Title

Matroids and convex geometry in combinatorics and algebra

Author

Gotti, Felix

Publication Date

2019

Peer reviewed|Thesis/dissertation

Matroids and convex geometry in combinatorics and algebra

by

Felix Gotti

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Lauren K. Williams, Chair
Assistant Professor Richard Bamler
Professor Bernd Sturmfels
Associate Professor Nike Sun

Spring 2019

Matroids and convex geometry in combinatorics and algebra

Abstract

Matroids and convex geometry in combinatorics and algebra

by

Felix Gotti

Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Lauren K. Williams, Chair

This thesis is a compendium of three studies on which matroids and convex geometry play a central role and show their connections to Catalan combinatorics, tiling theory, and factorization theory. First, we study positroids in connection with rational Dyck paths. Then, we study certain matroids on the lattice points of a regular triangle in connection with lozenge tilings. Finally, we explore the connection between the atomic structure of submonoids of $(\mathbb{N}^d, +)$ and the geometric properties of the cones they generate.

Positroids, first studied by Postnikov in 2006, are matroids that parameterize the cells of the totally nonnegative part of a Grassmannian variety. The first part of this thesis concerns with the study of a family of positroids that can be parameterized by (rational) Dyck paths. We call such positroids (rational) Dyck positroids. Using work of Reed and Skandera, we show that Dyck positroids on the ground set $[2n]$ are in natural bijection with unit interval orders of size n. We also offer recipes to read the decorated permutation of a Dyck positroid directly from either the antiadjacency matrix representation or the interval representation of the corresponding unit interval order. Finally, for the family of rational Dyck positroids, we provide combinatorial descriptions for some of the most relevant combinatorial objects that are in bijection with positroids.

The second part of this thesis pertains to the study of certain class of matroids which naturally appear in the set of 1-dimensional intersections of complete complex flag arrangements. More specifically, these matroids encode the dependency relations among the lines of such flag arrangements. The bases of such matroids can be thought of as certain n-subsets of lattice points of a regular n-simplex. For dimension 2, we provide various cryptomorphic characterizations of these matroids in connection with lozenge tilings of a regular triangle. We also study the connectivity of members of this family of matroids in any dimension.

Finitely generated submonoids of $(\mathbb{N}^d, +)$, also known as affine monoids, are crucial in the study of combinatorial commutative algebra and, in particular, toric geometry. Let $\mathcal{C}$ denote the class consisting of all submonoids of $(\mathbb{N}^d, +)$ (not necessarily finitely generated). The last part of this thesis is devoted to explore how atomic properties of a monoid M in $\mathcal{C}$ (and the monoid algebras M induces) are connected with the geometry of its conic hull $\mathsf{cone}(M)$

and with the combinatorial structure of the face lattice of $\mathsf{cone}(M)$. For monoids in $\mathcal{C}$, we investigate two of the most important arithmetic invariants in factorization theory: the system of sets of lengths and the elasticity. We conclude this thesis studying the atomicity of monoid algebras, including the algebras induced by monoids in $\mathcal{C}$. We shall provide a partial answer to a fundamental question about the atomicity of monoid algebras that Gilmer asked back in the 1980's.

To my wife and my parents

Contents

List of Figures

Acknowledgments

Many people contributed towards my mathematical background before my enrollment at Berkeley. It is my pleasure to thank my great friend and math coach Evidio Quintana for teaching me to love mathematics. I am also thankful to many professors at Matcom, especially Rita Roldán and Ramiro Piñeiro, for the tremendous amount of work trying to teach me math in a time where I was not that willing to learn (please, know that your invested energy was not wasted, but transformed into a potential motivation that exploded a few years later). In addition, I am grateful to my dear professor Alex Turull for helping me discover my love for algebra as well as to my outstanding mentor Scott Chapman for introducing me to research and proposing the project of my undergraduate thesis.

I would like to express my deepest gratitude to my PhD advisor, Lauren Williams, for her constant support during my years as a graduate student at Berkeley. Her encouragement, guidance, and advice have significantly influenced my development as a young mathematician. She not only taught me beautiful aspects of combinatorics and algebra but also made a profound positive impact on my conception of mathematical research. To her, I will be always grateful.

During my PhD period, I was fortunate to have other great math and career mentors. I would like to thank Scott Chapman and Alfred Geroldinger for their guidance on many of my papers, for their trust, and for sharing their expertise with me. I would also like to thank Bernd Sturmfels for his availability and many advices. In addition, I want to thank Alfred Geroldinger, Nathan Kaplan, Peter Sin, and Jan Vondrak for their willingness to be my postdoctoral sponsor scientists. Finally, I am grateful to Ira Young for his support as a career mentor and Colette Patt for many motivational conversations.

I would like to thank all my coauthors for the wonderful mathematics we have discovered together. I am particularly thankful to Marly Gotti and Harold Polo for their constant peer-review and their many challenging questions. I would also like to thank Winfried Bruns, Jim Coykendall, Pedro García Sánchez, and Roger Wiegand for their responsiveness and availability to answer my questions, as well as Federico Ardila and Vera Serganova for stimulating mathematical conversations. Finally, I am thankful to Richard Bamler, Bernd Sturmfels, and Nike Sun for accepting to be part of both my qualifying exams committee and my PhD dissertation committee.

I express my gratitude to my professors at UC Berkeley for all I have learned from them, as well as the faculty administration and staff of the department for the past years of service, especially Vicky Lee, for her constant help in logistics and excellent care of administrative matters. In addition, I thank Martin Olsson and Jon Wilkening for their availability as heads of the department. I am also thankful to the combinatorics group of the University of Miami, in particular, Bruno Benedetti, Jose Samper, Richard Stanley, and Michelle Wachs, for hosting me as a visiting graduate student during the Spring semester of 2018. In addition, my thankfulness extends to the people of the Harvard math department for hosting me as a research exchange scholar during the academic year 2018-2019 and to the combinatorics group of MIT for welcoming to their seminars during the same period.

I am glad to acknowledge generous financial support I have received from my advisor, the UC Berkeley Math Department, the Chancellor Fellowship, the NSF-AGEP, the Dissertation Fellowship, and the California Alliance.

In my friends I have found a valuable source of joy and encouragement as they have enriched my life during the past five years and have given me a healthy dose of distraction from my research. I especially thank my friends Dayron and Melina as well as my mother-in-law, Marlén, for showing significant enthusiasm for the details and progress of my career. My warmest gratitude goes to Harold, who has been like a brother to me and has shared each step of the last ten years of my life constantly motivating me to follow my/our dreams.

And now my lovely family... without whom neither the current version of me nor this PhD dissertation would have come to exist. I am eternally indebted and unmeasurable grateful to my parents, Juan Gotti and Blanca Rosa, and my grandparents Felix Gotti and María de Jesús for the unconditional love and early education they gave me. They are, indeed, the quiet heroes of my journey. Finally, I am endlessly grateful to Marly for being my colleague, my friend, my wife, my light, my inspiration... and my everything.

Chapter 1

Introduction

Matroids, posets, and lattices are combinatorial objects that have found connections to many other objects with geometric and algebraic flavors. This thesis is a compendium of three projects on which certain classes of matroids, posets, and lattices play a central role. The first part of this thesis consists of Chapter 2 and Chapter 3, which are based on the papers [50] (joint work with Anastasia Chavez) and [96], respectively. Here we study a family of matroids, called positroids, along with their connection with unit interval orders and rational Dyck paths. The second part of this thesis consists of Chapter 4 and Chapter 5, which are based on the paper [102] (joint work with Harold Polo). In this part, we investigate certain matroids on the lattice points of a regular triangle in connection with lozenge tilings. Lastly, the third part of this thesis consists of Chapter 6, Chapter 7, and Chapter 8, which are based on the papers [95], [91] and [56], respectively ([56] is a joint work with Jim Coykendall). This last part is dedicated to the study of the atomic and factorization structure of submonoids of $(\mathbb{N}^d, +)$ (and their monoid algebras) in connection with combinatorial and geometric properties of their conic hulls.

The classical theory of total positivity, introduced by Gantmacher, Krein, and Schoenberg in the 1930's, has been recently revitalized as a result of the many connections it has with Lusztig's work, in particular, with the introduction of the totally nonnegative part of a real flag variety. Consequently, an exploration of the combinatorial structure of the totally nonnegative part of the Grassmannian was initiated. Motivated by the work of Lusztig and the work of Fomin and Zelevinsky, in 2006 Postnikov initiated the study of positroids [127], matroids represented by elements of $(\mathrm{Gr}_{d,n})_{\geq 0}$. He proved that they are in bijection with various families of elegant combinatorial objects, including Grassmann necklaces, decorated permutations, Ꞁ-diagrams, and certain classes of plabic graphs (all of them to be introduced later).

In the first part of this thesis we study a class of positroids which can be parameterized by (rational) Dyck paths. We call such positroids (rational) Dyck positroids. We distinguish Dyck positroids from its generalized counterpart, rational Dyck positroids, mainly because the former are in a natural bijection with unit interval orders, as we shall reveal in Chapter 2. We will also offer recipes to read the decorated permutation of a Dyck positroid either from the antiadjacency matrix representation or from the interval representation of the corresponding unit interval order. Then, in Chapter 3 we provide combinatorial descriptions of the decorated permutation, Grassmann necklace, Le-diagram, and (homotopic classes of) plabic graphs corresponding to a rational Dyck positroid. Finally, we present a description by inequalities for the matroid polytope of a rational Dyck positroid, which improves the number of inequalities used in the description of a positroid polytope given in [17]. The main question motivating Chapter 2 was kindly provided by Alejandro Morales. Extended abstracts of Chapter 2 and Chapter 3 can be found in [51] and [93], respectively.

The second part of this thesis is concerned with the study of a family of matroids that naturally appear in the set of 1-dimensional intersections of complete complex flag arrangements. Specifically, these matroids encode the dependency relations among the lines of such flag arrangements. It has been proved by Ardila and Billey in [15] that in dimension 2 the bases of such matroids are in bijection with lozenge tilings of a regular triangle. For this reason, in dimension 2 we call such matroids *tiling matroids*. In addition, the so-called Spread Out Conjecture states that a similar characterization should be possible in higher dimension, as long as fine mixed subdivisions play the role of lozenge tilings. In [33] Billey and Vakil introduced a criterion that efficiently identifies many structure constant of the cohomology rings of intersections of Schubert varieties. It turns out that such a criterion can be refined provided a better understanding of the matroidal structure of higher-dimension tiling matroids.

In Chapter 4, we provide various cryptomorphic characterizations of tiling matroids. In particular, we characterize the independent sets, the circuits, and the flats of such matroids in terms of lozenge-like tilings. We also study the rank function of tiling matroids in connection with certain extremal lozenge tilings. Then, in Chapter 5, we fully characterize the connectedness of the tiling matroids (in any dimension). In particular, when the dimension is 2, we show that the connectedness of such matroids can be proved using arguments related to lozenge tilings.

Finitely generated additive submonoids of $\mathbb{N}^d$, also known as (reduced) affine monoids, are crucial in the study of toric algebras [121, Part II] and K-theory [37, Part III]. The last part of this thesis is dedicated to the study of the class consisting of all (not necessarily finitely generated) additive submonoids of $\mathbb{N}^d$. We let $\mathcal{C}$ denote such a class. Our study will focus on the connection between the combinatorial and geometric structures of the cones of monoids in $\mathcal{C}$ and their atomic and factorization properties. Although a systematic study of the monoids in $\mathcal{C}$ has not been carried out yet, subclasses of $\mathcal{C}$ have appeared in the recent literature in connection with algebraic geometry [29] and commutative algebra [69].

The last part of this thesis is devoted to explore how the atomic and factorization properties of a monoid M in $\mathcal{C}$ are connected with the geometry of its conic hull $\mathsf{cone}(M)$ and the combinatorial structure of the face lattice of $\mathsf{cone}(M)$. In Chapter 6, we offer combinatorial and geometric characterizations of three important subclasses of $\mathcal{C}$, those consisting of factorial, half-factorial, and other-half-factorial monoids. Primary monoids [74] and finitary monoids [84] have been two of the most important classes of monoids in the development of factorization theory. For primary monoids and finitary monoids in $\mathcal{C}$ we investigate geometric aspects of the cones they generate as well as some combinatorial aspects of the face lattice of such cones. In Chapter 7, we study two factorization invariants of monoids in $\mathcal{C}$, the system of sets of lengths and the elasticity. We construct monoids in $\mathcal{C}$ having extremal systems of sets of lengths. In addition, we answer a question on the rationality of the elasticities of monoids in $\mathcal{C}$ that was recently asked in [141]. We start Chapter 8 showing how some of the properties we have studied for monoids in $\mathcal{C}$ reflect on their monoid algebras. Then we contrast the atomicity of the monoid algebras induced by members of the class $\mathcal{C}$ and the atomicity of monoid algebras induced by general atomic monoids as well as monoids in the class $\mathcal{Q}$ (which consists of all atomic submonoids of $(\mathbb{Q}_{\geq 0}, +)$). Finally, we show that atomicity does not transfer, in general, from a monoid M to the monoid algebras that M induces over fields of finite characteristic; this provides a partial answer to a fundamental question about the atomicity of monoid algebras that Gilmer asked back in the 1980's.

Part I

On Rational Dyck Positroids and Related Combinatorial Objects

Chapter 2

Unit Interval Orders and the Totally Nonnegative Grassmannian

2.1 Introduction

A *unit interval order* is a partially ordered set that captures the order relations among a collection of unit intervals on the real line. Unit interval orders originated in the study of psychological preferences, first appearing in the work of Wiener [145], and then in greater detail in the work of Armstrong [24] and others. They were also studied by Luce [119] to axiomatize a class of utilities in the theory of preferences. Since then they have been systematically studied (see [62, 63, 64, 65, 146, 134] and references therein). These posets exhibit many interesting properties; for example, they can be characterized as the posets that are simultaneously $(\mathbf{3}+\mathbf{1})$-free and $(\mathbf{2}+\mathbf{2})$-free. Moreover, it was first proved in [146] that the number of non-isomorphic unit interval orders on the set $\{1, 2, \ldots, n\}$ equals $\frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number (see also [62, Section 4]).

In [134], motivated by the desire to understand the f-vectors of various classes of posets, Skandera and Reed showed that a simple procedure for labeling a unit interval order yields the useful form of its $n \times n$ antiadjacency matrix which is totally nonnegative (i.e., has all its minors nonnegative) with its zero entries appearing in a right-justified Young diagram located strictly above the main diagonal and anchored in the upper-right corner. The zero entries of such a matrix are separated from the one entries by a Dyck path joining the upper-left corner to the lower-right corner. Motivated by this observation, we call such matrices *Dyck matrices*. The Hasse diagram and the antiadjacency (Dyck) matrix of a canonically labeled unit interval order are shown in Figure 2.1.

On the other hand, it follows from work of Postnikov [127] that the $n \times n$ antiadjacency (Dyck) matrix of a (properly labeled) unit interval order P can be regarded as representing a rank n *positroid* on the ground set $\{1, 2, \ldots, 2n\}$. We will say that such a positroid is *induced* by P. Positroids, which are special matroids, were introduced and classified by Postnikov in his study of the totally nonnegative part of the Grassmannian [127]. He showed that there

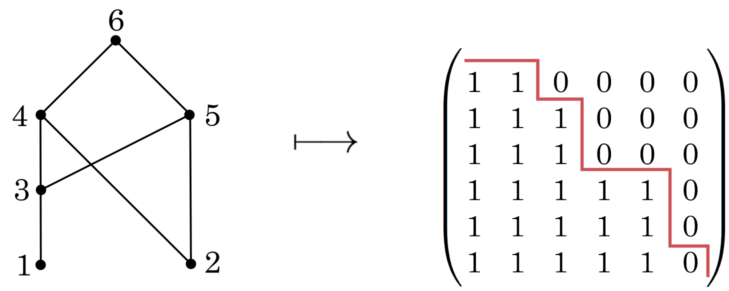

Figure 2.1: A canonically labeled unit interval order on the ground set $\{1, 2, \ldots, 6\}$ and its antiadjacency matrix, in which one entries and zero entries are separated by a Dyck path.

is a cell decomposition of the totally nonnegative part of the Grassmannian so that cells are indexed by positroids (or equivalent combinatorial objects). Positroids and the nonnegative Grassmannian have been the subject of a great deal of recent work, with connections and applications to cluster algebras [133], scattering amplitudes [19], soliton solutions to the Kadomtsev-Petviashvili equation [115], and free probability [17].

In this chapter we characterize a family of positroids on the ground set $\{1, 2, \ldots, 2n\}$ that bijectively arise from unit interval orders of size n. We call such positroids *Dyck positroids*. Positroids, in general, are in bijection with certain generalized permutations, which are known as decorated permutations. We shall see that the decorated permutations corresponding to Dyck positroids are standard permutations in the symmetric group S_{2n} on $2n$ letters, where n is the size of the corresponding unit interval order. The permutations corresponding to Dyck positroids have the following description.

Description of the Permutation. A (decorated) permutation $\pi \in S_{2n}$ represents a Dyck positroid on the set $\{1, 2, \ldots, 2n\}$ if and only if when 1 is fixed as the first entry of π, the following two conditions hold:

- the elements $1, \ldots, n$ appear in increasing order while the elements $n+1, \ldots, 2n$ appear in decreasing order;
- for every $1 \leq k \leq 2n$, there are at least as many elements of the set $\{1, \ldots, n\}$ as elements of the set $\{n+1, \ldots, 2n\}$ in the first k entries of π.

As indicated in the description above, the permutation corresponding to a Dyck positroid on the ground set $\{1, 2, \ldots, 2n\}$ naturally encodes a Dyck path of length $2n$. In particular, Dyck positroids are in bijection with Dyck paths of length $2n$ and, therefore, there are $\frac{1}{n+1}\binom{2n}{n}$ Dyck positroids on the ground set $\{1, 2, \ldots, 2n\}$. In this chapter we also provide a recipe to decode the permutation of a Dyck positroid directly from the antiadjacency matrix A of the corresponding unit interval order P. When the unit interval order is appropriately labeled, A shows a Dyck path (separating its zero entries from its one entries), which we call the *semiorder path* of A. The semiorder path of A coincides with the Dyck path encoded in the permutation corresponding to the Dyck positroid induced by P, and this fact yields the following recipe to obtain the permutation directly from A.

Recipe. Let P be a canonically labeled unit interval order on the set $\{1, 2, \ldots, n\}$, and let A be its antiadjacency matrix. Number the n vertical steps of the semiorder path of A from bottom to top by $1, \ldots, n$ and label the n horizontal steps from left to right by $n+1, \ldots, 2n$. Then the sequence of $2n$ labels, read in the northwest direction, is the decorated permutation associated to the Dyck positroid induced by P.

Example 2.1.1. The vertical assignment on the left of Figure 2.2 shows a set $\mathcal{I}$ of unit intervals along with a canonically labeled unit interval order P on the set $\{1, 2, \ldots, 5\}$ describing the order relations among the intervals in $\mathcal{I}$ (see Theorem 2.2.4). The vertical assignment on the right illustrates the recipe given before to read the decorated permutation $\pi = (1\ 2\ 10\ 3\ 9\ 4\ 8\ 7\ 5\ 6)$ corresponding to the Dyck positroid induced by P directly from the antiadjacency matrix. Note that the decorated permutation π is a 10-cycle satisfying both conditions given in description of the permutation we have given before. The solid and dashed assignment signs represent functions that we shall introduce later.

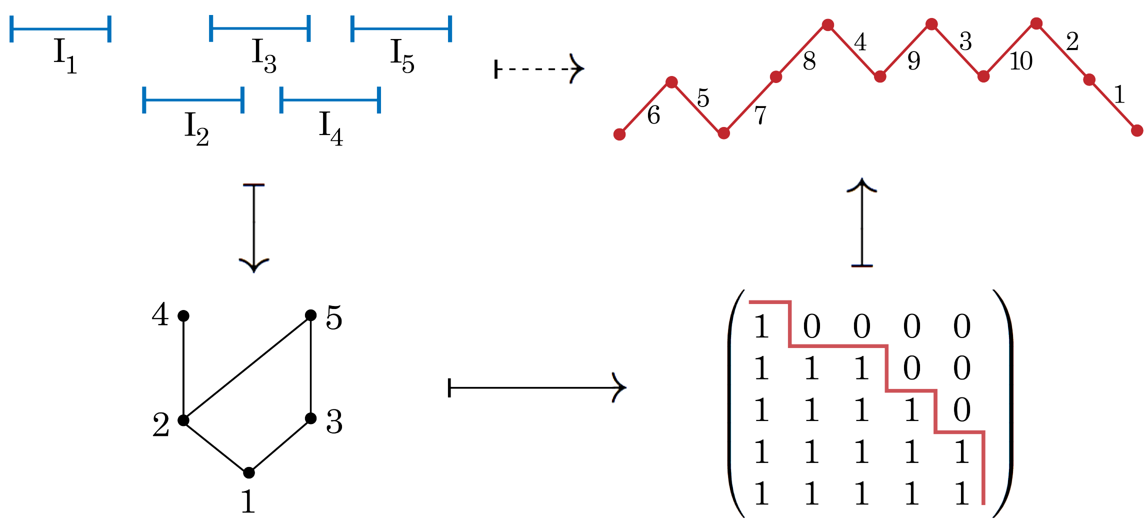

Figure 2.2: Following the solid assignments: unit interval representation $\mathcal{I}$, its unit interval order P, the antiadjacency matrix of P, and the Dyck path that separates the one entries from the zero entries of the antiadjacency matrix of P showing the decorated permutation $\pi = (1\ 2\ 10\ 3\ 9\ 4\ 8\ 7\ 5\ 6)$.

2.2 Posets and Positroids

General Notation: We let $\mathbb{N}$ denote the set of nonnegative integers. For every integer $n \geq 1$, we set $[n] := \{1, 2, \ldots, n\}$. In addition, for a set S and $k \in \mathbb{N}$, we let $\binom{S}{k}$ denote the collection consisting of all subsets of S of cardinality k, and we call an element of $\binom{S}{k}$ a *k-subset* of S. Finally, we let $\mathrm{Mat}_{d,m}(\mathbb{R})$ denote the set of all $d \times m$ real matrices, and let $\mathrm{Mat}^{+}_{d,m}(\mathbb{R})$ be the subset of $\mathrm{Mat}_{d,m}(\mathbb{R})$ consisting of those full-rank matrices with nonnegative maximal minors.

Unit Interval Orders

For ease of notation, when $(P, \leq_P)$ is a partially ordered set (*poset* for short), we just write P, tacitly assuming that the order relation on P is to be denoted by the symbol $\leq_P$. For $x, y \in P$, we will write $x <_P y$ when $x \leq_P y$ and $x \neq y$. In addition, every poset showing up in this thesis is assumed to be finite unless we specify otherwise.

An *order ideal* of a poset P is a subset I of P such that if $x \in I$ and $y \leq_P x$, then $y \in I$. Similarly, a *dual order ideal* is a subset I of P such that if $x \in I$ and $x \leq_P y$, then $y \in I$. For any $x \in P$, it is clear that the sets

$$\Lambda_x = \{y \in P \mid y \leq_P x\} \quad \text{and} \quad \mathrm{V}_x = \{y \in P \mid x \leq_P y\}$$

are an order ideal and a dual order ideal, respectively. They are called, respectively, the *principal ideal* and the *principal dual ideal* generated by x.

If the poset P has cardinality n, then a bijective function $\ell\colon P \to [n]$ is called an *n-labeling* of P; after identifying P with $[n]$ via ℓ, we say that P is an *n-labeled* poset. The n-labeled poset P is *naturally labeled* if $i \leq_P j$ implies that $i \leq j$ as integers for all $i, j \in P$.

Definition 2.2.1. A poset P of size n is a *unit interval order* if there exists a bijective map $i \mapsto [q_i, q_i + 1]$ from P to a set $S = \{[q_i, q_i + 1] \mid 1 \leq i \leq n,\ q_i \in \mathbb{R}\}$ of closed unit intervals of the real line such that for $i, j \in P$, $i <_P j$ if and only if $q_i + 1 < q_j$. We then say that S is an *interval representation* of P.

Example 2.2.2. The figure below depicts the 6-labeled unit interval order introduced in Figure 2.1 with a corresponding interval representation.

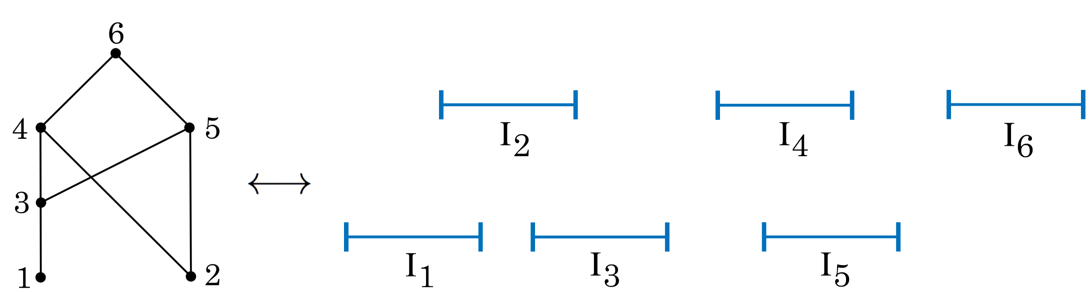

Figure 2.3: A 6-labeled unit interval order and one of its interval representations.

A useful way of representing an n-labeled unit interval order is through its antiadjacency matrix. If P is an n-labeled poset, then the *antiadjacency matrix* of P is the $n \times n$ binary matrix $A = (a_{i,j})$ with $a_{i,j} = 0$ if and only if $i <_P j$. The 6-labeled unit interval order of Example 2.2.2 along with its antiadjacency matrix are illustrated in Figure 2.1.

For each $n \in \mathbb{N}$, we denote by $\mathcal{U}_n$ the set of all non-isomorphic unit interval orders of cardinality n. For nonnegative integers n and m, let $\mathbf{n} + \mathbf{m}$ denote the poset which is the disjoint sum of an n-element chain and an m-element chain. Let P and Q be two posets. We say that Q is an *induced* subposet of P if there exists an injective map $f\colon Q \to P$ such that for all $r, s \in Q$ one has $r \leq_Q s$ if and only if $f(r) \leq_P f(s)$. By contrast, P is a *Q-free* poset if P does not contain any induced subposet isomorphic to Q.

Example 2.2.3. None of the properties of being (**3**+**1**)-free or being (**2**+**2**)-free imply the other one. For instance, Figure 2.4 shows, from left to right, a poset having **3**+**1** as an induced subposet (in red) and having **2**+**2** as an induced subposet (in blue), a (**2**+**2**)-free poset having **3**+**1** as an induced subposet (in red), a (**3**+**1**)-free poset having **2**+**2** as an induced subposet (in blue), and a poset that is both (**3**+**1**)-free and (**2**+**2**)-free.

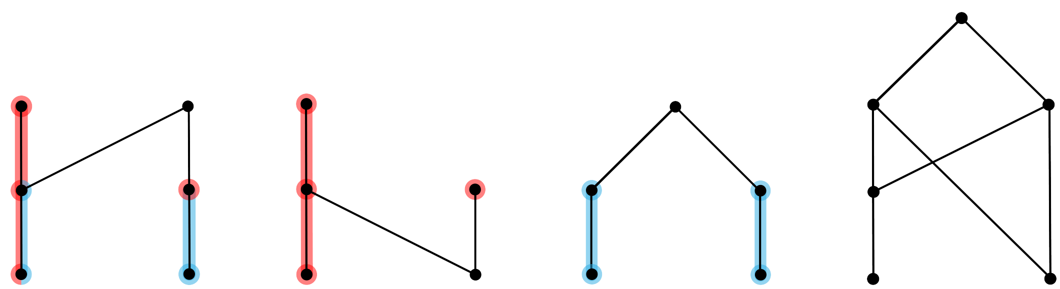

Figure 2.4: From left to right: a poset that is neither (**3**+**1**)-free nor (**2**+**2**)-free, a poset that is (**2**+**2**)-free but not (**3**+**1**)-free, a poset that is (**3**+**1**)-free but not (**2**+**2**)-free, and a poset that is both (**3**+**1**)-free and (**2**+**2**)-free.

The following theorem provides a useful characterization of the elements of $\mathcal{U}_n$.

Theorem 2.2.4. *[132, Theorem 2.1] A poset is a unit interval order if and only if it is simultaneously* $(\mathbf{3} + \mathbf{1})$*-free and* $(\mathbf{2} + \mathbf{2})$*-free.*

A binary square matrix A is said to be a *Dyck matrix* if its zero entries are separated from its one entries by a Dyck path joining the upper-left corner to the lower-right corner. We call such a Dyck path the *semiorder path* of A. All minors of a Dyck matrix are nonnegative (see, for instance, [1]). We denote by $\mathcal{D}_n$ the set of all $n \times n$ Dyck matrices. As presented in [134], every unit interval order can be naturally labeled so that its antiadjacency matrix is a Dyck matrix (details provided in Section 2.3).

Positroids

We proceed to introduce the first class of matroids we shall be studying in this thesis. From now on we shall be using the following notation.

Definition 2.2.5. Let E be a finite set, and let $\mathcal{B}$ be a nonempty collection of subsets of E. The pair $M = (E, \mathcal{B})$ is a *matroid* if for all $B, B' \in \mathcal{B}$ and $b \in B \setminus B'$, there exists $b' \in B' \setminus B$ such that $(B \setminus \{b\}) \cup \{b'\} \in \mathcal{B}$.

If $M = (E, \mathcal{B})$ is a matroid, then E is called the *ground set* of M and the elements of $\mathcal{B}$ are called *bases* of M. Any two bases of M have the same size, which we denote by $r(M)$ and call the *rank* of M. If $r(M) = d$ and $E = [n]$, then we say that M is *representable* if there exists $A \in \mathrm{Mat}_{d,n}(\mathbb{R})$ with columns $A_1, \ldots, A_n$ such that $B \in \mathcal{B}$ precisely when $\{A_b \mid b \in B\}$ is a basis for the vector space $\mathbb{R}^d$.

Definition 2.2.6. The matroid of rank d on the ground set $[n]$ that is represented by a matrix $A \in \mathrm{Mat}^+_{d,n}(\mathbb{R})$ is denoted by $\rho(A)$ and called a *positroid*.

Several families of combinatorial objects, in bijection with positroids, were introduced in [127] to study the totally nonnegative Grassmannian, including decorated permutations, Grassmann necklaces, Le-diagrams, and plabic graphs.

Definition 2.2.7. An n-tuple $(I_1, \ldots, I_n)$ of d-subsets of $[n]$ is called a *Grassmann necklace* of type (d, n) if for every $i \in [n]$ the following two conditions hold:

- $i \in I_i$ implies $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$ for some $j \in [n]$,
- $i \notin I_i$ implies $I_{i+1} = I_i$,

where $I_{n+1} = I_1$.

For $i \in [n]$, the total order $([n], \leq_i)$ is defined by

$$i \leq_i \cdots \leq_i n \leq_i 1 \leq_i \cdots \leq_i i - 1.$$

Given a matroid $M = ([n], \mathcal{B})$ of rank d, one can define the sequence $\mathcal{I}(M) = (I_1, \ldots, I_n)$, where I_i is the lexicographically minimal ordered basis of M with respect to the order $\leq_i$. The sequence $\mathcal{I}(M)$ is a Grassmann necklace of type (d, n) (see [127]). Moreover, when M is a positroid, we can recover M from $\mathcal{I}(M)$ as we will describe now. For $i \in [n]$, consider the partial order $\preceq_i$ on $\binom{[n]}{d}$ defined in the following way: if

$$S = \{s_1 \leq_i \cdots \leq_i s_d\} \quad \text{and} \quad T = \{t_1 \leq_i \cdots \leq_i t_d\}$$

are subsets of $[n]$, then $S \preceq_i T$ if $s_j \leq_i t_j$ for each $j \in [d]$.

Theorem 2.2.8. *[124, Theorem 6] If $\mathcal{I} = (I_1, \ldots, I_n)$ is a Grassmann necklace of type (d, n), then*

$$\mathcal{B}(\mathcal{I}) = \left\{ B \in \binom{[n]}{d} \;\middle|\; I_j \preceq_j B \text{ for each } j \in [n] \right\}$$

is the collection of bases of a positroid $M(\mathcal{I}) = ([n], \mathcal{B}(\mathcal{I}))$. Moreover, $M(\mathcal{I}(M)) = M$ for all positroids M.

By Theorem 2.2.8, the map $P \mapsto \mathcal{I}(P)$ is a one-to-one correspondence between the set of rank d positroids on the ground set $[n]$ and the set of Grassmann necklaces of type (d, n). For a positroid P, we call $\mathcal{I}(P)$ its *corresponding* Grassmann necklace.

Like Grassmann necklaces, decorated permutations are combinatorial objects that can be used to parameterize positroids. Decorated permutations have the extra advantage of offering a more compact parameterization.

Definition 2.2.9. A *decorated permutation* π on n letters is an element $\pi \in S_n$ in which fixed points j are marked either "clockwise"(denoted by $\pi(j) = \underline{j}$) or "counterclockwise" (denoted by $\pi(j) = \overline{j}$). A position $j \in [n]$ is called a *weak excedance* of π if $j < \pi(j)$ or $\pi(j) = \overline{j}$.

Following the next recipe, one can assign a decorated permutation $\pi_{\mathcal{I}}$ to each Grassmann necklace $\mathcal{I} = (I_1, \ldots, I_n)$:

(1) if $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$ for $i \neq j$, then $\pi_{\mathcal{I}}(j) = i$,

(2) if $I_{i+1} = I_i$ and $i \notin I_i$, then $\pi_{\mathcal{I}}(i) = \underline{i}$,

(3) if $I_{i+1} = I_i$ and $i \in I_i$, then $\pi_{\mathcal{I}}(i) = \overline{i}$,

where $I_{n+1} = I_1$.

Moreover, the map $\mathcal{I} \mapsto \pi_{\mathcal{I}}$ is a bijection from the set of Grassmann necklaces of type (d, n) to the set of decorated permutations of n letters having d weak excedances. Indeed, it is not hard to verify that the map $\pi \mapsto (I_1, \ldots, I_n)$, where

$$I_i = \{j \in [n] \mid j \leq_i \pi^{-1}(j) \text{ or } \pi(j) = \overline{j}\},$$

is the inverse of $\mathcal{I} \mapsto \pi_{\mathcal{I}}$. See [17, Proposition 4.6] for more details. The *corresponding* decorated permutation of a positroid P is $\pi_{\mathcal{I}(P)}$, where $\mathcal{I}(P)$ is the corresponding Grassmann necklace of P.

As it is the case for Grassmann necklaces and decorated permutations, Ⅎ-diagrams are in natural bijection with positroids, and they explicitly show the dimensions of the Grassmann cell of their corresponding positroids.

Definition 2.2.10. Let d and m be positive integers, and let Y_λ be the Young diagram associated to a given partition λ contained in a $d \times m$ rectangle. A Ⅎ-*diagram* (or *Le-diagram*) L of shape λ and type $(d, d+m)$ is obtained by filling the boxes of Y_λ with zeros and pluses so that no zero entry has simultaneously a plus entry above it in the same column and a plus entry to its left in the same row.

With notation as in the above definition, the southeast border of Y_λ determines a path of length $d+m$ from the northeast to the southwest corner of the $d \times m$ rectangle; we call such a path the *boundary path* of L.

It is well known that there is a natural bijection Φ from the set of Ⅎ-diagrams of type $(d, d+m)$ to the set of decorated permutations on $[d+m]$ having exactly d excedances (see [127, Section 20]). Thus, Ⅎ-diagrams of type $(d, d+m)$ also parameterize rank d positroids on the ground set $[d+m]$. Moreover, if $\Phi \colon L \to \pi$ and we label the steps of the boundary path of L in southwest direction, then $i \in [d+m]$ labels a vertical step of the boundary path of L if and only if i is a weak excedance of π (see [139, Lemma 5]).

Example 2.2.11. The picture below shows a Ⅎ-diagram L of type $(5, 12)$ with its boundary path highlighted. The decorated permutation $\Phi(L)$ is $(1\ 12\ 9\ 2)(3\ 10\ 11\ 7)(4\ 5)(6\ 8)$.

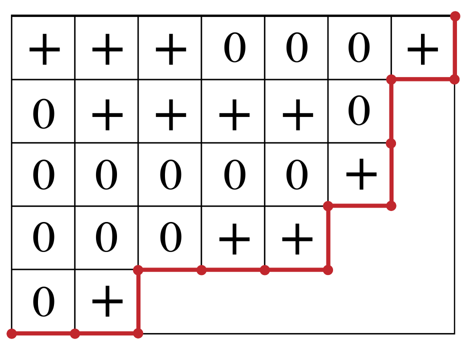

Figure 2.5: A Le-diagram of type $(5, 12)$ and shape $\lambda = (7, 6, 6, 5, 2)$.

Let λ be a partition, and let Y_λ be the Young diagram associated to λ. We call a *pipe dream* of shape λ to a tiling of Y_λ by elbow joints ⌝⌞ and crosses ┼. The next lemma yields a method (illustrated in Figure 2.6) to find the decorated permutation $\pi = \Phi(L)$ corresponding to a positroid directly from its Ⅎ-diagram.

Lemma 2.2.12. *[17, Lemma 4.8] Let L be the Ⅎ-diagram corresponding to a rank d positroid P on the ground set $[d+m]$. We can compute the decorated permutation π of P as follows.*

(1) Replace the pluses in the Ⅎ-diagram L with elbow joints ⌝⌞ and the zeros in L with crosses ┼ to obtain a pipe dream.

(2) Label the steps of the boundary path with $1, \dots, d+m$ in southwest direction, and then label the edges of the north and west border of Y_λ also with $1, \dots, d+m$ in such a way that labels of opposite border steps coincide.

(3) Set $\pi(i) = j$ if the pipe starting at the step labeled by i in the northwest border ends at the step labeled by j in the boundary path. If π fixes j write $\pi(j) = \underline{j}$ (resp., $\pi(j) = \overline{j}$) if j labels a horizontal (resp., vertical) step of the boundary path.

Example 2.2.13. Let P be the rank 5 positroid on the ground set $[13]$ having decorated permutation $\pi = (1\ 2\ 13\ 12\ 3\ 11\ 10\ 4\ 9\ 5\ 8\ 7\ 6)$. The following picture showing the ⅃-diagram corresponding to P along with its associated pipe dream sheds light upon the recipe described in Lemma 2.2.12.

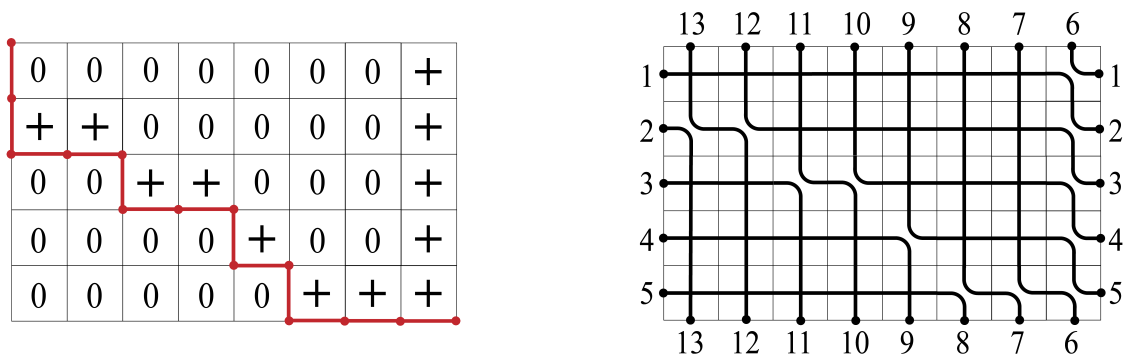

Figure 2.6: The Le-diagram of P on the left and the corresponding pipe dream giving rise to π on the right.

2.3 Canonical Labelings on Unit Interval Orders

In this section we introduce the concept of a *canonically* labeled poset, and we use it to exhibit an explicit bijection from the set $\mathcal{U}_n$ of non-isomorphic unit interval orders of cardinality n to the set $\mathcal{D}_n$ of $n \times n$ Dyck matrices.

We define the *altitude* function of P to be the map

$$\alpha \colon P \to \mathbb{Z} \quad \text{defined by} \quad i \mapsto |\Lambda_i| - |\mathrm{V}_i|.$$

We say that an n-labeled poset P *respects* altitude if for all $i, j \in P$, the fact that $\alpha(i) < \alpha(j)$ implies $i < j$ (as integers). Notice that every poset can be labeled by the set $[n]$ such that, as an n-labeled poset, it respects altitude (see [63, p. 33]).

Definition 2.3.1. An n-labeled poset is *canonically labeled* if it respects altitude.

Each canonically n-labeled poset is, in particular, naturally labeled. The next proposition, extending [142, proof of Theorem 2.11], characterizes canonically n-labeled unit interval orders in terms of their antiadjacency matrices.

Proposition 2.3.2. *[134, Proposition 5] An n-labeled unit interval order is canonically labeled if and only if its antiadjacency matrix is a Dyck matrix.*

The above proposition indicates that the antiadjacency matrices of canonically labeled unit interval orders are quite special. In addition, canonically labeled unit interval orders have very convenient interval representations.

Proposition 2.3.3. *Let P be an n-labeled unit interval order. Then the labeling of P is canonical if and only if there exists an interval representation*

$$\{[q_i, q_i + 1] \mid 1 \le i \le n\}$$

of P such that $q_1 < \cdots < q_n$.

Proof. Let $\alpha\colon P \to \mathbb{Z}$ be the altitude map of P. For the forward implication, suppose that the n-labeling of P is canonical. Among all the interval representations of P, assume that $\{[q_i, q_i + 1] \mid 1 \le i \le n\}$ gives the maximum $m \in [n]$ such that $q_1 < \cdots < q_m$. Suppose, by way of contradiction, that $m < n$. The maximality of m implies that $q_m > q_{m+1}$. This, along with the fact that $\alpha(m) \le \alpha(m+1)$, ensures that $q_m \in (q_{m+1}, q_{m+1} + 1)$. Similarly, $q_i + 1 \notin (q_{m+1}, q_m)$ for any $i \in [n]$; otherwise

$$\alpha(m+1) = |\Lambda_{m+1}| - |\mathrm{V}_{m+1}| < |\Lambda_m| - |\mathrm{V}_{m+1}| \le |\Lambda_m| - |\mathrm{V}_m| = \alpha(m)$$

would contradict that the n-labeling of P respects altitude. An analogous argument guarantees that $q_i \notin (q_{m+1} + 1, q_m + 1)$ for any $i \in [n]$.

Now take k to be the smallest natural number in $[m]$ such that $q_j > q_{m+1}$ for all $j \ge k$, and take $\sigma = (k \;\; k{+}1 \;\; \dots \;\; m \;\; m{+}1) \in S_n$. We will show that $S = \{[p_i, p_i + 1] \mid 1 \le i \le n\}$, where $p_i = q_{\sigma(i)}$, is an interval representation of P. Take $i, j \in P$ such that $i \le_P j$. Since i and j are comparable in P, at least one of them must be fixed by σ; say $\sigma(i) = i$. If $\sigma(j) = j$, then $p_i + 1 = q_i + 1 < q_j = p_j$. Also, if $\sigma(j) \ne j$, then $q_i + 1 < q_j \in (q_{m+1}, q_m)$. It follows from $q_i + 1 < q_m$ that

$$p_i + 1 = q_i + 1 < q_{m+1} < q_{\sigma(j)} = p_j.$$

The case of $\sigma(j) = j$ can be argued similarly. Thus, S is an interval representation of P. As $q_1 < \cdots < q_m$, the definition of k implies that $p_1 < \cdots < p_{m+1}$, which contradicts the maximality of m. Hence $m = n$, and the direct implication follows.

Conversely, note that if $\{[q_i, q_i + 1] \mid 1 \le i \le n\}$ is an interval representation of P satisfying $q_1 < \cdots < q_n$, then for every $m \in [n-1]$ we have

$$\alpha(m) = |\Lambda_m| - |\mathrm{V}_m| \le |\Lambda_{m+1}| - |\mathrm{V}_{m+1}| = \alpha(m+1),$$

which means that the labeling of P is canonical. □

If P is a canonically n-labeled unit interval order, and

$$\mathcal{I} = \{[q_i, q_i + 1] \mid 1 \leq i \leq n\}$$

is an interval representation of P satisfying $q_1 < \cdots < q_n$, then we say that $\mathcal{I}$ is a *canonical* interval representation of P.

Note that the image (as a multiset) of the altitude map does not depend on the labels but only on the isomorphism class of the corresponding poset. On the other hand, the altitude map α_P of a canonically n-labeled unit interval order P satisfies $\alpha_P(1) \leq \cdots \leq \alpha_P(n)$. Thus, if Q is a canonically n-labeled unit interval order isomorphic to P, then

$$(\alpha_P(1), \ldots, \alpha_P(n)) = (\alpha_Q(1), \ldots, \alpha_Q(n)), \tag{2.1}$$

where α_Q is the altitude map of Q. Let A_P and A_Q be the antiadjacency matrices of P and Q, respectively. As $\alpha_P(1) = \alpha_Q(1)$, the first rows of A_P and A_Q are equal. Since the number of zeros in the i-th column (resp., i-th row) of A_P is precisely $|\mathrm{V}_i(P)| - 1$ (resp., $|\Lambda_i(P)| - 1$), and similar statement holds for Q, the next lemma follows immediately by using (2.1) and induction on the row index of A_P and A_Q.

Lemma 2.3.4. *If two canonically labeled unit interval orders are isomorphic, then they have the same antiadjacency matrix.*

The Bijection φ

Now we can define a map $\varphi\colon \mathcal{U}_n \to \mathcal{D}_n$, by assigning to each unit interval order its antiadjacency matrix with respect to any of its canonical labelings. By Lemma 2.3.4, this map is well defined.

Theorem 2.3.5. *For each natural number n, the map $\varphi\colon \mathcal{U}_n \to \mathcal{D}_n$ is a bijection.*

Proof. Since $|\mathcal{U}_n| = |\mathcal{D}_n| = \frac{1}{n+1}\binom{2n}{n}$, it suffices to argue that φ is surjective. We proceed by induction on n. The case $n = 1$ is immediate as $|\mathcal{U}_1| = |\mathcal{D}_1| = 1$. Suppose that surjectivity holds for every $k \leq n$ and, to check that $\varphi\colon \mathcal{U}_{n+1} \to \mathcal{D}_{n+1}$ is surjective, take $D = (d_{i,j}) \in \mathcal{D}_{n+1}$. Let D' be the submatrix of D consisting of the first n columns and the first n rows. As D' is an $n \times n$ Dyck matrix, there is a canonically n-labeled unit interval order P' whose antiadjacency matrix is D'. Define P to be the $(n+1)$-labeled poset obtained by adding an element labeled by $n+1$ to P' with exactly the following order relations: $i \leq_P n+1$ if and only if either $i = n + 1$ or $d_{i,n+1} = 0$. Note that $n + 1$ is a maximal element in P and that the antiadjacency matrix of P is precisely D.

We are done once we check that P is a canonically labeled unit interval order. Since $\alpha_P(1) \leq \cdots \leq \alpha_P(n + 1)$, the labeling of P is canonical. Finally, let us show that P is, indeed, a unit interval order. Because P' happens to be a unit interval order, it suffices to check that for any $i, j, k \in [n]$ none of the posets

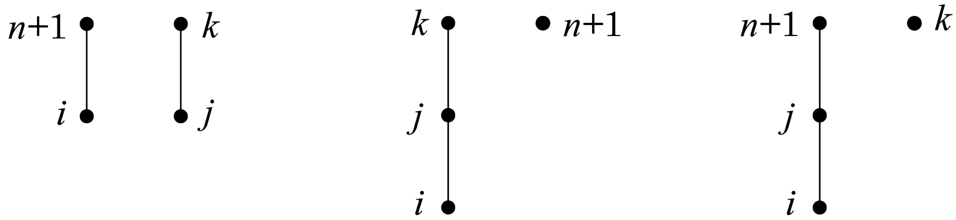

Figure 2.7: $\mathbf{2}+\mathbf{2}$ and $\mathbf{3}+\mathbf{1}$ posets.

is an induced subposet of P. The first and the second subposets in Figure 2.7 cannot be induced because $j \leq_P n+1$ for every non-maximal element j of P'. Let Q denote the third subposet shown above. If $k \leq_P n+1$, then Q cannot be induced. Suppose then that k is not comparable with $n+1$ in P. In this case, k is maximal in P. As j is not maximal in P and the labeling of P is canonical, $i < j < k$ as integers. Since $i \leq_P j$, one has that $d_{i,j} = 0$ and so $d_{i,k} = 0$. Thus, $i \leq_P k$, which implies that Q is not an induced subposet of P. Hence P is a canonically $(n+1)$-labeled unit interval order, which concludes the proof. □

2.4 Dyck Positroids

Rational Dyck Paths and Matrices

In the first two chapters of this thesis, we are mostly interested in certain class of positroids that can be parameterized by rational Dyck paths.

Definition 2.4.1. For each pair of nonnegative integers (m, d), a *rational Dyck path* of type (m, d) is a lattice path from $(0, 0)$ to (m, d) that only uses unit steps $(1, 0)$ or $(0, 1)$ and never goes above the diagonal line $y = (d/m)x$.

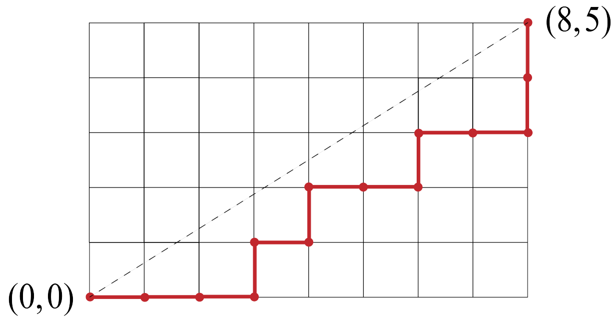

Figure 2.8: A $(8, 5)$-rational Dyck path.

When there is no risk of ambiguity, we will abuse notation by referring to a rational Dyck path without specifying the copy of $\mathbb{R}^2$ in which it is embedded. Figure 2.8 depicts a rational Dyck path of type $(8,5)$. Note that a rational Dyck path of type (m,m) is just an ordinary Dyck path of length $2m$. The number of Dyck paths of length $2m$ is precisely the m-th Catalan number (many other families of relevant combinatorial objects can also be counted by the Catalan numbers; see [137]). The number $\mathrm{Cat}(m,d)$ of rational Dyck paths of type (m,d) is the *rational Catalan number* associated to the pair (m,d). It is known that

$$\mathrm{Cat}(m,d) = \frac{1}{d+m}\binom{d+m}{d} \tag{2.2}$$

when $\gcd(d,m)=1$. A general formula for the rational Catalan numbers (without assuming co-primeness) was first conjectured by Grossman [106] and then proved by Bizley [34]. This general formula is more involved than the one stated in (2.2), as the next generating function shows:

$$\sum_{n=0}^{\infty}\mathrm{Cat}(nm,nd)x^n = \exp\left(\sum_{j=1}^{\infty}\frac{1}{d+m}\binom{jd+jm}{jd}\frac{x^j}{j}\right),$$

where, as before, $\gcd(d,m)=1$. The combinatorics associated to the rational Catalan numbers, also known as rational Catalan combinatorics, has received considerable attention during the last decade. In particular, rational Dyck paths have been studied in connection with core partitions [13], parking functions [21], noncrossing partitions [22], and rational associahedra [23]. For several results and conjectures on (m,d)-cores, the reader can consult [20] and [138].

Let us use the fact that that rational Dyck paths are natural generalization of Dyck paths to generalize the class of Dyck matrices.

Definition 2.4.2. A $d\times m$ binary matrix is called a *rational Dyck matrix* if its zero entries are separated from its one entries by a vertically-reflected rational Dyck path of type (m,d).

Observe that square rational Dyck matrices are precisely those that we have called before Dyck matrices. Here is an example of a 5×8 rational Dyck matrix:

$$\begin{pmatrix}1&1&1&0&0&0&0&0\\1&1&1&1&0&0&0&0\\1&1&1&1&1&1&0&0\\1&1&1&1&1&1&1&1\\1&1&1&1&1&1&1&1\end{pmatrix}.$$

Let $\mathcal{D}_{d,m}$ denote the set of $d\times m$ rational Dyck matrices. It is clear that $\mathcal{D}_n=\mathcal{D}_{n,n}$ for every $n\in\mathbb{N}$. Each rational Dyck path d of type (m,d) induces the $d\times m$ rational Dyck matrix whose zero entries are separated from its one entries via the vertically-reflected path of d. It is well known that standard Dyck matrices are *totally nonnegative*, i.e., all their minors are nonnegative (see, for instance, [1]).

Notation: If X is an $n \times n$ real matrix and $I, J \subseteq [n]$ satisfy $|I| = |J|$, then we let $\Delta_{I,J}(X)$ denote the minor of X determined by the set of rows indexed by I and the set of columns indexed by J. Besides, if Y is a $k \times n$ matrix and $K \subseteq [n]$ satisfies $|K| = k$, then we let $\Delta_K(Y)$ denote the maximal minor of Y determined by the set of columns indexed by K.

Consider the assignment $\phi_{d,m} \colon \mathrm{Mat}_{d,m}(\mathbb{R}) \to \mathrm{Mat}_{d,d+m}(\mathbb{R})$ defined by

$$\begin{pmatrix} a_{1,1} & \dots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{d-1,1} & \dots & a_{d-1,m} \\ a_{d,1} & \dots & a_{d,m} \end{pmatrix} \overset{\phi_{d,m}}{\longmapsto} \begin{pmatrix} 1 & \dots & 0 & 0 & (-1)^{d-1}a_{d,1} & \dots & (-1)^{d-1}a_{d,m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & -a_{2,1} & \dots & -a_{2,m} \\ 0 & \dots & 0 & 1 & a_{1,1} & \dots & a_{1,m} \end{pmatrix}.$$

The map $\phi_{d,m}$ somehow respects the minors of any given matrix:

Lemma 2.4.3. *[127, Lemma 3.9]*[1] *If $A \in \mathrm{Mat}_{d,m}(\mathbb{R})$ and $B = \phi_{d,m}(A)$, then*

$$\Delta_{I,J}(A) = \Delta_{(d+1-[d]\setminus I)\cup(d+J)}(B)$$

for all $I \subseteq [d]$ and $J \subseteq [m]$ satisfying $|I| = |J|$.

The next lemma, which can be immediately argued by induction, is used in the proof of Proposition 2.4.5.

Lemma 2.4.4. *Every square binary matrix whose zero entries form a Young diagram anchored in the upper-right corner is totally nonnegative.*

Proposition 2.4.5. *The inclusion $\phi_{d,m}(\mathcal{D}_{d,m}) \subseteq \mathrm{Mat}^+_{d,d+m}(\mathbb{R})$ holds.*

Proof. Take $D \in \mathcal{D}_{d,m}$, and set $A = \phi_{d,m}(D)$. As A has obviously full rank, it suffices to verify that each of its maximal minors is nonnegative. For $S \in \binom{[d+m]}{d}$ let A' be the submatrix of A determined by the set of columns indexed by S. Set $I = S \cap [d]$ and $J = \{j_1, \dots, j_{|S\setminus I|}\} = S \setminus I$ with $j_1 < \dots < j_{|J|}$. Note that $|J| \leq d$. Let B_J be the $d \times d$ matrix whose first $|I|$ columns are all equal to the vector $((-1)^{d-1}, \dots, -1, 1)^t$ and whose $(|I|+k)$-th column is equal to A_{j_k} for $1 \leq k \leq |J|$. Notice now that Lemma 2.4.4 ensures that the matrix $B = (I_d \mid B_J)$ is the image under $\phi_{d,d}$ of a totally nonnegative matrix of size d. As Dyck matrices are totally nonnegative, Lemma 2.4.3 ensures that every maximal minor of B is nonnegative. In particular, the maximal minor $\det A'$ of B is nonnegative. Hence $A \in \mathrm{Mat}^+_{d,d+m}(\mathbb{R})$, as desired. $\square$

Proposition 2.4.5 will allow us to produce positroids from rational Dyck paths.

[1]There is a typo in the entries of the matrix B in [127, Lemma 3.9].

Dyck Positroids

Using Lemma 2.4.3 and the map $\varphi\colon \mathcal{U}_n \to \mathcal{D}_n$ introduced at the end of Section 2.3, we can assign via $\phi_{n,n} \circ \varphi$ a matrix in $\text{Mat}^+_{n,2n}(\mathbb{R})$ to each unit interval order of cardinality n. In turn, every real matrix of $\text{Mat}^+_{n,2n}(\mathbb{R})$ gives rise to a positroid, a special representable matroid which has a very rich combinatorial structure.

Each unit interval order P (labeled so that its antiadjacency matrix is a Dyck matrix) induces a positroid via Lemma 2.4.3, namely, the positroid represented by the matrix $\phi_{n,n}(\varphi(P))$.

Definition 2.4.6. A positroid on $[2n]$ induced by a unit interval order is called a *Dyck positroid*.

We denote by $\mathcal{P}_n$ the set of all Dyck positroids on the ground set $[2n]$. The function $\rho \circ \phi_{n,n} \circ \varphi\colon \mathcal{U}_n \to \mathcal{P}_n$ plays a fundamental role in this chapter. Indeed, we will end up proving that this function is a bijection (see Theorem 2.5.4).

We use decorated permutations to provide a compact and elegant description of Dyck positroids.

The Decorated Permutation

We proceed to describe the decorated permutation associated to a Dyck positroid. Throughout this section A is an $n \times n$ Dyck matrix and

$$B = (b_{i,j}) = \phi_{n,n}(A)$$

is as in Lemma 2.4.3. We will consider the indices of the columns of B modulo $2n$. Furthermore, let P be the Dyck positroid represented by B, and let $\mathcal{I}_P$ and π^{-1} be the Grassmann necklace and the decorated permutation associated to P.

Lemma 2.4.7. *For $1 < i \le 2n$, the i-th coordinate set of $\mathcal{I}_P$ does not contain $i-1$.*

Proof. It is not hard to verify that every matrix resulting from removing one column from B still has rank n. As the matrix obtained by removing the $(i-1)$-st column from B has rank n, it contains n linearly independent columns. Therefore the lemma follows straightforwardly from the $<_i$-minimality of the i-th coordinate set of $\mathcal{I}_P$. □

For the rest of this section let B_j denote the j-th column of B. As a direct consequence of Lemma 2.4.7, we have that π does not have any counterclockwise fixed point. On the other hand, π cannot have any clockwise fixed point because every column of B is nonzero. Hence π (and therefore π^{-1}) does not fix any point. The next lemma immediately follows from the way π^{-1} is produced from the Grassmann necklace $\mathcal{I}_P$ (see the end of Section 2.2).

Lemma 2.4.8. *For $i \in \{1, \dots, 2n\}$, $\pi(i)$ equals the minimum $j \in [2n]$ with respect to the i-order such that $B_i \in \text{span}(B_{i+1}, \dots, B_j)$.*

Now we find an explicit expression for the function representing the inverse π of the decorated permutation π^{-1} associated to P. In order to do so, we will find it convenient to associate an index set and a map to the matrix B. We define the *set of principal indices* of B to be the subset of $\{n+1,\dots,2n\}$ defined by

$$J = \{j \in \{n+1,\dots,2n\} \mid B_j \neq B_{j-1}\}.$$

We associate to B the *weight* map $\omega\colon [2n] \to [n]$ defined by $\omega(j) = \max\{i \mid b_{i,j} \neq 0\}$; more explicitly, we obtain that

$$\omega(j) = \begin{cases} j & \text{if } j \in \{1,\dots,n\} \\ |b_{1,j}| + \cdots + |b_{n,j}| & \text{if } j \in \{n+1,\dots,2n\}. \end{cases}$$

Since the last row of the antiadjacency matrix A has all its entries equal to 1, the map ω is well defined. If $j \in \{n+1,\dots,2n\}$, then $\omega(j)$ is the number of nonzero entries in the column B_j. Now we have the following formula for π.

Proposition 2.4.9. *For $i \in \{1,\dots,2n\}$, we have*

$$\pi(i) = \begin{cases} i+1 & \text{if } n < i < 2n \text{ and } i+1 \notin J, \\ \omega(i) & \text{if } n < i < 2n \text{ and } i+1 \in J, \text{ or } i = 2n, \\ n+1 & \text{if } i = 1, \\ i-1 & \text{if } 1 < i \leq n \text{ and } i-1 \notin \omega(J), \\ j & \text{if } 1 < i \leq n \text{ and } i-1 = \omega(j) \text{ for some } j \in J. \end{cases}$$

The index j in the final case is necessarily unique.

Proof. First, suppose that $n < i < 2n$ and $i+1 \notin J$. Then we have $B_i = B_{i+1}$ and the set $\{B_i, B_{i+1}\}$ is linearly dependent. Lemma 2.4.8 then implies that $\pi(i) = i+1$.

Now suppose that $n < i < 2n$ and $i+1 \in J$. Then B_{i+1} results from replacing m ($m > 0$) of the last nonzero entries of B_i by zeros. Since $i+1 \in J$, the indices i and $i+1$ both appear in the i-th coordinate set of $\mathcal{I}_P$. Also, because the columns $B_i, B_{i+1}, B_{\omega(i+1)+1}, \dots, B_{\omega(i)}$ are linearly dependent, not all the indices $\omega(i+1)+1,\dots,\omega(i)$ appear in the i-th coordinate set of $\mathcal{I}_P$. On the other hand, at most one index in $\omega(i+1)+1,\dots,\omega(i)$ is missing from the i-th coordinate of $\mathcal{I}_P$; this is because the submatrix of B determined by the row-index set $\{\omega(i+1)+1,\dots,\omega(i)\}$ and the column-index set $\{n+1,\dots,2n\}$ has rank 1. By the minimality of the i-th coordinate set of $\mathcal{I}_P$ with respect to the i-order, the index of $\{\omega(i+1)+1,\dots,\omega(i)\}$ missing in the i-th coordinate set of $\mathcal{I}_P$ is $\omega(i)$. As a result, we have $\pi(i) = \omega(i)$; otherwise, in the submatrix of B whose columns are indexed by the $(i+1)$-st coordinate set of $\mathcal{I}_P$, the $\omega(i)$-th row would consist entirely of zeros, which, in turn, would contradict the fact that such a coordinate set represents a basis of the positroid P.

The above argument also applies when $i = 2n$ provided that we extend the domain of ω to $[2n+1]$ and set $\omega(2n+1) = 0$.

Note that $\pi(1) = n+1$ follows immediately from the minimality of the second coordinate set of $\mathcal{I}_P$ and the fact that $B_2,\dots,B_n,B_{n+1}$ are linearly independent.

Now suppose that $1 < i \leq n$ and $i-1 \notin \omega(J)$. The minimality of the coordinate sets of $\mathcal{I}_P$ implies that all the indices $i, \dots, n$ appear in the i-th coordinate set. Furthermore, Lemma 2.4.7 implies that $i-1$ does not belong to the i-th coordinate set of $\mathcal{I}_P$. Since no $j \in J$ has weight $i-1$, the $(i-1)$-st and i-th rows of the maximal submatrix of B determined by the column index set $\{n+1, \dots, 2n\}$ are equal. Consequently, we have $\pi(i) = i-1$; otherwise the associated maximal submatrix of B determined by the indices of the i-th coordinate set of $\mathcal{I}_P$ would have the i-th and $(i+1)$-st rows identical, which would contradict the fact that the i-th coordinate set of $\mathcal{I}_P$ represents a basis of P.

Finally, suppose that $1 < i \leq n$ and $i-1 \in \omega(J)$. Since not two elements of J have the same weight, there is at most one $j \in J$ such that $\omega(j) = i-1$. As before, all the indices $i, \dots, n+1$ appear in the i-th coordinate set of $\mathcal{I}_P$ (because $i > 1$). Each column B_k, for $n < k \leq 2n$ such that $\omega(k) = i-1$, is a linear combination of the columns $B_i, \dots, B_{n+1}$. Therefore such indices k do not appear in the i-th coordinate set of $\mathcal{I}_P$. By Lemma 2.4.7, it follows that $i-1$ does not appear in the i-th coordinate set of $\mathcal{I}_P$. Thus, $\pi(i) = j$, where $j \in [2n]$ satisfies that $\omega(j) = i-1$; otherwise, in the submatrix of B whose columns are indexed by the $(i+1)$-st coordinate set of $\mathcal{I}_P$, the $(i-1)$-st row would consist entirely of zeros, which would contradict that the $(i-1)$-st coordinate set of $\mathcal{I}_P$ represents a basis of P. By minimality of the $(i+1)$-st coordinate set of $\mathcal{I}_P$ one finds that $j \in J$. □

As the next theorem indicates, π^{-1} is a $2n$-cycle satisfying a very special property.

Theorem 2.4.10. *π^{-1} is a $2n$-cycle $(1\ j_1\ \dots\ j_{2n-1})$ satisfying the next two conditions:*

(1) in the sequence $(1, j_1, \dots, j_{2n-1})$ the elements $1, \dots, n$ appear in increasing order while the elements $n+1, \dots, 2n$ appear in decreasing order;

(2) for every $1 \leq k \leq 2n-1$, the set $\{1, j_1, \dots, j_k\}$ contains at least as many elements of the set $\{1, \dots, n\}$ as elements of the set $\{n+1, \dots, 2n\}$.

Proof. From Proposition 2.4.9 we immediately deduce that if $\pi(i) = j$ for $1 < i \leq 2n$, then $\omega(i) = \omega(j)$ when $i > n$ and $\omega(i) = \omega(j) + 1$ when $i \leq n$. This implies, in particular, that $\omega(i) \geq \omega(j)$. Suppose, by way of contradiction, that π^{-1}, and so π, is not a $2n$-cycle. Then there is a cycle $(i_1\ i_2\ \dots\ i_k)$ in the canonical cycle-type decomposition of π that does not contain 1. Therefore one has

$$\omega(i_1) \geq \omega(i_2) \geq \cdots \geq \omega(i_k) \geq \omega(i_1),$$

which implies $\omega(i_1) = \omega(i_2) = \cdots = \omega(i_k)$. Since $\{i_1, \dots, i_k\}$ does not contain 1, it follows that $\{i_1, \dots, i_k\} \subseteq \{n+1, \dots, 2n\}$, which is a contradiction. Hence the cycle-type decomposition of π^{-1} contains only one cycle, which has length $2n$.

Since $\pi(1) = n+1$, one gets that $\pi = (1\ n{+}1\ i_1\ i_2\ \dots\ i_{2n-2})$, where $\{i_1, \dots, i_{2n-2}\}$ is precisely the set $[2n] \setminus \{1, n+1\}$. As

$$\omega(i_1) \geq \omega(i_2) \geq \cdots \geq \omega(i_{2n-2}),$$

and $\omega(i) = i$ for every $i \in [n]$, the elements of the set $\{2, \dots, n\}$ must appear in the cycle $(1\ n{+}1\ i_1\ i_2\ \dots\ i_{2n-2})$ in decreasing order. On the other hand, by Proposition 2.4.9 the indices of equal columns of B (but perhaps the first one) show in increasing order and consecutively in the sequence $(1, n+1, i_1, i_2, \dots, i_{2n-2})$. Also, as the weight map ω is strictly decreasing when restricted to J, the elements of the set $\{n+1, \dots, 2n\}$ must appear in increasing order in the cycle $(1\ n{+}1\ i_1\ i_2\ \dots\ i_{2n-2})$. Thus, condition (1) holds.

To show condition (2), write $\pi = (n{+}1\ i_1\ i_2\ \dots\ i_{2n-2}\ 1)$ and suppose, by way of contradiction, that there exists $m \in \{1, \dots, 2n-2\}$ such that

$$\Big|\big\{1 \le j \le m \mid i_j \in \{2, \dots, n\}\big\}\Big| - 1 > \Big|\big\{1 \le j \le m \mid i_j \in \{n+1, \dots, 2n\}\big\}\Big|. \tag{2.3}$$

Let m be the minimal such index. By the minimality of m, one obtains that $i_m \in \{2, \dots, n\}$. Let k be the maximum index such that $m \le k$ and $i_j \in \{2, \dots, n\}$ for each $j = m, \dots, k$. Note that $k < 2n-2$ and $\pi(i_k) \in \{n+2, \dots, 2n\}$. Since

$$|\{j \le k \mid 2 \le i_j \le n\}| = |\{i_k, \dots, n\}|$$

and

$$|\{j \le k \mid n+2 \le i_j \le 2n\}| = |\{n+2, \dots, \pi(i_k) - 1\}|,$$

it follows by (2.3) that

$$(n - i_k + 1) - 1 > (\pi(i_k) - 1) - (n+2) + 1 = \pi(i_k) - n - 2,$$

which implies $2n - \pi(i_k) + 1 > i_k - 1$. On the other hand, the fact that all the entries of A below and on the main diagonal equal 1 implies that $\omega(j) \ge 2n{-}j{+}1$ for every $n{+}1 \le j \le 2n$. Since $1 < i_k \le n$, one finds that $i_k = \omega(i_k) = \omega(\pi(i_k)) + 1$. As $n+1 \le \pi(i_k) \le 2n$, we have

$$i_k - 1 = \omega(\pi(i_k)) \ge 2n - \pi(i_k) + 1 > i_k - 1,$$

which is a contradiction. Hence, writing $\pi^{-1} = (1\ j_1\ \dots\ j_{2n-1})$, we will obtain that for $k = 1, \dots, 2n-1$, the set $\{1, j_1, \dots, j_k\}$ contains at least as many elements of the set $[n]$ as elements of the set $\{n+1, \dots, 2n\}$, which is condition (2). ☐

2.5 How to Decode a Dyck Positroid from Its Unit Interval Order

Decoding a Dyck Positroid from the Antiadjacency Matrix of its Unit Interval Order

Throughout this section, let P be a canonically n-labeled unit interval order with antiadjacency matrix A. Also, let

$$\mathcal{I} = \{[q_i, q_i + 1] \mid 1 \le i \le n\}$$

be a canonical interval representation of P (i.e., $q_1 < \cdots < q_n$); Proposition 2.3.3 ensures the existence of such an interval representation. In this section we describe a way to obtain the decorated permutation associated to the Dyck positroid induced by P directly from either A or $\mathcal{I}$. Such a description will reveal that the function $\rho \circ \phi_{n,n} \circ \varphi \colon \mathcal{U}_n \to \mathcal{P}_n$ introduced in Section 2.2 is a bijection (Theorem 2.5.4).

Recall that the north and east borders of the Young diagram formed by the nonzero entries of A give a path of length $2n$ that we call the *semiorder path* of A. Let $B = (I_n|A') = \phi_{n,n}(A)$, where $\phi_{n,n}$ is the map introduced in Lemma 2.4.3. We will also associate a second path to A. Let the *inverted path* of A be the path consisting of the south and east borders of the Young diagram formed by the nonzero entries of A'. Note that the inverted path of A is just the reflection over a horizontal line of the semiorder path of A. Example 2.5.2 sheds light upon the statement of the next theorem, which describes a way to find the decorated permutation associated to the Dyck positroid induced by P directly from A.

Theorem 2.5.1. *If we number the n vertical steps of the semiorder path of A from bottom to top in increasing order with $\{1, \dots, n\}$ and the n horizontal steps from left to right in increasing order with $\{n+1, \dots, 2n\}$, then by reading the semiorder path in the northwest direction, we obtain the decorated permutation associated to the Dyck positroid induced by P.*

Proof. Let π^{-1} be the decorated permutation associated to the Dyck positroid induced by P. We label the n vertical steps of the inverted path of P from top to bottom in increasing order using the label set $[n]$, and we label the n horizontal steps from left to right in increasing order using the label set $\{n+1, \dots, 2n\}$ (see Example 2.5.2). Proving the theorem amounts to showing that we can obtain π (the inverse of the decorated permutation) by reading the inverted path in the northeast direction. Let $(s_1, s_2, \dots, s_{2n})$ be the finite sequence obtained by reading the inverted path in the northeast direction. Since the first step of the inverted path is horizontal and the last step of the inverted path is vertical, $s_1 = n+1$ and $s_{2n} = 1$. Thus, it suffices to check that $\pi(s_k) = s_{k+1}$ for $k = 1, \dots, 2n-1$.

Suppose first that the k-th step of the inverted path is horizontal, and so located right below the last nonzero entry of the s_k-th column of B. If the $(k+1)$-st step is also horizontal, then $s_{k+1} = s_k + 1$, which means that $\pi(s_k) = s_k + 1$ and so $\pi(s_k) = s_{k+1}$. On the other hand, if the $(k+1)$-st step is vertical, then $s_k = 2n$ or $s_k + 1$ is in the set of principal indices J of B; in both cases, $\pi(s_k) = \omega(s_k)$, the number of vertical steps from the top to s_k, namely, s_{k+1}. Hence $\pi(s_k) = s_{k+1}$.

Suppose now that the k-th step of the inverted path is vertical. Clearly, this implies that $1 \le s_k \le n$. If the $(k+1)$-st step is also vertical, then $s_{k+1} = s_k - 1$. Because steps k and $k+1$ are both vertical, A' does not contain any column with weight $s_k - 1$. As a result, $\pi(s_k) = s_k - 1 = s_{k+1}$. Finally, if the $(k+1)$-st step is horizontal, then $\{s_{k+1}\} = J \cap \omega^{-1}(s_k - 1)$ and, by Proposition 2.4.9, we find that $\pi(s_k) = s_{k+1}$. ☐

Example 2.5.2. In Figure 2.9, we can see displayed the antiadjacency matrix A of the canonically 5-labeled unit interval order P introduced in Example 2.1.1 and the matrix

$\phi_{5,5}(A)$ both showing their respective semiorder and inverted path encoding the decorated permutation $\pi = (1\ 2\ 10\ 3\ 9\ 4\ 8\ 7\ 5\ 6)$ associated to the positroid induced by P.

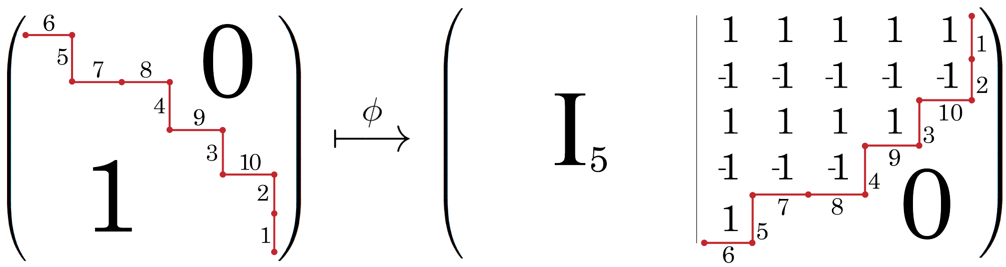

Figure 2.9: Dyck matrix A and its image $\phi_{5,5}(A)$ exhibiting the decorated permutation π along their semiorder path and inverted path, respectively.

As a consequence of Theorem 2.5.1, we can deduce that the map $\rho \circ \phi_{n,n} \circ \varphi \colon \mathcal{U}_n \to \mathcal{P}_n$ is indeed a bijection.

Lemma 2.5.3. *The set of $2n$-cycles $(1\ j_1\ \dots\ j_{2n-1})$ satisfying conditions (1) and (2) of Theorem 2.4.10 is in bijection with the set of Dyck paths of length $2n$.*

Proof. We can assign a Dyck path D of length $2n$ to the $2n$-cycle $(1{=}j_0\ j_1\ \dots\ j_{2n-1})$ by thinking of the entries $j_i \in \{1, \dots, n\}$ as ascending steps of the Dyck path D and the entries $j_i \in \{n+1, \dots, 2n\}$ as descending steps of D. The fact that such an assignment yields the desired bijection is straightforward. □

Theorem 2.5.4. *The map $\rho \circ \phi_{n,n} \circ \varphi \colon \mathcal{U}_n \to \mathcal{P}_n$ is a bijection.*

Proof. By definition of $\mathcal{P}_n$, it follows that $\rho \circ \phi_{n,n} \circ \varphi$ is surjective. Since $|\mathcal{U}_n|$ is the n-th Catalan number, it suffices to show that

$$|\mathcal{P}_n| \geq \frac{1}{n+1}\binom{2n}{n}.$$

To see this, one can take a $2n$-cycle $\sigma = (1\ j_1\ \dots\ j_{2n-1})$ satisfying conditions (1) and (2) of Theorem 2.4.10, and consider the Dyck path D specified by σ as in Lemma 2.5.3. By Theorem 2.5.1, the Dyck matrix whose semiorder path is the reverse of D induces a Dyck positroid with decorated permutation σ. Because the decorated permutation associated to a positroid is unique, Lemma 2.5.3 guarantees that $|\mathcal{P}_n| \geq \frac{1}{n+1}\binom{2n}{n}$. Hence $\rho \circ \phi_{n,n} \circ \varphi$ is bijective. □

Corollary 2.5.5. *The number of Dyck positroids on the ground set $[2n]$ equals the n-th Catalan number.*

Decoding a Dyck Positroid from the Interval Representation of its Unit Interval Order

We conclude this section by describing how to decode the decorated permutation associated to the Dyck positroid induced by P directly from its canonical interval representation $\mathcal{I}$. Labeling the left and right endpoints of the intervals $[q_i, q_i + 1] \in \mathcal{I}$ by the signs $-$ and $+$, respectively, we obtain a $2n$-tuple consisting of pluses and minuses by reading from the real line the labels of the endpoints of all such intervals. On the other hand, we can have another *plus-minus* $2n$-tuple if we replace the horizontal and vertical steps of the semiorder path of A by the signs $-$ and $+$, respectively, and then read it in southeast direction as indicated in the following example.

Example 2.5.6. The figure below shows the antiadjacency matrix of the canonically 5-labeled unit interval order P from Example 2.1.1 and a canonical interval representation of P, both encoding the plus-minus 10-tuple $(-,+,-,-,+,-,+,-,+,+)$, as described in the previous paragraph.

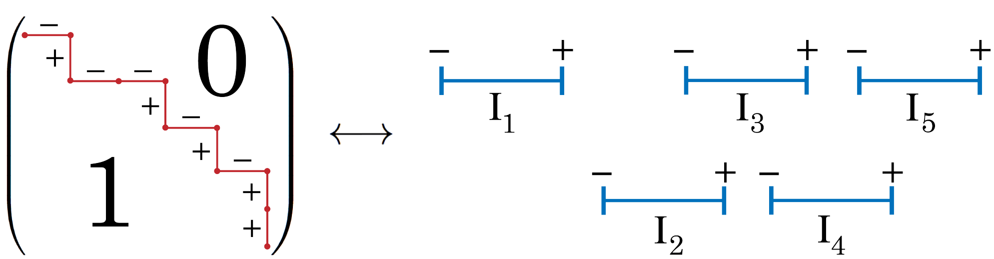

Figure 2.10: Dyck matrix and canonical interval representation of P encoding the 10-tuple $(-,+,-,-,+,-,+,-,+,+)$.

Lemma 2.5.7. *Let* $\mathbf{a}_n = (a_1, \ldots, a_{2n})$ *and* $\mathbf{b}_n = (b_1, \ldots, b_{2n})$ *be the* $2n$*-tuples with entries in* $\{+, -\}$ *obtained by labeling the steps of the semiorder path of* A *and the endpoints of all intervals in* $\mathcal{I}$*, respectively, in the way described above. Then* $\mathbf{a}_n = \mathbf{b}_n$*.*

Proof. Let us proceed by induction on the cardinality n of P. When $n = 1$, both $\mathbf{a}_1$ and $\mathbf{b}_1$ are equal to $(-,+)$ and so $\mathbf{a}_1 = \mathbf{b}_1$. Suppose now that the statement of the lemma is true for every canonically n-labeled unit interval order, and assume that P is a unit interval order canonically labeled by $[n+1]$ with antiadjacency matrix A and canonical interval representation $\mathcal{I}$. Set $m = |\Lambda_{n+1}| - 1$. By Proposition 2.3.2, the poset $P \setminus \{n+1\}$ is a unit interval order canonically labeled by $[n]$; therefore its associated plus-minus $2n$-tuples $\mathbf{a}'_n$ and $\mathbf{b}'_n$ are equal. Observe, in addition, that $\mathbf{b}_{n+1}$ can be recovered from $\mathbf{b}'_n$ by inserting the sign $-$ corresponding to the left endpoint of q_{n+1} (labeled by $2n+2$) in the position $m + n + 1$ (there are n left interval endpoints and m right interval endpoints to the left of q_{m+1} in $\mathcal{I}$) and adding the sign $+$ corresponding to the right endpoint of q_{n+1} (labeled by 1) at the end. On the other hand, $\mathbf{a}_{n+1}$ can be recovered from $\mathbf{a}'_n$ by inserting the sign $-$ corresponding to the rightmost horizontal step of the semiorder path of A in the position $m+n+1$ (there are n horizontal steps and m vertical steps before the last horizontal step of the semiorder path) and placing the sign $+$ corresponding to the vertical step labeled by 1 in the last position. Hence $\mathbf{a}_{n+1} = \mathbf{b}_{n+1}$, and the lemma follows by induction. □

As a consequence of Theorem 2.5.1 and Lemma 2.5.7, one obtains a way of reading the decorated permutation associated to the Dyck positroid induced by P directly from $\mathcal{I}$.

Corollary 2.5.8. *Labeling the left and right endpoints of the intervals $[q_i, q_i + 1]$ by $n + i$ and $n + 1 - i$, respectively, we obtain the decorated permutation associated to the positroid induced by P by reading these $2n$ labels from right to left on the real line.*

Proof. By Lemma 2.5.7, the $2n$-tuple resulting from reading the set $\{1, \dots, 2n\}$ as indicated in Corollary 2.5.8 equals the $2n$-tuple resulting from reading the same set from the semiorder path of A in northwest direction, as described in Theorem 2.5.1. Hence the corollary follows immediately from Theorem 2.5.1. □

Example 2.5.9. The diagram below illustrates how to label the endpoints of a canonical interval representation of the 6-labeled unit interval order P shown in Figure 2.1 to obtain the decorated permutation

$$\pi = (1\ 12\ 2\ 3\ 11\ 10\ 4\ 5\ 9\ 6\ 8\ 7)$$

associated to the positroid induced by P by reading such labels from the real line (from right to left).

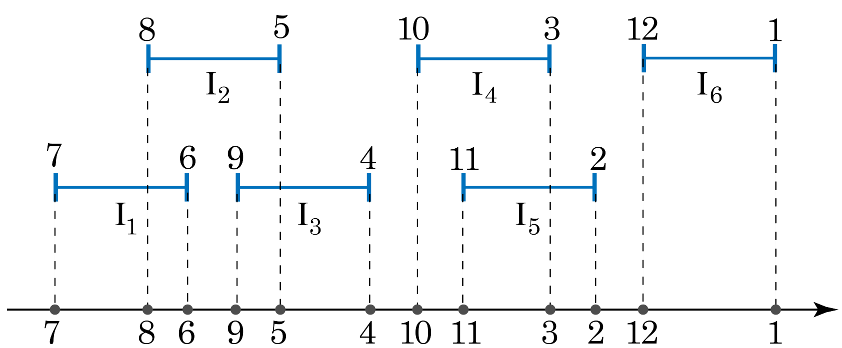

Figure 2.11: Decorated permutation π encoded in a canonical interval representation of P.

2.6 Le-diagrams

The set consisting of all d-dimensional subspaces of $\mathbb{R}^n$, denoted by $\mathrm{Gr}_{d,n}(\mathbb{R})$, is called the *real Grassmannian*. Elements in $\mathrm{Gr}_{d,n}(\mathbb{R})$ can also be understood as the orbits of the set of full-rank $d \times n$ real matrices under the left action of $\mathrm{GL}_d(\mathbb{R})$. For $A \in \mathrm{Mat}_{d,n}(\mathbb{R})$ and $I \in \binom{[n]}{d}$, the *Plücker coordinate* $\Delta_I(A)$ is the maximal minor of A determined by the column set I. The embedding $\mathrm{Gr}_{d,n}(\mathbb{R}) \hookrightarrow \mathbb{RP}^{\binom{n}{d}-1}$ induced by the map $A \mapsto (\Delta_I(A))$ makes $\mathrm{Gr}_{d,n}(\mathbb{R})$ a projective variety. Let $\mathrm{GL}_d^+(\mathbb{R})$ denote the set of real $d \times d$ matrices of positive determinant, and recall that $\mathrm{Mat}_{d,n}^+(\mathbb{R})$ is the set of real $d \times n$ matrices of rank d having nonnegative maximal minors.

Definition 2.6.1. The *totally nonnegative Grassmannian*, denoted by $\mathrm{Gr}_{d,n}^+(\mathbb{R})$, is the set of orbits of $\mathrm{Mat}_{d,n}^+(\mathbb{R})$ under the left action of $\mathrm{GL}_d^+(\mathbb{R})$, i.e., $\mathrm{Gr}_{d,n}^+(\mathbb{R}) = \mathrm{GL}_d^+(\mathbb{R}) \backslash \mathrm{Mat}_{d,n}^+(\mathbb{R})$.

For a full-rank $d \times n$ real matrix A, let $M(A)$ denote the matroid represented by A, and let $[A]$ denote the element of $\mathrm{Gr}_{d,n}(\mathbb{R})$ represented by A. The *matroid stratification* or *Gelfand-Serganova stratification* of $\mathrm{Gr}_{d,n}(\mathbb{R})$ is the collection of all *strata*

$$S_{\mathcal{M}} := \{[A] \in \mathrm{Gr}_{d,n}(\mathbb{R}) \mid M(A) = \mathcal{M}\},$$

where $\mathcal{M}$ runs over the set of rank k representable matroids on the ground set $[n]$. For each stratum $S_{\mathcal{M}}$, we define a *positroid cell* in $\mathrm{Gr}_{d,n}^+(\mathbb{R})$ by

$$S_{\mathcal{M}}^+ = S_{\mathcal{M}} \cap \mathrm{Gr}_{d,n}^+(\mathbb{R}).$$

Note that a representable matroid $\mathcal{M}$ is a positroid precisely when $S_{\mathcal{M}}^+$ is nonempty. The collection of nonempty positroid cells is called the *cellular decomposition* of $\mathrm{Gr}_{d,n}^+(\mathbb{R})$. For further details, see [127, Sections 2 and 3].

Let us proceed to characterize the Le-diagrams corresponding to Dyck positroids.

Theorem 2.6.2. *A* Ⅎ*-diagram L of type $(n, 2n)$ parameterizes a Dyck positroid on $[2n]$ if and only if its shape λ is a square of size n and L satisfies the following two conditions:*

(1) every column has exactly one plus except the last one that has n pluses;

(2) the horizontal unit steps right below the bottom-most pluses are the horizontal steps of a length $2n$ Dyck path supported on the main diagonal of L.

Proof. Suppose first that L satisfies (1) and (2). To verify that L corresponds to a Dyck positroid, let us use Lemma 2.2.12 to compute its decorated permutation π and show that π^{-1} satisfies Proposition 2.4.9. Note that $\pi^{-1}(1) = n+1$. For $i \in [2n] \backslash \{1\}$, we find $\pi^{-1}(i)$.

Assume first that $i \in \{2, \dots, n\}$. If there is only one plus in the $(i-1)$-st row of L (which means that $\omega(j) \neq i-1$ for each $j \in J$), it follows by Lemma 2.2.12 that $\pi^{-1}(i) = i-1$. On the other hand (i.e., there is exactly one principal element j in $\omega^{-1}(i-1)$), one obtains that $\pi^{-1}(i)$ is the label of the first column (from right to left) of L having a plus in the $(i-1)$-st row (which means $\pi^{-1}(i) = j$).

Assume now that $i \in \{n+1, \dots, 2n\}$. If the bottom-most plus in the column of L labeled by i is the last plus from right to left in its row, which is labeled by $\omega(i)$, then by Lemma 2.2.12 it follows that $\pi^{-1}(i) = \omega(i)$ (note, in this case, that $i = 2n$ or $i+1$ is a principal index). On the other hand, the columns of L labeled by i and $i+1$ are identical (i.e., $i+1$ is not a principal index), and Lemma 2.2.12 yields $\pi^{-1}(i) = i+1$.

Thus, π^{-1} is as described in Proposition 2.4.9, and so π is the decorated permutation of a Dyck positroid on the ground set $[2n]$. As the number of ⅃-diagrams satisfying the conditions above and the number of decorated permutations corresponding to Dyck positroids on the ground set $[2n]$ are equal to the n-th Catalan number, the proof follows. □

As a result of Theorem 2.6.2, each Dyck positroid cell in $\mathrm{Gr}^+_{k,n}(\mathbb{R})$ can be indexed by a ⅃-diagrams described in the same theorem. Postnikov proved that the positroid cell indexed by a ⅃-diagram L has dimension equal to the number of pluses of L [127, Theorem 4.6]. This immediately implies the following corollary.

Corollary 2.6.3. *The positroid cell of a Dyck positroid on the ground set* $[2n]$ *inside the cell decomposition of* $\mathrm{Gr}_{n,2n}(\mathbb{R})$ *has dimension* $2n-1$.

The next example illustrates the characterization established in Theorem 2.6.2.

Example 2.6.4. Figure 2.12 shows the ⅃-diagram corresponding to the positroid induced by the unit interval order displayed in Figure 2.1.

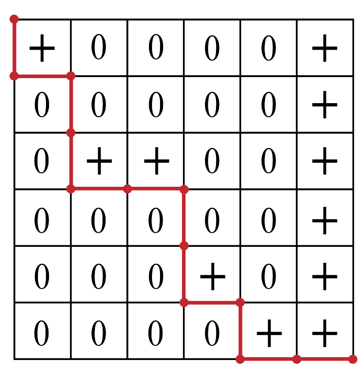

Figure 2.12: A Le-diagram of a Dyck positroid.

2.7 Adjacency of Dyck Positroid Cells

Given a decorated permutation π on n letters, its *chord diagram* is constructed in the following way. First, place n points labeled by $[n]$ in clockwise order around a circle. For all $i, j \in [n]$ with $i \neq j$ and $\pi(i) = j$, draw a directed chord from i to j. If π fixes i, then draw a directed chord from i to i, oriented counterclockwise if and only if $\pi(i) = \bar{i}$. For $i, j \in [n]$, let $\mathrm{Arc}(i, j)$ denote the set of points in the boundary circle of the chord diagram from i to j (both included) in clockwise order. Figure 2.13 shows an example of a chord diagram.

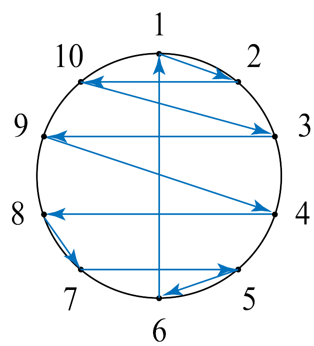

Figure 2.13: Chord diagram of the decorated permutation in Example 2.1.1.

Let AD and CB be two chords in the chord diagram of a decorated permutation π. We say that AD and CB form a *crossing* if they intersect inside the circle or on its boundary, and this crossing is *simple* if there are no other chords from $\mathrm{Arc}(C, A)$ to $\mathrm{Arc}(B, D)$. The left diagram in Figure 2.14 shows a simple crossing. On the other hand, two chords AB and CD form an *alignment* if they do not intersect and have a parallel orientation as shown in the right diagram of Figure 2.14. Notice that if A and B coincided in the right diagram below, then in order for AB and CD to have parallel orientation AB must be a loop oriented counterclockwise. An alignment, as shown in the right side of the picture below, is said to be *simple* if there are no other chords from $\mathrm{Arc}(C, A)$ to $\mathrm{Arc}(B, D)$.

Figure 2.14: A simple crossing on the left and a simple alignment on the right.

Let π_1 and π_2 be two decorated permutations of the same size n. We say that π_1 *covers* π_2, and write $\pi_1 \to \pi_2$, if the chord diagram of π_2 is obtained by turning a simple crossing of π_1 into a simple alignment. This is depicted in Figure 2.15.

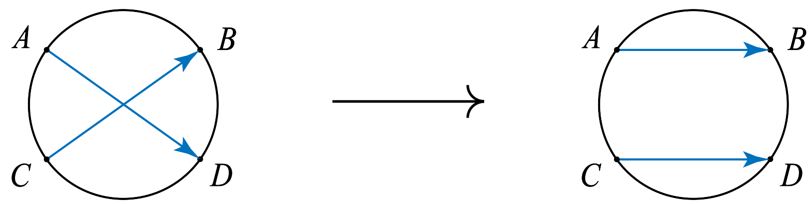

Figure 2.15: A covering relation.

If the points A and B happen to coincide, then the chord from A to B in the chord diagram of π_2 degenerates to a counterclockwise loop. Similarly, if the points C and D coincide, then the chord from C to D in the chord diagram of π_2 becomes a clockwise loop. Finally, if $A = B$ or $C = D$, then the loops at A and C in the chord diagram of π_2 must be counterclockwise and clockwise, respectively. These three types of covering relations, illustrated in Figure 2.16, are said to be *degenerate*.

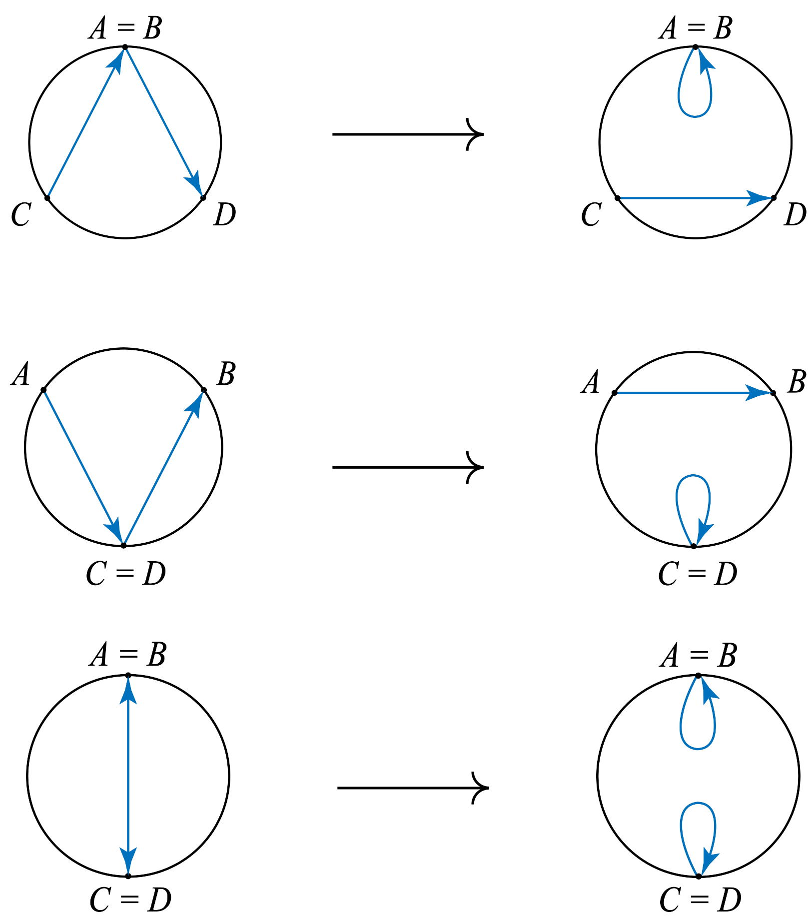

Figure 2.16: The three degenerate covering relations.

Two positroid cells are *adjacent* if the decorated permutation parameterizing them cover a common decorated permutation. Here is a necessary and sufficient condition for two Dyck positroid cells to be adjacent.

Proposition 2.7.1. *Let P_1 and P_2 be two distinct rank n Dyck positroids and π_1 and π_2 their respective decorated permutations. Then P_1 and P_2 label adjacent positroid cells if and only if there exists $i \in [2n] \setminus \{1, n+1\}$ such that when i is removed from the cycle decomposition of π_1 and π_2 the resulting cycles are equal.*

Proof. Let C_1 and C_2 be the chord diagrams of π_1 and π_2, respectively. Assume first that P_1 and P_2 label adjacent positroid cells whose decorated permutations both cover a permutation π. Let C denote the chord diagram of π. Theorem 2.4.10 ensures that C_1 and C_2 have a directed edge from $n+1$ to 1 and their non-degenerate simple crossings occur only along this edge. Unlike non-degenerate coverings, degenerate coverings increase the number of fixed points; therefore $\pi_1 \to \pi$ is a degenerate covering relation if and only if so is $\pi_2 \to \pi$. If $\pi_1 \to \pi$ and $\pi_2 \to \pi$ were both non-degenerate coverings, then the fact that both covering relations uncross the chord from $n+1$ to 1 would imply that both π_1 and π_2 can be uniquely recovered from π, as the other chord being uncrossed in both covering relations must be the chord from $\pi^{-1}(1)$ to $\pi(n+1)$. This, in turn, would contradict that $\pi_1 \neq \pi_2$. As a result, both $\pi_1 \to \pi$ and $\pi_2 \to \pi$ are degenerate coverings. As π_1 and π_2 are $2n$-cycles, π fixes exactly one element $i \in [2n] \setminus \{1, n+1\}$. Moreover, π is the result of removing i from the cycle decomposition of any of the permutations π_1 or π_2.

Conversely, suppose that for some $i \in [2n] \setminus \{1, n+1\}$, removing i from the cycle decomposition of either π_1 or π_2 produces the same $(2n-1)$-cycle π. In this case, $\pi_1 \to \pi$ and $\pi_2 \to \pi$ are degenerate covering relations. Hence π_1 and π_2 are adjacent and the proof follows. □

Example 2.7.2. There are a total of five Dyck positroids on the ground set [6]. Let $\pi_1, \dots, \pi_5$ be their five corresponding decorated permutations. These permutations are illustrated in the top row of Figure 2.17 via their chord diagrams. The bottom row of the same figure shows the chord diagrams of four of the decorated permutations covered by the π_i's. Although there are more than four decorated permutations covered by the π_i's, those depicted at the bottom of Figure 2.17 are enough to obtain all possible adjacency relations between the positroid cells parameterized by the π_i's. The exterior long arrows in Figure 2.17 represent covering relations.

It was proved in [127] that if π_1 and π_2 are two decorated permutations such that $\pi_1 \to \pi_2$, then they both have the same number of weak excedances. Thus, the set of all decorated permutations of $[2n]$ having n excedances can be regarded as a poset with order given by the transitive closure of the covering relation "$\to$"; this poset is called the *cyclic Bruhat order* and is denoted by $\mathrm{CB}_{n,2n}$. Given that the adjacency relations of Dyck positroid cells can be described so nicely, we believe the subposet of $\mathrm{CB}_{n,2n}$ consisting of those decorated permutations representing positroids in the closures of Dyck positroid cells of $\mathrm{Gr}^+_{n,2n}(\mathbb{R})$ may have an interesting description. Here we propose a problem stemming from Proposition 2.7.1.

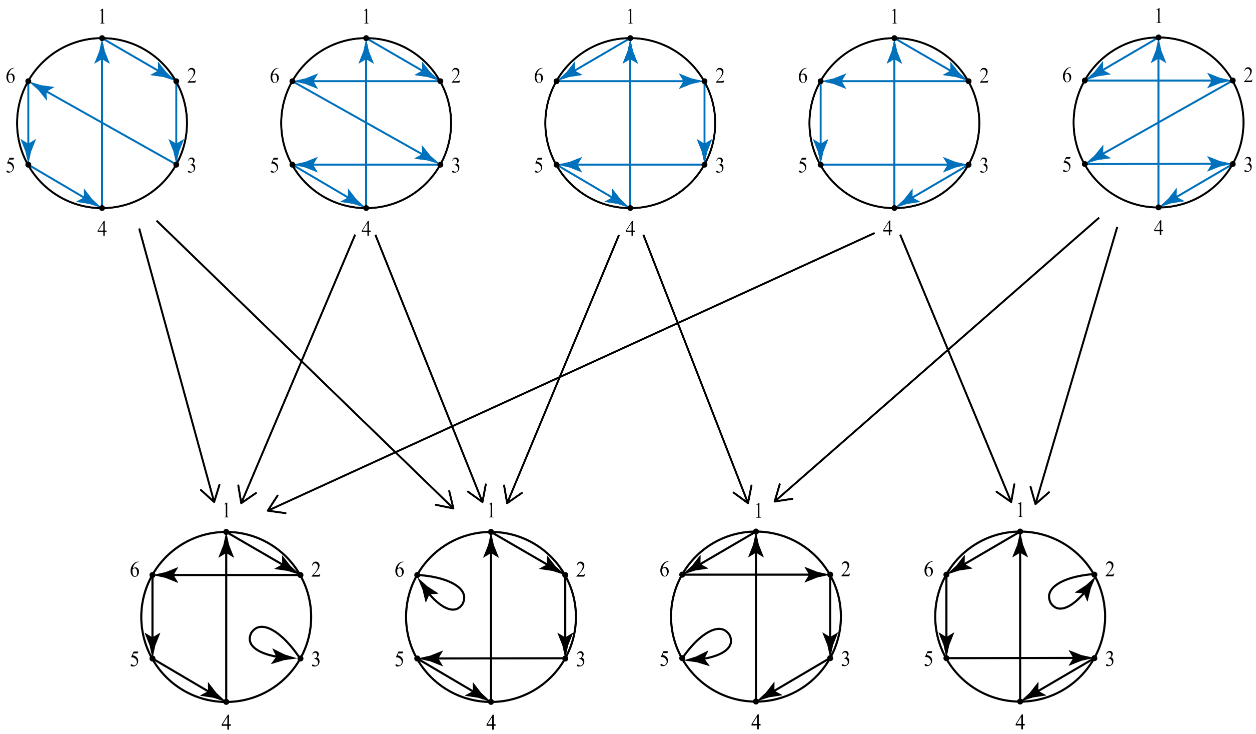

Figure 2.17: Subposet of $\mathrm{CB}_{3,6}$ illustrating the adjacency relations among the Dyck positroid cells of dimension 5.

Problem 2.7.3. *Describe the subposet of* $\mathrm{CB}_{n,2n}$ *consisting of those decorated permutations representing positroids in the closures of Dyck positroid cells of* $\mathrm{Gr}_{n,2n}^{+}(\mathbb{R})$.

2.8 An Interpretation of the f-vector of a Unit Interval Order

In hopes of a more thorough understanding of the *f-vectors* of $(\mathbf{3}+\mathbf{1})$-free posets, Skandera and Reed in [134] posed the following open problem: characterize the f-vectors of unit interval orders. With this goal in mind, we provide a combinatorial interpretation for the f-vector of a naturally labeled poset in terms of its antiadjacency matrix. Throughout this section, P is assumed to be a naturally labeled poset of cardinality n with antiadjacency matrix $A_P = (a_{i,j})$.

Definition 2.8.1. The *f-vector* of P is the sequence $f = (f_0, f_1, \dots, f_{n-1})$, where f_k is the number of $(k+1)$-element chains of P.

We wish to interpret the k-element chains of P in terms of some special Dyck paths inside A_P. To do this, define a *valley Dyck path* of A_P to be a Dyck path drawn inside A_P that has its endpoints and all its valleys on the main diagonal and all its peaks in positions (i, j) such that $a_{i,j} = 0$. Figure 2.18 illustrates a valley Dyck path with three peaks.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Figure 2.18: A valley Dyck path with three peaks inside the antiadjacency matrix of the poset displayed in Figure 2.19.

Proposition 2.8.2. *The entries of the f-vector of P are $f_0 = n$ and f_k equals the number of valley Dyck paths of A_P having exactly k peaks.*

Proof. To each $(k+1)$-element chain $\mathsf{c} : i_1 <_P \cdots <_P i_{k+1}$ we can assign a valley Dyck path v_{c} with k peaks as follows: the j-th peak begins at (i_j, i_j), heads east to (i_j, i_{j+1}), and then heads south to (i_{j+1}, i_{j+1}). To see that v_{c} is a valley Dyck path, it suffices to notice that every peak of v_{c} occurs at a zero entry of A_P since $i_j <_P i_{j+1}$ for each $j = 1, \dots, k$. On the other hand, suppose that v is a valley Dyck path with k peaks, namely $(i_1, i'_1), \dots, (i_k, i'_k)$. Then every valley of v is supported on the main diagonal, which means that $i'_j = i_{j+1}$ for each $j = 1, \dots, k$. Setting $i_{k+1} = i'_k$, we obtain that $v = v_{\mathsf{c}}$, where c is the $(k+1)$-element chain $i_1 <_P \cdots <_P i_{k+1}$. Thus, we have established a bijection that yields the desired result. □

Remark 2.8.3. Proposition 2.8.2 provides, in particular, an interpretation of the f-vector of any unit interval order. Given that a unit interval order can be labeled so that its antiadjacency matrix is a Dyck matrix, we think that the interpretation of the f-vector in Proposition 2.8.2 might be useful to find an explicit formula for the f_k's. This is because zero and one entries in a Dyck matrix are nicely separated, which could facilitate counting the valley Dyck paths having exactly k peaks.

Example 2.8.4. Let P be the naturally labeled poset on the set $[6]$ whose Hasse diagram is illustrated in Figure 2.19. It can be readily verified that the f-vector of P is $f = (6, 9, 4, 1, 0, 0)$. Valley Dyck paths realized on A_P with one, two, and three peaks are illustrated in Figure 2.20.

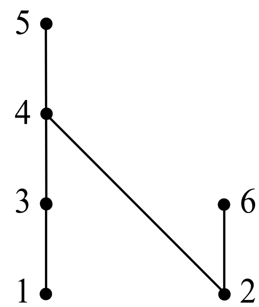

Figure 2.19: Naturally 6-labeled poset.

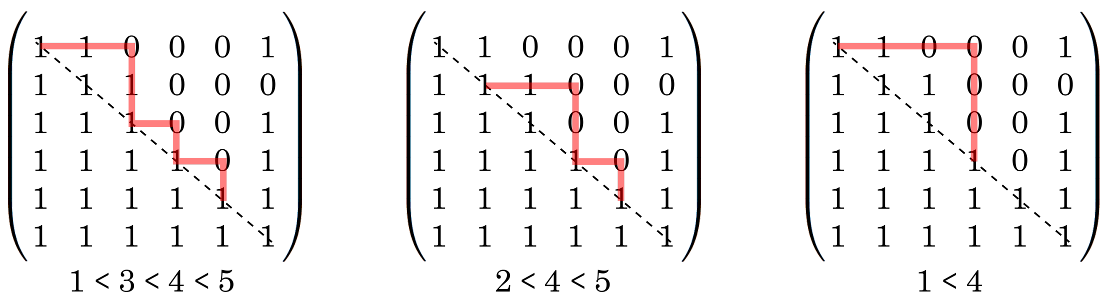

Figure 2.20: From left to right, antiadjacency matrices of the poset P illustrating valley Dyck paths with three peaks, two peaks, and one peak.

Problem 2.8.5. *Given an $n \times n$ Dyck matrix A, let r_i be the number of one entries in the i-th row of A. For $k \in [n-1]$, can we find, in terms of the r_i's, an explicit formula for the number of valley Dyck paths of A containing exactly k peaks?*

Chapter 3

Positroids Induced by Rational Dyck Paths

3.1 Introduction

The main purpose of this chapter is to extend the characterization of the decorated permutation corresponding to a Dyck positroid to a more general characterization comprising all the rational Dyck positroids. In addition, in this section we provide descriptions of the Grassmann necklace and the plabic graphs of any rational Dyck positroid. Then we study the classes of plabic graphs parameterizing rational Dyck positroids and provide a recipe to construct them from the rational Dyck path encoded in the decorated permutation corresponding to the rational Dyck positroid. Finally, we provide a description of the matroid polytope of a rational Dyck positroid.

3.2 Rational Dyck Positroids

In this section we generalize the characterizations of the decorated permutation and the Le-diagram of a Dyck positroid that we established in the previous chapter. First, let us introduce the following notation.

Notation: We denote by $\mathcal{P}_{d,m}$ the set of all rank d rational Dyck positroids on the ground set $[d+m]$.

Back to Decorated Permutations

For $D \in \mathcal{D}_{d,m}$, set $A = (a_{i,j}) = \phi_{d,m}(D)$. Let P be the positroid represented by A, and let π be the decorated permutation corresponding to P. Let us proceed to generalize the definition of principal indices and the weight map given in the previous chapter. Define the *set of principal indices* I_A of A to be

$$I_A = \{i \in \{d+1, \dots, d+m\} \mid A_i \neq A_{i-1}\},$$

where A_i denotes the i-th column of A. In addition, we associate to the matrix A the *weight map* $\omega_A \colon [d+m] \to [d]$ defined by

$$\omega_A(j) = \max\{i \mid a_{i,j} \neq 0\}$$

(notice that there is at least a nonzero entry in each column of A). For $d+1 \le j \le d+m$, we observe that the number of nonzero entries in the column A_j is precisely $\omega_A(j)$.

Observe now that if we remove one column from A the resulting matrix still has rank d. Thus, for each $i \in [d+m]$ the i-th entry of the Grassmann necklace corresponding to P does not contain $i-1$. As a result, π has no fixed points, which implies that it is a standard permutation. The next proposition, whose proof follows *mutatis mutandis* from that one of Proposition 2.4.9, gives an explicit description of the inverse of π.

Lemma 3.2.1. *(cf. [50, Proposition 4.3]) If A, I_A, ω_A, and π are defined as before, then for each $i \in [d+m]$,*

$$\pi^{-1}(i) = \begin{cases} i+1 & \text{if } d < i < d+m \text{ and } i+1 \notin I_A \\ \omega_A(i) & \text{if } d < i \text{ and either } i = d+m \text{ or } i+1 \in I_A \\ d+1 & \text{if } i = 1, \\ i-1 & \text{if } 1 < i \le d \text{ and } \omega_A(j) \neq i-1 \text{ for all } j \in I_A \\ j & \text{if } 1 < i \le d \text{ and } \{j\} = I_A \cap \omega_A^{-1}(i-1). \end{cases}$$

The disjoint cycle decomposition of π consists of only one full cycle, as the following proposition illustrates.

Proposition 3.2.2. *The decorated permutation π of a rank d rational Dyck positroid on the ground set $[d+m]$ is a $(d+m)$-cycle. Moreover, the set of weak excedances of π is $[d]$.*

Proof. Let P be a rational Dyck positroid with decorated permutation π, and let A be the matrix in $\phi_{d,m}(\mathcal{D}_{d,m})$ representing P. It follows from Lemma 3.2.1 that $\omega_A(i) \le \omega_A(j)$ provided that $\pi(i) = j$ and $j \neq 1$. Let σ be a nontrivial cycle of length ℓ in the disjoint cycle decomposition of π. Notice that σ cannot fix 1; otherwise, for $i \in [d+m]$ such that $\sigma(i) \neq i$ we would obtain $\sigma^j(i) \neq d+1$ for $j = 1, \dots, \ell$ and so

$$\omega_A(i) \le \omega_A(\sigma(i)) \le \cdots \le \omega_A(\sigma^\ell(i)) = \omega_A(i),$$

contradicting the fact that σ is nontrivial. Therefore π is a $(d+m)$-cycle.

Now let us proceed to show that the set of weak excedances of π is precisely $[d]$. To do this, write

$$\pi^{-1} = (d+1 \;\; \pi^{-1}(d+1) \;\; \dots \;\; (\pi^{-1})^{d+m-2}(d+1) \;\; 1).$$

As the map ω_A fixes each element of $[d]$, the fact that $\omega_A(i) \le \omega_A(j)$ whenever $\pi^{-1}(j) = i$ and $j \neq 1$ ensures that the sequence

$$(\pi^{-1}(d+1), \dots, (\pi^{-1})^{d+m-2}(d+1))$$

contains the elements $2, \dots, d$ in decreasing order. On the other hand, the fact that ω_A is strictly decreasing on I_A guarantees that the sequence

$$(\pi^{-1}(d+1), \dots, (\pi^{-1})^{d+m-2}(d+1))$$

contains the elements $d+2, \dots, d+m$ in increasing order. Since π does not fix any element, its weak excedances are those $j \in [d+m]$ such that $j < \pi(j)$. Because the elements of $[d]$ are the d smallest elements in $[d+m]$ and show increasingly in

$$\pi = (1 \ \pi(1) \ \dots \ \pi^{d+m-1}(1)),$$

each element of $[d]$ is a weak excedance of π. Besides, no element greater than d can be a weak excedance of π as $d+1, \dots, d+m$ show decreasingly in $\pi = (1 \ \pi(1) \ \dots \ \pi^{d+m-1}(1))$. Hence $[d]$ is the set of weak excedances of π. □

We shall proceed to verify that rational Dyck positroids are connected matroids. First, let us formally introduce the notion of connectedness in the context of matroids. We shall investigate this concept in Chapter 5 for another class of matroids.

Definition 3.2.3. A matroid $(E, \mathcal{B})$ is said to be *connected* if for every $b, b' \in E$ there exist $B, B' \in \mathcal{B}$ such that $B' = (B \setminus \{b\}) \cup \{b'\}$.

It follows from [17, Corollary 7.9] that a positroid on $[n]$ is connected if and only if its corresponding decorated permutation does not stabilize any proper cyclic interval of $[n]$. Hence the following result is an immediate consequence of Proposition 3.2.2.

Corollary 3.2.4. *Every rational Dyck positroid is connected.*

Before proving the main theorem of this section, let us collect the next technical result.

Lemma 3.2.5. *If $A \in \phi_{d,m}(\mathcal{D}_{d,m})$, then for each $j \in \{d+1, \dots, d+m\}$ the following inequality holds:*

$$\omega_A(j) \ge \frac{d}{m}(d+m-j+1).$$

Proof. Let $A = (I_d \mid A')$, and place A into $\mathbb{R}^2$ in such a way that the rational Dyck path d of type (m, d) separating the zero and nonzero entries of A' goes from $T = (0,0)$ to $R = (m, d)$. Let $S = (0, d)$ and let S' and T' be the intersection of the vertical line passing through the right endpoint of the $(j-d)$-th horizontal step of d with the segments $\overline{SR}$ and $\overline{TR}$, respectively. This description is illustrated in Figure 3.1.

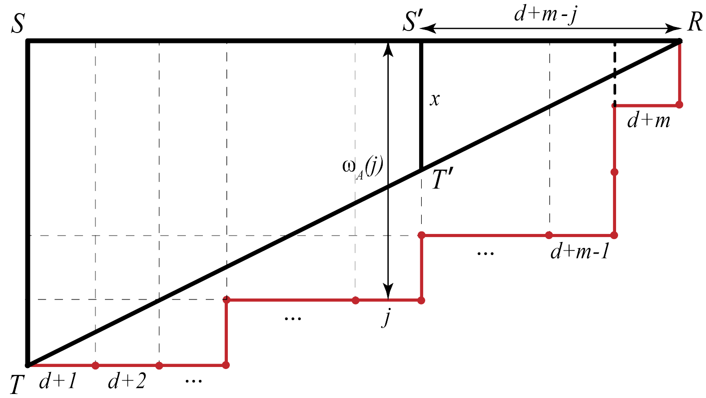

Figure 3.1: Illustration of the geometric inequality of Lemma 3.2.5.

If x is the length of the segment $S'T'$, then the similarity of the triangles RST and $RS'T'$ implies that

$$\frac{x}{d} = \frac{d+m-j}{m}.$$

As the rational Dyck path d never goes above the diagonal line $y = (d/m)x$, we obtain that

$$\omega_A(j) \geq x + \frac{d}{m} = \frac{d}{m}(d+m-j+1),$$

and the lemma follows. □

Theorem 3.2.6. *There is a bijection between the set of rational Dyck paths of type* (m, d) *and the set of rank* d *rational Dyck positroids on the ground set* $[d+m]$.

Proof. Identify the set of rational Dyck paths of type (m, d) with $\mathcal{D}_{d,m}$. Let $\alpha \colon \mathcal{D}_{d,m} \to \mathcal{P}_{d,m}$ be the map assigning to each rational Dyck path of type (m, d) its corresponding rational Dyck positroid via the map $\phi_{d,m}$ in Lemma 2.4.3. We will find a map $\beta \colon \mathcal{P}_{d,m} \to \mathcal{D}_{d,m}$ which is a left inverse of α. Let P be a positroid in $\mathcal{P}_{d,m}$ with corresponding decorated permutation π such that $\pi^{-1} = (i_1 \ \dots \ i_{d+m})$, where $i_1 = d+1$. Define $\beta(P)$ to be the lattice path $(s_1, \dots, s_{d+m})$ where $s_j = (1, 0)$ if $i_j \in \{d+1, \dots, d+m\}$ and $s_j = (0, 1)$ if $i_j \in [d]$. Showing that β is well defined, i.e., that $\beta(P)$ is a rational Dyck path, will be the fundamental part of this proof.

Let P be as in the previous paragraph, and let $A \in \phi_{d,m}(\mathcal{D}_{d,m})$ be a matrix representing P. For $j = 1, \dots, d+m$, set

$$S_j = s_1 + \cdots + s_j,$$

and take ℓ_j to be the slope of the line determined by the points $(0, 0)$ and S_j. As $\beta(P)$ consists of d vertical unit steps and m horizontal unit steps, it is a lattice path from $(0, 0)$ to (m, d). Suppose, by way of contradiction, that $\beta(P)$ is not a rational Dyck path. Then there exists a minimum $n \in [d+m-1]$ such that $\ell_n > d/m$. The minimality of n implies that $s_n = (0,1)$. Take

$$k = \max\{j \in [d+m] \mid j \geq n \text{ and } s_{j'} = (0,1) \text{ for all } n \leq j' \leq j\}.$$

As $s_k = (0,1)$ and the elements labeling vertical steps of $\beta(P)$, namely $1, \dots, d$, show decreasingly in $(i_1, \dots, i_{d+m})$, the number of vertical steps in $\{s_1, \dots, s_k\}$ is $d - i_k + 1$. In addition, as $s_{k+1} = (1,0)$ and the elements labeling horizontal steps, namely $d+1, \dots, d+m$, show increasingly in $(i_1, \dots, i_{d+m})$, there are $i_{k+1} - d - 1 = \pi^{-1}(i_k) - d - 1$ horizontal steps in $\{s_1, \dots, s_k\}$. The fact that $\ell_n \leq \ell_k$ now implies

$$\frac{d}{m} < \ell_k = \frac{d - i_k + 1}{\pi^{-1}(i_k) - d - 1}. \tag{3.1}$$

After applying some algebraic manipulations to the inequality (3.1), one finds that

$$\frac{d}{m}\big(d + m - \pi^{-1}(i_k) + 1\big) > i_k - 1. \tag{3.2}$$

Since $s_{k+1} = (1,0)$ and $i_{k+1} = \pi^{-1}(i_k)$, we have $d+1 \leq \pi^{-1}(i_k) \leq d+m$; thus, an application of Lemma 3.2.5 yields

$$\omega_A(\pi^{-1}(i_k)) \geq \frac{d}{m}\big(d + m - \pi^{-1}(i_k) + 1\big). \tag{3.3}$$

On the other hand, the fact that $i_k \in [d]$, along with Lemma 3.2.1, implies that

$$i_k = \omega_A(i_k) = \omega_A(\pi^{-1}(i_k)) + 1. \tag{3.4}$$

Now we combine (3.2), (3.3), and (3.4) to obtain

$$i_k - 1 = \omega_A(\pi^{-1}(i_k)) \geq \frac{d}{m}\big(d + m - \pi^{-1}(i_k) + 1\big) > i_k - 1,$$

which is a contradiction. Hence $\beta(P)$ is indeed a rational Dyck path and, therefore, β is a well-defined function.

To verify that β is a left inverse of α, take

$$\mathsf{d} = (d_1, \dots, d_{d+m}) \in \{(1,0),(0,1)\}^{d+m}$$

to be a rational Dyck path in $\mathcal{D}_{d,m}$ such that $\alpha(\mathsf{d}) = P$, and set

$$\mathsf{d}' = \beta(P) = (d'_1, \dots, d'_{d+m}).$$

As before, let $\pi^{-1} = (i_1 \ \dots \ i_{d+m})$, where $i_1 = d+1$. Let us verify that $\mathsf{d}' = \mathsf{d}$. Suppose inductively that $d'_j = d_j$ for each $j \in [d+m-1]$ (notice that $d'_1 = d_1 = (1,0)$). As $j < d+m$, it follows that $i_j \neq 1$. Assume first that $d_j = (1,0)$, which means that $i_j \in \{d+1,\dots,d+m\}$. If $\pi^{-1}(i_j) = i_j + 1$, then d_{j+1} is also a horizontal step by Lemma 3.2.1. Also, the fact that

$$i_{j+1} = i_j + 1 \in \{d+1,\dots,d+m\}$$

guarantees that d'_{j+1} is a horizontal step too. On the other hand, if $\pi^{-1}(i_j) \neq i_j + 1$ (which means that $\pi^{-1}(i_j) = \omega_A(i_j)$), then d_{j+1} is vertical by Lemma 3.2.1. In addition,

$$\pi^{-1}(i_j) = \omega_A(i_j) \leq d$$

implies that d'_{j+1} is also a vertical step. Hence $d'_{j+1} = d_{j+1}$. In a similar fashion the reader can verify that $d'_{j+1} = d_{j+1}$ when $d_j = (0,1)$. The fact that α has a left inverse function, along with $|\mathcal{D}_{d,m}| \geq |\mathcal{P}_{d,m}|$, yields that α is a bijection. □

Corollary 3.2.7. *The number of rank d rational Dyck positroids on the ground set $[d+m]$ equals the rational Catalan number* $\operatorname{Cat}(m,d)$*. In particular, there are*

$$\frac{1}{d+m}\binom{d+m}{d}$$

rank d rational Dyck positroids on the ground set $[d+m]$ when $\gcd(d,m) = 1$*.*

The following proposition provides a very straightforward recipe to compute the decorated permutation of a given rational Dyck positroid directly from its corresponding rational Dyck path. We will omit the proof as it follows with no substantial changes from the proof of [50, Proposition 5.1].

Proposition 3.2.8. *Let d be a rational Dyck path of type (m,d). Labeling the d vertical steps of d from top to bottom in increasing order with $1,\dots,d$ and the m horizontal steps from left to right in increasing order with $d+1,\dots,d+m$, we obtain the decorated permutation of the rational Dyck positroid induced by d by reading the step labels of d in southwest direction.*

Example 3.2.9. Let P be the rational Dyck positroid induced by the rational Dyck path d of type $(8,5)$ illustrated in Figure 3.2. The path d is labeled as indicated in Proposition 3.2.8. Therefore the decorated permutation of P is

$$\pi = (1\ 2\ 13\ 12\ 3\ 11\ 10\ 4\ 9\ 5\ 8\ 7\ 6),$$

which is obtained by reading the labels of d in southwest direction.

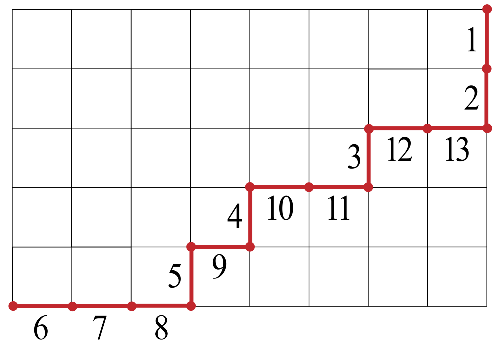

Figure 3.2: A rational Dyck path of type $(8, 5)$ encoding the decorated permutation of the rational Dyck positroid it induces.

Back to Le-diagrams

Now that we have characterized the decorated permutation of every rational Dyck positroid, we are in a good position to generalize the characterization of Ꞁ-diagrams we gave in Section 2.6.

We have seen in Proposition 3.2.2 that the decorated permutation of a rank d rational Dyck positroid on the ground set $[d+m]$ has $[d]$ as its set of weak excedances. Thus, the Ꞁ-diagram of a rational Dyck positroid has a rectangular shape, namely m^d. The next description, which gives a complete characterization of the Ꞁ-diagrams parameterizing rational Dyck positroids, has been proved in [50] for the case $d = m$. However, the argument for proving the general case is basically a reproduction of the case $d = m$ and, therefore, we decided to omit it.

Proposition 3.2.10. *A Ꞁ-diagram L of type $(d, d+m)$ corresponds to a rank d rational Dyck positroid on the ground set $[d+m]$ if and only if its shape λ is the full $d \times m$ rectangle and L satisfies the following two conditions:*

(1) every column has exactly one plus except the last one that has d pluses;

(2) the horizontal unit steps right below the bottom-most plus of each column are the horizontal steps of a horizontally-reflected rational Dyck path of type (m, d) (see Figure 2.6).

Corollary 3.2.11. *The positroid cell parameterized by a rank d rational Dyck positroid on the ground set $[d+m]$ inside the corresponding Grassmannian cell complex has dimension $d+m-1$.*

Example 3.2.12. Let P be the rank 5 rational Dyck positroid on the ground set $[13]$ having decorated permutation

$$\pi = (1\ 2\ 13\ 12\ 3\ 11\ 10\ 4\ 9\ 5\ 8\ 7\ 6).$$

The Le-diagram of P and the corresponding pipe dream giving rise to the decorated permutation π is the one shown in Figure 2.6 in the previous chapter.

3.3 Grassmann Necklaces

Let us proceed to describe the Grassmann necklaces corresponding to rational Dyck positroids. We will use this description in Section 3.5 to describe the matroid polytopes of rational Dyck positroids.

Proposition 3.3.1. *Let P be a rational Dyck positroid represented by $A \in \phi_{d,m}(D_{d,m})$, and let*

$$I_A = \{p_1 < \cdots < p_t\}$$

and

$$E_A = \{q_1, \ldots, q_u\} = [d] \setminus \{\omega_A(i) \mid d < i \leq d+m\}.$$

The Grassmann necklace $\mathcal{I}(P) = (I_1, \ldots, I_{d+m})$ of P, where $I_j = (a_1^j, \ldots, a_d^j)$, is characterized as follows.

(1) $I_1 = (1, 2, \ldots, d)$.

(2) If $j \in [d] \setminus \{1\}$ and $(d-j+1) + |\{p_i \mid \omega_A(p_i) < j-1\}| \geq d$, then $a_i^j = j+i-1$ for $i = 1, \ldots, d-j+2$, while $a_{d-j+2+i}^j = p_{s+i}$ for $s = \max\{k \mid \omega_A(k) \geq j-1\}$ and $i = 1, \ldots, j-2$.

(3) If $j \in [d] \setminus \{1\}$ and $(d-j+1) + |\{p_i \mid \omega_A(p_i) < j-1\}| < d$, then $a_i^j = j+i-1$ for $i = 1, \ldots, d-j+2$, while $a_{d-j+2+i}^j = p_{s+i}$ for $s = \max\{k \mid \omega_A(k) \geq j-1\}$ and $i = 1, \ldots, t-s$; also, $a_{d-j+2+i}^j = q_{i-(t-s)}$ for $i = t-s+1, \ldots, j-2$.

(4) If $j \in \{d+1, \ldots, d+m\}$, then $a_1^j = j$ and $a_i^j = p_{s+i-1}$ for $s = \max\{k \mid p_k \leq j\}$ and $i = 2, \ldots, t-s+1$ while $a_i^j = q_{i-(t-s+1)}$ for $i = t-s+2, \ldots, d+m$.

Proof. The statement (1) is straightforward. Let us check (2). The lexicographical minimality of I_j with respect to the order $\leq_j$ implies that $a_i^j = j+i-1$ for $i = 1, \ldots, d-j+1$ as the set $\{A_j, \ldots, A_d\}$ consists of $d-j+1$ distinct canonical vectors and so is linearly independent. Also, $A_{d+1} \notin \text{span}(A_j, \ldots, A_d)$, yielding $a_{d-j+2}^j = d+1$. Since

$$(d-j+1) + |\{p_i \mid \omega_A(p_i) < j-1\}| \geq d,$$

there are enough vectors in $\{A_{p_i} \mid 2 \leq i \leq t\}$ to complete the basis I_j. Here let us make two observations. If $\omega_A(p_i) \geq j-1$, then A_{p_i} is a linear combination of the columns already chosen. As $A_i = A_{p_j}$ when $p_j \leq i < p_{j+1}$, the minimality of I_j forces us to complete the basis taking indices in I_A. Hence completing I_j amounts to collecting the $j-2$ minimal elements in I_A indexing columns with weights less than $j-1$.

Notice that the first part of (3) follows similarly to (2); therefore it suffices to argue that $a^j_{d-j+2+i} = q_{i-(t-s)}$ for $i = t-s+1, \dots, j-2$. To do so we should take in a minimal way some vectors from $\{A_1, \dots, A_d\}$ to complete I_j; it suffices to take the first $j+s-t-2$ indices of $[d]$ which are not in the set $\{\omega_A(a^j_i) \mid 1 \le i \le (d-j+2)+(t-s)\}$. Those are precisely the first $j+s-t-2$ smallest elements of E_A.

Finally, let us verify (4). Since every column of A is different from the zero vector, $a^j_1 = j$. The fact that $a^j_i = p_{s+i-1}$ when $i = 2, \dots, t-s+1$ is an immediate consequence of the minimality of I_j; this is because equal columns of A are located consecutively and, for each $i \in [t]$, the column A_{p_i} is located all the way to the left in the block of identical columns it belongs. The equalities $a^j_{d-j+2+i} = q_{i-(t-s)}$ can be argued in the same manner we did in the previous paragraph. □

3.4 Plabic Graphs

Let us now proceed to characterize the move-equivalence classes of plabic graphs (up to homotopy) corresponding to rational Dyck positroids.

Definition 3.4.1. A *plabic graph* is an undirected finite graph G drawn inside a closed disk D (up to homotopy) satisfying the following two conditions:

(1) Every vertex of G in the interior of D is colored either black or white.

(2) Every vertex of G on the boundary of D is incident to a single edge.

With notation as in the above definition, the vertices in the interior of D are called *internal vertices* of G while the vertices on the boundary of D are called *boundary vertices* of G. In the context of this chapter, the boundary vertices of G are labeled clockwise starting by 1. Also, every plabic graph here is assumed to be *leafless* (i.e., there are no internal vertices of degree one) and without *isolated components* (i.e., every connected component must contain at least one boundary vertex). For the rest of this section, let G denote a plabic graph with n boundary vertices.

A *perfect orientation* $\mathcal{O}$ of G is a choice of directions for every edge of G in such a way that black vertices have outdegree one and white vertices have indegree one. If G admits a perfect orientation $\mathcal{O}$, we call G *perfectly orientable* and let $G_{\mathcal{O}}$ denote the directed graph on G determined by $\mathcal{O}$. A boundary vertex v of an oriented plabic graph $G_{\mathcal{O}}$ is a *source* (resp., *sink*) if v has indegree (resp., outdegree) zero. The set of boundary vertices that are sources (resp., sinks) of $G_{\mathcal{O}}$ is denoted by $I_{\mathcal{O}}$ (resp., $\bar{I}_{\mathcal{O}}$). It is known that any two perfect orientations of the same plabic graph G have source sets of the same size

$$d := \frac{1}{2}\left(n + \sum_{v \text{ black}} \big(\deg(v) - 2\big) + \sum_{v \text{ white}} \big(2 - \deg(v)\big)\right). \tag{3.5}$$

The *type* of G is defined to be (d, n). See [127] for more details.

Example 3.4.2. Figure 3.3 shows an oriented plabic graph $G_{\mathcal{O}}$. Here the source set of $G_{\mathcal{O}}$ is $I_{\mathcal{O}} = \{1, 5, 6, 8, 10\}$ and the sink set of $G_{\mathcal{O}}$ is $\bar{I}_{\mathcal{O}} = \{2, 3, 4, 7, 9, 11, 12\}$. Therefore G has type $(5, 12)$.

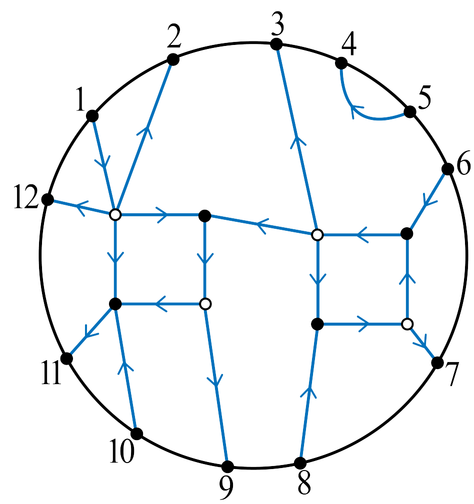

Figure 3.3: A plabic graph with a perfect orientation.

The next local transformations will partition the set of plabic graphs into equivalence classes. We will see later that such a set of equivalence classes is in one-to-one correspondence with the set of positroids.

(M1) **Square move:** If G has a square consisting of four trivalent vertices whose colors alternate, then the colors of these four vertices can be simultaneously switched. This local transformation is illustrated in Figure 3.4.

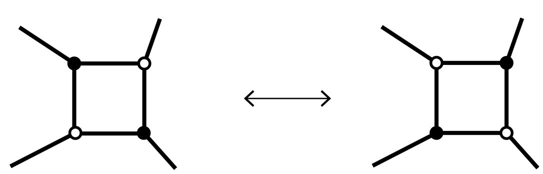

Figure 3.4: Local transformation M1.

(M2) **Unicolored edge contraction/uncontraction:** If G contains two adjacent vertices of the same color, then any edge joining these two vertices can be contracted into a single vertex with the same color of the two initial vertices. Conversely, a given vertex of G can be

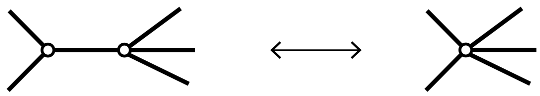

Figure 3.5: Local transformation M2.

uncontracted into an edge joining vertices of the same color as the given vertex. This local transformation is illustrated in Figure 3.5.

(M3) **Middle vertex insertion/removal:** If G contains a vertex of degree 2, then this vertex can be removed and its incident edges can be glued together. Conversely, a vertex (of any color) can be inserted in the middle of any edge of G. This local transformation is illustrated in Figure 3.6.

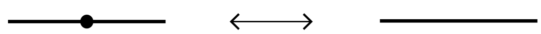

Figure 3.6: Local transformation M3.

(R1) **Parallel edge reduction:** If G contains two trivalent vertices of different colors connected by a pair of parallel edges, then these vertices and edges can be deleted, and the remaining two edges can be glued together. This local transformation is illustrated in Figure 3.7.

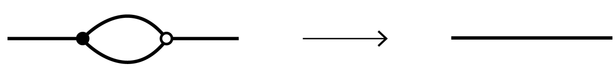

Figure 3.7: Local transformation R1.

Two plabic graphs are called *move-equivalent* if they can be obtained from each other by applying the local transformations (M1), (M2), and (M3); this defines an equivalence relation on the set of plabic graphs. A leafless plabic graph G without isolated components is said to be *reduced* if the local transformation (R1) cannot be applied to any plabic graph in the move-equivalence class of G. Any reduced plabic graph is known to be perfectly orientable.

Let G be a reduced plabic graph as above with boundary vertices $1, \ldots, n$. The trip from i is the path obtained by starting from i and traveling along edges of G according to the rule that each time we reach an internal white vertex we turn right, and each time we reach an internal black vertex we turn left (this rule is also known as "the rule of the road"). This trip ends at some boundary vertex $\pi(i)$. If the starting and ending points of the trip are the same vertex j, we set the color of the fixed point $\pi(j) = j$ to match the orientation of the trip (clockwise or counterclockwise.) In this way we associate a decorated permutation $\pi_G = (\pi(1), \ldots, \pi(n))$ to each reduced plabic graph G.

Definition 3.4.3. For a plabic graph G, the trip π_G described above is called the *decorated trip permutation* of G.

Example 3.4.4. Figure 3.8 shows the plabic graph introduced in Example 3.4.2 with a directed path shadowed. This path represents the trip obtained by starting from the boundary vertex labeled by 3 and following the rule of the road. As a result, if π_G is the decorated trip permutation of G, then $\pi_G(3) = 10$.

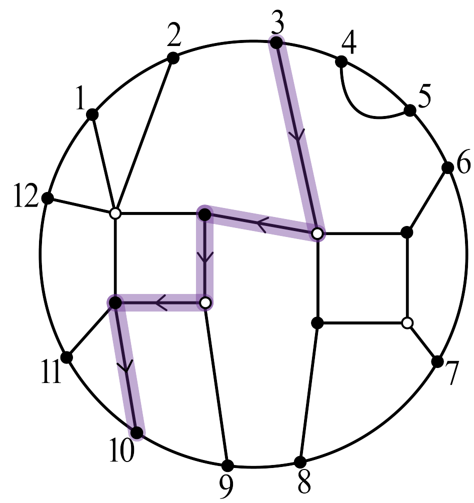

Figure 3.8: A plabic graph G showing a trip from 3 to 10 (following the rule of the road) corresponding to $\pi_G(3) = 10$, where π_G is the decorated trip permutation of G.

Theorem 3.4.5. *[127, Theorem 13.4] Two reduced plabic graphs are move-equivalent if and only if they have the same decorated trip permutation.*

As Grassmann necklaces, decorated permutations, and Ⅎ-diagrams, move-equivalence classes of plabic graphs of type (d, n) also parameterize rank d positroids on the ground set $[n]$.

Proposition 3.4.6. *[127, Section 11] For each pair of positive integers d and n with $d \leq n$ the assignment $G \mapsto P_G = ([n], \mathcal{B}_G)$, where*

$$\mathcal{B}_G = \{I_{\mathcal{O}} \mid \mathcal{O} \text{ is a perfect orientation of } G\},$$

is a one-to-one correspondence between move-equivalence classes of perfectly orientable plabic graphs of type (d, n) and rank d positroids on the ground set $[n]$.

Example 3.4.7. Let P be the rank 5 positroid on the ground set $[12]$ whose Ⅎ-diagram and decorated permutation were introduced in Example 2.2.11. The plabic graph illustrated in Figure 3.3 is an oriented plabic graph corresponding to P. The perfect orientation $\mathcal{O}$, also illustrated in Figure 3.3 gives the basis

$$I_{\mathcal{O}} = \{1, 5, 6, 8, 10\}$$

of P. We have seen before that the decorated permutation π corresponding to P can be written in disjoint cycle decomposition as follows:

$$\pi = (1\ 12\ 9\ 2)(3\ 10\ 11\ 7)(4\ 5)(6\ 8).$$

In particular, $\pi(3) = 10$, which is indicated by the directed path from 3 to 10 highlighted in Figure 3.8.

We conclude this section characterizing the plabic graphs corresponding to rational Dyck positroids.

Proposition 3.4.8. *A rational Dyck path* d *of type* (m, d) *induces a reduced plabic graph* G_{d} *of type* $(d, d+m)$ *as follows:*

(1) Draw a circle with $(0, 0)$ *and* (m, d) *diametrically opposed, and draw a black (resp., white) vertex in the middle of each vertical (resp., horizontal) step of* d.

(2) Draw a horizontal segment from each black vertex to the circle (going east) and label the intersections by $1, \dots, d$ *(clockwise). Similarly, draw a vertical segment from each white vertex to the circle (going north) and label the intersections by* $d+1, \dots, d+m$ *(clockwise).*

(3) Finally, join consecutive internal vertices in d *by segments and ignore the initial rational Dyck path* d *(see Figure 3.9).*

Proof. It follows immediately that the given recipe yields a plabic graph with $d+m$ boundary vertices. To find the type of G_{d}, notice that all internal vertices have degree 3, except the first internal white vertex and the last internal black vertex on d (in northeast direction) which have degree 2. Therefore, using the formula (3.5), one obtains

$$\frac{1}{2}\Bigg(n + \sum_{v \text{ black}} \big(\deg(v) - 2\big) + \sum_{v \text{ white}} \big(2 - \deg(v)\big)\Bigg) = \frac{1}{2}\big(d + m + (d-1) - (m-1)\big) = d.$$

Thus, the type of G_{d} is $(d, d+m)$.

Let us verify now that the graphs in the move-equivalence class of plabic graphs of a rational Dyck positroid are trees. Let G be a plabic graph representing a rational Dyck positroid P of type $(d, d+m)$, and let us check that G is a tree. As any two graphs in the same move-equivalence class of plabic graphs corresponding to P are homotopic, it suffices to assume that G is the representative described by the three steps in the statement of the proposition. Suppose, by way of contradiction, that G is not a tree, meaning that it has a cycle consisting of the vertices $v_1, \dots, v_k$ for some $k \geq 2$. Because every boundary vertex has degree one, each v_i must be an internal vertex. It immediately follows from the three steps described above that each internal vertex is connected to exactly one boundary vertex, which implies that $\deg(v_i) \geq 3$ for each $i = 1, \dots, k$. It is also clear from the three steps above that every vertex of G has degree at most 3. Thus, $\deg(v_i) = 3$ for each $i = 1, \ldots, k$. As G is connected the set of internal vertices of G is $\{v_1, \ldots, v_k\}$, contradicting the fact that G has internal vertices of degree 2, for instance, the black internal vertex adjacent to the boundary vertex 1.

The fact that no graph in the move-equivalence class of the plabic graph described in the three steps of the proposition has an internal cycle immediately implies that (R1) cannot be applied to any of such graphs. Hence the plabic graph we have described in the statement of the proposition must be reduced. □

Corollary 3.4.9. *The graphs in the move-equivalence class of plabic graphs of a rational Dyck positroid are trees.*

Theorem 3.4.10. *A rank d positroid on the ground set $[d+m]$ is a rational Dyck positroid if and only if it can be represented by one of the plabic graphs G_{d} described in Proposition 3.4.8.*

Proof. Let us prove first that for every rational Dyck path d of type (m, d), the plabic graph G_{d} represents a rational Dyck positroid. To do this we will show that the decorated trip permutation π_G of G_{d} is precisely the decorated permutation $\pi_{\mathsf{d}} = (1\ j_2\ \ldots j_{d+m})$ of the rational Dyck positroid P induced by d. Note that, if we label the internal vertices of G_{d}, which lie on d, as in Proposition 3.2.8, then the endpoints of each edge of G_{d} incident to the boundary have the same label.

First, we suppose that for $n < d + m$ the n-th step (going southwest) of d, which is labeled by j_n, is vertical. If the $(n+1)$-th step of d is also vertical, then $\pi_G(j_n) = j_{n+1}$ as there is a path in G_{d} from j_n to j_{n+1} following the rules of the road, namely the path going from the boundary vertex j_n west to the black internal vertex in the middle of the n-th step of d, turning left to the black internal vertex in the middle of the $(n+1)$-th step of d, and turning left to the boundary vertex j_{n+1}. On the other hand, if the $(n+1)$-th step of d is horizontal, then there is also a path from j_n to j_{n+1} following the rules of the road, namely the one going from the boundary vertex j_n to the black internal vertex in the middle of the n-th step of d, turning left to the white internal vertex in the middle of the $(n+1)$-th step of d, and turning right to the boundary vertex j_{n+1}, yielding again $\pi_G(j_n) = j_{n+1}$. In a similar way we can argue that $\pi_G(j_n) = j_{n+1}$ when the n-th step of d is horizontal; the verification is left to the reader.

Also notice that the path in G_{d} starting at the boundary vertex labeled by $d+1$ must travel in northeast direction through all the internal vertices until it reaches the boundary vertex labeled by 1; this is because every time it visits a black (resp., white) internal vertex it must turn left (resp., right) and this forces the path to avoid the edges incident to the boundary (except the first one and last one). Hence $\pi_G(d+1) = 1$ and, therefore, we can conclude that π_G is the decorated permutation of the rational Dyck positroid induced by d.

We have proved that each of the plabic graphs of type $(d, d+m)$ described in Proposition 3.4.8 represents a rational Dyck positroid of rank d on the ground set $[d+m]$. On the other hand, given a positroid induced by a rational Dyck path d of type (m, d), we can find its decorated permutation π_{d} by reading d in southwest direction as in Proposition 3.2.8 and, by the argument just explained above, one finds that $\pi_G = \pi_{\mathsf{d}}$, where π_G is the decorated trip permutation of the plabic graph G_{d} obtained from d by following the recipe in the statement of Proposition 3.4.8. The proof now follows. □

Example 3.4.11. Let P be the positroid induced by the rational Dyck path d of type $(8,5)$ shown in Figure 2.8. The following picture illustrates the plabic graph G_{d} corresponding to P described in Proposition 3.4.8 (on the left) and a minimal bipartite graph in the move-equivalence class of G_{d} (on the right).

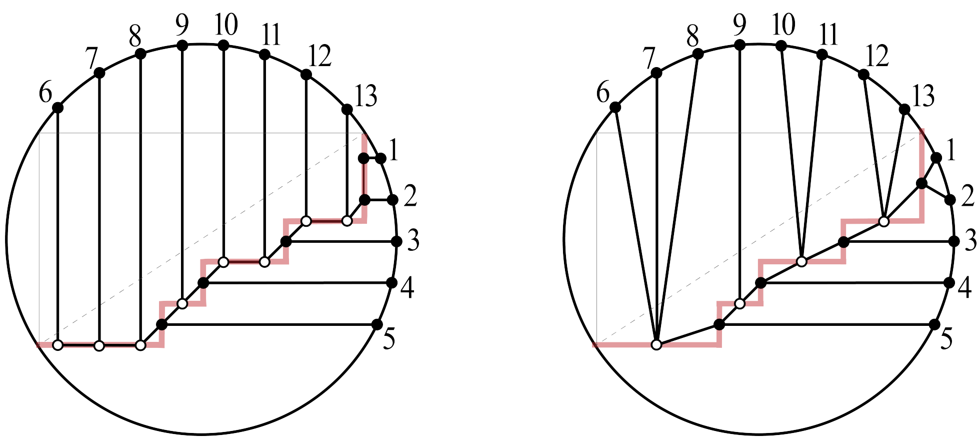

Figure 3.9: The plabic graph of a rank 5 rational Dyck positroid on the ground set [13] and a minimal bipartite plabic graph of its move-equivalence class.

3.5 The Polytope of a Rational Dyck Positroid

We conclude this chapter characterizing the matroid polytope of a rational Dyck positroid.

The *indicator vector* of a subset B of $[n]$ is defined to be $e_B := \sum_{j\in B} e_j$, where $e_1, \dots, e_n$ are the standard basic vectors of $\mathbb{R}^n$. Also, for a subset S of $\mathbb{R}^n$, we let $\operatorname{conv}(S)$ denote the convex hull of S.

Definition 3.5.1. The *matroid polytope* Γ_M of the matroid $M = ([n], \mathcal{B})$ is the convex hull of all indicator vectors of subsets in $\mathcal{B}$, namely

$$\Gamma_M := \operatorname{conv}\big(\{e_B \mid B \in \mathcal{B}\}\big).$$

When M happens to be a positroid, we call Γ_M the *positroid polytope* of M.

Matroid polytopes have been extensively studied in the literature; see, for example, [14] and references therein. In particular, the reader will find the next elegant characterization.

Theorem 3.5.2. *[73, Theorem 4.1] Let $\mathcal{B}$ be a collection of subsets of $[n]$, and let $\Gamma_{\mathcal{B}}$ denote $\operatorname{conv}\big(\{e_B \mid B \in \mathcal{B}\}\big) \subset \mathbb{R}^n$. Then $\mathcal{B}$ is the collection of bases of a matroid if and only if every edge of $\Gamma_{\mathcal{B}}$ is a parallel translate of $e_i - e_j$ for some $i, j \in [n]$.*

Descriptions of matroid polytopes by sets of inequalities have also been established. For instance, in [144] Welsh describes a general matroid polytope $M = ([n], \mathcal{B})$ by using $O(2^n)$ inequalities. When the matroid M happens to be a positroid, its positroid polytope can be described by using only $O(n^2)$ inequalities.

Proposition 3.5.3. *[17, Proposition 5.5] Let $\mathcal{I} = (I_1, \dots, I_n)$ be a Grassmann necklace of type (d, n), and let M be its corresponding positroid. For any $j \in [n]$, suppose the elements of I_j are $a_1^j \le_j \cdots \le_j a_d^j$. Then the positroid polytope Γ_M can be described by the inequalities*

$$x_1 + x_2 + \cdots + x_n = d, \tag{3.6}$$

$$x_j \ge 0 \qquad \text{for each } j \in [n], \tag{3.7}$$

$$x_j + x_{j+1} + \cdots + x_{a_k^j - 1} \le k - 1 \qquad \text{for each } j \in [n] \text{ and } k \in [d], \tag{3.8}$$

where all the subindices are taken modulo n.

Our next task consists in refining Proposition 3.5.3 for those positroids induced by rational Dyck paths; we will accomplish this by detecting redundant inequalities.

Proposition 3.5.4. *Let P be a rational Dyck positroid represented by the real $d \times (d+m)$ matrix $A \in \phi_{d,m}(\mathcal{D}_{d,m})$, and let $I_A = \{p_1 < \cdots < p_t\}$ be the set of principal indices of A. Then the positroid polytope Γ_P is described by the inequalities*

$$x_1 + \cdots + x_{d+m} = d, \tag{3.9}$$

$$x_i \ge 0 \qquad \text{for } i \in [d+m], \tag{3.10}$$

$$x_i \le 1 \qquad \text{for } i \in [d], \tag{3.11}$$

$$x_{p_i} + \cdots + x_{p_{i+1}-1} \le 1 \qquad \text{for } i \in [t], \tag{3.12}$$

$$x_i + \cdots + x_{p_{m(i)}-1} \le (d-i) + m(i) \qquad \text{for } i \in [d], \tag{3.13}$$

$$x_{p_i} + \cdots + x_{\omega_A(p_i)} \le \omega_A(p_i) \qquad \text{for } i \in [t]\backslash\{1\}, \tag{3.14}$$

where $m(i) = \max\{r \in [t] \mid \omega_A(p_r) \ge i \text{ and } r < i\}$.

Proof. Let $\mathcal{I} = (I_1, \dots, I_{d+m})$ be the Grassmann necklace corresponding to P, and let Γ be the polytope determined by (3.9)–(3.14). Take $x = (x_1, \dots, x_{d+m})$ in Γ_P. Both (3.9) and (3.10) hold by Proposition 3.5.3. Besides, taking $j \in [d]$ and $k = 2$ in (3.8), we obtain the inequalities (3.11), while taking $j \in I_A$ and $k = 2$ we get the inequalities (3.12). By Proposition 3.3.1, for $i \in [d]$, the first $d - i + 1$ entries of I_i are $i, \dots, d$ and the next entries are some of the indices $p_1, \dots, p_t$. Therefore taking $j = i$ and $k = (d - j + 1) + m(i)$ in the inequality (3.8), one gets (3.13). Again, by part (4) of Proposition 3.3.1, the first $t - i + 1$

entries of I_{p_i} are $p_i, \ldots, p_t$ and the next $\omega_A(p_i) + 1 - (t - i + 1)$ entries of I_{p_i} are the indices in $[\omega_A(p_i)] \setminus \{\omega_A(p_r) \mid i \leq r \leq t\}$. Hence (3.14) follows.

Now we show that every element $x = (x_1, \ldots, x_{d+m}) \in \Gamma$ must satisfy (3.6)–(3.8). As (3.6) and (3.7) hold by the definition of Γ, it suffices to verify the inequality (3.8). For $j \in [d+m]$ let $I_j = (a_1^j, \ldots, a_d^j)$. We always have $a_1^j = j$. Suppose first that $j \in [d]$. If $j < a_k^j \leq d+1$, then (3.8) results from adding $k-1$ of the inequalities (3.11). If $a_k^j = 1$ or $a_k^j > d+1$, then $a_{k-1}^j = p_{k'}$ for $k' \in [t]$. In this case, (3.8) results from adding (3.13) for $i = j$ and $k' - 2$ inequalities (3.12). Now suppose that $a_k^j = r$, where $1 < r < j$. Then we can obtain (3.8) by adding (3.13) for $i = j$, $k - 3$ inequalities (3.12), and (3.14) for the index i such that $\omega_A(p_i) = a_k^j - 1$.

Finally, suppose that $j \in \{d+1, \ldots, d+m\}$. We can always assume that $a_1^j \in I_A$ because $a_1^j \in \{d+1, \ldots, d+m\} \setminus I_A$ gives redundant inequalities. Let $a_1^j = p_s$ for some $s \in [t]$. If

$$a_k^j \in \{p_s + 1, \ldots, d+m\} \cup \{1\},$$

then by Proposition 3.3.1, it follows that $a_k^j \in I_A \cup \{1\}$; in this case (3.8) is the addition of $k-1$ inequalities (3.12). It only remains to consider $a_k^j \in \{2, \ldots, d\}$. Suppose first that $a_k^j \leq \omega_A(p_s)$. If $a_k^j \leq \omega_A(p_n)$ for each $n \in [t]$, then (3.8) is the addition of $t-s+1$ inequalities (3.12) and $(k-1)-(t-s+1)$ inequalities (3.11). Otherwise, there exists a smallest index r in $[t]$ such that $a_k^j > \omega_A(p_r)$. Taking $i = p_r$, we observe that (3.8) is obtained from adding $i-s$ inequalities (3.12), the inequality (3.14), and enough inequalities (3.11). Lastly, suppose that $a_k^j > \omega_A(p_s)$. In this case it is not hard to see that (3.8) is implied by the addition of (3.14) for $i = s$ and enough inequalities (3.11). □

Remark 3.5.5. Although the description of the rational Dyck positroid given in Proposition 3.5.4 is not as simple as the one presented in Proposition 3.5.3, the reader should observe that the number of inequalities in Proposition 3.5.4 is $O(d+m)$ while the number of inequalities in Proposition 3.5.3 is $O(d^2 + dm)$.

Part II

On Tilings and Matroids on the Lattice Points of a Regular Simplex

Chapter 4

Matroids and Tilings on Regular Subdivisions of a Triangle

4.1 Introduction

For $n, d \in \mathbb{Z}_{\geq 2}$, consider the $(d-1)$-dimensional simplex

$$S_{n,d} := \left\{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1 + \cdots + x_d = n-1 \text{ and } x_i \geq 0 \text{ for } 1 \leq i \leq d\right\},$$

and let $P_{n,d}$ denote the set of lattice points contained in $S_{n,d}$. The set $P_{4,3}$ is illustrated in Figure 4.1. We denote by $\mathcal{I}_{n,d}$ the collection of all subsets I of $P_{n,d}$ such that, for each $k \leq n$, every parallel translate of $P_{k,d}$ contains at most k lattice points of I. It has been proved in [15] that $\mathcal{I}_{n,d}$ is the collection of independent sets of a matroid $\mathcal{T}_{n,d}$ with ground set $P_{n,d}$. In this chapter, which is based on a joint work with Harold Polo [102], we study some aspects of the combinatorial structure of the matroids $\mathcal{T}_{n,3}$. The case $d = 3$ is particularly important because, as it was proved in [15], the matroid $\mathcal{T}_{n,3}$ is cotransversal and the cotransversality property of $\mathcal{T}_{n,3}$ allows to construct the Schubert-generic line arrangement $\mathbf{E}_{n,3}$ explicitly (see [15, Proposition 9.2]). Finally, it is worthy to notice that although cotransversal matroids have been the subject of a great deal of investigation, cotransversal matroids other than $\mathcal{T}_{n,3}$ do not seem to have been studied before in connection with tilings.

The original motivation to study the matroids $\mathcal{T}_{n,d}$ comes from [15], where the authors were interested in understanding the set $\mathbf{E}_{n,d}$ of 1-dimensional intersections of complete complex flag arrangements. It turns out that the dependence relations among the lines in $\mathbf{E}_{n,d}$ are encoded in $\mathcal{T}_{n,d}$. As a result, the structure of $\mathcal{T}_{n,d}$ was crucial to understand the linear dependence of line arrangements resulting from intersecting complete flags of $\mathbb{C}^n$ and, as a byproduct, to facilitate certain computations on the cohomology ring of the flag manifold. The matroids $\mathcal{T}_{3,d}$ have been studied in [16] in connection with acyclic permutations and fine mixed subdivisions of simplices [131].

Questions about tilings come in many diverse flavors, from problems about existence and enumeration to problems about computational complexity and feasibility, and they have

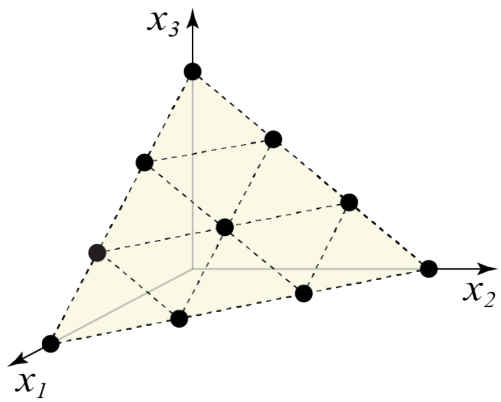

Figure 4.1: The set of lattice points $P_{4,3}$ and a (shaded) region of the plane in $\mathbb{R}^3$ containing it. The 10 lattice points of $P_{4,3}$ are depicted by dark black dots.

been investigated in connection with many fields, including combinatorial group theory [53], algebraic geometry [27], computational complexity theory [32], and stochastic processes [118]. Furthermore, tiling theory also finds applications to perfect matching [40], classical geometric problems [113], genetic [35], *etc.* See [18] for a friendly survey on tileability. Here we establish various cryptomorphic characterizations of the matroids $\mathcal{T}_{n,3}$ in terms of tilings of $S_{n,3}$ into unit triangles, rhombi, and trapezoids.

4.2 Tiling Matroids

As mentioned in the introduction, it was proved in [15] that for each $n, d \in \mathbb{N}$, the pair $\mathcal{T}_{n,d} = (P_{n,d}, \mathcal{I}_{n,d})$ is a matroid.

Definition 4.2.1. For $n \in \mathbb{N}$, we call $\mathcal{T}_{n,3}$ a *tiling matroid* and denote it simply by $\mathcal{T}_n$.

It is not hard to verify that the matroid $\mathcal{T}_n$ has rank n and its ground set has size $n(n+1)/2$. From now on we let T_n denote the convex hull of $P_{n,3}$ and think of elements in $P_{n,3}$ as triangles in a regular subdivision of T_n as follows. We tacitly assume that T_n is placed as in the top-right picture of Figure 4.2, namely that T_n is in the plane of the paper and has a horizontal base. In addition, we think of a lattice point p of $P_{n,3}$ as a closed equilateral triangle pointing upward, centered at p, whose side length is the minimal distance between lattice points in T_n. Finally, we rescale T_n so that the ground set of $\mathcal{T}_n$ consists of unit triangles. This transition from lattice points to unit triangles is illustrated in Figure 4.2.

Definition 4.2.2. We call the unit triangles representing lattice points of $P_{n,3}$ *unit upward triangles* of T_n and we call the unit triangles inside $T_n \setminus \cup_{\Delta \in P_{n,3}} \Delta$ *unit downward triangles* of T_n. Let $\mathfrak{u}(T_n)$ (resp., $\mathfrak{d}(T_n)$) denote the set of unit upward (resp., downward) triangles of T_n.

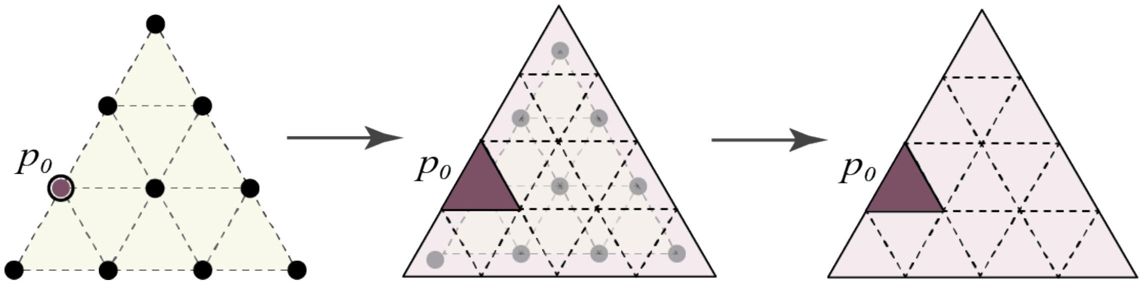

Figure 4.2: The leftmost picture shows $P_{4,3}$ and an element $p_0 \in P_{4,3}$. The next two pictures show the triangular representation of p_0. The rightmost picture in Figure 4.2 illustrates T_4 and the unit triangle p_0.

Then $\mathfrak{u}(T_n)$ is the ground set of $\mathcal{T}_n$. We say that a nonempty subset of T_n is a *lattice region* if its closure is the union of unit triangles of T_n. Note that any lattice upward triangle T of T_n is a parallel translate of T_ℓ for some $\ell \leq n$; in this case we call ℓ the *size* of T and set $\text{size}(T) := \ell$. If $A \subseteq T_n$ is a lattice region, then we define

$$\mathfrak{u}(A) := \{X \in \mathfrak{u}(T_n) \mid X \subseteq A\} \quad \text{and} \quad \mathfrak{d}(A) := \{X \in \mathfrak{d}(T_n) \mid X \subseteq A\}.$$

On the other hand, given a collection $\mathfrak{s}$ of lattice regions of T_n, we set

$$A(\mathfrak{s}) := \bigcup_{R \in \mathfrak{s}} R.$$

The *triangular hull* of $\mathfrak{s}$ is the smallest lattice upward triangle of T_n containing all lattice regions in $\mathfrak{s}$. The concepts in the following two definitions are central in our exposition.

Definition 4.2.3. For $\mathfrak{s} \subseteq \mathfrak{u}(T_n)$, we call $T_n \setminus A(\mathfrak{s})$ the *holey region* corresponding to $\mathfrak{s}$.

Definition 4.2.4. For a lattice region R of T_n, we call $\mathfrak{t}$ a *tiling* of R provided that $\mathfrak{t}$ consists of closed lattice regions of T_n whose interiors are pairwise disjoint and $\cup_{T \in \mathfrak{t}} T$ equals the closure of R.

Additionally, let us introduce notation for one of the most important lattice regions and tilings we will consider in this paper.

Definition 4.2.5. Given a tiling matroid $\mathcal{T}_n$, we call the union of two adjacent unit triangles of T_n a *unit rhombus*. A tiling into unit rhombi of a lattice region R of T_n is called a *lozenge tiling* of R.

Figure 4.3 illustrates all possible unit rhombi of T_n (up to translation): one vertical and two (symmetric) horizontal. We say that a unit rhombus R of T_n is *horizontal* provided that one of its sides is horizontal; otherwise, we say that R is *vertical*.

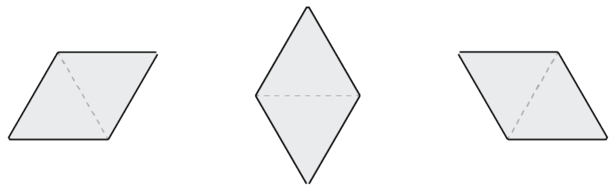

Figure 4.3: The three unit rhombi of T_n up to translation.

A lozenge tiling of the holey region corresponding to a 4-element subset of $\mathfrak{u}(T_4)$ is illustrated in the left picture of Figure 4.4. The following characterization of the bases of $\mathcal{T}_n$ was established in [15].

Theorem 4.2.6. *[15, Theorem 6.2] For the tiling matroid $\mathcal{T}_n$, let $\mathfrak{b}$ be a subset of $\mathfrak{u}(T_n)$. Then $\mathfrak{b}$ is a basis of $\mathcal{T}_n$ if and only if there exists a lozenge tiling of $T_n \setminus A(\mathfrak{b})$.*

The following example illustrates Theorem 4.2.6.

Example 4.2.7. Consider the tiling matroid $\mathcal{T}_4$. Let $\mathfrak{b}$ consist of the dark unit upward triangles in the left picture of Figure 4.4. Note that $\mathfrak{b}$ is a basis of $\mathcal{T}_4$. A lozenge tiling of the holey region $T_4 \setminus A(\mathfrak{s})$ is shown. On the other hand, let $\mathfrak{s}$ be the set of dark unit upward triangles in the right picture of Figure 4.4. One can easily see that the holey region $T_4 \setminus A(\mathfrak{s})$ cannot be tiled into unit rhombi.

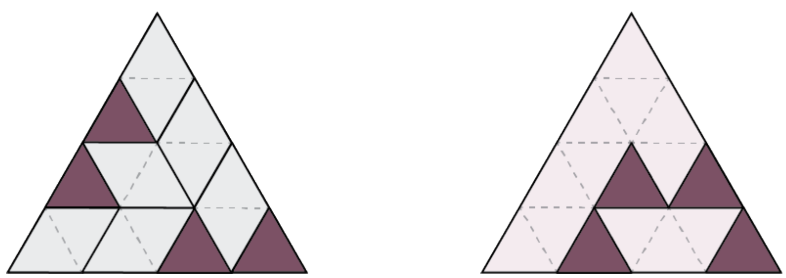

Figure 4.4: On the left, a basis of $\mathcal{T}_4$. On the right, a size-4 subset of $\mathfrak{u}(T_4)$ that is not a basis of $\mathcal{T}_4$.

4.3 Tiling Characterization of Independent Sets

In this section, we characterize the independent sets $\mathfrak{s}$ of the matroid $\mathcal{T}_n$ in terms of certain tilings of $T_n \setminus A(\mathfrak{s})$. This characterization generalizes that one of bases given in Theorem 4.2.6.

Definition 4.3.1. A *type-1 trapezoid* of T_n is a lattice trapezoid of T_n that is the union of two unit upward triangles and one unit downward triangle.

As in the case of unit rhombi, we say that a type-1 trapezoid T of T_n is *horizontal* if it has its two parallel sides horizontal. Up to translation, there are three type-1 trapezoids, as depicted in Figure 4.5.

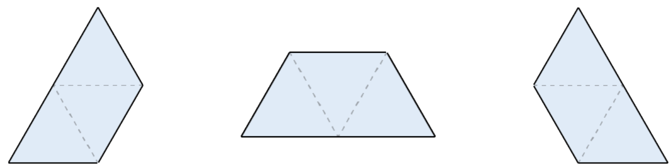

Figure 4.5: The three type-1 trapezoids up to translation.

Theorem 4.3.2. *Let $\mathfrak{s}$ be a subset of $\mathfrak{u}(T_n)$. Then $\mathfrak{s}$ is independent if and only if the lattice region $T_n \setminus A(\mathfrak{s})$ can be tiled using unit rhombi and exactly $n - |\mathfrak{s}|$ type-1 trapezoids.*

Proof. To prove the direct implication, suppose that $\mathfrak{s}$ is an independent set of $\mathcal{T}_n$. If $|\mathfrak{s}| = n$, then $\mathfrak{s}$ is a basis of $\mathcal{T}_n$, and we are done by Theorem 4.2.6. So we assume that $|\mathfrak{s}| < n$. Take a basis $\mathfrak{b}$ of $\mathcal{T}_n$ containing $\mathfrak{s}$. Theorem 4.2.6 ensures the existence of a lozenge tiling of $T_n \setminus A(\mathfrak{b})$. Let $\mathfrak{t}$ be one of such tilings. Merging some of the unit upward triangles in $\mathfrak{b} \setminus \mathfrak{s}$ with some of their adjacent rhombi, we create new tilings $\mathfrak{t}'$ of $T_n \setminus A(\mathfrak{s})$ consisting of m type-1 trapezoids ($m \le n - |\mathfrak{s}|$), some unit rhombi, and some unit upward triangles. Among all such tilings, let $\mathfrak{t}'$ be one maximizing m and, suppose, by way of contradiction, that $m < n - |\mathfrak{s}|$. Then there is a unit upward triangle $X \in \mathfrak{b} \setminus \mathfrak{s}$ that did not merge to any unit rhombus of $\mathfrak{t}$. By rotating T_n if necessary, we can assume that X is adjacent to a vertical unit rhombus in $\mathfrak{t}$. Now we consider two cases.

CASE 1. The horizontal side a of X is not in the horizontal border of T_n. The fact that X is adjacent to a vertical unit rhombus in $\mathfrak{t}$ forces $n \ge 3$. Let R denote one of the vertical unit rhombus adjacent to X in $\mathfrak{t}$. Because X did not merge in $\mathfrak{t}'$, another unit upward triangle $Y \in \mathfrak{b} \setminus \mathfrak{s}$ merged to R in $\mathfrak{t}'$ creating a type-1 trapezoid. Assume, without loss of generality, that Y is right after X in the same row. In this case, a is a side of the horizontal unit rhombus R' right below X in $\mathfrak{t}$. As X did not merge in $\mathfrak{t}'$, another unit upward triangle $Z \in \mathfrak{b} \setminus \mathfrak{s}$ merged to R' in $\mathfrak{t}'$ creating a type-1 trapezoid. Note that X, Y, and Z belong to the same size-3 lattice upward triangle of T_n, which can be re-tiled by using exactly three type-1 trapezoids in such a way that no two triangles in $\{X, Y, Z\}$ are part of the same trapezoid. But this yields a tiling of $T_n \setminus A(\mathfrak{s})$ containing more type-1 trapezoids than $\mathfrak{t}'$ does, which contradicts the maximality of m.

CASE 2. The horizontal side a of X is in the horizontal border of T_n. As the cases when $n < 3$ follow straightforwardly, we assume $n \ge 3$. Let $\mathfrak{x} = \{X_1, \dots, X_t\}$ be the maximal set of consecutive unit upward triangles in the bottom row of T_n such that $X \in \mathfrak{x} \subseteq \mathfrak{b} \setminus \mathfrak{s}$. Notice that if $t = 1$, then the maximality of $\mathfrak{x}$ would make one of the vertical unit rhombi adjacent to X in $\mathfrak{t}$ available to merge with X to form a tiling of $T_n \setminus A(\mathfrak{s})$ containing more type-1 trapezoids than $\mathfrak{t}'$ does, which is a contradiction. Hence we also assume that $t > 1$.

CASE 2.1. $X \in \{X_1, X_t\}$. Assume, without loss of generality, that $X = X_1$. We first suppose that X is not a corner of T_n. Then the unit rhombus R adjacent from the left to X in $\mathfrak{t}$ must be horizontal by the maximality of $\mathfrak{r}$. In addition, as X did not merge in $\mathfrak{t}'$, the unit rhombus R must have merged in $\mathfrak{t}'$ with the unit upward triangle Y right on top of it. Also, note the rhombus R' adjacent from the right to X in $\mathfrak{t}$ is vertical and must have merged to X_2 in $\mathfrak{t}'$ (by the maximality of m). Now the size-3 lattice upward triangle containing X, X_2, and Y can be re-tiled into three type-1 trapezoids such that not two triangles in $\{X, X_2, Y\}$ are part of the same trapezoid. However, this results in a new tiling of $T_n \setminus A(\mathfrak{s})$ having more type-1 trapezoids than $\mathfrak{t}'$ does, which is a contradiction.

On the other hand, suppose that $X = X_1$ is a corner of T_n. It follows from $|\mathfrak{s}| \geq 1$ that $t < n$. As a result, X_t cannot be a corner of T_n. Notice that for every $i \in [t-1]$ the common adjacent tile to X_i and X_{i+1} in $\mathfrak{t}$ is a vertical unit rhombus. If for some $i \in [t] \setminus \{1\}$ the triangle X_i did not merge with its left adjacent rhombus in $\mathfrak{t}'$, then we can re-tile $T_n \setminus A(\mathfrak{s})$ by merging X_j to its right adjacent vertical unit rhombus for all $j \in [i-1]$ to create a tiling of $T_n \setminus A(\mathfrak{s})$ containing more type-1 trapezoids than $\mathfrak{t}'$ does, which is not possible. Then we can assume that X_t merges in $\mathfrak{t}'$ to its left vertical unit rhombus. By the maximality of $\mathfrak{r}$, the unit rhombus R'' adjacent from the right to X_t in $\mathfrak{t}$ must be horizontal. If R'' does not merge in $\mathfrak{t}'$, then we can re-tile $T \setminus A(\mathfrak{s})$ by merging each of the X_i to its right adjacent unit rhombus, obtaining once again a tiling of $T_n \setminus A(\mathfrak{s})$ with more than m type-1 trapezoids. Then suppose that R'' merges in $\mathfrak{t}'$ to the unit upward triangle Z right on top of it. In this case, we can merge X_t with R'', Z with its top-left vertical unit rhombus, and X_i with its top-right unit rhombus for each $i \in [t-1]$ to obtain a tiling of $T_n \setminus A(\mathfrak{s})$ containing more type-1 trapezoids than $\mathfrak{t}'$ does. However, this contradicts the maximality of m.

CASE 2.2. $X \notin \{X_1, X_t\}$. Let $X = X_j$. Because $t < n$, either X_1 or X_t is not a corner of T_n. Suppose, without loss of generality, that X_t is not a corner of T_n. In this case, we can proceed as we did in the second paragraph of CASE 2.1 to obtain a new tiling of $T_n \setminus A(\mathfrak{s})$ having more than m type-1 trapezoids, which once again contradicts the maximality of m.

To check the reverse implication, suppose that $\mathfrak{t}$ is a tiling of the region $T_n \setminus A(\mathfrak{s})$ consisting of unit rhombi and $n - |\mathfrak{s}|$ type-1 trapezoids. Now split each type-1 trapezoid of $\mathfrak{t}$ into a unit rhombus and a unit upward triangle, and let $\mathfrak{s}'$ denote the set of all unit upward triangle resulting from such splittings. Then we have a tiling $\mathfrak{t}'$ of $T_n \setminus A(\mathfrak{s} \cup \mathfrak{s}')$ using only unit rhombi. By Theorem 4.2.6, the set $\mathfrak{s} \cup \mathfrak{s}'$ is a basis of $\mathcal{T}_n$. Hence $\mathfrak{s}$ is an independent set of $\mathcal{T}_n$, which completes the proof. □

The following example illustrates the characterization established in Theorem 4.3.2.

Example 4.3.3. Consider the tiling matroid $\mathcal{T}_4$. The left picture of Figure 4.6 shows an independent set $\mathfrak{s}$ of $\mathcal{T}_4$ whose elements are depicted by the three dark unit upward triangles. It also shows a tiling of the holey region corresponding to $\mathfrak{s}$ into unit rhombi and a type-1 trapezoid. By contrast, the right picture of Figure 4.6 illustrates a dependent set $\mathfrak{s}'$ of $\mathcal{T}_4$. Notice that the holey region corresponding to $\mathfrak{s}'$ cannot be tiled into unit rhombi and type-1 trapezoids as it consists of more unit downward triangles than unit upward triangles.

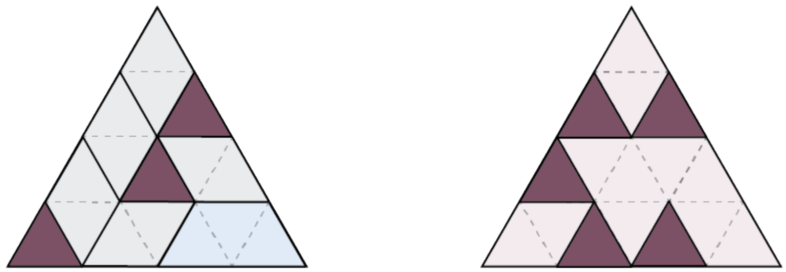

Figure 4.6: On the left, an independent set of $\mathcal{T}_4$. On the right, a dependent set of $\mathcal{T}_4$.

4.4 Rank and Tilings

Given a subset $\mathfrak{s}$ of $\mathfrak{u}(T_n)$, we now study how the size and rank of $\mathfrak{s}$ can be used to count the number of congruent pieces of any tiling of $T_n \setminus A(\mathfrak{s})$ into unit triangles and unit rhombi that maximizes the number of rhombi.

Let R be a lattice region of T_n, and let $\mathfrak{t}$ be a tiling of R consisting of unit triangles and unit rhombi. We call a tiling $\mathfrak{t}'$ of R a *reconfiguration* of $\mathfrak{t}$ provided that $\mathfrak{t}'$ contains the same numbers of unit upward triangles, unit downward triangles, and unit rhombi that $\mathfrak{t}$ does. Before presenting the main result of this section, let us collect the following lemma.

Lemma 4.4.1. *Let $\mathfrak{s}$ be a subset of $\mathfrak{u}(T_n)$, and let $\mathfrak{t}$ be a tiling of $T_n \setminus A(\mathfrak{s})$ into unit rhombi and unit triangles. Then there exists a reconfiguration of $\mathfrak{t}$ all whose unit downward triangles are adjacent from below to unit upward triangles.*

Proof. Among all reconfigurations of $\mathfrak{t}$, let $\mathfrak{t}'$ be one maximizing the number N of unit downward triangles that are adjacent from below to unit upward triangles. Let $\mathfrak{d}$ and $\mathfrak{u}$ be the sets of unit downward triangles and unit upward triangles in $\mathfrak{t}'$, respectively. Suppose, by way of contradiction, that there exists $X \in \mathfrak{d}$ that is not adjacent from below to any unit upward triangle. Then X must be adjacent from below to a horizontal unit rhombus of $\mathfrak{t}'$ and, therefore, we can move X one unit row up in T_n by using the moves illustrated in Figure 4.7, obtaining a reconfiguration $\mathfrak{t}''$ of $\mathfrak{t}'$. Notice that the triangles in $\mathfrak{d} \setminus \{X\}$ that were

Figure 4.7: The two moves used to switch a unit downward triangle with any potential adjacent horizontal unit rhombus.

adjacent from below in $\mathfrak{t}'$ to a unit rhombus (or a unit upward triangle) keep this property in $\mathfrak{t}''$. As N is maximal, X is still adjacent from below in $\mathfrak{t}''$ to a unit rhombus. Then we can actually move between reconfigurations of $\mathfrak{t}'$ by performing the move in Figure 4.7 to X until it reaches the second unit row of T_n (from top to bottom), obtaining a final reconfiguration of $\mathfrak{t}'$ where X is adjacent from below to the top unit triangle of T_n. However, this contradicts the maximality of N. Hence each unit downward triangle in $\mathfrak{t}'$ must be adjacent from below to a unit upward triangle, and the proof follows. □

The next result establishes a relation between the size and rank of subsets $\mathfrak{s}$ of $\mathfrak{u}(T_n)$ and certain tilings of $T_n \setminus A(\mathfrak{s})$ into unit triangles and unit rhombi.

Proposition 4.4.2. *If $\mathfrak{s} \subseteq \mathfrak{u}(T_n)$, then a tiling of $T_n \setminus A(\mathfrak{s})$ into unit rhombi and unit triangles with maximum number of rhombi must contain $|\mathfrak{s}| - r(\mathfrak{s})$ unit downward triangles.*

Proof. Let us first argue that there exists a tiling of $T_n \setminus A(\mathfrak{s})$ into unit rhombi and unit triangles having exactly $|\mathfrak{s}| - r(\mathfrak{s})$ unit downward triangles. Let $\mathfrak{r}$ be an independent set of $\mathcal{T}_n$ contained in $\mathfrak{s}$ such that $|\mathfrak{r}| = r(\mathfrak{s})$. Now take a basis $\mathfrak{b}$ of $\mathcal{T}_n$ containing $\mathfrak{r}$. As any subset of $\mathfrak{b}$ is an independent set of $\mathcal{T}_n$, the sets $\mathfrak{b} \setminus \mathfrak{r}$ and $\mathfrak{s} \setminus \mathfrak{r}$ are disjoint. By Theorem 4.2.6, there exists a lozenge tiling $\mathfrak{t}$ of $T_n \setminus A(\mathfrak{b})$. This, in turns, gives us a tiling $\mathfrak{t}'$ of $T_n \setminus A(\mathfrak{r})$ consisting of unit rhombi and the unit upward triangles in $\mathfrak{b} \setminus \mathfrak{r}$. Since $\mathfrak{b} \setminus \mathfrak{r}$ and $\mathfrak{s} \setminus \mathfrak{r}$ are disjoint, each $X \in \mathfrak{s} \setminus \mathfrak{r}$ must be covered by a rhombus R_X of $\mathfrak{t}'$. After splitting each rhombus R_X into two unit triangles, one obtains the desired tiling of $T_n \setminus A(\mathfrak{s})$ into unit rhombi and unit triangles containing exactly $|\mathfrak{s}| - r(\mathfrak{s})$ unit downward triangles.

Now observe that the number of rhombi in a tiling $\mathfrak{t}$ of $T_n \setminus A(\mathfrak{s})$ into unit rhombi and unit triangles determines the number of unit upward triangles and the number of unit downward triangles in $\mathfrak{t}$. Indeed, it is easy to verify that if $\mathfrak{t}$ contains m unit rhombi, then it must contain $\binom{n}{2} - m$ unit downward triangles and $\binom{n+1}{2} - m - |\mathfrak{s}|$ unit upward triangles. In particular, $\mathfrak{t}$ maximizes the number of unit rhombi if and only if it minimizes the number of unit downward triangles. Hence we are done once we prove that every tiling of $T_n \setminus A(\mathfrak{s})$ into unit rhombi and unit triangles contains at least $|\mathfrak{s}| - r(\mathfrak{s})$ unit downward triangles.

Take a tiling $\mathfrak{t}$ of $T_n \setminus A(\mathfrak{s})$ into unit rhombi and unit triangles minimizing the number of unit downward triangles. Let $\mathfrak{d}$ and $\mathfrak{u}$ be the sets of unit downward triangles and unit upward triangles of $\mathfrak{t}$, respectively. By Lemma 4.4.1, there is no loss in assuming that all triangles in $\mathfrak{d}$ are adjacent from below to unit upward triangles. On the other hand, the minimality of $|\mathfrak{d}|$ ensures that no triangle in $\mathfrak{d}$ is adjacent from below to a triangle in $\mathfrak{u}$. Hence each triangle of $\mathfrak{d}$ is adjacent from below to a triangle of $\mathfrak{s}$. Merging each triangle in $\mathfrak{d}$ with its corresponding adjacent from above triangle of $\mathfrak{s}$, one obtains a lozenge tiling $\mathfrak{t}'$ of a lattice region R satisfying that $\mathfrak{u}(R) \cap \mathfrak{s}$ consists precisely of those triangles in $\mathfrak{s}$ that did not merge with the $|\mathfrak{d}|$ unit downward triangles of $\mathfrak{t}$. By Theorem 4.2.6, the subset $\mathfrak{u}(R) \cap \mathfrak{s}$ of $\mathfrak{s}$ is independent. This implies that $|\mathfrak{s}| - |\mathfrak{d}| = |\mathfrak{u}(R) \cap \mathfrak{s}| \le r(\mathfrak{s})$ and, therefore, $|\mathfrak{d}| \ge |\mathfrak{s}| - r(\mathfrak{s})$, which completes the proof. □

The following corollary is an immediate consequence of Proposition 4.4.2.

Corollary 4.4.3. *If $\mathfrak{s} \subseteq \mathfrak{u}(T_n)$ and $\mathfrak{t}$ is a tiling of $T_n \setminus A(\mathfrak{s})$ into unit triangles and unit rhombi that maximizes the number of unit rhombi, then $\mathfrak{t}$ contains $\binom{n}{2} - (|\mathfrak{s}| - r(\mathfrak{s}))$ unit rhombi, $|\mathfrak{s}| - r(\mathfrak{s})$ unit downward triangles, and $n - r(\mathfrak{s})$ unit upward triangles.*

Remark 4.4.4. Notice that Corollary 4.4.3 does not imply Theorem 4.3.2 as some of the $n - r(\mathfrak{s})$ unit upward triangles in the tiling $\mathfrak{t}$ might be seated at the bottom line of the lattice region T_n and, therefore, do not have rhombi right below them to merge.

4.5 Tiling Characterization of Circuits

We proceed to characterize the circuits $\mathfrak{c}$ of $\mathcal{T}_n$ in terms of certain tilings of $T_n \setminus A(\mathfrak{c})$ into unit rhombi and (possibly reflected) type-1 trapezoids.

Let M be a matroid. A subset C of the ground set of M is called a *circuit* of M if $|C| - r(C) = 1$ and $C \setminus \{x\}$ is an independent set of M for each $x \in C$, that is, C is a minimal dependent set of M. On the other hand, a *loop* (resp., *parallel*) of M is a dependent set of M of size one (resp., two).

Remark 4.5.1. Tiling matroids are *simple*, i.e., they contain neither loops nor parallels.

Example 4.5.2. Consider the tiling matroid $\mathcal{T}_4$. The left picture of Figure 4.8 shows a circuit of rank 2 (and size 3) whose elements are depicted by the three dark unit upward triangles. It is easy to argue that all circuits of $\mathcal{T}_n$ have the same shape, meaning that they are geometrically equal up to translation. On the other hand, the right picture of Figure 4.8 shows a subset $\mathfrak{s}$ of $\mathfrak{u}(T_4)$ of rank 4 that is not a circuit even though $|\mathfrak{s}| - r(\mathfrak{s}) = 1$. Note that $\mathfrak{s}$ contains a circuit of rank 2.

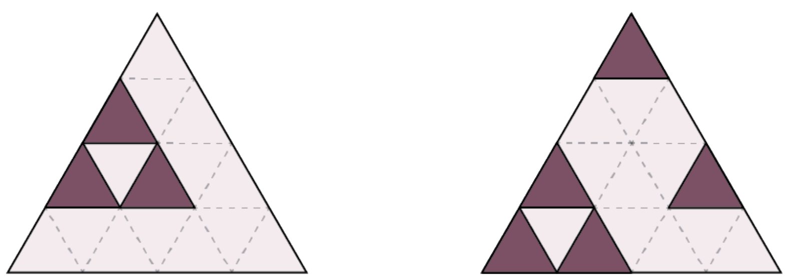

Figure 4.8: On the left, a circuit of rank 2. On the right, a non-circuit of rank 4.

Definition 4.5.3. A *type-2 trapezoid* is a lattice trapezoid of T_n which is the union of one unit upward triangle and two unit downward triangles (cf. Definition 4.3.1).

As in the case of type-1 trapezoids, one has three possible type-2 trapezoids (up to translation), one of them being horizontal. Observe that they are the reflection of the type-1 trapezoids through their largest sides.

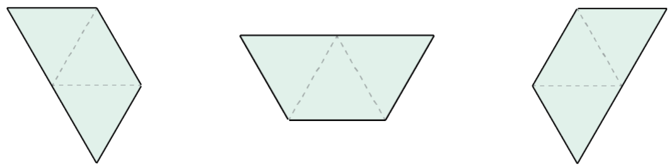

Figure 4.9: The three type-2 trapezoids up to translation.

To characterize further distinguished subsets $\mathfrak{s}$ of the ground set of $\mathcal{T}_n$ in terms of tiling of $T_n \setminus A(\mathfrak{s})$ into unit rhombi and unit trapezoids, $\mathfrak{s}$ cannot contain any circuit of rank 2 because such circuits isolate unit triangles (see Example 4.5.2). However, this requirement does not suffice as the following example illustrates.

Example 4.5.4. Consider the tiling matroid $\mathcal{T}_4$ along with the subset $\mathfrak{s}$ of $\mathfrak{u}(T_4)$ given by the dark unit upward triangles illustrated in Figure 4.10. As the lattice region $T_4 \setminus A(\mathfrak{s})$ consists of only one unit upward triangle and three unit downward triangles, it cannot be tiled into unit rhombi and unit trapezoids.

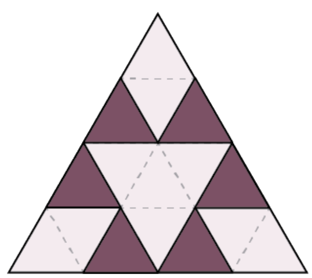

Figure 4.10: A subset of $\mathfrak{u}(T_4)$ whose corresponding holey region cannot be tiled into unit rhombi and unit trapezoids.

The holey region corresponding to most circuits, however, can be tiled into unit rhombi and unit trapezoids. Indeed, we will see in Theorem 4.5.8 a tiling characterization of circuits of rank greater than 2. First, we collect a few results we shall be using later. For a subset $\mathfrak{s}$ of $\mathfrak{u}(T_n)$, we say that a lattice upward triangle T of T_n of size k is *saturated* by $\mathfrak{s}$ if $|\mathfrak{s} \cap \mathfrak{u}(T)| = k$, *over-saturated* by $\mathfrak{s}$ if $|\mathfrak{s} \cap \mathfrak{u}(T)| \geq k$, and *strictly over-saturated* by $\mathfrak{s}$ if $|\mathfrak{s} \cap \mathfrak{u}(T)| > k$.

Proposition 4.5.5. *Let $\mathfrak{c}$ be a circuit of $\mathcal{T}_n$. Then there exists exactly one lattice upward triangle of T_n that is strictly over-saturated by $\mathfrak{c}$, namely the triangular hull of $\mathfrak{c}$.*

Proof. Since $\mathfrak{c}$ is a dependent set of $\mathcal{T}_n$, there must be a lattice upward triangle T of T_n that is strictly over-saturated by $\mathfrak{c}$. Notice that each $X \in \mathfrak{c}$ must be inside T; otherwise, T would be strictly over-saturated by $\mathfrak{c} \setminus \{X\}$, contradicting that $\mathfrak{c} \setminus \{X\}$ is an independent set of $\mathcal{T}_n$. Among all lattice upward triangles strictly over-saturated by $\mathfrak{c}$, assume that T is minimal under inclusion. The minimality of T implies now that T is the triangular hull of $\mathfrak{c}$. As $\mathfrak{c}$ is contained in $\mathfrak{u}(T)$, it follows that $\text{size}(T) < |\mathfrak{c}|$. The fact that $\mathfrak{c}$ contains an independent subset of $\mathcal{T}_n$ of size $|\mathfrak{c}| - 1$ yields $|\mathfrak{c}| - 1 \leq \text{size}(T)$. Hence $\text{size}(T) = |\mathfrak{c}| - 1$. Finally, let T' be a lattice upward triangle of T_n strictly over-saturated by $\mathfrak{c}$. Since T' contains $A(\mathfrak{c})$, we have that $T \subseteq T'$. This, along with the fact that T' is strictly over-saturated by $\mathfrak{c}$, guarantees that $\text{size}(T) \leq \text{size}(T') \leq |\mathfrak{c}| - 1 = \text{size}(T)$. Thus, $T = T'$, and the uniqueness follows. □

Lemma 4.5.6. *Each lattice (isosceles) trapezoid of T_n of side length k can be tiled using unit rhombi and k type-1 trapezoids.*

Proof. We can tile each unit row of such a lattice trapezoid by placing a horizontal type-1 trapezoid covering its three leftmost unit triangles and covering the rest of the row with unit rhombi, as illustrated in Figure 4.11.

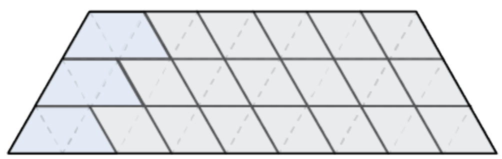

Figure 4.11: A lattice trapezoid of side length 3 tiled using unit rhombi and three type-1 trapezoids.

□

Lemma 4.5.7. *Let T be a lattice upward triangle of T_n. Then each unit rhombus or type-1 trapezoid of T_n covers at least the same number of unit upward triangles inside T as unit downward triangles inside T.*

Proof. Let S be either a unit rhombi or a type-1 trapezoid of T_n. Set $I = S \cap T$, and let t be the number of unit triangles inside I, i.e., $t = |\mathfrak{d}(I)| + |\mathfrak{u}(I)|$. Clearly, $t \in \{0, 1, 2, 3\}$. If $t = 0$, then $|\mathfrak{d}(I)| = |\mathfrak{u}(I)| = 0$. If $t = 1$, then I must be a unit upward triangle and so $0 = |\mathfrak{d}(I)| = |\mathfrak{u}(I)| - 1$. If $t = 2$, then I must be a unit rhombi and, therefore, $1 = |\mathfrak{d}(I)| = |\mathfrak{u}(I)|$. Finally, if $t = 3$, then S must be a type-1 trapezoid and $I = S$, which implies that $1 = |\mathfrak{d}(I)| = |\mathfrak{u}(I)| - 1$. As in any case we have verified that $|\mathfrak{d}(I)| \leq |\mathfrak{u}(I)|$, the lemma follows. □

We are now in a position to give a characterization of the circuits of $\mathcal{T}_n$.

Theorem 4.5.8. *Let $\mathfrak{c}$ be a subset of $\mathfrak{u}(T_n)$ such that $|\mathfrak{c}| \geq 4$. Then $\mathfrak{c}$ is a circuit of $\mathcal{T}_n$ if and only if the following two conditions hold:*

(1) the triangular hull of $\mathfrak{c}$ *is the only lattice upward triangle strictly over-saturated by* $\mathfrak{c}$*;*

*(2) the minimum number of type-*2 *trapezoids we can use to tile* $T_n \setminus A(\mathfrak{c})$ *into unit rhombi and unit trapezoids is* 1*.*

Proof. For the forward implication, let $\mathfrak{c}$ be a circuit. By Proposition 4.5.5, the triangular hull T of $\mathfrak{c}$ is the unique lattice upward triangle strictly over-saturated by $\mathfrak{c}$. We proceed to show that condition (2) also holds. As $\mathfrak{c}$ is a circuit, $\text{size}(T) = r(\mathfrak{c})$. Now fix $X \in \mathfrak{c}$. Note that $\mathfrak{c} \setminus \{X\}$ is a basis of the tiling matroid $\mathcal{T}_{r(\mathfrak{c})}$ with $T_{r(\mathfrak{c})} = T$. Thus, one can use Theorem 4.2.6 to obtain a tiling $\mathfrak{t}$ of $T \setminus A(\mathfrak{c} \setminus \{X\})$ consisting of unit rhombi. In such a tiling, X must be covered by a rhombus R_X. Notice that the unit downward triangle of R_X must be adjacent to a unit upward triangle Y not contained in $\mathfrak{c}$; otherwise, the triangular hull of R_X (which has size 2) would be strictly over-saturated by $\mathfrak{c}$, and Proposition 4.5.5 would force $|\mathfrak{c}| = 3$. Let R_Y be the unit rhombus of $\mathfrak{t}$ covering Y. Now we can obtain a tiling $\mathfrak{t}'$ of $T \setminus A(\mathfrak{c})$ as follows. First keep the tiling configuration of $\mathfrak{t}$ outside $R_X \cup R_Y$, then make X hollow, and finally merge the unit downward triangle of R_X to R_Y. Clearly, $\mathfrak{t}'$ consists of unit rhombi and one type-2 trapezoid, namely the new tile containing Y.

From the tiling $\mathfrak{t}'$ of $T \setminus A(\mathfrak{c})$, we can construct the desired tiling of $T_n \setminus A(\mathfrak{c})$ provided we tile $T_n \setminus T$ into unit rhombi and type-1 trapezoids. To do this, first split the lattice region $T_n \setminus T$ into three lattice trapezoids as illustrated in Figure 4.12. Clearly, the sum of the side

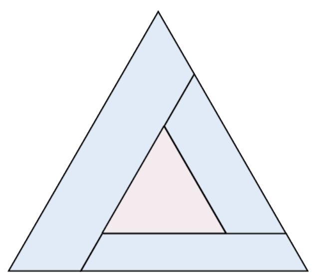

Figure 4.12: The lattice region $T_n \setminus T$ split into three lattice trapezoids.

lengths of these three lattice trapezoids is $n - r(\mathfrak{c})$. Therefore one can use Lemma 4.5.6 to tile $T_n \setminus T$ into unit rhombi and $n - r(\mathfrak{c})$ type-1 trapezoids, obtaining thereby the desired tiling of $T \setminus A(\mathfrak{c})$. Finally, notice that 1 is the minimal number of type-2 trapezoids we can use to tile $T_n \setminus A(\mathfrak{c})$ since using 0 type-2 trapezoids would imply, by Theorem 4.3.2, that $\mathfrak{c}$ is an independent set of $\mathcal{T}_n$.

To prove the backward implication, we first show that $|\mathfrak{u}(T) \cap \mathfrak{c}| \leq \ell + 1$ for each lattice upward triangle T of T_n of size ℓ. Suppose, by way of contradiction, that $|\mathfrak{u}(T) \cap \mathfrak{c}| \geq k+2$ for some lattice upward triangle T of size k. As $|\mathfrak{u}(T)| = |\mathfrak{d}(T)| + k$ and $|\mathfrak{d}(T)| = |\mathfrak{d}(T \setminus A(\mathfrak{c}))|$, one finds that

$$|\mathfrak{u}(T \setminus A(\mathfrak{c}))| = |\mathfrak{u}(T)| - |\mathfrak{u}(T) \cap \mathfrak{c}| \leq |\mathfrak{u}(T)| - (k+2) = |\mathfrak{d}(T \setminus A(\mathfrak{c}))| - 2.$$

Then $T \setminus A(\mathfrak{c})$ contains at least two more unit downward triangles than unit upward triangles. By Lemma 4.5.7, every unit rhombus or type-1 trapezoid in any tiling of $T_n \setminus A(\mathfrak{c})$ covers at least the same number of unit upward triangles as unit downward triangles of $\mathfrak{u}(T)$. Thus, we would need at least two type-2 trapezoids to tile $T \setminus A(\mathfrak{c})$ into unit rhombi and unit trapezoids, which contradicts condition (2).

It is clear that $\mathfrak{c}$ cannot be an independent set of $\mathcal{T}_n$; otherwise, we could use Theorem 4.3.2 to tile $T_n \setminus A(\mathfrak{c})$ using 0 type-2 trapezoids, contradicting that the minimum number of type-2 trapezoids needed to tile $T_n \setminus A(\mathfrak{c})$ into unit rhombi and unit trapezoids is 1. This, along with the fact that $|\mathfrak{u}(T) \cap \mathfrak{c}| \leq \ell + 1$ for each lattice upward triangle T of T_n of size ℓ, implies that $|\mathfrak{u}(T') \cap \mathfrak{c}| = \ell + 1$ for some lattice upward triangle T' of T_n of size ℓ.

We finally verify that $\mathfrak{c}$ is a circuit of $\mathcal{T}_n$. Take $X \in \mathfrak{c}$. By condition (1), the only lattice upward triangle of T_n strictly over-saturated by $\mathfrak{c}$ is the triangular hull T of $\mathfrak{c}$. Set $\ell = \text{size}(T)$. The existence of a lattice upward triangle of T_n over-saturated by $\mathfrak{c}$ by exactly one unit upward triangle forces $|\mathfrak{u}(T) \cap \mathfrak{c}| = \ell + 1$. Since T is the triangular hull of $\mathfrak{c}$, it follows that $X \in \mathfrak{u}(T)$ and, therefore, T is not strictly over-saturated by $\mathfrak{c} \setminus \{X\}$. Because no lattice upward triangle of T_n is strictly over-saturated by $\mathfrak{c} \setminus \{X\}$, the latter is an independent set of $\mathcal{T}_n$. As X was arbitrarily chosen, $\mathfrak{c}$ is a circuit. □

The following example illustrates that neither of the two conditions in Theorem 4.5.8 by itself suffices to ensure that $\mathfrak{c}$ is a circuit of $\mathcal{T}_n$.

Example 4.5.9. Consider the tiling matroid $\mathcal{T}_4$. Let $\mathfrak{s} \subset \mathfrak{u}(T_4)$ consists of the dark unit upward triangles in the left picture of Figure 4.13. Note that $\mathfrak{s}$ satisfies condition (1) of Theorem 4.5.8 since the only lattice triangle of T_4 that is strictly over-saturated by $\mathfrak{s}$ is T_4, which is the triangular hull of $\mathfrak{s}$. However, $\mathfrak{s}$ is not a circuit of $\mathcal{T}_4$ (observe that we need at least two type-2 trapezoids to tile $T_4 \setminus A(\mathfrak{s})$). Now let $\mathfrak{s}'$ be the set of dark unit upward triangles in the right picture of Figure 4.13. Observe that $\mathfrak{s}'$ satisfies condition (2) of Theorem 4.5.8; a tiling of $T_4 \setminus A(\mathfrak{s}')$ as described in condition (2) is shown in the picture. However, $\mathfrak{s}'$ is not a circuit of $\mathcal{T}_4$ (one can see that the triangular hull of $\mathfrak{s}'$ is not the only lattice upward triangle of T_4 strictly over-saturated by $\mathfrak{s}'$).

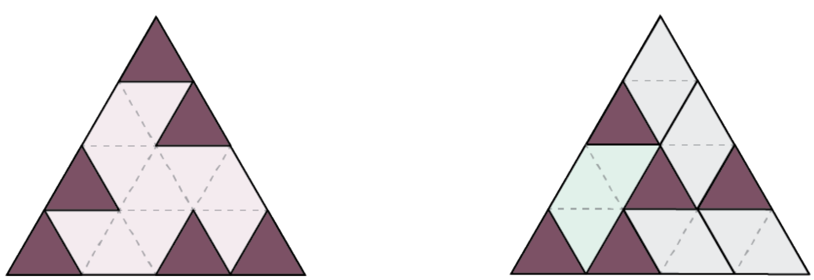

Figure 4.13: Two subsets of $\mathfrak{u}(T_4)$ that are not circuits of $\mathcal{T}_4$.

4.6 Geometric Characterization of Flats

Let M be a matroid with ground set E. The *closure operator* $\mathrm{cl}\colon 2^E \to 2^E$ of M is defined as follows:

$$\mathrm{cl}(S) = \{x \in E \mid r(S \cup \{x\}) = r(S)\}.$$

Matroids can be characterized in terms of their closure operators; see [125, Section 1.4]. A subset S of E is called a *flat* of M provided that $\mathrm{cl}(S) = S$. In this section, we give a geometric description of the flats of tiling matroids. To facilitate this, let us first introduce some notation.

Let $\mathfrak{s}$ be a subset of $\mathfrak{u}(T_n)$. Given two lattice upward triangles T and T' of T_n, we let $T \vee T'$ denote the triangular hull of $T \cup T'$. On the other hand, we say that the lattice upward triangle T of T_n of size k is *completely over-saturated* by $\mathfrak{s}$ if $\mathfrak{u}(T) \subseteq \mathfrak{s}$. We use the following result in the proof of Proposition 4.6.2

Lemma 4.6.1. *[15, Lemma 4.2] Let $\mathfrak{s}$ be an independent set of $\mathcal{T}_n$, and let T and T' be two lattice upward triangles of T_n saturated by $\mathfrak{s}$. If $T \cap T' \neq \emptyset$, then the lattice upward triangles $T \cap T'$ and $T \vee T'$ of T_n are also saturated by $\mathfrak{s}$.*

Proposition 4.6.2. *A subset $\mathfrak{f}$ of $\mathfrak{u}(T_n)$ is a flat of $\mathcal{T}_n$ if and only if every lattice upward triangle of T_n over-saturated by $\mathfrak{f}$ is also completely over-saturated by $\mathfrak{f}$.*

Proof. Suppose first that $\mathfrak{f}$ is a flat of $\mathcal{T}_n$. It suffices to prove that every maximal lattice upward triangle of T_n over-saturated by $\mathfrak{f}$ is completely over-saturated by $\mathfrak{f}$. Let $M_1, \dots, M_m$ be the maximal lattice upward triangles of T_n over-saturated by $\mathfrak{f}$. It follows by Lemma 4.6.1 that the M_i's are pairwise disjoint. Then the independent subsets of $\mathfrak{f}$ are unions $\mathfrak{i}_1 \cup \dots \cup \mathfrak{i}_m$, where each $\mathfrak{i}_j$ is an independent set of $\mathcal{T}_n$ contained in $\mathfrak{u}(M_j)$. Fix $j \in [m]$, and let us check that M_j is completely over-saturated by $\mathfrak{f}$. To do so, take $X \in \mathfrak{u}(M_j)$. Note that for each independent set $\mathfrak{i}$ contained in $\mathfrak{f} \cup \{X\}$,

$$r(\mathfrak{i}) = \sum_{i=1}^{m} r(\mathfrak{i} \cap \mathfrak{u}(M_i)) \leq \sum_{i=1}^{m} r(\mathfrak{u}(M_i)) \leq \sum_{i=1}^{m} \mathrm{size}(M_i) = r(\mathfrak{f}).$$

Thus, $r(\mathfrak{f} \cup \{X\}) = r(\mathfrak{f})$. Since $\mathfrak{f}$ is a flat of $\mathcal{T}_n$, it follows that $X \in \mathfrak{f}$. This implies that $\mathfrak{u}(M_j) \subseteq \mathfrak{f}$. Hence M_i is completely over-saturated by $\mathfrak{f}$ for every $i \in [m]$.

For the backward implication, let $M_1, \dots, M_m$ be the maximal lattice upward triangles of T_n that are over-saturated by $\mathfrak{f}$ (and, therefore, completely over-saturated by $\mathfrak{f}$). By Lemma 4.6.1, the M_i's are pairwise disjoint. For each $j \in [m]$, let $\mathfrak{i}_j$ be an independent set of $\mathcal{T}_n$ contained in $\mathfrak{u}(M_j)$. Take now $X \in \mathfrak{u}(T_n) \setminus \mathfrak{f}$, and let us verify that $\mathfrak{s} := \{X\} \cup \mathfrak{i}_1 \cup \dots \cup \mathfrak{i}_m$ is also an independent set of $\mathcal{T}_n$. Suppose, by contradiction, that T is a lattice upward triangle that is strictly over-saturated by $\mathfrak{s}$. As $\mathfrak{s} \setminus \{X\} \subseteq \mathfrak{f}$, one has that T is over-saturated by $\mathfrak{f}$. Clearly, $X \in \mathfrak{u}(T)$. This implies that T is a lattice upward triangle over-saturated by $\mathfrak{f}$ not contained in any of the M_i's, which is a contradiction. Hence $\mathfrak{s}$ is an independent set of $\mathcal{T}_n$ and, as a result,

$$r(\mathfrak{f} \cup \{X\}) \geq r(\mathfrak{s}) = 1 + \sum_{j=1}^{m} r(\mathfrak{i}_j) = 1 + r(\mathfrak{f}) > r(\mathfrak{f}).$$

Since X was arbitrarily taken in $\mathfrak{u}(T_n) \setminus \mathfrak{f}$, it follows that $\mathfrak{f}$ is a flat of the tiling matroid $\mathcal{T}_n$, which concludes the proof. □

Corollary 4.6.3. *Each flat of $\mathcal{T}_n$ consists of all unit upward triangles in the union of a disjoint collection of lattice upward triangles.*

Example 4.6.4. Figure 4.14 shows a flat (on the left) and a non-flat $\mathfrak{s}$ (on the right). Even though $\mathfrak{s}$ is not a flat, note that it consists of all the unit upward triangles inside a disjoint union of lattice upward triangles (the ones completely over-saturated by $\mathfrak{s}$).

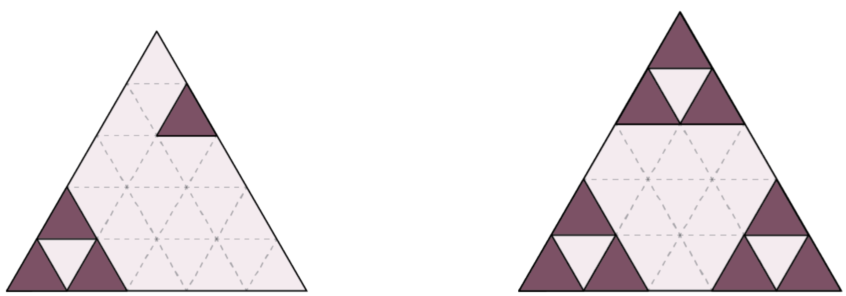

Figure 4.14: On the left, a flat of $\mathcal{T}_5$. On the right, a subset of $\mathfrak{u}(T_5)$ that is not a flat of $\mathcal{T}_5$.

Chapter 5

Connectedness of Matroids on the Lattice Points of a Regular Simplex

5.1 Connectedness of $\mathcal{T}_{n,3}$ Via Tilings

In order to illustrate how the tilings can be useful to prove matroidal properties of $\mathcal{T}_n$, we start tackling the 2-dimensional case, i.e., $d = 3$. Recall that a matroid $(E, \mathcal{B})$ is said to be *connected* if for all $x, y \in E$ there exist $B_1, B_2 \in \mathcal{B}$ such that $B_2 = (B_1 \setminus \{x\}) \cup \{y\}$.

We now introduce some terminology that will facilitate the proof of our next result. A *unit strip* of T_n is the lattice region occupied by all unit triangles of T_n that are at the same distance of a fixed border of T_n. A *unit row* of T_n is a horizontal unit strip. We number the unit rows of T_n by $1, \dots, n$ from bottom to top. For $X \in \mathfrak{u}(T_n)$, we define $h(X)$ to be the number of the unit row of T_n containing X. In addition, we let $\mathfrak{c}(X)$ denote the set of unit upward triangles inside the maximal lattice upward triangle of T_n having X as its top unit triangle.

Proposition 5.1.1. *For each $n \in \mathbb{N}$, the matroid $\mathcal{T}_n$ is connected.*

Proof. If $n = 1$, then $\mathcal{T}_n$ is trivially connected. So we assume that $n \geq 2$. Take $X, Y \in \mathfrak{u}(T_n)$ such that $X \neq Y$. We will find bases $\mathfrak{b}_1$ and $\mathfrak{b}_2$ of $\mathcal{T}_n$ satisfying that $\mathfrak{b}_2 = (\mathfrak{b}_1 \setminus \{X\}) \cup \{Y\}$. First, suppose that X and Y are in the same unit strip of T_n. After rotating T_n if necessary, we can assume that X and Y are both in the same unit row of T_n, say the k-th row. Now take $\mathfrak{s} \subset \mathfrak{u}(T_n)$ with $|\mathfrak{s}| = n - 1$ satisfying that there is exactly one member of $\mathfrak{s}$ in each row of T_n except in the k-th row. Clearly, $\mathfrak{s} \cap \{X, Y\}$ is empty. Set $\mathfrak{b}_1 = \mathfrak{s} \cup \{X\}$ and $\mathfrak{b}_2 = \mathfrak{s} \cup \{Y\}$. The set $\mathfrak{b}_1$ is a basis of $\mathcal{T}_n$ because each of the n rows of T_n contains exactly one member of $\mathfrak{b}_1$. Similarly, $\mathfrak{b}_2$ is a basis of $\mathcal{T}_n$. In addition, $\mathfrak{b}_2 = (\mathfrak{b}_1 \setminus \{X\}) \cup \{Y\}$, as desired.

Suppose now that X and Y are not in the same unit row of T_n. Then there is no loss in assuming that $h(X) > h(Y)$. It is not hard to see that in case of $Y \in \mathfrak{c}(X)$, we can perform a counterclockwise $\pi/3$-rotation to T_n and exchange the names of X and Y to get $h(X) > h(Y)$

and $Y \notin \mathfrak{c}(X)$. Hence we can assume, without loss of generality, that $h(X) > h(Y)$ and $Y \notin \mathfrak{c}(X)$.

Clearly, $\{X, Y\}$ is an independent set of $\mathcal{T}_n$. Let $\mathfrak{b}$ be a basis of $\mathcal{T}_n$ containing X and Y and having exactly one unit upward triangle in each unit row of T_n. By Theorem 4.2.6, there exists a lozenge tiling $\mathfrak{t}$ of the lattice region $T_n \setminus A(\mathfrak{b})$. As each unit downward triangle is part of a unit rhombus of $\mathfrak{t}$, each horizontal rhombus of $\mathfrak{t}$ is either in the bottom row of T_n or is adjacent from above to another unit horizontal rhombus. Therefore there exist unit horizontal rhombi $R_1, \dots, R_d$, where $d = h(X) - h(Y)$, such that X is supported on R_1 and R_i is supported on R_{i+1} for each $i \in [d-1]$ (see the left picture of Figure 5.1). For

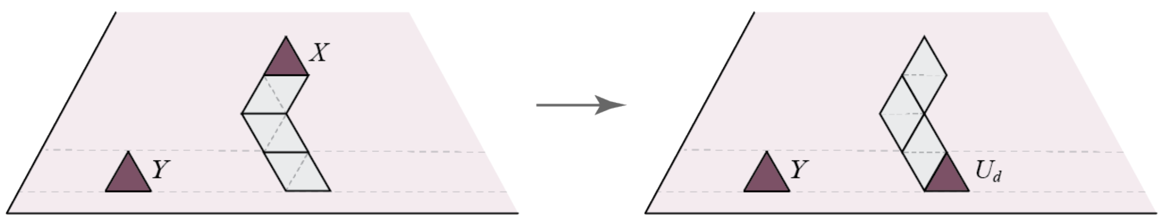

Figure 5.1: Illustration of how to re-tile a strip consisting of adjacent unit horizontal rhombi and a unit upward triangle adjacent on the top.

$i \in [d]$, let $R_i = U_i \cup D_i$, where U_i and D_i denote respectively the unit upward triangle and the unit downward triangle whose union is R_i. Define

$$\mathfrak{b}_1 = (\mathfrak{b} \setminus \{Y\}) \cup \{U_d\} \quad \text{and} \quad \mathfrak{b}_2 = (\mathfrak{b} \setminus \{X\}) \cup \{U_d\}.$$

As Y and U_d are in the same row of T_n, the fact that $\mathfrak{b}$ has exactly one unit upward triangle in each row of T_n ensures that $\mathfrak{b}_1$ is a basis of $\mathcal{T}_n$. To verify that $\mathfrak{b}_2$ is also a basis, let us find a lozenge tiling of $T_n \setminus A(\mathfrak{b}_2)$. To do so we keep the configuration of $\mathfrak{t}$ in the complement of the lattice region $R := X \cup R_1 \cup \dots \cup R_d$ and re-tile R by merging X with D_1, and U_i with D_{i+1} for each $i \in [d-1]$. It follows immediately that this produces a lozenge tiling of $T_n \setminus A(\mathfrak{b}_2)$. Theorem 4.2.6 now guarantees that $\mathfrak{b}_2$ is a basis of $\mathcal{T}_n$. The fact that $\mathfrak{b}_2 = (\mathfrak{b}_1 \setminus \{X\}) \cup \{Y\}$ completes the proof. ☐

5.2 Connectedness of $\mathcal{T}_{n,d}$

We conclude this chapter proving that $\mathcal{T}_{n,d}$ is connected when $d \geq 3$. Note that the matroid $\mathcal{T}_{n,2}$ contains only one basis and, therefore, it is connected if and only if $n = 1$.

Theorem 5.2.1. *For each $n, d \in \mathbb{N}$ such that $d \geq 3$, the matroid $\mathcal{T}_{n,d}$ is connected.*

Proof. Let $y = (y_1, \dots, y_d)$ and $y' = (y'_1, \dots, y'_d)$ be two elements in the ground set $T_{n,d}$ of $\mathcal{T}_{n,d}$, and suppose first that $y_i = y'_i = h - 1$ for some $i \in [d]$. Now take a subset S of $T_{n,d}$

such that $|S| = n-1$ and S intersects the plane determined by the equation $x_i = j-1$ exactly at one lattice point for each $j \in [n] \setminus \{h\}$ (clearly, such a set S exists). Following an idea similar to that one used in the proof of Proposition 5.1.1, the reader is welcome to verify that $B_y = S \cup \{y\}$ and $B_{y'} = S \cup \{y'\}$ are two bases of $\mathcal{T}_{n,d}$ satisfying that $B_{y'} = (B_y \setminus \{y\}) \cup \{y'\}$.

Now suppose that $y_i \neq y'_i$ for every $i \in [d]$. As $d \geq 3$, there exist $j, k \in [d]$ with $j \neq k$ such that $y_j < y'_j$ and $y_k < y'_k$. Take $B \subset T_{n,d}$ satisfying that $\{y, y'\} \subset B$ and intersecting the plane determined by the equation $x_k = i - 1$ exactly at one lattice point for each $i \in [n]$ (as $y_k \neq y'_k$, such set B exists). In addition, it is clear that B is a basis of $\mathcal{T}_{n,d}$. Now take $z = (z_1, \dots, z_d)$ with $z_k = y_k$, $z_i = y'_i$ for each $i \notin \{j, k\}$, and $z_1 + \cdots + z_d = n - 1$. Since

$$z_j = (n-1) - \sum_{i \in [d] \setminus \{j\}} z_i = (n-1) - y_k - \sum_{i \in [d] \setminus \{j,k\}} y'_i > (n-1) - \sum_{i \in [d] \setminus \{j\}} y'_i \geq 0,$$

it follows that z is a lattice point in the ground set of $\mathcal{T}_{n,d}$ (i.e., $z \in T_{n,d}$). Figure 5.2 shows the relative positions of y, y', and z.

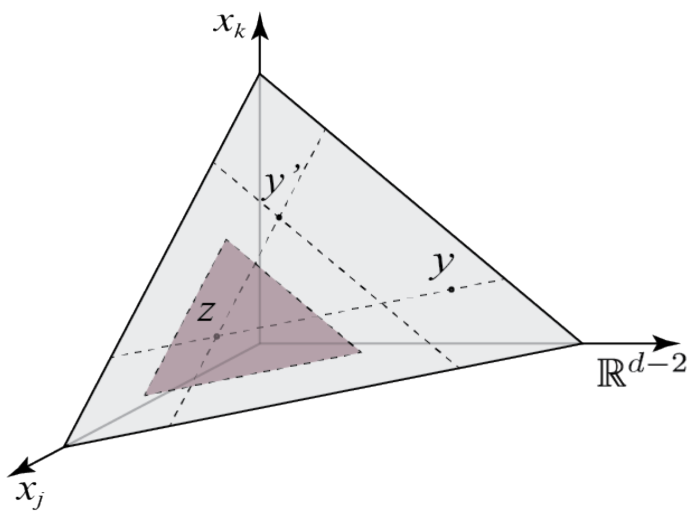

Figure 5.2: Relative positions of the lattice points y, y', and z.

Lastly, take

$$B_1 = (B \setminus \{y\}) \cup \{z\} \quad \text{and} \quad B_2 = (B \setminus \{y'\}) \cup \{z\}.$$

Since $z_k = y_k$, it follows that B_1 is a basis of $\mathcal{T}_{n,d}$. Suppose, by way of contradiction, that B_2 is not a basis of $\mathcal{T}_{n,d}$. Then for some $\ell \in [n]$ there exists a parallel translate T of the lattice simplex $T_{\ell,d}$ contained in $T_{n,d}$ and satisfying that $z \in T$ and $|T \cap B_2| > \ell$, which means that T is strictly over-saturated by B_2. Take $w = (w_1, \dots, w_d) \in \mathbb{Z}^d_{\geq 0}$ such that $T = w + T_{\ell,d}$. The fact that $z \in T$ implies that $z_i \geq w_i$ for every $i \in [d]$. As B is a basis of $\mathcal{T}_{n,d}$, it follows that $y' \notin T$. Thus, at least one coordinate of the vector $y' - w$ must be strictly negative.

As $y'_i - w_i = z_i - w_i \geq 0$ for each $i \in [d] \setminus \{j, k\}$, either $y'_j < w_j$ or $y'_k < w_k$. Now, because $y_j < y'_j$ and $y_k < y'_k$, the vector $y - w$ also has a strictly negative coordinate. Therefore $y \notin T$. This in turns implies that for every $i \in [n]$ the set $T \cap B_2$ intersects each plane determined by the equation $x_k = i - 1$ at most at one lattice point. This contradicts that T is strictly over-saturated by B_2. Hence B_2 is also a basis of $\mathcal{T}_{n,d}$. On the other hand, it is clear that $B_2 = (B_1 \setminus \{y'\}) \cup \{y\}$, which completes the proof. □

.

Part III

On the Geometry of Finite-Rank Submonoids of a Free Commutative Monoid and Their Monoid Algebras

Chapter 6

Geometry of Submonoids of a Finite-Rank Free Commutative Monoid

6.1 Introduction

Let $\mathcal{C}$ denote the class containing, up to isomorphism, all monoids that can be embedded into finite-rank free commutative monoids. If $\mathbb{F}$ is one of the fields $\mathbb{Q}$ or $\mathbb{R}$ and M is a monoid in $\mathcal{C}$, then the chain of natural inclusions

$$M \hookrightarrow \text{gp}(M) \hookrightarrow \mathbb{F} \otimes_{\mathbb{Z}} \text{gp}(M)$$

yields an embedding of M into the finite-dimensional vector space $\mathbb{F} \otimes_{\mathbb{Z}} \text{gp}(M)$, where $\text{gp}(M)$ is the Grothendieck group of M. Here we provide a systematic study on the connection between atomic and factorization aspects of monoids M in $\mathcal{C}$ and both the geometry of the conic hull $\mathsf{cone}(M)$ and the combinatorics of the face lattice of $\mathsf{cone}(M)$.

A commutative cancellative monoid is called atomic if any non-invertible element can be expressed as a product of irreducible elements. All monoids in $\mathcal{C}$ are atomic. After settling down the necessary terminology and recalling a few standard concepts in factorization theory and convex geometry, we begin the main core of this chapter giving some characterizations of monoids in $\mathcal{C}$. Right after this, we will exhibit some motivating examples of monoids in $\mathcal{C}$, and then we show that the geometric and combinatorial aspects of the conic hulls of monoids in $\mathcal{C}$ do not depend on the vector space such monoids are embedded into.

As for integral domains, an atomic monoid is called a UFM (or a unique factorization monoid) if every element has an essentially unique factorization into irreducibles. UFMs are the simplest monoids in the realm of factorization theory, as the main goal of this field is to study the deviation of an atomic monoid (resp., integral domain) from being a UFM (resp., a UFD). A huge variety of atomic conditions between being an atomic monoid (resp., domain) and being a unique factorization monoid (resp., domain) have been considered in the literature during the last four decades, including half-factoriality, other-half-factoriality, being finitary, and being strongly primary. We study here some of these intermediate atomic conditions for monoids in $\mathcal{C}$ and their monoid algebras.

An atomic monoid M is called half-factorial (or an HFM) provided that for all $x \in M^\bullet$, any two factorizations of x have the same number of irreducibles (counting repetitions). In addition, an integral domain is called half-factorial (or an HFD) if its multiplicative monoid is an HFM. The concept of half-factoriality was first investigated by Carlitz in the context of algebraic number fields, who proved that an algebraic number field is an HFD if and only if its class group has size at most two [39]. However, the term "half-factorial domain" is due to Zaks [148]; he also studied Krull domains that are HFDs in terms of their divisor class groups [149]. Parallel to this, Skula [135] and Śliwa [136], motivated by some questions of Narkiewicz on algebraic number theory [122, Chapter 9], also carried out systematic studies of HFDs. Since then HFMs and HFDs have been actively studied (see [41] and reference therein).

Other-half-factoriality, on the other hand, is a dual version of half-factoriality, and it was introduced by Coykendall and Smith in [58]. An atomic monoid M is called other-half-factorial (or an OHFM) provided that for all $x \in M^\bullet$ and $z, z' \in \mathsf{Z}(x)$ with $|z| = |z'|$, we have that $z = z'$. Although an integral domain is a UFD if and only if its multiplicative monoid is an OHFM [58, Corollary 2.11], OHFMs are not always factorial or half-factorial, even in the class $\mathcal{C}$. In the second part of this chapter, we offer geometric and combinatorial characterizations for the HFMs in $\mathcal{C}$ and for the OHFMs in $\mathcal{C}$.

The study of primary monoids was initiated by T. Tamura [140] and M. Petrich [126] in the 1970's and has received a great deal of attention since then [109, 110, 74]. Primary monoids naturally appear in commutative algebra: an integral domain is 1-dimensional and local if and only if its multiplicative monoid is primary. One of the most useful subclasses of primary monoids in factorization theory is that one consisting of finitely primary monoids. The initial interest on this subclass also comes from commutative algebra: the multiplicative monoid of a 1-dimensional local Mori domain with nonempty conductor is finitely primary [79, Proposition 2.10.7.6]. Finitely primary monoids were introduced in [74]. Definitions of primary and finitely primary monoids will be given in Section 6.6.

Motivated by the non-unique factorization phenomenon of certain noetherian domains, Geroldinger et al. introduced in [84] the class of finitary monoids. More precisely, the multiplicative monoid of a noetherian domain R is finitary if and only if R is 1-dimensional and semilocal [84, Proposition 4.14]. In addition, finitary monoids conveniently capture certain aspects of the arithmetic and factorization structure of more sophisticated monoids, including v-noetherian G-monoids [76] and congruence monoids [77]. Strongly primary monoids are those that are simultaneously primary and finitary. The class of strongly primary monoids plays an important role in factorization theory and commutative algebra, as it comprises various classes of well-studied monoids and integral domains. For instance, numerical monoids and v-noetherian primary monoids are strongly primary. On the other hand, the multiplicative monoid of a 1-dimensional local Mori domain is strongly primary. In the last section of this chapter, we study the conic hull $\mathsf{cone}(M)$ of monoids M in $\mathcal{C}$ that are either primary or finitary. We conclude this chapter with a few words about strongly primary monoids in $\mathcal{C}$.

6.2 Atomic Monoids and Convex Cones

In this section we introduce most of the relevant concepts on commutative monoids, factorization theory, and convex geometry required to follow the results presented in this chapter. General references for any undefined terminology or notation can be found in [105] for commutative monoids, in [79] for atomic monoids and factorization theory, and in [128] for convex geometry.

General Notation

Recall that $\mathbb{N} := \{0, 1, 2, \dots\}$. If $a, b \in \mathbb{Z}$ and $a \le b$, then we let $[[a, b]]$ denote the interval of integers between a and b, i.e.,

$$[[a, b]] := \{z \in \mathbb{Z} \mid a \le z \le b\}.$$

In addition, for $X \subseteq \mathbb{R}$ and $r \in \mathbb{R}$, we set

$$X_{\ge r} := \{x \in X \mid x \ge r\}$$

and we use the notation $X_{>r}$ in a similar way. Lastly, if $Y \subseteq \mathbb{R}^d$ for some $d \in \mathbb{N} \setminus \{0\}$, then we set $Y^\bullet := Y \setminus \{0\}$.

Atomic Monoids

A *monoid* is commonly defined in the literature as a semigroup along with an identity element. However, in what follows all monoids are also assumed to be commutative and cancellative, and we omit these two attributes accordingly. As we only consider commutative monoids, unless otherwise specified we will use additive notation. In particular, the identity element of a monoid M will be denoted by 0, and we let $M^\bullet$ denote the set $M \setminus \{0\}$. A monoid is called *reduced* if its only invertible element is the identity element. Unless we specify otherwise, monoids in this chapter are assumed to be reduced. For $x, y \in M$, we say that y *divides* x *in* M and write $y \mid_M x$ provided that $x = y + z$ for some $z \in M$. A submonoid N of M is called *divisor-closed* if for all $y \in N$ and $x \in M$ the condition $x \mid_M y$ implies that $x \in N$.

We write $M = \langle S \rangle$ when M is generated by a set S. If M can be generated by a finite set, we say that M is *finitely generated*. An element $a \in M^\bullet$ is called an *atom* provided that for each pair of elements $y, z \in M$ such that $a = y + z$ either $y = 0$ or $z = 0$. The set consisting of all atoms of M is denoted by $\mathcal{A}(M)$, that is,

$$\mathcal{A}(M) := M^\bullet \setminus \big(M^\bullet + M^\bullet\big).$$

Since M is reduced, it follows that $\mathcal{A}(M)$ will be contained in each generating set of M. If $\mathcal{A}(M)$ generates M, then M is said to be *atomic*. All monoids addressed in this paper are atomic. We say that $p \in M^\bullet$ is prime if whenever $p \mid_M x + y$ for some $x, y \in M$ either $p \mid_M x$ or $p \mid_M y$. The monoid M is called a *UFM* (or a *unique factorization monoid*) if every nonzero element can be written as a sum of primes (up to permutation). Clearly, every prime element of M is an atom. Thus, if M is a UFM, then it is, in particular, an atomic monoid.

A subset I of M is an *ideal* of M if $I + M \subseteq I$. An ideal I is *principal* if $I = x + M$ for some $x \in M$. Furthermore, M satisfies the *ascending chain condition on principal ideals* (or the *ACCP*) provided that every increasing sequence of principal ideals of M eventually stabilizes. It is well known that every monoid satisfying the ACCP is atomic [79, Proposition 1.1.4]. The Gram's monoid, exhibited in Section 8.3, is an atomic monoid that does not satisfy the ACCP.

For any monoid M there exist an abelian group $\text{gp}(M)$ and a monoid homomorphism $\iota \colon M \hookrightarrow \text{gp}(M)$ such that any monoid homomorphism $\phi \colon M \to G$ (where G is a group) uniquely factors through ι. The group $\text{gp}(M)$, which is unique up to isomorphism, is called the *difference group* (or *Grothendieck group*) of M. If M is a monoid in $\mathcal{C}$, then the *rank* of M, denoted by $\text{rank}(M)$, is the rank of the abelian group $\text{gp}(M)$, that is, the dimension of the $\mathbb{Q}$-space $\mathbb{Q} \otimes_{\mathbb{Z}} \text{gp}(M)$. The monoid M is *torsion-free* if $nx = ny$ for some $n \in \mathbb{N}$ and $x, y \in M$ implies that $x = y$. A monoid is torsion-free if and only if its difference group is torsion-free (see [37, Section 2.A]).

A multiplicative commutative monoid F is *free on* a subset A of F if every element $x \in F$ can be written uniquely in the form

$$x = \prod_{a \in A} a^{\mathsf{v}_a(x)},$$

where $\mathsf{v}_a(x) \in \mathbb{N}$ and $\mathsf{v}_a(x) > 0$ only for finitely many $a \in A$. It is well known that for each set A, there exists a unique (up to isomorphism) monoid F such that F is a free commutative monoid on A. For a monoid M, the free commutative monoid on $\mathcal{A}(M)$, denoted by $\mathsf{Z}(M)$, is called the *factorization monoid* of M, and the elements of $\mathsf{Z}(M)$ are called *factorizations*. If $z = a_1 \dots a_n$ is a factorization in $\mathsf{Z}(M)$ for some $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathcal{A}(M)$, then n is called the *length* of z and is denoted by $|z|$. In addition, the unique monoid homomorphism $\phi \colon \mathsf{Z}(M) \to M$ satisfying $\phi(a) = a$ for all $a \in \mathcal{A}(M)$ is called the *factorization homomorphism* of M. For each $x \in M$ the set

$$\mathsf{Z}(x) := \mathsf{Z}_M(x) := \phi^{-1}(x) \subseteq \mathsf{Z}(M)$$

is called the *set of factorizations* of x. In addition, for $k \in \mathbb{N}$ we set

$$\mathsf{Z}_k(x) := \{z \in \mathsf{Z}(x) : |z| = k\} \subseteq \mathsf{Z}(M).$$

Observe that the monoid M is atomic if and only if $\mathsf{Z}(x)$ is nonempty for all $x \in M$ (notice that $\mathsf{Z}(0) = \{\emptyset\}$). The monoid M is called an *FFM* (or *finite factorization monoid* provided that $|\mathsf{Z}(x)| < \infty$ for all $x \in M$. For each $x \in M$, the *set of lengths* of x is defined by

$$\mathsf{L}(x) := \mathsf{L}_M(x) := \{|z| : z \in \mathsf{Z}(x)\}.$$

If $|\mathsf{L}(x)| < \infty$ for all $x \in M$, then M is called a *BFM* (or a *bounded factorization monoid*). Clearly, every FFM is a BFM.

A very special family of atomic monoids is that of all *numerical monoids*, i.e., cofinite additive submonoids of $\mathbb{N}$. Each numerical monoid M has a unique minimal set of generators, which is finite; such a unique minimal generating set is precisely $\mathcal{A}(M)$. As a result, every numerical monoid is atomic and contains only finitely many atoms. A friendly introduction to numerical monoids can be found in [72]. The class of finitely generated submonoids of $(\mathbb{N}^d, +)$ naturally generalizes that one of numerical monoids. Although members of the former class are finitely generated and, therefore, finitary, numerical monoids are the only primary monoids in this class (Proposition 6.6.1(2)). However, we shall see later that there are many non-finitely generated submonoids of $(\mathbb{N}^d, +)$ that are primary. In addition, we will provide necessary conditions and sufficient conditions for a submonoid of $(\mathbb{N}^d, +)$ to be finitary.

Convex Cones

We let $e_1, \dots, e_d$ denote the canonical basic vectors of $\mathbb{R}^d$. In addition, we denote the standard inner product of $\mathbb{R}^d$ by $\langle\, , \rangle$, that is, $\langle x, y\rangle = \sum_{i=1}^d x_i y_i$ for all $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ in $\mathbb{R}^d$. As usual, for $x \in \mathbb{R}^d$ we let $||x||$ denote the Euclidean norm of x. We always consider the space $\mathbb{R}^d$ endowed with the topology induced by the Euclidean norm. Finally, we let the $\mathbb{Q}$-space $\mathbb{Q}^d$ inherit the inner product and the topology of $\mathbb{R}^d$. For a subset S of $\mathbb{R}^d$, we let $\mathsf{int}\, S$, $\bar{S}$, and $\mathsf{bd}\, S$ denote the interior, closure, and boundary of S, respectively.

Let V be a vector space over an ordered field. A nonempty convex subset C of V is called a *cone* provided that C is closed under linear combinations with nonnegative coefficients. A cone C is called *pointed* if $C \cap -C = \{0\}$. Unless otherwise stated, we assume that the cones we consider here are pointed. If X is a nonempty subset of V, the *conic hull* of X, denoted by $\mathsf{cone}(X)$, is defined as

$$\mathsf{cone}(X) := \{c_1 x_1 + \dots + c_n x_n \mid x_i \in X \;\text{ and }\; c_i \ge 0 \text{ for all } i \in [[1, n]]\},$$

i.e., $\mathsf{cone}(X)$ is the smallest cone in V containing X. A cone in V is called *simplicial*, if it is the conic hull of a linearly independent set of vectors. In addition, a cone in $\mathbb{R}^d$ is called *rational* if it is the conic hull of vectors with integer coordinates.

A *face* of C is a cone F contained in C satisfying the following condition: for all $x, y \in C$ the fact that the open line segment $\{tx + (1-t)y \mid 0 < t < 1\}$ intersects F implies that both x and y belong to F. If F is a face of C and F' is a face of F, then it is clear that F' must be a face of C. Now suppose that $\mathbb{F}$ is either $\mathbb{Q}$ or $\mathbb{R}$. For a nonzero vector $u \in \mathbb{F}^d$, consider the hyperplane $H := \{x \in \mathbb{F}^d \mid \langle x, u\rangle = 0\}$, and denote the closed half-spaces

$\{x \in \mathbb{F}^d \mid \langle x, u \rangle \le 0\}$ and $\{x \in \mathbb{F}^d \mid \langle x, u \rangle \ge 0\}$ by H^- and H^+, respectively. If a cone C satisfies that $C \subseteq H^-$ (resp., $C \subseteq H^+$), then H is called a *supporting hyperplane* of C and H^- (resp., H^+) is called a *supporting half-space* of C. A face F of C is called *exposed* if there exists a supporting hyperplane H of C such that $F = C \cap H$. The cone C is called *polyhedral* provided that it can be expressed as the intersection of finitely many half-spaces. The Farkas-Minkowski-Weyl Theorem states that a convex cone is polyhedral if and only if it is the conic hull of a finite set. On the other hand, Gordan's Lemma states that if C is a rational polyhedral cone in $\mathbb{R}^d$ and G is an additive subgroup of $\mathbb{Q}^d$, then $C \cap G$ is finitely generated (see [37, Lemma 2.9]).

A subset S of $\mathbb{R}^n$ is called an *affine set* (or an *affine subspace*) provided that for all $x, y \in S$ with $x \neq y$, the line determined by x and y is contained in S. Affine sets are translations of subspaces, and an $(n-1)$-dimensional affine set is called an *affine hyperplane*. The *affine hull* of S, denoted by $\mathsf{aff}(S)$, is the smallest affine set containing S. The relative interior of S, denoted by $\mathsf{relin}(S)$, is the Euclidean interior of S when considered as a subset of $\mathsf{aff}(S)$. If C is a cone, then C is the disjoint union of all the relative interiors of its nonempty faces [128, Theorem 18.2].

6.3 Monoids in $\mathcal{C}$

The class $\mathcal{C}$

In this section we introduce the class of monoids we shall be concerned with for the rest of this thesis. We also introduce the cones associated to such monoids.

Theorem 6.3.1. *For a monoid M, the following conditions are equivalent.*

(1) M can be embedded into a finite-rank free commutative monoid.

(2) M has finite rank and can be embedded into a free commutative monoid.

(3) There exists $d \in \mathbb{N}$ such that M can be embedded in $(\mathbb{N}^d, +)$ as a maximal-rank submonoid.

Proof. Let us verify first that (1) implies (2). Suppose that F is a finite-rank commutative monoid containing M. Assuming that $\text{gp}(M) \subset \text{gp}(F)$, one can consider $\text{gp}(M)$ as a $\mathbb{Z}$-submodule of $\text{gp}(F)$. Since $\text{gp}(F)$ is a finite-rank $\mathbb{Z}$-module, so is $\text{gp}(M)$. Hence M has finite rank, which yields (2).

Now we argue that (2) implies (1), consider a set X such that M is embedded into the free commutative monoid $\oplus_{x \in X} \mathbb{N}x$. After identifying M with its image, we can assume that $M \subseteq \oplus_{x \in X} \mathbb{N}x$ and also that $\text{gp}(M)$ is a subgroup of $\oplus_{x \in X} \mathbb{Z}x$. Since M has finite rank, the dimension of the subspace W of $V := \oplus_{x \in X} \mathbb{Q}x$ generated by $\text{gp}(M)$ is finite. Let

$\{b_1, \dots, b_k\}$ be a basis for W. For each $i \in [[1, k]]$ there exists a finite subset Y_i of X such that $b_i \in \oplus_{x \in Y_i} \mathbb{Q}x$. As a result, $W \subseteq \oplus_{y \in Y} \mathbb{Q}y$, where $Y = \cup_{i=1}^k Y_i$. Then

$$M \subseteq \Big(\bigoplus_{y \in Y} \mathbb{Q}y\Big) \bigcap \Big(\bigoplus_{x \in X} \mathbb{N}x\Big) = \bigoplus_{y \in Y} \mathbb{N}y.$$

Since Y is a finite set, $\oplus_{y \in Y} \mathbb{N}y$ is a finite-rank free commutative monoid, and so (1) holds.

Clearly, (3) implies (1). So it suffices to prove that (1) implies (3). To do this, let M be a monoid of rank d, and suppose that M is a submonoid of a free commutative monoid of rank r for some $r \in \mathbb{N}^\bullet$ with $r \geq d$. There is no loss of generality in assuming that M is a submonoid of $(\mathbb{N}^r, +)$. Let V be the subspace of the $\mathbb{Q}$-space $\mathbb{Q}^r$ generated by M. Since M has rank d, the subspace V has dimension d. Now consider the submonoid $M' := \mathbb{N}^r \cap V$ of $(\mathbb{N}^r, +)$. As M' is the intersection of the rational cone $\mathsf{cone}(\mathbb{N}^r \cap V)$ and the lattice $\mathbb{Z}^r \cap V \cong \mathbb{Z}^d$, it follows by Gordan's Lemma that M' is finitely generated. On the other hand, $M \subseteq M' \subseteq V$ guarantees that $\text{rank}(M') = d$. Since M' is a finitely generated additive submonoid of $\mathbb{N}^r$ of rank d, it follows by [37, Proposition 2.17] that M' is isomorphic to a submonoid of $(\mathbb{N}^d, +)$. This, in turn, implies that M is isomorphic to a submonoid of $(\mathbb{N}^d, +)$, which concludes our argument. □

As we are interested in studying monoids satisfying the equivalent conditions of Theorem 6.3.1, we introduce the following notation.

Notation: Let $\mathcal{C}$ denote the class consisting of all monoids (up to isomorphism) satisfying the conditions in Theorem 6.3.1. In addition, for every $d \in \mathbb{N}^\bullet$, we set

$$\mathcal{C}_d := \{M \in \mathcal{C} \mid \text{rank}(M) = d\}.$$

A monoid is *affine* if it is isomorphic to a finitely generated submonoid of the free abelian group $\mathbb{Z}^d$ for some $d \in \mathbb{N}$. The interested reader may find a self-contained treatment of affine monoids in [37]. Clearly, the class $\mathcal{C}$ contains all affine monoids. Computational aspects of affine monoids and factorization invariants of half-factorial affine monoids have been studied in [71] and [70], respectively. Diophantine monoids form a special subclass of that one consisting of affine monoids and has been studied in [46]. Monoids in $\mathcal{C}$ of small rank have been recently studied in [57]. Some other special subclasses of $\mathcal{C}$ have been previously consider in the literature as they naturally arise in the study of algebraic curves, toric geometry, and homological algebra. Here we offer a few examples.

Example 6.3.2. If M is finitely primary, then M is primary and satisfies that $\widehat{M} \cong \mathbb{N}^d$ [79, Theorem 2.9.2]. Hence $\mathcal{C}$ contains all finitary primary monoids.

Example 6.3.3. Good semigroups were introduced in [29] in the context of algebraic curves. Good semigroups are submonoids of $(\mathbb{N}^d, +)$ that naturally generalize value semigroups of an algebraic curve in the sense that monoids on both classes satisfy certain common "good" properties. For instance, the value semigroup S of the ring

$$R := \mathbb{C}[[x,y]]/\big((x^7 - x^6 + 4x^5y + 2x^3y^2 - y^4)(x^3 - y^2)\big)$$

is represented in Figure 6.1. As $\{(x,y) \in S \mid x < 13\}$ is finite, the affine line $x = 13$ of $\mathbb{R}^2$

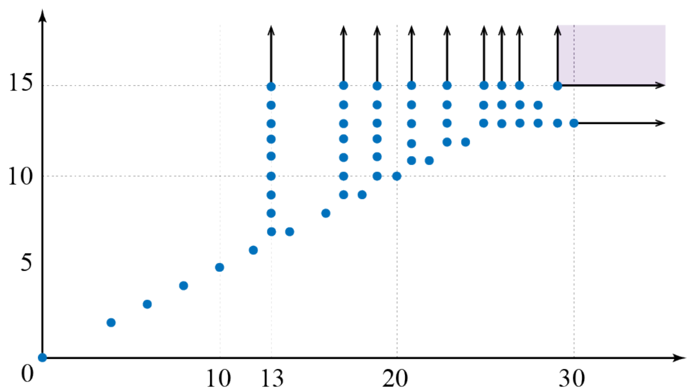

Figure 6.1: A non-finitely generated good semigroup.

contains infinitely many atoms of S. Hence the good semigroup S is not finitely generated (for more details on this example, see [30, page 8]). In addition, it has been verified in [29, Example 2.16] that the good semigroup

$$\{(x,y) \in \mathbb{N}^2 \mid x \geq 25 \text{ and } y \geq 27\}$$

is not the semigroup value of any algebraic curve. Clearly, the class $\mathcal{C}$ contains all good semigroups. Good semigroups have received substantial attention since they were introduced; see for example [29, 30, 59] and see [60, 120] for more recent studies.

Example 6.3.4. From the structure theorem for modules over a PID, we have that if R is a 1-dimensional integrally-closed local domain and M is a finitely generated torsion-free R-module, then M is free if and only if $M \otimes_R \text{Hom}(M,R)$ is torsion-free. It has been conjectured by C. Huneke and R. Wiegand that this property also holds for any 1-dimensional Gorenstein domain. Given a numerical monoid Γ and $s \in \mathbb{N} \setminus \Gamma$, consider the collection

$$M_\Gamma^s := \{(0,0)\} \cup \{(x,n) \mid \{x, x+s, x+2s, \dots, x+ns\} \subseteq \Gamma\}$$

consisting of all arithmetic sequences of step size s contained in Γ. It is clear that M_Γ^s is a monoid; it is a called a *Leamer monoid*. The atomic structure of Leamer monoids is connected to the Huneke-Wiegand conjecture via [69, Corollary 7]. Notice that Leamer monoids are non-finitely generated rank 2 monoids contained in the class $\mathcal{C}$. Factorization properties of Leamer monoids have been considered in [108].

The following example has been kindly provided by Roger Wiegand, and will appear in [25].

Example 6.3.5. Let α and β be two positive irrational numbers such that $\alpha < \beta$, and consider the monoid $M_{\alpha,\beta}$ defined as follows:

$$M_{\alpha,\beta} := \{(0,0)\} \bigcup \left\{ (m,n) \in \mathbb{N}^2 \;\middle|\; \alpha < \frac{n}{m} < \beta \right\}.$$

It follows from Farkas-Minkowski-Weyl Theorem that $M_{\alpha,\beta}$ is not finitely generated and, therefore, $|\mathcal{A}(M_{\alpha,\beta})| = \infty$. In addition, $M_{\alpha,\beta}$ is a primary FFM (see Proposition 6.6.1 and Proposition 6.5.7). For every $n \geq 3$, the sequence of monoids $\{M_n\}$ obtained by setting

$$\alpha = \frac{2}{n + \sqrt{n^2 - 4}} \quad \text{and} \quad \beta = \frac{n + \sqrt{n^2 - 4}}{2}$$

shows up in the study of Betti tables of short Gorenstein algebras. In an ongoing project, Avramov, Gibbons, and Wiegand have proved that

$$\mathcal{A}(M_n) = \{\omega^{1-a}(1,b) \mid (a,b) \in \Gamma\},$$

where $\omega \colon (p,q) \mapsto (np - q, p)$ is an automorphism of M_n and $\Gamma := \mathbb{Z} \times [[1, n-2]]$. This suggests the following question.

Question 6.3.6. *For any irrational (or algebraic) numbers α and β with $\alpha < \beta$, can we generalize Example 6.3.5 to describe the set of atoms of $M_{\alpha,\beta}$?*

The Cones of Monoids in $\mathcal{C}$

A *lattice* is a partially ordered set L, in which every two elements have a unique *join* (i.e., least upper bound) and a unique *meet* (i.e., greatest lower bound). The lattice L is *complete* if each $S \subseteq L$ has both a join and a meet. Two complete lattices are isomorphic if there is a bijection between them that preserves joints and meets. For background information on (complete) lattices and lattice homomorphisms, see [61, Chapter 2]. For a cone C, the collection of all its faces, denoted by $\mathsf{F}(C)$, is a complete lattice (under inclusion) [128, page 164], where the meet is given by intersection and the join of a given set of faces is the smallest face in $\mathsf{F}(C)$ containing all the given faces. The lattice $\mathsf{F}(C)$ is called the *face lattice* of C. Two cones C and C' are *combinatorially equivalent* provided that their face lattices are isomorphic.

Let $\mathbb{F}$ denote either $\mathbb{Q}$ or $\mathbb{R}$. As mentioned in the introduction, a monoid M in $\mathcal{C}$ of rank d can be embedded in a d-dimensional vector space over $\mathbb{F}$ via

$$M \hookrightarrow \operatorname{gp}(M) \hookrightarrow \mathbb{F} \otimes_{\mathbb{Z}} \operatorname{gp}(M) =: V,$$

where the flatness of $\mathbb{F}$ as a $\mathbb{Z}$-module ensures the injectivity of the second map. Then we can consider the conic hull $\mathsf{cone}_V(M)$ of M in V. It turns out that the combinatorial and geometric structures of $\mathsf{cone}_V(M)$ do not depend on the proposed embedding $M \hookrightarrow V$, as we proceed to show.

Proposition 6.3.7. *Let $\mathbb{F} \in \{\mathbb{Q}, \mathbb{R}\}$. Let M and M' be two monoids in $\mathcal{C}$, and let V and V' be two finite-dimensional vector spaces over $\mathbb{F}$ containing M and M', respectively. If the monoids M and M' are isomorphic, then*

(1) $\mathsf{cone}_V(M)$ *is homeomorphic to* $\mathsf{cone}_{V'}(M')$*;*

(2) $\mathsf{cone}_V(M)$ *is combinatorially equivalent to* $\mathsf{cone}_{V'}(M')$*.*

Proof. Let d be the rank of M. By Theorem 6.3.1 the monoid M can be embedded in $(\mathbb{N}^d, +)$. After identifying M with its image, we can assume that $M \subseteq \mathbb{N}^d$ and $\text{gp}(M)$ is a subgroup of $\mathbb{Z}^d$. Let $\varphi\colon M \to M'$ be a monoid isomorphism. Then φ extends to an injective group homomorphism $\text{gp}(M) \to V'$ with image $\text{gp}(M')$. By tensoring $\text{gp}(M)$ and V' with the flat $\mathbb{Z}$-module $\mathbb{F}$, such a group homomorphism extends to a linear transformation

$$\bar{\varphi}\colon V := \mathbb{F} \otimes \text{gp}(M) \to \mathbb{F} \otimes_{\mathbb{Z}} V' = V'.$$

Since $\mathbb{F}$ is flat, $\ker \bar{\varphi}$ is trivial and, therefore, $\bar{\varphi}$ is a linear embedding. Hence $\bar{\varphi}$ is a homeomorphism onto its image. As

$$\bar{\varphi}(\mathsf{cone}_V(M)) = \mathsf{cone}_{\bar{\varphi}(V)}(M') = \mathsf{cone}_{V'}(M'),$$

the cones $\mathsf{cone}_V(M)$ and $\mathsf{cone}_{V'}(M')$ are homeomorphic. Notice that we have chosen the vector space V but not V'. This, along the fact that being homeomorphic is a transitive relation, yields (1).

To argue (2), it suffices to observe that the fact that $\bar{\varphi}$ is a linear bijection taking $\mathsf{cone}_{\mathbb{F}^d}(M)$ onto $\mathsf{cone}_{V'}(M')$ guarantees that the map given by the assignment $F \mapsto \bar{\varphi}(F)$ is an order-preserving bijection from $\mathsf{F}(\mathsf{cone}_{\mathbb{F}^d}(M))$ to $\mathsf{F}(\mathsf{cone}_{V'}(M'))$ and, therefore, a lattice isomorphism. □

From now on we shall tacitly assume Proposition 6.3.7 when referring to the cone of a monoid M in $\mathcal{C}$ over a field $\mathbb{F} \in \{\mathbb{Q}, \mathbb{R}\}$, and feel free to choose (or let unspecified) the finite-dimensional $\mathbb{F}$-vector space in which M is embedded into.

Corollary 6.3.8. *If M is a monoid in $\mathcal{C}$, then* $\dim \mathsf{cone}(M) = \text{rank}(M)$.

Proof. Set $d = \text{rank}(M)$. By Theorem 6.3.1, we can assume that $M \subseteq \mathbb{N}^d$. Then we have that $\mathsf{cone}(M) \subseteq \mathbb{R}^d$ and, therefore, $\dim \mathsf{cone}(M) \le d$. On the other hand, $\text{rank}\,\text{gp}(M) = d$, along with the fact that $\text{gp}(M)$ is contained in the subspace of $\mathbb{F}^d$ generated by M, implies that M contains d linearly independent vectors. Hence $\dim \mathsf{cone}_{\mathbb{F}^d}(M) \ge d$, which concludes the proof. □

Proposition 6.3.9. *Let M be a cone in $\mathcal{C}$. Then $\overline{\mathsf{cone}(M)}$ is a pointed cone.*

Proof. Set $k = \text{rank}(M)$, and for $\mathbb{F} \in \{\mathbb{Q}, \mathbb{R}\}$ set $V := \mathbb{R} \otimes_{\mathbb{Z}} \text{gp}(M)$. Suppose, by way of contradiction, that $\overline{\mathsf{cone}_V(M)}$ is not pointed. Using Theorem 6.3.1, one can assume that M can be embedded into $(\mathbb{N}^k, +)$. Let $\iota\colon M \to (\mathbb{N}^k, +)$ be an injective monoid homomorphism. After tensoring both $\text{gp}(M)$ and $\text{gp}(\mathbb{N}^k) = \mathbb{Z}^k$ with the flat $\mathbb{Z}$-module $\mathbb{F}$, the homomorphism ι extends to a linear transformation $\bar{\iota}\colon \mathbb{R} \otimes_{\mathbb{Z}} \text{gp}(M) \to \mathbb{F}^k$. Since $\overline{\mathsf{cone}_V(M)}$ is not pointed, it contains a 1-dimensional subspace L. As $\bar{\iota}$ is linear, it must be continuous and, therefore,

$$\bar{\iota}(L) \subseteq \bar{\iota}\big(\,\overline{\mathsf{cone}_V(M)}\,\big) \subseteq \overline{\bar{\iota}(\mathsf{cone}_V(M))} = \overline{\iota(\mathsf{cone}_V(M))}.$$

This, along with the fact that $\iota(\mathsf{cone}_V(M)) \subseteq \mathbb{R}^k_{\ge 0}$, implies that $\bar{\iota}(L) \subseteq \mathbb{F}^k_{\ge 0}$. Since $\bar{\iota}(L)$ is a subspace of $\mathbb{F}^k$, it must be trivial, which contradicts the injectivity of $\bar{\iota}$. Thus, $\overline{\mathsf{cone}_V(M)}$ must be pointed, which completes our argument. □

Members of $\mathcal{C}$ are finite-rank torsion-free monoids. However, not every finite-rank torsion-free monoid is in $\mathcal{C}$. The next two examples shed some light upon this observation.

Example 6.3.10. A nontrivial submonoid M of $(\mathbb{Q}_{\ge 0}, +)$ is obviously a rank 1 torsion-free monoid. It follows by Theorem 6.3.1 that M belongs to $\mathcal{C}$ if and only if M is isomorphic to a numerical monoid. Hence [94, Proposition 3.2] guarantees that M is in $\mathcal{C}$ if and only if M is finitely generated. As a result, non-finitely generated submonoids of $(\mathbb{Q}_{\ge 0}, +)$ such as $\langle 1/p \mid p \text{ is prime}\rangle$ are finite-rank torsion-free monoids that do not belong to the class $\mathcal{C}$. Clearly, the Grothendieck groups of such monoids cannot be free.

The following example, courtesy of Winfried Bruns, shows that a finite-rank torsion-free monoid might not belong to $\mathcal{C}$ even though its Grothendieck group is free.

Example 6.3.11. Consider the additive monoid

$$M := \{(0,0)\} \cup \{(m,n) \in \mathbb{Z}^2 \mid n > 0\} \subseteq \mathbb{Z}^2.$$

It is clear that M is an additive submonoid of $\mathbb{Z}^2$ and, therefore, it has finite rank. In addition, it is clear that M is torsion-free. On the other hand,

$$\overline{\mathsf{cone}_{\mathbb{R}^2}(M)} = \{(x,y) \in \mathbb{R}^2 \mid y \ge 0\},$$

which is not a pointed cone. As a consequence, it follows from Proposition 6.3.9 that M does not belong to the class $\mathcal{C}$.

Cones Realized by Monoids in $\mathcal{C}$

We conclude this section characterizing the positive cones that can be realized by the monoids in $\mathcal{C}$. First, let us argue the following lemma.

Lemma 6.3.12. *For $d \in \mathbb{N}$, let C be a d-dimension positive cone in $\mathbb{R}^d$ and let $x \in \operatorname{int} C$. Then there exists a d-dimensional rational simplicial cone C_p such that $\mathbb{R}_{>0}x \subset \operatorname{int} C_p$ and $C_p \subseteq \{0\} \cup \operatorname{int} C$.*

Proof. For $d = 1$, take $C_p = \mathbb{R}_{\geq 0}x$. Then suppose that $d \geq 2$ and write $x = (x_1, \dots, x_d)$. As C is positive and $x \in \operatorname{int} C$, we have that $x_i > 0$ for $i \in [[1, d]]$. Let ℓ be the distance from x to the complement of $\operatorname{int} C$. Since the complement of $\operatorname{int} C$ is closed and $\{x\}$ is compact, $\ell > 0$. Consider the d-dimensional regular simplex $\Delta_n := \mathsf{conv}(e_1, \dots, e_d)$, and choose $N \in \mathbb{N}$ large enough such that $\operatorname{diam}(\Delta_n/N) = \sqrt{2}/N < \ell$. In addition, take $q = (q_1, \dots, q_d) \in \mathbb{Q}^d_{>0}$ such that $q_i < x_i$ for $i \in [[1, d]]$ and $\sum_{i=1}^{d}(x_i - d_i) < 1/N$. Now set $\Delta := q + \Delta_n/N$. Clearly, $x - q$ is an interior point of Δ_n/N and, therefore, x is an interior point of Δ. This, along with the fact that $\operatorname{diam}(\Delta) = \operatorname{diam}(\Delta_n/N) < \ell$, ensures that $\Delta \subset \operatorname{int} C$. Lastly take $C_p := \mathsf{cone}(\Delta)$. It is clear that C_p is a closed cone contained in $\{0\} \cup \operatorname{int} C$. In addition, $x \in \operatorname{int} \Delta$ implies that $\mathbb{R}_{>0}x \subset \operatorname{int} C_p$. As $\dim \Delta = d$, we have that $\dim C_p = d$. Hence the set of 1-dimensional edges of the polyhedral C_p has size at least d. On the other hand, the 1-dimensional faces of C_p are determined by some of the vertices of Δ. As $\mathbb{R}_{>0}q \in \operatorname{int} C_p$, the 1-dimensional faces of C_p are precisely the d nonnegative rays containing the points $q + e_i/N$ for $i \in [[1, d]]$. Thus, C_p is a d-dimensional rational simplicial cone. $\square$

Notation: We call a 1-dimensional subspace of $\mathbb{R}^d$ (resp., an infinite ray) a *rational line* (resp., a *rational ray*) if it contains a nonzero point with rational components.

Theorem 6.3.13. *For $d \in \mathbb{N}$, let C be a positive cone in $\mathbb{R}^d$. Then C can be generated by a monoid in $\mathcal{C}$ if and only if each 1-dimensional face of C is a rational ray.*

Proof. For the direct implication, suppose that C is generated by a monoid in $\mathcal{C}$. Then one can assume that $C = \mathsf{cone}(M)$, where M is a rank d submonoid of $(\mathbb{N}^d, +)$. Let L be a 1-dimensional face of C, and let x be a nonzero point in L. Now take $c_1, \dots, c_k \in \mathbb{R}_{>0}$ and $x_1, \dots, x_k \in M^\bullet$ such that $x = c_1x_1 + \dots + c_kx_k$. If $k = 1$, then $x_1 = \frac{1}{c_1}x \in L$ and, therefore, L is a rational ray. If $k > 1$, then $x_1' := c_2x_2 + \dots + c_kx_k \in \mathsf{cone}(M)^\bullet$ and

$$\frac{c_1}{1+c_1}x_1 + \frac{1}{1+c_1}x_1' = \frac{1}{1+c_1}x \in L.$$

As L is a face of C and the segment line from x_1 to x_1' intersects L, it follows that the whole segment is contained in L. In particular, $x_1 \in L$. Hence L is a rational ray.

For the reverse implication, assume that all 1-dimensional faces of C are rational rays. Consider the set $M := C \cap \mathbb{N}^d$. Clearly, M is an additive submonoid of $\mathbb{N}^d$ and $\mathsf{cone}(M) \subseteq C$. Take $x \in C^\bullet$, and set $\ell := \mathbb{R}_{>0}x$. Since C is the disjoint union of all the relative interiors of its nonempty faces, there exists a face C' of C such that $x \in \mathsf{relin}\, C'$. Suppose that C' is d'-dimensional. Then by Lemma 6.3.12 there exists a rational cone $C_x \subseteq \mathsf{relin}\, C'$ with d' 1-dimensional faces such that $\ell \subset \mathsf{relin}\, C_x$. Now take $v_1, \dots, v_{d'} \in \mathbb{N}^d \setminus \{0\}$ such that $\mathbb{R}_{\ge 0} v_i$ for $i \in [[1, d']]$ are the 1-dimensional faces of C_x. As $x \in \mathsf{relin}\, C_x$ we can write $x = c_1 v_1 + \cdots + c_{d'} v_{d'}$ for some $c_1, \dots, c_{d'} \in \mathbb{R}_{\ge 0}$. Because $v_i \in C \cap \mathbb{N}^d$ for $i \in [[1, d']]$, it follows that $x \in \mathsf{cone}(M)$. □

Not every positive cone in $\mathbb{R}^d$ can be generated by a monoid in $\mathcal{C}$. The following example sheds some light upon this observation.

Example 6.3.14. Let C be the cone in $\mathbb{R}^d$ generated by the set of vectors $\{e_1, \dots, e_{d-1}, v_d\}$, where $v_d := \pi e_d + \sum_{i=1}^{d-1} e_i$. It is clear that C is a positive cone. Note, in addition, that $\mathbb{R}_{\ge 0} v_d$ is a 1-dimensional face of C. Finally, observe that $\mathbb{R}_{>0} v_d$ contains no point with rational components. Hence it follows by Theorem 6.3.13 that C cannot be generated by any monoid in $\mathcal{C}$.

6.4 Face Submonoids

Face Submonoids

For M in $\mathcal{C}$ we would like to understand the structure of the face lattice of $\mathsf{cone}(M)$ in connection with the divisibility aspects of M. In particular, the submonoids of M obtained by intersecting M with the faces of $\mathsf{cone}(M)$ are very special as they inherit many divisibility and atomic properties from M, as we shall see in the next three sections.

Definition 6.4.1. Let M be a nontrivial monoid in $\mathcal{C}$. A submonoid N of M is called a *face submonoid* of M provided that $N = M \cap F$ for some face F of $\mathsf{cone}(M) \subseteq \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{gp}(M)$.

It follows from Proposition 6.3.7 that the definition of a face submonoid only depends on M.

Proposition 6.4.2. *Let M be a monoid in $\mathcal{C}$, and let N be the face submonoid of M determined by the face F. Then the following conditions hold.*

(1) N is a monoid in $\mathcal{C}$ satisfying that $\mathcal{A}(N) = \mathcal{A}(M) \cap F$.

(2) $\mathsf{cone}(N) = F$.

Proof. Let us argue (1) first. The fact that N is a monoid in $\mathcal{C}$ is a direct implication of Theorem 6.3.1. Since $\mathcal{A}(M) \cap F \subseteq N$, one has that $\mathcal{A}(M) \cap F \subseteq \mathcal{A}(N)$. To verify the reverse inclusion, take $a' \in \mathcal{A}(N)$, and let $a \in \mathcal{A}(M)$ such that $a \mid_M a'$. Take $b \in M$ such that $a' = a+b$. Then $a'/2$ belongs to the intersection of F and $\mathsf{relin}\,\{ta+(1-t)b \mid 0 \le t \le 1\}$. This implies that both a and b belong to F. As a result,

$$a \in \mathcal{A}(M) \cap F \subseteq N \subseteq \mathcal{A}(N).$$

Then $a' = a \in \mathcal{A}(M) \cap F$, which yields the desired inclusion. Hence (1) holds.

To argue (2), it suffices to assume that M is a submonoid of $(\mathbb{N}^d, +)$ of rank d for some $d \in \mathbb{N}^\bullet$ and F is a face of $\mathsf{cone}_{\mathbb{Q}^d}(M)$. Since $N \subseteq F$ and F is a cone, $\mathsf{cone}_{\mathbb{Q}^d}(N) \subseteq F$. To show the reverse inclusion, take $x \in F^\bullet$. Then $x = \sum_{i=1}^k q_i a_i$ for $a_1, \dots, a_k \in \mathcal{A}(M)$ and $q_1, \dots, q_k \in \mathbb{Q}_{>0}$. Then $mx \in M \cap F = N$, where m is the least common multiple of the denominators of the $q_i's$. So $x \in \mathsf{cone}(N)$. As a result, $F \subseteq \mathsf{cone}(N)$, and then (2) follows. □

Remark 6.4.3. With notation as in Proposition 6.4.2, the condition that the submonoid N is a face submonoid of M is needed to guarantee that $\mathcal{A}(N) = \mathcal{A}(M) \cap F$. To see this consider, for instance, any submonoid N of M such that $N \cap \mathcal{A}(M)$ is an empty set. It is clear in this case that $\mathcal{A}(N) \neq \mathcal{A}(M) \cap F$.

For a monoid M in $\mathcal{C}$, there might be submonoids of M obtained by intersecting M with certain non-supporting hyperplanes whose sets of atoms can be obtained as in Proposition 6.4.2.

Example 6.4.4. Consider the submonoid $M = \langle 2e_1, 2e_2, e_1+e_2 \rangle$ of $(\mathbb{N}^2, +)$. It can be easily checked that $\mathcal{A}(M) = \{2e_1, 2e_2, e_1+e_2\}$. Now consider the hyperplane $H = \mathbb{R}(e_1+e_2)$ of $\mathbb{R}^2$ and set $N = M \cap H$. It is clear that N is a submonoid of M satisfying that

$$\mathcal{A}(N) = \{e_1 + e_2\} = \mathcal{A}(M) \cap H.$$

However, notice that N is not a face submonoid of M.

Recall that a submonoid N of a monoid M is said to be divisor-closed provided that for all $y \in N$ and $x \in M$ the condition $x \mid_M y$ implies that $x \in N$. For any monoid M in $\mathcal{C}$, the concepts of a face submonoid and a divisor-closed submonoid coincide.

Theorem 6.4.5. *Let M be a monoid in $\mathcal{C}$. Then a submonoid N of M is divisor-closed in M if and only if N is a face submonoid of M.*

Proof. Suppose that $M \in \mathcal{C}_k$, and assume that $M \subseteq \mathbb{N}^k \subset \mathbb{R}^k$. We verify first that face submonoids of M are divisor-closed. To do so, take a face F of $\mathsf{cone}(M)$ and set $N := M \cap F$. To argue that N is a divisor-closed submonoid of M, take $x \in N$ and $y \in M \setminus \{x\}$ such that $y \mid_M x$. Then $x = y + y'$ for some $y' \in M$, which implies that

$$x/2 \in F \cap \mathsf{relin}\,\{ty + (1-t)y' \mid 0 \le t \le 1\}.$$

As F is a face both y and y' belong to F, and so $y \in N$. Hence N is divisor-closed.

Let us argue the reverse implication by induction. Notice that when M has rank 1, it is isomorphic to a numerical monoid and the only submonoids of M that are divisor-closed are the trivial and M itself, which are the face submonoids of M corresponding to the origin and to $\mathsf{cone}(M)$, respectively. Fix now $k > 1$ and assume that the divisor-closed submonoids of any monoid in $\mathcal{C}$ with rank less than k are face submonoids. Let M be a maximal-rank submonoid of $(\mathbb{N}^k, +)$ and let N be a submonoid of M that is not a face submonoid.

CASE 1. $\text{rank}(N) = k$. Since N is not a face submonoid of M, it follows that $N \neq M$. Take $v \in M \setminus N$ and a basis $v_1, \dots, v_k \in N$ of $\mathbb{Q}^k$ such that $v = \sum_{i=1}^k q_i v_i$, where the rational coefficients satisfy that $q_1, \dots, q_j \le 0$ and $q_{j+1}, \dots, q_k > 0$ (not all zeros) for some index $j \in \mathbb{N}^\bullet$. Then

$$dv + \sum_{i=1}^{j} (dq_i) v_i = \sum_{i=j+1}^{k} (dq_i) v_i \in N,$$

where d is the least common multiple of the denominators of all the nonzero q_i's. Since $v \notin N$, the monoid N cannot be divisor-closed.

CASE 2. $\text{rank}(N) < k$. Take $u \in \mathbb{Q}^k$ such that the hyperplane

$$H := \{h \in \mathbb{R}^k \mid \langle h, u \rangle = 0\}$$

of $\mathbb{R}^k$ contains linearly independent vectors $v_1, \dots, v_{k-1} \in M$ such that $v_1, \dots, v_r \in N$, where $r = \text{rank}(N)$. Consider the following two subcases.

CASE 2.1. H is a supporting hyperplane of $\mathsf{cone}(M)$. Because N is not a face submonoid of M, the face $F := H \cap \mathsf{cone}(M)$ of $\mathsf{cone}(M)$ must contain an element of $M \setminus N$. Since $\mathsf{cone}(M \cap F) = \mathsf{cone}(M) \cap F = F$, we have that N is not a face submonoid of $M \cap F$. As $\text{rank}(M \cap F) < \text{rank}(M)$, it follows by induction that N is not a divisor-closed submonoid of $M \cap F$. Therefore N cannot be a divisor-closed submonoid of M.

CASE 2.2. H is not a supporting hyperplane of $\mathsf{cone}(M)$. So there exist $w_{r+1}, w'_{r+1} \in M$ such that $\langle w_{r+1}, u \rangle > 0$ and $\langle w'_{r+1}, u \rangle < 0$. As $\{v_1, \dots, v_{k-1}, w'_{r+1}\}$ is a basis for $\mathbb{R}^k$ there exists w_{r+2} in $M \cap \mathsf{int}\,\mathsf{cone}(v_1, \dots, v_{k-1}, w'_{r+1})$ such that $S := \{v_1, \dots, v_r\} \cup \{w_{r+1}, w_{r+2}\}$ is linearly dependent. Clearly, $\langle w_{r+2}, u \rangle < 0$. After relabeling the vectors $v_1, \dots, v_r$ (if necessary), we have that

$$\sum_{i=1}^{j} q_i v_i = \left(\sum_{i=j+1}^{r} q_i v_i \right) + q_{r+1} w_{r+1} + q_{r+2} w_{r+2} \tag{6.1}$$

for some $j \in [[1, r]]$, and coefficients $q_1, \dots, q_{r+1} \in \mathbb{Q}_{\ge 0}$, and $q_{r+2} \in \mathbb{Q}$ (not all zeros). Observe that both coefficients q_{r+1} and q_{r+2} are different from zero. After taking the scalar product with u in both sides of (6.1), one obtains that

$$q_{r+2} \langle w_{r+2}, u \rangle = -q_{r+1} \langle w_{r+1}, u \rangle.$$

Hence q_{r+1} and q_{r+2} are both positive. Now we can multiply (6.1) by the common denominator d of all nonzero q_i, to obtain that $w_{r+1} \mid_M \sum_{i=1}^{j} (dq_i) v_i$. Since $w_{r+1} \notin N$, we have that N is not divisor-closed, which concludes the proof. □

6.5 Factoriality

Unique Factorization Monoids

In this section we study the factoriality of members of $\mathcal{C}$ in connection with the geometric properties of their corresponding cones. We shall provide geometric characterizations of the UFMs, HFMs, and OHFMs in $\mathcal{C}$.

To begin with, let us characterize the UFMs in $\mathcal{C}$.

Proposition 6.5.1. *For a monoid in $\mathcal{C}$, the following conditions are equivalent.*

(1) M is a UFM.

(2) Each face submonoid of M is a UFM.

(3) $|\mathcal{A}(M)| = \dim \mathsf{cone}(M)$.

Proof. To prove that (1) implies (3), we will first verify that $|\mathcal{A}(M)| \ge \dim \mathsf{cone}(M)$. Such inequality holds trivially if M contains infinitely many atoms. Then suppose that $\mathcal{A}(M)$ is finite. By Farkas-Minkowski-Weyl Theorem, $\mathsf{cone}(M)$ is polyhedral. As a result, $\mathsf{cone}(M)$ contains at least $\dim \mathsf{cone}(M)$ 1-dimensional edges. Since $\mathsf{cone}(M) = \mathsf{cone}(\mathcal{A}(M))$, any 1-dimensional edge of $\mathsf{cone}(M)$ must contain an atom of M. Thus, $|\mathcal{A}(M)| \ge \dim \mathsf{cone}(M)$, as desired. Suppose now, by way of contradiction, that $|\mathcal{A}(M)| > \dim \mathsf{cone}(M)$. Let $a_1, \dots, a_{d+1} \in \mathcal{A}(M)$ be distinct atoms. Then $\sum_{i=1}^{d+1} \beta_i a_i = 0$ for some $\beta_1, \dots, \beta_{d+1} \in \mathbb{Q}$ not all zeros. There is no loss in assuming that there exists an index $k \in [[1, d]]$ such that $\beta_i < 0$ for $i \in [[1, k]]$ and $\beta_i \ge 0$ for $i \in [[k+1, d+1]]$. Hence

$$\sum_{i=1}^{k} \beta_i a_i \quad \text{and} \quad \sum_{i=k+1}^{d+1} (-\beta_i) a_i$$

are two distinct factorizations of the same element of M, contradicting that M is a UFM.

Now we show that (3) implies (2). Set $d := \dim \mathsf{cone}(M)$ and suppose that $|\mathcal{A}(M)| = d$. Let N be a face submonoid of M. Since $|\mathcal{A}(M)| = d$, it follows by Farkas-Minkowski-Weyl Theorem that $\mathsf{cone}(M)$ is polyhedral. Then $N = M \cap H$ for some supporting hyperplane $H = \{x \in \mathbb{R}^d \mid \langle x, u \rangle = 0\}$ determined by $u \in \mathbb{R}^d$. Suppose that $\mathsf{cone}(M) \subseteq H^-$. Then if $x \in N$ and $\sum_{i=1}^{t} a_i \in \mathsf{Z}_M(x)$, one has that $\sum_{i=1}^{t} \langle a_i, u \rangle = 0$ and, therefore, $\langle a_i, u \rangle = 0$ for $i \in [[1, t]]$. This implies that $\sum_{i=1}^{t} a_i \in \mathsf{Z}_N(x)$. As a consequence, $\mathcal{A}(N) = \mathcal{A}(M) \cap N$. Thus, N is a UFM, and (2) follows.

As (2) trivially implies (1), our proof is complete. □

Corollary 6.5.2. *Let M be a UFM in $\mathcal{C}$. Then $\mathsf{cone}(M)$ is rational and polyhedral.*

Proof. By Proposition 6.5.1, the monoid M is finitely generated and so $\mathsf{cone}(M)$ is the conic hull of a finite set. Now the corollary follows by Farkas-Minkowski-Weyl Theorem. □

Half-Factorial Monoids

The concept of half-factoriality is a weaker version of that one of factoriality (or being a UFD). We proceed to offer characterizations of half-factorial monoids in the class $\mathcal{C}$ in terms of their face submonoids and in terms of the convex hull of their sets of atoms.

Definition 6.5.3. An atomic monoid M is called an *HFM* (or a *half-factorial monoid*) provided that for all $x \in M^\bullet$ and $z, z' \in \mathsf{Z}(x)$, we have that $|z| = |z'|$.

HFMs in $\mathcal{C}$ can be characterized as follows.

Proposition 6.5.4. *For a monoid M in $\mathcal{C}$ the next conditions are equivalent.*

(1) M is an HFM.

(2) Each face submonoid of M is an HFM.

(3) $\dim \mathsf{conv}(\mathcal{A}(M)) < \dim \mathsf{cone}(M)$*.*

Proof. First, we show that (1) implies (3). To do this, suppose that M is an HFM. Set $d := \dim \mathsf{cone}(M)$. Since $\mathsf{cone}(M) = \mathsf{cone}(\mathcal{A}(M))$, one can take linearly independent vectors $a_1, \dots, a_d$ in $\mathcal{A}(M)$. Take also $u \in \mathbb{Q}^d$ and $\alpha \in \mathbb{Q}$ such that the polytope $\mathsf{conv}(a_1, \dots, a_d)$ is contained in the affine hyperplane $H := \{q \in \mathbb{Q}^d \mid \langle q, u \rangle = \alpha\}$. In addition, fix $a \in \mathcal{A}(M)$, and write $a = \sum_{i=1}^d \beta_i a_i$ for some $\beta_1, \dots, \beta_d \in \mathbb{Q}$. From the fact that M is an HFM, we can deduce that $\sum_{i=1}^d \beta_i = 1$. As a result,

$$\langle a, u \rangle = \sum_{i=1}^d \beta_i \langle a_i, u \rangle = \alpha \sum_{i=1}^d \beta_i = \alpha,$$

which means that $a \in H$. Hence $\mathcal{A}(M) \subset H$, which implies that $\dim \mathsf{conv}(\mathcal{A}(M))$ is at most $d-1$. Then we have that $\dim \mathsf{conv}(\mathcal{A}(M)) < \dim \mathsf{cone}(M)$, as desired.

To argue that (3) implies (2), suppose that $\dim \mathsf{conv}(\mathcal{A}(M)) < \dim \mathsf{cone}(M)$. Then there exists an affine hyperplane H containing $\mathsf{conv}(\mathcal{A}(M))$. As in the previous paragraph, take $u \in \mathbb{Q}^d$ and $\alpha \in \mathbb{Q}$ such that $H = \{q \in \mathbb{Q}^d \mid \langle q, u \rangle = \alpha\}$. Now if $x \in M$ and

$$z := \sum_{a \in \mathcal{A}(M)} \beta_a a \in \mathsf{Z}(x),$$

then

$$|z| = \sum_{a \in \mathcal{A}(M)} \beta_a = \frac{1}{\alpha} \sum_{a \in \mathcal{A}(M)} \beta_a \langle a, u \rangle = \frac{1}{\alpha} \Big\langle \sum_{a \in \mathcal{A}(M)} \beta_a a, u \Big\rangle = \frac{1}{\alpha} \langle x, u \rangle.$$

Hence $\mathsf{L}(\mathsf{x}) = \{1/\alpha \langle x, u \rangle\}$ for all $x \in M^\bullet$, and so M is an HFM.

That (2) implies (1) follows trivially. $\square$

Corollary 6.5.5. *A monoid M in $\mathcal{C}_d$ is an HFM if and only if $\mathcal{A}(M)$ is contained in an affine hyperplane of $\mathbb{Q} \otimes_{\mathbb{Z}} \text{gp}(M)$.*

Remark 6.5.6. Corollary 6.5.5 has been previously established by Kainrath and Lettl in [112]. Fairly similar versions of the same result were first given by Zaks [149] and Narkiewicz [123].

The chain of implications (6.2), where being a UFM, an HFM, and an atomic monoid are included, has received a great deal of attention since it was first studied (in the context of integral domains) by Anderson, Anderson, and Zafrullah [3]:

$$\text{UFM} \Rightarrow \text{HFM} \Rightarrow \text{FFM} \Rightarrow \text{BFM} \Rightarrow \text{ACCP monoid} \Rightarrow \text{atomic monoid.} \qquad (6.2)$$

The first three implications above are obvious, while the last two implications follow from [79, Proposition 1.1.4] and [79, Corollary 1.3.3]. In addition, all the implications above are strict, and examples witnessing this observation (in the context of integral domains) can be found in [3]. We have already seen that not every monoid in $\mathcal{C}$ is an HFM. However, each monoid in $\mathcal{C}$ is an FFM, as the next proposition illustrates.

Proposition 6.5.7. *Each monoid in $\mathcal{C}$ is an FFM.*

Proof. By Theorem 6.3.1, it suffices to show that for every $d \in \mathbb{Z}_{\geq 1}$, any additive submonoid M of $\mathbb{N}^d$ is an FFM. Fix $x \in M$. It is clear that $\langle x, y \rangle \geq 0$ for all $y \in M$. Thus, $y \mid_M x$ implies that $||y|| \leq ||x||$. As a result, the set

$$\{a \in \mathcal{A}(M) : a \mid_M x\}$$

is finite, which implies that $\mathsf{Z}(x)$ is also finite. Hence M is an FFM. □

As an immediate consequence of Proposition 6.5.7, every monoid in $\mathcal{C}$ satisfies the last four conditions in the chain of implications (6.2).

Other-Half-Factorial Monoids

Other-half-factoriality is a dual version of half-factoriality and was introduced by Coykendall and Smith in [58].

Definition 6.5.8. An atomic monoid M is called an *OHFM* (or an *other-half-factorial monoid*) provided that for all $x \in M^\bullet$ and $z, z' \in \mathsf{Z}(x)$ with $|z| = |z'|$, we have that $z = z'$.

Although an integral domain is a UFD if and only if its multiplicative monoid is an OHFM [58, Corollary 2.11], an OHFM is not, in general, a UFM or an HFM, as one can deduce from the next theorem.

A set of points in a d-dimensional $\mathbb{F}$-space V (where $\mathbb{F}$ is either $\mathbb{Q}$ or $\mathbb{R}$) is said to be *affinely independent* provided that no k of such points lie in a $(k-2)$-dimensional affine subspace of V for $k \in [[2, d+1]]$. If a set is affinely independent, its points are said to be in *general linear position*.

Theorem 6.5.9. *Let M be a nontrivial monoid in $\mathcal{C}$. Then the following statements are equivalent.*

(1) M is an OHFM.

(2) every face submonoid of M is an OHFM.

(3) The points in $\mathcal{A}(M)$ are affinely independent.

(4) $\mathsf{conv}(\mathcal{A}(M))$ is a simplex with dimension either $\mathsf{rank}(M) - 1$ or $\mathsf{rank}(M)$.

Proof. To begin with, we verify that (1) and (3) are equivalent statements. For this, set $d = \dim \mathsf{cone}(M)$ and $V := \mathbb{Q} \otimes_{\mathbb{Z}} \text{gp}(M)$. Let us first prove that (1) implies (3). Assume, therefore, that M is an OHFM and suppose, by way of contradiction, that the points in $\mathcal{A}(M)$ are not affinely independent. Then there exist $k \in [[2, d+1]]$ and pairwise distinct vectors $a_1, \dots, a_k \in \mathcal{A}(M)$ contained in a $(k-2)$-dimensional affine subspace W of V. Let M' denote the submonoid of M generated by $a_1, \dots, a_k$. Since $W - a_k$ is a $(k-2)$-dimensional subspace of V, the vectors $a_1 - a_k, \dots, a_{k-1} - a_k$ are linearly dependent in $W - a_k$ and so $\sum_{i=1}^{k-1} q_i(a_i - a_k) = 0$ for some rational coefficients $q_1, \dots, q_{k-1}$ (not all zeros). After relabeling vectors and coefficients, we can assume the existence of $j \in [[1, k-2]]$ such that $q_1, \dots, q_j$ are negative and $q_{j+1}, \dots, q_{k-1}$ are nonnegative. Set

$$x := \sum_{i=1}^{j} (-mq_i) a_k + \sum_{i=j+1}^{k-1} (mq_i) a_i \in M',$$

where m is the least common multiple of the denominators of the nonzero q_i's. Then

$$z := \left(\sum_{i=1}^{j} (-mq_i) \right) a_k + \sum_{i=j+1}^{k-1} (mq_i) a_i \quad \text{and } z' := \quad \sum_{i=1}^{j} (mq_i) a_i + \left(\sum_{i=j+1}^{k-1} (-mq_i) \right) a_k,$$

are two factorizations in $\mathsf{Z}_{M'}(x)$ having the same length. As $a_1, \dots, a_k$ are also atoms of M, it follows that z and z' are also factorizations in $\mathsf{Z}_M(x)$, which contradicts that M is an OHFM. Thus, (3) follows.

To prove that (3) implies (1), suppose that the points in $\mathcal{A}(M)$ are affinely independent in V. We have seen in the proof of Proposition 6.5.1 that $|\mathcal{A}(M)| \ge \dim \mathsf{cone}(M)$. As a result, $|\mathcal{A}(M)| \in \{d, d+1\}$. If $|\mathcal{A}(M)| = d$, then Proposition 6.5.1 ensures that M is an UFM and, therefore, an HFM. Thus, we finally assume that $|\mathcal{A}(M)| = d + 1$. Let $\mathcal{A}(M) =: \{a_0, a_1, \dots, a_d\}$ and suppose, by way of contradiction, that M is not an OHFM. This implies the existence of $m_i, n_i \in \mathbb{N}$ for $i \in [[0, d]]$ such that

$$\sum_{i=0}^{d} m_i a_i = \sum_{i=0}^{d} n_i a_i \quad \text{and} \quad \sum_{i=0}^{d} (m_i - n_i) = 0.$$

Assume, without loss of generality, that $m_0 \neq n_0$. Let H be an affine hyperplane in V containing $a_1, \dots, a_n$. Take $u \in \mathbb{Q}^d$ and $\alpha \in \mathbb{Q}$ such that $H = \{q \in \mathbb{Q}^d \mid \langle q, u \rangle = \alpha\}$. As the points $a_0, a_2, \dots, a_n$ are affinely independent, $a_0 \notin H$. Then

$$\begin{aligned} 0 &= \sum_{i=0}^{d} (m_i - n_i)\langle u, a_i \rangle \\ &= (m_0 - n_0)\langle u, a_0 \rangle + \alpha \sum_{i=1}^{d} (m_i - n_i) \\ &= (m_0 - n_0)(\langle u, a_0 \rangle - \alpha). \end{aligned}$$

As a result, $\langle u, a_0 \rangle = \alpha$, which contradicts the fact that a_0 does not belong to H. Hence M must be an OHFM, yielding statement (1).

To argue that (3) implies (2), suppose that the points in $\mathcal{A}(M)$ are affinely independent. Assume that M is a submonoid of $(\mathbb{N}^d, +)$ of rank d for some $d \in \mathbb{N}$. Now suppose that N is a face submonoid of M, and let F be a face of $\mathsf{cone}(M)$ such that $N = M \cap F$. Proposition 6.4.2 ensures that $\mathcal{A}(N) = \mathcal{A}(M) \cap F$. Since the set of points $\mathcal{A}(M)$ is affinely independent, the set of points $\mathcal{A}(N)$ is also affinely independent. As we have already proved the equivalence of (1) and (3), we can conclude that N is an OHFM. Hence statement (2) follows.

It is clear, on the other hand, that (2) implies (1). Now the fact that (4) is a restatement of (3) completes our proof. □

Corollary 6.5.10. *Let N be a numerical monoid. Then N is an OHFM if and only if the embedding dimension of N is at most* 2*.*

Remark 6.5.11. The characterization proposed in Theorem 6.5.9 was indeed motivated by Corollary 6.5.10, which was first proved by Coykendall and Smith in [58].

The fact that every proper face submonoid of a monoid M in $\mathcal{C}$ is an OHFM does not guarantee that M is an OHFM, as one can see in the following example.

Example 6.5.12. Consider the submonoid $M := \langle 2e_1, 3e_1, 2e_2, 3e_2 \rangle$ of $(\mathbb{N}^2, +)$. It is easy to argue that $\mathcal{A}(M) = \{2e_1, 3e_1, 2e_2, 3e_2\}$. Notice that the 1-dimensional faces of $\mathsf{cone}_{\mathbb{R}^2}(M)$ are $\mathbb{R}_{\geq 0}e_1$ and $\mathbb{R}_{\geq 0}e_2$. Then there are two face submonoids of M corresponding to 1-dimensional faces of $\mathsf{cone}(M)$, and they are both isomorphic to the numerical monoid $\langle 2, 3 \rangle$, which is an OHFM by Corollary 6.5.10. Hence every proper face submonoid of M is an OHFM. However, $\mathsf{conv}(\mathcal{A}(M))$ is not a simplex and, therefore, it follows by Theorem 6.5.9 that M is not an OHFM.

We conclude this section with the following proposition.

Proposition 6.5.13. *Let M be an OHFM in $\mathcal{C}$. Then the faces of $\mathsf{cone}(M)$ whose corresponding face submonoids are not UFMs form a (possibly empty) interval in the face lattice $\mathsf{F}(\mathsf{cone}(M))$.*

Proof. Let $\mathcal{N}$ consist of all faces of $\mathsf{cone}(M)$ whose corresponding face submonoids are not UFMs. If M is a UFM, it follows by Proposition 6.5.1 that every face submonoid of M is also a UFM and, therefore, $\mathcal{N}$ is empty. Then we assume that M is not a UFM.

Among all the faces in $\mathcal{N}$, let F and F' be minimal in $\mathsf{F}(\mathsf{cone}(M))$. Suppose, by way of contradiction, that $F \neq F'$. Set $N := M \cap F$ and $N' := M \cap F'$. It follows by Proposition 6.4.2 that $F = \mathsf{cone}(N)$ and $F' = \mathsf{cone}(N')$. Since F and F' are minimal, they are not comparable and so we can take $a \in \mathcal{A}(N) \setminus \mathcal{A}(N')$ and $a' \in \mathcal{A}(N') \setminus \mathcal{A}(N)$. Once again, one can rely on the minimality of F and F' to obtain

$$\mathsf{rank}(\mathcal{A}(N)) = \mathsf{rank}(\mathcal{A}(N) \setminus \{a\}) \quad \text{and} \quad \mathsf{rank}(\mathcal{A}(N')) = \mathsf{rank}(\mathcal{A}(N') \setminus \{a'\}).$$

As a result, the rank of the set $\mathcal{A} := \mathcal{A}(N) \cup \mathcal{A}(N')$ is at most $|\mathcal{A}| - 2$. Set $n := |\mathcal{A}|$, and let A be the $d \times n$ matrix whose columns are the vectors in $\mathcal{A}$ (after some order is fixed). Then $\text{rank}\, A = \text{rank}(\mathcal{A}) \le n - 2$. Thus, $\dim \ker A \ge 2$. Consider the hyperplane of $\mathbb{Q}^n$ defined by

$$H := \{(x_1, \dots, x_n) \in \mathbb{Q}^n \mid x_1 + \dots + x_n = 0\}$$

and note that

$$\dim(H \cap \ker A) = \dim(H) + \dim(\ker A) - \dim(\text{span}(H \cup \ker A)) \ge (n-1) + 2 - n \ge 1.$$

Therefore there is a nonzero vector $(q_1, \dots, q_n)$ in $\ker A$ satisfying that $q_1 + \dots + q_n = 0$. First, taking $j \in [[1, n]]$ such that $q_1, \dots, q_j \le 0$ and $q_{j+1}, \dots, q_n > 0$, then taking m to be the least common multiple of the denominators of the nonzero q_i's, and finally proceeding as we did in the second paragraph of the proof of Theorem 6.5.9, we can obtain two factorizations of the same element of M having the same length. However, this contradicts that M is an OHFM. Hence there exists only one minimal face of $\mathsf{cone}(M)$ whose face submonoid is not a UFM, namely, F. Clearly, the face submonoid of any face containing F cannot be a UFM. This implies that $[F, \mathsf{cone}(M)] \subseteq \mathcal{N}$. The reverse inclusion follows from the uniqueness of a minimal face in $\mathcal{N}$. Hence $\mathcal{N}$ is the interval $[F, \mathsf{cone}(M)]$, which concludes our argument. $\square$

The reverse implication of Proposition 6.5.13 does not hold, as the next example illustrates.

Example 6.5.14. Consider the submonoid $M := \langle 3e_1, 3e_2, 2e_3, 3e_3 \rangle$ of $(\mathbb{N}^3, +)$. It can be readily verified that $\mathcal{A}(M) = \{3e_1, 3e_2, 2e_3, 3e_3\}$. Since $\{2e_3, 3e_3\}$ is an affinely dependent set, it follows by Theorem 6.5.9 that M is not an OHFM. However, the non-UFM face submonoids of M are precisely those determined by the faces of $\mathsf{cone}(M)$ contained in the interval $[\mathbb{R}e_3, \mathsf{cone}(M)]$.

6.6 Primary Monoids and Finitary Monoids

As mentioned at the beginning of this chapter, primary monoids and finitary monoids have been crucial in the development of non-unique factorization theory as the factorization structure of members in these two classes abstracts certain properties of important classes of integral domains. In the first part of this section, we investigate some geometric aspects of primary monoids in $\mathcal{C}$. Then we shift our focus to the study of finitary monoids of $\mathcal{C}$.

Primary Monoids

A monoid M is called *primary* provided that M is nontrivial and for all $a, b \in M^\bullet$ there exists $n \in \mathbb{N}$ such that $nb \in a + M$. The primary monoids in $\mathcal{C}$ are precisely those minimizing the number of face submonoids.

Proposition 6.6.1. *For a nontrivial monoid M in $\mathcal{C}$, the following conditions are equivalent.*

(1) M is primary.

(2) The only face submonoids of M are $\{0\}$ and M.

(3) $\mathsf{cone}(M)^\bullet$ is an open subset of $\mathbb{R} \otimes_{\mathbb{Z}} \text{gp}(M)$.

Proof. It follows from [79, Lemma 2.7.7] that M is primary if and only if the only divisor-closed submonoids of M are $\{0\}$ and M. This, along with Theorem 6.4.5, implies that the conditions (1) and (2) are equivalent.

To argue that (2) implies (3), take $x \in \mathsf{cone}(M)^\bullet$. Since $\mathsf{cone}(M)$ is the disjoint union of the relative interiors of all its faces, there exists a face F of $\mathsf{cone}(M)$ such that $x \in \mathsf{relin}\, F$. As $x \neq 0$, the dimension of F is at least 1 and, therefore, $M \cap F$ is a nontrivial face submonoid of M. It follows now by (2) that $M \cap F = M$ and, therefore, $x \in \mathsf{relin}\,\mathsf{cone}(M)$. Hence $\mathsf{cone}(M)^\bullet$ is open.

Finally, let us verify that (3) implies (2). Since every proper face of $\mathsf{cone}(M)$ is contained in the boundary of $\mathsf{cone}(M)$, the fact that $\mathsf{cone}(M)^\bullet$ is open implies that the only proper face of $\mathsf{cone}(M)$ is the origin, from which (2) follows. □

Remark 6.6.2. We want to emphasize that the fact that (1) and (3) are equivalent conditions in Proposition 6.6.1 was first established by Geroldinger, Halter-Koch, and Lettl [80, Theorem 2.4]. However, we obtain such a result here from the poset structure of the face lattice of $\mathsf{cone}(M)$.

Primary monoids in $\mathcal{C}$ account for all primary submonoids of any (non-necessarily finite-rank) free commutative monoid, as the next proposition illustrates.

Proposition 6.6.3. *Let M be a primary submonoid of a free commutative monoid. Then M has finite rank, and M can be embedded into $(\mathbb{N}^r, +)$, where $r = \mathsf{rank}(M)$.*

Proof. Let F_P be a free commutative monoid on an infinite set P such that M is a submonoid of F_P. For $s \in F_P$ and $S \subseteq F_P$, write

$$\text{Spec}(s) := \big\{p \in P \mid p \text{ divides } s \text{ in } F_P\big\} \quad \text{and} \quad \text{Spec}(S) := \bigcup_{s \in S} \text{Spec}(s).$$

Suppose for a contradiction that $\text{Spec}(M)$ contains infinitely many elements. Fix $x \in M^\bullet$, and take $p \in P$ such that $p \in \text{Spec}(M) \setminus \text{Spec}(x)$. Since p is a prime element of F_P, it is clear that the set

$$S := \{x \in M \mid p \text{ does not divide } x \text{ in } F_P\}$$

is a divisor-closed submonoid of M. The fact that $p \notin \text{Spec}(x)$ implies that S is a nontrivial submonoid of M, and the fact that $p \in \text{Spec}(M)$ implies that $S \neq M$. Thus, S is a proper nontrivial divisor-closed submonoid of M, which contradicts that M is primary. Hence $\text{Spec}(M)$ is finite and, as a result, M can be naturally embedded into $\mathbb{N}p_1 \oplus \cdots \oplus \mathbb{N}p_t$, where $p_1, \dots, p_t$ are the prime elements $\text{Spec}(M)$. It follows now from Theorem 6.3.1 that M can be embedded into $(\mathbb{N}^r, +)$. □

Finitely Primary Monoids

Now we restrict our attention to a special subclass of primary monoids, that one consisting of finitely primary monoids. The *complete integral closure* of a monoid M, denoted by $\widehat{M}$, is defined as follows:

$$\widehat{M} := \big\{x \in \text{gp}(M) \mid \text{ there exists } y \in M \text{ such that } nx + y \in M \text{ for every } n \in \mathbb{N}\big\}.$$

Clearly, $\widehat{M}$ is a submonoid of $\text{gp}(M)$ containing M, and so $\mathsf{rank}(\widehat{M}) = \mathsf{rank}(M)$. A monoid M is called *finitely primary* if there exist $d \in \mathbb{N}$ and a UFM $F := \langle p_1, \dots, p_d \rangle$, where $p_1, \dots, p_d$ are pairwise distinct prime elements, such that

(1) M is a submonoid of F,

(2) $M^\bullet \subseteq p_1 + \cdots + p_d + F$, and

(3) $\alpha(p_1 + \cdots + p_d) + F \subseteq M$ for some $\alpha \in \mathbb{N}^\bullet$.

In this case, it follows by [79, Theorem 2.9.2] that $\widehat{M} \cong (\mathbb{N}^d, +)$. Then $\text{rank}(M) = d$ and, moreover, any finitely primary monoid of rank d is in $\mathcal{C}_d$. On the other hand, it also follows from [79, Theorem 2.9.2] that finitely primary monoids are primary. Therefore, it follows from Proposition 6.6.1 that for any finitely primary monoid M the set $\mathsf{cone}(M)^\bullet$ is open. As the next proposition reveals, the closure of the same set happens to be a simplicial cone.

Proposition 6.6.4. *If M is a finitely primary monoid, then M is in $\mathcal{C}$ and $\overline{\mathsf{cone}(M)}$ is a rational simplicial cone.*

Proof. Let d be the rank of M. We have already observed that M is in the class $\mathcal{C}$. For the rest of the proof, assume that $M \subseteq \widehat{M} \subseteq \mathbb{N}^d$. Because $\widehat{M} \cong (\mathbb{N}^d, +)$, one can take distinct prime elements $p_1, \dots, p_d$ of $\widehat{M}$ such that $\widehat{M} = \langle p_1, \dots, p_d \rangle = \mathbb{N}p_1 \oplus \cdots \oplus \mathbb{N}p_d$. It follows from [79, Theorem 2.9.2] that

$$M^\bullet \subseteq p_1 + \cdots + p_d + \widehat{M} \quad \text{and} \quad \alpha(p_1 + \cdots + p_d) + \widehat{M} \subseteq M,$$

for some $\alpha \in \mathbb{N}^\bullet$. Let C_p be the cone in $\mathbb{R}^d$ generated by $p_1, \dots, p_d$. Clearly, C_p is a rational simplicial cone of dimension d. We claim that $\overline{\mathsf{cone}(M)} = C_p$. Since

$$M^\bullet \subseteq p_1 + \cdots + p_d + \widehat{M} \subset \mathsf{int}\, C_p,$$

we have that $M \subseteq C_p$. Therefore $\mathsf{cone}(M) \subseteq C_p$ and, as the cone C_p is closed, $\overline{\mathsf{cone}(M)} \subseteq C_p$. Let us proceed to argue that $C_p \subseteq \overline{\mathsf{cone}(M)}$. To do so, fix $\epsilon > 0$ and fix also an index $j \in [[1, d]]$. Let L be the 1-dimensional face of C_p in the direction of the vector p_j, and consider the conical open ball with central axis L given by

$$B(p_j, \epsilon) := \left\{ w \in \mathbb{R}^d \setminus \{0\} \;\middle|\; \frac{||w - \mathsf{p}_L(w)||}{||w||} < \epsilon \right\},$$

where $\mathsf{p}_L \colon \mathbb{R}^d \to \mathbb{R}p_j$ is the linear projection of $\mathbb{R}^d$ onto its subspace $\mathbb{R}p_j$. It is clear that the set

$$\{0\} \cup \big(B(p_j, \epsilon) \cap \mathsf{int}\, C_p\big)$$

is a d-dimensional subcone of C_p and, therefore, it must intersect $\widehat{M}$. Then one can take $y \in \widehat{M} \cap \mathsf{int}\, C_p$ such that $\mathbb{R}_{>0} y \subset B(p_j, \epsilon)$. Because

$$\alpha\big(\widehat{M} \cap \mathsf{int}\, C_p\big) \subseteq \alpha(p_1 + \cdots + p_d) + \widehat{M} \subseteq M,$$

we have that $\alpha y \in M$. As a result, $\mathbb{R}_{>0} y \subset \mathsf{cone}(M)$. As $\mathsf{cone}(M)$ and every open conical ball with central axis L have an open ray in common, $p_j \in L \subseteq \overline{\mathsf{cone}(M)}$. As the index j was arbitrarily taken, $p_j \in \overline{\mathsf{cone}(M)}$ for every $j \in [[1, \dots, d]]$, and so $C_p \subseteq \overline{\mathsf{cone}(M)}$. Hence $\overline{\mathsf{cone}(M)}$ is a rational simplicial cone. □

For a primary monoid M in $\mathcal{C}$, the fact that $\overline{\mathsf{cone}(M)}$ is rational and simplicial does not imply that M is finitely primary. The following example sheds some light upon this observation.

Example 6.6.5. Consider the subset M of $\mathbb{N}^2$ defined by

$$M := \{(0,0)\} \cup \big\{(n,m) \in \mathbb{N}^2 \mid n, m \in \mathbb{N}^\bullet \text{ and } m \leq 2^n\big\}.$$

From the fact that $f(x) = 2^x$ is a convex function, one can readily verify that M is a submonoid of $(\mathbb{N}^2, +)$. Since M contains $(n, 1)$ for every $n \in \mathbb{N}^\bullet$ and M, the ray $\mathbb{R}_{\geq 0} e_1$ is contained in $\overline{\mathsf{cone}_{\mathbb{R}^2}(M)}$. On the other hand, the fact that $\{(n, 2^n) \mid n \in \mathbb{N}^\bullet\} \subset M$, along with $\lim_{n\to\infty} 2^n/n = \infty$, guarantees that the ray $\mathbb{R}_{\geq 0} e_2$ is contained in $\overline{\mathsf{cone}_{\mathbb{R}^2}(M)}$. Thus,

$$\mathsf{cone}_{\mathbb{R}^2}(M) = \big\{(0,0)\big\} \cup \big\{(x,y) \in \mathbb{R}^2 \mid x, y > 0\big\} = \{(0,0)\} \cup \mathbb{R}^2_{>0}.$$

As $\mathsf{cone}(M)^\bullet$ is open, Proposition 6.6.1 ensures that M is a primary monoid. On the other hand, $\overline{\mathsf{cone}(M)} = \mathbb{R}^2_{\geq 0}$ is a rational simplicial cone.

To argue that M is not finitely primary, it suffices to verify that $\widehat{M} \ncong (\mathbb{N}^2, +)$. To do so, fix $m \in \mathbb{N}$, and then take $N \in \mathbb{N}$ large enough so that $nm \leq 2^n$ for every $n \geq N$. Note that $y := (N, Nm)$ belongs to M. Moreover,

$$n(1, m) + y = (n + N, (n + N)m) \in M$$

for every $n \in \mathbb{N}$. Therefore $(1, m) \in \widehat{M}$ for every $m \in \mathbb{N}$. On the other hand, for any $m \in \mathbb{N}^\bullet$ and $(a, b) \in M^\bullet$,

$$2^a(0, m) + (a, b) = (a, 2^a m + b) \notin M.$$

Hence $(n, m) \in \widehat{M}^\bullet$ implies that $n > 0$. As a result,

$$\widehat{M} = \{(n, m) \in \mathbb{N}^2 \mid n > 0\}.$$

Since $\mathcal{A}(\widehat{M}) = \{(1, n) \mid n \in \mathbb{N}\}$ contains infinitely many elements, $\widehat{M} \ncong (\mathbb{N}^2, +)$. Hence M cannot be finitely primary.

Finitary Monoids

Let M be a monoid. We say that M is *weakly finitary* if there exist a finite subset S of M and $n \in \mathbb{N}^\bullet$ such that $nx \in S + M$ for all $x \in M^\bullet$. In addition, a BFM M is called *finitary* if there exist a finite subset S of M and $n \in \mathbb{N}^\bullet$ such that $nM^\bullet \subseteq S + M$. Clearly, every finitary monoid is weakly finitary. In addition, every finitely generated monoid is finitary. Also, affine monoids are finitary.

The face submonoids of a monoid in $\mathcal{C}$ inherit the condition of being (weakly) finitary.

Proposition 6.6.6. *Let M be a monoid in $\mathcal{C}$. Then M is finitary (resp., weakly finitary) if and only if each face submonoid of M is finitary (resp., weakly finitary).*

Proof. We will prove only the finitary version of the proposition as the weakly finitary version follows similarly. Suppose that M is finitary, and let d be the rank of M. Take F to be a face of $\mathsf{cone}(M)$, and consider the face submonoid $N := M \cap F$. Since M is finitary, there exist $n \in \mathbb{N}$ and a finite subset S of M such that $nM^\bullet \subseteq S + M$. We claim that $nN^\bullet \subseteq S_F + N$, where $S_F := S \cap F$. Take $x_1, \dots, x_n \in N^\bullet = M^\bullet \cap F$. As

$$n(M^\bullet \cap F) \subseteq nM^\bullet \subseteq S + M,$$

there exist $s \in S$ and $y \in M$ such that $x_1 + \cdots + x_n = s + y$. Since $M \cap F$ is a divisor-closed submonoid of M, we find that $s, y \in F$. Therefore $s \in S_F$ and $y \in N$, which implies that $x_1 + \cdots + x_n \in S_F + N$. Hence N is a finitary monoid. The reverse implication follows trivially as $\mathsf{cone}(M)$ is a face of itself. □

Our next goal is to give a sufficient geometric condition for a monoid in $\mathcal{C}$ to be finitary. First, let us recall the concept of triangulation. A *conical polyhedral complex* p in $\mathbb{R}^d$ is a collection of polyhedral cones in $\mathbb{R}^d$ satisfying the following conditions:

(1) Every face of a polyhedron in p is also in p;

(2) The intersection of any two polyhedral cones C_1 and C_2 in p is a face of both C_1 and C_2.

Clearly, the underlying set of the face lattice of a given polyhedral cone is a conical polyhedral complex. For a conical polyhedral complex p in $\mathbb{R}^d$, we set $|\mathsf{p}| := \cup_{C \in \mathsf{p}} C$. Let p and p' be two conical polyhedral complexes. We say that p' is a *polyhedral subdivision* of p provided that $|\mathsf{p}| = |\mathsf{p}'|$ and each face of p is the union of faces of p'. A polyhedral subdivision p' of p is called a *triangulation* of p if p' consists of simplicial cones. Every conical polyhedral complex has certain special triangulations.

Theorem 6.6.7. *[37, Theorem 1.54] Let p be a conical polyhedral complex, and let $S \subset |\mathsf{p}|$ be a finite set of nonzero vectors such that $S \cap C$ generates C for each $C \in \mathsf{p}$. Then there exists a triangulation p' of p such that $\{\mathbb{R}_{\ge 0} v \mid v \in S\}$ is the set of 1-dimensional faces of p'.*

We are in a position now to offer a sufficient geometric condition for a monoid in $\mathcal{C}$ to be finitary.

Theorem 6.6.8. *Let M be a monoid in $\mathcal{C}$. If $\mathsf{cone}(M)$ is polyhedral, then M is finitary.*

Proof. Let d be the rank of M, and assume that $M \subseteq \mathbb{N}^d$. Since $\mathsf{cone}(M)$ is polyhedral, it follows by Farkas-Minkowski-Weyl Theorem that $\mathsf{cone}(M)$ is the conic hull of a finite set of vectors. As the vectors in such a generating set are nonnegative rational linear combinations of vectors in M, there exists $S = \{v_1, \dots, v_k\} \subset M$ with $k \ge d$ such that $\mathsf{cone}(M) = \mathsf{cone}(S)$. By Theorem 6.6.7, there exists a triangulation $\mathcal{T}$ of the face lattice of $\mathsf{cone}(M)$ whose set of 1-dimensional faces is $\{\mathbb{R}_{\ge 0} v_i \mid i \in [[1, k]]\}$. Then for any $T \in \mathcal{T}$ there are unique indices $t_1, \dots, t_d$ satisfying that

$$1 \le t_1 < \cdots < t_d \le k \quad \text{and} \quad T = \mathsf{cone}(v_{t_1}, \dots, v_{t_d}),$$

and we can use this to assign to T the parallelepiped

$$\Pi_T := \big\{ \alpha_1 v_{t_1} + \cdots + \alpha_d v_{t_d} \mid 0 \le \alpha_i < 1 \text{ for every } i \in [[1, d]] \big\}.$$

It is clear that

$$|\Pi_T \cap \mathbb{Z}^d| < \infty \quad \text{and} \quad \Pi_T \cap \mathbb{Z}^d \subset \mathbb{Q}_{\ge 0} v_{t_1} + \cdots + \mathbb{Q}_{\ge 0} v_{t_d}.$$

Then we can choose $N_T \in \mathbb{N}$ large enough so that $N_T v \in \mathbb{N}v_{t_1} + \cdots + \mathbb{N}v_{t_d}$ for every $v \in \Pi_T \cap \mathbb{Z}^d$. Now take

$$m := \max\{N_T\,|\Pi_T \cap \mathbb{Z}^d| : T \in \mathcal{T}\}$$

and set $n := m\,|\mathcal{T}|$. In order to show that M is finitary, it suffices to verify that $nM^\bullet \subseteq S+M$.

To do so, take (possibly repeated) elements $x_1, \dots, x_n \in M^\bullet$. For every $x \in \{x_1, \dots, x_n\}$, there exists $T \in \mathcal{T}$ with $x \in T$. Let $T = \mathsf{cone}(v_{t_1}, \dots, v_{t_d})$ for $t_1 < \cdots < t_d$ be a simplicial cone in $\mathcal{T}$. Observe that we can naturally partition T into (translated) copies of the parallelepiped Π_T, that is, T equals the disjoint union of the sets $v + \Pi_T$ for $v \in \mathbb{N}v_{t_1} + \cdots + \mathbb{N}v_{t_d}$. As a result, there exist $z \in \Pi_T \cap \mathbb{Z}^d$ and coefficients $\alpha_1, \dots, \alpha_d \in \mathbb{N}$ such that

$$x = z + \sum_{i=1}^{d} \alpha_i v_{t_i}. \tag{6.3}$$

Hence for $i \in [[1,n]]$, we can write $x_i = z_i + m_i$ for some $z_i \in \cup_{T \in \mathcal{T}} \Pi_T \cap \mathbb{Z}^d$ and $m_i \in M$. Since $n = m|\mathcal{T}|$, there exists $T_0 \in \mathcal{T}$ such that

$$|\{i \in [[1,n]] \mid z_i \in \Pi_{T_0} \cap \mathbb{Z}^d\}| \geq m.$$

Consider now the equivalence relation on the set of indices $\{i \in [[1,n]] \mid z_i \in T_0\}$ defined by $i \sim j$ whenever $z_i = z_j$. The fact that $m \geq N_{T_0}\,|\Pi_{T_0} \cap \mathbb{Z}^d|$ guarantees the existence of a class I determined by the relation $\sim$ and containing at least N_{T_0} distinct indices. Take $I_0 \subseteq I$ such that $|I_0| = N_{T_0}$. Setting $z := z_i$ for some $i \in I_0$, one has that

$$\sum_{i \in I_0} z_i = N_{T_0} z \in \mathbb{N}v_1 + \cdots + \mathbb{N}v_n \in S + M$$

and, therefore, there exist $v \in S$ and $m \in M$ such that $\sum_{i \in I_0} z_i = v + m$. As a result, one can set $m' = \sum_{i=1}^n x_i - \sum_{i \in I_0} x_i \in M$ to obtain that

$$\sum_{i=1}^{n} x_i = \Big(\sum_{i \in I_0} x_i\Big) + m' = \sum_{i \in I_0} z_i + m' + \sum_{i \in I_0} m_i = v + \Big(m + m' + \sum_{i \in I_0} m_i\Big) \in S + M.$$

Since the elements $x_1, \dots, x_n$ were arbitrarily taken in $M^\bullet$, the inclusion $nM^\bullet \subseteq S + M$ holds. Hence the monoid M is finitary, as desired. □

According to the characterization of cones generated by monoids in $\mathcal{C}$ we have provided in Theorem 6.3.13, every d-dimensional positive cone C of $\mathbb{R}^d$ with $C^\bullet$ open can be generated by a monoid in $\mathcal{C}$. Indeed, any such a cone can be generated by a finitary monoid in $\mathcal{C}$.

Proposition 6.6.9. *For $d \in \mathbb{N}$, let C be a positive cone in $\mathbb{R}^d$. If $C^\bullet$ is open in $\mathbb{R}^d$, then C can be generated by a finitary monoid in $\mathcal{C}$.*

Proof. Assume that $C^\bullet$ is open in $\mathbb{R}^d$. Take $M = \mathbb{N}^d \cap C$. It is clear that $C = \mathsf{cone}(M)$. Now take $v_0 \in M^\bullet$, and consider the monoid $M' := \{0\} \cup (v_0 + M)$. Let C' be the cone generated by M'. Notice that $\mathbb{Q}^d \cap C$ and $\mathbb{Q}^d \cap C'$ are the cones generated by M and M' over $\mathbb{Q}$, respectively. So proving that $C' = C$ amounts to showing that $\mathbb{Q}^d \cap C' = \mathbb{Q}^d \cap C$ (see [37, Proposition 1.70]). Since $M' \subseteq M$ it follows that $\mathbb{Q}^d \cap C' \subseteq \mathbb{Q}^d \cap C$. Now let ℓ_0 be the distance from $\{v_0\}$ to $\mathbb{R}^d \setminus C$. As $\mathbb{R}^d \setminus C$ is closed and $\{v_0\}$ is compact, $\ell_0 > 0$. Now take $v \in \mathbb{Q}^d \cap C^\bullet$ such that $||v|| > 1$, and let ℓ be the distance from v to $\mathbb{R}^d \setminus C$. By a similar argument, $\ell > 0$. Notice that the conical ball

$$B(v, \ell) := \left\{ w \in \mathbb{Q}^d \;\middle|\; \frac{||w - \mathsf{p}_v(w)||}{||w||} < \frac{\ell}{2} \right\}$$

is contained in $\mathbb{Q}^d \cap C$. Take $N \in \mathbb{N}$ such that

$$N > \max\Big\{ \frac{||v_0||}{||v|| - 1}, \frac{2||v_0||}{\ell} \Big\}$$

and $Nv \in \mathbb{Q}^d$. Now set $w_0 := Nv - v_0$. Notice that $||w_0|| \ge N||v|| - ||v_0|| > N$. Then we have that

$$\frac{||w_0 - \mathsf{p}_v(w_0)||}{||w_0||} < \frac{||v_0 - \mathsf{p}_v(v_0)||}{N} \le \frac{||v_0||}{N} < \frac{\ell}{2}.$$

Hence $w_0 \in \mathbb{Q}^d \cap B(v, \ell) \subseteq \mathbb{Q}^d \cap C$, and so there exist $c_1, \dots, c_k \in \mathbb{Q}_{>0}$ and $v_1, \dots, v_k \in M^\bullet$ such that $w_0 = \sum_{i=1}^k c_i v_i$. This implies that $nv \in M' \subseteq \mathsf{cone}(M')$ for some $n \in \mathbb{N}$ and, therefore, $v \in \mathbb{Q}^d \cap \mathsf{cone}(M')$. Hence $\mathbb{Q}^d \cap C^\bullet \subseteq \mathbb{Q}^d \cap C$.

As M' generates C, we only need to verify that M' is finitary. Take $w_1, w_2 \in M'^\bullet$, and then $v_1, v_2 \in M$ such that $w_1 = v_0 + v_1$ and $w_2 = v_0 + v_2$. Then

$$w_1 + w_2 = v_0 + (v_0 + v_1 + v_2) \in v_0 + (v_0 + M) \subset v_0 + M'.$$

As a result, $2M'^\bullet \subseteq v_0 + M'$, which implies that M' is a finitary monoid, as desired. □

Theorem 6.6.8 and Proposition 6.6.9 indicate that there is a huge variety of finitary monoids in $\mathcal{C}$. We proceed to exhibit a monoid in $\mathcal{C}_2$ that is not even weakly finitary. First, let us introduce the following notation.

Notation: For $x \in \mathbb{R}^2_{\ge 0} \setminus \{0\}$, we let $\text{slope}(x) \in \mathbb{R}_{\ge 0} \cup \{\infty\}$ denote the slope of the line $\mathbb{R}x$, and for $X \subset \mathbb{R}^2_{\ge 0}$ we set

$$\text{slope}(X) := \{\text{slope}(x) \mid x \in X^\bullet\}.$$

Example 6.6.10. Construct a sequence $\{v_n\}$ of vectors in $\mathbb{N}^\bullet \times \mathbb{N}^\bullet$ as follows. Set $v_1 = (1, 1)$ and suppose that, for $n \in \mathbb{N}$, we have chosen vectors $v_i = (x_i, y_i) \in \mathbb{N}^\bullet \times \mathbb{N}^\bullet$ such that $\text{slope}(v_i) < \text{slope}(v_{i+1})$ and $i||v_i|| < ||v_{i+1}||$ for $i \in [[1, n-1]]$. Let $v_{n+1} = (x_{n+1}, y_{n+1}) \in \mathbb{N}^2$ such that $x_{n+1} > 0$, $\text{slope}(v_{n+1}) > \text{slope}(v_n)$, and $||v_{n+1}|| > n||v_n||$. Now consider the submonoid $M := \langle v_n \mid n \in \mathbb{N}^\bullet \rangle$ of $(\mathbb{N}^2, +)$. Clearly, $\mathcal{A}(M) \subseteq \{v_n \mid n \in \mathbb{N}\}$. On the other hand, the fact that $||v_m|| > ||v_n||$ when $m > n$ implies that only atoms in $\{v_1, \dots, v_{n-1}\}$ can divide v_n in M. This, along with the fact that

$$\text{slope}(v_n) > \max\big\{\text{slope}(v_i) \mid i \in [[1, n-1]]\big\}$$

for every $n \in \mathbb{N}$, ensures that

$$\mathcal{A}(M) = \{v_n \mid n \in \mathbb{N}\}.$$

Finally, let us verify that M is not weakly finitary. Assume for a contradiction that there exist $n \in \mathbb{N}$ and a finite subset S of M such that $nx \in S + M$ for all $x \in M^\bullet$. We can assume without loss of generality that $S \subseteq \mathcal{A}(M)$, so we let $S = \{v_{n_1}, \dots, v_{n_k}\}$, where $n_1 < \cdots < n_k$. Take $N > \max\{n, n_k\}$. Then write $n'v_N = v_{n_i} + m$ for some $i \in [[1, k]]$ and $m \in M$ such that $n' \le n$ and $v_N \nmid_M m$. Since $\text{slope}(n'v_N) > \text{slope}(v_{n_i})$, there exists $j > N$ such that $v_j \mid_M m$. Therefore

$$||Nv_N|| > ||n'v_N|| = ||v_{n_i} + m|| > ||m|| \ge ||v_j|| \ge ||v_{N+1}||,$$

which is a contradiction. Hence M is not weakly finitary.

Strongly Primary Monoids

We conclude this section with a few words about strongly primary monoids in $\mathcal{C}$. A monoid is called *strongly primary* if it is simultaneously primary and finitary. The class of strongly primary monoids contains that one of finitely primary monoids [79, Theorem 2.9.2]. Let M be a monoid. For $x \in M^\bullet$ the smallest $n \in \mathbb{N}$ satisfying that $nM^\bullet \subseteq x + M$ is denoted by $\mathcal{M}(x)$. When such n does not exist, we set $\mathcal{M}(x) = \infty$. If M is strongly primary, then $\mathcal{M}(x) < \infty$ for all $x \in M^\bullet$ [79, Lemma 2.7.7]. In addition, set

$$\mathcal{M}(M) := \sup\{\mathcal{M}(a) \mid a \in \mathcal{A}(M)\} \subseteq \mathbb{N} \cup \{\infty\}.$$

Example 6.6.11. Consider the monoid

$$M := \{(0,0)\} \cup \{(x,y) \in \mathbb{N}^2 \mid x, y > 0\}.$$

It is clear that

$$A := \{(a,b) \in M \mid a = 1 \quad \text{or} \quad b = 1\} \subseteq \mathcal{A}(M)$$

On the other hand, if $(x,y) \in M^\bullet \setminus A$, then $x, y \ge 2$ and, therefore,

$$(x,y) = (1,1) + (x-1, y-1) \in M^\bullet + M^\bullet.$$

Hence $\mathcal{A}(M) = A$. In addition, the fact that $(1,1) \mid_M (x,y)$ for all $(x,y) \in M^\bullet \setminus A$ implies that $\mathcal{M}((1,1)) = 2$. The inclusion $2M^\bullet \subseteq (1,1) + M$ implies that M is a finitary monoid. On the other hand, $\mathsf{cone}(M)^\bullet$ is the open first quadrant, which implies via Proposition 6.6.1

that M is a primary monoid. As a result, M is strongly primary. Now fix $n \in \mathbb{Z}_{\geq 2}$. Note that if $(n,1) \mid_M m(1,1)$ for some $m \in \mathbb{N}$, then $m \geq n+1$. Thus, $\mathcal{M}((n,1)) \geq n+1$. On the other hand, if $(x,y) \in (n+1)M^\bullet$, then $x \geq n+1$ and $y \geq 2$, which implies that $(x,y)-(n,1) \in M$. As a result, $\mathcal{M}((n,1)) = n+1$ and, by a similar argument, $\mathcal{M}((1,n)) = n+1$. Hence $\mathcal{M}((a,b)) = a+b$ for every $(a,b) \in A$ and, in particular, $\mathcal{M}(M) = \infty$.

Unlike the computations in Example 6.6.11, an explicit computation of $\{\mathcal{M}(a) \mid a \in \mathcal{M}\}$ for a monoid M in $\mathcal{C}$ can be hard to carry out. However, for most monoids M in $\mathcal{C}$ one can argue that $\mathcal{M}(M) = \infty$ without performing such computations.

Proposition 6.6.12. *Let M be a strongly primary monoid in $\mathcal{C}$. Then the following conditions are equivalent.*

(1) $\mathcal{M}(M) < \infty$.

(2) $\dim \mathsf{cone}(M) = 1$.

(3) *M is isomorphic to a numerical monoid.*

Proof. Conditions (2) and (3) are obviously equivalent. Therefore it suffices to verify that (1) and (2) are equivalent. To argue that (1) implies (2) suppose, by way of contradiction, that $\dim \mathsf{cone}(M) \neq 1$. Since M is strongly primary $M^\bullet$ is not empty and, thus, $\dim \mathsf{cone}(M) \geq 2$. As M is primary, $\mathsf{cone}(M)$ is open by Proposition 6.6.1. Therefore M cannot be finitely generated, which means that $|\mathcal{A}(M)| = \infty$. Since

$$\{a \in \mathcal{A}(M) \mid ||a|| < n\}$$

is a finite set for every $n \in \mathbb{N}$, there exists a sequence $\{a_n\}$ of atoms of M satisfying that $\lim_{n\to\infty} ||a_n|| = \infty$. Now fix $x \in M^\bullet$. Because $\mathcal{M}(a_n)x = a_n + b$ for some $b \in M$, we have that

$$\lim_{n\to\infty} \mathcal{M}(a_n) = \lim_{n\to\infty} \frac{||a_n + b||}{||x||} \geq \frac{1}{||x||} \lim_{n\to\infty} ||a_n|| = \infty.$$

Hence $\mathcal{M}(M) = \infty$, which is a contradiction. For the reverse implication, suppose that $\dim \mathsf{cone}(M) = 1$. In this case, M is isomorphic to a numerical monoid. Since numerical monoids are finitely generated, $\mathcal{M}(M) < \infty$, and the proof follows. $\square$

Chapter 7

Two Factorization Invariants

7.1 Introduction

In 1960 L. Carlitz proved that a ring of integers is an HFD (i.e., a half-factorial domain) if and only if the size of its class group is at most 2 [39]. The phenomenon of non-unique factorizations of many other families of integral domains has been systematically studied since then (see [9, 11] and references therein). The study of the non-uniqueness of factorizations on commutative cancellative monoids has also earned significant attention during the last few decades (see [44, 49, 83]). This is mainly because several factorization properties of an integral domain R are purely multiplicative in nature and, therefore, can be completely understood by studying only its multiplicative monoid $R \setminus \{0\}$. To measure how far an integral domain or a commutative cancellative monoid is from being a UFD (or an HFD), many algebraic and arithmetic invariants have proved to be useful. Such invariants include the class group (of a Krull domain/monoid) [39], the system of sets of lengths [75], the elasticity [9], and the set of distances [44]. In this chapter we investigate the phenomenon of non-unique factorizations of monoids in $\mathcal{C}$ by using two important non-factoriality measures: the system of sets of lengths and the elasticity. The interested reader can find further factorization invariants as well as the role they play in non-unique factorization theory in the survey [78]. See [43] for a friendly insight to non-unique factorization theory through a number theoretical perspective.

The system of sets of lengths $\mathcal{L}(M)$ of an atomic monoid M encodes significant information about the arithmetic of factorizations in M. As a result, the system of sets of lengths is perhaps the most investigated factorization invariant in the context of atomic monoids. In particular, the search for examples of families of atomic monoids having extremal systems of sets of lengths has been frequently explored in the recent literature (see [111] and [68]). In the first part of this chapter we exhibit, for every $d \geq 2$, a monoid M_d in $\mathcal{C}_d$ having full system of sets of lengths.

In the 1970's, Narkiewicz posed the question of whether the arithmetic describing the non-uniqueness of factorizations in a Krull domain could be used to characterize its class group (for affirmative answers to this, see [79, Sections 7.1 and 7.2]). In general, the question of whether $\mathcal{L}(M)$ completely determines a monoid M (up to isomorphism) inside a distinguished family of atomic monoids has been previously studied (see [2], [98, Section 4], and [75, Section 6]). In the context of Krull monoids, this question is known as the Characterization Problem, which is still open and being actively investigated. Similar questions have been answered for numerical monoids [2] and for monoids in the class $\mathcal{Q}$ [98]. Here, we argue that, for any $d \geq 2$, the system of sets of lengths does not characterize (up to isomorphism) the monoids in the class $\mathcal{C}_d$.

The concept of elasticity was introduced by R. Valenza [143] in the context of algebraic number theory. The elasticity $\rho(M) \in \mathbb{R}_{\geq 1} \cup \{\infty\}$ of an atomic monoid M measures how far is M from being an HFM; in particular, M is an HFM if and only if $\rho(M) = 1$. Although the elasticity encodes substantially less amount of information than the system of sets of lengths does, the former is, in general, much easier to compute. The elasticity of integral domains and atomic monoids has been considered by many authors (see, for instance, [9], [28], [47], and [101]). In the second part of this chapter, we turn to study the elasticity of monoids in $\mathcal{C}$. In particular, we prove that the elasticity of any monoid in $\mathcal{C}_2$ is either rational or infinite. Then, we show that if the convex cone of $M \in \mathcal{C}$ is a polyhedral cone, then $\rho(M)$ is also rational or infinite. We conclude this chapter exploring how our study of the geometry of monoids in $\mathcal{C}$ reflects on their monoid algebras.

7.2 Sets of Lengths

The System of Sets of Lengths

As in the previous chapters, monoids are assumed to be commutative and cancellative and are written additively, unless we specify otherwise. Also, until Section 8.1 all monoids are tacitly assumed to be reduced. Recall that $\mathcal{C}$ denotes the class of all finite-rank submonoids of any free commutative monoid (up to isomorphism), while $\mathcal{C}_d$ denotes the subclass of monoids in $\mathcal{C}$ of rank d. Given an atomic monoid M and $x \in M$, recall that $\mathsf{Z}(x)$ denotes the set of factorizations of x and $\mathsf{L}(x)$ is the set of lengths of x.

Definition 7.2.1. Let M be a atomic monoid. Then the *system of sets of lengths* of M, denoted by $\mathcal{L}(M)$, is the collection of all the sets of lengths of elements of M, i.e.,

$$\mathcal{L}(M) := \{\mathsf{L}(x) \mid x \in M\}.$$

We say that a BFM M has *full system of sets of lengths* if $\mathcal{L}(M) = \mathbb{P}_{\text{fin}}$, where

$$\mathbb{P}_{\text{fin}} := \big\{\{0\}, \{1\}\big\} \cup \big\{S \subset \mathbb{Z}_{\geq 2} \mid S \text{ is finite}\big\}.$$

Note that $\mathbb{P}_{\text{fin}}$ is the largest (under inclusion) system of sets of lengths a BFM can have. In [75] the interested reader can find a friendly introduction to the system of sets of lengths and the role such invariant plays in factorization theory.

The Elasticity

Another important factorization invariant related to the system of sets of lengths of an atomic monoid M is the elasticity.

Definition 7.2.2. The *elasticity* $\rho(x)$ of an element $x \in M^\bullet$ is defined as

$$\rho(x) := \rho_M(x) := \frac{\sup \mathsf{L}(x)}{\inf \mathsf{L}(x)}.$$

Note that $\rho(x) \in \mathbb{Q}_{\ge 1} \cup \{\infty\}$ for all $x \in M^\bullet$. On the other hand, the *elasticity* of M is defined to be

$$\rho(M) := \sup\{\rho(x) \mid x \in M\}.$$

The *set of elasticities* of M is $\mathcal{R}(M) := \{\rho(x) \mid x \in M^\bullet\}$. We say that M is *fully elastic* provided that $\mathcal{R}(M) = \{q \in \mathbb{Q} \mid 1 \le q \le \rho(M)\}$. The elasticity was first used by R. Valenza [13] as a tool to measure the phenomenon of non-unique factorizations in the context of algebraic number theory. The system of sets of lengths and the elasticity have received a great deal of attention in the literature in recent years; see, for instance, [2, 47, 66, 101, 103].

Numerical monoids, introduced in the previous chapter, plays an important role in our study of the system of sets of lengths of monoids in $\mathcal{C}$. We end this section with the following realization theorem of Geroldinger and Schmid, which will be crucial in the proof of Theorem 7.3.4.

Theorem 7.2.3. *[82, Theorem 3.3] Let L be a finite and nonempty subset of $\mathbb{Z}_{\ge 2}$, and let $f \colon L \to \mathbb{Z}_{\ge 1}$ be a map. Then there exist a numerical monoid M and a squarefree element $x \in M$ such that*

$$\mathsf{L}(x) = L \text{ and } |\mathsf{Z}_k(x)| = f(k) \text{ for every } k \in L,$$

where $\mathsf{Z}_k(x) := \{z \in \mathsf{Z}(x) \mid |z| = k\}$.

7.3 The Systems of Sets of Lengths of Monoids in $\mathcal{C}$

The Kainrath Property

After [111], an atomic monoid is said to satisfy the Kainrath Property provided it is a BFM and it has full system of sets of lengths. In this section we construct, for each $d \ge 2$, a monoid M in $\mathcal{C}_d$ having the Kainrath Property.

As every monoid in $\mathcal{C}$ is a BFM, to show that a monoid M in $\mathcal{C}$ satisfies the Kainrath Property, it suffices to verify that $\mathbb{P}_{\text{fin}} \subseteq \mathcal{L}(M)$. Before proceeding with our main result, let us exhibit some examples of families of atomic monoids and domains that have recently been proved to satisfy the Kainrath Property.

The first class of atomic monoids satisfying the Kainrath Property was given by Kainrath [111] in the context of Krull monoids. A monoid K is called a *Krull monoid* if there exists a monoid homomorphism $\phi\colon K \to D$, where D is a free commutative monoid, satisfying the next two conditions:

(1) if $a, b \in K$ and $\phi(a) \mid_D \phi(b)$, then $a \mid_K b$;

(2) for every $d \in D$ there exist $a_1, \dots, a_n \in K$ with $d = \gcd\{\phi(a_1), \dots, \phi(a_n)\}$.

The basis elements of D are called the *prime divisors* of K, and the group $\mathrm{Cl}(K) := D/\phi(K)$ is called the *class group* of K. As Krull monoids are isomorphic to submonoids of free commutative monoids, Krull monoids are atomic (see [79, Section 2.3] for further details about Krull monoids).

Theorem 7.3.1. *[111, Theorem 1] Let M be a Krull monoid with infinite class group in which every divisor class contains a prime divisor. For a finite subset L of $\mathbb{Z}_{\ge 2}$ there exists some $x \in M$ such that $\mathsf{L}(x) = L$.*

In the same direction, Frisch has proved that the multiplicative monoid of the domain of integer-valued polynomials $\mathrm{Int}(\mathbb{Z})$ also satisfies the Kainrath Property (see [67]). This result was recently generalized in [68] to the domain $\mathrm{Int}(\mathcal{O}_K)$ of polynomials over a given number field K stabilizing the ring of integers $\mathcal{O}_K$.

Theorem 7.3.2. *[68, Theorem 1] Let K be a number field with ring of integers $\mathcal{O}_K$. Moreover, let $1 \le m_1 \le \cdots \le m_n$ be natural numbers. Then there exists a polynomial in $\mathrm{Int}(\mathcal{O}_K)$ with n essentially different factorizations into irreducible polynomials in $\mathrm{Int}(\mathcal{O}_K)$ where the lengths of these factorizations are $m_1 + 1, \dots, m_n + 1$.*

Let $\mathcal{Q}$ denote the class consisting of all submonoids of $(\mathbb{Q}_{\ge 0}, +)$. Although monoids in $\mathcal{Q}$ are natural generalizations of numerical monoids, the former are not necessarily finitely generated or atomic. Moreover, if an atomic monoid M of $\mathcal{Q}$ is not isomorphic to a numerical monoid, then $|\mathcal{A}(M)| = \infty$. The atomic structure and factorization theory of monoids in $\mathcal{Q}$ and their monoid algebras have only been studied recently (see [94, 99, 92, 42, 56]). In particular, it has been proved the existence of monoids in $\mathcal{Q}$ satisfying the Kainrath Property.

Theorem 7.3.3. *[98, Theorem 3.6] There exists a monoid in $\mathcal{Q}$ satisfying the Kainrath Property.*

It was proved in [81] that for d large enough there exists a primary monoid in $\mathcal{C}_d$ satisfying the Kainrath Property. Now we exhibit a primary monoid in $\mathcal{C}_2$ satisfying the Kainrath Property. Then we use such a monoid, to construct, for every $d \in \mathbb{Z}_{\ge 2}$, a monoid in $\mathcal{C}_d$ satisfying the Kainrath Property.

Theorem 7.3.4. *There exists a primary monoid in $\mathcal{C}_2$ satisfying the Kainrath Property.*

Proof. As $\mathbb{P}_{\text{fin}}$ is a countable collection, we can list its members. Let $S_1, S_2, \dots$ be an enumeration of the members of $\mathbb{P}_{\text{fin}}$. Fix $\ell, L \in \mathbb{Q}_{>0}$ such that $\ell < L$. Now take a sequence $\{a_n\}$ of elements in $\mathbb{N}^2$ such that the sequence $\{\text{slope}(a_{2n-1})\}$ strictly decreases to ℓ and the sequence $\{\text{slope}(a_{2n})\}$ strictly increases to L. In addition, assume that

$$\max\{\text{slope}(a_{2n-1}) \mid n \in \mathbb{N}\} < \min\{\text{slope}(a_{2n}) \mid n \in \mathbb{N}\}.$$

Now for every $n \in \mathbb{N}$, we use Theorem 7.2.3 to obtain an additive submonoid M_n of $\mathbb{N}a_n$ and an element $x_n \in M_n$ such that $\mathsf{L}_{M_n}(x_n) = S_n$ (note that $\mathbb{N}a_n$ contains an isomorphic copy of every numerical monoid). After rescaling each $M_1, M_2, \dots$ (in this order) one can guarantee that

$$\min\{||a|| \mid a \in \mathcal{A}(M_{n+1})\} > \max\big\{||x_n||, \max\{||a|| \mid a \in \mathcal{A}(M_n)\}\big\} \tag{7.1}$$

for every $n \in \mathbb{N}$. Take M to be the smallest additive submonoid of $\mathbb{N}^2$ containing every monoid M_n. Clearly, M is generated by the set

$$A := \cup_{n\in\mathbb{N}}\mathcal{A}(M_n).$$

Let us prove that each S_n is contained in $\mathcal{L}(M)$.

Claim : If $a \in \mathcal{A}(M)$ divides x_n in M, then $\text{slope}(a) = \text{slope}(x_n)$.

Proof of Claim : Suppose, by way of contradiction, that there exist $n \in \mathbb{N}$ and $a \in \mathcal{A}(M)$ such that $a \mid_M x_n$ and $\text{slope}(a) \neq \text{slope}(x_n)$. Assume first that n is even, say $n = 2k$. Since $a \mid_M x_{2k}$, there exists $b \in M$ such that $a + b = x_{2k}$. Because a and b are vectors located in the interior of the first quadrant, x_{2k} is the longest diagonal of the lattice parallelogram determined by the vectors a and b. Hence $||a|| < ||x_{2k}||$. Observe that if $\text{slope}(a) < \text{slope}(x_{2k})$, then we can take $a' \in \mathcal{A}(M)$ satisfying that $a' \mid_M x_{2k}$ and $\text{slope}(a') > \text{slope}(x_{2k})$. Then we can assume without loss of generality that $\text{slope}(a) > \text{slope}(x_{2k})$. This, along with the fact that x_{2k} has an even index, ensures that $a \in \mathcal{A}(M_m)$ for some index $m > 2k$. Now the inequality (7.1) guarantees that $||a|| > ||x_{2k}||$, which contradicts the already-established inequality $||a|| < ||x_{2k}||$. Hence every atom a of M dividing x_{2k} in M must satisfy that $\text{slope}(a) = \text{slope}(x_{2k})$. The case when n is odd can be argued similarly. Thus, the claim follows.

As a direct consequence of the above claim, $\mathsf{L}_M(x_n) = \mathsf{L}_{M_n}(x_n) = S_n$ for every $n \in \mathbb{N}$. Hence $\mathbb{P}_{\text{fin}} \subseteq \mathcal{L}(M)$, and so M satisfies the Kainrath Property. Finally, notice that

$$\mathsf{cone}(M)^\bullet := \{x \in \mathbb{Q}^2 \mid x \neq 0 \;\text{ and }\; \ell < \text{slope}(x) < L\},$$

which is an open subset of $\mathbb{Q}^2$. Thus, it follows by Proposition 6.6.1 that M is primary, which concludes the proof. □

The reader might have noticed that the argument we presented in the proof of Theorem 7.3.4 can be simplified by using only one limit slope instead of two. We record this parallel result in the next proposition for future reference. However, it is not hard to see that the monoid resulting from using only one slope does not have the extra desirable property of being primary.

Proposition 7.3.5. *There exists a monoid M in $\mathcal{C}_2$ satisfying the Kainrath Property such that* slope(M) *has only one limit point.*

Proof. It is left to the reader as it follows the same argument as the proof of Theorem 7.3.4. □

As every submonoid of $\mathbb{N}$ is isomorphic to a numerical monoid, and the elasticity of a numerical monoid is finite [45, Theorem 2.1], no monoid in $\mathcal{C}_1$ can satisfy the Kainrath Property. However, Theorem 7.3.4 can be used to construct, for each $d \geq 2$, a maximal-rank submonoid of $\mathbb{N}^d$ satisfying the Kainrath Property.

Corollary 7.3.6. *For every $d \geq 2$, there exists a monoid in $\mathcal{C}_d$ satisfying the Kainrath Property.*

Proof. The case $d = 2$ is Theorem 7.3.4. Suppose, therefore, that $d \geq 3$. By Theorem 7.3.4 there exists a submonoid M' of $\mathbb{N}^d$ with $\text{rank}(M') = 2$ such that $p_i(M') = \{0\}$ for every $i \in [[3, d]]$. Take vectors $v_3, \dots, v_d \in \mathbb{N}^d$ such that the rank of the submonoid

$$M := \langle M' \cup \{v_3, \dots, v_d\} \rangle$$

of $\mathbb{N}^d$ is d. Since $v_i \notin M'$ for each $i \in [[3, d]]$, it follows that M' is a divisor-closed submonoid of M. Therefore $\mathsf{L}_{M'}(x) = \mathsf{L}_M(x)$ for all $x \in M'$. As a result, $\mathcal{L}(M') = \mathbb{P}_{\text{fin}}$ implies that $\mathcal{L}(M) = \mathbb{P}_{\text{fin}}$. Thus, M satisfies the Kainrath Property. □

We would like to remark that the submonoid M of $\mathbb{N}^d$ (for $d \geq 3$) constructed in Corollary 7.3.6 is not primary. Notice, for instance, that

$$M \cap \{x \in \mathbb{Q}^d \mid p_d(x) = 0\}$$

is a nonempty proper divisor-closed submonoid of M. However, the reader is invited to prove the following conjecture, which we believe to be true.

Conjecture 7.3.7. *For every dimension $d \geq 3$, there exists a primary monoid in $\mathcal{C}_d$ satisfying the Kainrath Property.*

The Characterization Problem

We conclude this section answering the characterization problem for sets of lengths in each class $\mathcal{C}_d$. Let $\phi \colon M \to M'$ be a monoid isomorphism, where M and M' are monoids in $\mathcal{C}$. Then ϕ extends to a group isomorphism $\text{gp}(M) \to \text{gp}(M')$. In particular, if two monoids in $\mathcal{C}$ are isomorphic, they have the same rank. Therefore Corollary 7.3.6 immediately implies that the system of sets of lengths does not characterize monoids in $\mathcal{C}_{\geq 2}$. On the other hand, it was proved in [2] that the system of sets of lengths does not characterize monoids in $\mathcal{C}_1$. Now we extend these two observations.

Proposition 7.3.8. *For any $d \geq 2$, the system of sets of lengths does not characterize monoids inside the class $\mathcal{C}_d$.*

Proof. First, suppose that $d = 2$. Take $M, M' \in \mathcal{C}_d$, and let $\phi \colon M \to M'$ be a monoid isomorphism. Then ϕ extends to a group isomorphism $\text{gp}(M) \to \text{gp}(M')$ and, since $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, ϕ also extends to an isomorphism $\bar{\phi} \colon \mathbb{Q} \otimes_{\mathbb{Z}} \text{gp}(M) \to \mathbb{Q} \otimes_{\mathbb{Z}} \text{gp}(M')$ of $\mathbb{Q}$-spaces. As $\bar{\phi}$ is a linear transformation, it must be continuous. Thus, if the monoids M and M' are isomorphic, then $\text{slope}(\mathcal{A}(M))$ and $\text{slope}(\mathcal{A}(M'))$ have the same number of limit points. Now if M and M' are the monoids in $\mathcal{C}_2$ constructed in Proposition 7.3.5 and in the proof of Theorem 7.3.4, respectively, then $\text{slope}(\mathcal{A}(M)) \subset \mathbb{R}$ has one limit point and $\text{slope}(\mathcal{A}(M')) \subset \mathbb{R}$ has two limit points. Hence $\mathcal{L}(M) = \mathbb{P}_{\text{fin}} = \mathcal{L}(M')$, but M and M' are not isomorphic.

Suppose, on the other hand, that $d > 2$. Notice that the monoid M of $\mathcal{C}_d$ constructed in Corollary 7.3.6 satisfies that $\mathsf{cone}(M)$ is polyhedral. This is because there exists one supporting plane of $\mathsf{cone}(M)$ containing all but finitely many elements of $\mathcal{A}(M)$. Slightly modifying the proof of Corollary 7.3.6, we can construct a monoid M' in $\mathcal{C}_d$ with one of its 1-dimensional extreme rays containing two atoms. As an isomorphism $\phi \colon M \to M'$ would send atoms to atoms and its $\mathbb{Q}$-linear extension $\bar{\phi} \colon \mathbb{Q} \otimes_{\mathbb{Z}} \text{gp}(M) \to \mathbb{Q} \otimes_{\mathbb{Z}} \text{gp}(M')$ would send 1-dimensional faces of $\mathsf{cone}(M)$ to 1-dimensional faces of $\mathsf{cone}(M')$, such isomorphism ϕ cannot exist. Hence M and M' are not isomorphic monoids even though $\mathcal{L}(M) = \mathbb{P}_{\text{fin}} = \mathcal{L}(M')$. □

7.4 Rationality of the Elasticity of Monoids in $\mathcal{C}$

The Rational-Infinite Elasticity Problem

We now turn our attention to the elasticity of monoids in $\mathcal{C}$. In [141], S. Tringali posed the following question.

Question 7.4.1. *Is always the elasticity of a submonoid of a free commutative monoid of finite rank either rational or infinite?*

Clearly, the submonoids of free commutative monoids of finite rank are precisely those in $\mathcal{C}$. In Theorem 7.4.7 and Theorem 7.4.8, we shall provide two positive partial answers to Question 7.4.1.

Before delving into the actual question, let us extend the notion of the set of lengths and the elasticity for submonoids M of $\mathbb{N}$ so they are both defined in terms of any fixed finite generating set of M (not necessarily $\mathcal{A}(M)$). Similar generalizations of other arithmetic invariants have been useful in the past to study aspects of the non-unique factorization theory of certain classes of monoids, including arithmetical congruence monoids [26]. In particular, the generalized set of lengths was first used in the context of numerical monoids in [48], where a similar relaxation of the set of distances was studied.

Definition 7.4.2. Let $k \in \mathbb{N}$ and $n_1, \dots, n_k \in \mathbb{N}^\bullet$. Then for any nonzero $x \in \langle n_1, \dots, n_k \rangle$, we define the *generalized set of lengths* of x with respect to the distinguished generators $n_1, \dots, n_k$ to be

$$\mathsf{L}_g(x) = \left\{ c_1 + \cdots + c_k \;\middle|\; c_1, \dots, c_k \in \mathbb{N} \text{ and } \sum_{i=1}^{k} c_i n_i = x \right\} \subset \mathbb{N}^\bullet.$$

Similarly, the *generalized elasticity* of x with respect to $n_1, \dots, n_k$ is defined to be

$$\rho_g(x) = \frac{\max \mathsf{L}_g(x)}{\min \mathsf{L}_g(x)}.$$

Lemma 7.4.3. *For $k \in \mathbb{Z}_{\geq 2}$, take $n_1, \dots, n_k \in \mathbb{N}$ such that $n_1 < \cdots < n_k$. If the set*

$$\left\{ \rho_g(x) \mid x \in \langle n_1, \dots, n_k \rangle^\bullet \right\}$$

has a limit point, then it must be n_k/n_1.

Proof. Set $M = \langle n_1, \dots, n_k \rangle$. It is not hard to see that we can take $N \in \mathbb{N}$ large enough such that for all $x \in M^\bullet$ there exists $r_x \in [[1, N]] \cap M$ satisfying that $x = r_x + m_x n_1 n_k$ for some $m_x \in \mathbb{N}$. Now fix $x_0 \in M$ with $x_0 > N$, and write $x_0 = r + m n_1 n_k$ for $r \in [[1, N]] \cap M$ and $m \in \mathbb{N}$. As $x_0 > N$, it follows that $m \geq 1$. Therefore any formal sum of copies of $n_1, \dots, n_k$ adding to x_0 and maximizing the number of distinguished generators (counting repetitions) must contain at least $m n_k$ copies of n_1 and so

$$\max \mathsf{L}_g(x_0) = \max \mathsf{L}_g(r + m n_1 n_k) = \max \mathsf{L}_g(r) + m n_k. \tag{7.2}$$

Similarly, any formal sum of copies of $n_1, \dots, n_k$ adding to x_0 and minimizing the number of distinguished generators must contain at least $m n_1$ copies of n_k and so

$$\min \mathsf{L}_g(x_0) = \min \mathsf{L}_g(r + m n_1 n_k) = \min \mathsf{L}_g(r) + m n_1. \tag{7.3}$$

Using (7.2) and (7.3), we obtain that

$$\rho_g(x_0) = \frac{\max \mathsf{L}_g(r) + m n_k}{\min \mathsf{L}_g(r) + m n_1}.$$

As a result,

$$\left\{ \rho_g(x) \mid x \in M \setminus [[1, N]] \right\} \subseteq \left\{ \frac{n_k + \frac{1}{m} \max \mathsf{L}_g(r)}{n_1 + \frac{1}{m} \min \mathsf{L}_g(r)} \;\middle|\; r \in [[1, N]] \cap M \text{ and } m \in \mathbb{Z}_{\geq 1} \right\}.$$

From the above inclusion of sets, it immediately follows that $\left\{ \rho_g(x) \mid x \in M^\bullet \right\}$ can have at most one limit point, namely n_k/n_1. ∎

Remark 7.4.4. Lemma 7.4.3 is essentially a generalization of [45, Corollary 2.3], which states that the only limit point of the set of elasticities of a numerical monoid minimally generated by the elements $a_1 < a_2 < \cdots < a_k$ (for $k \geq 2$) is a_k/a_1.

Remark 7.4.5. Another result similar to Lemma 7.4.3 was previously established in [28, Corollary 4.5]. We decided to reprove it here not only for the sake of completeness, but also because we need to work in a more general context, meaning that our distinguished set of generators $\{n_1, \dots, n_k\}$ is not necessarily minimal, and $n_1, \dots, n_k$ are not necessarily relatively prime.

The Case of Dimension Two

For a nonzero vector $a \in \mathbb{R}^d$, we let $\mathsf{p}_a \colon \mathbb{R}^d \to \mathbb{R}a$ be the linear transformation that projects a vector of $\mathbb{R}^d$ onto the 1-dimensional space $\mathbb{R}a$. Also, for each $j \in [[1, d]]$, we let $p_j(x)$ denote the j-th component of x. In particular, for nonzero vectors $a, b \in \mathbb{N}^2$, we have that

$$\frac{\langle a, b\rangle}{||a||} = \frac{p_1(a)p_1(b) + p_2(a)p_2(b)}{||a||}$$

is the Fourier coefficient of b with respect to the unit vector $a/||a||$. Therefore the projection of b on a is given by

$$\mathsf{p}_a(b) = \frac{p_1(a)p_1(b) + p_2(a)p_2(b)}{||a||^2}\, a. \tag{7.4}$$

Lemma 7.4.6. *Let $a, x, y \in \mathbb{N}^2$ such that slope$(x) <$ slope$(a) <$ slope(y). Also, let α be the acute angle between x and a, and let β be the acute angle between a and y. Then the following identity holds:*

$$\big(||a||\,||y||\sin\beta\big)x + \big(||a||\,||x||\sin\alpha\big)y = ||x||\,||y||\sin(\alpha+\beta)a.$$

Moreover, the coefficients of x, y, and a in the above identity are nonnegative integers.

Proof. Set $a^\perp := (-p_2(a), p_1(a))$, and note that

$$\mathsf{p}_{a^\perp}(x) = -(||x||\sin\alpha)\, a^\perp/||a^\perp||$$

and

$$\mathsf{p}_{a^\perp}(y) = (||y||\sin\beta)\, a^\perp/||a^\perp||.$$

Taking

$$z = (||y||\sin\beta)x + (||x||\sin\alpha)y,$$

we obtain that $\mathsf{p}_{a^\perp}(z) = 0$, which implies that z and a are colinear, i.e., $\mathsf{p}_a(z) = z$. Since

$$\mathsf{p}_a(x) = (||x||\cos\alpha)a/||a|| \quad \text{and} \quad \mathsf{p}_a(y) = (||y||\cos\beta)a/||a||,$$

it follows that

$$\begin{aligned} z = \mathsf{p}_a(z) &= (||y|| \sin\beta)\mathsf{p}_a(x) + (||x|| \sin\alpha)\mathsf{p}_a(y) \\ &= ||x||\, ||y|| \big(\sin\alpha\cos\beta + \sin\beta\cos\alpha\big)\frac{a}{||a||} \\ &= ||x||\, ||y|| \sin(\alpha+\beta)\frac{a}{||a||}. \end{aligned}$$

Hence $||a||z = ||x||\,||y|| \sin(\alpha+\beta)a$, which is the desired trigonometric identity. Finally, observe that the coefficients of a, x, and y represent areas of lattice parallelograms. Hence such coefficients must be nonnegative integers. □

We are now in a position to prove that the elasticity of each monoid in $\mathcal{C}_2$ is either rational or infinite.

Theorem 7.4.7. *Let M be a monoid in $\mathcal{C}_2$. Then $\rho(M)$ is either rational or infinite.*

Proof. If M is finitely generated, then it follows by [8, Theorem 7] that $\rho(M)$ is rational. So we assume that M is not finitely generated. Note that for every $v \in \mathbb{N}^2$, the submonoid $\mathbb{N}v \cap M$ of M is isomorphic to an additive submonoid of $\mathbb{N}$ and is, therefore, finitely generated. This, along with the fact that $|\mathcal{A}(M)| = \infty$, implies that the set $\text{slope}(\mathcal{A}(M))$ must have at least one limit point (maybe ∞). By reflecting M with respect to the line $y = x$ if necessary, we can assume that $\text{slope}(\mathcal{A}(M))$ has a finite limit point.

CASE 1. The set $\text{slope}(\mathcal{A}(M))$ has at least two limit points. In this case, we will argue that M has infinite elasticity. To do so, take $N \in \mathbb{N}$.

CASE 1.1. There exists $a \in \mathcal{A}(M)$ such that the vector $\text{slope}(a)$ is strictly between two limit points of $\text{slope}(M)$. Then the sets

$$X := \{x \in \mathcal{A}(M) \mid \text{slope}(x) < \text{slope}(a)\}$$

and

$$Y := \{y \in \mathcal{A}(M) \mid \text{slope}(y) > \text{slope}(a)\}$$

are both infinite. So there exist atoms $x \in X$ and $y \in Y$ such that $\min\{||x||, ||y||\} \geq 2N||a||$. By Lemma 7.4.6, one has that

$$\big(||a||\,||y|| \sin\beta\big)x + \big(||a||\,||x|| \sin\alpha\big)y = ||x||\,||y|| \sin(\alpha+\beta)a, \tag{7.5}$$

where α is the acute angle between x and a, and β is the acute angle between a and y. Using the fact that $\min\{||x||, ||y||\} \geq 2N||a||$, we obtain

$$||x||\,||y|| \sin(\alpha+\beta) > N(||a||\,||y|| \sin\beta + ||a||\,||x|| \sin\alpha). \tag{7.6}$$

Because the coefficients of x, y, and a in the identity (7.5) are positive integers, the element $h_0 := ||x||\,||y||\sin(\alpha+\beta)a$ belongs to M. Moreover, applying the inequality (7.6), one obtains that

$$\rho(M) \ge \rho(h_0) \ge \frac{||x||\,||y||\sin(\alpha+\beta)}{||a||\,||y||\sin\beta + ||a||\,||x||\sin\alpha} \ge N.$$

Hence $\rho(M) = \infty$.

CASE 1.2. There is no $a \in \mathcal{A}(M)$ such that slope(a) is strictly between two limit points. Observe that, in this case, slope(M) contains exactly two limit points. As a consequence, one can choose $x, y, a \in \mathcal{A}(M)$ such that slope$(x) <$ slope$(a) <$ slope(y). In addition, note that we can assume that $||a||$ is large enough that the inequalities $||a||\sin\beta > N||x||/2$ and $|a|\sin\alpha > N||y||/2$ hold, where the angles α and β are defined as in the CASE 1.1. By Lemma 7.4.6, we again obtain the identity (7.5). Because the coefficients of x, y, and a in (7.5) are positive integers, $h_1 = 2||x||\,||y||\sin(\alpha+\beta)a$ belongs to M. Using the inequalities $||a||\sin\beta > N||x||/2$ and $||a||\sin\alpha > N||y||/2$, we get

$$\rho(M) \ge \rho(h_1) \ge \frac{||a||\,||y||\sin\beta + ||a||\,||x||\sin\alpha}{||x||\,||y||\sin(\alpha+\beta)} > N.$$

Thus, in this case, $\rho(M) = \infty$.

CASE 2. The set slope$(\mathcal{A}(M))$ contains only one limit point. Let ℓ be the limit point of slope$(\mathcal{A}(M))$. Now consider the set $X_\ell := \{x \in \mathcal{A}(M) \mid \text{slope}(x) < \ell\}$ and the set $Y_\ell := \{y \in \mathcal{A}(M) \mid \text{slope}(y) > \ell\}$.

CASE 2.1. The sets X_ℓ and Y_ℓ are both nonempty. Fix $N \in \mathbb{N}$. Take $x \in X_\ell$ and $y \in Y_\ell$. Since ℓ is a limit point of slope(M), we can choose $a \in \mathcal{A}(M)$ satisfying that slope$(x) <$ slope$(a) <$ slope(y) and large enough such that $||a||\sin\beta > N||x||/2$ and $||a||\sin\alpha > N||y||/2$. Proceeding exactly as we did in CASE 1.2, we can conclude that $\rho(M) > N$. Hence $\rho(M) = \infty$ in this case again.

CASE 2.2. One of the sets X_ℓ and Y_ℓ is empty. Assume, without loss of generality, that X_ℓ is not empty. Take a nonzero vector $v \in \mathbb{R}^2_{\ge 0}$ such that slope$(v) = \ell$, and set

$$v^\perp = (-p_2(v), p_1(v))/||(-p_2(v), p_1(v))||.$$

Now define the set

$$S_\ell = \{||\mathsf{p}_{v^\perp}(a)|| : a \in \mathcal{A}(M)\}.$$

CASE 2.2.1. The set S_ℓ is not finite. Fix $x \in X_\ell$. As S_ℓ is infinite and the Fourier coefficient of each vector in X_ℓ with respect to the normal vector $v^\perp$ is an integer, there exists $a \in X_\ell$ such that $||\mathsf{p}_{v^\perp}(a)|| > 2N||x||$. Note that $||\mathsf{p}_{v^\perp}(a)|| = ||a||\sin\beta'$, where β' is the acute angle between a and v. Now take $y \in X_\ell$ such that the acute angle β between y and a is close enough to β' that both inequalities slope$(y) >$ slope(a) and $||a||\sin\beta > N||x||$ hold. Now, we can apply Lemma 7.4.6 to obtain once again the identity (7.5). Since the coefficients of x, y, and a in (7.5) are positive integers, the element $h_2 := 2||x||\,||y||\sin(\alpha+\beta)a$ belongs to M. On the other hand, using the fact that $||a||\sin\beta > N||x||$, one finds that

$$\rho(M) \ge \rho(h_2) \ge \frac{||a||\,||y||\sin\beta + ||a||\,||x||\sin\alpha}{||x||\,||y||\sin(\alpha+\beta)} > \frac{||a||\sin\beta}{||x||\sin(\alpha+\beta)} > N.$$

This allows us to conclude again that $\rho(M) = \infty$.

CASE 2.2.2. The set S_ℓ is finite. Take $M_\ell = \langle s_1, \dots, s_k\rangle$, where $S_\ell = \{s_1, \dots, s_k\}$ and $s_1 < \cdots < s_k$. Among all the atoms of M minimizing the set S_ℓ, let x be the one of largest slope. On the other hand, among all the atoms of M maximizing S_ℓ, let a be the one with largest slope. Fix $\epsilon > 0$.

First, suppose that $\text{slope}(a) < \text{slope}(x)$, and let α be the acute angle between x and a. Now take $y \in \mathcal{A}(M)$, and let β denote the acute angle between a and y. Since

$$\lim_{||y||\to\infty} \frac{||a||\sin\beta}{||x||\sin(\alpha+\beta)} = \frac{||\mathsf{p}_{v^\perp}(a)||}{||\mathsf{p}_{v^\perp}(x)||} = \frac{s_k}{s_1},$$

we can assume that $||y||$ is large enough such that the inequalities $\text{slope}(a) > \text{slope}(y)$ and

$$\frac{||a||\sin\beta}{||x||\sin(\alpha+\beta)} > \frac{s_k}{s_1} - \epsilon \tag{7.7}$$

both hold. Now, Lemma 7.4.6 allows us to use identity (7.5) once again. This, along with the fact that $h_3 := ||x||\,||y||\sin(\alpha+\beta)a \in M$, implies that

$$\rho(M) \ge \rho(h_3) \ge \frac{||a||\sin\beta + \big(||x||/||y||\big)||a||\sin\alpha}{||x||\sin(\alpha+\beta)} > \frac{s_k}{s_1} - \epsilon.$$

Thus, $\rho(M) \ge s_k/s_1 \ge \rho_g(M_\ell)$.

Now suppose that $\text{slope}(a) > \text{slope}(x)$. Let α be defined as before, take $y \in \mathcal{A}(M)$, and let β now denote the acute angle between x and y. Since

$$\lim_{||y||\to\infty} \frac{||a||\sin(\alpha+\beta)}{(||x||/||y||)||a||\sin\alpha + ||x||\sin\beta} = \frac{||\mathsf{p}_{v^\perp}(a)||}{||\mathsf{p}_{v^\perp}(x)||} = \frac{s_k}{s_1}$$

and ℓ is a limit point of $\text{slope}(M)$, we can assume that $||y||$ large enough so that $\text{slope}(x) > \text{slope}(y)$ and

$$\frac{||a||\sin(\alpha+\beta)}{(||x||/||y||)||a||\sin\alpha + ||x||\sin\beta} > \frac{s_k}{s_1} - \epsilon.$$

By Lemma 7.4.6,

$$\big(||x||\,||y||\sin\beta\big)a + \big(||x||\,||a||\sin\alpha\big)y = ||a||\,||y||\sin(\alpha+\beta)x. \tag{7.8}$$

Since the coefficients in (7.8) are positive integers, $h_4 := ||a||\,||y||\sin(\alpha+\beta)x$ is an element of M and, therefore,

$$\rho(M) \ge \rho(h_4) \ge \frac{||a||\sin(\alpha+\beta)}{\big(||x||/||y||\big)||a||\sin\alpha + ||x||\sin\beta} \ge \frac{s_k}{s_1} - \epsilon.$$

As a consequence, $\rho(M) \ge s_k/s_1 \ge \rho_g(M_\ell)$.

Finally, suppose that $\text{slope}(a) = \text{slope}(x)$. In this case, it is not hard to see that the element $h_5 := p_1(a)p_2(a)x \in M$ can also be written in the form $h_5 = p_1(a)p_2(x)a$. Since $a, x \in \mathcal{A}(M)$, it follows that

$$\rho(M) = \rho(h_5) \ge \frac{p_2(a)}{p_2(x)} = \frac{||a||}{||x||} = \frac{s_k}{s_1} \ge \rho_g(M_\ell).$$

Thus, we always have $\rho(M) \ge \rho_g(M_\ell)$. On the other hand, if $a_1 + \cdots + a_n \in \mathsf{Z}_M(w)$ is an n-length factorization of $w \in M$, then $||\mathsf{p}_{v^\perp}(a_1)|| + \cdots + ||\mathsf{p}_{v^\perp}(a_n)||$ is an n-length generalized factorization of $||\mathsf{p}_{v^\perp}(w)||$ in M_ℓ. Therefore $\mathsf{L}_M(w) \subseteq \mathsf{L}_{g(M_\ell)}(||\mathsf{p}_{v^\perp}(w)||)$ for all $w \in M$, where $\mathsf{L}_{g(M_\ell)}(h)$ denotes the generalized set of lengths of h in M_ℓ with respect to the distinguished set of generators $s_1, \dots, s_k$. This implies that $\rho(M) \le \rho_g(M_\ell)$. Hence $\rho(M) = \rho_g(M_\ell)$, which is rational by Lemma 7.4.3. This completes the proof. □

Higher Dimension

We conclude our exposition providing a subclass of $\mathcal{C}_d$ (for $d \ge 3$) whose members have rational or infinite elasticity.

Theorem 7.4.8. *Let M be a monoid in $\mathcal{C}_d$ with $d \ge 3$. If the cone of M in the $\mathbb{Q}$-space $\mathbb{Q} \otimes_{\mathbb{Z}} \text{gp}(M)$ is polyhedral, then $\rho(M)$ is either rational or infinite.*

Proof. Set $d = \text{rank}(M)$. After identifying $\mathbb{Q} \otimes_{\mathbb{Z}} \text{gp}(M)$ with $\mathbb{Q}^d$, we can assume that M is a submonoid of $\mathbb{N}^d$. If M is finitely generated, then we can argue that $\rho(M) \in \mathbb{Q}$ as we did at the beginning of the proof of Theorem 7.4.7. Then there is no loss in assuming that M is not finitely generated, i.e., $|\mathcal{A}(M)| = \infty$.

Fix $N \in \mathbb{N}$. As $\mathsf{cone}(M)$ is polyhedral, it must have finitely many 1-dimensional faces; call them $L_1, \dots, L_n$. Since each L_i is a 1-dimensional face, we can take $a_i \in L_i \cap \mathcal{A}(M)$ for each $i \in [[1, n]]$. Clearly, $\mathsf{cone}(M) = \mathsf{cone}(a_1, \dots, a_n)$. Consider the parallelepiped

$$\Pi := \big\{\alpha_1 a_1 + \cdots + \alpha_n a_n \mid 0 \le \alpha_i \le 1 \text{ for every } i \in [[1, n]]\big\}.$$

Since $\Pi \cap \mathbb{Z}^d$ is finite and

$$\Pi \cap \mathbb{Z}^d \subset \mathbb{Q}_{\ge 0} a_1 + \cdots + \mathbb{Q}_{\ge 0} a_n,$$

we can choose $N_0 \in \mathbb{N}^\bullet$ large enough that $N_0 z \in \mathbb{Z}_{\ge 0} a_1 + \cdots + \mathbb{Z}_{\ge 0} a_n$ for each $z \in \Pi \cap \mathbb{Z}^d$. Notice that for each $n \in \mathbb{N}$, there exist only finitely many atoms of H whose norms are at most n. As a result, there exists $a \in \mathcal{A}(M)$ such that $||a|| > (N+1)(||a_1|| + \cdots + ||a_n||)$.

As we can naturally partition $\mathsf{cone}(M)$ into copies of the parallelepiped Π, we can write $a = v + c_1 a_1 + \cdots + c_n a_n$ for some $v \in \Pi \cap \mathbb{Z}^d$ and nonnegative integer coefficients $c_1, \dots, c_n$.

Since the diameter of Π is $||a_1 + \cdots + a_n||$ and $v \in \Pi$, it follows that $||v|| \leq ||a_1|| + \cdots + ||a_n||$ and, therefore,

$$1 + \sum_{i=1}^{n} c_i \geq \frac{||v||}{\sum_{i=1}^{n} ||a_i||} + \frac{|| \sum_{i=1}^{n} c_i a_i ||}{\sum_{i=1}^{n} ||a_i||} \geq \frac{||a||}{\sum_{i=1}^{n} ||a_i||} > 1 + N.$$

Hence $c_1 + \cdots + c_n > N$. Taking $c'_1, \ldots, c'_n \in \mathbb{Z}_{\geq 0}$ such that $N_0 v = c'_1 a_1 + \cdots + c'_n a_n$, we obtain that

$$N_0 a = N_0 v + \sum_{i=1}^{n} N_0 c_i a_i = \sum_{i=1}^{n} (c'_i + N_0 c_i) a_i.$$

Therefore

$$\rho(M) \geq \rho(N_0 a) \geq \frac{\sum_{i=1}^{n} (c'_i + N_0 c_i)}{N_0} \geq \sum_{i=1}^{n} c_i > N.$$

As N was arbitrarily taken, $\rho(M) = \infty$, which concludes the proof. □

Recall that an atomic monoid M is fully elastic if

$$\mathcal{R}(M) = \{q \in \mathbb{Q} \mid 1 \leq q \leq \rho(M)\}.$$

Each monoid in $\mathcal{C}_1$ fails to be fully elastic (see [45, Theorem 2.2]). However, because every atomic monoid satisfying the Kainrath Property is, obviously, fully elastic, we have the following direct implication of Corollary 7.3.6.

Proposition 7.4.9. *For each $d \geq 2$, there exists a monoid in $\mathcal{C}_d$ that is fully elastic.*

Chapter 8

Atomicity of Monoid Algebras

8.1 Introduction

For a monoid M and an integral domain R, let $R[M]$ denote the algebra consisting of all polynomial expressions with coefficients in R and exponents in M. In [86] R. Gilmer offers a very comprehensive summary of the theory of commutative semigroup rings developed until the 1980's. Many algebraic properties of an integral domain R and a monoid M implying the corresponding property on the algebra $R[M]$ had been studied by that time. In the 1990's there was a flurry of papers investigating whether factorization properties of an integral domain R (including being an atomic domain, being a UFD, HFD, FFD, BFD, and satisfying the ACCP) are inherited by the ring of polynomials $R[X]$, the ring of power series $R[[X]]$, or certain special subrings of $R[X]$ and $R[[X]]$; see, for instance, [3, 5, 4, 10, 31]. We conclude this thesis exploring whether some of the atomic properties of R are inherited by $R[M]$ for all monoids M in the class $\mathcal{C}$.

Factorizations (revisited)

From now on, we no longer assume that monoids are, by default, reduced. This is mainly because in this section we study the multiplicative monoids of the monoid algebras induced by monoids in $\mathcal{C}$, which are not in general reduced. Therefore we need to revise some of the concepts about factorizations introduced in Section 6.2.

8.2 Factorizations and Monoid Algebras

Monoids and Factorizations (revisited)

For a monoid M, we let $M^\times$ denote the set of all invertible elements of M, which we also call *units* of M. Two elements $x, y \in M$ are called *associates* if $y = ux$ for some $u \in M^\times$. It can be readily verified that being associates defines an equivalence relation $\simeq$ on M that is compatible with the monoid operation. Therefore the operation of M induces a well defined operation on the set of classes $M/\simeq$ of $\simeq$, and with this operation $M/\simeq$ becomes a monoid, which is denoted by M_{red}. The monoid M_{red} is reduced, and it is called the *reduced monoid* of M. An element $a \in M$ is called an atom if whenever $a = x + y$ for some $x, y \in M$, then either $x \in M^\times$ or $y \in M^\times$. The set of all atoms of M is denoted by $\mathcal{A}(M)$, and M is called *atomic* if $\mathcal{A}(M)$ generates M. It is easy to check that M is atomic if and only if M_{red} is atomic.

Let M be a monoid. The *factorization monoid* $\mathsf{Z}(M)$ of M is the (multiplicatively-written) free commutative monoid on the set $\mathcal{A}(M_{\text{red}})$, while the *factorization homomorphism* of M is the unique monoid homomorphism $\phi\colon \mathsf{Z}(M) \to M_{\text{red}}$ fixing every atom of M_{red}. For an element $x \in M$, the sets

$$\mathsf{Z}(x) := \phi^{-1}(xM^\times) \subseteq \mathsf{Z}(M) \quad \text{and} \quad \mathsf{L}(x) = \{|z| : z \in \mathsf{Z}(x)\} \subseteq \mathbb{N},$$

are called the *set of factorizations* and the *set of lengths* of x, respectively. The monoid M is called an *HFM* (resp., a *UFM*) provided that $\mathsf{L}(x)$ (resp., $\mathsf{Z}(x)$) is a singleton for every $x \in M \setminus M^\times$. On the other hand, M is said to be a *BFM* (resp., an *FFM*) provided that $\mathsf{L}(x)$ (resp., $\mathsf{Z}(x)$) is a finite set for every $x \in M \setminus M^\times$. Clearly, M is a UFM (resp., an HFM, a BFM, an FFM) if and only if M_{red} is a UFM (resp., an HFM, a BFM, an FFM).

For an integral domain R, we let $R^\bullet$ denote the multiplicative monoid of R. By simplicity, we let $\mathcal{A}(R)$, $\mathsf{Z}(R)$, and ϕ_R denote $\mathcal{A}(R^\bullet)$, $\mathsf{Z}(R^\bullet)$, and $\phi_{R^\bullet}$, respectively. In addition, for a nonzero non-unit $r \in R$, we let $\mathsf{Z}_R(r)$ and $\mathsf{L}_R(r)$ denote $\mathsf{Z}_{R^\bullet}(r)$ and $\mathsf{L}_{R^\bullet}(r)$, respectively.

Monoid Algebras

For an integral domain R and a monoid M, let $R[X; M]$ denote the set consisting of all functions $f\colon M \to R$ satisfying that $\{s \in M \mid f(s) \neq 0\}$ is finite. We shall conveniently represent an element $f \in R[X; M]$ by

$$f = \sum_{s \in M} f(s) X^s = \sum_{i=1}^{n} f(s_i) X^{s_i},$$

where $s_1, \dots, s_n$ are those elements $s \in M$ satisfying that $f(s) \neq 0$. After defining addition and multiplication on $R[X; M]$ as in a ring of polynomials, $R[X; M]$ becomes an integral domain [86, Theorem 8.1] with set of units $R^\times$ [86, Theorem 11.1]. Following Gilmer [86], we will write $R[M]$ instead of $R[X; M]$. The domain $R[M]$ is called the *monoid algebra* of M *over* R, while M is called the *monoid of exponents* of $R[M]$. Monoid algebras have been studied by Gilmer et al. in [85, 87, 88] and more recently in [114]. In addition, numerical monoid algebras have been studied in [12, 36], while affine monoid algebras have been studied in [38, 107]. Finally, monoid algebras with monoids of exponents in the class $\mathcal{Q}$ have been investigated in [7] and more recently in [89, 90, 100]. For background information on monoid algebras see [86].

As we are particularly interested in monoid algebras induced by monoids in the class $\mathcal{C}$, the following notation will make our statements more succinct:

$$\mathcal{C}_A := \{R[M] \mid R \text{ is an integral domain and } M \in \mathcal{C}\}.$$

An integral domain R is called an *atomic domain* (resp., an *ACCP domain*, a *BFD*, an *FFD*, an *HFD*) if its multiplicative monoid is an atomic monoid (resp., an ACCP monoid, a BFM, an FFM, an HFM). Although we have previously introduced this variety of non-factoriality types in the context of monoids, it was indeed in the context of integral domains where most of these concepts were introduced. The definition of an atomic domain is due to Cohn [52], the definitions of a BFD and an FFD are due to Anderson, Anderson, and Zafrullah [3] and, as we have pointed out before, the definition of an HFD is due to Zaks [148]. As the multiplicative monoid of an integral domain may not be reduced, we have to slightly modify the chain of implication (6.2) so it holds in the context of integral domains:

$$\text{UFD} \Rightarrow \big(\text{HFD, FFD}\big) \Rightarrow \text{BFD} \Rightarrow \text{ACCP domain} \Rightarrow \text{atomic domain}. \tag{8.1}$$

Notice that in the chain of implications (8.1), being an HFD does not imply being an FFD. Like in the class of all monoids, in the class of all integral domains each implication in (8.1) is strict. Examples witnessing this observation can be found in [3].

8.3 Algebras in $\mathcal{C}_A$

Classic Types of Integral Domains in $\mathcal{C}_A$

We would like to study the class of algebras $\mathcal{C}_A$ (up to isomorphism) induced by monoids in $\mathcal{C}$. To begin with, let us determine which algebras in $\mathcal{C}_A$ are Dedekind domains, UFDs, PIDs, or Euclidean domains. Gilmer and Parker proved in [87] (see also [86, Theorem 13.8]) that for a torsion-free monoid M and an integral domain R the following statements are equivalent:

(1) $R[M]$ is a Euclidean domain;

(2) $R[M]$ is a PID domain;

(3) $R[M]$ is a Dedekind domain;

(4) R is a field and $M \in \{\mathbb{N}, \mathbb{Z}\}$.

As a result, the only algebras in $\mathcal{C}_A$ which are Dedekind domains, PIDs and/or Euclidean domains are, up to isomorphism, the polynomial rings $F[X]$, where F is a field. On the other hand, Gilmer and Parker proved that $R[M]$ is a UFD if and only if M is a UFM and R is a UFD [86, Theorem 14.16] (when M is a monoid in $\mathcal{C}$). Therefore, as an immediate consequence of the characterization of UFMs in $\mathcal{C}$ given in Proposition 6.5.1, one can deduce that the only UFD monoid algebras induced by monoids in $\mathcal{C}$ are polynomial rings $R[X_1, \dots, X_n]$, where $n \in \mathbb{N}$ and R is a UFD.

HFDs in $\mathcal{C}_A$

Any UFD is clearly an HFD, and a Krull domain is an HFD provided that its class group is $\mathbb{Z}/2\mathbb{Z}$ [39].

Unlike we have seen for UFDs, it is not true in general that $R[M]$ is an HFD when M is in $\mathcal{C}$ and R is an HFD. This is illustrated in Example 8.3.1 below, which has been taken from [12]. Indeed, this is not true even if we take M to be $(\mathbb{N}_0, +)$ (see [5, Example 5.4]): it has been proved by Coykendall in [54] that if R is an integral domain such that the polynomial ring $R[X]$ is an HFD, then R is an HFD and integrally closed. Coykendall has also proved that the integral closure of an HFD is not in general an atomic domain [55].

Example 8.3.1. Take R to be the field $\mathbb{Z}_2$, which is obviously an HFD. Now consider the monoid algebra $\mathbb{Z}_2[N]$, where N is the numerical monoid $\langle 3, 4\rangle$. It can be readily verified that the polynomial expressions

$$f = X^8(1+X)^3(1+X^2+X^5) \quad \text{and} \quad g = X^8(1+X+X^2)^3(1+X^2+X^5)$$

are both irreducible elements in $\mathbb{Z}_2[N]$. Now consider the polynomial expression $h := (fg)^3 \in \mathbb{Z}_2[N]$. It is clear that $6 \in \mathsf{Z}_{\mathbb{Z}_2[N]}(h)$. In addition, we can also write

$$h = (X^3)^{16}(1+X^3)^9(1+X^4+X^{10})^3.$$

Therefore 28 is also an element of $\mathsf{Z}_{\mathbb{Z}_2[N]}(h)$. As $|\mathsf{Z}_{\mathbb{Z}_2[N]}(h)| \geq 2$, the monoid algebra $\mathbb{Z}_2[N]$ is not an HFD.

BFDs and FFDs in $\mathcal{C}_A$

Let us proceed to show that if an integral domain R is a BFD (resp., an FFD), then the monoid algebra $R[M]$ is a BFD (resp., an FFD) for each monoid M in the class $\mathcal{C}$.

Clearly, every UFD is an FFD, and every FFD is a BFD. In addition, every Krull domain is an FFD [3, Theorem 5.1] and every Noetherian domain is a BFD [3, Proposition 2.2]. However, neither FFDs are necessarily HFDs nor HFDs are necessarily FFDs; see [3, Section 5] for examples. An integral domain R is a BFD (resp., an FFD) if and only if its ring of polynomials $R[X]$ is a BFD (resp., an FFD) [3, Proposition 2.5 and Proposition 5.3]. In addition, when R is a BFD, the ring of power series $R[[X]]$ is also a BFD [3, Proposition 2.5]. It is not true, however, that $R[[X]]$ is an FFD provided that R is an FFD [6, Example 10].

Notation. For an integral domain R and $d \in \mathbb{N}^\bullet$, let $R[X_1, \dots, X_d]$ denote the polynomial ring on d variables. If $f = \sum_{i=1}^n r_i X^{\alpha_i} \in R[X_1, \dots, X_d]$ for some $n \in \mathbb{N}$, where the X^{α_i}'s are pairwise distinct monomials, then we set

$$\deg(f) := \max \deg_s X^{\alpha_i}, \quad \text{where} \quad \deg_s(X_1^{\alpha_{i1}} \dots X_d^{\alpha_{id}}) = \sum_{j=1}^d \alpha_{ij}.$$

Proposition 8.3.2. *Let M be a monoid in $\mathcal{C}$, and let R be an integral domain. Then the following statements hold.*

(1) $R[M]$ is a BFD if and only if R is a BFD.

(2) $R[M]$ is an FFD if and only if R is an FFD.

Proof. Set $d := \text{rank}(M)$ and think of $R[M]$ as a subdomain of $R[X_1, \dots, X_d]$. The direct implication of (1) follows from the fact that M is torsion-free, along with [114, Proposition 1.4]. To argue the reverse implication of (1), suppose, by way of contradiction, that $R[M]$ is not a BFD. Then there exists $f \in R[M]$ such that $|\mathsf{L}_{R[M]}(f)| = \infty$. Since R is a BFD, it is also a GCD domain. Let m be a greatest common divisor of all nonzero coefficients of f, and let $\ell := \max \mathsf{L}_R(m) \in \mathbb{N}$ (as m might be a unit of R). Now set $v := \deg f$, and take $g_1 \dots g_N \in \mathsf{Z}_{R[M]}(f)$ for some $N > \ell + v$ and irreducible elements $g_1, \dots, g_N$ of $R[M]$. Notice that at most ℓ irreducible elements in $\{g_1, \dots, g_N\}$ belong to R. Suppose, without loss of generality, that $g_1, \dots, g_{N-\ell} \in R[M] \setminus R$. Then

$$\deg f \geq \deg g_1 \dots g_{N-\ell} = \sum_{i=1}^{N-\ell} \deg g_i \geq N - \ell > v,$$

which contradicts the fact that $\deg f = v$. Hence $R[M]$ is a UFD, and the reverse implication follows.

Since M is a torsion-free monoid, the direct implication is an immediate consequence of [114, Proposition 1.4]. For the reverse implication, assume that R is an FFD. Since $R[M]$ is atomic, [3, Theorem 5.1] ensures that $R[M]$ is an FFD if and only if each nonzero element of $R[M]$ has at most finitely many non-associate irreducible divisors. Let us verify this for an arbitrary element $f \in R[M]^\bullet$. Because R is an FFD and $R[\mathbb{N}^d] \cong R[X_1, \dots, X_d]$, the algebra $R[\mathbb{N}^d]$ is also an FFD. Thus, there are only finitely many non-associate irreducible elements in $R[\mathbb{N}^d]$ dividing f (i.e., f is contained in only finitely many principal ideals of $R[\mathbb{N}^d]$). Clearly, for $g \in R[M]^\bullet$ one has that $g \mid_{R[M]} f$ if and only if $f \in (g)_{R[M]}$, which implies that $f \in (g)_{R[\mathbb{N}^d]}$; here $(g)_{R[M]}$ and $(g)_{R[\mathbb{N}^d]}$ denote the principal ideals generated by g in $R[M]$ and $R[\mathbb{N}^d]$, respectively. If $g' \in R[M]$ satisfies that $(g')_{R[\mathbb{N}^d]} = (g)_{R[\mathbb{N}^d]}$, then $g' = ug$ for some $u \in R[\mathbb{N}^d]^\times = R^\times = R[M]^\times$ and, therefore, $(g')_{R[M]} = (g)_{R[M]}$. This, along with the fact that f is only contained in finitely many principal ideals of $R[\mathbb{N}^d]$, ensures that f is contained in only finitely many principal ideals of $R[M]$. Hence f has only finitely many non-associate irreducible divisors in $R[M]$, which concludes the proof. □

ACCP domains in $\mathcal{C}_A$

It is clear that an integral domain is an ACCP domain if and only if it satisfies the ACCP (i.e., every ascending chain of principal ideals eventually stabilizes). So it follows from the chain of implications (6.2) that if an integral domain is a BFD, then it must satisfy the ACCP. In a similar manner, we have that if an integral domain satisfies the ACCP, then it must be atomic. On the other hand, not every integral domain satisfying the ACCP is atomic. However, examples witnessing this observation are not easy to construct. The first such an example was given by Grams [104] using the monoid we exhibit in Example 8.3.3. Further examples of atomic domains not satisfying the ACCP were given by Zaks in [147].

Example 8.3.3. Let p_n denote the n^{th} odd prime. The monoid

$$G = \left\langle \frac{1}{2^n \cdot p_n} \;\middle|\; n \in \mathbb{N} \right\rangle$$

was introduced by A. Grams in [104] to construct the first atomic integral domain that does not satisfy the ACCP. It is not hard to check that G is an atomic monoid. However, the increasing chain of principal ideals $\{1/2^n + G\}$ does not stabilize and, therefore, G does not satisfy the ACCP. We call G the *Gram's monoid*.

It is well known that an integral domain R satisfies the ACCP if and only if the ring of polynomials $R[X]$ satisfies the ACCP, and the same is true if we replace $R[X]$ by the ring of power series $R[[X]]$ (for a stronger version of these two statements, see [10, Proposition 1.1]). However, the problem of determining whether a monoid algebra $R[M]$ satisfies the ACCP is more subtle and not settled yet in its full generality. As the following result indicates, there are many algebras in $\mathcal{C}_A$ satisfying the ACCP.

Proposition 8.3.4. *Let M be a monoid and let F be a field. Then M satisfies the ACCP if and only if $F[M]$ satisfies the ACCP.*

Proof. For the direct implication, suppose that M satisfies the ACCP. Let $(f_1) \subseteq (f_2) \subseteq \cdots$ be an ascending chain of principal ideals of $F[M]$. Then the sequence $\{\deg(f_n) + M\}$ of subsets of M is an ascending chain of principal ideals, which stabilizes because M satisfies the ACCP. As M is reduced, there exists $N \in \mathbb{N}$ such that $\deg(f_n) = \deg(f_N)$ for all $n \geq N$. This, along with the fact that f_n divides f_N in $F[M]$ for each $n \geq N$, implies that $(f_n) = (f_N)$ for each $n \geq N$. The reverse implication follows immediately from the fact that $a_1 + M \subsetneq a_2 + M \subsetneq \cdots$ is an increasing chain of principal ideals of M if and only if $(X^{a_1}) \subsetneq (X^{a_2}) \subsetneq \cdots$ is an increasing chain of principal ideals in $R[M]$. □

Corollary 8.3.5. *For any monoid in $\mathcal{C}$ and any field F, the monoid algebra $F[M]$ satisfies the ACCP.*

If $R \subseteq T$ is an extension of integral domains with $T^\times \cap R = R^\times$, then R satisfies the ACCP provided that T satisfies the ACCP. This can be used to argue the following simple result.

Proposition 8.3.6. *Let M be a monoid in $\mathcal{C}$ and let R be an integral domain. If $R[M]$ satisfies the ACCP, then R satisfies the ACCP.*

Proof. We have that $R \subseteq R[M]$ is an integral extension, and it follows from [86, Theorem 11.1] that the set of units of $R[M]$ is $R^\times$. Then R satisfies the ACCP when $R[M]$ does. □

8.4 Algebras in $\mathcal{C}_A$ with Extremal Systems of Sets of Lengths

Now that we have characterized the monoid algebras of members of $\mathcal{C}$ that are BFDs, let us verify that some of these monoid algebras satisfy the Kainrath Property. Recall that a nontrivial BFM M satisfies the Kainrath Property provided that for each $L \subseteq \mathbb{N}_{\geq 2}$, there exists $x \in M^\bullet$ with $\mathsf{L}(x) = L$. In addition, an integral domain satisfies the *Kainrath Property* if its multiplicative monoid satisfies the Kainrath Property.

Proposition 8.4.1. *For a BFD R, the following statements hold.*

(1) For every $d \in \mathbb{N}_{\geq 2}$, there exists a rank-$d$ monoid M in $\mathcal{C}$ such that $R[M]$ satisfies the Kainrath Property.

(2) For every $d \in \mathbb{N}_{\geq 3}$, there exists a rank-$d$ primary monoid M in $\mathcal{C}$ such that $R[M]$ satisfies the Kainrath Property.

Proof. To prove (1), suppose that $d \in \mathbb{N}_{\geq 2}$. By [95, Corollary 4.8], there exists a rank-d monoid M in $\mathcal{C}$ satisfying the Kainrath Property. For $q \in M^\bullet$, let us argue that the equality $\mathsf{L}_{R[M]}(X^q) = \mathsf{L}_M(q)$ holds. Notice that if an element $f \in R[M]$ divides X^q in $R[M]$, then f must be a monomial. i.e., there exist $r \in R$ and $b \in M$ such that $f = rX^b$. Since X^b is irreducible in $R[M]$ if and only if b is an atom of M, each factorization $z_{R[M]}$ of X^q in $R[M]$ determines a unique factorization of q in M having length $|z_{R[M]}|$. Conversely, it follows immediately that each factorization z_M of q in M uniquely determines a factorization of X^q in $R[M]$ of length $|z_M|$. Hence $\mathsf{L}_{R[M]}(X^q) = \mathsf{L}_M(q)$, as desired.

Now suppose that $L \subseteq \mathbb{N}_{\geq 2}$. Since the monoid M satisfies the Kainrath Property, there exists $b \in M$ such that $\mathsf{L}_M(b) = L$. This, in turn, implies that $\mathsf{L}_{R[M]}(X^b) = L$. On the other hand, it follows by Proposition 8.3.2 that $R[M]$ is a BFD. Hence $R[M]$ satisfies the Kainrath Property. The statement (2) follows in a similar way, once we invoke [95, Corollary 4.9] to find, for each $d \in \mathbb{N}_{\geq 3}$, a rank-$d$ primary monoid in $\mathcal{C}$ satisfying the Kainrath Property. □

8.5 Atomicity of Monoid Algebras over Fields of Finite Characteristic

Unlike some of the atomic properties presented before, atomicity does not transfer from an integral domain to its polynomial ring. In 1993, M. Roitman constructed a class of atomic integral domains whose polynomial rings fail to be atomic [130]. This implies, in particular, that atomicity does not transfer, in general, from an integral domain R to the algebras in $\mathcal{C}_A$ with coefficients in R. However, we have already seen that for any monoid M in $\mathcal{C}$ and any field F, the monoid algebra $F[M]$ is an FFD and, therefore, an atomic domain. By contrast, it is not true in general that $F[M]$ is an atomic domain provided that M is an atomic monoid. This is a question we will answer in this chapter. Furthermore, even if we take M in $\mathcal{Q}$, it is not true in general that $F[M]$ is atomic over any field F. We will also construct a monoid M in $\mathcal{Q}$ such that $\mathbb{Z}_2[M]$ is not atomic. Note, however, that the members of both classes $\mathcal{C}$ and $\mathcal{Q}$ are natural (non-finitely generated) generalizations of numerical monoids.

The Atomicity Transfer Problem

The following fundamental question about the atomicity of monoid algebras was stated by R. Gilmer as an open problem back in the 1980's.

Question 8.5.1. *[86, p. 189] Let M be a commutative cancellative monoid and let R be an integral domain. Is the monoid algebra $R[M]$ atomic provided that both M and R are atomic?*

In 1990, D. Anderson et al. restated a special version of the above question in the context of polynomial rings, namely the case of $M = (\mathbb{N}_0, +)$.

Question 8.5.2. *[3, Question 1] If R is an atomic integral domain, is the integral domain $R[X]$ also atomic?*

Question 8.5.2 was answered negatively by M. Roitman in 1993. He constructed a class of atomic integral domains whose polynomial rings fail to be atomic [130]. In a similar direction, Roitman constructed examples of atomic integral domains whose corresponding power series rings fail to be atomic as well as examples of atomic power series rings over non-atomic domains [129]. This illustrates that, in general, atomicity is not easily inherited. In the positive side, it was proved in [3, Section 1] that for an integral domain R, the polynomial ring $R[X]$ is atomic if and only if $R[\{X_\alpha\}]$ is atomic for any family $\{X_\alpha\}$ of indeterminates.

Observe that Roitman's negative answer to Question 8.5.2 gives a striking answer to Question 8.5.1, showing that $R[M]$ can fail to be atomic even if one takes M to be the simplest nontrivial atomic monoid, namely $M = (\mathbb{N}_0, +)$. This naturally suggests the question of whether the atomicity of M implies the atomicity of $R[M]$ provided that R is the taken to be one of the simplest nontrivial atomic integral domains, a field. Clearly, this is another special version of Question 8.5.1.

Question 8.5.3. *Let F be a field. If M is an atomic monoid, is the monoid algebra $F[M]$ also atomic?*

There are many known classes of atomic monoids whose monoid algebras (over any field) happen to be atomic. For instance, we have seen in Proposition 8.3.4 that if a monoid M

satisfies the ACCP (and, therefore, is atomic), then for any field F the monoid algebra $F[M]$ also satisfies the ACCP. Therefore $F[M]$ inherits the atomicity of M as it is well known that integral domains satisfying the ACCP are atomic. In particular, every finitely generated monoid satisfies the ACCP and, thus, induces atomic monoid algebras. As BFMs also satisfy the ACCP [79, Corollary 1.3.3], one can use them to obtain many non-finitely generated atomic monoid algebras with rational (or even real) exponents; this is because submonoids of $(\mathbb{R}_{\geq 0}, +)$ not having 0 as a limit point are BFMs [92, Proposition 4.5]. Furthermore, an infinite class of atomic submonoids of $(\mathbb{Q}_{\geq 0}, +)$ (which are not BFMs) with atomic monoid algebras was exhibited in [97, Theorem 5.4].

Let us proceed to provide a negative answer for Question 8.5.3. First we find, for every field F of finite characteristic, a rank-2 totally ordered atomic monoid M such that the monoid algebra $F[M]$ is not atomic. Then we construct a rank-1 totally ordered atomic monoid M (i.e., an additive submonoid of $\mathbb{Q}_{\geq 0}$) such that the monoid algebra $\mathbb{Z}_2[M]$ is not atomic.

Non-atomic Monoid Algebras of Finite Characteristic

We proceed to find, for any given field F of finite characteristic, a rank-2 totally ordered atomic monoid M such that the monoid algebra $F[M]$ is not atomic.

Proposition 8.5.4. *For reduced monoids M and N the following statements hold.*

(1) If M and N are atomic, then $M \times N$ is atomic.

(2) If M and N satisfy the ACCP, then $M \times N$ satisfies the ACCP.

Proof. To argue (1), suppose that M and N are atomic. Clearly, $(a, 0)$ and $(0, b)$ belong to $\mathcal{A}(M \times N)$ when $a \in \mathcal{A}(M)$ and $b \in \mathcal{A}(N)$. Therefore, for any atomic decompositions $r = \sum_{i=1}^{k} a_i$ and $s = \sum_{j=1}^{\ell} b_j$ of $r \in M$ and $s \in N$,

$$(r, s) = \sum_{i=1}^{k} (a_i, 0) + \sum_{j=1}^{\ell} (0, b_j)$$

is an atomic decomposition of (r, s) in $M \times N$. Hence $M \times N$ is atomic, and (1) follows. To argue (2), assume that M and N both satisfy the ACCP. Let $\{(a_n, b_n) + M \times N\}$ be an increasing chain of principal ideals in $M \times N$. Then $\{a_n + M\}$ and $\{b_n + N\}$ are increasing chains of principal ideals in M and N, respectively. As $\{a_n + M\}$ and $\{b_n + N\}$ stabilize, $\{(a_n, b_n) + M \times N\}$ must also stabilize. As a result, $M \times N$ satisfies the ACCP, which completes the proof. □

Let r and m be integers with $m > 0$ and $\gcd(r, m) = 1$. Recall that the order of r modulo m is the smallest $n \in \mathbb{N}^{\bullet}$ for which $r^n \equiv 1 \pmod{m}$, and that r is a primitive root modulo m if its order modulo m equals $\phi(m)$, where ϕ is the Euler's totient function. It is well known that for any odd prime p and positive integer k, there exists a primitive root modulo p^k.

Lemma 8.5.5. *Let F be a field of characteristic p and $n \in \mathbb{N}$ be such that $\gcd(p,n) = 1$. Then the polynomial $X^n + Y^n + X^nY^n$ is irreducible in $F[X,Y]$.*

Proof. Set $f(X,Y) = Y^n(1+X^n) + X^n$. Because there are no $a, b \in F$ satisfying that $X+Y+XY = (X+a)(Y+b)$, it is clear that $f(X,Y)$ is irreducible in $F[X,Y]$ when $n=1$. Then we assume that $n \geq 2$. Since $\gcd(p,n) = 1$, there exists a primitive n^{th} root of unity ω over F. Let α be a root of $X^n + 1 \in F[X]$ in some extension field of F. Now set

$$K := F(\omega, \alpha, X, (1+X^n)^{\frac{1}{n}}),$$

where X is an indeterminate. Note that $F[Y]$ is a UFD containing $F[X,Y]$. In addition,

$$f(X,Y) = \prod_{i=1}^{n} \big(Y(1+X^n)^{\frac{1}{n}} - \omega^i \alpha X\big)$$

in $K[Y]$. So in order to show that f is irreducible in $F[X,Y]$ it suffices to argue that $(1+X^n)^{\frac{m}{n}} \notin F[X]$ for any $m \in [[1, n-1]]$.

Let $\gcd(m,n) = d$. Take $n', m' \in \mathbb{N}$ such that $n = n'd$ and $m = m'd$. Since $F[X]$ is a UFD, there exist $k \in \mathbb{N}$ and irreducible polynomials $p_1(X), \dots, p_k(X) \in F[X]$ such that

$$(X^n+1)^{\frac{m}{n}} = (X^n+1)^{\frac{m'}{n'}} = p_1(X)^{a_1} p_2(X)^{a_2} \cdots p_k(X)^{a_k},$$

for some exponents $a_1, \dots, a_k \in \mathbb{N}$. Similarly, there exist $t \in \mathbb{N}$ and irreducible polynomials $q_1(X), \dots, q_t(X) \in F[X]$ such that $(X^n+1) = q_1(X)^{b_1} q_2(X)^{b_2} \cdots q_t(X)^{b_t}$ for some exponents $b_1, \dots, b_t \in \mathbb{N}$. Combining the above observations, we find that

$$(X^n+1)^{m'} = p_1(X)^{n'a_1} p_2(X)^{n'a_2} \cdots p_k(X)^{n'a_k} = q_1(X)^{m'b_1} q_2(X)^{m'b_2} \cdots q_t(X)^{m'b_t}.$$

Since $\gcd(n', m') = 1$ and $F[X]$ is a UFD, it follows that $t = k$, $p_i(X) = q_i(X)$ (without loss of generality), and $n'a_i = m'b_i$. Hence $m' \mid a_i$ for all $1 \leq i \leq k$. Writing $a_i = m'a_i'$ one obtains

$$X^n + 1 = \big(p_1(X)^{a_1'} p_2(X)^{a_2'} \dots p_k(X)^{a_k'}\big)^{n'}. \tag{8.2}$$

For simplicity, we write the product $p_1(X)^{a_1'} p_2(X)^{a_2'} \dots p_k(X)^{a_k'}$ more concisely as $p(X)$. As a result, (8.2) becomes

$$X^n + 1 = p(X)^{n'}.$$

Taking the formal derivatives on both sides of the previous equality, one obtains that $nX^{n-1} = n'p(X)^{n'-1}p'(X)$. As $n \neq 0$ in a field of characteristic p, we must have that $p(x)$ is a monomial. However, this contradicts the equality $X^n + 1 = p(X)^{n'}$. □

Motivated by the Gram's monoid, in the next example we exhibit a family of monoids in the class $\mathcal{Q}$ indexed by prime numbers whose members are atomic but do not satisfy the ACCP.

Example 8.5.6. Let $\{p_n\}$ be a sequence consisting of all prime numbers ordered increasingly. For each prime p consider the monoid

$$M_p := \left\langle \frac{1}{p^n p_n} \;\middle|\; p_n \neq p \right\rangle.$$

A very elementary argument of divisibility can be used to check that M_p is atomic for each prime p. On the other hand, M_p contains the strictly increasing sequence of principal ideals $\{1/p^n + M_p\}$ and, therefore, M_p does not satisfy the ACCP. Notice that M_2 is precisely the Gram's monoid.

For any field F and any monoids M and N in $\mathcal{Q}$, there is a canonical F-algebra isomorphism

$$F[M \times N] \cong F[X; M][Y; N]$$

induced by the assignment $x^{(a,b)} \mapsto X^a Y^b$. To avoid having ordered pairs as exponents, we will identify $F[M \times N]$ with $F[X; M][Y; N]$ and write the elements of $F[M \times N]$ as polynomial expressions in two variables.

Theorem 8.5.7. *For each field F of finite characteristic p, there exists an atomic monoid M such that the monoid algebra $F[M]$ is not atomic.*

Proof. Let $M := M_p \times M_p$, where M_p is the atomic monoid parametrized by p exhibited in Example 8.5.6. We first claim that each nonunit factor of $f := X + Y + XY \in F[M]$ has the form

$$\big(X^{\frac{1}{p^k}} + Y^{\frac{1}{p^k}} + X^{\frac{1}{p^k}} Y^{\frac{1}{p^k}}\big)^t$$

for some $k \in \mathbb{N}_0$ and $t \in \mathbb{N}$. To prove our claim, let $g \in F[M]$ be a nonunit factor of f, and take $h \in F[M]$ such that $f = g\,h$. Then there exist $k \in \mathbb{N}_0$ and $a \in \mathbb{N}$ with $p \nmid a$ such that $g(X^{ap^k}, Y^{ap^k})$ and $h(X^{ap^k}, Y^{ap^k})$ are both in the polynomial ring $F[X, Y]$. After changing variables, we obtain

$$g(X^{ap^k}, Y^{ap^k})h(X^{ap^k}, Y^{ap^k}) = X^{ap^k} + Y^{ap^k} + X^{ap^k} Y^{ap^k} = (X^a + Y^a + X^a Y^a)^{p^k}.$$

By Lemma 8.5.5, the polynomial $X^a + Y^a + X^a Y^a$ is irreducible in the polynomial ring $F[X, Y]$. Since $F[X, Y]$ is a UFD, there exists $t \in \mathbb{N}$ such that

$$g\big(X^{ap^k}, Y^{ap^k}\big) = \big(X^a + Y^a + X^a Y^a\big)^t.$$

Going back to the original variables, we obtain $g(X, Y) = \big(X^{\frac{1}{p^k}} + Y^{\frac{1}{p^k}} + X^{\frac{1}{p^k}} Y^{\frac{1}{p^k}}\big)^t$, which establishes our initial claim.

Now we verify that $F[M]$ is not atomic. As $f = \big(X^{\frac{1}{p}} + Y^{\frac{1}{p}} + X^{\frac{1}{p}}Y^{\frac{1}{p}}\big)^p$, the polynomial expression f is not irreducible. By the argument given in the previous paragraph, any factor g of f in a potential decomposition into irreducibles of $F[M]$ must be of the form $\big(X^{\frac{1}{p^k}} + Y^{\frac{1}{p^k}} + X^{\frac{1}{p^k}}Y^{\frac{1}{p^k}}\big)^t$ and, therefore,

$$g = \big(X^{\frac{1}{p^{k+1}}} + Y^{\frac{1}{p^{k+1}}} + X^{\frac{1}{p^{k+1}}}Y^{\frac{1}{p^{k+1}}}\big)^{pt}. \tag{8.3}$$

Since $X^{\frac{1}{p^{k+1}}} + Y^{\frac{1}{p^{k+1}}} + X^{\frac{1}{p^{k+1}}}Y^{\frac{1}{p^{k+1}}} \in F[M]$, the equality (8.3) would contradict that g is an irreducible element of $F[M]$. Thus, the algebra $F[M]$ is not atomic. □

It is clear that any submonoid M of $(\mathbb{R}, +)$ is totally ordered with respect to the natural order inherited from $\mathbb{R}$.

Corollary 8.5.8. *For each field F of finite characteristic p, there exists a rank-2 totally ordered atomic monoid M such that $F[M]$ is not atomic.*

Proof. Let M_p be the monoid introduced in Example 8.5.6 corresponding to the prime p. Since π is irrational, for any $a, a', b, b' \in \mathbb{Q}$ the fact that $a + \pi b = a' + \pi b'$ immediately implies that $a = a'$ and $b = b'$. Using this observation, one can easily verify that the map

$$\psi \colon M_p \times M_p \to M_p + \pi M_p \quad \text{defined by} \quad \psi(a, b) = a + \pi b$$

is a monoid isomorphism. On the other hand, the fact that $\{1, \pi\}$ is a linearly independent set of $\mathbb{R}$ (seen as a $\mathbb{Q}$-vector space) implies that $\{1, \pi\}$ is also a linearly independent set of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \text{gp}(M_p + \pi M_p)$. Thus, $M_p + \pi M_p$ is a rank-2 totally ordered monoid. Because M_p is atomic, Proposition 8.5.4(1) ensures that $M_p + \pi M_p$ is also atomic. Since $M_p \times M_p$ and $M_p + \pi M_p$ are isomorphic monoids, [86, Theorem 7.2(2)] ensures that $F[M_p \times M_p]$ and $F[M_p + \pi M_p]$ are isomorphic F-algebras. Finally, it follows by Theorem 8.5.7 that $F[M_p + \pi M_p]$ is not an atomic domain. □

A Non-atomic Monoid Algebra with Rational Exponents

The purpose of this section is to construct an atomic monoid M in $\mathcal{Q}$ such that the algebra $\mathbb{Z}_2[M]$ fails to be atomic. Since each monoid in $\mathcal{Q}$ is totally ordered and has rank 1, this result will complement Corollary 8.5.8.

Proposition 8.5.9. *There exists an atomic monoid M in $\mathcal{Q}$ satisfying the following two conditions:*

(1) M is contained in the ring $\mathbb{Z}\big[\frac{1}{2}, \frac{1}{3}\big]$, and

(2) $\frac{1}{2^n} \in M$ for every $n \in \mathbb{N}$.

Proof. Let $\{\ell_n\}$ be a strictly increasing sequence of positive integers satisfying that

$$3^{\ell_n - \ell_{n-1}} > 2^{n+1}$$

for every $n \in \mathbb{N}$. Now set $A = \{a_n, b_n \mid n \in \mathbb{N}\}$, where

$$a_n := \frac{2^n 3^{\ell_n} - 1}{2^{2n} 3^{\ell_n}} \quad \text{and} \quad b_n := \frac{2^n 3^{\ell_n} + 1}{2^{2n} 3^{\ell_n}}.$$

It is clear that $1 > b_n > a_n$ for every $n \in \mathbb{N}$. In addition,

$$a_n = \frac{1}{2^n} - \frac{1}{2^{2n}3^{\ell_n}} = \frac{1}{2^{n+1}} + \left(\frac{1}{2^{n+1}} - \frac{1}{2^{2n}3^{\ell_n}}\right) > \frac{1}{2^{n+1}} + \frac{1}{2^{2n+2}3^{\ell_{n+1}}} = b_{n+1}$$

for every $n \in \mathbb{N}$. Therefore the sequence $b_1, a_1, b_2, a_2, \dots$ is strictly decreasing and bounded from above by 1. Consider now the monoid $M = \langle A \rangle$. Clearly, M satisfies condition (1). On the other hand, $\frac{1}{2^n} = a_{n+1} + b_{n+1} \in M$ for every $n \in \mathbb{N}_0$. Thus, M also satisfies condition (2).

All we need to prove is that M is atomic. It suffices to verify that A is a minimal generating set of M [79, Proposition 1.1.7]. Suppose, by way of contradiction, that this is not the case. Then there exists $n \in \mathbb{N}$ such that $M = \langle A \setminus \{a_n\} \rangle$ or $M = \langle A \setminus \{b_n\} \rangle$.

CASE 1. $M = \langle A \setminus \{a_n\} \rangle$. In this case,

$$a_n = \sum_{i=1}^{N} \alpha_i a_i + \sum_{i=1}^{N} \beta_i b_i \tag{8.4}$$

for some $N \in \mathbb{N}_{\ge n}$ and nonnegative integer coefficients α_i's and β_i's ($i \in [[1, N]]$) such that $\alpha_n = 0$ and either $\alpha_N > 0$ or $\beta_N > 0$. Since the sequence $b_1, a_1, b_2, a_2, \dots$ is strictly decreasing, $\alpha_i = \beta_i = 0$ for $i \in [[1, n]]$. Notice that $\alpha_i = \beta_i$ cannot hold for all $i \in [[n+1, N]]$; otherwise,

$$a_n = \sum_{i=n+1}^{N} \alpha_i a_i + \sum_{i=n+1}^{N} \alpha_i b_i = \sum_{i=n+1}^{N} \alpha_i \frac{1}{2^{i-1}},$$

which is impossible because $3 \mid \mathsf{d}(a_n)$. Set

$$m = \max\big\{ i \in [[n+1, N]] \mid \alpha_i \neq \beta_i \big\}.$$

First assume that $\alpha_m > \beta_m$. Then we can rewrite (8.4) as follows:

$$a_n = (\alpha_m - \beta_m)\frac{2^m 3^{\ell_m} - 1}{2^{2m}3^{\ell_m}} + \sum_{i=m}^{N} \beta_i \frac{1}{2^{i-1}} + \sum_{i=n+1}^{m-1} \frac{\alpha_i(2^i 3^{\ell_i} - 1) + \beta_i(2^i 3^{\ell_i} + 1)}{2^{2i}3^{\ell_i}}. \tag{8.5}$$

After multiplying both sides of the equality (8.5) by $2^{2N}3^{\ell_m}$, one can easily see that each summand involved in such an equality except perhaps $2^{2N-2m}(\alpha_m - \beta_m)(2^m 3^{\ell_m} - 1)$ is divisible by $3^{\ell_m - \ell_{m-1}}$. Therefore $3^{\ell_m - \ell_{m-1}}$ must also divide $\alpha_m - \beta_m$. Now since $a_m > b_{m+1} > \frac{1}{2^{m+1}}$, we find that

$$a_n \geq \alpha_m a_m \geq (\alpha_m - \beta_m) b_{m+1} \geq 3^{\ell_m - \ell_{m-1}} b_{m+1} > \frac{3^{\ell_m - \ell_{m-1}}}{2^{m+1}} > 1, \tag{8.6}$$

which is a contradiction. In a similar way we arrive at a contradiction if we assume that $\beta_m > \alpha_m$.

CASE 2: $M = \langle A \setminus \{b_n\} \rangle$. In this case, it is not hard to see that

$$b_n - \alpha_n a_n = \sum_{i=n+1}^{N} \alpha_i a_i + \sum_{i=n+1}^{N} \beta_i b_i \tag{8.7}$$

for some nonnegative coefficients α_i's ($i \in [[n, N]]$) and β_j's ($j \in [[n+1, N]]$) such that either $\alpha_N > 0$ or $\beta_N > 0$. Observe that $\alpha_n \in \{0, 1\}$ as it is obvious that $2a_n > b_n$. As before, there exists $m \in [[n+1, N]]$ such that $\alpha_m \neq \beta_m$, and we can assume that such m is as large as possible. If $\alpha_m > \beta_m$, then

$$b_n - \alpha_n a_n = (\alpha_m - \beta_m) \frac{2^m 3^{\ell_m} - 1}{2^{2m} 3^{\ell_m}} + \sum_{i=m}^{N} \beta_i \frac{1}{2^{i-1}} + \sum_{i=n+1}^{m-1} \frac{\alpha_i (2^i 3^{\ell_i} - 1) + \beta_i (2^i 3^{\ell_i} + 1)}{2^{2i} 3^{\ell_i}}.$$

Since $\mathsf{d}(b_n - \alpha_n a_n) \in \{2^{n-1} 3^{\ell_n}, 2^{2n} 3^{\ell_n}\}$, after multiplying the previous equation by $2^{2N} 3^{\ell_m}$ we can see that $3^{\ell_m - \ell_{m-1}}$ divides $\alpha_m - \beta_m$. Now, an argument similar to that one given in CASE 1 can be used to obtain that $b_n > 1$, which is a contradiction. We can proceed in a similar manner to obtain a contradiction if we assume that $\alpha_m < \beta_m$. Hence we have proved that A is a minimal generating set of M, which means that M is atomic with $\mathcal{A}(M) = A$. □

The following result will be used in the proof of Lemma 8.5.11.

Lemma 8.5.10. *[116, page 179] If p is an odd prime and r is a primitive root modulo p^2, then r is a primitive root modulo p^n for every $n \geq 2$.*

The next lemma is proposed as an exercise in [117, Chapter 3]. For the convenience of the reader, we provide a proof here.

Lemma 8.5.11. *For each $n \in \mathbb{N}$, the polynomial $X^{2 \cdot 3^n} + X^{3^n} + 1$ is irreducible in $\mathbb{Z}_2[X]$.*

Proof. One can verify that the order of 2 modulo 9 is $\phi(9) = 6$. Then 2 is a primitive root modulo 3^2. It follows now by Lemma 8.5.10 that 2 is also a primitive root modulo 3^n for every $n \geq 2$, that is, $2 + 3^n\mathbb{Z}$ generates the multiplicative group of units of $\mathbb{Z}/3^n\mathbb{Z}$ for each $n \geq 2$.

Set $f(X) = X^{2 \cdot 3^n} + X^{3^n} + 1$, and suppose that $f(X) = g(X)h(X)$ in $\mathbb{Z}_2[X]$, where $g(X)$ is irreducible. Since

$$X^{3^{n+1}} + 1 = \big(X^{3^n} + 1\big) f(X),$$

each primitive root of unity modulo 3^{n+1} must be a root of $f(X)$. Take r to be a primitive root modulo 3^{n+1}. As r is a root of $f(X)$, either $g(r) = 0$ or $h(r) = 0$. Suppose, without loss of generality, that $g(r) = 0$. Let $m \in [[1, 3^{n+1}]]$ such that $3 \nmid m$. Since 2 is a generator of the multiplicative group $(\mathbb{Z}/3^{n+1}\mathbb{Z})^\times$, there exists $k \in \mathbb{N}^\bullet$ such that

$$2^k = m \pmod{3^{n+1}}.$$

Taking $\ell \in \mathbb{N}_0$ so that $m = 2^k + 3^{n+1}\ell$, one obtains

$$g\big(r^m\big) = g\big(r^{2^k+3^{n+1}\ell}\big) = g\big(r^{2^k}\big(r^{3^{n+1}}\big)^\ell\big) = g\big(r^{2^k}\big) = g(r)^{2^k} = 0.$$

Thus, the polynomial $g(X)$ contains at least $\phi(3^{n+1}) = 2 \cdot 3^n$ distinct roots. This implies that $\deg g(X) = \deg f(X)$. Hence $f(X) = g(X)$, and so $f(X)$ is irreducible over $\mathbb{Z}_2$. □

Theorem 8.5.12. *There exists an atomic monoid M in $\mathcal{Q}$ such that $\mathbb{Z}_2[M]$ is not atomic.*

Proof. Let M be an atomic monoid in $\mathcal{Q}$ satisfying conditions (1) and (2) of Proposition 8.5.9. First, we will argue that each factor of the element $X^2 + X + 1$ in $\mathbb{Z}_2[M]$ has the form $\big(X^{2\frac{1}{2^k}} + X^{\frac{1}{2^k}} + 1\big)^t$ for some $k \in \mathbb{N}$ and $t \in \mathbb{N}^\bullet$. First, note that because M contains $\langle 1/2^k \mid k \in \mathbb{N}\rangle$, it follows that

$$X^{2\frac{1}{2^k}} + X^{\frac{1}{2^k}} + 1 \in \mathbb{Z}_2[M]$$

for all $k \in \mathbb{N}_0$. Now suppose that $f(X)$ is a factor of $X^2 + X + 1$ in $\mathbb{Z}_2[M]$, and take $g(X) \in \mathbb{Z}_2[M]$ such that

$$X^2 + X + 1 = f(X)g(X).$$

Then there exists $k \in \mathbb{N}$ such that

$$f\big(X^{6^k}\big)g\big(X^{6^k}\big) = \big(X^{6^k}\big)^2 + X^{6^k} + 1 = \big(X^{2\cdot 3^k} + X^{3^k} + 1\big)^{2^k}$$

in the polynomial ring $\mathbb{Z}_2[X]$. By Lemma 8.5.11, the polynomial $X^{2\cdot 3^k} + X^{3^k} + 1$ is irreducible in $\mathbb{Z}_2[X]$. Since $\mathbb{Z}_2[X]$ is a UFD, there exists $t \in \mathbb{N}^\bullet$ such that

$$f\big(X^{6^k}\big) = \big(X^{2\cdot 3^k} + X^{3^k} + 1\big)^t = \Big(\big(X^{6^k}\big)^{2\frac{1}{2^k}} + \big(X^{6^k}\big)^{\frac{1}{2^k}} + 1\Big)^t. \tag{8.8}$$

After changing variables in (8.8), one obtains that $f(X) = \big(X^{2\frac{1}{2^k}} + X^{\frac{1}{2^k}} + 1\big)^t$. Thus, each factor of $X^2 + X + 1$ in $\mathbb{Z}_2[M]$ has the desired form.

Now suppose, by way of contradiction, that the domain $\mathbb{Z}_2[M]$ is atomic. Then

$$X^2 + X + 1 = \prod_{i=1}^n f_i(X)$$

for some $n \in \mathbb{N}^{\bullet}$ and irreducible elements $f_1(X), \dots, f_n(X)$ in $\mathbb{Z}_2[M]$. Since $f_1(X)$ is a factor of $X^2 + X + 1$, there exist $k \in \mathbb{N}$ and $t \in \mathbb{N}^{\bullet}$ such that

$$f_1(X) = \big(X^{2\frac{1}{2^k}} + X^{\frac{1}{2^k}} + 1\big)^t.$$

As $f_1(X)$ is irreducible, $t = 1$. Now the equality

$$f_1(X) = \Big(X^{2\frac{1}{2^{k+1}}} + X^{\frac{1}{2^{k+1}}} + 1\Big)^2$$

contradicts the fact that $f_1(X)$ is irreducible in $\mathbb{Z}_2[M]$. Hence $\mathbb{Z}_2[M]$ is not atomic. $\square$

Bibliography

[1] M. Aissen, I. J. Schoenberg, and A. Whitney. "On generating functions of totally positive sequences". In: *J. Anal. Math.* 2 (1952), pp. 93–103.

[2] J. Amos et al. "Sets of lengths do not characterize numerical monoids". In: *Integers* 7 (2007), A50.

[3] D. D. Anderson, D. F. Anderson, and M. Zafrullah. "Factorizations in integral domains". In: *J. Pure Appl. Algebra* 69 (1990), pp. 1–19.

[4] D. D. Anderson, D. F. Anderson, and M. Zafrullah. "Factorizations in integral domains II". In: *J. Algebra* 152 (1992), pp. 78–93.

[5] D. D. Anderson, D. F. Anderson, and M. Zafrullah. "Rings between D[X] and K[X]". In: *Houston J. Math.* 17 (1991), pp. 109–129.

[6] D. D. Anderson and B. Mullins. "Finite factorization domains". In: *Proc. Amer. Math. Soc.* 124 (1996), pp. 389–396.

[7] D. D. Anderson et al. "Monoid domain constructions of antimatter domains". In: *Comm. Alg.* 35 (2007), pp. 3236–3241.

[8] D. D. Anderson et al. "Rational elasticity of factorizations in Krull domains". In: *Proc. Amer. Math. Soc.* 117 (1993), pp. 37–43.

[9] D. F. Anderson. "Elasticity of factorizations in integral domains: a survey". In: *Factorization in Integral Domains*. Ed. by D. D. Anderson. Vol. 189. Lect. Notes Pure Appl. Math. New York: Marcel Dekker, 1997, pp. 1–29.

[10] D. F. Anderson and D. N. El Abidine. "Factorization in integral domains, III". In: *J. Pure Appl. Algebra* 135 (1999), pp. 107–127.

[11] D. F. Anderson, S. T. Chapman, and W. W. Smith. "Some factorization properties of Krull domains with infinite cyclic divisor class group". In: *J. Pure Appl. Algebra* 96 (1994), pp. 97–112.

[12] D. F. Anderson and D. Scherpenisse. "Factorization in K[S]". In: *Factorization in Integral Domains*. Ed. by D. D. Anderson. Vol. 189. Lect. Notes Pure Appl. Math. New York: Marcel Dekker, 1997, pp. 45–56.

[13] J. Anderson. "Partitions which are simultaneously $t1$- and $t2$-core". In: *Discrete Math.* 248 (2002), pp. 237–243.

[14] F. Ardila, C. Benedetti, and J. Doker. "Matroid polytopes and their volumes". In: *Discrete Comput. Geom.* 43 (2010), pp. 841–854.

[15] F. Ardila and S. Billey. "Flag arrangements and triangulations of products of simplices". In: *Adv. Math.* 214 (2007), pp. 495–524.

[16] F. Ardila and C. Ceballos. "Acyclic systems of permutations and fine mixed subdivisions of simplices". In: *Discrete Comput. Geom.* 49 (2013), pp. 485–510.

[17] F. Ardila, F. Rincón, and L. K. Williams. "Positroids and non-crossing partitions". In: *Trans. Amer. Math. Soc.* 368 (2016), pp. 337–363.

[18] F. Ardila and R. P. Stanley. "Tilings". In: *Math. Intelligencer* 32 (2010), pp. 32–43.

[19] N. Arkani-Hamed et al. *Grassmannian Geometry of Scattering Amplitudes*. Cambridge: Cambridge University Press, 2016.

[20] D. Armstrong, C. R. H. Hanusa, and B. C. Jones. "Results and conjectures on simultaneous core partitions". In: *European J. Combin.* 41 (2014), pp. 205–220.

[21] D. Armstrong, N. A. Loehr, and G. S. Warrington. "Rational parking functions and Catalan numbers". In: *Ann. Comb.* 20 (2016), pp. 21–58.

[22] D. Armstrong, B. Rhoades, and N. Williams. "Rational associahedra and noncrossing partitions". In: *Electron. J. Combin.* 20 (2013), #54.

[23] D. Armstrong, B. Rhoades, and N. Williams. "Rational Catalan combinatorics: the associahedron". In: *Discrete Math. Theor. Comput. Sci.* 20 (2013), pp. 933–944.

[24] W. E. Armstrong. "The determinateness of the utility function". In: *Econom. J.* 49 (1939), pp. 453–467.

[25] L. L. Avramov, C. Gibbons, and R. Wiegand. "Monoids of Betti tables over short Gorenstein algebras". Under preparation.

[26] P. Baginski, S. T. Chapman, and G. J. Schaeffer. "On the delta-set of a singular arithmetical congruence monoid". In: *J. Théor. Nombres Bordeaux* 20 (2008), pp. 45–59.

[27] F. W. Barnes. "Algebraic theory of brick packing I". In: *Discrete Math.* 42 (1982), pp. 7–26.

[28] T. Barron, C. O'Neil, and R. Pelayo. "On the set of elasticities in numerical monoids". In: *Semigroup Forum* 94 (2017), pp. 94–37.

[29] V. Barucci, M. D'Anna, and R. Fröberg. "Analitically unramifed one-dimensional semilocal rings and their value semigroups". In: *J. Pure Appl. Algebra* 147 (2000), pp. 215–254.

[30] V. Barucci, M. D'Anna, and R. Fröberg. "The Apery Algorithm for a Plane Singularity with Two Branches". In: *Beiträge zur Algebra und Geometrie* 46 (2005), pp. 1–18.

[31] V. Barucci, L. Izelgue, and S. Kabbaj. "Some factorization properties of A + XB[X] domains". In: *Commutative Ring Theory*. Vol. 185. Lect. Notes Pure Appl. Math. Marcel Dekker, 1997, pp. 69–78.

[32] D. Beauquier et al. "Tiling figures of the plane with two bars". In: *Comput. Geom.* 5 (1995), pp. 1–25.

[33] S. Billey and R. Vakil. "Intersections of Schubert varieties and other permutation array schemes". In: *Algorithms in Algebraic Geometry*. Ed. by F. O. Schreyer A. Dickenstein and A J. Sommese. Vol. 146. New York: The IMA Vol. Math. Appl., Springer, 2008, pp. 21–54.

[34] M. T. L. Bizley. "Derivation of a new formula for the number of minimal lattice paths from $(0, 0)$ to (km, kn) having just t contacts with the line $my = nx$ and having no points above this line; and a proof of Grossmans formula for the number of paths which may touch but do not rise above this line". In: *J. Inst. Actuar.* 80 (1954), pp. 55–62.

[35] M. Bodner, J. Patera, and M. Peterson. "Affine reflection groups for tiling applications: Knot theory and DNA". In: *J. Math. Phys.* 53 (2012), p. 013516.

[36] E. Briales et al. "Combinatorics and syzygies for semigroup algebras". In: *Coll. Math.* 49 (1998), pp. 239–256.

[37] W. Bruns and J. Gubeladze. *Polytopes, Rings and K-theory*. New York: Springer Monographs in Mathematics, Springer, 2009.

[38] W. Bruns and J. Gubeladze. "Semigroup algebras and discrete geometry". In: *Geometry of toric varieties*. Ed. by L. Bonavero and M. Biron. Vol. 6. Sémin. Congr. Paris: Soc. Math. France, 2002, pp. 43–127.

[39] L. Carlitz. "A characterization of algebraic number fields with class number two". In: *Proc. Amer. Math. Soc.* 11 (1960), pp. 391–392.

[40] T. Chaboud. "Domino tiling in planar graphs with regular and bipartite dual". In: *Theor. Comp. Sci.* 159 (1996), pp. 137–142.

[41] S. T. Chapman and J. Coykendall. "Half-factorial domains, a survey". In: *Non-Noetherian Commutative Ring Theory*. Vol. 520. Mathematics and Its Applications. Kluwer Academic Publishers, 2000, pp. 97–115.

[42] S. T. Chapman, F. Gotti, and M. Gotti. "Factorization invariants of Puiseux monoids generated by geometric sequences". Submitted. [arXiv:1904.00219]. 2019.

[43] S. T. Chapman, F. Gotti, and M. Gotti. "How do elements really factor in $Z[\sqrt{-5}]$?" In: *Advances in Commutative Algebra*. Ed. by A. Badawi and J. Coykendall. Springer Trends in Mathematics. Singapore: Birkhäuser, 2019, pp. 171–195.

[44] S. T. Chapman, F. Gotti, and R. Pelayo. "On delta sets and their realizable subsets in Krull monoids with cyclic class groups". In: *Colloq. Math.* 137 (2014), pp. 137–146.

[45] S. T. Chapman, M. T. Holden, and T. A. Moore. “Full elasticity in atomic monoids and integral domains”. In: *Rocky Mountain J. Math.* 36 (2006), pp. 1437–1455.

[46] S. T. Chapman, U. Krause, and E. Oeljeklaus. “On Diophantine monoids and their class groups”. In: *Pacific J. Math.* 207 (2002), pp. 125–147.

[47] S. T. Chapman et al. “Computing the elasticity of a Krull monoid”. In: *Linear Algebra Appl.* 336 (2001), pp. 191–200.

[48] S. T. Chapman et al. “Delta sets of numerical monoids using nonminimal sets of generators”. In: *Comm. Algebra* 38 (2010), pp. 2622–2634.

[49] S. T. Chapman et al. “Omega-primality in embedding dimension three numerical semigroups”. In: *J. Algebra Appl.* 63 (2016), p. 1650007.

[50] A. Chavez and F. Gotti. “Dyck paths and positroids from unit interval orders”. In: *J. Combin. Theory Ser. A* 154 (2018), pp. 507–532.

[51] A. Chavez and F. Gotti. “On positroids induced by rational Dyck paths (Extended Abstract)”. In: *Séminaire Lotharingien de Combinatoire* 78B (2018), 12pp.

[52] P. M. Cohn. “Bézout rings and their subrings”. In: *Math. Proc. Cambridge Philos. Soc.* 64 (1968), pp. 251–264.

[53] J. Conway and J. Lagarias. “Tiling with polyominoes and combinatorial group theory”. In: *J. Combin. Theory Ser. A* 53 (1990), pp. 183–208.

[54] J. Coykendall. “A characterization of polynomial rings with the half-factorial property”. In: *Factorization in Integral Domains*. Vol. 189. Lect. Notes Pure Appl. Math. Marcel Dekker, 1997, pp. 291–294.

[55] J. Coykendall. “On the integral closure of a half-factorial domain”. In: *J. Pure Appl. Algebra* 180 (2003), pp. 25–34.

[56] J. Coykendall and F. Gotti. “On the atomicity of monoid algebras”. In: *J. Algebra* (2019). To appear.

[57] J. Coykendall and G. Oman. “Factorization theory of root closed monoids of small rank”. In: *Comm. Algebra* 45 (2017), pp. 2795–2808.

[58] J. Coykendall and W. W. Smith. “On unique factorization domains”. In: *J. Algebra* 332 (2011), pp. 62–70.

[59] M. D’Anna. “The canonical module of a one-dimensional reduced local ring”. In: *Comm. Algebra* 25 (1997), pp. 2939–2965.

[60] M. D’Anna et al. “Good subsemigroups of $\mathbb{N}^n$”. In: *Internat. J. Algebra Comput.* 28 (2018), pp. 179–206.

[61] B. A. Davey and H. A. Priestley. *Introduction to Lattices and Orders*. Cambridge: Cambridge University Press, 2002.

[62] R. A. Dean and G. Keller. “Natural partial orders”. In: *Canad. J. Math.* 20 (1968), pp. 535–554.

[63] P. C. Fishburn. *Interval Graphs and Interval Orders*. New York: Wiley, 1985.

[64] P. C. Fishburn. "Interval representations for interval orders and semiorders". In: *J. Math. Psych.* 10 (1973), pp. 91–105.

[65] P. C. Fishburn and B. Monjardet. "Norbert Wiener on the theory of measurement (1914, 1915, 1921)". In: *J. Math. Psych.* 36 (1992), pp. 165–184.

[66] M. Freeze and A. Geroldinger. "Unions of sets of lengths". In: *Funct. Approx. Comment. Math.* 39 (2008), pp. 149–162.

[67] S. Frisch. "A construction of integer-valued polynomials with prescribed sets of lengths of factorizations". In: *Monatsh. Math.* 171 (2013), pp. 341–350.

[68] S. Frisch, S. Nakato, and R. Rissner. "Integer-valued polynomials on ring of algebraic integers of number fields with prescribed sets of lengths of factorizations". In: abs/1710.06783 (2017).

[69] P A. García-Sánchez and M. J. Leamer. "Huneke-Wiegand Conjecture for complete intersection numerical semigroup". In: *J. Algebra* 391 (2013), pp. 114–124.

[70] P. A. García-Sánchez, I. Ojeda, and A. Sánchez-R.-Navarro. "Factorization invariants in half-factorial affine semigroups". In: *Internat. J. Algebra Comput.* 23 (2013), pp. 111–122.

[71] P. A. García-Sánchez, C. O'Neill, and G. Webb. "On the computation of factorization invariants for affine semigroups". In: *J. Algebra Appl.* 18 (2019), p. 1950019.

[72] P. A. García-Sánchez and J. C. Rosales. *Numerical Semigroups*. Vol. 20. Developments in Mathematics. New York: Springer-Verlag, 2009.

[73] I. M. Gelfand et al. "Combinatorial geometries, convex polyhedra, and Schubert cells". In: *Adv. in Math.* 63 (1987), pp. 301–316.

[74] A. Geroldinger. "On the structure and arithmetic of finitely primary monoids". In: *Czech. Math. J.* 46 (1996), pp. 677–695.

[75] A. Geroldinger. "Sets of lengths". In: *Amer. Math. Monthly* 123 (2016), pp. 960–988.

[76] A. Geroldinger. "The complete integral closure of monoids and domains". In: *PU.M.A.* 4 (1993), pp. 147–165.

[77] A. Geroldinger and F. Halter-Koch. "Congruence monoids". In: *Acta Arith.* 112 (2004), pp. 263–296.

[78] A. Geroldinger and F. Halter-Koch. "Non-unique factorizations: a survey". In: *Multiplicative Ideal Theory in Commutative Algebra*. Ed. by J. W. Brewer et al. Advanced Courses in Mathematics CRM Barcelona. New York: Springer, 2006, pp. 207–226.

[79] A. Geroldinger and F. Halter-Koch. *Non-unique Factorizations: Algebraic, Combinatorial and Analytic Theory*. Vol. 278. Pure and Applied Mathematics. Boca Raton: Chapman & Hall/CRC, 2006.

[80] A. Geroldinger, F. Halter-Koch, and G. Lettl. "The complete integral closure of monoids and domains II". In: *Rendiconti di Matematica Serie VII* 15 (1995), pp. 281–292.

[81] A. Geroldinger, W. Hassler, and G. Lettl. "On the arithmetic of strongly primary monoids". In: *Semigroup Forum* 75 (2007), pp. 567–587.

[82] A. Geroldinger and W. Schmid. "A realization theorem for sets of lengths in numerical monoids". In: *Forum Math.* 30 (2018), pp. 1111–1118.

[83] A. Geroldinger and Q. Zhong. "Sets of arithmetical invariants in transfer Krull monoids". In: *J. Pure Appl. Algebra* 223 (2019), pp. 3889–3918.

[84] A. Geroldinger et al. "Finitary monoids". In: *Semigroup Forum* 67 (2003), pp. 1–21.

[85] R. Gilmer. "A two-dimensional non-Noetherian factorial ring". In: *Proc. Amer. Math. Soc.* 44 (1974), pp. 25–30.

[86] R. Gilmer. *Commutative Semigroup Rings*. Chicago Lectures in Mathematics. London: The University of Chicago Press, 1984.

[87] R. Gilmer and T. Parker. "Divisibility properties of semigroup rings". In: *Mich. Math. J.* 21 (1974), pp. 65–86.

[88] R. Gilmer and T. Parker. "Nilpotent elements of commutative semigroup rings". In: *Mich. Math. J.* 22 (1975), pp. 97–108.

[89] R. Gipson and H. Kulosman. "Atomic and AP semigroup rings F[X;M], where M is a submonoid of the additive monoid of nonnegative rational numbers". In: *Int. Electron. J. Algebra* 22 (2017), pp. 133–146.

[90] F. Gotti. "Atomic and antimatter semigroup algebras with rational exponents". In: *Proceedings of the International Meeting of Numerical Semigroups*. Ed. by V. Barucci et al. To appear. Available at https://arxiv.org/abs/1801.06779. Switzerland: Springer Nature, 2019.

[91] F. Gotti. "Geometric aspects of finite-rank submonoids of a free commutative monoid". Under preparation. 2019.

[92] F. Gotti. "Increasing positive monoids of ordered fields are FF-monoids". In: *J. Algebra* 518 (2019), pp. 40–56.

[93] F. Gotti. "On positroids induced by rational Dyck paths (Extended Abstract)". In: *Séminaire Lotharingien de Combinatoire* 80B (2018), 12pp.

[94] F. Gotti. "On the atomic structure of Puiseux monoids". In: *J. Algebra Appl.* 16 (2017), p. 1750126.

[95] F. Gotti. "On the system of sets of lengths and the elasticity of submonoids of a finite-rank free commutative monoid". In: *J. Algebra Appl.* (2019). To appear. Available at https://arxiv.org/abs/1806.11273.

[96] F. Gotti. "Positroids induced by rational Dyck paths". Submitted. [arXiv:1706.09921]. 2018.

[97] F. Gotti. "Puiseux monoids and transfer homomorphisms". In: *J. Algebra* 516 (2018), pp. 95–114.

[98] F. Gotti. "Systems of sets of lengths of Puiseux monoids". In: *J. Pure Appl. Algebra* 223 (2019), pp. 1856–1868.

[99] F. Gotti and M. Gotti. "Atomicity and boundedness of monotone Puiseux monoids". In: *Semigroup Forum* 96 (2018), pp. 536–552.

[100] F. Gotti and M. Gotti. "On the molecules of numerical semigroups, Puiseux monoids, and Puiseux algebras". In: *Proceedings of the International Meeting of Numerical Semigroups*. Ed. by V. Barucci et al. To appear. [arXiv:1702.08270v2]. Switzerland: Springer Nature, 2019.

[101] F. Gotti and C. O'Neil: "The elasticity of Puiseux monoids". In: *J. Commut. Algebra* (2019). To appear. Available at https://arxiv.org/abs/1703.04207.

[102] F. Gotti and H. Polo. "Tilings and matroids on regular subdivisions of a triangle". Submitted. Available at https://arxiv.org/abs/1802.05633. 2018.

[103] M. Gotti. "On the local k-elasticities of Puiseux monoids". In: *Internat. J. Algebra Comput.* 29 (2019), pp. 147–158.

[104] A. Grams. "Atomic domains and the ascending chain condition for principal ideals". In: *Math. Proc. Cambridge Philos. Soc.* 75 (1974), pp. 321–329.

[105] P. A. Grillet. *Commutative Semigroups*. Vol. 2. Advances in Mathematics. Boston: Kluwer Academic Publishers, 2001.

[106] H. D. Grossman. "Fun with lattice points: paths in a lattice triangle". In: *Scripta Math.* 16 (1950), pp. 207–212.

[107] J. Gubeladze. "The isomorphism problem for commutative monoid rings". In: *J. Pure Appl. Algebra* 129 (1998), pp. 35–65.

[108] J. Haarmann et al. "Factorization properties of Leamer monoids". In: *Semigroup Forum* 89 (2014), pp. 409–421.

[109] F. Halter-Koch. "Divisor theories with primary elements and weakly Krull domains". In: *Boll. Un. Mat. Ital. B* 9 (1995), pp. 417–441.

[110] F. Halter-Koch. "Elasticity of factorizations in atomic monoids and integral domains". In: *J. Théor. Nombres Bordeaux* 7 (1995), pp. 367–385.

[111] F. Kainrath. "Factorization in Krull monoids with infinite class group". In: *Colloq. Math.* 80 (1999), pp. 23–30.

[112] F. Kainrath and G. Lettl. "Geometric notes on monoids". In: *Semigroup Forum* 61 (2000), pp. 298–302.

[113] R. Kenyon. "A note on tiling with integer-sided rectangles". In: *J. Combin. Theory Ser. A* 74 (1996), pp. 321–332.

[114] H. Kim. "Factorizations in monoid domains". In: *Comm. Algebra* 29 (2001), pp. 1853–1869.

[115] Y. Kodama and L. K. Williams. "KP solitons and total positivity on the Grassmannian". In: *Invent. Math.* 198 (2014), pp. 637–699.

[116] R. Kumanduri and C. Romero. *Number Theory With Computer Applications*. New Jersey: Prentice Hall, 1998.

[117] R. Lidl and H. Niederreiter. *Introduction to Finite Fields and their Applications*. Cambridge: Cambridge University Press, 1986.

[118] M. Luby, D. Randall, and A. Sinclair. "Markov chain algorithms for planar lattice structures". In: *SIAM J. Comput.* 31 (2001), pp. 167–192.

[119] R. D. Luce. "Semiorders and a theory of utility discrimination". In: *Econometrica* 24 (1956), pp. 178–191.

[120] N. Maugeri and G. Zito. "Embedding dimension of a good semigroup". In: abs/1903.02057 (2019).

[121] E. Miller and B. Sturmfels. *Combinatorial Commutative Algebra*. Vol. 227. Graduate Texts in Mathematics. New York: Springer-Verlag, 2005.

[122] W. Narkiewicz. *Elementary and analytic theory of algebraic numbers*. Springer Monographs in Mathematics. third edition. Berlin: Springer-Verlag, 2004.

[123] W. Narkiewicz. "Finite abelian groups and factorization problems". In: *Coll. Math.* 42 (1979), pp. 319–330.

[124] S. Oh. "Positroids and Schubert matroids". In: *J. Combin. Theory Ser. A* 118 (2011), pp. 2426–2435.

[125] J. Oxley. *Matroid Theory*. New York: Oxford University Press, 1992.

[126] M. Petrich. *Introduction to Semigroups*. Charles E. Merrill Publishing Co., 1973.

[127] A. Postnikov. "Total positivity, Grassmannians, and networks". In: abs/math/0609764 (2006).

[128] R. T. Rockafellar. *Convex Analysis*. Vol. 28. Princeton Landmarks in Mathematics. New Jersey: Princeton Univ. Press, 1970.

[129] M. Roitman. "On the atomic property for power series rings". In: *J. Pure Appl. Algebra* 145 (2000), pp. 309–319.

[130] M. Roitman. "Polynomial extensions of atomic domains". In: *J. Pure Appl. Algebra* 87 (1993), pp. 187–199.

[131] F. Santos. "The Cayley Trick and triangulations of products of simplices". In: *Integer Points in Polyhedra – Geometry, Number Theory, Algebra, Optimization*. Ed. by A. Barvinok et al. Vol. 374. Providence, Rhode Island: Contemporary Mathematics, AMS, 2005, pp. 151–177.

[132] D. Scott. "Measurement structures and linear inequalities". In: *J. Math. Psych.* 1 (1964), pp. 233–247.

[133] J. S. Scott. "Grassmannians and cluster algebras". In: *Proc. London Math. Soc.* 92 (2006), pp. 345–380.

[134] M. Skandera and B. Reed. "Total nonnegativity and (3+1)-free posets". In: *J. Combin. Theory Ser. A* 103 (2003), pp. 237–256.

[135] L. Skula. "On c-semigroups". In: *Acta Arith.* 31 (1976), pp. 247–257.

[136] J. Śliwa. "Factorizations of distinct lengths in algebraic number fields". In: *Acta Arith.* 31 (1976), pp. 399–417.

[137] R. P. Stanley. *Catalan Numbers*. Cambridge: Cambridge University Press, 2015.

[138] R. P. Stanley and F. Zanello. "The Catalan case of Armstrongs conjecture on core partitions". In: *SIAM J. Discrete Math.* 29 (2015), pp. 658–666.

[139] E. Steingrimsson and L. K Williams. "Permutation tableaux and permutation patterns". In: *J. Combin. Theory Ser. A* 114 (2007), pp. 211–234.

[140] T. Tamura. "Basic study of N-semigroups and their homomorphisms". In: *Semigroup Forum* 8 (1974), pp. 21–50.

[141] S. Tringali. "Is the elasticity of a submonoid of the free abelian monoid over a finite set either rational or infinite?" Nov. 2016.

[142] W. T. Trotter. *Combinatorics and Partially Ordered Sets: Dimension Theory*. Baltimore: Johns Hopkins University Press, 1992.

[143] R. J. Valenza. "Elasticity of factorization in number fields". In: *J. Number Theory* 36 (1990), pp. 212–218.

[144] D. J. A. Welsh. *Matroid Theory*. London: Academic Press Inc., 1976.

[145] N. Wiener. "A contribution to the theory of relative position". In: *Proc. Cambridge Philos. Soc.* 17 (1914), pp. 441–449.

[146] R. L. Wine and J. E. Freund. "On the enumeration of decision patterns involving n means". In: *Ann. Math. Stat.* 28 (1957), pp. 256–259.

[147] A. Zaks. "Atomic rings without a.c.c. on principal ideals". In: *J. Algebra* 74 (1982), pp. 223–231.

[148] A. Zaks. "Half-factorial domains". In: *Bull. Amer. Math. Soc.* 82 (1976), pp. 721–723.

[149] A. Zaks. "Half-factorial domains". In: *Israel J. Math.* 37 (1980), pp. 281–302.