Title
PARTICLE ACCELERATION AND PHASE STABILITY IN A LINEAR ACCELERATOR - LECTURE X

Authors
Brobeck, W.M.
Mirk, K.
DuBois, A.

Publication Date
1953-03-04

UNIVERSITY OF CALIFORNIA
Radiation Laboratory
Berkeley, California

UCRL 2177

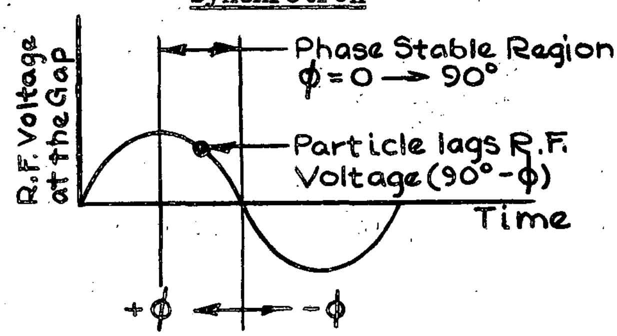
Particle Acceleration and Phase Stability in a Linear Accelerator

LECTURE X
March 4, 11, and 18, 1953

W. M. Brobeck
(Notes by: K. Mirk and A. DuBois)

I. Comparison of phase stability in a Synchrotron and in a Linear Accelerator.
See Page 13 for definition of symbols.

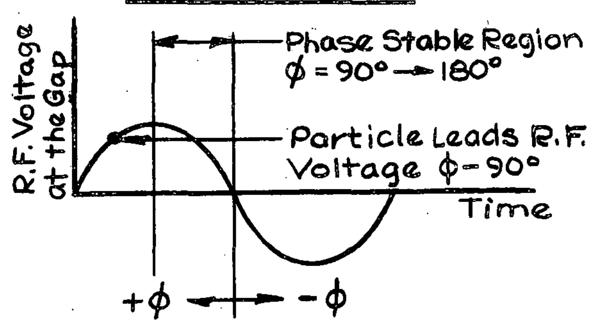
Synchrotron



(Figure 1a)

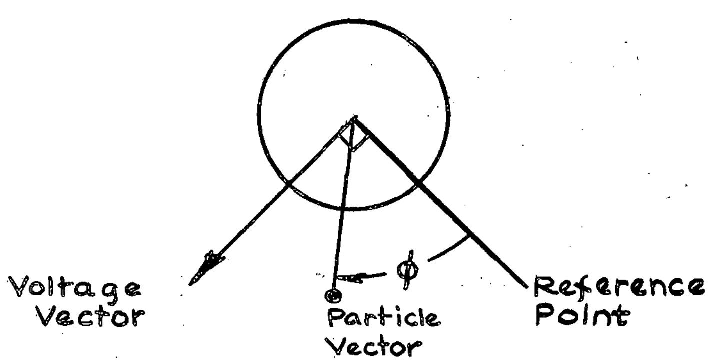
If $E > E_s$ then $V > V_s$ but the orbit radius increases more rapidly than the velocity so ω decreases and, therefore, ϕ decreases. Since $\phi < \phi_s$ the gap voltage is less than that seen by a synchronous particle and $\Delta E < \Delta E_s$.

Linear Accelerator

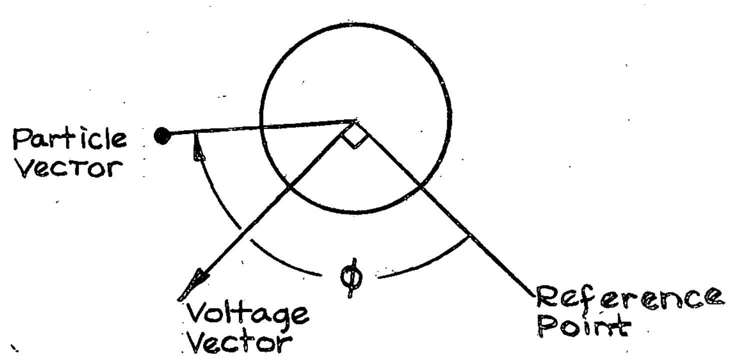


(Figure 1b)

If $E > E_s$ then $V > V_s$ and, since the path length is independent of the energy of the particle, ϕ increases. Since $\phi > \phi_s$ the gap voltage is less than that seen by a synchronous particle and $\Delta E < \Delta E_s$.



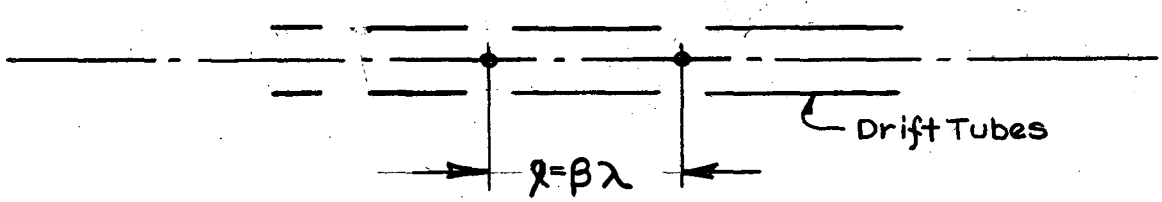
(Figure 2a)



(Figure 2b)

The following derivations pertain to ions in the non relativistic region in a cavity of constant voltage gradient.

II. Significance of ω in a Linear Accelerator.



(Figure 3)

During the time a synchronous particle travels from the center of one gap to the center of the next gap the R.F. voltage vector has gone through one cycle. Therefore, the particle in traveling this distance (l in Figure 3) has gone the equivalent of one revolution in a circular machine and one can define an equivalent particle angular velocity, ω, which is the velocity of the particle vector (see Figure 2b).

$$\omega = 2\pi f \tag{1}$$

$$\omega = 2\pi\frac{v}{l} = \frac{2\pi BC}{l} \tag{2}$$

$$\omega_s = \frac{2\pi v_s}{l} = 2\pi\frac{B_s C}{l} = \omega_o \tag{3}$$

III. Velocity of phase oscillation.

$$d\theta_o = \omega_o dt$$

$$d\theta = \omega dt$$

$$dn = \frac{d\theta}{2\pi} = \frac{\omega dt}{2\pi} \tag{4}$$

$$d\phi = d\theta - d\theta_o \tag{5}$$

Combining Equations (4), (5)

$$\frac{d\phi}{dn} = \frac{d\theta - d\theta_o}{dn} = \frac{d\theta - d\theta_o}{\omega\frac{dt}{2\pi}}$$

$$\frac{d\phi}{dn} = \frac{2\pi\ (\omega dt - \omega_o dt)}{\omega dt}$$

$$\frac{d\phi}{dn} = \frac{2\pi\ (\omega - \omega_o)}{\omega} \tag{6}$$

And from Equations (2), (3)

$$\frac{d\phi}{dn} = \frac{2\pi(B - B_s)}{B} \tag{7}$$

IV. Energy gain per gap.

If we assume a large number of drift tubes the energy gain per gap can be expressed in the differential form:

$$\frac{dE}{dn} = e\,V\,\mathrm{Sin}\,\phi \quad \text{(see Figure 1b)}$$

but

$$V = X\,l = X\,B_s\,\lambda$$

so

$$\frac{dE}{dn} = e\,X\,B_s\,\lambda\,\mathrm{Sin}\,\phi$$

since

$$E = E_o + E_k$$

$$dE = dE_k$$

we can write

$$\frac{dE_k}{dn} = e\,X\,B_s\,\lambda\,\mathrm{Sin}\,\phi \tag{8}$$

V. Relation between particle energy and particle velocity.

$$E_k = \frac{1}{2} M v^2 \tag{9}$$

for non relativistic region $M = M_o$

and

$$E_o = MC^2 \text{ (Einstein)}$$

so

$$E_k = \frac{1}{2} E_o\,B^2 \tag{10}$$

VI. Drift tube number.

Equation (8) may be rewritten as

$$dn = \frac{dE_k}{e\,X\,B_s\,\lambda\,\mathrm{Sin}\,\phi}$$

Considering the synchronous particle

$$dn = \frac{dE_{ks}}{e\,X\,B_s\,\lambda\,\mathrm{Sin}\,\phi_s}$$

from Equation (10)

$$dn = \frac{E_o\,B_s\,d\,B_s}{e\,X\,B_s\,\lambda\,\mathrm{Sin}\,\phi_s}$$

$$n = \int_0^{B_s} \frac{E_o \, d \, B_s}{e \, X \, \lambda \, \text{Sin} \, \phi_s} = \frac{E_o \, B_s}{e \, X \, \lambda \, \text{Sin} \, \phi_s} \qquad (11)$$

Note that n is zero at B = 0, therefore n is larger than the actual number of the drift tube in the machine counting from the injection end.

VII. General differential equation describing the path of the particle.

We have obtained the following four equations relating the five variable B, n, ϕ, ϕ_s, and E_k.

(7) $$\frac{d\phi}{dn} = \frac{2\pi \, (B - B_s)}{B}$$

(8) $$\frac{dE_k}{dn} = e \, X \, B_s \, \lambda \, \text{Sin} \, \phi$$

(10) $$E_k = \frac{1}{2} E_o \, B^2$$

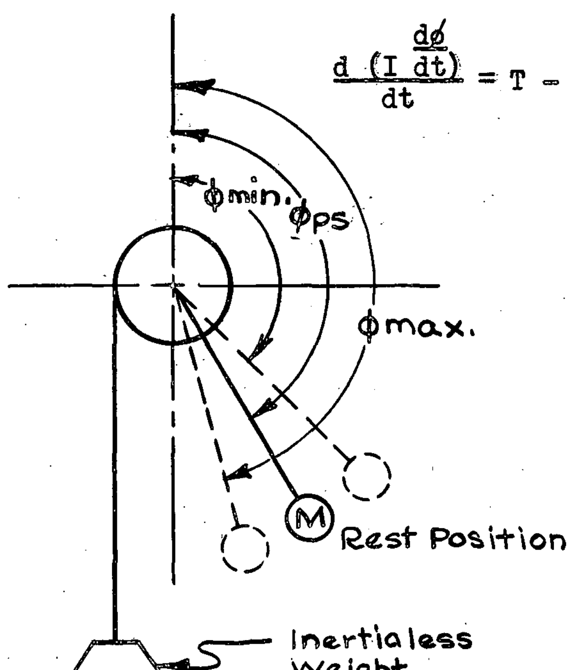
(11) $$n = \frac{E_o \, B_s}{e \, X \, \lambda \, \text{Sin} \, \phi_s}$$

These Equations can be combined to yield the following differential Equation:

$$\frac{d \left(n^2 \frac{d\phi}{dn}\right)}{dn} = \frac{-2\pi n}{\text{Sin} \, \phi_s} \left[\text{Sin} \, \phi_s - \text{Sin} \, \phi \right]$$

This Equation is similar in form to that of a pendulum with a constant torque (see below) for which a solution already exists.

$$\frac{d \left(I \frac{d\phi}{dt}\right)}{dt} = T - G \, \text{Sin} \, \phi$$



(Figure 4)

G = restoring moment due to force of gravity on mass M

I = moment of inertia of the pendulum

T = torque caused by the inertialess weight

t = time

Pendulum	analogous to	Linear Accelerator
G	- - - - - - - - - - - - - - -	$\frac{2\pi n}{\mathrm{Sin}\,\phi_s}$
I	- - - - - - - - - - - - - - -	n^2
T	- - - - - - - - - - - - - - -	$2\pi n$
t	- - - - - - - - - - - - - - -	n

for small amplitudes of oscillation, the pendulum frequency is:

$$f_p = \frac{1}{2\pi}\sqrt{\frac{-G\ \mathrm{Cos}\ \phi_{ps}}{I}}$$

and by analogy

$$f'_\phi = \frac{1}{2\pi}\sqrt{\frac{-\frac{2\pi n}{\sin\phi_s}\ \mathrm{Cos}\ \phi}{n^2}} \qquad \text{cycles/drift tube}$$

$$f'_\phi = \frac{1}{2\pi}\sqrt{\frac{-2\pi}{n\ \tan\phi_s}} \qquad \text{cycles/drift tube} \tag{12}$$

(as f' is less than one its reciprocal, drift tubes/cycle, is more convenient to use)

$$f_\phi = \frac{f_o}{2\pi}\sqrt{\frac{-2\pi}{n\ \tan\phi_s}} \qquad \text{cycles/second} \tag{13}$$

VIII. Amplitude of phase oscillation.

In our analogy I and G represent terms which vary with time. Provided I and G change slowly it can be shown, by application of the "Adiabatic theorem", that the amplitude of the pendulum oscillation varies with $[I\ G]^{-1/4}$. Therefore, by analogy the amplitude of phase angle oscillation in a linear accelerator is:

$$(\phi_{max} - \phi_{min}) \sim \left[(n^2)\left(\frac{2\pi n}{\mathrm{Sin}\ (\pi - \phi_s)}\right)\right]^{-1/4}$$

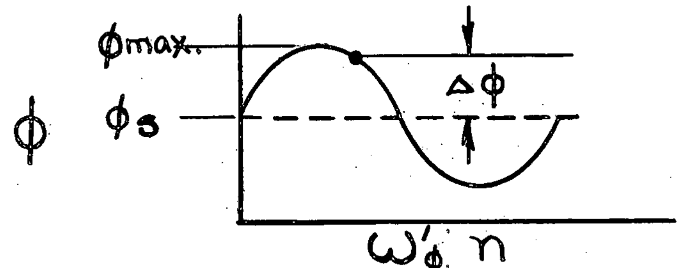
$$(\phi_{max} - \phi_{min}) \sim \left[\frac{2\pi n^3}{\mathrm{Sin}\ \phi_s}\right]^{-1/4}$$

$$(\phi_{max} - \phi_{min}) \sim \frac{\mathrm{Sin}^{1/4}\phi_s}{n^{3/4}} \tag{14}$$

or from Equations (10), (11)

$$(\phi_{max} - \phi_{min}) \sim \frac{\text{Sin}^{1/4}\, \phi_s}{E_k^{3/8}} \tag{15}$$

IX. Energy oscillations.



(Figure 5)

From Figure 5 it is apparent that

$$\Delta\phi = \Delta\phi_{max} \text{ Sin } \omega'_\phi n$$

$$\frac{d(\Delta\phi)}{dn} = \frac{d\phi}{dn} = \Delta\phi_{max}\, \omega'_\phi \text{ Cos } \omega'_\phi n$$

this function has its maximum value when

$$\cos \omega'_\phi n = 1$$

$$\therefore \left[\frac{d\phi}{dn}\right]_{max} = \Delta\phi_{max}\, \omega'_\phi$$

from Equation (12)

$$\omega'_\phi = \sqrt{\frac{-2\pi}{n \tan \phi_s}}$$

then

$$\left(\frac{d\phi}{dn}\right)_{max} = \Delta\phi_{max} \sqrt{\frac{-2\pi}{n \tan \phi_s}} \tag{16}$$

$$\frac{B - B_s}{B} = \frac{\Delta B}{B}$$

$$\frac{E_k - E_{ks}}{E_k} = \frac{\Delta E_k}{E_k}$$

Since

$$B \sim \sqrt{E_k}$$

$$\frac{dB}{B} \sim \frac{1}{2}\,\frac{dE_k}{E_k}$$

and, for small amplitudes,

$$\frac{\Delta B}{B} \sim \frac{1}{2}\,\frac{\Delta E_k}{E_k}$$

therefore

$$\frac{B - B_s}{B} = \frac{1}{2}\,\frac{\Delta E_k}{E_k}$$

substituting into Equation (7)

$$\frac{d\phi}{dn} = 2\pi \left(\frac{1}{2}\,\frac{\Delta E_k}{E_k}\right)$$

and

$$\left(\frac{d\phi}{dn}\right)_{max} = \frac{\pi\, \Delta E_{k\ max}}{E_k} \tag{17}$$

combining Equations (16), (17)

$$\frac{\pi\, \Delta E_{k\ max}}{E_k} = \Delta \phi_{max} \sqrt{\frac{-2\pi}{n \tan \phi_s}}$$

or

$$\Delta E_{k\ max} = \frac{\Delta \phi_{max}}{\pi}\, E_k \sqrt{\frac{-2\pi}{n \tan \phi_s}} \tag{18}$$

for small amplitudes only.

X. Phase and Energy oscillation for large amplitudes.

The expression for change in energy as derived for large amplitudes is the same as for the synchrotron

$$\frac{\Delta E_k}{E_k} = 2\sqrt{\frac{1}{\pi\, n \operatorname{Sin} \phi_s}}\ \sqrt{(\pi - \phi - \phi_s) \operatorname{Sin} \phi_s - \operatorname{Cos} \phi - \operatorname{Cos} \phi_s} \tag{19}$$

$$\frac{\Delta E_k}{E_k} = \text{maximum when } \phi = \phi_s$$

∴ this maximum excursion from the synchronous energy is:

$$\left(\frac{\Delta E}{E_k}\right)_{max} = 2\sqrt{\frac{1}{\pi\, n \operatorname{Sin} \phi_s}}\ \sqrt{(\pi - 2\,\phi_s) \operatorname{Sin} \phi_s - 2 \operatorname{Cos} \phi_s} \tag{20}$$

the maximum departure from the synchronous phase angle occurs when $E = E_s$ or $\Delta E = 0$.

$$\therefore \frac{\Delta E}{E_k} = 0$$

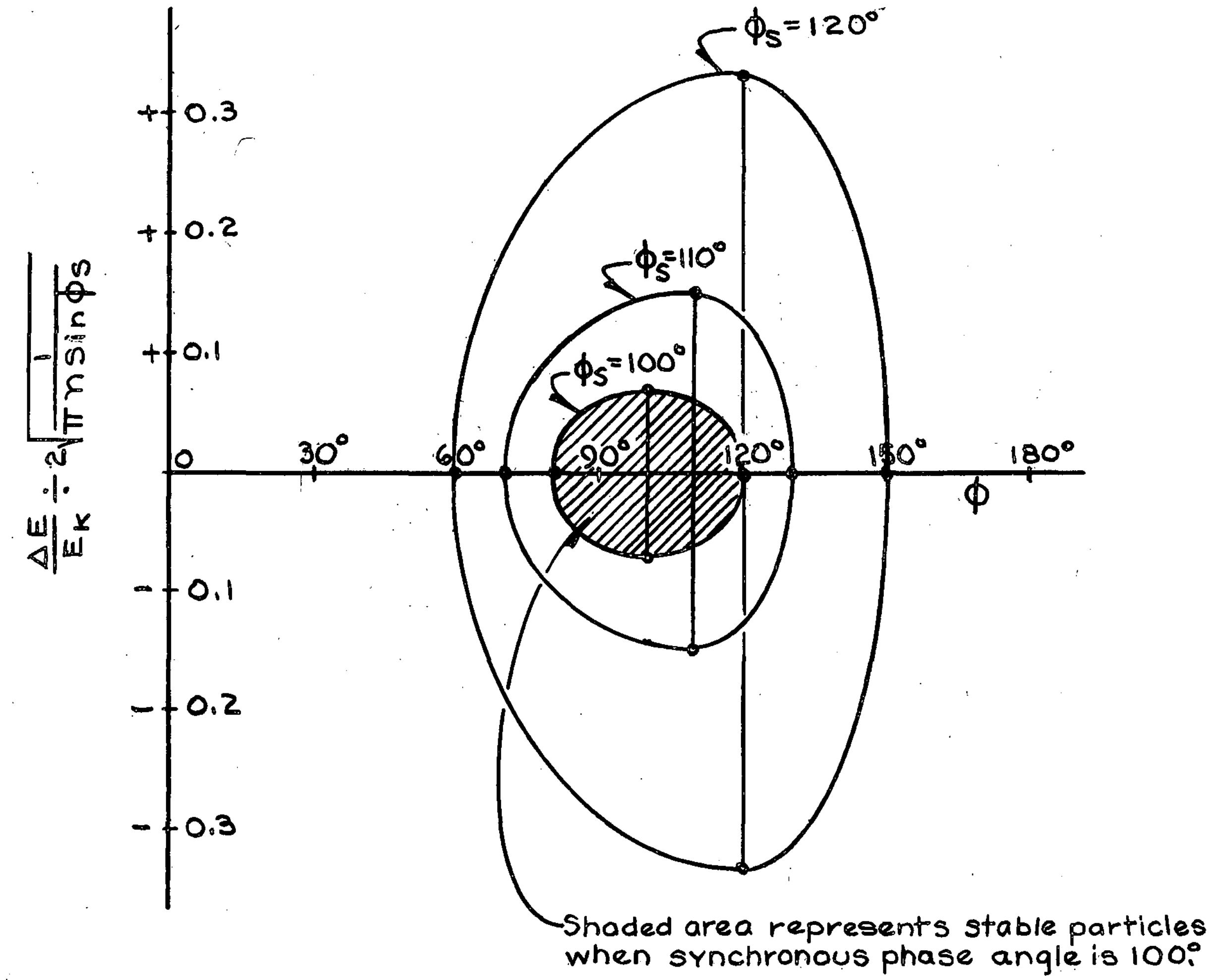
and

$$\sqrt{(\pi - \phi - \phi_s) \operatorname{Sin} \phi_s - \operatorname{Cos} \phi - \operatorname{Cos} \phi_s} = 0 \tag{21}$$

This Equation is satisfied when $\phi = \phi_{min}$ and when $\phi = \phi_{max}$. By inspection of the pendulum analogy it can be seen that $\phi_{min} = \pi - \phi_s$ and ϕ_{max} can be obtained by trial from Equation (21). Tabular and graphical solutions of equation (19) appear in Table 1 and in Figure 6.

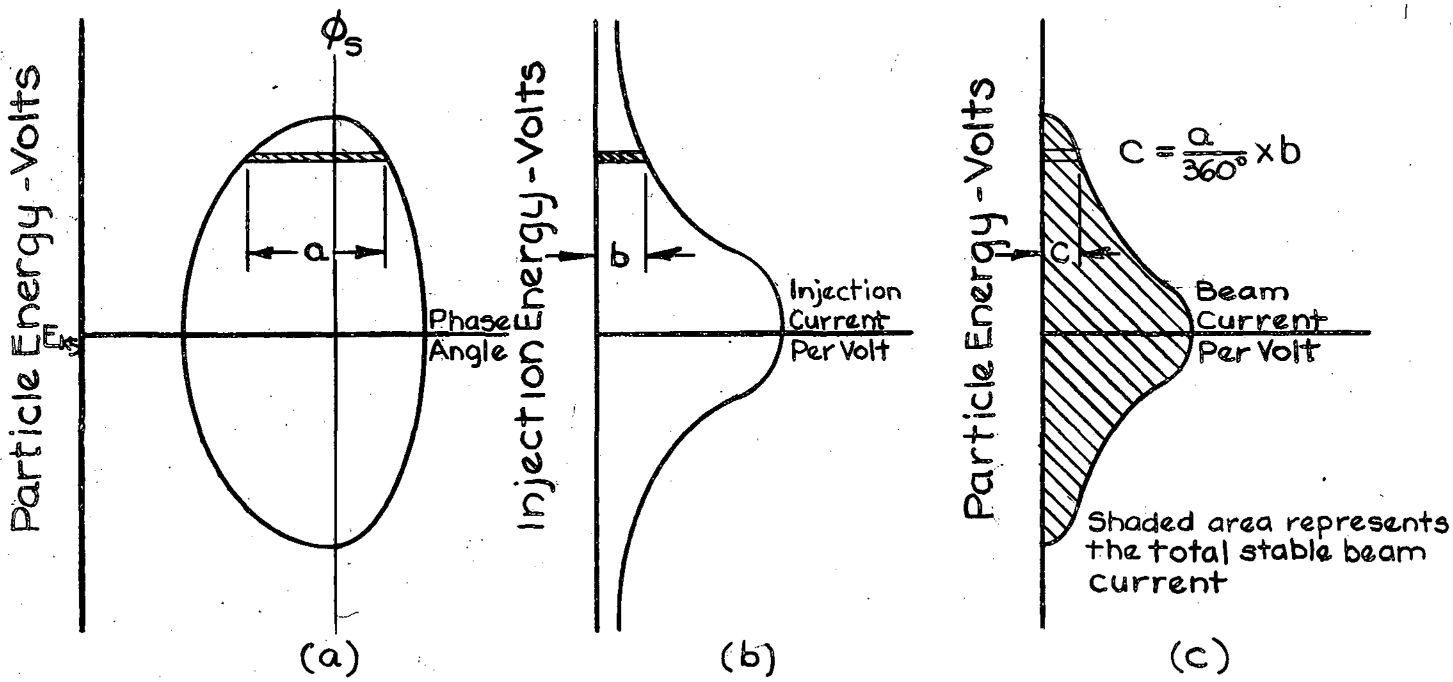
ϕ_s	$\phi_{min.}$	$\phi_{max.}$	$\phi_{MAX} - \phi_{min}$	$\frac{\Delta E}{E_K} \div 2\sqrt{\frac{1}{\pi n \sin\phi_s}}$
90	90	90	0	0
100	80	120	40	±.07
110	70	130	60	±.15
120	60	150	90	±.33

(Table 1)



(Figure 6)

If Figure 6 is replotted for the particular conditions at injection (Figure 7a) and if the energy spectrum of the injection beam is plotted (Figure 7b), one can graphically determine the resulting stable beam current as shown in Figure 7c.



(Figure 7)

XI. Graphical solution for drift tube spacing.

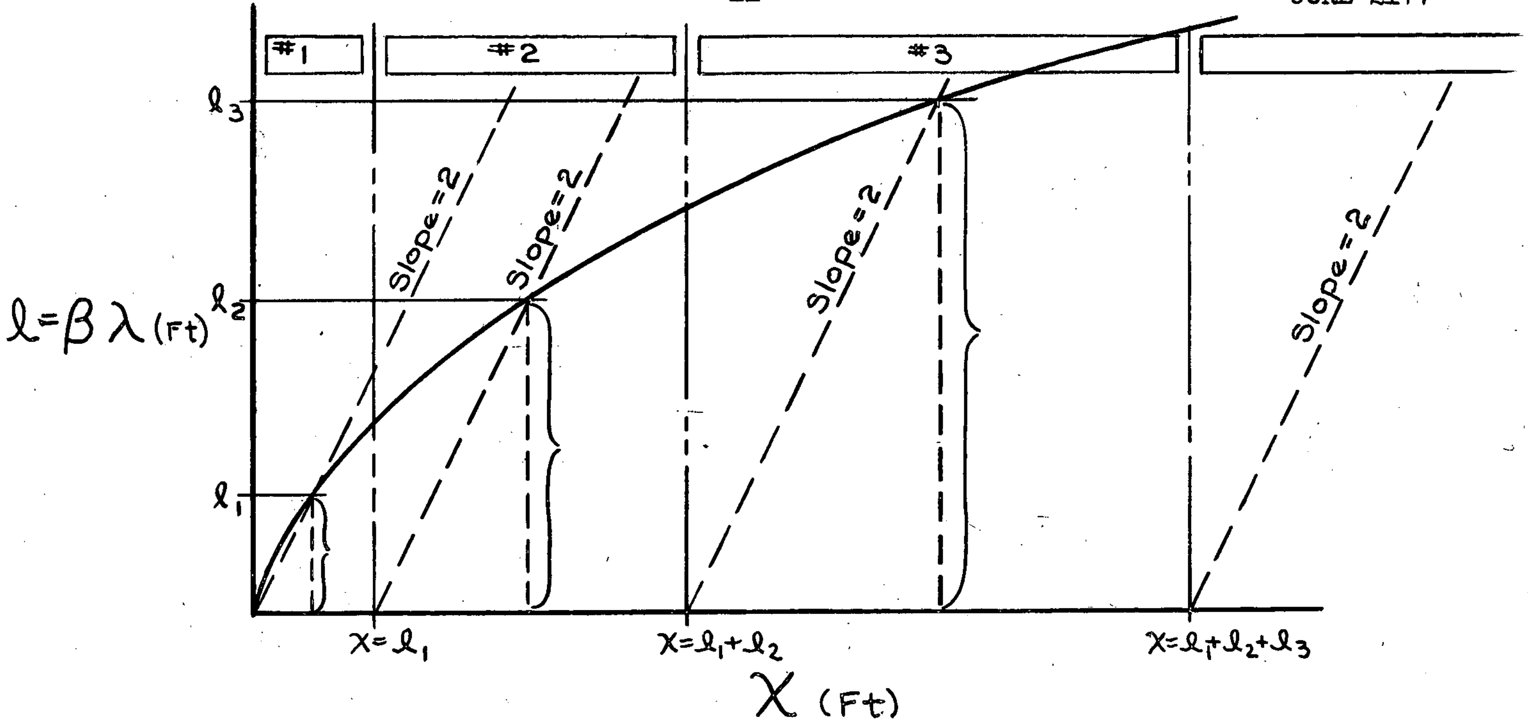
1. Assume X, λ, and ϕ_s or obtain from design basis.

2. Calculate $B = \sqrt{\frac{2\ e\ X\ x\ \mathrm{Sin}\ \phi}{E_o}}$ and plot Bλ as a function of x (distance along the cavity).

3. Starting at the origin draw a straight line of Slope 2.

4. The ordinate at the intersection of the straight line and the curve indicates the proper length of the first drift tube.

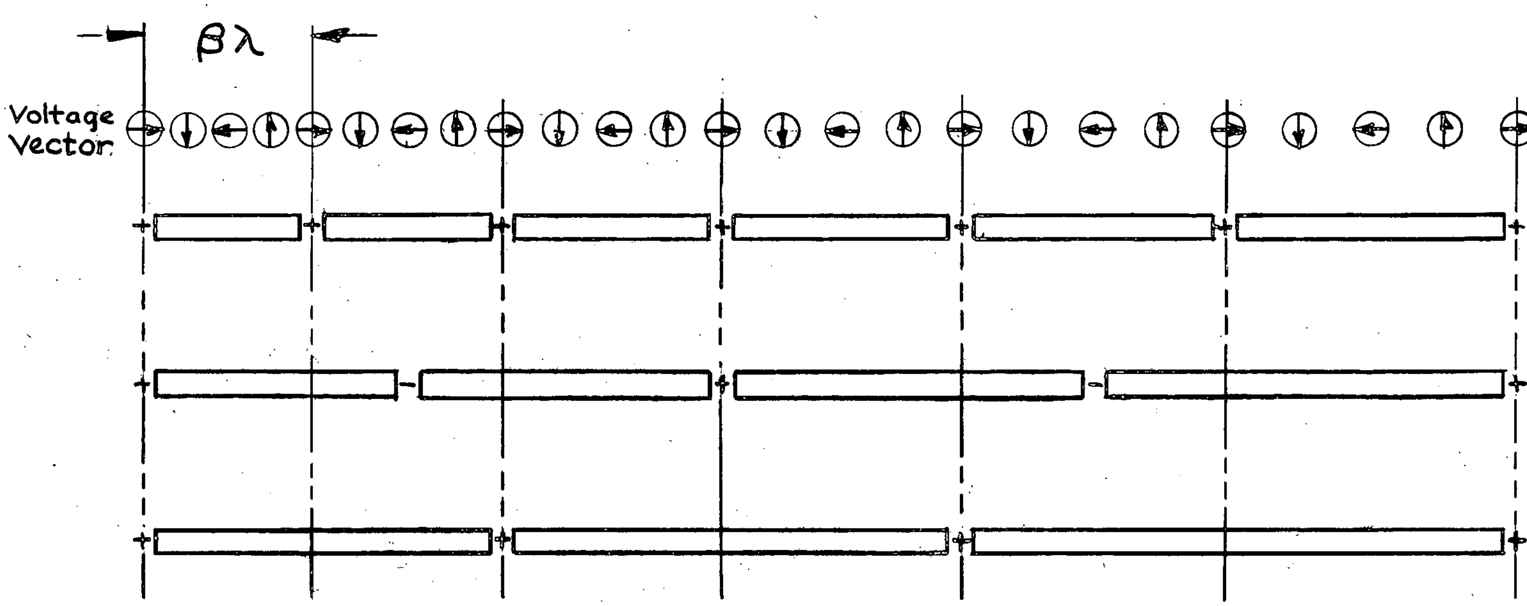
5. Repeat steps 3 and 4 as shown in Figure 8 for subsequent drift tubes.



(Figure 8)

This results from the fact that the value of Bλ at, or near the center of each drift tube, must equal the drift tube spacing.

As shown by the sketch below, if the distance between gaps is equal to Bλ or an integral multiple of Bλ the particle will be accelerated at each gap. However, if this distance is a non-integral multiple value of Bλ the particle may be alternately accelerated and decelerated.



(Figure 9)

If l is an integral multiple of $\frac{B\lambda}{2}$ acceleration is possible if the drift tubes are excited 180° out of phase. This cannot be done in the resonant cavity type design but is possible with other RF geometry. Three phase linacs, etc., are conceivable.

All the derivations have been made for $l = 1B\lambda$.

XII. Transit time factor.

The electric field at the gap is changing with time; therefore, the energy gained by a particle is slightly dependent upon the time required for it to cross the gap and thus upon the ratio $\frac{g}{l}$. To include this effect a transit-time factor, T, is introduced where:

$$T = \frac{\int_0^l X_i \, dx}{X\, l}$$

then

$$\frac{dE}{dn} = e\, X\, l\, T \,\mathrm{Sin}\, \phi$$

If the electric field gradient is constant across the gap at any given time (i.e. varies with time but not with distance):

$$T = \frac{\mathrm{Sin}\,(2\pi \frac{g}{l})}{2\pi \frac{g}{l}}$$

Definition of terms

Symbols

ϕ	=	phase angle measured as shown in Figure 2b
E_o	=	rest energy of particle
E	=	total energy of particle
E_k	=	kinetic energy of particle
v	=	linear velocity of the particle
θ	=	angular displacement of a vector
ω	=	angular velocity in radians/second
ω'	=	angular velocity in radians/drift tube
f	=	frequency in cycles per second
f'	=	frequency in cycles per drift tube
C	=	velocity of light
n	=	drift tube number
λ	=	R. F. wave length
V	=	maximum voltage between drift tubes
X	=	maximum voltage gradient along the cavity

Subscripts

ϕ designates the particle

o designates the oscillator

p designates the pendulum

s designates the synchronous value

Numerical Example

Sample Calculations for Linac Building No. 10, University of California Radiation Laboratory.

Particle : proton (E_0 = 937 MeV charge, e = 1)

Voltage gradient, X = 0.81 MV/ft. (practical volts)

R.F. wave length, λ = 146 CM. (4.8 ft.)

Synchronous phase angle, $\phi_s = 30° + 90° = 120°$

Particle energy at injection, E_k = 4 MeV

Exit energy, E_k = 32 MeV

Injection

Drift tube number.

$$n = \frac{E_0 \; B_s}{e \; x \; \lambda \; \text{Sin} \; \phi_s} \qquad \text{Equation 11}$$

$$B_s^2 = \frac{E^2 - E_0^2}{E^2} \qquad \text{(Lecture V, Page 3)}$$

or

$$B_s^2 = \frac{(E + E_0) \; (E - E_0)}{E^2}$$

$$B_s^2 = \frac{(941 + 937) \; (941 - 937)}{(941)^2}$$

$$B_s = 0.092$$

Substituting in Equation 11

$$n = \frac{(937)(0.092)}{(1)(0.81)(4.8)(0.86)}$$

$$n = 26 \text{ at injection}$$

This is the number of drift tubes required for particles starting from rest to attain a kinetic energy of 4 MeV (injection energy).

Phase Oscillation

$$f'_\phi = \frac{1}{2\pi}\sqrt{-\frac{2\pi}{n \tan \phi_s}} \qquad \text{Equation 12}$$

or

$$f'_\phi = \sqrt{-\frac{1}{2\pi \; n \tan \phi_s}}$$

$n = 26$
$\tan \phi_s = \tan 120° = -1.73$

$$f'_\phi = \sqrt{-\frac{1}{(6.28)(26)(-1.73)}}$$

$$f'_\phi = \frac{1}{16.8}$$

or one phase oscillation occurs in 16.8 drift tubes.

Amplitudes of energy oscillations

$$\frac{\Delta E}{E_k} = 2\sqrt{\frac{1}{\pi\ n\ \text{Sin}\ \phi_s}}\ \sqrt{(\pi - \phi - \phi_s)\ \text{Sin}\ \phi_s - \text{Cos}\ \phi - \text{Cos}\ \phi_s}$$

Equation 19

or

$$\frac{\frac{\Delta E}{E_k}}{2\sqrt{\frac{1}{\pi\ n\ \text{Sin}\ \phi_s}}} = \sqrt{(n - \phi - \phi_s)\ \text{Sin}\ \phi_s - \text{Cos}\ \phi - \text{Cos}\ \phi_s} = \pm\ 0.33$$

Table I for $\phi_s = 120°$

solving for ΔE

$$\Delta E = (\pm\ 0.33)(E_k)(2)\ \sqrt{\frac{1}{\pi\ n\ \text{Sin}\ \phi_s}}$$

$$\Delta E = (\pm\ 0.33)(4)(2)\ \sqrt{\frac{1}{\pi\ (26)(0.86)}} = \pm\ 0.31\ \text{MeV}$$

if $\phi = \phi_s$ at injection, particles of energies between 4.31 MeV and 3.69 MeV are accepted.

Exit

Drift tube number.

$$B_s^2 = \frac{E^2 - E_o^2}{E^2}$$

$$B_s^2 = \frac{(937 + 32)^2 - (937)^2}{(937 + 32)^2}$$

$$B_s = 0.25$$

$$n = \frac{(937)(0.25)}{(1)(0.81)(4.8)(0.86)} = 70$$

This is the number of drift tubes required for particles starting from rest to attain a kinetic energy of 32 MeV. However, since the particles are injected at 4 MeV (26 tubes) the actual number of drift tubes required is 70 - 26 or 44. This number is less than the actual number, (47), of drift tubes in the Building 10 linac since relativistic effects and the transit time factor were neglected in the calculations.

Phase oscillations

$$f'_\phi = \frac{1}{2\pi}\sqrt{-\frac{2\pi}{n\tan\phi_s}} \qquad \text{Equation 12}$$

$$f'_\phi = \sqrt{-\frac{1}{2\pi\, n\tan\phi_s}} \qquad n = 70,\ \phi_s = 120^\circ$$

$$f'_\phi = \sqrt{-\frac{1}{(6.28)(70)(-1.73)}} = \frac{1}{28.5}$$

or one phase oscillation occurs every 28 1/2 tubes.

Maximum amplitude of energy variations

$$\phi_{max} - \phi_{min} \sim \frac{\sin^{1/4}\phi_s}{n^{3/4}} \qquad \text{Equation 14}$$

or

$$\Delta\phi_1 \sim \frac{\sin^{1/4}\phi_s}{n_1^{3/4}} \quad \text{and} \quad \Delta\phi_2 \sim \frac{\sin^{1/4}\phi_s}{n_2^{3/4}}$$

$$\therefore \frac{\Delta\phi_1}{\Delta\phi_2} = \frac{n_2^{3/4}}{n_1^{3/4}}$$

solving for $\Delta\phi_2$

$$\Delta\phi_2 = \Delta\phi_1 \left(\frac{n_1}{n_2}\right)^{3/4}$$

$$\Delta\phi_2 = 90^\circ \left(\frac{26}{70}\right)^{3/4} \cong 40^\circ$$

$n_1 = 26$
$n_2 = 70$
$\Delta\phi_1 = 90^\circ$
(Table I, $\phi_s = 120^\circ$)

From Table I for ϕ range = 40°

$$\frac{\frac{\Delta E}{E_k}}{2\sqrt{\frac{1}{\pi\, n \sin\phi_s}}} = \pm\, 0.07$$

$$\therefore \quad \Delta E_{max} = (\pm\ 0.07)(2)\ E_k\sqrt{\frac{1}{\pi\ n\ \text{Sin}\ \phi_s}}$$

$$\Delta E_{max} = (\pm\ 0.07)(2)(32)\sqrt{\frac{1}{\pi\ (70)(.86)}} = \pm\ 0.32\ \text{MeV}$$

or the exit energy could vary between 32.32 MeV and 31.68 MeV. In an actual machine the variance in exit energy would be somewhat less due to the effects of variation of phase angle on defocusing.