximation

We now state the second-order approximation for $H_\varepsilon^{-1} u$ and $(H_\varepsilon^{-1} u, v)$. We assume that $\tilde{J}_\varepsilon$ is given by $\tilde{J}_\varepsilon \subset \tilde{J} + \varepsilon \tilde{J}' + \varepsilon^2 \tilde{J}_\varepsilon''$ with $\hat{J} \geq 1$, $\tilde{J}' \geq 0$, $\tilde{J}_\varepsilon'' \geq 0$.

Theorem 1.3 (Kato[16]). Assume $\tilde{J}_\varepsilon \subset \tilde{J} + \varepsilon \tilde{J}' + \varepsilon^2 \tilde{J}_\varepsilon''$, where $\tilde{J}_\varepsilon'' \geq 0$ is a closed form, nonincreasing as $\varepsilon \to +0$, such that $\inf_{\varepsilon\to+0} \tilde{J}_\varepsilon''$ is closable, and let $\tilde{J}'' = c - \inf_{\varepsilon\to+0} \tilde{J}_\varepsilon''$. Let $\tilde{J}_\varepsilon' = \tilde{J}' + \varepsilon \tilde{J}_\varepsilon''$ and let $\bigcup_{\varepsilon>0} \mathcal{D}[\tilde{J}_\varepsilon']$ be a core of J'. Let H_ε'', H'' be the self-adjoint operators belonging to $\tilde{J}_\varepsilon''$, $\tilde{J}''$. Let $A_\varepsilon' = H'^{\frac{1}{2}} H_\varepsilon^{-\frac{1}{2}}$, $B_\varepsilon'' = H_\varepsilon''^{\frac{1}{2}} H_\varepsilon^{-\frac{1}{2}}$.

Then

(i) A_ε', B_ε'' are bounded operators and

$$\| A_\varepsilon' \| \leq \varepsilon^{-\frac{1}{2}}, \quad \| B_\varepsilon'' \| \leq \varepsilon^{-1} ;$$

---

16 Kato (1) Theorem 16.1

(ii)

$$\varepsilon^{\frac{1}{2}} A'_\varepsilon \longrightarrow 0,\ \varepsilon^{\frac{1}{2}} A'^{*}_\varepsilon \longrightarrow 0,\ \varepsilon B''_\varepsilon \longrightarrow 0,\ \varepsilon B''^{*}_\varepsilon \longrightarrow 0,$$
$$\varepsilon \longrightarrow +0;$$

(iii)

$$\tilde{J}' = c - \inf \tilde{J}'_\varepsilon;$$

(iv) if u is such that $H^{-1}u$ is in $\mathcal{D}[\tilde{J}_{\varepsilon_1}] \cap \mathcal{D}[H']$ for some $\varepsilon_1 > 0$, then

$$H_\varepsilon^{-1} u = H^{-1} u - \varepsilon H_\varepsilon^{-1} H' H^{-1} u - \varepsilon^2 H^{-\frac{1}{2}} B''^{*}_\varepsilon H''^{\frac{1}{2}}_\varepsilon H^{-1} u,$$
$$0 < \varepsilon \leq \varepsilon_1,$$

$$H_\varepsilon^{-1} u = H^{-1} u - \varepsilon H^{-1} H' H^{-1} u + o(\varepsilon),\ \varepsilon \longrightarrow +0;$$

(v) if $H^{-1}u$, $H^{-1}v$ in $\mathcal{D}[\tilde{J}_{\varepsilon_1}] \cap \mathcal{D}[H']$ for $\varepsilon_1 > 0$, then

$$\begin{aligned}(H_\varepsilon^{-1} u, v) = (H^{-1} u, v) &- \varepsilon (H' H^{-1} u, H^{-1} v) \\ &+ \varepsilon^2 (H_\varepsilon^{-1} H' H^{-1} u, H' H^{-1} v) \\ &- \varepsilon^2 (H''^{\frac{1}{2}}_\varepsilon H^{-1} u, H''^{\frac{1}{2}}_\varepsilon H^{-1} v) \\ &+ \varepsilon^3 (H' H^{-1} u, H_\varepsilon^{-\frac{1}{2}} B''^{*}_\varepsilon H''^{\frac{1}{2}}_\varepsilon H^{-1} v) \\ &+ \varepsilon^3 (H_\varepsilon^{-\frac{1}{2}} B''^{*}_\varepsilon H''^{\frac{1}{2}}_\varepsilon H^{-1} u, H' H^{-1} v) \\ &+ \varepsilon^4 (B''^{*}_\varepsilon H''^{\frac{1}{2}}_\varepsilon H^{-1} u, B''^{*}_\varepsilon H''^{\frac{1}{2}}_\varepsilon H^{-1} v),\end{aligned}$$
$$0 < \varepsilon \leq \varepsilon_1,$$

$$\begin{aligned}(H_\varepsilon^{-1} u, v) = (H^{-1} u, v) &- \varepsilon (H' H^{-1} u, H^{-1} v) \\ &+ \varepsilon^2 (H^{-1} H' H^{-1} u, H' H^{-1} v) \\ &- \varepsilon^2 (H''^{\frac{1}{2}} H^{-1} u, H''^{\frac{1}{2}} H^{-1} v) + o(\varepsilon^2),\ \varepsilon \to +0.\end{aligned}$$

To conclude this chapter we give the following theorem which we use repeatedly.

Theorem 1.4. If we have $\tilde{\mathcal{J}}_\varepsilon = \tilde{\mathcal{J}} + \varepsilon \tilde{\mathcal{J}}_\varepsilon'$, then the condition $H^{-1}u$ in $\mathcal{D}[H_\varepsilon'^{\gamma}]$, $\frac{1}{2} \leq \gamma \leq 1$, implies $H^{-1}u$ in $\mathcal{D}[\tilde{\mathcal{J}}_\varepsilon]$.

Proof: $H^{-1}u$ in $\mathcal{D}[H_\varepsilon'^{\gamma}]$ implies $H^{-1}u$ in $\mathcal{D}[H_\varepsilon'^{\frac{1}{2}}] = \mathcal{D}[\tilde{\mathcal{J}}_\varepsilon']$. $H^{-1}u$ is clearly in $\mathcal{D}[H] \subset \mathcal{D}[\tilde{\mathcal{J}}]$. Hence $H^{-1}u$ is in $\mathcal{D}[\tilde{\mathcal{J}}] \cap \mathcal{D}[\tilde{\mathcal{J}}_{\varepsilon_1}'] = \mathcal{D}[\tilde{\mathcal{J}}_\varepsilon]$.

II.

PERTURBATION THEORY OF SINGULAR ORDINARY DIFFERENTIAL OPERATORS OF THE SECOND ORDER

1. Statement of the Problem and Results

Consider the differential equation

$$\frac{d^2y}{dx^2} + \{\lambda - q(x)\} y = 0 , \tag{2.1}$$

with $a < x < b$ where $-\infty \leq a < b \leq +\infty$ and λ is a complex number. The function $q(x)$ is a real-valued, continuous function in the open interval (a, b). When (a, b) is a finite interval and $q(x)$ tends to finite limits at a and b we have the classical Sturm-Liouville problem. If (a, b) is an infinite interval or $q(x) \rightarrow \infty$ at either end point we have the singular case. There exists considerable theory for this case.[17] We shall consider only the singular case, since most of the interesting applications to mathematical physics are of this type.

We consider the case where Eq. (2.1) is of the limit-circle[18] type at a and the limit-point[18] type at b, although the results to be shown will apply to the other possibilities, i.e. limit-circle type at both end points or limit-point type at both end points.

In Eq. (2.1) let $a = 0$ and $b = +\infty$; then for the case we are considering to be a solvable eigenvalue problem, i.e., for some λ, a $y(x, \lambda)$ in $L_2(0, \infty)$ satisfying Eq. (2.1), the solutions $y(x, \lambda)$ must satisfy a condition of the form

$$y(0) \cos\alpha + y'(0) \sin\alpha = 0. \tag{2.2}$$

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[17] Weyl (1), Stone (1), Titchmarsh (1), Kodaira (1)

[18] Weyl (1), Titchmarsh (1) Ch. II, Kodaira (1)

We shall further assume q(x) real-valued, continuous in $(0, \infty)$,

and $q(x) \geq q_0 > -\infty$ for all x in $(0, \infty)$. (2.3)

Equation (2.1) with conditions (2.2) and (2.3) will be called the unperturbed problem. As is usual in perturbation theory, we assume the unperturbed problem to be completely solved, i.e. there exists a complete set of eigenvalues and eigenfunctions (discrete or continuous) that can be calculated.

Now consider Eq. (2.1) to be modified,

$$\frac{d^2 y}{dx^2} + \{\lambda - q(x) - \varepsilon p(x)\} y = 0 \tag{2.4}$$

where $\varepsilon \rightarrow +0$ and x is in $(0, \infty)$. The term $\varepsilon p(x) y(x)$ is considered as a perturbation on Eq. (2.1). We assume

$$\begin{aligned} &p(x) \text{ real-valued, continuous for } x \text{ in } (0, \infty), \\ &p(x) \geq p_0 > -\infty \quad \text{for } x \text{ in } (0, \infty). \end{aligned} \tag{2.5}$$

The boundary conditions are given by Eq. (2.2). Equation (2.4) with conditions (2.2) will be called the perturbed problem. We wish to find expressions for the eigenvalues and eigenfunctions of the perturbed problem in terms of those of the unperturbed problem.

Before stating the results of this chapter we need to make some further assumptions. We assume the inequality

$$\int_0^\infty \left\{q(x) f(x) - \frac{d^2 f}{dx^2}\right\} f(x)\, dx \geq \gamma \int_0^\infty \{f(x)\}^2 dx\ , \quad \gamma > -\infty \tag{2.6}$$

for all f(x) in $L_2(0, \infty)$ satisfying conditions (2.2). We have from integration by parts,

$$\int_0^\infty \left\{q(x) f(x) - \frac{d^2 f}{dx^2}\right\} f(x)\, dx = \int_0^\infty \left[\left(\frac{df}{dx}\right)^2 + q(x)\{f(x)\}^2\right] dx - \{f(0)\}^2 \cot\alpha\, .$$

For the case $f(0) = 0$ we see that $\gamma = q_0$, as given in condition (2.3).

In addition we assume that at least the lower part of the spectrum of the unperturbed problem consists of discrete eigenvalues. That is, there exists a number $\lambda^* < \infty$ such that

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \lambda^*, \qquad (2.7)$$

where $\lambda_1, \ldots, \lambda_n$ are points of the discrete spectrum. Consequently if any continuous spectrum exists it lies above λ^*. The nature of the spectrum of eigenvalue problems in equations of type (2.1) is determined by the behavior of the function q(x) and also the boundary conditions. Discussions of this problem, including criteria for the discreteness of spectrum, are given by Friedrichs (3) and Titchmarsh (1), chapter V. We have, for example, the criterion that if $q(x) \longrightarrow \infty$ as $x \longrightarrow +\infty$, then there is pure point spectrum. In quantum mechanical problems this corresponds to a potential well with sides of infinite depth, which leads to only bound states.

With only Assumptions (2.5), (2.6), and (2.7), which in practice are not very restrictive, we can derive the zeroth-order approximation, which is given in the following.

<u>Theorem 2.1.</u> <u>Let</u> λ_0 <u>be an isolated eigenvalue of the unperturbed problem.</u> <u>Then in the neighborhood of</u> λ_0 <u>there exists an eigenvalue</u> λ_ε <u>of the perturbed problem, and in this neighborhood there are no other points of the spectrum of the perturbed problem provided</u> ε <u>is sufficiently small. Furthermore we have that</u>

$$\lambda_\varepsilon \longrightarrow \lambda_0 \quad , \quad \varepsilon \longrightarrow +0.$$

With the above theorem we establish the fact that for an eigenvalue of the unperturbed problem there exists a corresponding eigenvalue of the perturbed problem that converges to it, but we cannot make such a statement concerning the eigenfunctions at this point.

We shall assume that for the remainder of the chapter, conditions (2.5), (2.6), and (2.7) are satisfied. We now proceed to the first-order approximation and make the following additional assumption: if $\phi_0(x)$ is the eigenfunction corresponding to λ_0 of the unperturbed problem, then

$$\int_0^\infty p(x)\{\phi_0(x)\}^2 dx < \infty \quad . \tag{2.8}$$

This condition assures the existence of the first-order coefficient in the formal perturbation series.

With Eq. (2.8) we state

Theorem 2.2. Let λ_0 be an isolated eigenvalue of the unperturbed problem and let $\phi_0(x)$ be the corresponding eigenfunction with

$$\int_0^\infty \{\phi_0(x)\}^2 dx = 1 \quad ,$$

Let

$$\lambda^{(1)} = \int_0^\infty p(x)\{\phi_0(x)\}^2 dx < \infty$$

then

$$\lambda_\varepsilon = \lambda_0 + \varepsilon\lambda^{(1)} + o(\varepsilon) \ , \quad \varepsilon \to +0 . \tag{2.9}$$

If $\phi_\varepsilon(x)$ is the eigenfunction corresponding to λ_ε then we have, uniformly in any finite interval, $\phi_\varepsilon(x) - \phi_0(x) = o(\varepsilon^{\frac{1}{2}}) \ , \ \varepsilon \to +0 .$

With Eq. (2.9) of Theorem 2.2 we have the first-order approximation to the perturbed eigenvalue rigorously established. It is seen that the first two terms of Eq. (2.9) agree with the corresponding terms obtained by identification of coefficients in the formal series expansion that is usually assumed. As we shall see in the proof of Theorem 2.2, a bound can be computed for the error involved in taking the first two terms of the series to be the perturbed eigenvalue.

The order of approximation can be improved by introducing further restrictions. We shall state these in a new theorem.

Theorem 2.3. Let λ_0 be an isolated eigenvalue of the unperturbed problem and let $\phi_0(x)$ be the corresponding eigenfunction with

$$\int_0^\infty \{\phi_0(x)\}^2 dx = 1$$

Let

$$\lambda^{(1)} = \int_0^\infty p(x)\{\phi_0(x)\}^2 dx < \infty .$$

Furthermore, assume that

$$\int_0^\infty \{p(x)\}^{2\alpha}\{\phi_0(x)\}^2 dx < \infty , \tag{2.10}$$

where $\frac{1}{2} \leq \alpha < 1$ then

$$\lambda_\varepsilon = \lambda_0 + \varepsilon\lambda^{(1)} + o(\varepsilon^{2\alpha}) , \quad \varepsilon \to +0 .$$

If $\phi_\varepsilon(x)$ is the eigenfunction corresponding to λ_ε then we have, uniformly in any finite interval,

$$\phi_\varepsilon(x) - \phi_0(x) = o(\varepsilon^\alpha) , \quad \varepsilon \to +0 .$$

This theorem indicates that if the function p(x) satisfies the more restrictive condition (2.10), then the approximation to λ_ε by the first two terms of the series is a closer one.

We can now proceed to the second-order approximation of the eigenvalue and first-order approximation of the eigenfunction. Again we must introduce further restrictions that the function p(x) must satisfy. In the statement of the next theorem we assume that the unperturbed Eq. (2.1) has a complete set of discrete eigenvalues and eigenfunctions, i.e. pure point spectra. This assumption is not necessary for the proof of the second-order approximation, but is taken for convenience in expressing the coefficients of the higher-order terms in the perturbation series.

Theorem 2.4. Let $\{\lambda_\nu\}$ and $\{\psi_\nu(x)\}$, $\nu = 1, 2, 3, \ldots\ldots$, be the eigenvalues and orthonormal eigenfunctions of the unperturbed problem. Consider a fixed n.

Let $$\lambda_n^{(1)} = \int_0^\infty p(x)\{\psi_n(x)\}^2 dx < \infty .$$

Furthermore, we assume

$$\int_0^\infty \{p(x)\,\psi_n(x)\}^2 dx < \infty \quad ; \tag{2.11}$$

then

$$\lambda_{\varepsilon n} = \lambda_n + \varepsilon\lambda_n^{(1)} + \varepsilon^2\lambda_n^{(2)} + o(\varepsilon^2), \ \varepsilon \to +0,$$

where

$$\lambda_n^{(2)} = \sum_{\nu \neq n} \frac{1}{\lambda_n - \lambda_\nu}\left[\int_0^\infty p(x)\,\psi_n(x)\,\psi_\nu(x)\,dx\right]^2 .$$

If $\psi_{\varepsilon n}(x)$ is the eigenfunction corresponding to $\lambda_{\varepsilon n}$ then we have, uniformly in any finite interval,

$$\psi_{\varepsilon n}(x) = \psi_n(x) + \varepsilon\,\psi_n^{(1)}(x) + o(\varepsilon), \ \varepsilon \to +0,$$

where

$$\psi_n^{(1)}(x) = \sum_{\nu \neq n} \frac{\psi_\nu(x)}{\lambda_n - \lambda_\nu}\int_0^\infty p(y)\,\psi_n(y)\,\psi_\nu(y)\,dy .$$

Higher-order approximations to the eigenvalues and eigenfunctions can be obtained by the methods to be employed in this chapter. For purposes of applications to problems in mathematical physics the first- and second-order approximations are usually all that are employed, as the difficulty in computing the higher-order coefficients becomes great.

In fact in many practical problems only a few eigenvalues and eigenfunctions of the unperturbed problem are known from numerical integrations of the differential equation. In these cases only the first-order coefficient of the perturbed eigenvalue can be computed in the conventional manner. In a later section we give a method for determining the second-order coefficient even though the complete set of eigenfunctions for the unperturbed problem is not known.

The results of Theorem 2.2 and Theorem 2.4 are similar to those of Titchmarsh,[19] but Conditions (2.8) and (2.11) are less restrictive than his conditions. Theorem 2.3 is a new approximation.

2. Formulation of the Problem in the Theory of Quadratic Forms in Hilbert Space.

In order to establish the theorems stated in the preceding section we need to first formulate the problem as one in the spectral theory of linear operators in Hilbert space. As the Hilbert space, $\mathfrak{H}$; we have the space of real-valued measurable functions, f(x), defined on $(0, \infty)$ which are square-summable, i.e. the space $L_2(0,\infty)$. Measure and integration are in the sense of Lebesgue. In this space we have the norm,

$$\|f\| = \left(\int_0^\infty \{f(x)\}^2 dx\right)^{\frac{1}{2}} < \infty ,$$

and the inner product

$$(f, g) = \int_0^\infty f(x)\, g(x)\, dx$$

for f, g in $\mathfrak{H}$. Actually this space is a set of equivalence classes of functions. Two functions are equivalent in this sense if they differ only on a set of Lebesgue measure zero. So "$f_1(x) = f_2(x)$ almost everywhere" is equivalent to $\|f_1 - f_2\| = 0$.

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[19] Titchmarsh (2) Theorems 4, 5, 6, 7

For the unperturbed problem we are considering the differential operator, which is formally given by

$$Lu = q(x)u(x) - \frac{d^2u}{dx^2} \tag{2.12}$$

for u(x) in $L_2(0,\infty)$ and $0 < x < \infty$. The function q(x) satisfies Condition (2.3). Lu is not defined for all u in $\mathcal{H}$ and L is an unbounded operator. Consequently, for L to be properly defined as an operator in Hilbert space its domain must also be specified. The domain of L, $\mathcal{D}[L]$ is the set of functions u(x) such that u(x), u'(x) are absolutely continuous and u, Lu belong to $\mathcal{H}$. The unperturbed problem can be written as the following eigenvalue problem in $\mathcal{H}$:

$$Lu = \lambda u, \tag{2.13}$$

where λ is a complex number.

In order for Eq. (2.13) to be a solvable eigenvalue problem, i.e. a complete set of eigenfunctions corresponding to real eigenvalues that satisfy Eq. (2.13), we must have a self-adjoint operator. The operator L with domain $\mathcal{D}[L]$ is not self-adjoint. We must find the self-adjoint contraction $H \subset L$, i.e. the operator H given formally by Eq. (2.12) with $\mathcal{D}[H] \subset \mathcal{D}[L]$ and such that H is a self-adjoint operator.

The theory of singular second-order ordinary differential operators[17] indicates that since Eq. (2.12) is of the limit-point type at ∞ and of the limit-circle type at 0 a suitable boundary condition must be imposed at 0. Only conditions of a certain general form[20] are admissible. The boundary conditions for the unperturbed problem given by Eq. (2.2) are self-adjoint boundary conditions.[20]

[20] Weyl (1), Stone (1) Ch. X, Kodaira (1)

We have then that the operator H -- given formally by Eq. (2.12) and with domain $\mathcal{D}[H]$, where $\mathcal{D}[H]$ is the set of functions u(x), $0 < x < \infty$ such that u(x), u'(x) are absolutely continuous, u(x), H u belong to $\mathcal{H}$, and also $u(0)\cos\alpha + u'(0)\sin\alpha = 0$ -- is a self-adjoint operator in $\mathcal{H}$. And the unperturbed problem

$$H u = \lambda u \tag{2.14}$$

is a self-adjoint eigenvalue problem in Hilbert space.

Similarly we can define the self-adjoint operator

$$H_\varepsilon = q(x) + \varepsilon p(x) - \frac{d^2}{dx^2} \quad , \varepsilon > 0 \tag{2.15}$$

for $0<x<\infty$ and q(x) as before and p(x) satisfying Eq. (2.5) real-valued, continuous in $(0, \infty)$. The domain of H_ε, $\mathcal{D}[H_\varepsilon]$ is the set of functions u(x), $0 < x < \infty$, such that u(x), u'(x) are absolutely c ontinuous, and u(x), $H_\varepsilon u$ belong to $\mathcal{H}$ and also

$$u(0)\cos\alpha + u'(0)\sin\alpha = 0 .$$

Then H_ε is a self-adjoint operator and the perturbed problem

$$H_\varepsilon u = \lambda u \tag{2.16}$$

is a self-adjoint eigenvalue problem in Hilbert space.

We see that H_ε is given by

$$H_\varepsilon = H + \varepsilon H^{(1)} \tag{2.17}$$

where $H^{(1)}$ the perturbing operator, is just the multiplication operator given by $H^{(1)}u = p(x)u(x)$ and $\mathcal{D}[H^{(1)}]$ is the set of functions u(x) such that $\|H^{(1)}u\| = \left(\int_0^\infty \{p(x)u(x)\}^2 dx\right)^{1/2} < \infty$

We have $\mathcal{D}[H_\varepsilon] = \mathcal{D}[H] \cap \mathcal{D}[H^{(1)}] = \bar{\mathcal{D}}$. Let $\bar{H}$ be the restriction of the self-adjoint unperturbed operator H to the domain of the self-adjoint operator H_ε. We have for u, v in $\bar{\mathcal{D}}$

$$(\bar{H} u, v) = \int_0^\infty \left[q(x)u(x) - \frac{d^2u}{dx^2}\right] v(x)\, dx$$

$$= \int_0^\infty \left[\frac{du}{dx}\frac{dv}{dx} + q(x)u(x)v(x)\right] dx - u(0)v(0)\cot\alpha \quad ,$$

$$(u, \bar{H} v) = \int_0^\infty u(x)\left[q(x)v(x) - \frac{d^2v}{dx^2}\right] dx$$

$$= \int_0^\infty \left[\frac{du}{dx}\frac{dv}{dx} + q(x)u(x)v(x)\right] dx - u(0)v(0)\cot\alpha \, ,$$

$$\therefore \; (\bar{H} u, v) = (u, \bar{H} v) \quad , \; u, v \in \bar{\mathcal{D}} \, .$$

We have then that $\bar{H}$ is a linear symmetric operator. Now let us define the form $J[u,v] = (\bar{H} u, v)$ where $\mathcal{D}[J] = \bar{\mathcal{D}}$. Then the quadratic form is

$$J[u] = (\bar{H} u, u)$$

$$= \int_0^\infty \left[\left(\frac{du}{dx}\right)^2 + q(x)\{u(x)\}^2\right] dx - \{u(0)\}\cot\alpha \, ,$$

We have from condition (2.6) that J is a lower semibounded form. Furthermore, J is closable, since it is derived from a semibounded symmetric operator. Denote the closure of J by $\tilde{J}$. The self-adjoint operator associated with $\tilde{J}$, in the sense of Theorem D, is denoted $\check{H}$. It is the Friedrichs extension of $\bar{H}$. From the spectral theory of ordinary differential operators of second order,[21] we have that $\bar{H}$

[21] Rellich (6), Rellich (7) p 54, Stone (1) Ch. X.

with domain $\bar{\mathcal{D}}$ is an essentially self-adjoint operator. Therefore, $\tilde{\bar{H}}$ is its unique self-adjoint extension and consequently coincides with H. That is, we have

$$\tilde{\bar{H}} = H \qquad , \qquad \mathcal{D}[\tilde{\bar{H}}] = \mathcal{D}[H]. \tag{2.18}$$

Now let $\bar{H}^{(1)}$ be the restriction to $\bar{\mathcal{D}}$ of the perturbing operator $H^{(1)}$ defined in (2.17). We have u, v in $\bar{\mathcal{D}}$

$$(\bar{H}^{(1)}u, v) = \int_0^\infty p(x)\, u(x)\, v(x)\, dx = (u, \bar{H}^{(1)}v).$$

We have then that $\bar{H}^{(1)}$ is a linear symmetric operator. Now we define the form $J^{(1)}[u,v] = (\bar{H}^{(1)}u, v)$ with $\mathcal{D}[J^{(1)}] = \bar{\mathcal{D}}$. The quadratic form is

$$J^{(1)}[u] = (\bar{H}^{(1)}u, u) = \int_0^\infty p(x)\,\{u(x)\}^2\, dx$$

$$\geq p_0 \int_0^\infty \{u(x)\}^2\, dx ,$$

which is seen to be lower semibounded. The form $J^{(1)}$ is then closable and its closure is denoted by $\tilde{J}^{(1)}$. The self-adjoint operator associated with $\tilde{J}^{(1)}$ is $\tilde{\bar{H}}^{(1)}$ and is the Friedrichs extension of $\bar{H}^{(1)}$. Moreover, $\tilde{\bar{H}}^{(1)}$ coincides with $H^{(1)}$; a multiplication operator, with domain $\bar{\mathcal{D}}$ is clearly essentially self-adjoint. That is, we have

$$\tilde{\bar{H}}^{(1)} = H^{(1)} \qquad , \qquad \mathcal{D}[\tilde{\bar{H}}^{(1)}] = \mathcal{D}[H^{(1)}] .$$

We can define the form $J_\varepsilon = J + \varepsilon J^{(1)}$ for $\varepsilon > 0$ where $\mathcal{D}[J_\varepsilon] = \bar{\mathcal{D}}$. It is clear that J_ε is lower semibounded since J and $J^{(1)}$ are lower semibounded. The form J_ε is closable by Theorem B and its closure is denoted by $\tilde{J}_\varepsilon$. We then have from Theorem B that $\tilde{\tilde{J}}_\varepsilon \subset \tilde{\tilde{J}} + \varepsilon \tilde{\tilde{J}}^{(1)}$. We further have that $\tilde{\tilde{J}}_\varepsilon = \tilde{\tilde{J}} + \varepsilon \tilde{\tilde{J}}^{(1)}$ by Theorem C, since the operator $\bar{H} + \varepsilon \bar{H}^{(1)}$ is essentially self-adjoint. The self-adjoint operator associated with $\tilde{\tilde{J}}_\varepsilon$ is the Friedrichs extension of $\bar{H} + \varepsilon \bar{H}^{(1)}$ and coincides with H_ε defined in Eq. (2.17) since $\bar{H} + \varepsilon \bar{H}^{(1)}$ is essentially self-adjoint.

So we have that the self-adjoint operators associated with $\tilde{\tilde{J}}, \tilde{\tilde{J}}^{(1)}, \tilde{\tilde{J}}_\varepsilon$ are the operators $H, H^{(1)}, H_\varepsilon$ previously defined. Consequently, we have an equivalent formulation of the perturbation problem given in Eqs. (2.16) and (2.17), that is, to find the spectral properties of a self-adjoint operator H_ε associated with a quadratic form $\tilde{\tilde{J}}_\varepsilon$ where $\tilde{\tilde{J}}_\varepsilon = \tilde{\tilde{J}} + \varepsilon \tilde{\tilde{J}}^{(1)}$.

For convenience we can assume without loss of generality that $\tilde{\tilde{J}} \geq 1$, $\tilde{\tilde{J}}^{(1)} \geq 0$, and therefore $\tilde{J}_\varepsilon \geq 1$. (Kato[22]) This implies that

$$\begin{aligned} &(Hu, u) \geq (u, u), \\ &(H^{(1)}u, u) \geq 0 . \end{aligned} \tag{2.19}$$

These conditions appear to be much more restrictive than conditions (2.5) and (2.6); however, the condition $(Hu, u) \geq (u, u)$ may be assumed if H is only bounded below, for we only need to add a suitable constant to H and this means only a change of the origin of the spectra of H_ε by the same amount. We write condition (2.5)

$$(H^{(1)}u, u) \geq -\alpha(u, u) - \beta(Hu, u), \quad \alpha, \beta \geq 0.$$

If we set $H^{(1)'} = H^{(1)} + \alpha + \beta H$, then $(H^{(1)'}u, u) \geq 0$.

Also we see that
$$\begin{aligned} H + \varepsilon H^{(1)} &= H + \varepsilon (H^{(1)'} - \alpha - \beta H) \\ &= (1 - \varepsilon\beta)\left(H + \frac{\varepsilon}{1-\varepsilon\beta} H^{(1)'}\right) - \varepsilon\alpha \\ &= (1 - \varepsilon\beta)(H + \varepsilon' H^{(1)'}) - \varepsilon\alpha . \end{aligned}$$

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The factor $(1-\varepsilon\beta)$ and the added number $-\varepsilon\alpha$ imply only a change of scale and of origin of the spectra of H_ε. So we could consider the problem of $H' + \varepsilon' H^{(1)'}$ where we have $(H'u, u) \geq (u, u)$ and $(H^{(1)'}u, u) \geq 0$, and $H' = H + |\gamma| + 1$,

$$H^{(1)'} = H^{(1)} + \alpha + \beta H.$$

Consequently we shall assume that condition (2.19) is satisfied for purposes of convenience in proofs.

3. Zero-Order Approximation; Proof of Theorem 2.1.

In Section 2 of this chapter we formulated the perturbation problem in the language of semibounded closed quadratic forms. In particular we have the relationship

$$\tilde{J}_\varepsilon = \tilde{J} + \varepsilon \tilde{J}^{(1)},$$

and

$$\mathcal{D}[\tilde{J}_\varepsilon] = \mathcal{D}[\tilde{J}] \cap \mathcal{D}[\tilde{J}^{(1)}],$$

which is independent of ε. With this formulation of the problem we can make use of the asymptotic perturbation theory of such forms. Considering $\mathcal{D}[\tilde{J}_\varepsilon]$ as a set defined for the continuous parameter $\varepsilon > 0$, we note that $\mathcal{D}[\tilde{J}_\varepsilon]$ is independent of ε. We can show that $\tilde{J}_\varepsilon$ is nondecreasing in ε, that is nonincreasing as $\varepsilon \longrightarrow +0$ by the following simple argument. Consider the two equations for x in $(0, \infty)$,

$$\frac{d^2 u}{dx^2} + \{\lambda - q_1(x)\} u = 0,$$

$$\frac{d^2 u}{dx^2} + \{\lambda - q_2(x)\} u = 0,$$

with $u(0) = 0$.

Let

$$J_1[u] = \int_0^\infty \left[\left\{\frac{du}{dx}\right\}^2 + q_1(x)\{u(x)\}^2\right] dx ,$$

$$J_2[u] = \int_0^\infty \left[\left\{\frac{du}{dx}\right\}^2 + q_2(x)\{u(x)\}^2\right] dx ;$$

clearly $J_2[u] \geq J_1[u]$ for u in $\mathcal{D}[J_1] \cap \mathcal{D}[J_2]$

if $q_2(x) \geqq q_1(x)$ for all x in $(0, \infty)$.

In our case

$$q_1(x) = q(x) + \varepsilon_1 p(x),$$
$$q_2(x) = q(x) + \varepsilon_2 p(x),$$
$$\varepsilon_2 \geqq \varepsilon_1 .$$

Now from Theorem H, using a continuous parameter ε instead of the discrete index, we have the lemma.

Lemma 2.1. Let $\tilde{J}_\varepsilon \geq 1$ be a closed form, nonincreasing as $\varepsilon \to +0$. Let $\tilde{J} = C - \inf_{\varepsilon \to +0} \tilde{J}_\varepsilon$. If H_ε is the self-adjoint operator belonging to $\tilde{J}_\varepsilon$ and H the self-adjoint operator belonging to $\tilde{J}$, and if the spectrum of H consists of discrete eigenvalues at least in the lower part, then the same holds for H_ε and the eigenvalues of H_ε converge to the corresponding ones of H, i.e. $\lambda_{\varepsilon j} \longrightarrow \lambda_j$ as $\varepsilon \to +0$

for some $\lambda_j < \lambda^*$, the upper bound for discrete spectra.

In Section 2 we noted that $J_\varepsilon \geq 1$ could be assumed if conditions (2.5) and (2.6) are fulfilled. We assumed in Eq. (2.7) that the lower part of the spectrum of the unperturbed problem is discrete. Consequently, Lemma 2.1 applies directly to our problem. Theorem H provides that in each neighborhood of λ_j which is a simple eigenvalue of H, there is exactly one eigenvalue of H_ε -- $\lambda_{\varepsilon j}$ -- provided ε is sufficiently small. We have thus established Theorem 2.1, which gives the zero-order approximation to the perturbed eigenvalue.

4. First-Order Approximation; Proof of Theorem 2.2.

We proceed now to derive the first-order approximations. We have formulated the problem as one of finding the spectral properties of H_ε which is the self-adjoint operator associated with the closed form $\tilde{J}_\varepsilon$, defined by $\tilde{J}_\varepsilon = \tilde{J} + \varepsilon \tilde{J}^{(1)}$, where $\tilde{J}$, $\tilde{J}^{(1)}$ were defined in Section 2. H and $H^{(1)}$ are the self-adjoint operators belonging to $\tilde{J}$, $\tilde{J}^{(1)}$. We have assumed for convenience that $\tilde{J} \geq 1$ and $\tilde{J} \geq 0$. We have that

$$\mathcal{D}[\tilde{J}_\varepsilon] = \mathcal{D}[\tilde{J}] \cap \mathcal{D}[\tilde{J}^{(1)}]$$ is dense in $\mathcal{H}$ and independent of ε. Furthermore, in the previous section we showed that $\tilde{J}_\varepsilon$ is nonincreasing as $\varepsilon \to +0$.

With these considerations we can now make use of Theorem 1.1 from the perturbation theory of quadratic forms.

In order to derive the first-order and higher-order approximations to the perturbed eigenvalues and eigenvectors, we use some lemmas on estimating eigenvalues and eigenvectors developed by Kato.[23] Consider a self-adjoint operator H with the trial eigenvector w.

[23] Kato (2)

Lemma A (Kato[24]). Let w be in $\mathcal{D}[H]$, $\|w\| = 1$, and let $\eta = (Hw, w)$, $\theta = \|(H-\eta)w\|$. Then for any $\alpha < \eta$ the interval $\left(\alpha, \eta + \frac{\theta^2}{\eta - \alpha}\right]$ contains at least one point of the spectrum of H, and for any $\beta > \eta$ the interval

$$\left[\eta - \frac{\theta^2}{\beta - \eta}, \beta\right)$$

has the same property. $\theta \leq \|(H-\gamma)w\|$ for any scalar γ.

Lemma B (Kato[25]). Let (α, β) be an interval in which there is at most one nondegenerate eigenvalue of H but no other points of the spectrum. Let w, η, θ be as in Lemma A and further let $\theta^2 < (\eta - \alpha)(\beta - \eta)$. Then the interval (α, β) contains exactly one eigenvalue λ_0, which is contained in a smaller interval

$$\left[\eta - \frac{\theta^2}{\beta - \eta}, \eta + \frac{\theta^2}{\eta - \alpha}\right].$$

Let ϕ_0 be the eigenvector associated with λ_0. Then we have

$$\|\phi_0 - w\| \leq \frac{\theta/\delta}{\left(1 - \frac{\theta^2}{\delta^2}\right)^{1/4}},$$

provided

$$\theta < \delta = \min(\eta - \alpha, \beta - \eta) \quad ; \|\phi_0\| = 1, (\phi_0, w) \geq 0.$$

Proof of Theorem 2.2. From Theorem 2.1 we know that the interval (α, β) contains exactly one eigenvalue λ_ε^{-1} of H_ε^{-1}, so we can apply Lemma B to H_ε^{-1} with the trial vector $w = \phi_0$ the unperturbed eigenvector. Then $\eta = (H_\varepsilon^{-1}\phi_0, \phi_0)$.

From condition (2.8), $$\int_0^\infty p(x)\{\phi_0(x)\}^2 dx < \infty,$$

we have that ϕ_0 is in $\mathcal{D}[H^{(1)\frac{1}{2}}] = \mathcal{D}[\tilde{J}^{(1)}]$ and therefore $\lambda^{-1}\phi_0$ is in $\mathcal{D}[\tilde{J}']$ and also $H^{-1}\phi_0$; consequently $H^{-1}\phi_0$ is in $\mathcal{D}[\tilde{J}_\varepsilon]$

[24] Kato (1) Lemma 18.1

[25] Kato (1) Lemma 18.2

by Theorem 1.4 and we can use the asymptotic expansion of $(H_\varepsilon^{-1}\phi_0, \phi_0)$ given by Theorem 1.1 (iv). Then

$$\eta = (H^{-1}\phi_0, \phi_0) - \varepsilon \| H^{(1)\frac{1}{2}} H^{-1}\phi_0 \|^2 + o(\varepsilon),$$

$$\eta = \lambda_0^{-1} - \varepsilon \lambda_0^{-2} \| H^{(1)\frac{1}{2}} \phi_0 \|^2 + o(\varepsilon).$$

We have

$$\theta = \| (H_\varepsilon^{-1} - \eta) \phi_0 \|$$

$$\leq \| (H_\varepsilon^{-1} - \lambda_0^{-1}) \phi_0 \| = \| H_\varepsilon^{-1}\phi_0 - H^{-1}\phi_0 \| = o(\varepsilon^{\frac{1}{2}})$$

by Theorem 1.1 (iii). So $\theta = o(\varepsilon^{\frac{1}{2}})$ and $\theta^2 < (\eta - \alpha)(\beta - \eta)$ for sufficiently small ε. Using Lemma B, we have

$$-\frac{\theta^2}{\beta - \eta} \leq \lambda_\varepsilon^{-1} - \eta \leq \frac{\theta^2}{\eta - \alpha} .$$

Therefore

$$\lambda_\varepsilon^{-1} = \lambda_0^{-1} - \varepsilon \lambda_0^{-2} \| H^{(1)\frac{1}{2}} \phi_0 \|^2 + o(\varepsilon),$$

which gives

$$\lambda_\varepsilon = \lambda_0 + \varepsilon \lambda^{(1)} + o(\varepsilon), \quad \varepsilon \to +0,$$

where $\lambda^{(1)} = \| H^{(1)\frac{1}{2}} \phi_0 \|^2 = \int_0^\infty p(x) \{ \phi_0(x) \}^2 dx .$

Also by Lemma B we have

$$\| \phi_\varepsilon - \phi_0 \| \leq \frac{\frac{\theta}{\delta}}{\left[1 - \left(\frac{\theta}{\delta}\right)^2 \right]^{1/4}},$$

assuming $\|\phi_\varepsilon\| = 1$ and $(\phi_\varepsilon, \phi_0) \geq 0$, so we have

$$\|\phi_\varepsilon - \phi_0\| = o(\varepsilon^{\frac{1}{2}}), \quad \varepsilon \to +0.$$

It is interesting to note that by using Lemma B in this way we actually can get estimates of the errors involved in these approximations. We see that

$$\begin{aligned}
\theta^2 = \|(H_\varepsilon^{-1} - \eta)\phi_0\|^2 &\leq \|(H_\varepsilon^{-1} - \lambda_0^{-1})\phi_0\|^2 \\
&= \|\lambda_0^{-1}\phi_0 - \varepsilon\lambda_0^{-1} H_\varepsilon^{-\frac{1}{2}} B_\varepsilon'^{*} H^{(1)\frac{1}{2}}\phi_0 - \lambda_0^{-1}\phi_0\|^2 \\
&= \varepsilon^2\lambda_0^{-2}\|H_\varepsilon^{-\frac{1}{2}} B_\varepsilon'^{*} H^{(1)\frac{1}{2}}\phi_0\|^2 \\
&\leq \varepsilon^2\lambda_0^{-2}\|B_\varepsilon'^{*} H^{(1)\frac{1}{2}}\phi_0\|^2 = \varepsilon\lambda_0^{-2}\|\varepsilon^{\frac{1}{2}} B_\varepsilon'^{*} H^{(1)\frac{1}{2}}\phi_0\|^2 \\
&\leq \varepsilon\lambda_0^{-2}\|H^{(1)\frac{1}{2}}\phi_0\|^2 = \varepsilon\lambda_0^{-2}\lambda^{(1)},
\end{aligned}$$

because we have $\|H_\varepsilon^{-\frac{1}{2}}\| \leq 1$ and $\|\varepsilon^{\frac{1}{2}} B_\varepsilon'^{*}\| \leq 1$.

We also have

$$\eta^* = \lambda_0^{-1} - \varepsilon\lambda_0^{-2}\lambda^{(1)} \leq \eta,$$

where

$$\eta = \lambda_0^{-1} - \varepsilon\lambda_0^{-2}\lambda^{(1)} + \varepsilon^2\lambda_0^{-2}\|B_\varepsilon'^{*} H^{(1)\frac{1}{2}}\phi_0\|^2,$$

so

$$\frac{1}{\eta - \alpha} \leq \frac{1}{\eta^* - \alpha}.$$

Now, from $\lambda_\varepsilon^{-1} - \eta \leq \dfrac{\theta^2}{\eta - \alpha}$ and the above inequalities for θ^2 and $\dfrac{1}{\eta - \alpha}$, we have

$$\begin{aligned}
\lambda_\varepsilon^{-1} - \lambda_0^{-1} + \varepsilon\lambda_0^{-2}\lambda^{(1)} &\leq \varepsilon^2\lambda_0^{-2}\|B_\varepsilon'^{*} H^{(1)\frac{1}{2}}\phi_0\|^2 + \frac{\theta^2}{\eta - \alpha} \\
&\leq \varepsilon\lambda_0^{-2}\lambda^{(1)} + \frac{\varepsilon\lambda_0^{-2}\lambda^{(1)}}{\eta^* - \alpha}.
\end{aligned}$$

All the quantities on the right-hand side can be computed easily. We have a similar inequality for the other side from Lemma B.

We have established

$$\left(\int_0^\infty \{\phi_\varepsilon(x) - \phi_0(x)\}^2 dx\right)^{1/2} = o(\varepsilon^{\frac{1}{2}}) \quad , \quad \varepsilon \to +0,$$

and now wish to derive the uniform convergence over any finite interval. We make use of the following formula[26] which can be verified by integration by parts. For any continuous function u(x) with continuous first and second derivatives we have

$$u(x) = \frac{1}{(\Delta x)^2}\int_x^{x+\Delta x} \left[(x+\Delta x-y)^2(y-x)\,u''(y) - (6y-6x-4\Delta x)\,u(y)\right]dy$$

Using the above equation together with Eqs. (2.1) and (2.4), we have

$$\phi_\varepsilon(x) = \frac{1}{(\Delta x)^2}\int_x^{x+\Delta x} \left[(x+\Delta x-y)^2(y-x)\{q(y)+\varepsilon p(y)-\lambda_\varepsilon\}\,\phi_\varepsilon(y)dy\right.$$

$$- \frac{1}{(\Delta x)^2}\int_x^{x+\Delta x} (6y-6x-4\Delta x)\,\phi_\varepsilon(y)\,dy\;;$$

$$\phi_0(x) = \frac{1}{(\Delta x)^2}\int_x^{x+\Delta x} \left[(x+\Delta x-y)^2(y-x)\{q(y)-\lambda_0\}\,\phi_0(y)dy\right.$$

$$- \frac{1}{(\Delta x)^2}\int_x^{x+\Delta x} (6y-6x-4\Delta x)\,\phi_0(y)\,dy\;.$$

Hence $\phi_\varepsilon(x) - \phi_0(x)$

$$= \frac{1}{(\Delta x)^2}\int_x^{x+\Delta x} \left[(x+\Delta x-y)^2(y-x)\{q(y)-\lambda_0\} - (6y-6x-4\Delta x)\right]\{\phi_\varepsilon(y)-\phi_0(y)\}dy$$

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$$+\frac{1}{(\Delta x)^2}\int_x^{x+\Delta x}(x+\Delta x-y)^2(y-x)\{\varepsilon p(y)-(\lambda_\varepsilon-\lambda_0)\}\phi_\varepsilon(y)dy.$$

By the Schwartz inequality, the square of the first term is

$$O\|\phi_\varepsilon-\phi_0\|^2=o(\varepsilon)$$

So the first term is $o(\varepsilon^{\frac{1}{2}})$ and the second term is $O(\varepsilon)\|\phi_\varepsilon\|=O(\varepsilon)$.

Consequently, we have $\phi_\varepsilon(x)-\phi_0(x)=o(\varepsilon^{\frac{1}{2}})$, $\varepsilon\to+0$,

uniformly over any finite interval. We thus have established Theorem 2.2, giving the first-order approximation to the perturbed eigenvalue and the zero-order approximation to the perturbed eigenfunction.

5. Improved First-Order Approximation; Proof of Theorem 2.3.

We wish now to improve the approximations derived in the previous section by imposing further restrictions on the perturbing term p(x). In order to do this we use the results of Theorem 1.2.

Proof of Theorem 2.3. We can apply Lemma B, given in the previous section, to H_ε^{-1} with the trial vector ϕ_0, the unperturbed eigenvector. Then $\eta=(H_\varepsilon^{-1}\phi_0,\phi_0)$.

From condition (2.10) of Theorem 2.3 we have

$$\int_0^\infty\{p(x)\}^{2\alpha}\{\phi_0(x)\}^2dx<\infty,$$

which means ϕ_0 is in $\mathcal{D}[H^{1/\alpha}]$ also $\lambda_0^{-1}\phi_0$ in $\mathcal{D}[H^{1/\alpha}]$ consequently, $H^{-1}\phi_0$ is in $\mathcal{D}[H^{1/\alpha}]$ and we can use the expansion of $(H_\varepsilon^{-1}\phi_0,\phi_0)$ given by Theorem 1.2 (iv).

Then

$$\eta = (H^{-1}\phi_0, \phi_0) - \varepsilon \| H^{(1)\frac{1}{2}} H^{-1}\phi_0 \|^2 + \varepsilon^2 \| H_\varepsilon^{-\beta} C_\varepsilon^* H^{(1)\alpha} H^{-1}\phi_0 \|^2$$

$$= \lambda_0^{-1} - \varepsilon \lambda_0^{-2} \| H^{(1)\frac{1}{2}} \phi_0 \|^2 + \varepsilon^2 \lambda_0^{-2} \| H_\varepsilon^{-\beta} C_\varepsilon^* H^{(1)\alpha} \phi_0 \|^2 .$$

Also we have

$$\eta = \lambda_0^{-1} - \varepsilon \lambda_0^{-2} \| H^{(1)\frac{1}{2}} \phi_0 \|^2 + o(\varepsilon^{2\alpha}) .$$

We note that

$$\theta = \| (H_\varepsilon^{-1} - \eta) \phi_0 \|$$

$$\leq \| (H_\varepsilon^{-1} - \lambda_0^{-1}) \phi_0 \| = \| H_\varepsilon^{-1} \phi_0 - H^{-1}\phi_0 \| = o(\varepsilon^\alpha),$$

by Theorem 1.2 (iii). So $\theta = o(\varepsilon^\alpha)$ and $\theta^2 < (\eta - \alpha)(\beta - \eta)$ for sufficiently small ε, where the interval (α, β) contains exactly one eigenvalue λ_ε^{-1}.

Using Lemma B, we have

$$-\frac{\theta^2}{\beta - \eta} \leq \lambda_\varepsilon^{-1} - \eta \leq \frac{\theta^2}{\eta - \alpha} .$$

Hence

$$\lambda_\varepsilon^{-1} = \lambda_0^{-1} - \varepsilon \lambda_0^{-2} \| H^{(1)\frac{1}{2}} \phi_0 \|^2 + o(\varepsilon^{2\alpha}),$$

which gives

$$\lambda_\varepsilon = \lambda_0 + \varepsilon \lambda^{(1)} + o(\varepsilon^{2\alpha}) , \quad \varepsilon \to +0 ,$$

$$\lambda^{(1)} = \| H^{(1)\frac{1}{2}} \phi_0 \|^2 = \int_0^\infty p(x) \{ \phi_0(x) \}^2 dx .$$

Also by Lemma B we have

$$\|\phi_\varepsilon - \phi_0\| \leq \frac{\theta/\delta}{\left[1-(\theta/\delta)^2\right]^{1/4}},$$

assuming

$$\|\phi_\varepsilon\| = 1 \quad , \quad (\phi_\varepsilon, \phi_0) \geq 0 .$$

Then we have

$$\left(\int_0^\infty \{\phi_\varepsilon(x) - \phi_0(x)\}^2 dx\right)^{1/2} = o(\varepsilon^\alpha) .$$

We derive $\phi_\varepsilon(x) - \phi_0(x) = o(\varepsilon^\alpha)$, $\varepsilon \to +0$ uniformly over any finite interval from convergence in the mean exactly as in the proof of Theorem 2.2 in the previous section. We have thus established Theorem 2.2, giving an improved first-order approximation to the perturbed eigenvalue and an improved zero-order approximation to the perturbed eigenfunction.

6. Second-Order Approximation; Proof of Theorem 2.4.

We proceed now to derive the next order of approximation for eigenvalues and eigenfunctions by imposing further restrictions on the function p(x).

Proof of Theorem 2.4.

Let $\{E(\lambda)\}$ be the resolution of the identity for the self-adjoint operator H. We define the operator S,

$$S = \int' \frac{1}{\lambda - \lambda_0} \, dE(\lambda) ,$$

where $\int'$ means integration except for the point $\lambda = \lambda_0$, an eigenvalue of H. S is called the reduced resolvent of H and is a bounded, self-adjoint operator. We use the following lemma from the perturbation theory of quadratic forms.

Lemma 2.2 (Kato[27]) Let $\tilde{J} \geq 1$, $\tilde{J}' \geq 0$ be closed forms. Let $\mathcal{D}[\tilde{J}] \cap \mathcal{D}[\tilde{J}']$ be dense in $\mathcal{H}$. Let $\tilde{J}_\varepsilon = \tilde{J} + \varepsilon \tilde{J}'$ be a closed form nonincreasing as $\varepsilon \rightarrow +0$. Let H_ε, H', H be the self-adjoint operators belonging to $\tilde{J}_\varepsilon, \tilde{J}', \tilde{J}$ respectively. Let λ_0 be the eigenvalue of H and let ϕ_0 be the corresponding eigenvector, $\|\phi_0\| = 1$. Let ϕ_0 be in $\mathcal{D}[H']$. Then

$$\lambda_\varepsilon = \lambda_0 + \varepsilon\lambda^{(1)} + \varepsilon^2\lambda^{(2)} + o(\varepsilon^2),\ \varepsilon \rightarrow +0,$$

$$\phi_\varepsilon = \phi_0 + \varepsilon\phi^{(1)} + o(\varepsilon),\ \varepsilon \rightarrow +0,$$

$$\lambda^{(1)} = (H'\phi_0, \phi_0)$$

$$\lambda^{(2)} = -(SH'\phi_0, H'\phi_0),$$

$$\phi^{(1)} = -SH'\phi_0 .$$

We assume that the spectrum of the unperturbed operator is totally discrete and denote $\{\lambda_\nu\}$ and $\{\psi_\nu(x)\}$ as the complete set of eigenvalues and orthonormal eigenfunctions. We consider a fixed n, and from condition (2.11) of the theorem we have

$$\int_0^\infty \{p(x)\psi_n(x)\}^2 dx < \infty$$

Thus ψ_n is in $\mathcal{D}[H']$ and we apply Lemma 2.2. The spectral representation of H is

$$Hf = \int \lambda\, dE(\lambda) f = \sum_{\nu=1}^{\infty} (f, \psi_\nu)\, \psi_\nu$$

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[27] Kato (1) Theorem 20.1

for some f in $\mathfrak{D}[H]$. So we have

$$-SH'\psi_n = \sum_{\nu \neq n} \frac{(H'\psi_n, \psi_\nu)}{\lambda_n - \lambda_\nu} \psi_\nu = \psi_n^{(1)},$$

Also we have

$$-(SH'\psi_n, H'\psi_n) = \sum_{\nu \neq n} \frac{(H'\psi_n, \psi_\nu)^2}{\lambda_n - \lambda_\nu} = \lambda_n^{(2)},$$

From the lemma we then have

$$\lambda_{\varepsilon n} = \lambda_n + \varepsilon \lambda_n^{(1)} + \varepsilon^2 \lambda_n^{(2)} + o(\varepsilon^2), \ \varepsilon \to +0,$$

and
$$\| \psi_{\varepsilon n} - \psi_n - \varepsilon \psi_n^{(1)} \| = o(\varepsilon), \ \varepsilon \to +0,$$

where
$$\lambda_n^{(1)} = \int_0^\infty p(x) \{\psi_n(x)\}^2 dx,$$

$$\lambda_n^{(2)} = \sum_{\nu \neq n} \frac{1}{\lambda_n - \lambda_\nu} \left[\int_0^\infty p(x) \psi_n(x) \psi_\nu(x) dx \right]^2,$$

and

$$\psi_n^{(1)}(x) = \sum_{\nu \neq n} \frac{\psi_\nu(x)}{\lambda_n - \lambda_\nu} \left[\int_0^\infty p(y) \psi_n(y) \psi_\nu(y) dy \right].$$

We shall now establish the uniform convergence over any finite interval. The function $\psi_n^{(1)}(x)$ is twice differentiable and satisfies the following differential equation:

$$\frac{d^2\psi_n^{(1)}}{dx^2} + \{\lambda_n - q(x)\}\psi_n^{(1)} = -\{\lambda_n^{(1)} - p(x)\}\psi_n.$$

This is not an eigenvalue problem but $\psi_n^{(1)}(x)$ must satisfy conditions (2.2) at $x=0$ and belong to $L_2(0,\infty)$. We verify that $\psi_n^{(1)}(x)$ does satisfy the above differential equation by writing it in the form

$$H\psi_n^{(1)} = \lambda_n\psi_n^{(1)} - H'\psi_n + \lambda_n^{(1)}\psi_n$$

$$= \sum_{\nu\neq n}\frac{H'_{n\nu}\lambda_n}{\lambda_n-\lambda_\nu}\psi_\nu - \sum_{\nu=1}^{\infty} H'_{n\nu}\psi_\nu + H'_{nn}\psi_n$$

$$= \sum_{\nu\neq n}\frac{H'_{n\nu}\lambda_n}{\lambda_n-\lambda_\nu}\psi_\nu - \sum_{\nu\neq n} H'_{n\nu}\psi_\nu$$

$$= \sum_{\nu\neq n}\frac{H'_{n\nu}\lambda_n - H'_{n\nu}\lambda_n + H'_{n\nu}\lambda_\nu}{\lambda_n-\lambda_\nu}\psi_\nu$$

$$= \sum_{\nu\neq n}\frac{H'_{n\nu}\lambda_\nu}{\lambda_n-\lambda_\nu}\psi_\nu = \sum_{\nu\neq n}\frac{H'_{n\nu}}{\lambda_n-\lambda_\nu}H\psi_\nu$$

$$= H\sum_{\nu\neq n}\frac{H'_{n\nu}}{\lambda_n-\lambda_\nu}\psi_\nu = H\psi_n^{(1)}.$$

Now, using the representation for twice differentiable functions given in Theorem 2.2 and the differential equation for $\psi_n^{(1)}(x)$, we have

$$\psi_n^{(1)}(x) = \frac{1}{(\Delta x)^2}\int_x^{x+\Delta x}\left[(x+\Delta x-y)^2(y-x)\{q(y)-\lambda_n\}-(6y-6x-4\Delta x)\right]\psi_n^{(1)}(y)\,dy$$

$$+\frac{1}{(\Delta x)^2}\int_x^{x+\Delta x}(x+\Delta x-y)^2(y-x)\{p(y)-\lambda_n^{(1)}\}\psi_n(y)\,dy.$$

Also from Theorem 2.2 we have $\psi_{\varepsilon n}(x)-\psi_n(x)$

$$=\frac{1}{(\Delta x)^2}\int_x^{x+\Delta x}\left[(x+\Delta x-y)^2(y-x)\{q(y)-\lambda_n\}-(6y-6x-4\Delta x)\right]\{\psi_{\varepsilon n}(y)-\psi_n(y)\}\,dy$$

$$+\frac{1}{(\Delta x)^2}\int_x^{x+\Delta x}(x+\Delta x-y)^2(y-x)\{\varepsilon p(y)-(\lambda_{\varepsilon n}-\lambda_n)\}\psi_{\varepsilon n}(y)\,dy.$$

Hence we have $\psi_{\varepsilon n}(x)-\psi_n(x)-\varepsilon\psi_n^{(1)}(x)$

$$=\frac{1}{(\Delta x)^2}\int_x^{x+\Delta x}\left[(x+\Delta x-y)^2(y-x)\{q(y)-\lambda_n\}-(6y-6x-4\Delta x)\right]\{\psi_{\varepsilon n}(y)-\psi_n(y)-\psi_n^{(1)}(y)\}\,dy$$

$$+\frac{1}{(\Delta x)^2}\int_x^{x+\Delta x}(x+\Delta x-y)^2(y-x)\{\varepsilon p(y)-(\lambda_{\varepsilon n}-\lambda_n)\}\{\psi_{\varepsilon n}(y)-\psi_n(y)\}\,dy$$

$$+\frac{1}{(\Delta x)^2}\int_x^{x+\Delta x}(x+\Delta x-y)^2(y-x)(\lambda_{\varepsilon n}-\lambda_n-\varepsilon\lambda_n^{(1)})\,\psi_n(y)\,dy.$$

By the Schwartz inequality, the square of the first term is less than

$$O\|\psi_{\varepsilon n}-\psi_n-\varepsilon\psi_n^{(1)}\|^2=o(\varepsilon^2).$$

So the first term is $o(\varepsilon)$. The second term is $O(\varepsilon)\{\psi_{\varepsilon n}-\psi_n\}=o(\varepsilon)$ since $\{\psi_{\varepsilon n}-\psi_n\}$ tends to zero by Theorem 2.2. The third term is $O(\varepsilon^2)$ since

$$\lambda_{\varepsilon n}-\lambda_n-\varepsilon\lambda_n^{(1)}=\varepsilon^2\lambda_n^{(2)}+o(\varepsilon^2).$$

Consequently, we have

$$\psi_{\varepsilon n}(x)-\psi_n(x)-\varepsilon\psi_n^{(1)}(x)=o(\varepsilon),\ \varepsilon\to+0,$$

uniformly over any finite interval. We have thus established Theorem 2.4.

7. Higher-Order Perturbations

Until now we have considered only the case of a first-order perturbation. We consider now higher-order perturbations, that is, equations of the type

$$\frac{d^2u}{dx^2}+\{\lambda-q(x)-\sum_{j=1}^{n}\varepsilon^j p_j(x)\}u=0$$

on the interval $(0,\infty)$ with $\varepsilon>0$. The $p_j(x)$ are assumed to be continuous and real-valued in $(0,\infty)$ and $p_j(x)\geq p_j>-\infty$ for x in $(0,\infty)$. We wish to investigate whether the theorems giving first-and second-order approximations to the eigenvalues are valid for such problems. To do this let us consider the more general equation,

$$\frac{d^2u}{dx^2}+\{\lambda-q(x)-\varepsilon p_1(x)-\varepsilon^2 p_2(x,\varepsilon)\}u=0$$

in $(0,\infty)$ with condition (2.2). The functions $q(x), p_1(x), p_2(x,\varepsilon)$ are real-valued, continuous in $(0,\infty)$; also

$$q(x)\geq q_0,\quad p_1(x)\geq p_1,\quad p_2(x,\varepsilon)\geq p_2.$$

for all x in $(0, \infty)$ and $\varepsilon > 0$. Moreover, let $p_2(x,\varepsilon) \to p_2(x,0)$ as $\varepsilon \to +0$. We shall establish the following two theorems.

Theorem I. Let λ_0 be an isolated eigenvalue of the unperturbed problem and let $\phi_0(x)$ be the corresponding eigenfunction with

$$\int_0^\infty \{\phi_0(x)\}^2 dx = 1.$$

Let

$$\lambda^{(1)} = \int_0^\infty p_1(x)\{\phi_0(x)\}^2 dx < \infty ;$$

then

$$\lambda_\varepsilon = \lambda_0 + \varepsilon\lambda^{(1)} + o(\varepsilon), \quad \varepsilon \to +0.$$

If $\phi_\varepsilon(x)$ is the eigenfunction corresponding to λ_ε, then

$$\phi_\varepsilon(x) - \phi_0(x) = o(\varepsilon^{\frac{1}{2}}), \quad \varepsilon \to +0,$$

uniformly over any finite interval.

Theorem II. Let the conditions of Theorem I be satisfied. Furthermore, assume pure point spectra and

$$\int_0^\infty \{p_1(x)\phi_n(x)\}^2 dx < \infty, \quad \int_0^\infty p_2(x,\varepsilon)\{\phi_n(x)\}^2 dx < \infty, \quad \varepsilon \le \varepsilon_1,$$

for some $\varepsilon_1 > 0$ and fixed n. Then

$$\lambda_{\varepsilon n} = \lambda_n + \varepsilon\lambda_n^{(1)} + \varepsilon^2\lambda_n^{(2)} + o(\varepsilon^2), \quad \varepsilon \to +0,$$

where

$$\lambda_n^{(2)} = \sum_{\nu \ne n} \frac{1}{\lambda_n - \lambda_\nu}\left[\int_0^\infty p_1(x)\phi_n(x)\phi_\nu(x)\,dx\right]^2 + \int_0^\infty p_2(x,0)\{\phi_n(x)\}^2 dx.$$

If $\phi_{\varepsilon n}(x)$ is the eigenfunction corresponding to $\lambda_{\varepsilon n}$, then we have uniformly over any finite interval

$$\phi_{\varepsilon n}(x) = \phi_n(x) + \varepsilon\phi_n^{(1)}(x) + o(\varepsilon), \quad \varepsilon \to +0,$$

where $$\phi_n^{(1)}(x) = \sum_{\nu \neq n} \frac{1}{\lambda_n - \lambda_\nu} \left[\int_0^\infty p_1(y)\, \phi_n(y)\, \phi_\nu(y)\, dy \right].$$

As a problem in the Hilbert space $L_2(0,\infty)$ the perturbed operator is given by $H_\varepsilon = H + \varepsilon H' + \varepsilon^2 H''_\varepsilon$ with domain $\bar{\mathcal{D}} = \mathcal{D}[H] \cap \mathcal{D}[H'] \cap \mathcal{D}[H''_\varepsilon]$

The operators H, H' are defined in Section 2 and H''_ε is the multiplication operator, i.e.

$$H''_\varepsilon u = p_2(x,\varepsilon)\, u(x)$$

for u in $\mathcal{D}[H''_\varepsilon]$. $\mathcal{D}[H''_\varepsilon]$ is the set of functions such that

$$\int_0^\infty \{ p_2(x,\varepsilon)\, u(x) \}^2 dx < \infty$$

Let $\bar{H}''_\varepsilon$ be the restriction of H''_ε to $\bar{\mathcal{D}}$. We have for u,v in $\bar{\mathcal{D}}$

$$(\bar{H}''_\varepsilon u, v) = \int_0^\infty p_2(x,\varepsilon)\, u(x)\, v(x)\, dx = (u, \bar{H}''_\varepsilon v)$$

and the domain $\bar{\mathcal{D}}$ is dense in $\mathcal{H}$ so $\bar{H}''_\varepsilon$ is a symmetric operator. We define the form $J''_\varepsilon[u,v] = (\bar{H}''_\varepsilon u, v)$ with $\mathcal{D}[J''_\varepsilon] = \bar{\mathcal{D}}$. The quadratic form is

$$J''_\varepsilon[u] = (\bar{H}''_\varepsilon u, u) = \int_0^\infty p_2(x,\varepsilon) \{u(x)\}^2 dx$$
$$\geq p_2 \int_0^\infty \{u(x)\}^2 dx,$$

and is observed to be lower semibounded. The form J''_ε is then closable and its closure is denoted by $\tilde{J}''_\varepsilon$. The self-adjoint operator associated with $\tilde{J}''_\varepsilon$ is $\tilde{\bar{H}}''_\varepsilon$ and is the Friedrichs extension of $\bar{H}''_\varepsilon$. Furthermore $\tilde{\bar{H}}''_\varepsilon$ coincides with H''_ε since $\bar{H}''_\varepsilon$ is essentially self-adjoint.

We can now define the form

$$J_\varepsilon = J + \varepsilon J' + \varepsilon^2 J_\varepsilon'' \quad , \quad \varepsilon > 0 \; ,$$

where $\mathcal{D}[J_\varepsilon] = \overline{\mathcal{D}}$ and J and J' are defined in Section 2. J_ε is lower semibounded and closable as before, and

$$\tilde{J}_\varepsilon \subset \tilde{J} + \varepsilon \tilde{J}' + \varepsilon^2 \tilde{J}_\varepsilon'' \quad .$$

by Theorem B. We further have, by Theorem C, that

$$\tilde{J}_\varepsilon = \tilde{J} + \varepsilon \tilde{J}' + \varepsilon^2 \tilde{J}_\varepsilon'' \qquad \text{since } \overline{H} + \varepsilon \overline{H}' + \varepsilon^2 \overline{H}_\varepsilon''$$

is essentially self-adjoint. The self-adjoint operator associated with $\tilde{J}_\varepsilon$ is the Friedrichs extension of $\overline{H} + \varepsilon \overline{H}' + \varepsilon^2 \overline{H}_\varepsilon''$. It is the unique self-adjoint extension since $\overline{H} + \varepsilon \overline{H}' + \varepsilon^2 \overline{H}_\varepsilon''$ is essentially self-adjoint. So the operators $H_\varepsilon, H, H', H_\varepsilon''$ are the self-adjoint operators associated with $\tilde{J}_\varepsilon, \tilde{J}, \tilde{J}', \tilde{J}_\varepsilon''$. We then have the equivalent perturbation problem of finding the eigenvalues of a self-adjoint operator H_ε associated with a form $\tilde{J}_\varepsilon$ where

$$\tilde{J}_\varepsilon = \tilde{J} + \varepsilon \tilde{J}' + \varepsilon^2 \tilde{J}_\varepsilon'' .$$

We assume for convenience $\tilde{J} \geq 1$, $\tilde{J}' \geq 0$, $\tilde{J}_\varepsilon'' \geq 0$. Now let $\tilde{J}_\varepsilon' = \tilde{J}' + \varepsilon \tilde{J}_\varepsilon''$, then $\tilde{J}_\varepsilon = \tilde{J} + \varepsilon \tilde{J}_\varepsilon'$ and $\tilde{J}' = c - \inf_{\varepsilon \to +0} \tilde{J}_\varepsilon'$. We can then use the results of Theorem 1.1 to derive Theorem I exactly as in the proof of Theorem 2.2.

To derive the second-order approximation we use the following lemma, which is a gneralization of Lemma 2.2.

Lemma 2.3 (Kato[28]) Let the assumptions of Theorem 1.3 be satisfied. Let λ_0 be the eigenvalue of H and let ϕ_0 be the corresponding eigenvector, $\|\phi_0\| = 1$. Let ϕ_0 be in $\mathcal{D}[\tilde{J}_{\varepsilon_1}] \cap \mathcal{D}[H']$ for some $\varepsilon_1 > 0$. Then $\lambda_\varepsilon = \lambda_0 + \varepsilon\lambda^{(1)} + \varepsilon^2\lambda^{(2)} + o(\varepsilon^2)$, $\varepsilon \to +0$,

and $\phi_\varepsilon = \phi_0 + \varepsilon\phi^{(1)} + o(\varepsilon)$, $\varepsilon \to +0$,

where
$$\lambda^{(1)} = (H'\phi_0, \phi_0),$$
$$\lambda^{(2)} = -(SH'\phi_0, H'\phi_0) + \|H''^{\frac{1}{2}}\phi_0\|^2,$$
$$\phi^{(1)} = -SH'\phi_0.$$

The quadratic form $\tilde{J}_\varepsilon = \tilde{J} + \varepsilon\tilde{J}' + \varepsilon^2\tilde{J}''_\varepsilon$ satisfies the conditions of Theorem 1.3 and the convergence of the integrals,

$$\int_0^\infty \{p_1(x)\phi_n(x)\}^2 dx < \infty,$$

$$\int_0^\infty p_2(x,\varepsilon_1)\{\phi_n(x)\}^2 dx < \infty,$$

means that ϕ_n is in $\mathcal{D}[H'] \cap \mathcal{D}[H''^{\frac{1}{2}}_\varepsilon]$, but this implies ϕ_n in $\mathcal{D}[\tilde{J}_{\varepsilon_1}] \cap \mathcal{D}[H']$ by Theorem E and Theorem 1.4. We can then use Lemma 2.3 to prove Theorem II just as in the proof of Theorem 2.4. With the assumption of pure point spectra we have

$$\lambda_n^{(2)} = \sum_{\nu \neq n} \frac{(H'\phi_n, \phi_\nu)}{\lambda_n - \lambda_\nu} + \|H''^{\frac{1}{2}}\phi_n\|^2$$

$$= \sum_{\nu \neq n} \frac{1}{\lambda_n - \lambda_\nu}\left[\int_0^\infty p_1(x)\phi_n(x)\phi_\nu(x)dx\right]^2 + \int_0^\infty p_2(x,0)\{\phi_n(x)\}^2 dx,$$

[28] Kato (1) Theorem 20.1

$$\phi_n^{(1)}(x) = \sum_{\nu \neq n} \frac{\phi_\nu(x)}{\lambda_n - \lambda_\nu} \left[\int_0^\infty p_1(y)\, \phi_n(y)\, \phi_\nu(y)\, dy \right].$$

We have then the first- and second- order approximations for the case of higher-order perturbations.

8. Local behavior of $\lambda(\eta)$ of the equation

$$\frac{d^2u}{dx^2} + \{\lambda - q(x,\eta)\} u = 0 .$$

The theorems established in this chapter can be used to study the dependence of the eigenvalue λ upon a parameter η, assuming small changes in η. Consider the differential equation

$$\frac{d^2u}{dx^2} + \{\lambda - q(x,\eta)\} u = 0 ,$$

$0 < x < \infty$ with $q(x,\eta)$ a real-valued function continuous in x in $(0,\infty)$ and analytic in the real variable η. Furthermore, assume $q(x,\eta) \geq q^*$ for x in $(0,\infty)$, independent of η. We assume the limit-circle case at 0 and limit-point case at ∞. For a boundary condition at 0 we take u(0) = 0.

At the point $\eta = \eta_0$ we are given the lowest eigenvalue, $\lambda(\eta_0) = \lambda_0$, and the corresponding eigenfunction, $u(x,\eta_0) = u_0(x)$. We wish to find $\lambda(\eta)$ in the neighborhood of η_0. Let $\eta_1 = \eta_0 + \Delta\eta$. We have

$$q(x,\eta_1) = q(x,\eta_0) + \Delta\eta \frac{\partial q}{\partial \eta}(x,\eta_0) + \frac{(\Delta\eta)^2}{2} \frac{\partial^2 q}{\partial \eta^2}(x,\eta_0) + \cdots$$

Let

$$\varepsilon = \Delta\eta , \quad q(x,\eta_0) = q_0(x) , \quad q(x,\eta_1) = q_1(x).$$

We can write the above as

$$q_1(x) = q_0(x) + \varepsilon q^{(1)}(x) + \varepsilon^2 q^{(2)}(x, \varepsilon).$$

The problem then is to find λ and $u(x)$ for the equation

$$\frac{d^2u}{dx^2} + \{\lambda - q_0(x) - \varepsilon q^{(1)}(x) - \varepsilon^2 q^{(2)}(x,\varepsilon)\} u = 0$$

This equation is of the type considered in the last section; consequently, we have sufficient conditions that can be tested to establish the validity of first- and second-order approximations.

Denote $\lambda(\eta_1) = \lambda_\varepsilon$ and $u(x, \eta_1) = u_\varepsilon(x)$.

We have $q(x, \eta) \geq q^*$ for all x in $(0, \infty)$ independent of η, so $q_0(x)$, $q^{(1)}(x)$ and $q^{(2)}(x, \varepsilon)$ are all greater than q^*.

We assume further that the lower part of the spectrum for the problem with $q(x, \eta_0)$ is discrete. Then if

$$\lambda^{(1)} = \int_0^\infty q^{(1)}(x) \{u_0(x)\}^2 dx < \infty$$

we have the first-order approximation and

$$\lambda_\varepsilon = \lambda_0 + \varepsilon \lambda^{(1)} + o(\varepsilon), \quad \varepsilon \to +0,$$

$$u_\varepsilon(x) = u_0(x) + o(\varepsilon^{\frac{1}{2}}), \quad \varepsilon \to +0,$$

uniformly in any finite interval. If, furthermore, we have

$$\int_0^\infty \{q^{(1)}(x) u_0(x)\}^2 dx < \infty,$$

then the second-order approximation is valid and we have

$$\lambda_\varepsilon = \lambda_0 + \varepsilon \lambda^{(1)} + \varepsilon^2 \lambda^{(2)} + o(\varepsilon^2), \quad \varepsilon \to +0,$$

$$u_\varepsilon(x) = u_0(x) + \varepsilon u^{(1)}(x) + o(\varepsilon), \quad \varepsilon \to +0,$$

uniformly in any finite interval.

The difficulty with the second-order approximation is that we are given only the lowest eigenvalue and eigenfunction at η_0, so the coefficients $\lambda^{(2)}$ and $U^{(1)}(X)$ cannot be computed by the usual formulae. We know, however, that the function $U^{(1)}(X)$ satisfies the differential equation

$$\frac{d^2U^{(1)}}{dX^2} + \{\lambda_0 - q_0(X)\} U^{(1)} = -\{\lambda^{(1)} - q^{(1)}(X)\} U_0(X),$$

with the conditions $U^{(1)}(0) = 0$,

$$\int_0^\infty \{U^{(1)}(X)\}^2 dX < \infty .$$

This was verified in Section 7 of this chapter. This is a two-point boundary-value problem. However, once $U^{(1)}(X)$ is computed, then $U_\varepsilon(X)$ can be computed for any ε sufficiently small. We now wish to determine $\lambda^{(2)}$. Consider the two equations

$$U_0'' + \{\lambda_0 - q_0(X)\} U_0 = 0$$

$$U_\varepsilon'' + \{\lambda_\varepsilon - q_1(X)\} U_1 = 0$$

Multiplying the first equation by $U_\varepsilon(X)$ and the second by $U_0(X)$, subtracting the first from the second, and then integrating gives

$$\int_0^\infty U_0(X) U_\varepsilon''(X)\, dX - \int_0^\infty U_0''(X) U_\varepsilon(X)\, dX$$

$$+ (\lambda_\varepsilon - \lambda_0) \int_0^\infty U_0(X)\, U_\varepsilon(X) dX$$

$$= \int_0^\infty \{q_1(X) - q_0(X)\} U_0(X) U_\varepsilon(X) dX .$$

The first two terms cancel by integration by parts and by the vanishing of $U_0(X)$ and $U_\varepsilon(X)$ at 0 and ∞. So we have

$$(\lambda_\varepsilon-\lambda_0)\int_0^\infty U_0(X)U_\varepsilon(X)\,dX = \int_0^\infty \{q_1(X)-q_0(X)\}U_0(X)U_\varepsilon(X)\,dX$$

$$(\varepsilon\lambda^{(1)}+\varepsilon^2\lambda^{(2)}+o(\varepsilon^2))\int_0^\infty\left[\{U_0(X)\}^2+\varepsilon U_0(X)U^{(1)}(X)+o(\varepsilon)\right]dX$$

$$=\int_0^\infty\left[\varepsilon q^{(1)}(X)+\varepsilon^2 q^{(2)}(X,0)+O(\varepsilon^3)\right]\left[\{U_0(X)\}^2+\varepsilon U_0(X)U^{(1)}(X)+o(\varepsilon)\right]dX$$

Now, equating coefficients of like powers of ε and assuming $U_0(X)$ normalized so that

$$\int_0^\infty \{U_0(X)\}^2\,dX = 1$$

we have

$$\lambda^{(1)} = \int_0^\infty q^{(1)}(X)\{U_0(X)\}^2\,dX$$

and

$$\lambda^{(2)} = \int_0^\infty\left[q^{(1)}(X)-\lambda^{(1)}\right]U_0(X)U^{(1)}(X)\,dX+\int_0^\infty q^{(2)}(X,0)\{U_0(X)\}^2\,dX.$$

We thus have a direct method for computing the second-order coefficient without knowledge of the spectral decomposition of the unperturbed operator.

9. <u>An example from the quantum theory of liquids</u>[29]

As an example of the method of the previous section consider the Schrödinger equation for liquid helium (He^4). The one-dimensional equation to be considered is

$$\frac{d^2\psi}{dX^2}+\{E-V(X,\eta)\}\psi=0,\qquad 0<X<1,$$

$$V(X,\eta)=A(\eta)F_1(X)-B(\eta)F_2(X),$$

[29] Example suggested by Dr. Marshall Rosenbluth

$$A(\eta) = 21.8\,\eta^{-\frac{10}{3}} \quad , \eta > 0,$$
$$B(\eta) = 54.5\,\eta^{-\frac{4}{3}} \quad , \eta > 0,$$
$$F_1(X) = \frac{1}{X}\left[\frac{1}{(1-X)^{10}} - \frac{1}{(1+X)^{10}}\right] \quad , \quad 0 < X < 1,$$
$$F_2(X) = \frac{1}{X}\left[\frac{1}{(1-X)^{4}} - \frac{1}{(1+X)^{4}}\right] \quad , \quad 0 < X < 1.$$

We note that $F_1(x)$ and $F_2(x)$ tend to finite limits as $x \to +0$, in fact $F_1(x) \to 20$, $F_2(x) \to 8$ as $x \to +0$, and $F_1(x)$ and $F_2(x)$ tend to ∞ as $x \to 1$. So we have $V(x,\eta) \geq V^*$ for x in (0, 1). We have the boundary condition $\psi(0) = 0$. For some η we wish to find E such that

$$\int_0^1 \{\psi(X, E)\}^2 \, dX < \infty .$$

The physical problem is to calculate certain physically observable quantities by use of the prescribed potential $V(X,\eta)$ in the one-dimensional equation. The variable η is proportional to the density of liquid He^4. The value $\eta_0 = 3.75$ corresponds to normal density. From the quantity $\frac{d^2E}{d\eta^2}$ at $\eta = \eta_0$ we can calculate the speed of sound in liquid He^4. From the previous section we see that this quantity can be computed by second-order perturbation approximation.

By numerical integration or differential analyzer we can find $E(\eta_0)$ and $\psi(X, \eta_0)$ which will be considered as the unperturbed solution. Let $\eta_1 = \eta_0 + \Delta\eta$, then

$$V(x,\eta_1) = V(x,\eta_0) + \Delta\eta \frac{\partial V}{\partial \eta}(x,\eta_0) + \frac{(\Delta\eta)^2}{2} \frac{\partial^2 V}{\partial \eta^2}(x,\eta_0) + \cdots$$

Let

$$\varepsilon = \Delta\eta \quad , \quad V(x,\eta_0) = V_0(x) \quad , \quad \frac{\partial V}{\partial \eta}(x,\eta_0) = V^{(1)}(x),$$

and

$$V^{(2)}(x,\varepsilon) = \frac{1}{2!} \frac{\partial^2 V}{\partial \eta^2}(x,\eta_0) + \frac{\varepsilon}{3!} \frac{\partial^3 V}{\partial \eta^3}(x,\eta_0) + \cdots$$

We are then solving the perturbed equation,

$$\frac{d^2\psi}{dx^2} + \{E - V_0(x) - \varepsilon V^{(1)}(x) - \varepsilon^2 V^{(2)}(x,\varepsilon)\}\psi = 0 .$$

We find that

$$E^{(1)}(\eta_0) = \int_0^1 V^{(1)}(x)\{\psi(x,\eta_0)\}^2 dx < \infty, \quad \int_0^1 V^{(2)}(x,\varepsilon)\{\psi(x,\eta_0)\}^2 dx < \infty,$$

$$\int_0^1 \{V^{(1)}(x)\,\psi(x,\eta_0)\}^2 dx < \infty \quad , \text{ by}$$

numerical quadrature. Consequently, the second-order expansion is valid and

$$E(\eta_1) = E(\eta_0) + \varepsilon\, E^{(1)}(\eta_0) + \varepsilon^2 E^{(2)}(\eta_0) + o(\varepsilon^2) \,, \; \varepsilon \to +0,$$

$$\psi(x,\eta_1) = \psi(x,\eta_0) + \varepsilon\, \psi^{(1)}(x,\eta_0) + o(\varepsilon) \,, \; \varepsilon \longrightarrow +0,$$

uniformly in any finite interval. The function $\psi^{(1)}(x,\eta_0)$ can be computed by integration of the equation

$$\frac{d^2\psi^{(1)}}{dx^2} + \{E(\eta_0) - V(x,\eta_0)\}\,\psi^{(1)} = -\{E^{(1)}(\eta_0) - V^{(1)}(x)\}\psi(x,\eta_0),$$

with

$$\psi^{(1)}(0) = 0 \quad , \quad \int_0^1 \{\psi^{(1)}(x,\eta_0)\}^2 dx < \infty .$$

Then

$$E^{(2)}(\eta_0) = \int_0^1 [V^{(1)}(x) - E^{(1)}(\eta_0)]\, \psi(x, \eta_0)\, \psi^{(1)}(x, \eta_0)\, dx$$
$$+ \int_0^1 V^{(2)}(x, 0)\, \{\psi(x, \eta_0)\}^2\, dx.$$

We have

$$E^{(2)}(\eta_0) = \frac{1}{2} \frac{d^2 E}{d\eta^2}\Big|_{\eta = \eta_0}$$

which can be used to calculate the speed of sound in liquid He^4.

10. Solution of the equation

$$\frac{d^2 u}{dx^2} - q(x)\, u - \varepsilon p(x) u = - f(x)$$

by iterated integral equations.

In this section we digress from the perturbed eigenvalue problem considered in the first two sections of this chapter. We wish to find the solution of the equation

$$\frac{d^2 u}{dx^2} - q(x) u - \varepsilon p(x)\, u = - f(x) \tag{2.20}$$

satisfying the boundary conditions of Eq. (2.2), where q(x) and p(x) are defined as before and f(x) belongs to the class $L_2(0, \infty)$.

Consider first the equation

$$\frac{d^2 u}{dx^2} - q(x) u = - f(x) \tag{2.21}$$

The solution $u_0(x)$ of Eq. (2.21) satisfying the condition (2.2) can be written as the following integral equation,

$$u_0(x) = \int_0^\infty G(x, y)\, f(y)\, dy \tag{2.22}$$

where G(x, y) is the Green's function determined in the usual way from linearly independent solutions of the homogeneous equation

$$\frac{d^2u}{dx^2} - q(x)u = 0 \tag{2.23}$$

that satisfy the boundary conditions (2.2).

Considering this problem as one in the Hilbert space $\mathcal{H} = L_2(0,\infty)$ we write Eq. (2.21) as the following equation in $\mathcal{H}$:

$$Hu = f,$$

where H is the unperturbed self-adjoint operator defined in the previous section. The solution (2.22) is given by

$$u_0 = H^{-1}f.$$

H^{-1} is then the integral operator (2.22) with $G(x, y)$ as the kernel.

Consider now Eq. (2.20). We can rewrite this as

$$\frac{d^2u}{dx^2} - q(x)u = -f(x) + \varepsilon p(x)u(x)$$

In this form we see that the solution $u_\varepsilon(x)$ of Eq. (2.20) satisfying conditions (2.2) can be written as the integral equation

$$u_\varepsilon(x) = \int_0^\infty G(x,y)f(y)\,dy - \varepsilon\int_0^\infty G(x,y)p(y)u_\varepsilon(y)\,dy. \tag{2.24}$$

We see that

$$u_\varepsilon(x) = u_0(x) - \varepsilon\int_0^\infty G(x,y)p(y)u_\varepsilon(y)\,dy.$$

Proceeding formally, we can solve the above integral equation by iteration, i.e.,

$$u_\varepsilon^{(0)}(x) = u_0(x),$$

$$U_\varepsilon^{(1)}(x) = U_0(x) - \varepsilon \int_0^\infty G(x,y)p(y)U_0(y)\,dy,$$

$$U_\varepsilon^{(2)}(x) = U_0(x) - \varepsilon \int_0^\infty G(x,y)p(y)U_\varepsilon^{(1)}(y)\,dy,$$

. . . .

$$U_\varepsilon^{(n)}(x) = U_0(x) - \varepsilon \int_0^\infty G(x,y)p(y)U_\varepsilon^{(n-1)}(y)\,dy,$$

which gives the solution

$$U_\varepsilon(x) = U_0(x) - \sum_{n=1}^{\infty} \varepsilon^n \int_0^\infty K_n(x,y)U_0(y)\,dy,$$

where
$$K_1 = K, \quad K_n(x,y) = \int_0^\infty K(x,z)K_{n-1}(z,y)\,dz, \quad n>1,$$

$$K(x,z) = G(x,z)p(z).$$

Of course this procedure is purely formal, and we have no assurance that the series for $U_\varepsilon(x)$ is convergent in the ordinary sense, since we have not assumed anything about the smallness of p(x). We can, however, consider the series for $U_\varepsilon(x)$ from the point of view of asymptotic perturbation series.

Now, considering the perturbed equation as one in the Hilbert space $\mathcal{H} = L_2(0,\infty)$, we can write Eq. (2.20) as the following equation in $\mathcal{H}$

$$H_\varepsilon U = f$$

where H_ε is the perturbed self-adjoint operator defined in the previous section in Eq. (2.17). The solution is then

$$U_\varepsilon = H_\varepsilon^{-1} f.$$

Therefore the problem is to find the inverse operator H_ε^{-1}.

Theorem 2.5. Let $U_0(X)$ be the solution of the unperturbed Eq. (2.21) satisfying conditions (2.2). If p(x) in Eq. (2.20) satisfies the condition

$$\int_0^\infty p(X)\{U_0(X)\}^2 dX < \infty \ ,$$

then the solution $U_\varepsilon(X)$ of the perturbed Eq. (2.20) satisfying conditions (2.2) can be approximated as

$$U_\varepsilon(X) = U_0(X) + o(\varepsilon^{\frac{1}{2}}) \ , \ \varepsilon \to +0,$$

uniformly over any finite interval.

Proof of Theorem 2.5. In Theorem 1.1 (iii) let u be the function f(x) in $L_2(0,\infty)$ on the right-hand side of Eq. (2.20). Then if $H^{-1}f$ is in $\mathcal{D}[\tilde{J}_\varepsilon]$ we have

$$H_\varepsilon^{-1} f = H^{-1} f + o(\varepsilon^{\frac{1}{2}}) \ , \ \varepsilon \to +0.$$

But we know that $H_\varepsilon^{-1} f = U_\varepsilon$ the solution of Eq. (2.20), and $H^{-1}f = U_0$, the solution of Eq. (2.21), so, if U_0 is in $\mathcal{D}[\tilde{J}_\varepsilon]$ then we have

$$U_\varepsilon = U_0 + o(\varepsilon^{\frac{1}{2}}) \ , \ \varepsilon \to +0.$$

From Theorem 1.4 we have that U_0 in $\mathcal{D}[H^{(1)\frac{1}{2}}]$ implies U_0 in $\mathcal{D}[\tilde{J}_\varepsilon]$, so we must have U_0 in $\mathcal{D}[H^{(1)\frac{1}{2}}]$.

This means $\displaystyle\int_0^\infty p(X)\{U_0(X)\}^2 dX < \infty \ ,$

since p(x) is the perturbing multiplication operator and this is the condition stated in Theorem 2.5. So we have $\| U_\varepsilon - U_0 \| = o(\varepsilon^{\frac{1}{2}})$ as $\varepsilon \longrightarrow +0$. We wish to prove uniform convergence over any finite interval. We use the representation for twice-differentiable functions that was given in Theorem 2.2. Making use of Eq. (2.20) we have

$$U_\varepsilon(x) = \frac{1}{(\Delta x)^2} \int_x^{x+\Delta x} \left[(x+\Delta x - y)^2 (y-x) \{ q(y) - \varepsilon p(y) \} - (6y - 6x - 4\Delta x) \right] U_\varepsilon(y) \, dy$$

$$- \frac{1}{(\Delta x)^2} \int_x^{x+\Delta x} (x + \Delta x - y)^2 (y - x) f(y) \, dy,$$

and using Eq. (2.21), we have

$$U_0(x) = \frac{1}{(\Delta x)^2} \int_x^{x+\Delta x} \left[(x + \Delta x - y)^2 (y-x) q(y) - (6y - 6x - 4\Delta x) \right] U_0(y) \, dy$$

$$- \frac{1}{(\Delta x)^2} \int_x^{x+\Delta x} (x + \Delta x - y)^2 (y - x) f(y) \, dy.$$

Hence $U_\varepsilon(X) - U_0(X)$

$$= \frac{1}{(\Delta X)^2}\int_X^{X+\Delta X}\left[(X+\Delta X-y)^2(y-X)\,q(y)-(6y-6X-4\Delta X)\right]\{U_\varepsilon(y)-U_0(y)\}\,dy$$

$$+\frac{1}{(\Delta X)^2}\int_X^{X+\Delta X}(X+\Delta X-y)^2(y-X)\,\varepsilon\, p(y)\,U_\varepsilon(y)\,dy.$$

By the Schwartz inequality, the square of the first term does not exceed

$$\frac{1}{(\Delta X)^2}\int_X^{X+\Delta X}\left[(X+\Delta X-y)^2(y-X)\,q(y)-(6y-6X-4\Delta X)\right]^2 dy$$

$$\cdot\frac{1}{(\Delta X)^2}\int_X^{X+\Delta X}\{U_\varepsilon(y)-U_0(y)\}^2\,dy$$

$$= O\|U_\varepsilon - U_0\|^2 = o(\varepsilon)$$

So the first term is $o(\varepsilon^{\frac{1}{2}})$ and the second term is clearly $O(\varepsilon)$ and therefore we have

$$U_\varepsilon(X) - U_0(X) = o(\varepsilon^{\frac{1}{2}}) \quad , \quad \varepsilon \longrightarrow +0 ,$$

uniformly over any finite interval. Thus we have established Theorem 2.5.

We can improve this approximation by imposing a further restriction on p(x), as we see in the following theorem.

<u>Theorem 2.6.</u> <u>Let $U_0(X)$ be the solution of the unperturbed Eq. (2.21)</u>

satisfying conditions (2.2). If p(x) in Eq. (2.20) satisfies the condition

$$\int_0^\infty \{p(x)\}^{2\alpha}\{U_0(x)\}^2 dx < \infty \quad , \frac{1}{2} \leq \alpha < 1,$$

then the solution $U_\varepsilon(x)$ of the perturbed Eq. (2.20) satisfying conditions (2.2) can be approximated as

$$U_\varepsilon(x) = U_0(x) + o(\varepsilon^\alpha) \quad , \varepsilon \to +0,$$

uniformly over any finite interval.

Proof of Theorem 2.6. In Theorem 1.2 (iii) let u be the function f(x) in $L_2(0,\infty)$ on the right-hand side of Eq. (2.20). Then if $H^{-1}f$ is in $\mathcal{D}[H'^\alpha]$ we have

$$H_\varepsilon^{-1} f = H^{-1} f + o(\varepsilon^\alpha) \quad , \varepsilon \to +0.$$

But we know that $H_\varepsilon^{-1} f = U_\varepsilon$, the solution of Eq. (2.20), and $H^{-1}f = U_0$ the solution of Eq. (2.21), so if U_0 is in $\mathcal{D}[H'^\alpha]$ then

$$U_\varepsilon = U_0 + o(\varepsilon^\alpha) \quad , \varepsilon \to +0.$$

The condition U_0 in $\mathcal{D}[H'^\alpha]$ means

$$\int_0^\infty \{p(x)\}^{2\alpha}\{U_0(x)\}^2 dx < \infty,$$

which is the condition assumed in Theorem 2.6. So we have

$$\left(\int_0^\infty \{U_\varepsilon(x) - U_0(x)\}^2 dx\right)^{1/2} = o(\varepsilon^\alpha) \quad , \varepsilon \to +0.$$

We derive $$U_\varepsilon(x) - U_0(x) = o(\varepsilon^\alpha) \,, \varepsilon \to +0,$$

uniformly over any finite interval from the above convergence in the mean exactly as in the proof of Theorem 2.5.

By further restricting the function p(x) we can derive the next-order approximation.

Theorem 2.7. Let $u_0(x)$ be the solution of the unperturbed Eq. (2.21) satisfying the conditions (2.2) and let G (x, y) be the Green's function for the homogeneous Eq. (2.23). If p(x) in Eq. (2.20) satisfies the condition

$$\int_0^\infty \{p(x)\,u_0(x)\}^2\,dx < \infty ,$$

then the solution $u_\varepsilon(x)$ of the perturbed Eq. (2.20) satisfying conditions (2.2) can be approximated as

$$u_\varepsilon(x) = u_0(x) - \varepsilon\int_0^\infty G(x,y)\,p(y)\,u_0(y)\,dy + o(\varepsilon), \qquad \varepsilon \to +0,$$

uniformly over any finite interval.

For the proof of Theorem 2.7 we make use of the following lemma, which is a special case of Theorem 1.3

Lemma 2.4. Let $\tilde{\mathfrak{J}} \geq 1$, $\tilde{\mathfrak{J}}' \geq 0$ be closed forms. Let $\mathfrak{D}[\tilde{\mathfrak{J}}] \cap \mathfrak{D}[\tilde{\mathfrak{J}}']$ be dense in $\mathfrak{H}$. Let $\tilde{\mathfrak{J}}_\varepsilon = \tilde{\mathfrak{J}} + \varepsilon\,\tilde{\mathfrak{J}}'$ be a closed form, nonincreasing as $\varepsilon \to +0$. Let H_ε, H', H be the self-adjoint operators belonging to $\tilde{\mathfrak{J}}_\varepsilon$, $\tilde{\mathfrak{J}}'$, $\tilde{\mathfrak{J}}$ respectively. Then if u is such that $H^{-1}u$ is in $\mathfrak{D}[H']$ we have

$$H_\varepsilon^{-1}u = H^{-1}u - \varepsilon\,H^{-1}H'H^{-1}u + o(\varepsilon), \quad \varepsilon \to +0.$$

Proof of Theorem 2.7. In Lemma 2.4 let u be the function f(x) in $L_2(0,\infty)$ on the right-hand side of Eq. (2.20). Then if $H^{-1}f$ is in $\mathcal{D}[H']$ we have

$$H_\varepsilon^{-1}f = H^{-1}f - \varepsilon H^{-1}H'H^{-1}f + o(\varepsilon), \ \varepsilon \to +0.$$

We know that $H_\varepsilon^{-1}f = U_\varepsilon$ the solution of Eq. (2.20), and $H^{-1}f = U_0$ the solution of Eq. (2.21), so if U_0 is in $\mathcal{D}[H']$ then

$$U_\varepsilon = U_0 - \varepsilon H^{-1}H'U_0 + o(\varepsilon), \ \varepsilon \to +0.$$

From Theorem 2.7 we have the condition

$$\int_0^\infty \{p(x)\,U_0(x)\,dx\}^2\,dx < \infty,$$

which implies U_0 in $\mathcal{D}[H']$. We have

$$H^{-1}H'U_0 = \int_0^\infty G(x,y)\,p(y)\,U_0(y)\,dy = U^{(1)}(x).$$

Then we have proved so far that

$$\| U_\varepsilon - U_0 + \varepsilon U^{(1)} \| = o(\varepsilon), \ \varepsilon \to +0.$$

We wish to establish uniform convergence over any finite interval. From the proof of Theorem 2.5 we have

$$U_\varepsilon(x) - U_0(x)$$

$$= \frac{1}{(\Delta x)^2}\int_x^{x+\Delta x}\left[(x+\Delta x-y)^2(y-x)q(y) - (6y-6x-4\Delta x)\right]\{U_\varepsilon(y)-U_0(y)\}\,dy$$

$$+ \frac{1}{(\Delta x)^2}\int_x^{x+\Delta x}(x+\Delta x-y)^2(y-x)\,\varepsilon\,p(y)\,U_\varepsilon(y)\,dy.$$

We have that $u^{(1)} = H^{-1} H' u_0$

The function $u^{(1)}(x)$ is a solution of the differential equation

$$\frac{d^2 u}{dx^2} - q(x)\, u^{(1)} = - p(x) u_0(x)$$

with the boundary conditions (2.2). That is, we have $H u^{(1)} = H' u_0$.
The representation for twice-differentiable functions, which was given in the proof of Theorem 2.5, can be applied to $u^{(1)}(x)$. Hence

$$u^{(1)}(x) = \frac{1}{(\Delta x)^2} \int_x^{x+\Delta x} \left[(x+\Delta x - y)^2 (y-x)\, q(y) - (6y - 6x - 4\Delta x) \right] u^{(1)}(y)\, dy$$

$$- \frac{1}{(\Delta x)^2} \int_x^{x+\Delta x} (x+\Delta x - y)^2 (y-x)\, p(y)\, u_0(y)\, dy.$$

Then $u_\varepsilon(x) - u_0(x) + \varepsilon u^{(1)}(x)$

$$= \frac{1}{(\Delta x)^2} \int_x^{x+\Delta x} \left[(x+\Delta x - y)^2 (y-x)\, q(y) - (6y - 6x - 4\Delta x) \right] \left\{ u_\varepsilon(y) - u_0(y) + \varepsilon u^{(1)}(y) \right\} dy$$

$$+ \frac{1}{(\Delta x)^2} \int_x^{x+\Delta x} (x+\Delta x - y)^2 (y-x)\, \varepsilon p(y) \left\{ u_\varepsilon(y) - u_0(y) \right\} dy.$$

By the Schwartz inequality, the square of the first term is less than

$$O\| u_\varepsilon - u_0 + \varepsilon u^{(1)} \|^2 = o(\varepsilon^2) .$$

So the first term is $o(\varepsilon)$. The second term is $o(\varepsilon)$ since $u_\varepsilon(y) - u_0(y)$ tends to 0 by Theorem 2.5. Thus we have

$$u_\varepsilon(x) = u_0(x) - \varepsilon \int_0^\infty G(x, y) p(y) u_0(y) dy + o(\varepsilon),$$

as $\varepsilon \rightarrow +0$, uniformly over any finite interval.

III.

PERTURBATION THEORY OF PARTIAL DIFFERENTIAL OPERATORS OF SECOND ORDER

1. Statement of the Problem and Results

Consider the Hilbert space, $L_2(E_3)$ --that is, the space of real-valued, measurable functions, $f(x, y, z)$, defined in ordinary three-dimensional space, which are square summable; measure and integration are in the sense of Lebesgue. In $\mathcal{H} = L_2(E_3)$ we have the norm

$$\|f\| = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{f(x,y,z)\}^2 \, dx\, dy\, dz \right)^{1/2}$$

and the inner product

$$(f,g) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\, g(x,y,z)\, dx\, dy\, dz \,,$$

$$f, g \in \mathcal{H}.$$

The operator, H_0, is defined by

$$H_0 u = q(x,y,z)\, u(x,y,z) - \nabla^2 u \tag{3.1}$$

for u in $\mathcal{D}[H_0] \subset \mathcal{H}$. $\mathcal{D}[H_0]$ is the set of functions $u(x, y, z)$ such that u vanishes for $r \leq R_1$ and $r \geq R_2$ where $r = \sqrt{x^2 + y^2 + z^2}$ and R_1, R_2 are positive numbers; u, u_x, u_y, u_z are absolutely continuous in x, y, z and u, $H_0 u$ belong to $\mathcal{H}$. The function $q(x, y, z)$

is real-valued and continuous in x, y, z except possibly at the origin. We have

$$(H_0 u, v) = \iiint [u_x v_x + u_y v_y + u_z v_z + q u^2] \, dx\, dy\, dz$$
$$= (u, H_0 v)$$

for u; v in $\mathcal{D}[H_0]$ and $\mathcal{D}[H_0]$ is dense in $\mathcal{H}$, so H_0 is symmetric.

We assume that the function $q(x, y, z)$ is such that the operator H_0 is lower semibounded, that is,

$$(H_0 u, u) \geq c(u, u) \quad , c > -\infty \tag{3.2}$$

for all u in $\mathcal{D}[H_0]$. An example of a symmetric, semibounded operator is given by the Schrodinger operator for the hydrogen atom,

$$Au = -\nabla^2 u - \frac{a}{r} u,$$

which is defined for functions in the domain $\mathcal{D}[H_0]$ that we are considering. We note that[30]

$$(Au, u) = \iiint [u_x^2 + u_y^2 + u_z^2 - \frac{a}{r} u^2] \, dx\, dy\, dz \geq -2a^2 (u, u).$$

Every symmetric, semibounded operator has a self-adjoint extension, the Friedrichs extension, which preserves the semi-bound.

We take the Friedrich's extension of H_0, denoted H, to be the unperturbed operator. The unperturbed eigenvalue problem is then expressed by the equation

$$Hu = \lambda u$$

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30 Riesz-Sz. Nagy, p 328

for u in $\mathcal{D}[H]$. Then u satisfies the differential equation

$$\nabla^2 u + \{\lambda - q(x,y,z)\} u = 0 . \tag{3.3}$$

The above eigenvalue equation is solvable because H is self-adjoint, so we assume that the complete set of eigenvalues and eigenfunctions is known for the unperturbed problem.

Consider now the operator H' defined by

$$H'u = p(x,y,z)\,u(x,y,z) \tag{3.4}$$

for u in $\mathcal{D}[H'] \subset \mathcal{H}$. $\mathcal{D}[H']$ is the set of functions u in $\mathcal{H}$ such that

$$\iiint \{p(x,y,z)\,u(x,y,z)\}^2 \,dx\,dy\,dz < \infty .$$

The function $p(x,y,z)$ is a real-valued function, continuous in x, y, z. We assume that $p(x,y,z)$ is such that H' is a lower semibounded operator, i.e.,

$$(H'u, u) \geq \gamma\,(u, u) \quad , \quad \gamma > -\infty , \tag{3.5}$$

for all u in $\mathcal{D}[H']$.

Consider now the common domain $\bar{\mathcal{D}} = \mathcal{D}[H] \cap \mathcal{D}[H']$, which is dense in $\mathcal{H}$. Let $\bar{H}'$ be the restriction of H' to $\bar{\mathcal{D}}$. The operator $\bar{H}'$ on $\bar{\mathcal{D}}$ is a symmetric, semibounded operator. The Friedrichs extension of $\bar{H}'$ denoted $\tilde{\bar{H}}'$ coincides with H', i.e.,

$$\tilde{\bar{H}}' = H' \quad , \quad \mathcal{D}[\tilde{\bar{H}}'] = \mathcal{D}[H'] .$$

Now let $\bar{H}$ be the restriction of H , the unperturbed operator, to the common domain $\bar{\mathcal{D}}$. $\bar{H}$ on $\bar{\mathcal{D}}$ is a symmetric, semibounded operator; its Friedrichs extension is denoted by $\tilde{\bar{H}}$. We assume $\bar{H}$ on $\bar{\mathcal{D}}$ to be essentially self-adjoint, then $\tilde{\bar{H}}$ is its unique self-adjoint extension[31] and coincides with H, i.e.

$$\tilde{\bar{H}} = H \quad , \quad \mathcal{D}[\tilde{\bar{H}}] = \mathcal{D}[H].$$

The theory of partial differential operators in Hilbert space has been treated by several authors.[32] In particular, Kato[33] gives the conditions for the Schrödinger operator for N particle systems with Coulomb interaction to be an essentially self-adjoint operator. He also applies these conditions to the helium atom equation.[34]

The operator $\bar{H} + \varepsilon \bar{H}'$ defined on $\bar{\mathcal{D}}$ with $\varepsilon > 0$ is then a symmetric, semibounded operator. We denote the Friedrichs extension of $\bar{H} + \varepsilon \bar{H}'$ by H_ε. Moreover, $\bar{H} + \varepsilon \bar{H}'$ is essentially self-adjoint so H_ε is its unique self-adjoint extension. We take the self-adjoint operator H_ε to be the perturbed operator. The perturbed eigenvalue problem is then expressed by the equation

$$H_\varepsilon u = \lambda u$$

for u in $\mathcal{D}[H_\varepsilon]$. Then u satisfies the differential equation

$$\nabla^2 u + \{\lambda - q(x, y, z) - \varepsilon p(x, y, z)\} u = 0 \tag{3.6}$$

We wish to find the eigenvalues and eigenfunctions of Eq. (3.6) as asymptotic perturbation expansions.

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31 Stone page 51

32 Friedrichs (4), Halperin (1), Kato (3), Murray (1)

33 Kato (3)

34 Kato (4)

Using the methods employed in the preceding chapter, we can derive theorems analogous to Theorems 2.2, 2.3, 2.4.

A complication that arises in partial differential eigenvalue problems is the phenomen of degeneracy. That is, a particular eigenvalue may have more than one eigenfunction associated with it. We shall treat the case of first-order splitting. We assume the lower part of the spectrum of the unperturbed problem is discrete. We take the lowest eigenvalue to have multiplicity m, i.e.,

$$\lambda = \lambda_1 = \lambda_2 = \cdots = \lambda_m < \lambda_{m+1} . \qquad (3.7)$$

The functions $\phi_1(x, y, z), \cdots, \phi_m(x, y, z)$ are associated eigenfunctions. We shall show that the perturbed problem has m discrete eigenvalues $\lambda_{\varepsilon 1} \leq \lambda_{\varepsilon 2} \leq \cdots \leq \lambda_{\varepsilon m}$, which are perturbations of the value λ and we shall derive the first- and second-order approximations to these m eigenvalues.

We assume for the remainder of this chapter that conditions (3.2), (3.5), and (3.7) are satisfied. We now proceed to state the theorem that gives the first-order approximation and make the following assumption. If

$$\phi_1(x, y, z), \cdots, \phi_m(x, y, z)$$

are the eigenfunctions corresponding to λ of the unperturbed problem, we assume

$$\iiint p(x, y, z) \{ \phi_j(x, y, z) \}^2 \, dx \, dy \, dz < \infty \qquad (3.8)$$

for $j = 1, \cdots, m$.

With Condition (3.8) we can state:

Theorem 3.1. Let λ be an eigenvalue of the unperturbed problem multiplicity m; let $\phi_1(x, y, z), \ldots, \phi_m(x, y, z)$ be the corresponding eigenfunctions, that are determined so that

$$(\phi_j, \phi_k) = \delta_{jk},$$

$$\iiint p(x, y, z)\, \phi_j(x, y, z)\, \phi_k(x, y, z)\, dx\, dy\, dz = \lambda_j^{(1)} \delta_{jk}$$

where $\delta_{jk} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$

and $j, k = 1, \ldots, m$.

We take $\lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_m^{(1)}$.

Then $\lambda_{\varepsilon j} = \lambda + \varepsilon \lambda_j^{(1)} + o(\varepsilon),\ \varepsilon \to +0,\ j = 1, \cdots, m.$

Let $\phi_{\varepsilon j}(x, y, z)$ be the corresponding eigenfunctions with

$$\|\phi_{\varepsilon j}\| = 1, \quad (\phi_{\varepsilon j}, \phi_j) \geq 0,$$

then

$$\lim_{\varepsilon \to +0} \{\phi_{\varepsilon j}(x, y, z) - \phi_j(x, y, z)\} = 0$$

uniformly in any finite region.

The first two terms of the eigenvalue expansion agree with the corresponding terms obtained by identification of coefficients in the formal series expansion that is usually assumed. The condition (3.8) guarantees the existence of the first-order coefficient.

The order of approximation can be improved by introducing a further restriction on the function $p(x, y, z)$.

Theorem 3.2. Let all the conditions of Theorem 3.1 be satisfied. In addition assume that

$$\iiint \{p(x, y, z)\}^{2\alpha} \{\phi_j (x, y, z)\}^2 dx\,dy\,dz < \infty \qquad (3.9)$$

for $j = 1, \ldots, m$ where $\frac{1}{2} \leq \alpha < 1$.

Then we have

$$\lambda_{\varepsilon j} = \lambda + \varepsilon \lambda_j^{(1)} + o(\varepsilon^{2\alpha}), \ \varepsilon \to +0, \ j = 1, \cdots, m,$$

and

$$\phi_{\varepsilon j}(x, y, z) = \phi_j (x, y, z) + o(\varepsilon^{2\alpha - 1}), \ \varepsilon \to +0,$$

uniformly in any finite region.

We now proceed to the second-order approximation of the eigenvalue and first-order approximation of the eigenfunction. Again we must introduce a further restriction that the function $p(x, y, z)$ must satisfy. In the statement of the next theorem we assume the unperturbed Eq. (3.3) has pure point spectra. This assumption is not necessary for the proof of the second-order approximation, but is taken for convenience in expressing the coefficients of the higher-order terms in the perturbation series.

Theorem 3.3. Let $\{\lambda_\nu\}$ and $\{\phi_\nu(x, y, z)\}$, $\nu = 1, 2, 3, \cdots$ be the eigenvalues and orthonormal eigenfunctions of the unperturbed problem. Let λ be an eigenvalue of the unperturbed problem with

$$\lambda = \lambda_1 = \cdots = \lambda_m \quad \text{and let } \phi_1(x, y, z), \cdots, \phi_m(x, y, z)$$

be the corresponding eigenfunctions, which are determined so that

$$\iiint p(x, y, z)\, \phi_j(x, y, z)\, \phi_k(x, y, z)\, dx\, dy\, dz = \lambda_j^{(1)} \delta_{jk},$$
$$j, k = 1, \cdots, m.$$

In addition we assume

$$\iiint \{p(x, y, z)\, \phi_j(x, y, z)\}^2\, dx\, dy\, dz < \infty, \tag{3.10}$$
$$j = 1, \cdots, m.$$

Then

$$\lambda_{\varepsilon j} = \lambda + \varepsilon \lambda_j^{(1)} + \varepsilon^2 \lambda_j^{(2)} + o(\varepsilon^2), \quad \varepsilon \to +0,$$
$$j = 1, \cdots, m,$$

where

$$\lambda_j^{(2)} = \sum_{\nu=m+1}^{\infty} \frac{H_{j\nu}'^{\,2}}{\lambda_j - \lambda_\nu},$$

$$H_{j\nu}' = \iiint p(x, y, z)\, \phi_j(x, y, z)\, \phi_\nu(x, y, z)\, dx\, dy\, dz.$$

Let $\phi_{\varepsilon j}(x, y, z)$ be the eigenfunctions corresponding to $\lambda_{\varepsilon j}$ with $\|\phi_{\varepsilon j}\| = 1$ and $(\phi_{\varepsilon j}, \phi_j) \geq 0$, then

$$\phi_{\varepsilon j}(x, y, z) = \phi_j(x, y, z) + \varepsilon \phi_j^{(1)}(x, y, z) + o(\varepsilon), \quad \varepsilon \to +0,$$

uniformly in any finite region, where

$$\phi_j^{(1)}(x, y, z) = \sum_{\nu=m+1}^{\infty} \frac{H'_{j\nu}}{\lambda_j - \lambda_\nu} \phi_\nu(x, y, z)$$

$$+ \sum_{\substack{k < m+1 \\ k \neq j}} \frac{\phi_k(x, y, z)}{\lambda_j^{(1)} - \lambda_k^{(1)}} \sum_{\nu=m+1}^{\infty} \frac{H'_{j\nu} H'_{\nu k}}{\lambda_j - \lambda_\nu}$$

The results of Theorems 3.1 and 3.3 are similar to those of Titchmarsh,[35] but the conditions (3.8) and (3.10) are less restrictive than his conditions. Theorem 3.2 is new.

2. Formulation of the Perturbation Problem in the Theory of Quadratic Forms in Hilbert Space.

In order to establish the theorems stated in the previous section we make use of the theory given in Chapter I; consequently we formulate the perturbation problem in the language of quadratic forms.

Consider the restriction $\bar{H}$ of the self-adjoint, unperturbed operator H to the common domain $\bar{\mathfrak{D}} = \mathfrak{D}[H] \cap \mathfrak{D}[H']$. We noted in the previous section that $\bar{H}$ is a symmetric operator. Now we define the form

$$J[u, v] = (\bar{H}u, v), \qquad \mathfrak{D}[J] = \bar{\mathfrak{D}}.$$

[35] Titchmarsh (3)

Then

$$J[u] = (\bar{H}u, u) \geq c(u,u) \quad , \quad c > -\infty \, ,$$

by condition (3.2) so J is lower semibounded. Furthermore, J is closable, since it is defined by a semibounded, symmetric operator. Denote the closure of J by $\tilde{J}$. We assumed in the previous section that $\bar{H}$ is essentially self-adjoint. The self-adjoint operator associated with $\tilde{J}$ in the sense of Theorem D is then H because an essentially self-adjoint operator has a unique self-adjoint extension.

We define the form $J'[u,v] = (\bar{H}'u, v)$ where $\bar{H}'$ is the restriction of H', defined in Eq. (3.4), to the domain $\bar{\mathcal{D}}$. Then

$$J'[u] = (\bar{H}'u, u) \geq \gamma(u,u) \, , \, \gamma > -\infty \, ,$$

by condition (3.5) so J' is lower semibounded. J' is closable since H' is a lower semibounded, symmetric operator. Denote the closure of J' by $\tilde{J}'$. The self-adjoint operator associated with $\tilde{J}'$ is H', since $\bar{H}'$ is essentially self-adjoint.

We now define the form

$$J_\varepsilon = J + \varepsilon J' \quad , \quad \varepsilon > 0 \quad , \quad \mathcal{D}[J_\varepsilon] = \bar{\mathcal{D}} \, .$$

The form J_ε is lower semibounded, since J and J' are lower semibounded. J_ε is closable by Theorem B and its closure is denoted by $\tilde{J}_\varepsilon$. We have $\tilde{J}_\varepsilon \subset \tilde{J} + \varepsilon \tilde{J}'$. We also have $\tilde{J}_\varepsilon = \tilde{J} + \varepsilon \tilde{J}'$ by Theorem C, because the operator $\bar{H} + \varepsilon \bar{H}'$ is essentially self-adjoint. The self-adjoint operator associated with $\tilde{J}_\varepsilon$ is the Friedrichs extension of $\bar{H} + \varepsilon \bar{H}'$ and is the perturbed operator H_ε defined in the previous section.

As we noted in Section 2.2, we can assume without loss of generality that

$$\tilde{J} \geq 1 , \tilde{J}' \geq 0 , \tilde{J}_\varepsilon \geq 1 .$$

We now have the perturbation problem formulated as a problem in the theory of quadratic forms, that is, to find the spectral properties of a self-adjoint operator H_ε associated with a quadratic form $\tilde{J}_\varepsilon$ where

$$\tilde{J}_\varepsilon = \tilde{J} + \varepsilon \tilde{J}' \quad , \varepsilon > 0 ,$$

3. First-Order Approximation; Proof of Theorem 3.1

In the previous section we formulated the perturbation problem in the language of semibounded, closed forms. In particular we have

$$\tilde{J}_\varepsilon = \tilde{J} + \varepsilon \tilde{J}' \quad ,$$

With this formulation we can make use of the asymptotic perturbation theory of such forms.

We first establish the fact that for the m eigenvalues of the unperturbed problem $\lambda_1 = \cdots = \lambda_m$ there correspond exactly m eigenvalues of the perturbed problem $\lambda_{\varepsilon 1} \leq \lambda_{\varepsilon 2} \leq \cdots \leq \lambda_{\varepsilon m}$. To do this we wish to make use of Theorem H given in the first chapter.

We consider the form $\tilde{J}_\varepsilon$ defined for the real variable $\varepsilon > 0$. Considering $D[\tilde{J}_\varepsilon]$ as a set defined for $\varepsilon > 0$, we note that

$$D[\tilde{J}_\varepsilon] = D[\tilde{J}] \cap D[\tilde{J}']$$

and is independent of ε. We show that $\tilde{J}_\varepsilon$ is nondecreasing in ε by the following argument. Consider the forms

$$J_1[u] = \iiint [u_x^2 + u_y^2 + u_z^2 + q u^2 + \varepsilon_1 \rho u^2] \, dx\,dy\,dz ,$$

$$J_2[U]=\iiint[U_x^2+U_y^2+U_z^2+qU^2+\varepsilon_2 pU^2]\,dx\,dy\,dz.$$

We have for $\varepsilon_2 \geq \varepsilon_1 > 0$ that $J_2[U] \geq J_1[U]$ for u in $\mathcal{D}[J_1] \cap \mathcal{D}[J_2]$.

We now can apply Theorem H, using a continuous parameter ε instead of discrete index n, because we have shown that $\tilde{J}_\varepsilon$ is nonincreasing as $\varepsilon \to +0$. We thus have exactly m perturbed eigenvalues of H_ε corresponding to the m eigenvalues of H, and these m perturbed eigenvalues converge to the unperturbed eigenvalues as $\varepsilon \to +0$.

We shall now derive the first-order approximation to these m eigenvalues.

<u>Proof of Theorem 3.1</u> We make use of the following lemma from the perturbation theory of quadratic forms:

<u>Lemma 3.1 (Kato[36])</u>. Let $\tilde{J} \geq 1$, $\tilde{J}' \geq 0$ be closed forms with $\mathcal{D}[\tilde{J}] \cap \mathcal{D}[\tilde{J}']$ dense in $\mathcal{H}$. Let $\tilde{J}_\varepsilon \subset \tilde{J} + \varepsilon \tilde{J}'$, $\varepsilon > 0$, be a closed form nonincreasing as $\varepsilon \to +0$. Let H_ε, H, H' be the self-adjoint operators associated with $\tilde{J}_\varepsilon, \tilde{J}, \tilde{J}'$ respectively. Let λ be an eigenvalue of H with multiplicity m. Let $\phi_1, \dots, \phi_m$ the eigenspace of H corresponding to λ, be a subset of $\mathcal{D}[\tilde{J}_{\varepsilon_1}]$, $\varepsilon_1 > 0$. Let $\phi_1, \dots, \phi_m$ be determined so that $(\phi_j, \phi_k) = \delta_{jk}$ and $(H'^{\frac{1}{2}}\phi_j, H'^{\frac{1}{2}}\phi_k) = \lambda_j^{(1)}\delta_{jk}$ and let $\lambda_1^{(1)} < \lambda_2^{(1)} < \dots < \lambda_m^{(1)}$.

[36] Kato (1) Theorem 19.3

Then

$$\lambda_{\varepsilon j} = \lambda + \varepsilon \lambda_j^{(1)} + o(\varepsilon) \quad , \ \varepsilon \to +0 ,$$

$$\| \phi_{\varepsilon j} - \phi_j \| = o(1) \quad , \ \varepsilon \longrightarrow +0 ,$$

for $j = 1, \cdots , m$ where $\phi_{\varepsilon j}$ is the eigenfunction corresponding to $\lambda_{\varepsilon j}$ and $\| \phi_{\varepsilon j} \| = 1$, $(\phi_{\varepsilon j}, \phi_j) \geq 0$.

From condition (3.8) of Theorem 3.1 we have

$$\iiint p(x, y, z) \{ \phi_j (x, y, z) \}^2 dx dy dz < \infty$$

which means ϕ_j is in $\mathcal{D}[H'^{\frac{1}{2}}]$ so $\lambda^{-1} \phi_j = H^{-1} \phi_j$ is in $\mathcal{D}[H'^{\frac{1}{2}}]$ and thus in $\mathcal{D}[\tilde{J}_{\varepsilon_1}]$ by Theorem 1.4. From this we get that ϕ_j in $\mathcal{D}[\tilde{J}_\varepsilon]$.

The other conditions of Lemma 3.1 are fulfilled in Theorem 3.1, so we have the first-order expansion of the eigenvalue given by

$$\lambda_{\varepsilon j} = \lambda + \varepsilon \lambda_j^{(1)} + o(\varepsilon) \quad , \ \varepsilon \longrightarrow +0 ,$$

where

$$\lambda_j^{(1)} = \| H'^{\frac{1}{2}} \phi_j \|^2 = \iiint p(x, y, z) \{ \phi_j (x, y, z) \}^2 dx dy dz$$

From $\| \phi_{\varepsilon j} - \phi_j \| = o(1)$, $\varepsilon \to +0$ we wish to establish

$$\phi_{\varepsilon j} (x, y, z) - \phi_j (x, y, z) = o(1) \quad , \ \varepsilon \to +0 ,$$

uniformly in any finite region. From Courant-Hilbert[37] we have for $U(x, y, z)$ which satisfies $\nabla^2 U = -4 \pi \mu$, the following mean-value formula:

[37] Courant-Hilbert (1) page 250 (5)

$$u(x_0, y_0, z_0) = \frac{3}{4\pi R^3}\iiint\limits_{r\le R} u(r,\theta,\phi)\, r^2 \sin\theta \, dr\, d\theta\, d\phi$$
$$+ \iiint\limits_{r\le R} \frac{(R-r)^2(2R+r)}{r}\mu\, r^2 \sin\theta\, dr\, d\theta\, d\phi \tag{3.11}$$

which gives the value of u at the center (x_0, y_0, z_0) of a finite sphere with radius R . From the above formula and Eq. (3.6) we have

$$\phi_{\varepsilon j}(x_0, y_0, z_0) = \frac{3}{4\pi R^3}\iiint\limits_{r\le R} \phi_{\varepsilon j}(r,\theta,\phi)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$-\frac{1}{8\pi R^3}\iiint\limits_{r\le R} (R-r)^2(2R+r)\{q+\varepsilon p - \lambda_{\varepsilon j}\}\phi_{\varepsilon j}\, r \sin\theta\, dr\, d\theta\, d\phi$$

and from (3.3) we have

$$\phi_j(x_0, y_0, z_0) = \frac{3}{4\pi R^3}\iiint\limits_{r\le R} \phi_j(r,\theta,\phi)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$-\frac{1}{8\pi R^3}\iiint\limits_{r\le R} (R-r)^2(2R+r)\{q-\lambda_j\}\phi_j(r,\theta,\phi)\, r \sin\theta\, dr\, d\theta\, d\phi$$

$$\phi_{\varepsilon j}(x_0, y_0, z_0) - \phi_j(x_0, y_0, z_0)$$

$$= \frac{3}{4\pi R^3}\iiint\limits_{r\le R} \{\phi_{\varepsilon j}(r,\theta,\phi) - \phi_j(r,\theta,\phi)\}\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$-\frac{1}{8\pi R^3}\iiint_{r\le R}(R-r)^2(2R+r)\{q-\lambda_j\}\{\phi_{\varepsilon j}-\phi_j\}r\sin\theta\, dr\, d\theta\, d\phi$$

$$-\frac{1}{8\pi R^3}\iiint_{r\le R}(R-r)^2(2R+r)\{\varepsilon p-(\lambda_{\varepsilon j}-\lambda_j)\}\phi_{\varepsilon j}\, r\sin\theta\, dr\, d\theta\, d\phi .$$

By the Schwartz inequality the square of the first term and the square of the second term $\le O\|\phi_{\varepsilon j}-\phi_j\|^2 = o(1)$. So the first two terms are $o(1)$. The third term is $O(\varepsilon)$ since εp and $\lambda_{\varepsilon j}-\lambda_j$ are $O(\varepsilon)$. The above holds for any point (x_0, y_0, z_0) so we have $\phi_{\varepsilon j}(x, y, z) - \phi_j(x, y, z) = o(1)$, $\varepsilon \to +0$ uniformly in any finite region. We thus have established Theorem 3.1, giving the first-order approximation to the perturbed eigenvalues and the zero-order approximation to the perturbed eigenfunction.

4. Improved First-Order Approximation; Proof of Theorem 3.2.

We now improve the approximations derived in the previous section by imposing further restrictions on the function $p(x, y, z)$.

For the proof of Theorem 3.2 we make use of some additional lemmas for the estimating of eigenvalues and eigenvectors in Hilbert space.

Lemma C (Kato[38]). Let $\mathcal{H}_m$ be a finite dimensional Hilbert space with dimension m . Let H be a self-adjoint operator in $\mathcal{H}_m$ and let $w_1, \ldots, w_m$ be m vectors of $\mathcal{H}_m$ with $\|w_j\| = 1$.

Let

$$\eta_j = (Hw_j, w_j) \quad , \theta_j = \|(H-\eta_j)w_j\| \quad , \quad j = 1, \cdots, m.$$

[38] Kato (1) Ex. 18.1

We assume $\eta_1 < \eta_2 < \cdots < \eta_m$. Let

$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$ be the eigenvalues of H.

Then

$$\xi_j \leq \lambda_j \leq \mathcal{J}_j ,$$

where

$$\xi_j = \eta_j - (\xi_{j+1} - \eta_j)^{-1} \theta_j^2 ,$$
$$\mathcal{J}_j = \eta_j + (\eta_j - \mathcal{J}_{j-1})^{-1} \theta_j^2 ,$$

provided θ_j are sufficiently small.

<u>Lemma D (Kato[39])</u>. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Let H be a self-adjoint operator in $\mathcal{H}$. Let m vectors $w_1, \ldots, w_m$ in $\mathcal{D}[H]$ be such that

$$(Hw_j, w_k) = \eta_j \delta_{jk} , \quad (w_j, w_k) = \delta_{jk} .$$

We assume $\eta_1 \leq \eta_2 \leq \cdots \leq \eta_m$. Let

$\theta = \|(H - \eta_j) w_j\|$. Let α be a number such that $\alpha < \eta_1$ and let $\lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of H which are larger than α. Then

$$\lambda_k \leq \max(\eta_k, \rho_k) , \quad k = 1, \ldots, m,$$

[39] Kato (1) Lemma 18.3

where ρ_k is the largest root of the equation

$$\sum_{j=1}^{k} \frac{\theta_j^2}{(\eta_j-\alpha)(\rho-\eta_j)} = 1 .$$

In particular

$$\rho_k \le \eta_k + \sum_{j=1}^{k} \frac{\theta_j^2}{\eta_j - \alpha} .$$

Similarly let $\beta > \eta_m$ and let $\cdots \le \lambda'_{m-1} \le \lambda'_m$ be eigen-values of H smaller than β. Then

$$\lambda'_k \ge \min(\eta_k, \sigma_k) \quad , \quad k = 1, \cdots, m,$$

where σ_k is the smallest root of the equation

$$\sum_{j=k}^{m} \frac{\theta_j^2}{(\beta-\eta_j)(\eta_j-\sigma)} = 1 .$$

In particular we have

$$\sigma_k \ge \eta_k - \sum_{j=k}^{m} \frac{\theta_j^2}{\beta - \eta_j} .$$

<u>Lemma E (Kato[40])</u>. Let w_j, η_j, θ_j be as in Lemma D. Let (α, β) be an interval containing at most m eigenvalues of H, but no other points of the spectrum of H. If $\alpha < \eta_1$, $\beta > \eta_m$ and

$$\sum_{j=1}^{m} \frac{\theta_j^2}{(\eta_j-\alpha)(\beta-\eta_j)} < 1 ,$$

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[40] Kato (1) Lemma 18.4

then there are exactly m eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$ in (α, β) and

$$\min(\eta_k, \sigma_k) \leq \lambda_k \leq \max(\eta_k, \rho_k),$$

$k = 1, \ldots, m,$ with ρ_k, σ_k as defined in Lemma D. Let $\phi_1, \ldots, \phi_m$ be the eigenvectors of H belonging to $\lambda_1, \ldots, \lambda_m$ respectively, such that

$$(\phi_j, \phi_k) = \delta_{jk}.$$

Let

$$\delta_j = \min(\eta_j - \alpha, \beta - \eta_j)$$

$$\tau_j = \min_{k \neq j} |\eta_j - \lambda_k|$$

Then

$$1 - |(\phi_j, w_j)|^2 \leq \frac{\theta_j^2}{\delta_j^2} + \frac{\theta_j^2}{\tau_j^2}\left(\frac{\theta_1^2}{\delta_1^2} + \cdots + \frac{\theta_m^2}{\delta_m^2}\right),$$

for $j = 1, \ldots, m$.

<u>Proof of Theorem 3.2.</u> From condition (3.9) of Theorem 3.2 we have

$$\iiint \{p(x,y,z)\}^{2\alpha} \{\phi_j(x,y,z)\}^2 \, dx\, dy\, dz < \infty$$

for $j = 1, \ldots, m, \; \frac{1}{2} \leq \alpha < 1,$ which means ϕ_j is in $\mathcal{D}[H'^{\alpha}]$. Also $\lambda^{-1}\phi_j$ and thus $H^{-1}\phi_j$ are in $\mathcal{D}[H'^{\alpha}]$ and hence in $\mathcal{D}[H'^{\alpha}] \cap \mathcal{D}[\hat{J}_{\varepsilon_1}]$ by Theorem 1.4. We can then use the asymptotic expansions of $H_\varepsilon^{-1}\phi_j$ and $(H_\varepsilon^{-1}\phi_j, \phi_j)$ given by Theorem 1.2(iii) and (iv). In particular

$$H_\varepsilon^{-1}\phi_j = H^{-1}\phi_j + o(\varepsilon^\alpha) = \lambda^{-1}\phi_j + o(\varepsilon^\alpha),$$

$$\begin{aligned}(H_\varepsilon^{-1}\phi_j, \phi_k) &= (H^{-1}\phi_j, \phi_k) - \varepsilon(H'^{\frac{1}{2}}H^{-1}\phi_j, H'^{\frac{1}{2}}H^{-1}\phi_k) + o(\varepsilon^{2\alpha}) \\ &= \lambda^{-1}\delta_{jk} - \varepsilon\lambda^{-2}(H'^{\frac{1}{2}}\phi_j, H'^{\frac{1}{2}}\phi_k) + o(\varepsilon^{2\alpha}) \\ &= (\lambda^{-1} - \varepsilon\lambda^{-2}\lambda_j^{(1)})\delta_{jk} + o(\varepsilon^{2\alpha}).\end{aligned}$$

We wish to use Lemma E; however, we cannot take the ϕ_j to be the trial vectors because $(H_\varepsilon^{-1}\phi_j, \phi_k)$ is not a diagonal matrix, as is seen above. The proof is divided into two stages.

1. In the first stage we diagonalize the matrix $(H_\varepsilon^{-1}\phi_j, \phi_k)$ by diagonalizing the operator $EH_\varepsilon^{-1}E = K$, where $E = P\{\phi_1, \cdots, \phi_m\}$. K is a self-adjoint operator on an m-dimensional Hilbert space. We apply Lemma C to $-K$ with trial vectors $\phi_1, \cdots, \phi_m$.

We set $\eta_j = (-K\phi_j, \phi_j) = -(H_\varepsilon^{-1}\phi_j, \phi_j)$

$$= -\lambda^{-1} + \varepsilon\lambda^{-2}\lambda_j^{(1)} + o(\varepsilon^{2\alpha}),$$

$$\theta_j^2 = \|(-K - \eta_j)\phi_j\|^2$$

$$\leq \|(-K + \lambda^{-1} - \varepsilon\lambda^{-2}\lambda_j^{(1)})\phi_j\|^2.$$

We have

$$\|(-K + \lambda^{-1} - \varepsilon\lambda^{-2}\lambda_j^{(1)})\phi_j\|^2 = \sum_{k=1}^{m} |(\{-K + \lambda^{-1} - \varepsilon\lambda^{-2}\lambda_j^{(1)}\}\phi_j, \phi_k)|^2$$

by the Riesz-Fischer theorem because $\phi_1, \cdots, \phi_m$ is a complete orthonormal set in $E\mathfrak{H}$. Moreover

$$\sum_{k=1}^{m} |(\{-K + \lambda^{-1} - \varepsilon\lambda^{-2}\lambda_j^{(1)}\}\phi_j, \phi_k)|^2 = o(\varepsilon^{4\alpha}),$$

so $\theta_j = o(\varepsilon^{2\alpha})$. Consequently, ξ_j and $\mathcal{J}_j$ in Lemma C are defined and

$$\xi_j = \eta_j + o(\varepsilon^{4\alpha-1}),$$

$$\mathcal{J}_j = \eta_j + o(\varepsilon^{4\alpha-1}).$$

We prove this by induction. $\mathcal{J}_1 = \eta_1$, and assume

$$\mathcal{J}_j = \eta_j + o(\varepsilon^{4\alpha-1}).$$

Then

$$\zeta_{j+1} = \eta_{j+1} + \frac{\theta_{j+1}^2}{\eta_{j+1} - \zeta_j} = \eta_{j+1} + \frac{o(\varepsilon^{4\alpha})}{\varepsilon\lambda^{-2}(\lambda_{j+1}^{(1)} - \lambda_j^{(1)}) + o(\varepsilon^{2\alpha})}$$

$$= \eta_{j+1} + o(\varepsilon^{4\alpha-1}).$$

Similarly for ξ_j. If $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_m$ are the eigenvalues of $-K$, then

$$\xi_j \leq \mu_j \leq \zeta_j \qquad , \; j = 1, \cdots, m \; ;$$

therefore,

$$\mu_j = \eta_j + o(\varepsilon^{4\alpha-1}) = -\lambda^{-1} + \varepsilon\lambda^{-2}\lambda_j^{(1)} + o(\varepsilon^{2\alpha}).$$

Let ψ_j be the eigenvectors of $-K$ corresponding to μ_j such that

$(\psi_j, \psi_k) = \delta_{jk}$. Using Lemma B of the preceding chapter we have

$$\| \psi_j - \phi_j \| \leq \frac{\frac{\theta_j}{\delta_j}}{\left(1 - \left(\frac{\theta_j}{\delta_j}\right)^2\right)^{1/4}}$$

$$(\psi_j, \phi_j) \geq 0,$$

$$\delta_j = \min\,(\eta_j - \zeta_{j-1}, \xi_{j+1} - \eta_j) = O(\varepsilon).$$

Since $\theta_j = o(\varepsilon^{2\alpha})$ we have

$$\| \psi_j - \phi_j \| = o(\varepsilon^{2\alpha-1}).$$

2. In the second stage we apply Lemma E to $-H_\varepsilon^{-1}$, using $\psi_1, \dots, \psi_m$ as trial vectors. We have

$$(-H_\varepsilon^{-1}\psi_j, \psi_k) = (-K\psi_j, \psi_k) = \mu_j \delta_{jk}$$

and $(\psi_j, \psi_k) = \delta_{jk}$, so we can apply E.

In E let $\eta_j = \mu_j$ and

$$\theta_j = \|(-H_\varepsilon^{-1} - \eta_j)\psi_j\| = \|(-H_\varepsilon^{-1} - \mu_j)\psi_j\|.$$

We note that

$$E(-H_\varepsilon^{-1} - \mu_j)\psi_j = (-K - \mu_j)\psi_j = 0.$$

Hence

$$(-H_\varepsilon^{-1} - \mu_j)\psi_j = (I - E)(-H_\varepsilon^{-1} - \mu_j)\psi_j$$

$$= (I - E)(-H_\varepsilon^{-1})E\psi_j;$$

$$\sum_{j=1}^{m} \theta_j^2 = \sum_{j=1}^{m} \|(-H_\varepsilon^{-1} - \mu_j)\psi_j\|^2$$

$$= \sum_{j=1}^{m} \|(I - E)(-H_\varepsilon^{-1})E\psi_j\|^2$$

$$= \sum_{j=1}^{m} \|(I - E)(-H_\varepsilon^{-1})E\phi_j\|^2,$$

because the expression $\sum_j \| T \psi_j \|^2$, the trace of T, for T, a bounded operator, is independent of the orthonormal system. Then

$$\sum_{j=1}^{m} \theta_j^2 = \sum_{j=1}^{m} \| (I - E)(-H_\varepsilon^{-1}) \phi_j \|^2$$

$$= \sum_{j=1}^{m} \| (I - E)(-H_\varepsilon^{-1} + \lambda^{-1}) \phi_j \|^2$$

$$\leq \sum_{j=1}^{m} \| (H_\varepsilon^{-1} - \lambda^{-1}) \phi_j \|^2 = o(\varepsilon^{2\alpha}),$$

by Theorem 1.2 (iii). Thus we have $\theta_j = o(\varepsilon^\alpha)$ and we can apply Lemma E. We can choose $(\alpha', \beta') = (-\beta, -\alpha)$ in such a way that $\eta_j - \alpha'$, $\beta' - \eta_j$ are all > 0. We see by Lemma D that $\rho_k \leq \eta_k + o(\varepsilon^{2\alpha})$ and

$\sigma_k \geq \eta_k + o(\varepsilon^{2\alpha})$. Thus we have by Lemma E

$$-\lambda_{\varepsilon k}^{-1} = -\lambda + \varepsilon \lambda^{-2} \lambda_k^{(1)} + o(\varepsilon^{2\alpha}),$$

hence

$$\lambda_{\varepsilon k} = \lambda + \varepsilon \lambda_k^{(1)} + o(\varepsilon^{2\alpha}),$$

which is the required result.

Also by Lemma E we estimate $1 - |(\phi_{\varepsilon j}, \psi_j)|^2$. We have $\delta_j > 0$;

$$\tau_j = \varepsilon \min \left(| \lambda_j^{(1)} - \lambda_{j-1}^{(1)} |, | \lambda_j^{(1)} - \lambda_{j+1}^{(1)} | \right) + o(\varepsilon^{2\alpha})$$

is of the order $O(\varepsilon)$. Hence $1 - |(\phi_{\varepsilon j}, \psi_j)|^2 = o(\varepsilon^{2\alpha - 1})$, and $\| \phi_{\varepsilon_j} - \psi_j \| = o(\varepsilon^{2\alpha-1})$ if

$(\phi_{\varepsilon_j}, \psi_j) \geq 0$. We already have

$$\| \psi_j - \phi_j \| = o(\varepsilon^{2\alpha-1}).$$ Hence

$$\| \phi_{\varepsilon_j} - \phi_j \| = o(\varepsilon^{2\alpha-1}) \quad \text{when} \quad (\phi_{\varepsilon_j}, \phi_j) \geq 0.$$

We derive the uniform convergence over any finite region of

$$\phi_{\varepsilon_j}(x, y, z) - \phi_j(x, y, z) = o(\varepsilon^{2\alpha-1}), \quad \varepsilon \to +0,$$

from

$$\| \phi_{\varepsilon_j} - \phi_j \| = o(\varepsilon^{2\alpha-1}),$$

as in the proof of Theorem 3.1. We thus have established Theorem 3.2.

5. Second-Order Approximation; Proof of Theorem 3.3

We now derive the next order of approximation for eigenvalues and eigenfunctions by imposing a further restriction on the function $p(x, y, z)$.

Proof of Theorem 3.3. Let $\{E(\lambda')\}$ be the resolution of the identity for the self-adjoint operator H. We define the operator S,

$$S = \int' \frac{1}{\lambda' - \lambda} \, dE(\lambda')$$

where $\int'$ means integration except for the point $\lambda' = \lambda (= \lambda_1 = \cdots = \lambda_m)$, and eigenvalue of H. S is the reduced resolvent of H and is a bounded, self-adjoint operator. We next define the operators

$$S_j' = \sum_{k \neq j} \frac{1}{\lambda_k^{(1)} - \lambda_j^{(1)}} P\{\phi_k\}, \quad j = 1, \cdots, m,$$

where $P\{\phi_k\}$ is the projection on the one-dimensional subspace determined by ϕ_k. The S_j' are bounded, self-adjoint operators. We state now a lemma from the perturbation theory of quadratic forms.

Lemma 3.2 (Kato[41]). Let the conditions of Lemma 3.1 be satisfied. Let the eigenvectors $\phi_1, \ldots, \phi_m$ be in $\mathcal{D}[\tilde{J}_{\varepsilon_1}] \cap \mathcal{D}[H']$ for some $\varepsilon_1 > 0$

Then

$$\lambda_{\varepsilon j} = \lambda + \varepsilon \lambda_j^{(1)} + \varepsilon^2 \lambda_j^{(2)} + o(\varepsilon^2) \quad , \varepsilon \to +0,$$

$$\| \phi_{\varepsilon j} - \phi_j - \varepsilon \phi_j^{(1)} \| = o(\varepsilon) , \ \varepsilon \to +0 \quad , j = 1, \ldots, m,$$

where

$$\lambda_j^{(2)} = -(S H' \phi_j , H' \phi_j),$$

$$\phi_j^{(1)} = - S H' \phi_j + (H' S_j')^* S H' \phi_j .$$

From condition (3.10) of Theorem 3.3 we have

$$\iiint \{p(x,y,z)\, \phi_j(x,y,z)\}^2 dx\, dy\, dz < \infty ,$$

$j = 1, \ldots, m$ so ϕ_j is in $\mathcal{D}[H']$ and ϕ_j is in $\mathcal{D}[\tilde{J}_{\varepsilon_1}] \cap \mathcal{D}[H']$ by Theorem 1.4.

We then can apply Lemma 3.2. The spectral representation of H is

$$Hf = \sum_{\nu=1}^{\infty} (f, \phi_\nu)\, \phi_\nu \quad , \qquad f \text{ in } \mathcal{D}[H],$$

since we have assumed discrete spectra. So we have

[41] Kato (1) Theorem 20.2

$$-SH'\phi_j = \sum_{\nu=m+1}^{\infty} \frac{H'_{j\nu}}{\lambda_j - \lambda_\nu} \phi_\nu \quad ,$$ where

$$H'_{j\nu} = (H'\phi_j, \phi_\nu) \quad ,$$ hence

$$\lambda_j^{(2)} = \sum_{\nu=m+1}^{\infty} \frac{H'^2_{j\nu}}{\lambda_j - \lambda_\nu} \quad .$$ Also we calculate

$$(H'S'_j)^* SH'\phi_j = S'_j H'SH'\phi_j$$

$$= \sum_{\substack{k<m+1 \\ k\neq j}} \frac{\phi_k}{\lambda_j^{(1)} - \lambda_k^{(1)}} \sum_{\nu=m+1}^{\infty} \frac{H'_{j\nu} H'_{\nu k}}{\lambda_j - \lambda_\nu} \quad .$$

So we have

$$\phi_j^{(1)}(x,y,z) = \sum_{\nu=m+1}^{\infty} \frac{H'_{j\nu}}{\lambda_j - \lambda_\nu} \phi_\nu(x,y,z)$$

$$+ \sum_{\substack{k<m+1 \\ k\neq j}} \frac{\phi_k(x,y,z)}{\lambda_j^{(1)} - \lambda_k^{(1)}} \sum_{\nu=m+1}^{\infty} \frac{H'_{j\nu} H'_{\nu k}}{\lambda_j - \lambda_\nu}$$

We shall now establish the uniform convergence of

$$\phi_{\varepsilon j}(x,y,z) = \phi_j(x,y,z) + \varepsilon\, \phi_j^{(1)}(x,y,z) + o(\varepsilon),$$
$$\varepsilon \longrightarrow +0 .$$

The function $\phi_j^{(1)}(x, y, z)$ satisfies the differential equation

$$\nabla^2 \phi_j^{(1)} + \{\lambda_j - q(x, y, z)\} \phi_j^{(1)} = -\{\lambda_j^{(1)} - p(x, y, z)\} \phi_j$$

as in the previous chapter. Consequently, using the mean-value formula (3.11), we can represent $\phi_j^{(1)}(x, y, z)$

$$\phi_j^{(1)}(x_0, y_0, z_0) = \frac{3}{4\pi R^3} \iiint\limits_{r \le R} \phi_j^{(1)}(r, \theta, \phi)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$- \frac{1}{8\pi R^3} \iiint\limits_{r \le R} (R - r)^2 (2R + r) \{q - \lambda_j\} \phi_j^{(1)}\, r \sin\theta\, dr\, d\theta\, d\phi$$

$$- \frac{1}{8\pi R^3} \iiint\limits_{r \le R} (R - r)^2 (2R + r) \{p - \lambda_j^{(1)}\} \phi_j\, r \sin\theta\, dr\, d\theta\, d\phi.$$

Using the above representation and the expression for

$$\phi_{\varepsilon j}(x_0, y_0, z_0) - \phi_j(x_0, y_0, z_0)$$

from the proof of Theorem 3.1, we have

$$\phi_{\varepsilon j}(x_0, y_0, z_0) - \phi_j(x_0, y_0, z_0) - \varepsilon \phi_j^{(1)}(x_0, y_0, z_0)$$

$$= \frac{3}{4\pi R^3} \iiint\limits_{r \le R} \{\phi_{\varepsilon j} - \phi_j - \varepsilon \phi_j^{(1)}\} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$-\frac{1}{8\pi R^3}\iiint\limits_{r\leq R}(R-r)^2(2R+r)\{q-\lambda_j\}\{\phi_{\varepsilon j}-\phi_j-\varepsilon\phi_j^{(1)}\}r\sin\theta\, dr\, d\theta\, d\phi$$

$$-\frac{1}{8\pi R^3}\iiint\limits_{r\leq R}(R-r)^2(2R+r)\{\varepsilon p-(\lambda_{\varepsilon j}-\lambda_j)\}\{\phi_{\varepsilon j}-\phi_j\}r\sin\theta\, dr\, d\theta\, d\phi$$

$$-\frac{1}{8\pi R^3}\iiint\limits_{r\leq R}(R-r)^2(2R+r)(\lambda_{\varepsilon j}-\lambda_j-\varepsilon\lambda_j^{(1)})\,\phi_j\, r\sin\theta\, dr\, d\theta\, d\phi.$$

By the Schwartz inequality the squares of the first two terms are each less than $O\|\phi_{\varepsilon j}-\phi_j-\varepsilon\phi_j^{(1)}\|^2=o(\varepsilon^2)$,

so the first two terms are $o(\varepsilon)$. The third term is $O(\varepsilon)\{\phi_{\varepsilon j}-\phi_j\}=o(\varepsilon)$ because $\{\phi_{\varepsilon j}-\phi_j\}\longrightarrow 0$ as $\varepsilon\longrightarrow +0$.

The fourth term is $O(\varepsilon^2)$ since

$$\lambda_{\varepsilon j}-\lambda_j-\varepsilon\lambda_j^{(1)}=\varepsilon^2\lambda_2^{(2)}+o(\varepsilon^2).$$

Hence, we have

$$\phi_{\varepsilon j}(x,y,z)-\phi_j(x,y,z)-\varepsilon\,\phi_j^{(1)}(x,y,z)=o(\varepsilon),$$

$\varepsilon\longrightarrow +0$ uniformly over any finite region, which completes the proof of Theorem 3.3.

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