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Investigating the Language Demands and Resources for Multilingual Students in Inquiry-Based Undergraduate Mathematics Courses: The Case of Inquiry-Oriented Linear Algebra

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Investigating the Language Demands and Resources for Multilingual Students in Inquiry-Based Undergraduate Mathematics Courses: The Case of Inquiry-Oriented Linear Algebra

A dissertation submitted in partial satisfaction of the
requirements for the degree Doctor of Philosophy

in

Mathematics and Science Education

by

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2024

The Dissertation of Ernesto Daniel Calleros is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

Chair

University of California San Diego

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2024

DEDICATION

Dedico esta tesis a mi hermosa, inteligente y solidaria mamá, María Concepción Calleros Chavoya. Madre, eres mi más grande inspiración. Gracias por siempre creer en mí, apoyarme para lograr mis metas e inspirar mi amor por la comunidad a través de tu ejemplo. ¡Te amo con todo mi corazón!

I dedicate this dissertation to my beautiful, intelligent and supportive mom, María Concepción Calleros Chavoya. Mother, you are my greatest inspiration. Thank you for always believing in me, supporting me to achieve my goals, and inspiring my love for the community through your example. I love you with all my heart!

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

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ABSTRACT OF THE DISSERTATION

Investigating the Language Demands and Resources for Multilingual Students in Inquiry-Based Undergraduate Mathematics Courses: The Case of Inquiry-Oriented Linear Algebra

by

Ernesto Daniel Calleros

Doctor of Philosophy in Mathematics and Science Education

University of California San Diego, 2024
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Inquiry-based mathematics education (IBME) includes features such as: (a) Using tasks with authentic problem contexts, (b) building on students' everyday resources, (c) relying on small-group and whole-class discussions, and (d) establishing certain norms of participation for the inquiry classroom community (Laursen & Rasmussen, 2019). Prior research on active learning (which includes IBME) suggests inquiry approaches are more effective than lectures in undergraduate mathematics education (Freeman et al., 2014). However, inquiry-based approaches might not yield equal benefits for students from certain marginalized groups (e.g., women in Johnson et al., 2020). One important group to consider is multilingual students whose primary language differs from the language of instruction. In this dissertation, I conducted a multi-case study analysis to explore an overarching question related to equity for multilingual students: What language demands and resources do multilingual students experience in one inquiry-oriented linear algebra (IOLA) course?

Grounded in a situated sociocultural theory of learning, I constructed a framework that captures language demands and resources along three interrelated dimensions: lexico-grammatical, situational, and normative. The data collected were classroom observations of one IOLA course taught by an expert instructor, as well as semi-structured interviews with 4 multilingual students from the course. The 4 students spanned a diverse range of linguistic and cultural backgrounds (Korean, Vietnamese, Malaysian, and Latino) and comfort levels with English. The interview data was analyzed using inductive and deductive coding (Miles et al., 2019).

The study's findings show that (a) authentic problem contexts use complex language, especially for multilingual students; (b) "everyday" language resources for students from the dominant community might not function as such for multilingual students; (c) an emphasis on verbal participation can obscure multilingual students' communication resources; and (d) inquiry classrooms can induce norm tensions about communication for multilingual students when the norms of their communities outside the inquiry classroom are not explicitly considered. This study underscores the opportunity to continue improving IBME to address the language demands induced for multilingual students by incorporating instructional language resources and leveraging the students' language resources.

Chapter 1 Introduction

General Motivation

Based on prior research findings that indicate active learning is more effective overall than traditional lecturing (Freeman et al., 2014), undergraduate STEM courses are increasingly adopting active learning instructional approaches (Cooper et al., 2018). One active learning approach that has gained traction in undergraduate mathematics is inquiry-based mathematics education (Laursen & Rasmussen, 2019). As Laursen and Rasmussen (2019) noted, two major strands of inquiry-based mathematics education in the U.S. context are inquiry-based learning and inquiry-oriented instruction, but their "similarities are more important than the (apparent) differences" (p. 132). In both strands, instructors orchestrate whole-class and small-group discussions that build on students' mathematical thinking and lead students toward exploring and reinventing important mathematical ideas and procedures. Existing research suggests that inquiry-based mathematics education is more beneficial overall than traditional lecturing in several ways, including student learning, student retention of knowledge, and student empowerment (e.g., Ju & Kwon, 2007; Kogan & Laursen, 2014; Kwon et al., 2005; Laursen et al., 2014; Rasmussen et al., 2006; Zandieh et al., 2017).

However, while inquiry-based mathematics education may be beneficial on average, researchers have begun to explore the unequal distribution of learning opportunities for certain student groups in inquiry-based contexts. For example, examining content assessment data from 522 students, Johnson et al. (2020) found that inquiry-oriented abstract algebra benefited men more than women. In particular, mathematical performance was lower for women than for men in the inquiry-oriented classes represented in the study, while women performed equally well in inquiry-oriented and lecture-based courses. Johnson et al. (2020) hypothesized this difference might be due to gender-based microaggressions that emerge in inquiry-oriented classrooms during small-group discussions or due to implicit bias when the instructor seeks to build on students' mathematical ideas to orchestrate discussions. For instance, in a preliminary analysis, they found that instructors called less on women than on men to respond to mathematically substantive questions.

Findings such as the results of Johnson et al. (2020) raise questions about which other student groups might also be differentially impacted by inquiry-based undergraduate mathematics classes. One important group to consider is multilingual students who are learning the language of instruction. In the U.S. context, this group might include international students as well as domestic students who were classified as English Learners at some point in their prior schooling. Attending to the impact of inquiry-based mathematics education on multilingual undergraduate students is important because inquiry-based undergraduate math courses are usually taught only in English and these courses may induce different language requirements than lectures do. Simultaneously, the U.S. undergraduate population is becoming more linguistically diverse, as predicted by demographic shifts in the K-12 population and enrollment changes in the undergraduate population. According to the National Center for Educational Statistics (NCES, 2021a), the proportion of students classified as English Learners (ELs) in K-12 public schools in the U.S increased by 11% (half a million students) from fall 2010 to fall 2018. The same data source documented that in fall 2018, students classified as ELs made up 19.4% of student enrollment in K-12 public schools in California. Moreover, NCES (2021b) documented an increase of 48% in the U.S. undergraduate enrollment of Latino students (many of whom may be multilingual students who speak a language other than English at home) between fall 2009 and fall 2019, even when the total undergraduate enrollment decreased by 5%. Thus, attending to the increase of linguistic diversity is critical for making undergraduate mathematics education more equitable in order to address national calls to increase the number of college graduates with STEM degrees to meet the predicted needs of the U.S. workforce (President's Council of Advisors on Science and Technology, 2012).

Existing research on the language demands of inquiry-based mathematics education is limited. By language demands, I refer to how certain tools in the classroom limit or complicate the communication and interpretation of mathematical information. Furthermore, there is a significant gap in the literature on how language demands may specifically impact multilingual students. This gap is especially prevalent in research on undergraduate mathematics education. To address this gap, this dissertation study aims to identify the language demands of inquiry-based mathematics education in undergraduate classes. In addition, the study seeks to identify the language resources – the ways that certain tools amplify or facilitate communication or interpretation of mathematical information – that could address such language demands in these classes. I will investigate these research aims predominantly from the perspectives of multilingual undergraduate students to amplify their voices and lived experiences as multilingual learners.

To begin this investigation, this study will focus on an inquiry-oriented linear algebra course (based on an expanded version of the curriculum developed by Wawro et al. (2013)) as a case study of how multilingual learners may experience and engage with inquiry-based mathematics education. Using a combination of ethnographic classroom observations (Patton, 2002) and semi-structured interviews (diSessa, 2007) with 4 multilingual students, I will investigate the language demands and resources in an inquiry-oriented linear algebra class.

Personal Motivation and Positionality

Beyond demographic shifts and prior research, my educational and professional experiences have also motivated my interest in pursuing this dissertation study. These experiences include my past educational journey as a multilingual Spanish-speaking undergraduate student as well as my experiences teaching and conducting research about language and undergraduate multilingual learners.

As a former multilingual undergraduate student who was learning English while taking mathematics courses, I experienced first-hand many of the challenges that other multilingual students might face. When I took discussion-based mathematics courses, at times I felt mathematically incompetent and lost due to the fuzziness and lack of organization I perceived in these courses. It was difficult for me to follow seemingly never-ending conversations that did not appear to clearly signal when an important mathematical idea, procedure, or definition had been developed. Consequently, I did not know when or how to ask clarifying questions, join the class conversation, or take notes productively. An added challenge was that, while I was expected to communicate during whole-class discussions, I felt that Spanish (my home language) was not valued in these discussions. Besides, in these classes I was one of the few Spanish speakers, so the likelihood of being grouped with students who understood my home language during small-group discussions was low. In contrast, in lecture-based courses, I felt mathematically and communicatively competent. There were clear signals of what mathematical information was valuable to write down during class, so I felt confident taking notes and asking clarifying questions about important ideas that did not make sense to me.

I have gathered similar stories from other multilingual students about their experiences in discussion and lecture-based mathematics courses. These stories have come from my past undergraduate peers as well as my own former undergraduate students. My personal experiences and the stories from other multilingual students have led to my interest in investigating the language demands of inquiry-based mathematics education (special types of discussion-based courses) with a predominant focus on the perspectives of multilingual students, which is the goal of this dissertation study. In addition, being familiar with the experiences of many multilingual undergraduate students, including myself in the past, will uniquely position me to capture and interpret the lived experiences of multilingual students from an asset-based perspective during my dissertation study.

Personal experiences aside, I also have experience conducting NSF-funded research about language and K-12 students classified as ELs in mathematics classrooms. For example, I have conducted classroom observations in linguistically diverse classrooms, designed and conducted interviews with ELs, and designed discussion-based learning environments in K-12 mathematics classrooms that are more linguistically equitable for all students. These experiences have led me to conceptualize the constructs of language demands and resources in discussion-based undergraduate mathematics classes, such as inquiry-based classrooms. In addition, these experiences will better equip me to collect and analyze classroom observations and interviews in relation to the language demands and resources in an inquiry-based undergraduate mathematics class.

The Statement of Issue

To highlight why it is important to consider the role of language in undergraduate inquiry-based mathematics classes, I will first discuss the origin and nature of inquiry-based mathematics education. Then, I will argue how language might play a more critical role in inquiry-based undergraduate mathematics classes than lecture-based classes.

What is Inquiry-Based Mathematics Education?

Laursen and Rasmussen (2019) described the relationship between active learning and inquiry-based mathematics education. They stated that inquiry approaches emerged as a branch of active learning, which arose in contrast to a traditional lecturing approach (Freeman et al., 2014). Grounded on theories of learning that highlight the need for students to construct their own knowledge, active learning challenges the teacher-centered transmission perspective reflected in a lecturing approach. Active learning strategies engage students in doing and thinking as well as talking with their peers about what they are doing and thinking (Bonwell & Eison, as cited in Laursen & Rasmussen, 2019). This doing and thinking can be done through reading, writing, discussing, or solving problems, and must engage students in higher-order thinking tasks, including analyzing and synthesizing (Bonwell & Eison, as cited in Laursen & Rasmussen, 2019; CBMS, 2016). In active learning contexts, the teacher is responsible for orchestrating students' mathematical doing, thinking, and talking through selecting important mathematical ideas and crafting and implementing tasks that encourage students to think deeply about those ideas (Laursen & Rasmussen, 2019).

Laursen and Rasmussen (2019) noted that inquiry-based mathematics education exhibits all the features of active learning with several additional distinguishing characteristics. First, the inquiry-based instructional materials are based on learning trajectories that extend over relatively long periods of time (e.g., weeks). Such inquiry curricula consist of instructional sequences of tasks that cohere over several daily lessons of instruction to lead students toward big ideas, such as proving a major theorem or creating a definition or procedure. Instructors craft and enact these task sequences by inquiring into and capitalizing on students' mathematical ideas. Second, mathematical activity in inquiry-based classes engages students in authentic mathematical practices, akin to those of practicing mathematicians. These practices include exploring patterns, conjecturing, proving theorems, defining, creating and using algorithms, modeling, and comparing solutions (Moschkovich, 2007; Rasmussen et al., 2005). Moreover, Laursen and Rasmussen (2019) argued that inquiry-based mathematics classrooms tend to engage students in different communication types and settings, including small-group discussions as well as presentations and discussions in a whole-class setting.

Language Seen as an Important Factor in Inquiry-Based Mathematics Education

Focusing on an IOLA undergraduate class as a case study, I will share initial observations about language demand in inquiry-based undergraduate mathematics courses relative to lecture-based classes. To construct these observations, I will compare an instructional task designed as part of the inquiry-oriented linear algebra (IOLA) curriculum (Wawro et al., 2012) with a section of a traditional linear algebra textbook (Larson, 2016). How information is presented in a traditional textbook may resemble how information is presented in lecture-based classes.

I will first compare these two instructional materials in terms of their mathematical foci, situating them in their respective lessons or book sections. Then, I will compare these materials in relation to language demand, illustrating ways in which an inquiry-based undergraduate mathematics class may induce different language demands than a lecture-based class.

Mathematical Comparison: IOLA Curriculum Versus a Traditional Textbook. Both the lesson from the inquiry-oriented course and the book section from the traditional textbook have a similar mathematical focus in an undergraduate linear algebra course. Both instructional materials aim to build span and linear independence from the concept of linear combinations. The inquiry-oriented lesson relies on solving and discussing the solution of a single mathematical task called The Carpet Ride Problem: The Maiden Voyage (Wawro et al., 2012), shown in Figure 1.1. This task centers around the story of a young person who can travel with a hoverboard and a magic carpet, and it asks students to determine and justify whether the traveler can reach a given location using these two forms of transportation. Mathematically, each mode of transportation and any given destination can be represented as a vector. So, a particular journey with these modes of transportation can be represented with a linear combination of these two vectors. Thus, whether there is a combination of the two forms of transportation that can reach a particular location corresponds to whether there is a linear combination of vectors that can equal a particular vector–that is, whether there is a solution to a vector equation (and to a system of equations). This task is the first in an instructional sequence of four tasks targeting student reinvention of the concepts of span and linear independence (Wawro et al., 2012).

On the other hand, Larson's (2016) traditional linear algebra textbook devotes a particular book section (section 4.4), partially shown in Figures 1.2 and 1.3, to the topic of span and linear independence. Similar to The Carpet Ride Problem: The Maiden Voyage, this section builds from the concept of linear combinations, but with a different instructional approach. Rather than letting students reinvent the concept and definition of linear combinations through exploration of a problem, this section begins with a formal definition of linear combinations and then presents several examples, observations or theorems. The first example illustrates the definition of linear combinations by presenting two examples of linear combinations. The second example illustrates a procedure for finding the coefficients in a linear combination (through setting up a system of equations) by asking students to write a particular vector as a linear combination of three other vectors. The third example illustrates a non-example, that is, a case in which a vector is not a linear combination of the other vectors. Then the topics of spanning sets and linear independence are presented using a similar organization of information–a formal definition, basic examples and non-examples of the definition, examples of relevant procedures, and relevant observations or theorems.

THE MAGIC CARPET RIDE PROBLEM

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on thehover board's movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 miles East and 1 mile North of its starting location.

We denote the restriction on the *magic carpet's* movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 mile East and 2 miles North of its starting location.

PROBLEM ONE: THE MAIDEN VOYAGE

Your Uncle Cramersuggests that your first adventure should be to go visit the wise man, Old Man Gauss. Uncle Cramer tells you that Old Man Gauss lives in a cabin that is 107 miles East and 64 miles North of your home.

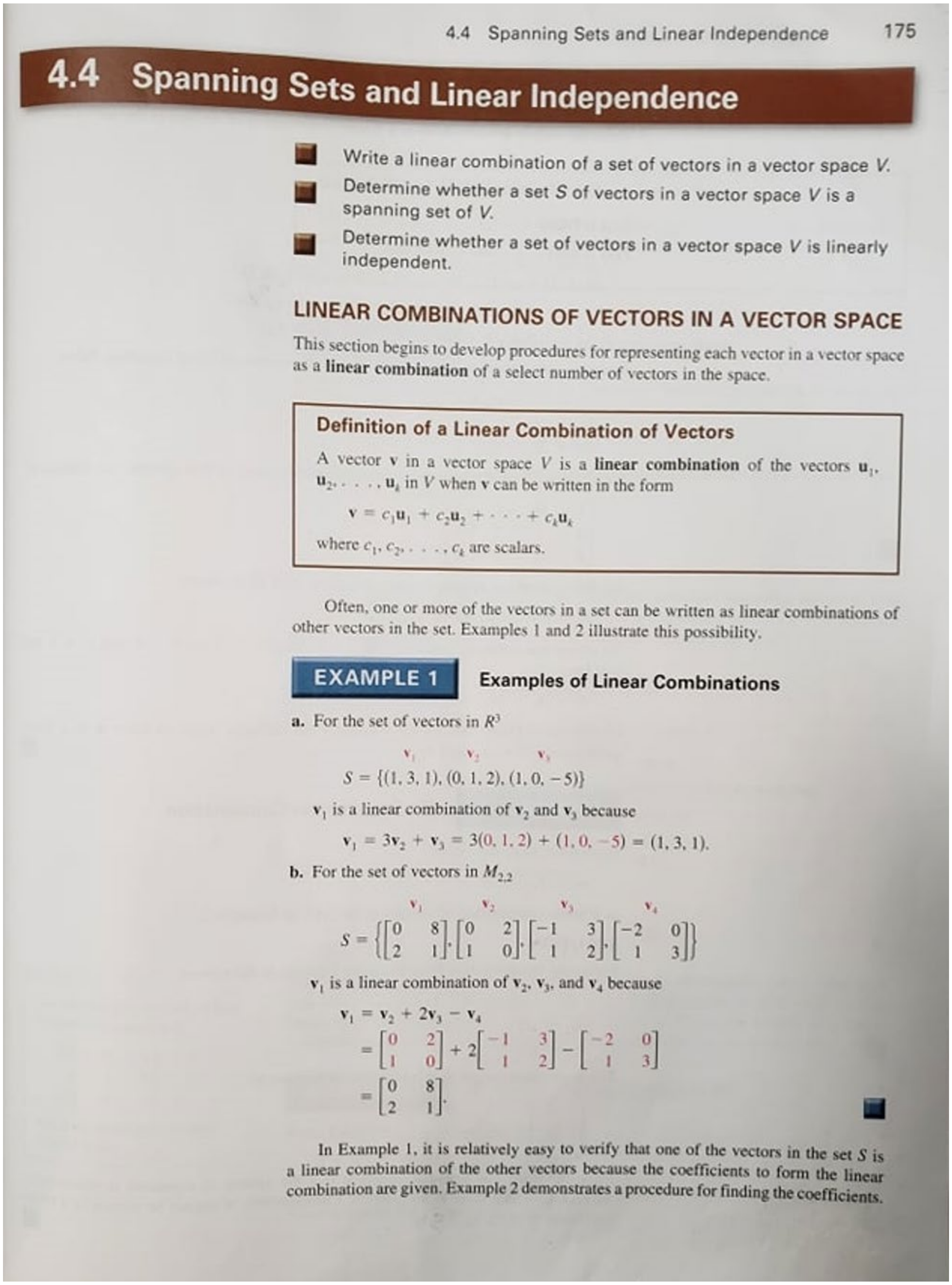
TASK:

Investigate whether or not you can use the hover board and the magic carpet to get to Gauss's cabin.

If so, how? If it is not possible to get to the cabin with these modes of transportation, why is that the case? Use the vector notation for each mode of transportation as part of your explanation. Use a diagram or graphic to help illustrate your point(s).

Figure 1.1:

The First Problem in an Inquiry-Oriented Linear Algebra Course (Wawro et al., 2012)



4.4 Spanning Sets and Linear Independence 175

4.4 Spanning Sets and Linear Independence

- Write a linear combination of a set of vectors in a vector space V.
- Determine whether a set S of vectors in a vector space V is a spanning set of V.
- Determine whether a set of vectors in a vector space V is linearly independent.

LINEAR COMBINATIONS OF VECTORS IN A VECTOR SPACE

This section begins to develop procedures for representing each vector in a vector space as a **linear combination** of a select number of vectors in the space.

Definition of a Linear Combination of Vectors

A vector $\mathbf{v}$ in a vector space V is a **linear combination** of the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ in V when $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$$

where $c_1, c_2, \ldots, c_k$ are scalars.

Often, one or more of the vectors in a set can be written as linear combinations of other vectors in the set. Examples 1 and 2 illustrate this possibility.

EXAMPLE 1 **Examples of Linear Combinations**

a. For the set of vectors in R^3

$$S = \{\underset{\mathbf{v}_1}{(1, 3, 1)}, \underset{\mathbf{v}_2}{(0, 1, 2)}, \underset{\mathbf{v}_3}{(1, 0, -5)}\}$$

$\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$ because

$$\mathbf{v}_1 = 3\mathbf{v}_2 + \mathbf{v}_3 = 3(0, 1, 2) + (1, 0, -5) = (1, 3, 1).$$

b. For the set of vectors in $M_{2,2}$

$$S = \left\{\underset{\mathbf{v}_1}{\begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix}}, \underset{\mathbf{v}_2}{\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}}, \underset{\mathbf{v}_3}{\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}}, \underset{\mathbf{v}_4}{\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}}\right\}$$

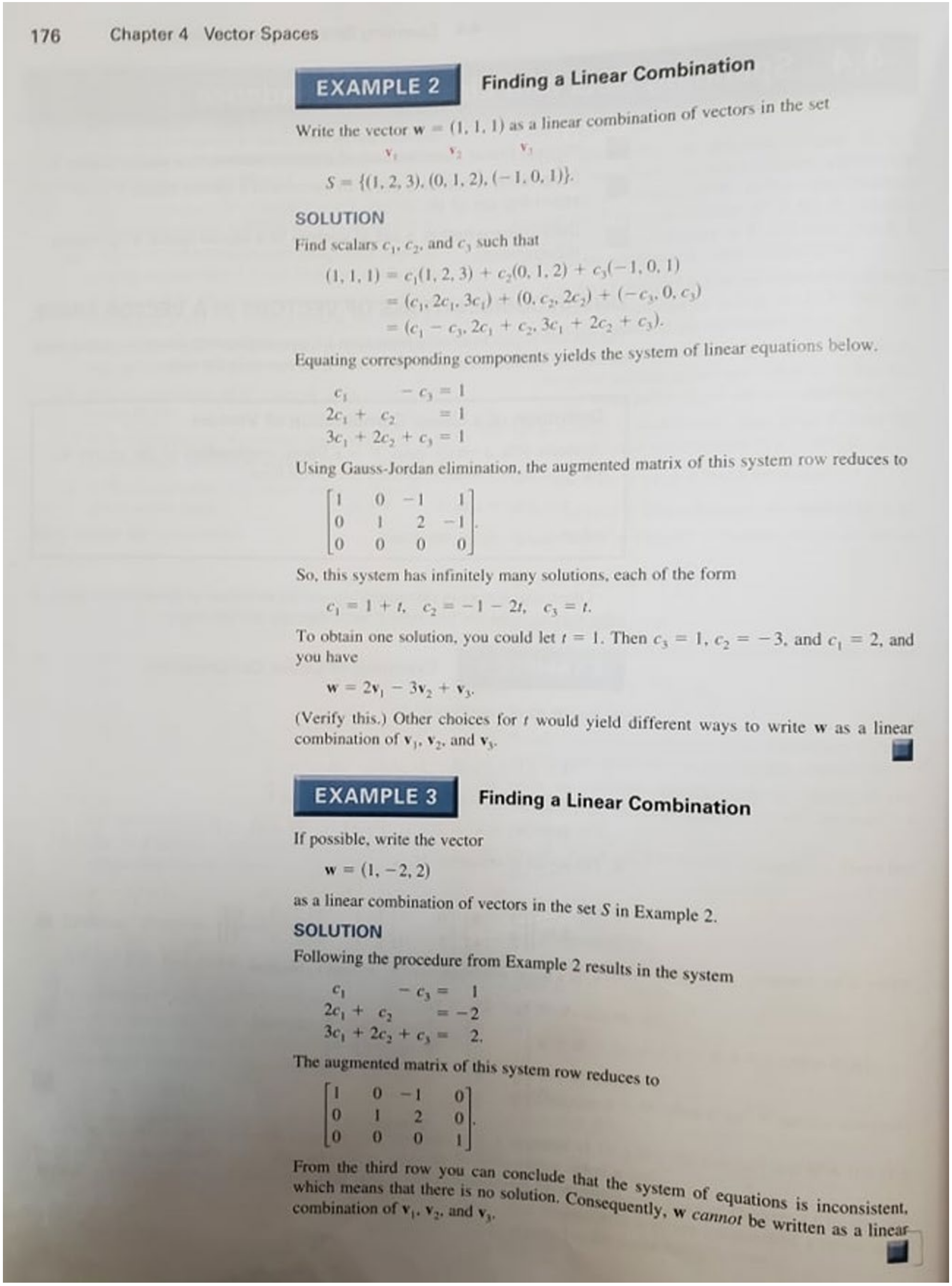
$\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ because

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{v}_2 + 2\mathbf{v}_3 - \mathbf{v}_4 \\ &= \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix}. \end{aligned}$$

In Example 1, it is relatively easy to verify that one of the vectors in the set S is a linear combination of the other vectors because the coefficients to form the linear combination are given. Example 2 demonstrates a procedure for finding the coefficients.

Figure 1.2:

First Page of a Section from a Traditional Textbook (Larson, 2016)



176 Chapter 4 Vector Spaces

EXAMPLE 2 **Finding a Linear Combination**

Write the vector $\mathbf{w} = (1, 1, 1)$ as a linear combination of vectors in the set

v_1 v_2 v_3

$$S = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}.$$

SOLUTION

Find scalars c_1, c_2, and c_3 such that

$$\begin{aligned}(1, 1, 1) &= c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-1, 0, 1)\\ &= (c_1, 2c_1, 3c_1) + (0, c_2, 2c_2) + (-c_3, 0, c_3)\\ &= (c_1 - c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3).\end{aligned}$$

Equating corresponding components yields the system of linear equations below.

$$\begin{aligned} c_1 \quad\quad\quad - c_3 &= 1\\ 2c_1 + c_2 \quad\quad\quad &= 1\\ 3c_1 + 2c_2 + c_3 &= 1\end{aligned}$$

Using Gauss-Jordan elimination, the augmented matrix of this system row reduces to

$$\begin{bmatrix} 1 & 0 & -1 & 1\\ 0 & 1 & 2 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

So, this system has infinitely many solutions, each of the form

$$c_1 = 1 + t, \quad c_2 = -1 - 2t, \quad c_3 = t.$$

To obtain one solution, you could let $t = 1$. Then $c_3 = 1$, $c_2 = -3$, and $c_1 = 2$, and you have

$$\mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3.$$

(Verify this.) Other choices for t would yield different ways to write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

EXAMPLE 3 **Finding a Linear Combination**

If possible, write the vector

$$\mathbf{w} = (1, -2, 2)$$

as a linear combination of vectors in the set S in Example 2.

SOLUTION

Following the procedure from Example 2 results in the system

$$\begin{aligned} c_1 \quad\quad\quad - c_3 &= 1\\ 2c_1 + c_2 \quad\quad\quad &= -2\\ 3c_1 + 2c_2 + c_3 &= 2.\end{aligned}$$

The augmented matrix of this system row reduces to

$$\begin{bmatrix} 1 & 0 & -1 & 0\\ 0 & 1 & 2 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

From the third row you can conclude that the system of equations is inconsistent, which means that there is no solution. Consequently, $\mathbf{w}$ *cannot* be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Figure 1.3:

Second Page of a Section from a Traditional Textbook (Larson, 2016)

Language Comparison: IOLA Task Versus Traditional Textbook Examples. Immediately, one can notice stark differences in language between the inquiry-oriented task and the examples in the traditional textbook. Some of these differences relate to the amount and familiarity of the language induced by the inquiry-oriented task or the textbook examples as well as to the imagined ways these instructional materials might be taught or learned.

Amount and Familiarity of Language. There are notable differences between the inquiry-oriented task and the examples in the traditional textbook in terms of the amount and familiarity of the language presented and requested. For example, the length of the text presented in The Carpet Ride Problem: The Maiden Voyage is several paragraphs, whereas each of the problem statements in the traditional textbook's examples are one or two sentences each. Thus, students need to interpret or receive much more text in The Carpet Ride Problem: The Maiden Voyage than in the examples from the traditional textbook to make sense of the problems. Yet, the entirety of the text in the inquiry-oriented task revolves around a single non-mathematical context (motion with two modes of transportation), so that students can draw on everyday resources to make sense of this task, whereas the problem statements in the traditional text examples are exclusively provided in mathematical language and draw on multiple forms of notation (e.g., vectors, matrices, and special symbols such as $\mathbf{R}^3$). These differences reflect the fact that inquiry-oriented instruction attempts to design tasks that build from contexts that are realistic, or imaginable, to students (Gravemeijer, 1999; Gravemeijer & Doorman, 1999), whereas traditional texts do not follow this task/example design heuristic. In terms of language demands, the issue is that the same context can be familiar to some students and unfamiliar to others. In a pilot interview with a Chinese-English multilingual engineering graduate student at a university in Southern California, I found that the terms *hover board* and *magic carpet* in The

Carpet Ride Problem: The Maiden Voyage were unfamiliar to him and initially got him stuck on the problem. This observation indicates that embedding mathematics in seemingly "realistic" problem contexts, as inquiry-oriented tasks generally do, may induce an unanticipated language demand for some students, in particular multilingual students. On the other hand, the examples in the traditional text, which may resemble the examples presented in lecture-based courses, did not require students to make sense of any non-mathematical contexts. However, these examples also don't allow students to draw on everyday resources, thereby potentially creating a different yet critical linguistic challenge for students who may not already be familiar with the exhibited mathematical language and notation in these examples. Another difference is that, while the examples in the traditional text only explicitly ask for answers, The Carpet Ride Problem: The Maiden Voyage requests answers with elaborated explanation and illustration that engage students in language-intensive mathematical practices. For example, through questions and directions such as "Why is that the case?", "Use … as part of your explanation", and "Illustrate your point," the problem statement in The Carpet Ride Problem: The Maiden Voyage might engage students in the practices of explaining, justifying, and exemplifying. Moreover, by asking students to include "vector notation" and a "diagram or graphic" in their justification, this task also encourages students to participate in the practice of connecting multiple representations. Thus, students are asked to produce more language in The Carpet Ride Problem: The Maiden Voyage than in the traditional textbook's examples.

Language Related to Ways These Materials Might be Enacted. Differences in language demand may become more visible when The Carpet Ride Problem: The Maiden Voyage is enacted in a live inquiry-based class and the examples from the traditional text are embedded in a lecture-based class due to differences in the expected forms of participation across these different class types. Students in an inquiry-based class are expected to engage in live and authentic mathematical argumentation about the task with their peers and the teacher (Laursen & Rasmussen, 2019), whereas students in a lecture-based class are typically expected to stay relatively quiet and take notes while the teacher presents solutions to the examples in a "chalk talk" routine (Artemeva & Fox, 2011). In this manner, the students in an inquiry-based class must make sense of ideas, arguments, and procedures *as* they are being co-constructed by students in the moment (when they might not yet be in a coherent, accepted, or organized form). In contrast, the students in a lecture-based class are usually presented with information that is pre-planned and tightly organized by the teacher, so it might be easier to process the receipt of information in a lecture-based undergraduate mathematics class than in an inquiry-based class.

Similarly, it may be easier for students to figure out the norms for participating in a lecture-based class than in an inquiry-based class. First, lecturing has been the most widely used approach in mathematics education (Freeman et al., 2014; Ridder-Symoens & Rüegg, 1992), so students enrolled in a course such as linear algebra likely have become more familiar with the norms of participation in lecture-based courses than with the norms in inquiry-based courses. Second, student engagement is less dynamic (and thus potentially easier to manage) in a lecture-based course than in an inquiry-based class. Indeed, in lecture-based courses, the instructor is typically the main authority deciding who talks, when, and how. Furthermore, students in a lecture-based class are generally expected to stay quiet, listen, and take notes, so participating verbally (and hence knowing the norms of verbal participation) may not be required for a student to be construed as a competent participant (Cazden, 2001) in a lecture-based class. In contrast, the norms of participation in an inquiry-based class are co-constructed over time as students, with guidance from the instructor, negotiate and renegotiate who can speak, when, and how

(Yackel et al., 2000; Yackel & Cobb, 1996). For example, it may take multiple iterations of negotiation to establish what constitutes an acceptable explanation in a particular undergraduate mathematics class (Yackel et al., 2000). Establishing and discerning each norm in a given inquiry-based undergraduate mathematics class may require that students both produce and interpret language.

Adding to the linguistic complexity found in an inquiry-based classroom, each communication structure used in an inquiry-based class (e.g., small-group versus whole-class discussion) may require different types of productive and receptive language. Whole-class discussions may be used to launch and consolidate ideas with some direct support from the instructor, whereas small-group discussions may focus on exploring ideas more freely, without much expert help. Thus, the students may need to produce and interpret exploratory talk in their small-group interactions but expository talk in whole-class discussions (Bunch, 2014; Mercer, 2004). In addition, since the instructor may not be present in every small-group discussion, students may sometimes get off topic in their small groups (as I have seen in my observations as an instructor and prior experiences as a learner). So, at times, students may need to discern mathematical talk from non-mathematical talk during small-group discussions. This language demand can impact many students, especially those who speak a primary language that differs from the language used during small-group discussions.

Research Questions

The initial observations in language noted above indicate that IOLA instruction may induce different language demands than lecturing. These demands may be more complex in certain ways. This intuition about language demands warrants the need for a rigorous and systematic investigation of the language demands in inquiry-based undergraduate mathematics classes. Moreover, since there may be language resources in these classes that address some of these language demands, an investigation of language resources is also warranted. As previously stated, language resources are the ways that certain tools in the classroom amplify or facilitate communication or interpretation of mathematical information. To honor the assets of multilingual students, such an investigation would not only explore instructional language resources but also student language resources. Instructional language resources are those that are primarily conveyed through communication within instructional tasks (e.g., the IOLA tasks) or through broader communication from the instructor. Student language resources are those conveyed through communication primarily initiated or brought by a student based on their prior knowledge and experiences.

Focusing on an IOLA course as a case study of inquiry-based mathematics education, this study begins undertaking this warranted investigation by addressing the following research questions:

(RQ1) What language demands do multilingual students experience in an undergraduate IOLA course?

(RQ2) What instructional language resources do multilingual students experience in an undergraduate IOLA course?

(RQ3) What student language resources do multilingual students use in an undergraduate IOLA course?

Through investigating RQ1, I will begin to explore the different ways that inquiry-based undergraduate mathematics classes may be experienced as challenges by multilingual students, especially those who are learning the language of instruction. In addition, through investigating RQ2 and RQ3, I will explore both instructional and student language resources that could be leveraged in these classes for making inquiry-based mathematics education more linguistically accessible for all undergraduate students, including emergent multilingual students.

Chapter 2 Literature Review

This chapter contextualizes inquiry-based mathematics education, and reviews the literature related to language demands and resources, with a focus on the perspectives of multilingual students. Reviewing these areas in the literature will build a foundation to address the goals of this study—identifying the language demands and resources of inquiry-based undergraduate mathematics education—while highlighting the resources of multilingual undergraduate students.

This chapter has four sections. To understand why this study focuses on inquiry-based mathematics education as an alternative to traditional lecturing, the first section of this chapter highlights the strengths and areas of growth of an inquiry-based approach relative to lecturing. The second section introduces this study's overarching and conceptual frameworks, which highlight the relationship between language and mathematics to conceptualize the constructs of language demands and resources. The third section reviews the literature in relation to the components of the study's conceptual framework. Finally, related to the goal of the study to highlight the perspectives of multilingual students, the fourth section discusses the literature on the resources of this student population.

Inquiry-Based Mathematics Education as an Alternative to Lecturing

This study focuses on inquiry-based mathematics education (using IOLA as a case example) as an alternative to traditional lecturing because scholars have shown that inquiry-based mathematics education is a better approach in several ways. At the same time, researchers have identified areas of growth for inquiry-based mathematics education. Below, I discuss the ways inquiry-oriented mathematics education has been shown to be better than lecturing, followed by a presentation of equity issues related to inquiry-based mathematics education.

Why is Inquiry-Based Mathematics Education a Better Alternative Than Lecturing?

Previous research has established that inquiry-based mathematics education is a better instructional approach than lecturing. Three metrics along which IBME has been claimed to be better than lecturing are: (a) overall effectiveness, (b) theoretical basis, and (c) empirical basis.

Overall Effectiveness or Promise for Student Success. Evidence of effectiveness for inquiry-based mathematics education comes both from broad studies of active learning approaches and from studies that focus specifically on inquiry-based instruction relative to lecturing. As noted earlier, as a branch of active learning (Laursen & Rasmussen, 2019), inquiry-based mathematics education can be expected to improve learning outcomes and decrease the likelihood of failure for students on average. Indeed, in a meta-analysis of 225 studies reporting data on examination scores or failure rates in STEM active learning courses versus traditional courses, Freeman et al. (2014) found that average examination scores improved by about 6% in active learning courses in comparison to lecture-based courses. Similarly, students in traditional lecturing were 1.5 times more likely to fail than students in active learning courses. Furthermore, research has shown that active learning has the potential to narrow achievement gaps for underrepresented students. For instance, Theobald et al. (2020) conducted a comprehensive search for both published and unpublished research comparing the performance of underrepresented students to their overrepresented peers in active learning and traditional lecturing classrooms. They found that when instructors spent a high proportion of class time engaging students in active learning strategies, the achievement gaps in exam scores narrowed by 33%, and the gaps in passing rates narrowed by 45%.

Similar results have been found in large-scale studies that compared the effectiveness of inquiry-based classes to lecture-based classes. For example, Laursen et al. (2014) examined aggregate student learning gains in the following measures: “cognitive gains in understanding and thinking; affective gains in confidence, persistence, and positive attitude about mathematics; and collaborative gains in working with others, seeking help, and appreciating different perspectives” (p. 409) for large numbers of students in different inquiry-based learning (IBL) and non-IBL courses across four universities. Laursen et al. (2014) found that student learning gains were greater across all measures in inquiry-based learning courses in comparison to traditional lecture courses. Moreover, analyzing the same data described above, Kogan and Laursen (2014) found that inquiry-based learning has a sizable and persistent impact on previously low-achieving students’ grades. More specifically, the grades of the students in IBL courses “improved by about 0.3 grade points in the next term, and even more in further IBL courses,” unlike the grades of their lecture peers and higher-achieving classmates (Kogan & Laursen, 2014, p. 194). Thus, inquiry-based instruction can increase equity in higher education for underrepresented and previously low-achieving students.

In addition to findings from large-scale comparative studies, comparable results have been found when looking at content-based assessments in small numbers of inquiry-oriented classrooms (Laursen & Rasmussen, 2019). The results from these small-scale studies indicate that inquiry-oriented instruction is more effective than lecturing on two fronts: (a) student learning and (b) retention of knowledge. For example, Rasmussen et al. (2006) found that students in the inquiry-oriented classes had developed deeper conceptual understanding than students in the lecture-based classes, whereas no significant difference was found in the attained procedural knowledge between the two groups. Moreover, one year later, in a follow-up comparison study with a subset of the same students, Kwon et al. (2005) found that students in the inquiry-oriented classes retained more conceptual knowledge than students in the lecture-based classes, with no significant difference in the retention of procedural knowledge.

Furthermore, research indicates that students in these inquiry-oriented classes not only develop and retain more conceptual knowledge but also become empowered in creating mathematics through engaging in authentic mathematical practices (Ju & Kwon, 2007; Zandieh et al., 2017). For instance, Ju and Kwon (2007) found that students in inquiry-oriented differential equations classes shifted from viewing themselves as passive recipients and consumers of mathematics to positioning themselves as active producers of mathematics.

Theoretical Basis. Inquiry-based mathematics education reflects a more expansive model of learning than does lecture-based instruction. As Struyven et al. (2010) noted, a lecturing approach usually reflects an "information transmission/teacher-focused" perspective to instruction, which assumes that students passively acquire knowledge by receiving information transmitted to them by the teacher. These assumptions align with traditional cognitive theory (Fox, 1997). In contrast, according to Laursen and Rasmussen (2019), inquiry-oriented instruction is explicitly grounded in both cognitive and social learning theories that assume students actively construct knowledge in a reflexive relationship with the social processes (e.g., communication about mathematics with peers and the teacher) occurring during classroom instruction (Cobb & Yackel, 1996).

In addition, as noted by Laursen and Rasmussen (2019), whereas the curricula for a lecture-based class are typically created solely based on expert knowledge of mathematics, inquiry-oriented instruction builds from students' known or predicted knowledge and reinvention of mathematics (with guidance from the instructor). In particular, inquiry-oriented instruction is grounded in the instructional design theory of Realistic Mathematics Education (Gravemeijer, 1999), which suggests designing instructional tasks that build on contexts that are realistic–that is, imaginable–to learners (Gravemeijer, 1999; Gravemeijer & Doorman, 1999; van den Heuvel-Panhuizen, 2003; Wubbels et al., 1997). By building on theories that acknowledge and leverage students' prior knowledge and active role in their own learning process, inquiry-based instruction might be a better approach than lecturing.

Empirical Basis. Whereas the empirical basis of a lecture-based approach may rely primarily on static assessments of students' content knowledge, inquiry-based mathematics education reflects and addresses research that captures not only students' attained content knowledge, but also the processes in attaining that knowledge. For example, inquiry-oriented mathematics instruction and corresponding curricular materials have been developed through design-based research (Cobb, 2000) that is conducted in actual classrooms as instruction unfolds (Laursen & Rasmussen, 2019). On the other hand, studies that have validated the effectiveness of lecturing come primarily from static assessments or survey data collected outside of the instructional process (e.g., Carriger, 2016). In this manner, in comparison to a lecturing approach, inquiry-based mathematics education has a more dynamic empirical basis that captures the teaching and learning process.

For example, as Laursen and Rasmussen (2019) documented, inquiry-oriented instruction stems from processes identified by researchers in classrooms where "students learn to speak and act mathematically" (p. 133). For example, some processes that Yackel and Cobb (1996) identified in such classrooms included *social norms*, talk patterns that are not specific to mathematics (Yackel & Cobb, 1996). Two social norms they identified were that students "routinely explain[ed] their own thinking" and "listen[ed] and attempt[ed] to make sense of other's thinking" while solving novel problems (Laursen & Rasmussen, p. 133) developed two goals of inquiry-oriented instruction: (a) "students share their thinking," and (b) "students orient to and engage in other's thinking" (p. 133). This exemplifies how inquiry-oriented instruction is empirically guided by research on classroom processes.

Furthermore, whereas the basis of a lecturing approach and corresponding curricula might rely on one-time assessments of effectiveness, the design-based research grounding the development and refinement of inquiry-oriented instruction is conducted over multiple cycles of design, implementation, and analysis (Cobb, 2000). Data sources for such design-based research extend beyond student responses to surveys or written assessments to also include records of the following: class sessions (e.g., video recordings and field notes), interviews where students solve problems and answer questions related to their learning and classroom experiences, team meetings with the teacher, and student work (Laursen & Rasmussen, 2019). With this rich collection of data sources of what happens in classroom activity, inquiry-oriented instruction and curricula is developed and refined by investigating processes, such as "how students build particular ideas, …[how] teaching strategies promote students' mathematical progress, [and] how social aspects of classroom interaction relate to student identity and mathematical growth" (Laursen & Rasmussen, 2019, p. 13). Thus, the more extended empirical basis of inquiry-oriented instruction, relative to that of lecturing, may contribute to the argument that inquiry-based instruction is a better alternative than a lecture-based approach.

Equity Issues in Inquiry-Based Mathematics Education

As noted in the Introduction chapter, while inquiry-based mathematics education may be beneficial on average, it may also be inequitable for certain groups of students. For example, Johnson et al. (2020) found that inquiry-oriented abstract algebra benefited men more than women, while women did equally well in inquiry-oriented and lecture-based classes. The researchers conjectured a link between this result and the reliance of inquiry-oriented instruction on small-group and whole-class discussions. For example, they noted that small-group discussions open opportunities for gender-based microaggressions that could negatively impact women's mathematical performance. Also, the instructor's role to build on students' mathematical thinking for orchestrating whole-class discussions can present opportunities for implicit bias when selecting whose ideas are built on.

Indeed, in a preliminary analysis of 42 inquiry-oriented instructors in their study, the same researchers found that instructors called on women at lower rates than men to respond to mathematically substantive questions during whole-class discussions, and the instructors restated and elaborated women's ideas at lower rates than men's ideas (Johnson et al., 2020). Unfortunately, a similar result was found in a preliminary analysis comparing student performance in inquiry-oriented differential equations with student performance in lecture-based differential equations classes (Johnson et al., 2020). This suggests that inequity based on gender in the inquiry-oriented undergraduate mathematics setting is not a one-off phenomenon and thus requires further attention. In the same vein, it raises questions about which inequities based on other social markers, such as language background, may be present in the inquiry-oriented setting.

More generally, research on active learning approaches, including inquiry-based mathematics education, highlights the need to attend to equity and inclusive teaching. For example, in the meta-analysis conducted by Theobald et al. (2020) that I described earlier, there were some studies where achievements gaps for minoritized students widened in classes that spent more time using active learning, thereby suggesting that active learning does not guarantee equity. Theobald et al. (2020) hypothesized that the variation in the extent to which active learning reduced achievement gaps was largely due to the extent to which active learning was coupled with inclusive teaching.

This finding that active learning does not guarantee equity implies that there is room for making learning more equitable through more inclusive teaching in undergraduate STEM active learning classes, which include inquiry-based undergraduate mathematics classes. Through investigating language demands and resources in an inquiry-oriented linear algebra course with a focus on the experiences and perspectives of multilingual students, this dissertation study provides one avenue for making inquiry-based undergraduate mathematics classes more equitable for all students, including multilingual students who are learning the language of instruction.

Theoretical Perspectives and Conceptual Framework

This dissertation study draws on three theoretical perspectives to build a conceptual framework for analyzing language demands and resources in an undergraduate mathematics classroom. First, I draw on a sociocultural perspective (Moschkovich, 2015) to define what mathematics learning is and how it happens. Second, I incorporate the lens of language socialization (Schecter & Bayley, 2004) to situate uses of language in the context of mathematics learning. Third, because this study is concerned with the resources of multilingual students, I draw on a translanguaging stance (García, 2009) to highlight the language practices of multilingual students from an asset perspective. Building on these three perspectives, I construct a conceptual framework, that I refer to as the Lexico-Grammatical-Situational-Normative (LSN) framework, for analyzing language demands and resources in an inquiry-oriented linear algebra class. Below, I will discuss each of these perspectives in relation to my dissertation study, and I will describe my study's conceptual framework.

Sociocultural Perspective

In a sociocultural perspective, learning involves appropriating socially and culturally shared processes as tools for thinking (Confrey, 1995; Culligan, 2013; John-Steiner & Mahn, 1996; Walshaw, 2016). As socially shared activity becomes internalized, it can mediate thinking–that is, it can constrain, afford, or even spur the formation of certain ways of thinking. Mediating activity involves a transformation from cultural resources into mental resources (Berger, 2005; Confrey, 1995; Friesen, 2012; Pugh, 2017; Walshaw, 2016). Cultural resources are tools that humans within communities have created over time to help people in thinking about and symbolizing their values, ideas, and practices. These tools include language, graphs, gestures, technology, and art (Walshaw, 2016). Culturally shared resources are not just external aids to cognition; they are a necessary part of concept formation and can, in fact, significantly change cognition (Berger, 2005; Steele, 2001). Unfortunately, this means that a tool can not only support learning but also hinder it, implying that learning does not rely solely on the tool, but on how the tool is used. The relationship between learning and the uses of a tool are consequential for conceptualizing language demands and resources, which is the focus of this dissertation study. Under a sociocultural perspective, language demands can be conceptualized as tools that may constrain learning through limiting communication or interpretation of mathematical information, whereas language resources can be seen as tools that may support learning through facilitating communication or interpretation of mathematical information.

In addition, the assumptions behind a sociocultural perspective align with another goal of the study–to highlight the resources of multilingual students. Moschkovich (2015) emphasized how a sociocultural perspective assumes that mathematical activity integrates elements from different aspects of learning: cognitive, cultural, and discursive. For example, this perspective suggests that learning mathematics involves not only understanding word meanings (cognitive) but also appropriating valued mathematical practices (cultural) and developing sociomathematical norms (Cobb et al., 1993; discursive). The cultural and discursive views draw on situated perspectives of learning mathematics (Brown et al., 1989; Greeno, 1998) where learning involves participating in a community of practice (Forman, 1996; Lave & Wenger, 1991) and using a variety of material, linguistic, and social tools (Greeno, 1998).

In contrast to traditionally cognitive perspectives, a sociocultural perspective of mathematics learning is consequential for highlighting the assets of multilingual students. Traditional cognitive perspectives (e.g., Anderson, 1996; Bruer, 2001; Wenger, 1987) may assume that "meanings are static and given by definitions" (Moschkovich, 2015, p. 44) and hence judge student learning relative to formal meanings as well as standard terminology and pronunciation. Thus, cognitive perspectives might focus on multilingual students' lack and misuse of English for expressing mathematical ideas, potentially dismissing multilingual students' alternative forms of communication (such as using objects, drawing, or gesturing) that might indicate their powerful engagement with mathematical ideas (Moschkovich, 2015). Thus, by expanding beyond word meanings to include what students say and do, a sociocultural perspective has the potential to focus on the assets, not the deficits, of multilingual students.

Language Socialization

Given the dissertation study's focus on language use, I will now describe Schecter and Bayley's (2004) perspective on language socialization theory to highlight my assumptions about language practices of teachers and students. At the core, this framework assumes that people "are socialized into the norms and patterns of their culture by and through language" (Schecter & Bayley, 2004, p. 605). In other words, language socialization is at the same time discursive and cultural. In addition, the socialization process develops as a "composite phenomenon of cognitive-linguistic and sociocultural factors" (Schecter & Bayley, 2004, p. 609). That is, language socialization is both cognitive and cultural. In all, language socialization is at the same time cognitive, cultural, and discursive, in alignment with the three views of learning embedded in a sociocultural perspective.

In their study, Schecter and Bayley (2004) shared the stories of two multilingual speakers, highlighting their language practices during interactions. The authors argued that the participants' language practices were mediated by "both the context of interaction and the culturally sanctioned roles of the participants" (p. 609). In addition, focusing on contexts in which multilingual participants have a choice between using the minority or the dominant language, Schecter and Bayley (2004) highlighted the dynamic and interactive nature of the participants' language practices. The researchers showed that, while people can be socialized into the norms and patterns of a community, they also act as agents in their uses of language as they define and redefine their roles and take up or reject roles assigned by others.

The dynamic and interactive nature of language socialization will allow me in this study to contextualize my interpretation of the language practices of teachers and multilingual students. For example, some teachers may promote the use of English for teaching and learning for various reasons, including the institution's sanctioning of English as the language of instruction. In turn, this positioning of English in the class could mark multilingual students' multilingualism as a deficit. More generally, depending on how multilingual students are positioned by others and how they position themselves in the class, some multilingual students may limit themselves to speaking only English during whole-class discussions. Also, during small-group discussions, depending on who these students are grouped with, they may or may not make visible their full linguistic repertoire, which from an external perspective may also include features of languages other than English.

A Translanguaging Stance

Given the focus of the dissertation study on the linguistic resources of multilingual students, this study takes a translanguaging stance (García, 2009; García et al., 2017). This stance represents an asset-based approach to multilingualism centered on the language practices of multilingual students. In contrast to the view that a multilingual student's multilingualism is simply an addition of multiple, autonomous, and bounded named languages (such as English and Spanish), a translanguaging stance assumes that the language practices of a multilingual student are all part of a single language system, the student's language (García, 2009). While from an external perspective these language practices may at times include features of English only, Spanish only, or from both languages simultaneously, from an internal perspective (the student's perspective) these practices come from the student's single and diverse linguistic repertoire. In particular, while a student may *deploy* only features of a named language (e.g., English) in a given setting, the student internally *uses* their full linguistic repertoire to distinguish which linguistic features to use in that setting, and they are capable of using all the features of their repertoire (García et al., 2017).

This stance legitimizes fluid and dynamic language practices (translanguaging) as the norm of multilingual students (García, 2009; The Translanguaging Study Group, 2020). This stance positions multilingual students' fluid translanguaging practices as "unique resources and practices that bilinguals [and multilinguals] can access, instead of perceiving them as a deficit or a sign of incompetence" (The Translanguaging Study Group, 2020, p. 11). Thus, a translanguaging perspective is an asset-based approach of multilingualism with the potential to dismantle persistent deficit-oriented narratives about the language and literacy of culturally and linguistically minoritized students (Wang et al., 2021). In this study, I take a translanguaging stance as I collect and interpret multilingual students' language practices. I consider students' "dynamic…and fluid language practices as valuable for their own sake and for the purpose of expanding students' semiotic repertoires and facilitating content-area learning [such as mathematics learning]" (Wang et al, 2021, p. 11). Taking this theoretical lens will help me meet the study's research goal of highlighting the perspectives of multilingual students from an asset-based standpoint.

Conceptual Framework

Corresponding to the cognitive, cultural, and discursive views of learning and language use embedded in both a sociocultural perspective (Moschkovich, 2015) and a language socialization framework (Schecter & Bayley, 2004), I devised a conceptual framework (see Figure 2.1) for analyzing language use along three respective and interrelated dimensions: lexico-grammatical, situational, and normative. I use this framework in this study to identify the language demands and resources in an IOLA class. Recall that a language demand is a tool that constrains communication or interpretation of mathematical information, whereas a language resource is a tool that facilitates communication or interpretation of mathematical information. Below, I describe the three dimensions of the study's conceptual framework that can be used for analyzing language use in the mathematics classroom.

Table 2.1:

A Framework for Analyzing Language Use at the Classroom Level

Dimension	Definition	Components
Lexico-Grammatical	Uses of form, grammar, and meanings of words and phrases from the mathematics register in a setting	• Syntactic • Semantic
Situational	Daily uses of tools within the situational aspects of a setting	• Material • Activity • Semiotic • Sociocultural
Normative	Relatively constant (normative) aspects of tools in a setting over a period of time	• Social norms • Sociomathematical norms

Lexico-Grammatical Dimension. The lexico-grammatical dimension attends to the uses of words and phrases in the mathematics register (Spanos et al., 1988) that may constrain or facilitate communication or interpretation of information in a situation (in this study, a mathematics teaching and learning session). This dimension involves two components: syntactic and semantic (Moschkovich, 2015; Spanos et al., 1988). The syntactic component pertains to "how linguistic signs, or symbols, behave in relation to each other" (Spanos et al., 1988, p. 224). On the other hand, the semantic component pertains to "how linguistic signs behave in relation to objects or concepts they refer to (their denotations) or their senses (their connotations)" (Spanos et al., 1988, p. 224). In other words, the syntactic component focuses on the form and grammar of words and phrases, whereas the semantic component focuses on the meanings of words or phrases. For example, the syntactic component may involve attending to words or phrases written in passive voice and to words or phrases that mark a logical relationship between two or more parts of a statement. On the other hand, the semantic component may involve unfamiliar phrases as well as unfamiliar denotations and connotations of words with multiple meanings.

The meanings of words and phrases can serve as references or vocabulary (Spanos et al., 1988), so the semantic component consists of two roles: vocabulary and referential. For instance, linguistic uses related to the vocabulary role of mathematical words include presenting new technical vocabulary, natural language vocabulary (which has a different meaning in mathematics), complex strings of words and phrases, synonymous words and phrases, and symbols and mathematical notation as "vocabulary." Linguistic uses related to the referential role of words and phrases include using articles as pre-modifiers and employing variables.

Situational Dimension. The second dimension of the conceptual framework attends to the daily situational aspects of a mathematics teaching and learning session that may constrain or facilitate communication or interpretation of information. For this dimension, I draw on Gee and Green's (1998) MASS (material, activity, semiotic, sociocultural) system, which identifies four inextricably connected aspects of a situation: a material component, an activity component, a semiotic component, and a sociocultural component. The material component involves the "actors, place (space), time, and objects present (or referred to) during interaction" (p. 134). The activity component consists of the "specific social activity or interconnected chains of activity (events) in which the participants are engaging; activities (events) are, in turn, made up of a sequence of actions" (Gee & Green, 1998 p. 134). The semiotic component refers to the "situated meanings and cultural models connected to various 'sign systems' such as language, gestures, images, or other symbolic systems" (p. 134-135). Finally, the sociocultural component attends to the "personal, social, and cultural knowledge, feelings, and identities (cognition, affect, and identity are all equally important here) relevant in the interaction, including sociocultural knowledge about sign systems, activities, and the material world (i.e., all of the other aspects just described)" (p. 135).

The MASS system was not specifically designed for the purpose of analyzing language demands and resources, but the goals that the system was designed to meet align with this purpose. This system was created with the goals to explore "the relationships among discourse, social practices, and learning and … to analyze written artifacts from a classroom" (Gee & Green, 1998, p. 134). More generally, it was created to provide a qualitative study approach that combined discourse analysis with ethnography (Gee & Green, 1998). These goals are related to the purpose of this study. Indeed, one motivation for identifying language demands and resources (particular types of language uses) is the assumption that there is a relationship between language or tool uses (discourse and social practices) and learning opportunities. Embedded in this assumption is the supposition that both discourse and social practices be conceptualized and related as language or tool uses. Thus, this system can be used to organize and identify language or tool uses beyond words and phrases (and hence potential demands and resources) in the mathematics classroom.

Now I provide examples of how the MASS system can be used as a lens for exploring language or tool uses that may function as language demands and resources in the mathematics classroom. The material component can be used to draw attention to how certain material aspects may facilitate or constrain communication. These material aspects include seating arrangements (and student grouping). It may also involve the use or banning of certain technologies. For example, banning cell phones can prevent multilingual students from accessing apps, like Google Translate, that can help them translate terms from English into their primary language(s).

The activity component can be used to consider the mathematical problems presented in the classroom and their associated mathematical activity. Such activity includes reading and interpreting problems of various lengths and problem contexts. To some students, the length may seem too long, or the problem contexts may be unfamiliar to some students (Zahner et al., 2018).

Another way the activity component can be used is to attend to the communication genres requested and used to solve the problems, such as procedural account, mathematical proof, mathematical explanation, mathematical reports, narrative, and short answer. These genres were adapted from Lyon et al. (2012)'s research in science education. A *procedural account* involves listing detailed and coherent procedures. A *mathematical proof* involves writing a proof of a proposed statement. A *mathematical explanation* involves explaining, justifying, or generalizing a mathematical pattern. A *mathematical report* involves synthesizing observed patterns and properties into a report that is at least a few paragraphs long. A *narrative* genre involves personalizing mathematical phenomena. Finally, a *short answer* genre involves generating and providing a single numerical/algebraic answer or selecting a multiple-choice answer without the need to explain or justify the answer chosen.

Finally, the activity component also includes attention to teacher and student engagement in mathematical practices (Rasmussen et al., 2005), mathematical language routines (Zwiers et al., 2017), and teacher moves that might make mathematical practices explicit (e.g., see Selling, 2016). Mathematical practices include defining, algorithmatizing, symbolizing, and theoremizing (Rasmussen et al., 2005; Rasmussen et al., 2015; Zandieh et al., 2017).

The semiotic component helps to consider the following different sets of sign systems as aspects of a mathematics class session that may induce language demands and resources: *communication mediums*, *mathematical language systems*, and *linguistic repertoires*.

Communication mediums include speaking, writing, gesturing, and displaying (Moschkovich, 2015). Mathematical language systems include natural language, symbol system, and visual display (O'Halloran, 1998). Linguistic repertoires refer to teacher and student uses of linguistic features directly associated with named languages such as English, Spanish, and Chinese, as well as to multilinguals' translanguaging, that is, "the deployment of…their…full linguistic repertoire without regard for watchful adherence to the socially and politically defined boundaries of named (and usually national and state) languages" (Otheguy et al., 2015, p. 283). Certain ways of using these sign systems can induce language demands and resources for students (as will be noted in the next section of this chapter).

The sociocultural component can be used to draw attention to *participant structures* and *communication modes.* Participant structures include whole class, small group, pair, and individual (Lyon et al., 2012). These structures within a classroom environment that either reflect or contradict beliefs about learning from a sociocultural perspective (Moschkovich, 2015). More specifically, each participant structure is based on who a student is expected to learn from/with. For example, in an "individual" participant structure, the student is expected to learn alone or passively from the instructor, whereas in the other participant structures the student is expected to learn through interacting with others, including peers and the teacher. Yet, a whole-class participant structure differs from all the other participant structures in that it traditionally includes the teacher as the most powerful member (e.g., as an evaluator or facilitator). While these participant structures could support student learning (e.g., small-group work in Zahner, 2012), they could also constrain students' interpretation or communication of information during class (e.g., small-group work in Hwang et al., 2021).

Within each participant structure, students can participate in different communication modes: *interpersonal*, *presentational*, and *interpretive* (Lyon et al., 2012). Whereas the interpersonal mode involves face-to face, two-way communication, the other two modes involve only one-way communication, with the presentational mode focusing on message delivery and the interpretive on deriving meaning. Each mode differentially reflects how a student learns–i.e., by engaging in conversation with others, by delivering messages to others, or by receiving information.

Each communicative mode may induce a different language demand. For example, "if students engage in the interpersonal mode with the teacher or students during a task or discussion, they may receive immediate feedback not afforded to them while just listening to the teacher or presenting in front of the class" (Lyon et al., 2012, p. 634). Conversely, a student can be quiet in a lecture and still appear to participate competently.

It is important to note the four MASS situational aspects are presented as distinct categories for heuristic purposes. In reality, these aspects cannot be disconnected (Gee & Green, 1988). For example, the act of writing notes involves both a *semiotic* aspect (the communicative medium of writing) and a *material* aspect (e.g., a writing device and a notebook). However, as Gee and Green (1988) noted, using distinct categories during analysis can help a researcher to focus their attention on a particular aspect of the data. At the end, the researcher can construct a more holistic picture of the data by synthesizing findings across the different aspects.

Normative Dimension. Finally, the third dimension of the conceptual framework is the normative dimension, which attends to norms. I define norms as explicit or implicit rules (Much & Shweder, 1978) that govern student engagement in a mathematics teaching and learning session. I consider two types of norms: social and sociomathematical. Social norms refer to rules that are not specific to mathematics, including remaining silent, writing notes, and raising hands to request permission to speak. Sociomathematical norms refer to rules specific to mathematics and include criteria on what constitutes a mathematically productive solution.

These definitions were inspired by the work of Yackel and Cobb (1996), who defined social norms as patterns of mathematics classroom talk that are not specific to mathematics. On the other hand, they defined sociomathematical norms as patterns of talk specific to mathematics (Yackel & Cobb, 1996). Sociomathematical norms include criteria of what constitutes a different, acceptable, or elegant solution (Yackel & Cobb, 1996). For instance, Yackel et al. (2000) found that, in an inquiry-oriented differential equations class, there was the sociomathematical norm that an acceptable explanation extended beyond simply recounting procedures to include an explicit interpretation of rate of change. Note that Yackel and Cobb's (1996) definitions of social and sociomathematical norms were formulated in terms of "talk" (speaking) during class discussions. However, student talk might not be the only, nor the most common, form of communication by multilingual students, the target student population investigated in this study. Thus, I offer different (expanded) definitions of social and sociomathematical norms that include patterns of "engagement," which could involve communication beyond speaking.

Both social and sociomathematical norms could be organized in terms of Gee and Green's (1988) MASS (material, activity, semiotic, sociocultural) aspects of a situation. For example, the expectation that students keep their phones away might exemplify a social norm in the context of a material aspect of a situation, whereas the expectation that students contribute ideas to the whole class only in the presentational mode might exemplify a social norm within a sociocultural aspect. For an additional example, consider the expectation that students define explicitly the parameters, units, and the function represented by a graph when they reference a graph in their explanation. This expectation might constitute a sociomathematical norm within the semiotic aspect. On the other hand, the expectation that an acceptable student explanation includes an interpretation of rate of change, as opposed to simply being a recount of a procedure (Yackel et al., 2000) might constitute a sociomathematical norm within the activity aspect.

Thus, the MASS system appears promising for organizing social and sociomathematical norms that have been identified in the data. Conversely, in pilot data, I was able to identify norms more easily through searching for patterned engagement within each situational aspect in the MASS system. Hence, I integrate the MASS system as an intermediate lens for attending to social and sociomathematical norms.

Interrelated Dimensions. It is important to note that the three dimensions of the conceptual framework (lexico-grammatical, situational, and normative) are interrelated in two main ways. One way relates the dimensions horizontally, that is, across the rows of the table in Figure 2.1. A second way relates the dimensions horizontally and diagonally across language demands and resources, as illustrated in the same figure.

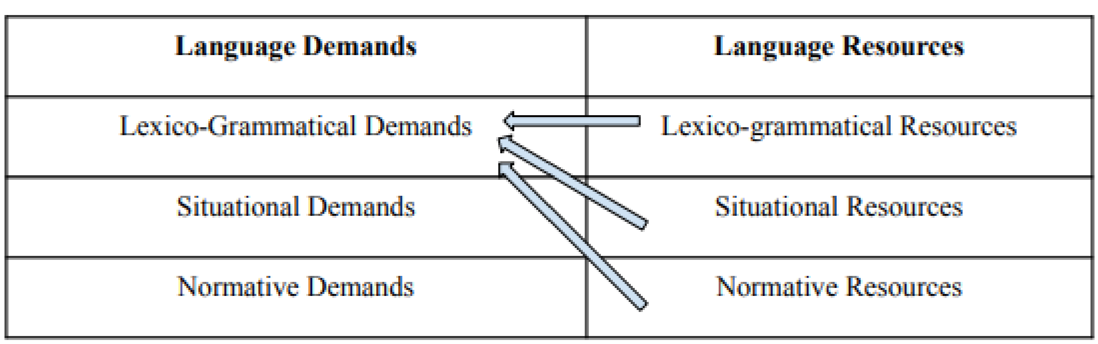


Figure 2.1:

A Framework for Analyzing Language Demands and Resources at the Classroom Level

Note. The arrows illustrate that a language demand in one dimension can be balanced by a language resource in a different dimension.

For an example of the vertical relationships between the dimensions, consider the sociomathematical norm that an acceptable student explanation in an inquiry-oriented differential equations class includes an interpretation of rate of change (Yackel et al., 2000). This sociomathematical norm represents a *normative* feature of the *situational* activity of "explaining" and involves using particular *lexico-grammatical* terms (words or phrases) to represent a meaning of rate of change. Another example of this vertical relationship can be seen in how the situational and normative dimensions are related through the MASS system. While the situational dimension attends to the variety of language uses (and their associated demands and resources) within each element of the MASS system, the normative dimension attends to the relative constancy–i.e., the normativity–of language uses (and their associated demands and resources) within each element of the MASS system. With this integrated nature, this framework will help me identify and relate language uses at different grain sizes (i.e., from words in the lexico-grammatical dimension to sets of utterances and entire classroom structures in the situational and normative dimensions) and along different foci (from focusing on variety in the lexico-grammatical and situational dimensions to constancy in the normative dimension). Thus, this framework will help me construct a more complete picture of the language demands and resources in the IOLA class that I investigate in this study.

As noted above, a second way in which the dimensions of the framework are interrelated is through horizontal or diagonal relationships across language demand and language resource constructs. This way of relating highlights that, while a particular language demand within a given dimension can be addressed by a language resource in the same dimension, such a language demand can also be addressed by a language resource lying in a different dimension of the framework. For instance, as indicated in Figure 2.1, a lexico-grammatical demand can be addressed by a lexico-grammatical, situational, or a normative resource. To see this, consider the mathematical term "linear independence," which may function as a *lexico-grammatical demand* for students when the term is introduced as a new technical term in a linear algebra class. One *lexico-grammatical resource* that may help to address this lexico-grammatical demand, according to the inquiry-oriented linear algebra curriculum (Wawro et al., 2013), may be for the instructor to tag the phrase "going back home" with the term "linear independence" (e.g., see Wawro et al., 2012). However, there may be students who, at that point in the lesson, may not yet grasp the intended mathematical meaning embedded in the phrase "going back home" and hence fail to construct the intended meaning of the term "linear independence." To help such students make sense of this term, the instructor can place these and other students into groups, so they can have a small-group discussion (a *situational resource*) where they connect the meanings of "going back home" and "linear independence." While this pedagogical strategy might work well for some students in some groups, it may not work in groups where a single student with an incorrect interpretation of linear independence dominates the discussion. Thus, the norms that all members in a group share their reasoning, ask questions when they don't understand someone's contribution, and help each other might be useful (*normative resources*) to have in some groups.

To sum up, this study proposes a conceptual framework for analyzing language demands and resources across three interrelated dimensions: lexico-grammatical, situational, and normative. This framework reflects a sociocultural theoretical perspective by integrating a cognitive dimension (lexico-grammatical) with a discursive dimension (normative) and cultural dimension (situational). It is important to note that the operationalization of the lexico-grammatical dimension was heavily inspired by Spanos et al.'s (1988) framework, which was historically developed from traditional cognitive theory. However, this study adapts Spanos et al.'s framework within a sociocultural perspective. As Erath et al. (2021) noted, what is important about lexico-grammatical features is not to stop attending to them but to consider them "as *means* to communicate, think, and learn mathematics topics" rather than as "ends in themselves" (p. 246). This study aligns with Erath et al.'s (2021) recommendations. For example, rather than considering lexico-grammatical features as static features of the mathematics register, I view them as situated in the context of how they are used as tools in the cultures and discourses of the mathematics classroom community and its members. In addition, the analysis of demands and resources in this study does not stop at the lexico-grammatical dimension; this analysis also considers other dimensions that derive from more cultural and discursive research traditions. In the next section, I elaborate on and further illustrate the dimensions of this framework (and their interrelated nature) through a literature review.

Research on Language and Mathematics

While there is little research on language and mathematics at the undergraduate level, the research in K-12 mathematics education has developed diverse and rich conceptualizations of language and related findings that can be built on for identifying language demands and resources in the undergraduate mathematics classroom. In K-12 mathematics education research on language and mathematics, perspectives have expanded from framing language solely in terms of a cognitive view to integrating discursive and cultural views (Erath et al., 2021). Recall that these three views relate to the three dimensions of the proposed conceptual framework: lexico-grammatical, situational, and normative. In this section, I review mathematics education research related to each dimension.

Research Related to the Lexico-Grammatical Dimension

Early K-12 mathematics research on language has focused on the form and grammatical structure of words and phrases in mathematical activity that may constitute a linguistic challenge for English Learners (ELs). Studies of assessments from K-12 research have shown that ELs' mathematical competence may not be accurately measured by standardized assessments due to the linguistic complexity in the assessment items (Abedi et al., 1995; Abedi & Lord, 2001; Martiniello, 2008). The linguistic complexity found in the assessments included unfamiliar or infrequent words as well as grammatical constructions that were unnecessarily complex, including passive voice, long nominal phrases, conditional statements, relative clauses, question phrases, and abstract or impersonal presentations. For example, one assessment item included the more complex question phrase "which is the best approximation of the number" rather than the simpler phrase "approximately how many" (Abedi & Lord, 2001, p. 221).

More recently, Zahner et al. (2018) developed a framework for analyzing linguistic complexity in mathematical tasks. Among the factors they identified as affecting linguistic complexity were the presence or absence of unfamiliar non-mathematical words and the use of words with multiple meanings (lexical ambiguity). Wynn (2019) provided an example of a context term in a mathematical task that may have a denotation that is unfamiliar to some students in a secondary mathematics class. The task she considered revolves around building a rectangular dog "pen" and finding the perimeter of the pen. Wynn noted that the teacher of the class anticipated that the word "pen" would be problematic for her students because, while "pen" was intended to refer to an enclosure for holding an animal, "pen" is more commonly used to refer to a writing instrument. In fact, in her visit to the class, Wynn found evidence that students appealed more to the common meaning of "pen": she noticed the students motioning writing with a pen or saying the Spanish word “pluma” (which means “pen” as a writing instrument) while making sense of the problem context.

The literature on language and mathematics with a focus on college settings has identified types of linguistic complexity similar to those found in the K-12 literature as well as additional linguistic challenges. In particular, lexical ambiguity has also been noted as a linguistic challenge in undergraduate mathematics and statistics education research (e.g., Cornu, 1981; Kaplan et al., 2009; Lavy & Mashiach-Eizenberg, 2009). For example, research has shown that many students interpret the calculus term limit and several statistical terms such as standard deviation in mathematically incorrect ways based on their understanding of the everyday meanings of these terms. In addition, Mestre (1986) found that the presence of double negatives may cause difficulties for some college students.

Beyond documenting similar and additional linguistic challenges in a developmental college algebra setting, Spanos et al. (1988) synthesized the literature to develop a framework for identifying linguistic complexity in the mathematics register. His framework organized the features of the mathematics register into three categories: syntactic, semantic, and pragmatic. Relevant to the lexico-grammatical dimension, the syntactic and semantic features primarily encompassed challenges at the level of words and phrases. (The discussion of pragmatic features is reserved for a later subsection because it captures challenges at the level of entire word problems, rather than single words or phrases.) Syntactic features related to complex language forms, including comparative forms, prepositions, passive voice, reversal errors, and logical connectors. Semantic features referred to challenges associated with the denotative, connotative, and conceptual patterns of language. These challenges were organized according to four dimensions: (a) lexico-grammatical; (b) referential; (c) vagueness; and (d) similar terms with different functions. Lexico-grammatical challenges included the presence of new technical vocabulary, natural language vocabulary that has a different meaning in mathematics, complex strings of words and phrases, synonymous words and phrases, and symbols and mathematical notation as “vocabulary.” The referential dimension attended to the roles of articles as pre-modifiers and to the use of variables. The vagueness dimension attended to ambiguous context words or instructions in word problems. In summary, this synthesis of the literature indicates that form, grammar, and meanings of certain words and phrases may constitute a language demand in a mathematics class for some students.

On the other hand, the math education literature has latently indicated how certain words and phrases can be used as language resources. For example, Khisty and Chval (2002) illustrated how a teacher in a fifth-grade math classroom helped students construct meaning for the term quadrilateral by breaking up the word as quadri-lateral and then connecting the part “quadri” to the Spanish word “quadro,” which means square. She then led students to notice that a square has four sides and related the idea of having four sides to the meaning of quadrilateral. In addition, in a discussion about the relationship between multiplication and division, the same teacher introduced the term “opposite” for helping students construct meaning of the term “inverse” (as in division is the inverse of multiplication).

Language resources in the form of words and phrases are also implicitly present in the undergraduate mathematics education literature. For example, in an undergraduate inquiry-oriented linear algebra class investigated by Wawro et al. (2012), the instructor labeled the ability to “get back home” (in the context of a math problem about motion) with the mathematical term linear dependence. In this sense, the phrase “get back home” was used to support student understanding of the notion of linear independence and to connect their understanding to formal terminology in mathematics.

One can also imagine language resources at the undergraduate classroom akin to those documented in K-12. For instance, similar to how the fifth-grade teacher leveraged the structure of quadrilateral as quadri-lateral, the compound structure of a term such as linear independence, which combines the term linear with the term independence, could be leveraged for making sense of the notion of linear independence. This could be done, for example, by connecting to student's everyday understanding of independence as well as to their mathematical and everyday understandings of the term linear.

Research Related to the Situational Dimension

I will now discuss research related to the material, activity, semiotic, and sociocultural (MASS; Gee & Green, 1998) aspects of teaching and learning mathematics that may give rise to language demands and resources. Although none but one of the studies I discuss used the MASS system to analyze or report their findings, the MASS system provides me with a useful way to synthesize the literature on tools beyond words and phrases that could be used to constrain or facilitate communication or interpretation of information. Most research discussed is situated in mathematics and learning contexts, although I include a few studies from outside mathematics education when their results could be extended to mathematics education settings.

Material. In terms of materials, I review the research that addresses seating arrangements and technology uses in classrooms, primarily within mathematics education contexts. I focus on seating and technology because previous studies have indicated that these two material aspects of teaching and learning influence students' opportunities to communicate or interpret information, which may support or hinder learning of content and language.

The literature indicates that certain seating arrangements are more effective than others for allowing or promoting student engagement, with circle and small group arrangements preferred over traditional lecture-style seating. For example, Patton (2002) noticed that during conversations in childhood education programs, the arrangement of chairs affected participation. Though he didn't specify the details, he stated that "it is typically much easier to generate discussion when chairs are in a circle than in lecture style" (p. 282). Lecture style seating was also reported as the least effective seating arrangement in Harvey and Kenyon's (2013) quantitative, cross-sectional study that asked students to rate five different seating styles across three dimensions: comfort and space, learning engagement, and interactivity. The five seating styles were modern mobile chairs, table armchairs, fixed tiered seating with table arms, rectangle tables with standard chairs, and trapezoid tables with chairs on casters (see Harvey & Kenyon for images of these seating styles). Harvey and Kenyon found that, in every dimension, modern mobile chairs and trapezoid tables with chairs on casters were most frequently rated the highest, whereas traditional table armchairs and fixed tiered seating with table arms were rated lowest.

Further, certain ways of assigning students to small groups have been shown to either promote or hinder communication or interpretation of information, particularly for multilingual students. Overall, studies suggested attending to language background to avoid, as much as possible, leaving any student as the only one in their group who speaks their primary language (e.g., Hwang et al., 2021; Willet, 1995; Zahner, 2012). For example, in two undergraduate mathematics classrooms, Hwang et al. (2021) showcased the experiences of two students from two different classrooms during small-group work. One student, a Chinese-English speaker who felt more proficient in Chinese, was grouped with students who primarily spoke English. The other student, a monolingual English speaker, was placed in a group with students who primarily spoke Chinese during group discussions. Both students experienced linguistic challenges due to this grouping dynamic. The Chinese-English speaker decreased her participation in the group because she did not feel comfortable speaking in English, and the monolingual English speaker ultimately moved to a different group because she could not understand what her initial group members shared during group discussions in Chinese. By ensuring as much as possible that each student has at least one group peer that speaks their primary language, students can more easily use their primary language and their peers as resources for communicating about (and learning) mathematics.

Shifting gears to a different material, the use of technology has been documented as inducing a language resource in math education contexts. In particular, some math education research studies involving multilingual students have incorporated dynamic representational technology (DRT)—a technology offered through the computer which displays multiple, linked, editable representations and that may incorporate animations or feedback—for supporting student engagement with problem contexts (e.g., Zahner et al., 2012; Zahner, Calleros et al. 2021; Zahner, Pelaez et al., 2021). As Zahner, Pelaez et al., (2021) noted, DRT can help emergent multilingual students with connecting symbolic expressions and graphs with natural language, gestures, and actions (e.g., dragging and pointing).

Another technology tool that has been used to facilitate communication in linguistically diverse mathematics classrooms was mediated by calculator use. More specifically, Chval (2004) showcased how a fifth-grade teacher of second language learners promoted the use of calculator "keystrokes" as a language resource. Chval used the term "keystroke" in "two distinct ways: (1)

to denote the striking of a calculator key, and (2) to speak or write the symbol representing the calculator key" (Chval, 2004, p. 75). The teacher in the study had students present keystrokes both verbally and in writing to communicate their mathematical ideas, thereby creating a common language to mediate interactions in small-group and whole-class discussions. In this manner, calculator keystrokes served as a means for communicating and facilitating social activity, especially for the students who were learning English as a second language.

Finally, the app Google Translate, used through a cell phone, was another resource that my research team and I observed students using in the study described in Zahner, Calleros et al. (2021). In the study, one Spanish-speaking, ninth-grade student who had recently arrived in the USA with no English knowledge was paired with an English-Spanish speaking student peer. While solving a mathematical task which contained the term "average," the peer used Google Translate to translate the mathematical term "average" into Spanish as "promedio" for the newly arrived student. This appeared to help the two students make sense of the language in the task and to promote mathematical communication between them.

Activity. In terms of activities, the literature has focused on holistic features of math problems (or tasks) and mathematical practices (Moschkovich, 2015; Rasmussen et al., 2005) as language uses that may give rise to language demands or resources. For example, Zahner et al. (2018) identified the following features of mathematical tasks as affecting linguistic complexity: familiarity of the problem context for students and the relative number of words on a task. Tasks that were too wordy or that had problem contexts which were unfamiliar to students posed higher linguistic complexity. In addition, problems whose context may be difficult to imagine (due to the background knowledge required) can also pose linguistic challenges for students (Martiniello, 2008; Moschkovich, 2015).

Relatedly, in their framework for identifying linguistic complexity in the mathematics register, Spanos et al. (1988) organized the challenging features of the mathematics register into three main categories, one of which (the pragmatic category) related to features of math problems. More specifically, pragmatic features consisted of epistemological and textual challenges. Epistemological challenges attended to math problems requiring experience or knowledge that was either lacking, restricting, conflicting, or contradictory for students. Textual challenges related to the lack of real-life objects or activities in math tasks and the lack of natural interaction in the tasks.

Written text genres (Lyon et al., 2012; Veel, 1997) are additional language uses that have been identified in the inquiry-based science education literature and can be conceptualized as features of math problems or tasks when adapted to mathematics education contexts. The text genres identified in science education are: experimental accounts, science persuasion, science explanations, science reports, and narrative (Lyon et al., 2012; Veel, 1997). Experimental accounts referred to written texts or writing tasks that involved listing detailed, coherent procedures. Science persuasion involved arguing a point of view. Science explanations (respectively, science reports and narrative) involved explaining (respectively, describing and personalizing) natural phenomena. While these texts "can serve as opportunities for students to engage with disciplinary content and practices, ... depending on the particular texts they may also present challenges for ELs and struggling readers and writers" (Lyon et al., 2012, p. 634). Thus, depending on their genre and how they are implemented, math problems or tasks can function as language resources or demands.

As noted in the Conceptual and Analytical Framework subsection earlier in this chapter, I adapted the above genres from science education research to create the following analogous genres in math education: procedural accounts, mathematical proof, mathematical explanations, mathematical reports, and narrative. I also added the following genre to the list: short answer. This genre stemmed from Zahner et al. (2018)'s framework, which distinguished between the simple presence of a writing prompt (e.g., one that could involve a short response) with a writing prompt that requested a mathematical explanation, justification, or generalization as factors affecting linguistic complexity.

Core to the activity dimension is undergraduate mathematics education research related to mathematical practices, "the ways in which mathematicians go about their profession" (Rasmussen et al., 2005; Rasmussen et al., 2015, p. 264; Zandieh et al., 2017). Rasmussen et al. (2015) listed the following mathematical practices as central to the activity of professional mathematicians: defining, algorithmatizing, symbolizing, and theoremizing (Rasmussen et al., 2005; Rasmussen et al., 2015; Zandieh et al., 2017). It is important to note that there are alternative ways in which researchers might characterize mathematical practices. For example, Moschkovich (2015) listed the Standards for Mathematical Practices put out by the Common Core State Standards for Mathematics (Common Core State Standards Initiative, 2010), which make recommendations for K-12 mathematics. In this study, I choose to characterize mathematical practices as defined by Rasmussen et al., (2005), Rasmussen et al., (2015), and Zandieh et al. (2017), because such research was situated in undergraduate mathematics contexts, in alignment with the context explored in this study.

While mathematical practices can provide students opportunities to engage more authentically in mathematical communication, mathematical practices may function as language demands due to their implicit nature (Selling, 2014). As Moschkovich (2004) noted, appropriating a mathematical practice entails figuring out what's implicit, including making sense of particular symbols, signs, terms, or goals of participating in particular practices. Thus, Selling (2014) constructed a list of teacher moves for making mathematical practices explicit to students. For example, one move for teachers is to name the practice in which students engaged, and another move is to explain the goal/rationale of a mathematical practice.

Another aspect of mathematical practices that can make them induce language demand is that they require more communication and/or communication that is more sophisticated than students may be accustomed to (Moschkovich, 2015). For example, consider the practice of theoremizing, which is defined as:

> activity related to both conjecturing and proving; Theoremizing includes aspects of conjecturing, such as noticing relationships between mathematical entities and proposing statements that capture those relationships. It also consists of activities involving making arguments and justifications for or against proposed statements. (Rasmussen et al., 2015, p. 278)

One can see how language intensive this practice is: for example, it expects students to craft arguments, develop justifications, and engage in explaining, all of which are examples of mathematical discourse practices (Moschkovich, 2007).

To facilitate student engagement in such language-intensive practices, teachers can incorporate *mathematical language routines* (MLRs) into their lessons (Zwiers et al., 2017). MLRs are adaptable language mechanisms that can be used to amplify, evaluate, and develop students' language (Zwiers et al., 2017). Zwiers et al. (2017) and Driscoll et al. (2016) described a variety of MLRs. For example, one MLR is "Stronger and Clearer Each Time," which is defined as a structure that "provides an interactive opportunity for students to revise and refine both their ideas and their verbal and written output" (Zwiers et al., 2017, p. 9). This and other

MLRs were used in Zahner, Calleros et al. (2021)'s study to give students access to mathematical discourse.

Semiotic. Recall from the conceptual framework that the semiotic aspect attends to communication mediums (Moschkovich, 2015), mathematical language systems (Moschkovich, 2015; O'Halloran, 1998), and linguistic repertoires (Erath et al., 2021). Communication mediums include speaking, writing, gesturing, and displaying. There is not much research contemplating how each communication medium might constitute language demands or resources. However, Schleppegrell (2007) noted that all these mediums together construct meaning, and "it is only by cross-referring and integrating these thematically, by operating with them as if they were all component resources of a single semiotic system, that meanings actually get effectively made and shared in real life" (Lemke, 2003, p. 229). Thus, integrating multiple communication mediums can constitute a language resource. In addition, using or allowing multiple communication mediums can give students classified as ELs more resources to express and interpret information (e.g., Dominguez, 2005). For example, Lyon et al. (2012) documented how a teacher communicating in writing in addition to orally gave a student classified as an EL the written resources that allowed him time to process information and to which he could refer. On the other hand, limiting communication to a single communication medium could make it more difficult to construct shared meanings, hence potentially inducing a language demand.

Mathematical language systems include natural language, symbol system, and visual display (Moschkovich, 2015; O'Halloran, 1998). Each system has affordances and limitations for representing information. For example, natural language is apt for providing the contextual information about a situation in a mathematics problem but may lack the resources to express mathematical statements or relationships succinctly. On the other hand, symbol systems are effective at capturing relationships between entities in complete and closed form but may visually hide the connection between the problem's context and the problem's intended mathematical processes. Yet, visual systems are precisely fit for making visible the essence of a relationship or idea at a glance, while not being effective at capturing all the details of a problem context nor at encapsulating a mathematical relationship in compact and closed form. Thus, like the discussion around communication mediums, it may be important for instructors and students to use and integrate multiple mathematical language systems as a language resource. On the other hand, limiting communication to a single mathematical language system can leave out important contextual, mathematical, or holistic information, which could induce a language demand.

Research on language in mathematics classrooms emphasized that students' full linguistic repertoires are cultural tools for expressing ideas (Barwell et al., 2017). Thus, Erath et al. (2021) called for materials and instruction to include students' home languages as well as students' translanguaging practices to support student engagement in the classroom. For example, in Garza (2018)'s study, a teacher in a seventh-grade classroom with Latinx bilingual students used translanguaging practices, which included what may be seem externally as elements from both Spanish and English. By translanguaging themselves in the classroom, the teacher encouraged their students to develop mathematical biliteracy through drawing on their full linguistic repertoire.

While allowing all students to draw on their home languages and full linguistic repertoires for communicating and learning in the mathematics classroom can help promote social justice and serve as language resources for many students, the research shows that such a situation does not come without tensions. For example, recall Hwang et al. (2021)'s case study about two undergraduate students (one monolingual English speaker and one multilingual speaker who felt more confident speaking in Chinese) who expressed having trouble communicating about mathematics in their respective small groups in a mathematics class. Both case study students attributed this communication struggle to the fact that all their group members drew on their primary languages, which differed from the respective case study student's home language. Thus, while a situation where all students are allowed to draw on their entire linguistic repertoires in the math classroom can create language resources (for some students in a given moment), this situation can also create language demands (for other students) that may need to be addressed, say by attending to other classroom dynamics and tool uses (e.g., group composition).

Sociocultural. Relevant to the sociocultural component of the conceptual framework is research on participant structures and communication mediums. Participant structures include whole class, small group, pair, and individual (Lyon et al., 2012). There is not much research on the pair and individual participant structures, but Zahner, Pelaez et al. (2021) noted how the whole-class and small-group structures have been documented to have both benefits and limitations in the context of class discussions (e.g., Chapin et al., 2009; Zahner, Pelaez et al. 2021). For example, one benefit of whole-class discussions is that they allow for exploring a wide range of student thinking, since all students and the teacher are available at once during the discussion. However, whole-class discussions may seem too large and public for certain students to feel safe participating. This may especially affect multilingual students who don't yet feel proficient in the language of whole-class discussion (e.g., English). On the other hand, small-group discussions may provide a more private and low-stakes, collaborative, and personalized space, where multilingual students may use resources, such as their primary language(s) that they might not use in whole-class settings. However, within any given small-group discussion, the range of generated student ideas might not be large enough for students to compare, contrast, and deepen their understanding as well as connect to the broader mathematical community through the teacher. Thus, both whole-class and small-group formats fill the gap of the other, and it becomes important to include both participant structures in the math classroom. In addition, both participant structures build on each other. For example, the teacher can use the whole-class structure to efficiently launch a task (reaching all students at once), and then let the students discuss and solve the task in their small groups (where more flexible exploration of thinking may be allowed). Next, each small group can present their emergent findings to the whole class, allowing the teacher and all students together to develop shared and more solidified understandings, while connecting them to more standard mathematical conventions (through the teacher acting as "broker between the classroom community and the mathematical community;" Zandieh et al., 2017, p. 101). Thus, in their study, Zahner, Pelaez et al. (2021) implemented both whole-class and small-group discussions in an alternating fashion in the learning environment they designed.

Students can participate in different communication modes: *interpersonal*, *presentational*, and *interpretive* (Lyon et al., 2012). Each communicative mode may induce a different language demand; Lyon et al. (2012) illustrated this by contrasting the experiences of two students classified as ELs from two different classrooms. In their study, one student, Juan, did not engage interpersonally with the teacher, but the second student, Mia, did. During the interpersonal engagement between Mia and her teacher, the teacher probed into Mia's understanding about whether Mia understood what resources she could use during a performance assessment. Through this probing, Mia was able to figure out she was allowed to work with her peers, even though she was being assessed based on her performance. On the other hand, Juan only engaged in the presentational or interpretive mode, so he did not have the opportunity to figure out he could engage with his peers.

Research Related to the Normative Dimension

Recall that social norms capture the mutual expectations, not necessarily specific to mathematics, that co-develop during student and teacher engagement in mathematics classrooms, whereas sociomathematical norms express the mutual expectations specific to students' mathematical activity that arise in mathematics classroom interactions (Yackel et al., 2000). These norms "are not obligations or regulations for students to meet" but rather are established through a process of constant (re)negotiation during interactions (Güven & Dede, 2017, p. 267). Much of the existing mathematics education research on social and sociomathematical norms has focused on documenting and characterizing such norms during classroom discussions across both K-12 and undergraduate mathematics (e.g., Fifty, 2020; Yackel & Cobb, 1996; Yackel et al., 2000), without explicitly attending to how such norms may constitute a linguistic challenge or resource for students.

For example, situating their research in elementary inquiry-oriented mathematics classrooms where students regularly discussed and solved new or unfamiliar problems, Yackel and Cobb (1996) investigated social and sociomathematical norms for whole-class discussions. They found that these classrooms were characterized by the following social norms: (a) students routinely explain their own thinking, (b) students listen and attempt to make sense of each other's reasoning, (c) students ask questions if they don't understand someone's contribution, (d) students offer different solution strategies, and (e) students indicate if they agree or disagree, with justification (Laursen & Rasmussen, 2019). Yackel and Cobb (1996) also characterized the second-grade math classrooms they studied in terms of sociomathematical norms. For example, they found that the criteria for what constituted a *mathematically different* solution were mutually negotiated during whole-class teacher and student interaction. In addition, the criteria of what constituted *mathematical sophistication, efficiency, and elegance* as well as what counted as an *acceptable mathematical explanation and justification* were other sociomathematical norms during whole-class discussions in these classrooms.

Yackel et al. (2000) extended the analyses of social and sociomathematical norms to the undergraduate mathematics context, particularly inquiry-oriented differential equations. Focusing within the discourse practice of explaining during whole-class talk, they identified the following social norms: students were expected to explain their thinking and to try to make sense of each other's ideas. Within the same focus of explaining during whole-class discussions, Yackel et al. (2000) found the sociomathematical norm that an acceptable explanation was one that included an interpretation of rate of change. For clarifying and contrasting purposes, Yackel et al. (2000) briefly compared the characterizations of these norms in this classroom with the characterizations of these norms in another reform-oriented classroom. In contrast to the inquiry-oriented class, students in the comparison class were not expected to explain their reasoning nor to try to make sense of someone's reasoning. Moreover, in the comparison class, an explanation given solely in terms of procedures (without including an interpretation of rate of change) was acceptable.

Extending beyond whole-class talk, Fifty (2020) investigated the social and sociomathematical norms of small-group discussions. For example, he investigated the social norm of *what it means to analyze solutions* and "the sociomathematical norm of *what constitutes an acceptable solution*" (p. 96). He characterized these as well as other social and sociomathematical norms as they emerged in four different small groups, comparing the characterizations across the groups. For instance, consider “the sociomathematical norm of *what constitutes an acceptable solution*” (Fifty, 2020, p. 96). He found that in one group this sociomathematical norm was characterized as “an acceptable solution is one that utilizes any valid approach” (Fifty, 2020, p. 92), whereas in a second group this sociomathematical norm was characterized as “an acceptable solution is one that uses a familiar approach or leads to the correct answer” (p. 102). Yet, in the remaining two groups, this sociomathematical norm was characterized in a different way: An acceptable solution is one that uses a familiar approach and leads to the correct answer. This work showed that social and sociomathematical norms may differ among different small groups.

In addition to characterizing and comparing norms, Fifty (2020) explored the relationship between the two constructs of social and sociomathematical norms. He found evidence that characterizations of these two constructs mutually influence one another. For example, he found that one group’s sociomathematical norm of an acceptable solution as one that followed a familiar approach may have influenced, and may have been influenced by, the development of the social norm of avoiding engagement with non-familiar approaches.

Existing math education literature latently suggests that social and sociomathematical norms may constitute language demands and resources. Indeed, just as determining social and sociomathematical norms for norm analysis may require attending to implicit regularities in the patterns of social interaction (Güven & Dede, 2017), the same may be required for students to figure out these regularities as they engage in discourse. The implicit aspect of norms may thus present a language demand for students. Likewise, similar to how researchers must consider cases of dissonance with a conjectured norm for confirming or revisiting a conjecture during norm analysis, students must do the same, which could induce a language demand for students.

More generally, social and sociomathematical norms may induce language demands because they involve "goals, meanings for utterances, and focus of attention," which are multiple, situated, and connected to multiple discourse communities (Moschkovich, 2007, p. 25). In particular, the sociomathematical norms that teachers intend to establish in their classrooms stem from "socially, culturally, and historically produced … [aspects of engaging in discourse practices, and mathematical practices more broadly] that have become normative" (Moschkovich, 2007, p. 25). Thus, sociomathematical norms in some classrooms can induce language demands by representing ways of engaging in discourse practices that may differ from the ways that which some students might initially assume. For example, prior to enrolling in an inquiry-oriented differential equations course, many students may be used to interpreting what it means to explain as simply recounting procedures without giving conceptual justifications. In contrast, in the inquiry-oriented differential equations studied by Yackel et al. (2000), there was the sociomathematical norm that an acceptable explanation was one that was grounded in an interpretation of rate of change. Thus, sociomathematical norms call for ways of participating that may be different and perhaps more linguistically complex than the initial inclinations of students.

Another example of this potential difference and/or higher linguistic complexity can be seen in Moschkovich (2007)'s description of a classroom interaction in which a teacher presented a definition of a parallelogram and then sought to engage students in the sociomathematical norm of using a definition in a binary way (e.g., a trapezoid either fits or does not fit the definition of a parallelogram). While this was the teacher's intention, students used definitions in a non-binary way (e.g., claiming that a trapezoid is half a parallelogram), focusing instead on the relative number of parallel sides of two given objects. This illustrates that when students are engaged in discourse practices, multiple interpretations of the appropriate ways to engage in these practices are available, which could result in students "violating" the intended sociomathematical norms. As discussed earlier, such "violations" (or cases of dissonance with the intended norms), could produce language demands for students.

Similar to sociomathematical norms, social norms may also involve negotiating multiple and situated meanings for utterances in the establishment of social norms. This aspect of social norms can make such norms function as language demands for some students even when a teacher makes explicit cues about what the teacher's intended norms are. Ghosh (2022) illustrated this situation in a secondary mathematics classroom. In his study, the teacher often provided instructions to encourage student talk while simultaneously giving directions to maintain discipline in the classroom. For instance, Ghosh found that the teacher sat the students in pairs, telling students to "take a moment and think together with the person next to you" (Ghosh, 2022, n.p.) However, the teacher also gave seemingly contradictory statements to some pairs of students, telling them, for example, "If learning isn't happening, I may move you to somewhere less socially tempting, where you are not sitting next to your BFFs and talking. Because the point of you being here is to learn, OK? Not to chat about, you know, whatever it is you all talk about" (n.p.). While the intended norms may have been clear to the teacher (e.g., that students are expected to talk with others, but only about mathematics), these intentions may not have been clear to some students, for whom the line between what constitutes mathematical and non-mathematical talk may be blurrier than for the teacher.

One additional potential linguistic challenge associated with social and sociomathematical norms stems from the different discourse communities explicitly implemented in instruction. For example, each small group in a class may represent a different discourse community, each of which is embedded in a larger discourse community, the whole class. In turn, as Fifty's (2020) work implied, each discourse community may subscribe to different social and sociomathematical norms. Indeed, Fifty found that characterizations of the same social and sociomathematical norm categories differed among different small groups, which could also imply that norms might differ between whole-class talk and small-group talk. Thus, each time students alternate between whole-class engagement and small-group engagement in a given class session (which could often be the case in inquiry-oriented classrooms; Laursen & Rasmussen, 2019) or switch between different small groups throughout the semester, students may need to (re)negotiate and distinguish the social and sociomathematical norms connected to different discourse communities. This could add to the language demands associated with the (re)negotiation process embedded in establishing and interpreting social and sociomathematical norms.

Yet, social and sociomathematical norms could also function as language resources. Indeed, Fifty's (2020) result that social and sociomathematical norms mutually influence one another implies that social norms and sociomathematical norms may serve as language resources for each other. Overall, however, research that explicitly investigates how social and sociomathematical norms may give rise to language demands and resources is needed. Such investigations need to extend the analysis of social and sociomathematical norms to communication and behavior beyond the activity component (e.g., analyzing solutions and explaining one's thinking) of a situation and beyond the "speaking" communication medium in the semiotic component.

Research on Multilingual Learners in Mathematics

As we will see in this section, there is limited research in mathematics education about multilingual students at the undergraduate level. However, extensive research has been conducted about multilingual students' learning and participation in K-12 mathematics levels. In a review of the literature about K-12 students who are learning mathematics in English while simultaneously learning English, de Araujo et al. (2018) identified three themes: (a) the relationship between multilingual learners' language proficiencies and mathematical performance, (b) multilingual learners' linguistic resources, and (c) multilingual learners' cultural resources. In contrast, undergraduate mathematics education literature focusing on multilingual learners has been almost exclusively constrained to studies about only one of these themes: the relationship between language proficiency and mathematical performance. This section discusses the literature about multilingual students according to these three themes, primarily in K-12 mathematics levels and, when possible, at the undergraduate level.

Language Proficiency and Mathematical Performance

Studies in K-12 mathematics settings have documented a strong relationship between language background and mathematical performance (de Araujo et al., 2018). One important finding is that students who are proficient in more than one language tend to perform better on mathematics assessments than students who are proficient in a single language or none. For example, in a study in Papua New Guinea comparing a group of 232 bilingual (Melanesian Pidgin-English) speaking students to a student group of 69 monolingual English speakers in Grade 6, Clarkson (1992) found indications that bilingual students who were competent in both languages performed better than monolingual English speakers on mathematics assessments. These indications held firm even though the monolingual students attended schools with more instructional resources and English was the official language of the country. However, bilingualism was not the only important factor in this comparative finding; the proficiency level in each language was of crucial importance.

In the same study, Clarkson (1992) reported clear evidence that bilingual students with low competence in both languages were disadvantaged in mathematics when compared to both groups of bilingual students who were proficient in both languages and the group of monolingual students. A similar result was found in a different study in Australia involving 85 Vietnamese-English speaking students in Grade 4 (Clarkson, 2007).

The undergraduate literature on multilingual students has also heavily explored the relationship between language background and mathematical performance. A series of studies showed that multilingual undergraduate students outperformed their monolingual English counterparts during the first year of university, but this apparent success of multilingual students dwindled as students moved into the higher university years (Durand-Guerrier et al., 2012). For example, in a study involving 83 first-year undergraduates that compared the performance of second-language students (mainly Asian) to English first-language students in New Zealand, Barton and Neville-Barton (2003) found that second-language students received better grades than first-language students, and student grades were even higher for recently arrived students. This was attributed to the fact that the first-year math courses were not as intensive in terms of natural language and to the presumably more solid knowledge that multilingual students were coming with from their native country abroad. Indeed, Barton and Neville-Barton (2003) also found that second language students had a disadvantage in mathematical achievement, in comparison to first language students, in questions that were more language-intensive. Also, although undergraduate students in general experience difficulties using logic (e.g., Durand-Guerrier et al., 2012), Barton et al. (2005) showed that these difficulties were worse for second-language learners in advanced university levels.

Linguistic Resources

As de Araujo et al. (2018) showed, mathematics education research has elicited resources that K-12 multilingual students use beyond written and spoken English, including students' primary language(s) and nonverbal communication. For example, in a qualitative study involving one seventh/eighth dual-language classroom with a bilingual teacher and 24 multilingual students in Arizona, Rubinstein-Avila et al. (2014) investigated the intersection of bilingualism/biliteracy and the learning of mathematics. The 24 participants had been born either in Mexico or the U.S. and considered Spanish to be their dominant language, although some were comfortable using English. Most participants were classified with an intermediate English proficiency level by their school district. The data included video segments of a demonstration experiment, taught in English by Dr. McGraw (one of the researchers), who was an English monolingual, using a non-routine mathematics problem.

The results of Rubinstein-Avila et al. (2014)'s study showed that students were able to translanguage—that is, to move freely and deliberately between features that might externally correspond to Spanish and English—as they engaged in collaborative problem solving. Although small-group communication was carried out mostly in Spanish, most students had no problem switching to English to respond to Dr. McGraw's queries and switching back to Spanish to share their exchange with Dr. McGraw and their group peers. This study showed that allowing students to communicate in class in their primary language and translanguage does not necessarily stunt or interfere with students' English language development. Rather, it affords students the opportunity to draw on their linguistic resources strategically, for example, to be heard, clarify and verify their basic understanding, and elaborate on concepts. This work highlights the importance for teachers to allow students to draw on their students' entire linguistic repertoires, including their translanguaging practices (García, 2009).

In addition, math education researchers have found that K-12 multilingual learners also resort to nonverbal resources to communicate and learn mathematics. For instance, in a qualitative study involving one classroom of 7 second-grade multilingual students (bilingual in English and Spanish) in Austin, Texas, Dominguez (2005) investigated how multilingual learners used gestures to communicate their mathematical thinking. Two of the participants were proficient bilinguals and the remaining five were less proficient bilinguals. Dominguez conducted individual interviews with each of the students using a clinical-interview protocol and video equipment. The interviewer had each interviewee solve mathematical tasks and tell how he or she solved them.

In her analysis of the study, Dominguez (2005) found that nearly all students were able to communicate their mathematical reasoning both verbally and nonverbally as they solved problems. More precisely, four students executed a sweeping hand movement to indicate the total in a problem, a behavior that repeats or adds emphasis to verbal behavior. Two students pointed to numbers for which they did not recall the name, and one student made the shape of a square with both hands to request a chart. Six students simultaneously used nonverbal and verbal behavior when counting the number of elements in a set by pointing to or touching each element while naming the numeral that corresponded to each element counted. Dominguez concluded that these functions of gestures support the hypothesis that gestures and speech are correlated in meaning. Thus, in addition to attending to students' speech in English and in their primary language(s), teachers should attend to students' gestures and other nonverbal behavior to make sense of multilingual students' mathematical communication and understanding.

Rather than focusing on students' linguistic repertoires and communicative mediums, some undergraduate mathematics education researchers focused on identifying students' resources in relation to which mathematical language systems they preferred to use. For example, in a study involving 83 first year undergraduates, Barton and Neville-Barton (2003) found that the second language students preferred to express themselves mathematically in terms of mathematical symbols over natural language, diagrams, or graphs.

Cultural Resources

The literature has also documented K-12 multilingual students' cultural resources that extend beyond spoken, written, and nonverbal communication in their mathematics learning (de Araujo et al., 2018). In particular, many mathematics education researchers have sought to identify and build on multilingual students' funds of knowledge (Moll et al., 1992). For instance, in a qualitative study involving multilingual (Spanish-English bilingual) students in Grades 4 and 5 at a school in Austin, Texas, Dominguez (2011) investigated the relationship between students' mathematical activity and students' familiarity with the context and named language of the task. The participants spoke Spanish and English outside of school but interacted in English only during mathematics instruction, which was in English. The methods used in the study included visiting the homes of the students and interviewing parents to document students' everyday experiences. Based on these visits and interviews, Dominguez used students' everyday experiences (such as grocery shopping and making scrambled eggs) as contexts for school-based math problems. Then, he had the students solve these problems (as well as problems with similar mathematical demands but framed in a context that was unfamiliar to students) in pairs in their mathematics class and videotaped the student-student interactions during the problem-solving process. Dominguez found that when students worked on tasks with familiar contexts, they engaged in meaning making actions, as opposed to simply applying procedures.

In a related study involving one fourth grade classroom and one fifth grade classroom–each classroom with 100% bilingual Latino/a students in a school in Central Texas–Dominguez (2017) examined how students' familiarity with the task context and named language affected their tendency to take risks in the process of solving mathematical tasks. Using similar methods (as in his 2011 study) of visiting the students' homes, interviewing their parents, and having students solve problems with familiar and unfamiliar contexts, Dominguez (2017) found that "familiar contexts that students encountered in mathematical tasks supported them in leading discussions characterized by taking risks" (p. 41). In addition, he found that when familiar tasks were given in Spanish (students' primary language), students generated richer and longer conversations than when familiar tasks were given to them in English. Findings such as the results in Dominguez (2011, 2017) emphasize how important it is for teachers to leverage their students' cultural resources in their mathematics teaching and task design.

Semi-structured Interviews

To explore language demands and resources directly from the participants' perspectives, I designed semi-structured interviews (diSessa, 2007). In a semi-structured interview, the interviewer uses an interview protocol to control the flow and direction of the interaction while leaving room to explore emergent topics (Bernard, 2017). Semi-structured interviews allow researchers to expand beyond exploring external behavior to inquire into the feelings and thoughts of the participants (Patton, 2002). In this section, I describe how I designed and implemented semi-structured interviews with the instructor and the five selected student interviewees.

Each interviewee completed two interviews, labeled Interviews 1 and 2. Interview 1 was conducted over multiple interview sessions per participant, so in essence it represented multiple interviews. All student and teacher interviews followed a similar protocol that touched upon the same content goals and design (see Appendix B) except for two main differences. One difference was that the student interviews asked about their own experiences, while the teacher interviews asked about the experiences of her students. The second difference is that Student Interview 1 consisted of five components, while Teacher Interview 1 only included the first four components. For their first interview, each participant took about 2.5 hours to complete the first four components, and an average of about 30 minutes for the fifth component. For their second interview, each participant took an average of approximately 75 min.

All components of Student Interview 1 investigated both language demands and resources, each with a different focus. Component 1 requested general information about students' experiences in an inquiry-based mathematics course, especially in relation to a lecture-based class. Components 2, 3, and 4 respectively corresponded to the three L-S-N dimensions of the framework. The fifth component was designed to elicit textual aspects of mathematical tasks that induce language demands and resources. This component presented the first 12 mathematical tasks from the IOLA class and asked participants to arrange these tasks from most to least linguistically challenging and to explain their ranking of each task. Students had the option of assigning the same ranking to multiple tasks.

Student Interview 2 included a technique known as video-stimulated recall dialogue (VSRD) as a way of giving the interviewees an immediate visual to reflect on the classroom phenomena and a reference point to articulate their experiences in the IOLA class (Hargreaves et al., 2003). Applying this technique during the interview, I showed students video clips from their class that I had selected based on my preliminary analysis of the classroom observation data and asked them to describe any language challenges or supports.

I conducted all student and instructor interviews primarily in English, except for the interviews with Luis, a Latino multilingual student with whom I shared a home language. During my interview with Luis, I drew on my Spanish-speaking skills to translanguage and encouraged him to do the same. Given that I did not speak the home languages of the remaining multilingual students, I employed various language strategies during the interview to mitigate any language challenges for both myself and the participants. For instance, I attempted to speak English at a pace they could easily follow and encouraged them to use digital resources to translate any part of the conversation into their preferred languages. In addition, since the interviews were conducted over Zoom, I leveraged several Zoom features to facilitate communication. For example, when we discussed specific instructional materials or learning tools, I often shared my screen (and encouraged participants to share theirs) to provide a visual reference during our conversation. Additionally, we created visuals by utilizing the Zoom chat feature, which I frequently used to type both the interview questions as well as some of the interview responses I gathered from the participants. These practices enabled participants to visually confirm my questions and allowed me to verify my understanding of their responses.

Reflection

In summary, two important differences between the K-12 literature and the undergraduate math literature on multilingual students can be observed. First, there is relatively very little research at the undergraduate level focused on multilingual students. Second, whereas the existing research at the K-12 level spans all three themes substantively, the undergraduate level research lies primarily within the first theme: the relationship between language proficiency and mathematical performance.

Now, the first theme stems primarily from a cognitive view of learning, whereas the second and third themes stem from cultural views of learning. This reflects that, while the K-12 literature has in the past operated heavily from a cognitive perspective, now it primarily operates from a sociocultural perspective. On the other hand, the undergraduate literature on multilingual students appears to continue to operate from a cognitive perspective when it comes to studying multilingual students.

This is problematic because, as I noted earlier in the chapter, cognitive perspectives are more prone to framing multilingual students' learning in deficit ways. Thus, in this study, I seek to challenge this trend by explicitly adopting a sociocultural perspective (Moschkovich, 2015), which allows me to take an asset-based view of multilingual students. This is important for this study because one of its goals is to highlight the perspectives of multilingual students.

Conclusion

Overall, grounded in the literature on language and mathematics and a sociocultural perspective of learning, I developed a conceptual framework for identifying language demands and resources. The conceptual framework organizes language uses within three dimensions: lexico-grammatical, situational, and normative. The lexico-grammatical dimension seeks to capture words and phrases that function as language demands and resources (in terms of form, grammar, and meanings). The situational dimension seeks to capture materials, activities, semiotic systems, and sociocultural tools whose use may give rise to language demands and resources. Finally, the normative dimension attends to the relative constancy of communication and action within each situational aspect across class sessions.

By reviewing the research at both K-12 and undergraduate levels on mathematics, language, and multilingual students, I found several gaps in the literature. First, there is relatively little research situated in the classroom in the lexico-grammatical dimension. Much of the literature in math education in the lexico-grammatical dimension draws on data from assessment performance, task analysis, surveys, and clinical interviews. Second, there is relatively little research explicitly focused on language resources in the lexico-grammatical dimension; Most of the research in the lexico-grammatical dimension relates to language demands.

Third, there is relatively little substantive research at the undergraduate level within the situational dimension. Most substantive research in that dimension comes from K-12 levels. Fourth, although there is substantive research at the undergraduate level in the normative dimension, such research does not explicitly study and systematically relate their findings to language demands and resources. Moreover, such research is almost exclusively limited to analyzing norms related to verbal speech. Finally, I found there is relatively little math education research on multilingual undergraduate students, and the little research that exists is situated outside the U.S. Moreover, such research primarily operates from a cognitive perspective, which might induce a deficit view on multilingual students' resources.

This dissertation study will fill these gaps in the current literature on language and mathematics as follows. First, this study will explicitly, systematically, and substantively explore language demands at the undergraduate level in all three dimensions of the conceptual framework (lexico-grammatical, situational, and normative). Second, the research within all dimensions will be situated in the undergraduate mathematics classroom. Third, the normative dimension will be analyzed in communication mediums that extend beyond student speech in whole-class and small-group discussions. Finally, I will explicitly adopt a sociocultural perspective, which will allow me to take an asset-based approach when investigating and sharing the perspectives of multilingual students taking undergraduate mathematics classes in the U.S.

In sum, aligned with a sociocultural perspective, prior research, and the study's conceptual framework, I will conceptualize language demands and resources through the lens of the lexico-grammatical, situational, and normative dimensions. Using these dimensions as a framework, I will investigate each of the following research questions:

(RQ1) What language demands do multilingual students experience in an undergraduate IOLA course?

(RQ2) What instructional language resources do multilingual students experience in an undergraduate IOLA course?

(RQ3) What language resources do multilingual students use to interpret and communicate mathematical information in an IO linear algebra course?

Chapter 3 Methodology

This dissertation is a multi-case study of how four multilingual students experienced language demands and instructional language resources in one inquiry-oriented linear algebra (IOLA) class. It also analyzes how students utilized their own language resources during this class. In this chapter, I describe the design, data sources, and methods of my study. First, I present the study's methodological framework. Next, I describe the study's setting and participants, including my processes for selecting them. Then, I detail and justify the data sources I collected. Finally, I unpack the techniques I used for data analysis to address my research questions.

Methodological Framework

The overarching framework of this study was inspired by Moschkovich and Brenner's (2000) methodological approach, which integrated a naturalistic paradigm with more traditionally cognitive approaches. In particular, this research study combined the use of ethnographic classroom observations (Gee & Green, 1998; Patton, 2002) with semi-structured interviews with the teacher and students (diSessa, 2007; Hargreaves et al., 2003). This methodological framework is reflected in the timeline of the study's data collection, as shown in Table 3.1.

Table 3.1:

Timeline of Data Collection for My Dissertation Study

Week in Fall 2022 Semester	Weeks 1-6	Weeks 8-10	Weeks 12-13
Phase	I	II	III
Approach	Naturalistic	Cognitive	Cognitive
Data Source	Classroom observations	First round of student interviews; First teacher interview	Second round of student interviews; Second teacher interview
Modality	In-person	Over Zoom	Over Zoom

As noted in Table 3.1, this study involved three phases during the Fall 2022 academic term at Southwestern University (pseudonym) in the U.S. Each phase consisted of either a set of ethnographic observations of regular class sessions in a 16-week IOLA course or a set of semi-structured interviews. Phase I spanned approximately the first six weeks of the course. In this phase, I conducted classroom observations and wrote field notes (Patton, 2002). The class met twice a week on Tuesdays and Thursdays for 75 minutes each day. Then, in Phase II, I conducted a first round of one-on-one semi-structured interviews with the teacher and select multilingual students from the course. Finally, in Phase III, I conducted a second round of interviews with the teacher and the same select students. Immediately after each classroom observation or interview, I conducted a preliminary analysis of that data to inform subsequent data collection. All observations and interviews were videorecorded, transcribed, and analyzed.

A central framing of my dissertation study drew on case study methodology. The purpose of case-study research is to produce a detailed depiction and examination of a case or several cases, where a case is defined as a phenomenon with a focus and a boundary (Creswell & Poth, 2018; Miles et al., 2014). In my study, I considered two types of cases from one IOLA course. The first type of case focused on individual multilingual students who were learning the language of instruction, bounded by their cultural background and their experiences with language demands and resources in the IOLA class. More specifically, a case study for a multilingual student consisted of the following information:

(a) the student's cultural/linguistic background and prior history with inquiry-based mathematics education;
(b) the aspects of their IOLA class the student experienced as language demands;
(c) the instructional aspects of their IOLA class the student experienced as language resources; and
(d) the language resources the student drew from themselves and their peers in their IOLA class.

The other type of case was the instructor of the IOLA class with similar components as (a) through (d), but from her perspective as a teacher about her students' experiences.

This case study approach was apt for my study because it allowed me to capture the study's phenomena of interest (language demands and resources for multilingual students in one IOLA course) in the context of each multilingual student's unique background (culture, language, and prior history with inquiry-based mathematics education). This was accomplished by treating students' backgrounds not merely as demographic data but as lived experiences. Furthermore, I was able to analyze and report the language demands and resources for each individual student separately, thereby maintaining a close connection between the findings and the student's background. This approach enabled me to identify important relationships between a student's background and their linguistic experiences in the IOLA classroom.

Setting and Participants

This study took place at Southwestern University in the southwestern United States. The university is located in a conservative state with a history of passing anti-immigrant and anti-bilingual education laws, such as Arizona's Proposition 203, California's Proposition 187, and Texas' Senate Bill 4 (Arizona State Senate, 2019; Senate Research Center, 2017; Su, 2020). This context is relevant for understanding the perspectives and linguistic experiences of multilingual students and their instructors in and out of the classroom (e.g., Civil, 2018).

The study was situated in one linguistically diverse IOLA class with 35 students. Based on a survey I administered in the class, 20% of the students self-reported not having native proficiency in English. The class material was based on an expanded version of a nationally and publicly available IOLA curriculum developed by Wawro et al. (2013). This class was chosen for my study because it was taught by a recognized IOLA teacher-researcher. Below, I begin by describing my process and rationale for selecting the IOLA instructor as well as four multilingual students from her IOLA class for my case studies. Then I provide initial descriptions of the case study students.

Selection of Participants

The instructor was selected through convenience sampling (Patton, 1990) based on desired criteria. I sought an IOLA instructor with multiple years of experience. My intention was that choosing a highly established teacher would allow me to focus on language demands and resources in a more stable instructional approach. The more stable the approach, the easier it would be to abstract defining characteristics of the learning environment and ultimately ascertain the transferability of the study results (Moschkovich & Brenner, 2000). To find a suitable IOLA instructor for my study, I contacted various teacher-researchers who were recognized as expert inquiry-based instructors by some members of my dissertation committee. Of those instructors, only the instructor I selected was planning to teach an IOLA course in the U.S. in the semester I had designated for data collection. At the time of the study, the selected instructor had experience teaching IOLA for more than 10 years, and she was also a mathematics education researcher focusing on IOLA as one of her main research areas. She identified as a White woman and reported being a monolingual English speaker.

Student participants for the semi-structured interviews were selected using a criterion sampling method, a strategy of choosing cases among a pool of potential subjects based on a predetermined criterion of importance (Patton, 1990). Most importantly, I sought to recruit four student interviewees who spoke a primary language that was different from English. Additionally, if possible, I desired four interviewees that spanned a diverse range of linguistic, cultural, and gender backgrounds. On the first day of the classroom observations, I distributed a screening survey to all students in the selected IOLA class. The survey asked students to report the language(s) they spoke, their race or ethnicity, and their gender. The students were also asked to rate their proficiency in each of their language(s) from one to six, where one represented no proficiency and six represented native proficiency.

In the screening survey, seven out of 35 students self-reported a non-native proficiency in English. These students comprised one woman and six men. Unfortunately, the woman chose not to participate in the study. Two men agreed to be videorecorded during the classroom observations but not participate in the interviews. The remaining four men agreed to participate in all aspects of the study and were thus selected as participants for the semi-structured interviews. The four selected students spanned a diverse range of linguistic and cultural backgrounds. Three of the selected interviewees were international students who self-rated their

English proficiency as a four and self-reported native proficiency in Malaysian, Vietnamese, and Korean, respectively. The pseudonyms I assigned to these students were Johan, Nam, and Seok. The remaining multilingual interviewee, Luis (pseudonym), was a U.S. domestic, Latino student who self-rated his English proficiency as a five and self-reported native proficiency in Spanish. I invited Johan, Nam, and Luis to sit together in one small group during the IOLA class. The other multilingual student was already engaged with another group.

A few days into the study, I added Kaden (pseudonym), a predominantly monolingual English-speaking man, to the pool of interviewees. Although the focus of my study was on the experiences of multilingual students, I found it ethical and informative to include Kaden in the interviews for several reasons. Kaden expressed eagerness to participate in the interviews and voluntarily sat in the same small group as Johan, Nam, and Seok. So, Kaden's class experiences were tightly intertwined with three of the other student interviewees. Thus, his perspective as a group member could reveal insights into the experiences of Johan, Nam, and Seok. Ultimately, I leveraged Kaden's data as part of the context for making sense of the language demands and resources reported by Johan, Nam, and Seok. I did not create a case study for Kaden because his experiences as a monolingual student were outside the scope of this dissertation study. (In a future project with a different scope, I plan to create and report Kaden's case study.)

Overview of Case Study Students

Next, I provide initial descriptions for Johan, Nam, Seok, and Luis, identifying the aspects of their background–culture, language, and prior history with inquiry-based mathematics education–that were most relevant to their case. (More detailed descriptions of each case study's background are reported in Chapters 4 and 5.)

Johan was an international undergraduate student from Malaysia who identified himself as a man. He completed his K-12 education in Malaysia before he moved to the U.S. to pursue his undergraduate studies. Johan's language repertoire included Malaysian (or Malay), Arabic, Japanese, and English. On the screening form of the research study, he indicated his language proficiencies on a scale from one to six. He reported native proficiency (level six) in Malay, professional working proficiency (level four) in Arabic, limited working proficiency (level three) in Japanese, and professional working proficiency (level four) in English. During the observation period of this study, Johan was enrolled in an IOLA course, which served as both his first linear algebra and his first inquiry-based mathematics course. His previous education in Malaysia relied heavily on lectures and drill-and-practice methods.

Nam was an international undergraduate student from Vietnam who identified himself as a man. He completed his K-12 education in Vietnam before he moved to the U.S. to pursue his undergraduate studies. Before transferring to Southwestern University, he completed two years at a community college in the US. He reported native proficiency (level six) in Vietnamese and professional working proficiency (level four) in English. The IOLA course served as both his first linear algebra class and his first inquiry-based mathematics course. His previous education in Vietnam and the US relied heavily on lectures.

Seok was an international undergraduate student from South Korea who identified himself as a man. He completed part of his K-12 education in South Korea, and then moved to the U.S. to start 8th grade and eventually begin his undergraduate studies at Southwestern University. He reported native proficiency (level six) in Korean and professional working proficiency (level four) in English. The IOLA course served as both his first linear algebra class and his first inquiry-based mathematics course. His previous education in South Korea and the

US relied heavily on lectures, but he had some courses in Southwestern University where he engaged in project-based learning.

Luis was a Latino undergraduate domestic student who identified himself as a man. He was born in the US, where he completed his K-12 education. Before transferring to Southwestern University, he completed two years at a community college in the U.S. He reported native proficiency (level six) in Spanish and full professional proficiency (level five) in English. The IOLA course was his second linear algebra class, and he had prior experience with both active learning undergraduate mathematics courses and lecture-based math classes.

Data Sources

The primary sources utilized in this dissertation were classroom observations and semi-structured interviews. Additionally, informal sources emerged organically throughout the study, including email exchanges, text messages, and conversations before and after classroom observations. While initially intended solely for logistical purposes, these informal channels unexpectedly provided valuable insights into the background of the case study. Similarly, pre- and post-class conversations were primarily aimed at establishing rapport and understanding participants, yet they also yielded valuable information regarding students' backgrounds and their experiences in class. Subsequently, I further explored this additional gathered student information during the interviews (e.g., through follow-up and clarifying questions).

In this section, I focus on the classroom observations and semi-structured interviews. I structured these data sources with the anticipation that the class content would be derived from an expanded version of the curriculum formulated by Wawro et al. (2013). Consequently, I expected that my classroom observations would capture approximately two instructional units from this curriculum: (a) linear independence and span, and (b) systems of equations. Ultimately, my classroom observations spanned these two instructional units as well as one lesson from an additional unit in this curriculum focused on "matrices as transformations." Below I detail the design and data collection process of the classroom observations and semi-structured interviews. Then, I describe the preliminary processes of analysis, transcription, segmentation, and data reduction.

Classroom Observations

The study included observations in the IOLA classroom to capture the natural setting where students learned linear algebra. These observations allowed me to better understand the context in which students experienced any language demands and instructional resources or leveraged their own resources. During whole-class discussions, I primarily sat in the back of the room taking field notes and videorecording the lesson. I also moved around with the video camera during small-group discussions. Below, I describe how I took fieldnotes, videorecorded the classroom observations, and collected classroom artifacts.

Fieldnotes. The fieldnotes I took contained a description of what I observed and believed to be worth noting during each lesson. In line with Patton's (2002) suggestion, my fieldnotes were primarily descriptive. They included the date of the lesson, who was present, the physical arrangements, the social interactions that occurred, the activities that took place, and the possible meanings of what was observed from the position of both the teacher and the students. Although the primary focus of my fieldnotes was description, I also included my own feelings, reactions, and reflections about what I observed, so as to capture my own experiences as an observer. Finally, I allowed room for insights, interpretations, and beginning hypotheses about what happened during the observation and its potential meaning.

I organized the complex stimuli I experienced during the classroom observations according to the constructs in my conceptual framework. I did so by following Patton's (2002) recommendation of building my observations around activities or blocks of communication that have a beginning, middle, and end. In this way, I identified self-contained units of activity to break down my notetaking into manageable units. For example, some units of activity included whole-class and small-group discussion blocks related to answering different parts of a task (e.g., question 1 of Task 1), engaging in specific disciplinary practices (e.g., defining), or creating particular mathematical objects or processes (e.g., proving a conjecture). To keep track of the timing of these events, I marked the beginning of each unit with a timestamp. This gave me a time reference for where I could later locate a particular instance of the observation for further reflection or analysis.

Mathematical Tasks. The mathematical activity and discussion in the IOLA class revolved around solving mathematical tasks that were handed to students in printed form. The IOLA class in the study often alternated between whole-class and small-group discussions about the solutions to these tasks. In their small groups, students wrote their collective solutions on their group whiteboards and, sometimes, on their individual worksheets. During the classroom observations, I took pictures of the individual worksheets of the five student interviewees. Later, I followed up with the instructor over email to collect digital blank copies of all the tasks that were introduced during the classroom observations. There were 11 mathematical tasks in total (shown in Appendix A, Figures 2-18), each covering about one class day. I assigned a number to each task based on the first day of observation (for example, Task 2 was introduced on the second day of class) and named them to closely mirror their titles printed on the worksheets. Below I broadly describe the mathematical contexts and topics embedded in the tasks. I organized them based on my knowledge of Wawro et al.'s (2013) IOLA curriculum and the primary mathematical topics I identified during the classroom observations.

The first six tasks (Tasks 1, 2, 3a, 3b, 4, and 5) focused on span and linear independence. Task 1 (The Carpet Ride Problem: The Maiden Voyage) first asks students to imagine themselves as someone who could travel using a hoverboard and a magic carpet. The task then asks students to determine and justify whether this traveler can get to a specified destination (Old Man Gauss' cabin) employing these two modes of transportation. Mathematically, each mode of transportation and any given destination can be represented as a vector. So, a particular journey with these modes of transportation can be represented with a linear combination of these two vectors. Hence, determining the feasibility of reaching a specific location with a combination of these modes corresponds to establishing the existence of a linear combination of the two vectors that equals a specific vector. Task 2 (The Carpet Ride Problem: Hide and Seek) uses the same story context as Task 1, with one slight twist: Old Gauss intends to relocate to a cabin situated elsewhere. Then it asks students to determine whether there are any locations where he cannot be reached using the two modes of transportation. Mathematically, this task amounts to determining if there are any vectors that cannot be expressed as linear combinations of the two vectors representing the modes.

Task 3a (Span Worksheet) provides a formal definition of span, and it asks students to determine the span of various sets of vectors as well as describe each span using different representations. Task 3b (Group Quiz) asks students more focused questions related to determining the span of a set of vectors. For example, these questions ask students to identify the dimension of the vectors and to conceptualize their mathematical thinking about vectors using their prior knowledge of the equation $y = mx+b$. Task 3b has the potential to reinforce and connect the concepts embedded in Tasks 1, 2, and 3a. Task 4 returns to the carpet ride story context, but now in a three-dimensional world. So, the modes of transportation are now given as three-dimensional vectors. The task asks students to determine the feasibility of using the three modes of transportation to depart from and subsequently return home. Mathematically, home can be represented by the zero vector, so the essential student activity is to determine if there exists a nontrivial linear combination of the three vectors (representing the modes) that equals the zero vector. Task 5 asks students to create examples of linearly independent and linearly dependent sets of n vectors from $\mathbf{R}^n$, for different values of n. Students are also given the option to state if no such set of vectors can be created for a given dimension n.

Tasks 6-10 focused on systems of equations and their corresponding solution spaces. Task 6 asks students to solve a system of equations algebraically and interpret the solution space of the system geometrically. Task 7 (Meal Plans: Constraining the Number of Meals) asks students to imagine they are enrolled in a semester-long meal plan at a given university. The story context involves different constraints on the number of meals. For example, one constraint is that "each meal (breakfast, lunch, and dinner) can be purchased up to twice per day." This task asks students to identify feasible meal plan options that satisfy the constraints in the story by representing the constraints symbolically (e.g., as equations). Then students are then asked to interpret the constraints and feasible meal plan options geometrically in a 3D graph. Task 8 (Meal Plans: Constraining the Cost) uses the same story context as Task 7 but introduces a new constraint: The cost of the meals must equal a certain dollar amount. The student is then asked to explore these constraints and the corresponding feasible meal plan options symbolically and graphically.

Next, Task 9 (The Car Rental Problem) introduces a story about a car rental company in a given city, where customers are allowed to return a rental to any of the company's three locations. A diagram is provided that illustrates the pair-wise flow of rental cars between the three locations in terms of the percentage of cars rented in one location and returned in another. The task then asks students to imagine themselves being in a team of consultants and being hired by the car rental company to achieve their eventual goal: "To determine how the company might most efficiently manage its resources to meet demand and optimize profit." The task formulates this eventual goal into two main questions: (a) "If we [the company and team of consultants] stopped reshuffling the cars, how many cars would eventually end up in each location and does the answer to that depend on the number of cars we start with at each location?" (b) Is there a number of cars we could start with at each location that would give a relatively stable distribution?" The task then provides a starting point for accomplishing this goal by giving students specific values for the initial number of cars at each location and asking them to determine the number of cars at each location at the end of particular weeks. Task 10 (The Car Rental Problem: Follow-Up) follows up on Task 9 during the next day of class by co-constructing a table that fulfills the starting activity recommended in Task 9. Task 10 then presents a list of questions that collectively address the main questions described in Task 9.

Finally, Task 11 focuses on matrices as transformations. This task provides a capital letter F drawn on an unlabeled grid as well as different matrices that induce certain transformations on the F. It asks students to determine and graph the image of the F with respect to the different matrices by multiplying each point on the original F by a given matrix. The task encourages students to generalize this process by giving them certain transformations and asking them to determine the matrices that induce those transformations through matrix multiplication. For example, one transformation given is a rotation of 180 degrees.

Videorecording. The classroom observation video recordings aimed to capture both verbal and nonverbal communication (including gestures, body orientation, eye gaze, and actions such as moving a cursor) as well as the use of different tools of communication in the IOLA classroom. During the small-group discussions, I videorecorded the interactions among the four selected student interviewees who sat in the same small group (three multilingual students and one predominantly monolingual student), capturing the group's collective board work. I did not videorecord the additional multilingual student interviewee during small-group discussions because he was in a separate small group. In whole-class discussions, I pointed the video camera at the main speaker(s) or to any written work being displayed and discussed.

Preliminary Analyses

At the conclusion of each classroom observation and interview, I filled out a contact summary form (Miles et al., 2014) as a method of preliminary analysis. According to Miles et al. (2014), a contact summary form is a tool used by researchers to summarize the main points after contact with a data source. The template I used (see Appendix B) started with a section that identified the specific contact event (data source, event date, and event topic or interviewee). Then it included questions about interpretations (e.g., salient categories related to each research question and dimension of the L-S-N framework), preliminary findings (e.g., summaries of any identified categories), impressions (e.g., what struck the researcher as interesting), and wonders (e.g., what new questions to consider) related to the data gathered during the contact event.

Transcription and Segmentation

All classroom observations and interviews were transcribed for analysis. The text in each transcript was organized into a three-column table, where each row captured a new turn of talk, nonverbal communication, and/or a comment about something relevant that was present or happening in the moment of the observation or interview that was being transcribed. The left-most column is a timestamp for a given turn. The middle column notes the speaker or actor of the turn or the label "comment." Finally, the right-most column consists of the quoted utterance, a description of the actor's nonverbal behavior, and/or a comment about something relevant that was present or happening. For the text in the right-most column, I used the transcription conventions described in Table 3.2.

Table 3.2:

Transcription Conventions

Convention	Description
[ ]	Modification of part of an utterance was indicated in square brackets. Examples of modification included insertions of words or letters that grammatically became part of an utterance to make its meaning more comprehensible to the reader.
<< >>	An utterance in a language other than English was placed inside double angle brackets.
[[ ]]	Translation of an utterance from another language into English was placed inside double brackets.
( )	If there was an uncertain transcription, the questionable words appeared within single parentheses. An exception was for the titles of IOLA tasks, which appeared within single parentheses adjacent to their task labels.
(( ))	Comments about a part of an utterance and/or related nonverbal behavior, such as gestures, movements and looks, were indicated with double parentheses. Examples included insertions of words that provided additional context to part of an utterance, such as interpretations of what the pronoun "it" might refer to.
((inaudible))	Parts of utterances that could not be discerned for transcription were indicated with the word inaudible within double parentheses.

To organize the transcript data from each data source into manageable units, I divided the text in each transcript into episodes, segments of communication related to a common topic. In components 1 through 4 of Interview 1 (see Appendix B), the episodes were defined by each main interview question (the numbered questions in the interview protocol). In this interview's fifth component, the mathematical tasks sorting activity, the episodes were the explanations of each task ranking in terms of linguistic complexity. Similarly, for Interview 2, the episodes were defined by each video clip. Finally, for the classroom observations, each episode consisted of blocks of communication that had a beginning, middle, and end (e.g., whole-class and small-group discussion related to answering different parts of a task).

Data Reduction

To better highlight the categories of language demands and resources for multilingual students in the IOLA class, I engaged in data reduction (Miles et al., 2014). Because the interviews were explicitly designed to elicit language demands and resources from the perspectives of multilingual students, the interviews (relative to the classroom observations) provided the most relevant data for answering my research questions. Therefore, I focused on the interview data for my main analysis. In addition, after conducting preliminary analyses of the interviews, I found that Interview 2 did not add much to the outline of general findings from Interview 1. Interview 2 primarily elaborated and provided more examples for the Interview 1 analysis results. So, I decided to focus primarily on the data from Interview 1, which consisted of multiple interview sessions per participant, for systematic analysis.

It is worth noting that, although the classroom observation data was not systematically analyzed, it informed the data I gathered during Interview 1. During the interview, I had my field notes and contact summary forms from the classroom observations at hand. I used these observation notes and preliminary analyses to interpret the interview responses in real-time, as well as to ask follow-up questions or elicit information grounded in my classroom observations. Thus, the interview data expanded upon the insights from the classroom observations, rather than originating solely from isolated interviews.

Analysis

As noted in the previous section, I used the Interview 1 data (comprised of multiple interview sessions per participant) as my main source for analysis. For example, I contextualized my interpretations from Interview 1 with the fieldnotes, transcriptions, and preliminary analyses from the classroom observations. In this section, I describe how I systematically analyzed each

Interview 1 transcript to answer my three research questions, which focused on language demands (RQ1), instructional language resources (RQ2), and student language resources (RQ3) for multilingual students. I tackle this set of research questions case-by-case, the overarching unit of analysis. I address each research question through the dimensions of the L-S-N framework: lexico-grammatical, situational, and normative.

Identifying Language Demands (RQ1)

To answer RQ1 within the three L-S-N dimensions, I conducted *two-cycle coding* on the interview data of each student cases study, followed by a cross-case *clustering* technique (Miles et al., 2014). In brief, the first cycle consisted of assigning initial codes to the interview data of each student, whereas the second cycle focused on compiling and grouping those codes into more general categories. This coding process allowed me to systematically and efficiently identify tools of communication that each student experienced as language demands. After completing the two coding cycles for each student case study, I compiled and grouped the codes across all the case studies into the finalized categories. To add reliability to my coding, I discussed the interview data and coding with multiple researchers. For each coding cycle and application of clustering, I engaged with at least one researcher and confirmed whether they agreed with my interpretation and coding of the data. Disagreements were relatively few and were resolved through discussion. Below I describe my coding processes in more detail.

In the first cycle, I analyzed each student interview transcript by episode (the unit of analysis) to identify when and how a communication tool induced a language demand for each student. (Episodes were defined earlier in this chapter in Data Sources – Transcription and Segmentation.) For evidence of a language demand, I considered the presence of any of the following conditions about language challenges:

(a) Student expressed they didn't know how to interpret or use a communication tool from class;

(b) Student expressed they were confused about how a communication tool was used in class;

(c) Student used a communication tool in a way that conflicted with the intentions of the teacher or the classroom community; or

(d) Student expressed that the class used a communication tool in a way that conflicted with the student's intentions.

More generally, I attended to words that signaled a challenge (in relation to language and communication), such as: hard, difficult, complex, hindering, tension, frustrating, annoying, disturbing, and uncomfortable. I applied these criteria to communication tools in the IOLA class within each dimension. A lexico-grammatical tool could be a word, phrase, or grammatical feature, whereas a situational tool could be a particular material, activity, semiotic system, or sociocultural structure of communication. Lastly, a normative tool could be a social or sociomathematical norm.

To understand the nature of conditions (c) and (d) within the normative dimension, I conceptualized the linguistic conflicts associated with these conditions as *norm tensions*—which are conflicts between the norms of one community and the norms or situational aspects of another community—that constrained the case study student's communication or interpretation of mathematical information. In this study, I focused on the norm tensions between two kinds of communities: (1) the student's IOLA classroom community and (2) any other communities to which the student belonged, especially national, cultural, and linguistic. During my analysis of Johan's case study (the first case I analyzed), I discovered different ways in which he experienced norm tensions. This led me to the development of a Norm Tensions framework, illustrated in Figure 3.1, for characterizing three types of norm tensions a student may encounter in an inquiry-based classroom. This framework proved to be useful for capturing the norm tensions of all the case studies in my dissertation.

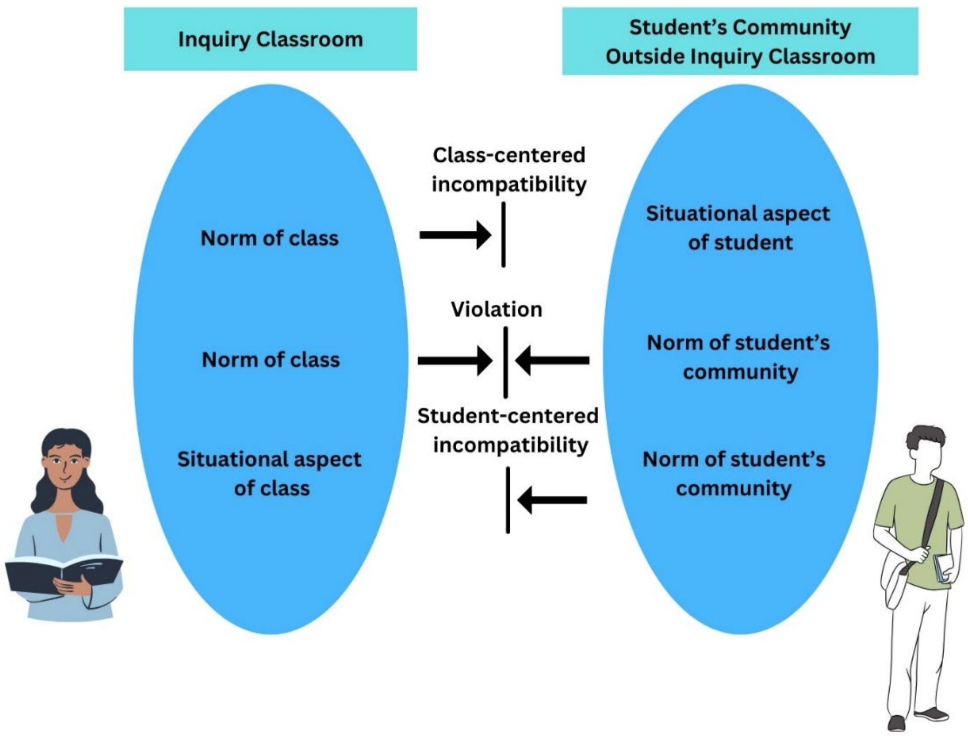


Figure 3.1:

Norm Tensions Framework

The framework consists of three types of norm tensions: violation, class-centered incompatibility, and student-centered incompatibility. A *norm violation* refers to when a norm of the student's inquiry classroom community competes with a norm of another community to which the student belongs. A *class-centered norm incompatibility* refers to when a norm of the student's inquiry class is inconsistent with a student's situational tool of communication, making it infeasible for a student to engage in the class norm. An example of a situational tool of communication for a given student is their linguistic repertoire (which includes student use of their primary language as well as any translanguaging practices), provided that the student does not impose their linguistic repertoire as an expectation in the IOLA classroom. As illustrated in Figure 3.1, I differentiate a class-centered norm incompatibility from a norm violation by requiring that, in a class-centered norm incompatibility, a student does not position their situational tool of communication as an expectation that should be legitimated in the inquiry classroom (that is, there is no evidence that the norm from the student's community is regulating the student's expectation of communication in the IOLA classroom). A *student-centered norm incompatibility* refers to a situational aspect of the inquiry class setting that makes it infeasible for the student to engage in a norm they have internalized from their other communities.

It is worth noting that the Norm Tensions Framework can be applied to norms of any kind, which may or may not induce language challenges. However, in my study, I applied this framework to identify the norm tensions that constrained the case study student's communication or interpretation of mathematical information. I described these norm tensions primarily from the perspective of each case study student (as opposed to the researcher's or the teacher's perspectives).

Once I determined the presence of a condition from (a)-(d) for a communication tool within any dimension in the context of a given episode, I identified the tool's primary dimension and component (see Conceptual Framework in Chapter 2 for the components within each dimension). Then, I assigned a code that captured the language challenge associated with the condition. To achieve this, I used a tactic of "subsuming particulars into the general," where I asked myself, "What is this specific thing [language challenge] an instance of? Does it belong to a more general class [category]?" (Miles et al., 2014, p. 285). For example, during the interview, Nam noted a specific challenge with discerning between numerator and denominator in English.

During my analysis, I recognized this as a language challenge because it fit condition (b) above. This challenge resided within the lexico-grammatical dimension because it was an issue with specific words, and I organized it under the semantic component because it dealt with discerning meanings. Applying the coding tactic described above, I coded this language challenge with the category of “antonym pairs.”

I identified categories from the data through a combination of inductive and deductive coding (Miles et al., 2014). That is, some of the categories emerged as a result of coding, while others resulted from an a priori orienting construct found in the breakdown of the components in the L-S-N conceptual framework (see Chapter 2). Other categories emerged as a hybrid of these two approaches. For instance, the category “antonym pairs” mentioned earlier was obtained inductively. In contrast, speech communication was another category that was identified in the data, but it was predefined as a construct under the semiotic component of the situational dimension.

In the second cycle of each interview analysis, I organized these codes into more general categories that captured the essence of each identified language challenge. To accomplish this objective, I employed a technique known as “clustering,” wherein I progressed towards broader conceptual levels by inductively grouping similar codes together (Miles et al., 2014). After completing the second cycle of coding for one case study student, I began the first coding cycle for the next case study student.

Finally, once I completed both cycles of coding for all student case studies, I again used the “clustering” technique described above to create common codes between case studies, when possible. For example, after completing the two cycles of coding for Johan’s case study, my findings included a code called “cardinal direction terms” for a lexico-grammatical demand. This code arose from Johan noting that he had trouble discerning between “north” and “south” and “east” and “west.” However, later when I identified the code “antonym pairs” to capture a lexico-grammatical demand that Nam experienced, I realized that this new code also captured the language demand that Johan reported. Therefore, “antonym pairs” served as a common code between Johan’s and Nam’s case studies. Appendix C shows the finalized codes that spanned all student case study interview data. Note that there is one table for each research question and each dimension of the L-S-N framework, except for the situational dimension. For each research question, the situational dimension has two tables, one for the communication within the tasks and one for general classroom communication.

Identifying Instructional and Student Language Resources (RQ2 and RQ3)

To address RQ2 and RQ3, I used similar analysis methods as with RQ1, with three key differences. First, for both RQ2 and RQ3, instead of searching for evidence of language demands (or challenges), I searched for evidence of language resources (or supports). I was focused on instructional language resources for RQ2, but on student language resources for RQ3.

Second, I considered different sets of conditions for evidence of language resources. For RQ2, the conditions to identify instructional language resources were:

(a) Student expressed the class used communication tools that the student knew how to interpret or utilize;

(b) Student expressed the class used tools in a way that was clear to the student; or

(c) Student expressed that the class used a communication tool in a way that aligned with the student’s intentions.

Similarly, for RQ3, the conditions for identifying student language resources were:

(a) Student expressed they drew on their prior knowledge to interpret or use a communication tool from class; or

(b) Student utilized a communication tool in a way that they preferred and aligned with their prior experiences.

More generally, while focusing on instructional language resources for RQ2 and on student language resources for RQ3, I attended to words that signaled a resource, such as: helpful, easy, clear, facilitating, useful, comfortable, beneficial, and simple. Finally, for both RQ2 and RQ3, there was no necessity to utilize a framework to analyze the nature of any normative conditions, as there were no conflicts present.

Cross-Case Comparison

After answering the research questions by case study, I addressed them in terms of similarities and differences across case studies. I compared the student case studies in various ways. One way was by noting the presence of any common codes within each dimension between student case studies. For example, as noted earlier, "antonym pairs" was a common code between Johan's and Nam's case studies. A second way of comparison was by highlighting the codes that were salient in the experiences of one or more case study student(s) in the IOLA classroom. To capture the salience of a code about communication within the tasks, I considered the student's task rankings of linguistic complexity that the student attributed to the language demand or resource encapsulated in the code. To capture the salience of a code about general classroom communication, I considered the intensity with which a student described the associated language demand or resource. Criteria that I used to gage intensity included student language that expressed strong emotion or that indicated a significant impact on the student's experience. For example, one code for normative demand that I identified for Johan was a norm violation related to respect towards the teacher. Johan described this norm violation intensely through the emotionally charged language he used to express it: "In Islam, we prioritize our manner with the teacher very, very high… don't interrupt when they are speaking…So in Malaysia, if you do that, you're dead!"

Finally, a third way through which I compared the student case studies was to count the number and types of codes per student for each research question and dimension. For example, Johan encountered a significantly larger number of normative demands than Nam, as evidenced by their relative numbers of codes encapsulating language demands in the normative dimension. Moreover, Nam reported one normative demand of each type (among violations, student-centered incompatibilities, and class-centered incompatibilities), while most of the norm tensions Johan faced were violations, which represent the highest degree of conflict between the IOLA class and a student's cultural communities outside of the classroom.

Conclusion

This chapter outlined the framing and methods I used in my study. I employed a case study approach to capture the linguistic experiences of multilingual students within the context of their unique backgrounds. Using convenience sampling, I selected an expert inquiry-based instructor with extensive teaching experience and background as a researcher of IOLA. She identified as White and was a monolingual English speaker. Subsequently, I used a criterion sampling method to select four multilingual students for the study. Three participants (Johan, Nam, and Seok) were international students who self-rated their English proficiency as four out of six and reported native proficiency in Malaysian, Vietnamese, and Korean, respectively. The fourth participant, Luis, was a U.S. domestic Latino student who self-rated his English proficiency as five and reported native proficiency in Spanish.

For data collection, I combined ethnographic classroom observations with two semi-structured interviews with the teacher and each multilingual student. I used the first interview with each participant, averaging 2.5 hours, as the primary source for analysis. Data analysis involved two-cycle coding for each participant case study data: the first cycle assigned initial codes to the interview data of each student, and the second cycle compiled and grouped these codes into broader categories. Additionally, I used a cross-case clustering technique to inductively group similar codes. Finally, I performed a cross-case comparison using various approaches, including capturing and comparing the prominence of certain codes across the case study students.

Table 3.3 summarizes the logic of this dissertation study by connecting the research questions with the primary data sources, analysis methods, and organization of the findings. The next three chapters present the research findings, addressing all three research questions by case study. Chapters 4 and 5 focus on the case studies of the four multilingual students. More specifically, Chapter 4 examines the cases of Johan and Nam – two international, multilingual undergraduate students who arrived in the U.S. immediately prior to starting their undergraduate studies. Chapter 5 covers the cases of Seok and Luis – two multilingual undergraduate students who completed part or all their K-12 education in the U.S. Finally, Chapter 6 presents the case of the IOLA instructor.

Table 3.3:

Summary of the Logic of This Dissertation Study

Research Questions	Primary Data Sources	Analysis Methods	Organization of the Findings
RQ1: Language Demands	Interview 1 Protocol (multiple interview sessions with each case study student and the instructor)	Two-cycle coding for each case study student and the instructor Cross-case clustering technique Cross-case comparison	Reported by case study
RQ2: Instructional Language Resources	Interview 1 Protocol (multiple interview sessions with each case study student and the instructor)	Two-cycle coding for each case study student and the instructor Cross-case clustering technique Cross-case comparison	Reported by case study
RQ3: Student Language Resources	Interview 1 Protocol (multiple interview sessions with each case study student and the instructor)	Two-cycle coding for each case study student and the instructor Cross-case clustering technique Cross-case comparison	Reported by case study

Chapter 4 Case Study Findings for International, Multilingual Undergraduate Students

In this chapter, I analyze the perspectives of international, multilingual undergraduate students who came to the U.S. at the start of their undergraduate studies. Each was taking part in one inquiry-oriented linear algebra (IOLA) course at Southwestern University in the U.S. In doing this research, I specifically focus on three key research questions:

(1) What language demands do multilingual students experience in an IOLA course?

(2) What instructional language resources do multilingual students experience in an IOLA course?

(3) What student language resources do multilingual students use in an IOLA course?

To address these questions, I present two cases: Johan (from Malaysia) and Nam (from Vietnam). Both students completed their high school education abroad before coming to the U.S to pursue their undergraduate studies in a STEM field. Each case study consists of the following information:

(a) the student's cultural/linguistic background and prior history with inquiry-based mathematics education;

(b) the aspects of their IOLA class the student experienced as language demands;

(c) the instructional aspects of their IOLA class the student experienced as linguistic resources; and

(d) the linguistic resources the student drew from themselves and their peers in their IOLA class.

I analyze the interview responses of each case study student with a focus on items (b) through (c) above, regarding language demands and resources. In Chapters 2 and 3, I provided theoretical and analytical descriptions of these phenomena. Applying those previous concepts to this chapter, I take language demands to mean tools that constrain the student's communication or interpretation of mathematical information. Similarly, I conceptualize language resources as tools that facilitate the student's communication or interpretation of mathematical information. I consider two types of language resources: student resources and instructional resources. Further, I analyze both language demands and resources along the three sociolinguistic dimensions in my L-S-N framework: lexico-grammatical, situational, and normative. In short, the lexico-grammatical dimension attends to words, phrases, and grammar, whereas the situational dimension attends to the broader material, activity, semiotic, and sociocultural (MASS) aspects of communication (Gee & Green, 1998). Lastly, the normative dimension attends to the norms built over time that create or reflect expectations about communication.

For each student's case study analysis, I operationalized normative language demands as norm tensions—between the student's IOLA classroom community and any other communities to which the student belonged, especially national, cultural, and linguistic—that constrained the case study student's communication or interpretation of mathematical information. Figure 4.1 illustrates three possible types of norm tensions a student may encounter in an inquiry classroom: violation, class-centered incompatibility, and student-centered incompatibility. A norm violation refers to when a norm of the student's inquiry classroom community competes with a norm of another community to which the student belongs. A class-centered norm incompatibility refers to when a norm of the student's inquiry class is inconsistent with a student's situational tool of communication, making it infeasible for a student to engage in the norm. A student-centered norm incompatibility refers to a situational aspect of the inquiry class setting that makes it infeasible for the student to engage in a norm they have internalized from their other communities. All three types of norm tensions were evident in the analysis of the two case study students, with examples of each type constituting language demands (that is, constraining the students' communication or interpretation of mathematical information).

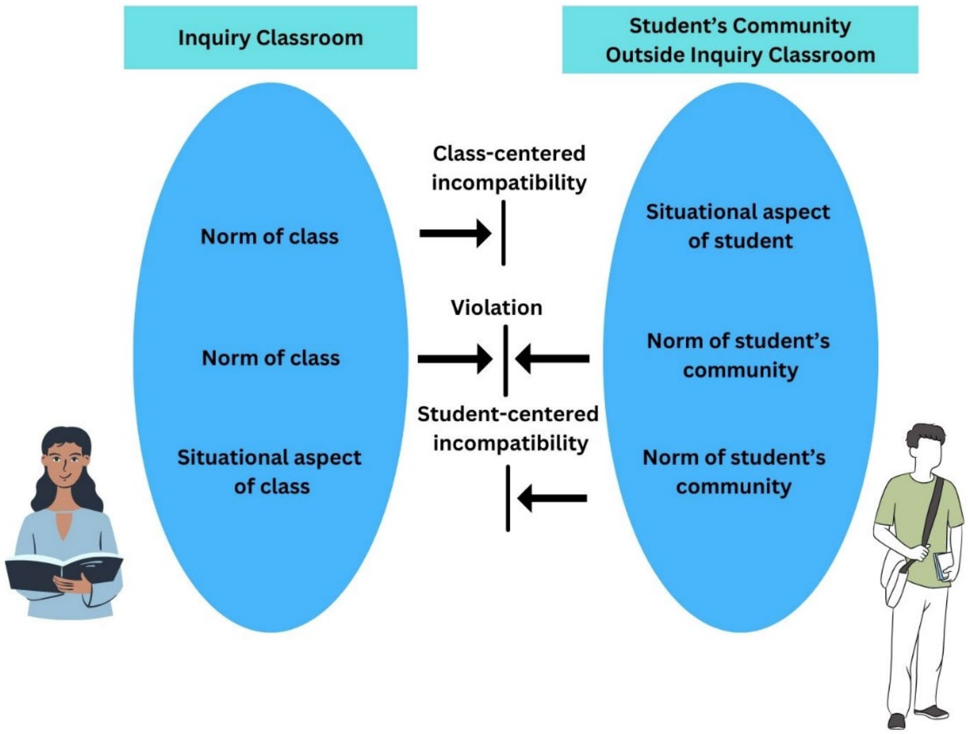


Figure 4.1:

The Norms Tensions Framework for Characterizing Norm Tensions

In each subsection of this chapter, I identify and unpack language demands, instructional language resources, and student language resources for each case study student using their interview data. Through my analysis, I found that each student had at least one language demand and resource within each dimension of the L-S-N framework. While each student's perspective consisted of a variety of demands and resources, situational and normative demands constituted the bulk of the language demands across the two case studies.

I begin this chapter by describing each case study's findings, focusing on three main subsections that correspond respectively with the three research questions: (a) language demands, (b) student language resources, and (c) instructional language resources. Each subsection is organized along the three dimensions of the L-S-N framework. I consider each dimension in relation to both communication on the tasks and more general classroom communication (that is, all other sources of communication—for example, teacher and student talk—in the IOLA classroom). Descriptions of the mathematical contexts and topics embedded in these tasks are included in Chapter 3—Data Sources—Classroom Observations—Mathematical Tasks, and images of these tasks are included in Appendix A. In the conclusion, I compare these case studies by research question and dimension, highlighting the most relevant similarities and differences.

Johan's Case Study

Johan was an international undergraduate student who self-identified as male and Malaysian. He completed his entire K-12 education in Malaysia, and he moved to the U.S. to begin his undergraduate studies. At the time of the study's data collection, he was majoring in software engineering and beginning his junior year at Southwestern University.

Johan's linguistic repertoire included Malaysian (Malay), Arabic, Japanese, and English. On the research study's screening form, which asked him to rate his language proficiencies from levels one through six, he self-reported native proficiency (level six) in Malay, professional working proficiency (level four) in Arabic, and limited working proficiency (level three) in Japanese. In addition, he also self-reported a professional working proficiency (level four) in English. However, during his first interview, he described his English proficiency as a five out of 10.

Johan's perspectives of and linguistic experiences with learning mathematics in his undergraduate studies were strongly mediated by his past educational and language history in Malaysia. He started learning English when he was seven years old. In his high school mathematics classes, he spoke 60% Malay and 40% English to learn the subject matter. His class lectures were primarily conducted in English, but the instructor and students used both English and Malay to explain the mathematics.

Johan spent his last two years of high school in Malaysia preparing for a national exam that he referred to as the “Malaysian National Certificate Examination.” During those years, he learned some linear algebra concepts, such as matrices, but he did not take a linear algebra course. The inquiry-oriented linear algebra (IOLA) course he took during my observations for this dissertation study was simultaneously his first linear algebra course and his first inquiry-based mathematics course. The classes he took in Malaysia to prepare for the national exam used a drill and practice approach.

Johan’s participation in the IOLA class was negatively impacted by language challenges induced by the classroom environment. This impact was so deep for Johan that when I asked him how easy or difficult it was to participate in the IOLA class, he said, “I think if I rate one to 10 with one being easy and 10 being difficult, I will put it around eight.” This case reveals the experience of a student who found the inquiry environment uncomfortable, and language and culture may be key aspects of this student’s discomfort.

Below I describe the demands and resources, spanning all three dimensions of the L-S-N framework, that Johan experienced in the IOLA class. For Johan, the most salient language demands and resources were captured in the normative dimension. The subsections that follow will describe the language demands, instructional language resources, and student language resources that Johan experienced.

Language Demands

In this section, I describe the language demands that Johan experienced across the three L-S-N dimensions: lexico-grammatical, situational, and normative.

Lexico-Grammatical Demands. Recall that the lexico-grammatical dimension consists of two components: syntactic and semantic. The syntactic level focuses on the form and grammar of words and phrases, whereas the semantic level focuses on the meanings of words or phrases. Below I describe the semantic and syntactic demands that Johan experienced, which are also captured in Table 4.1.

Table 4.1:

Lexico-Grammatical Demands Experienced by Johan in the IOLA Classroom

Component	Lexico-Grammatical Demands	Example	Context of Example
Semantic	Antonym pairs	"East" versus "West"	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
		"North" versus "South"	
	New mathematical terms	"Vector notation"	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
		"Span"	Task 3a (Span Worksheet)
	Words or phrases with unfamiliar meanings	"Up to twice"	Task 7 (Meal Plans: Constraining the Number of Meals)
Syntactic	Unfamiliar notation	Notation of a vector	Task 1 (The Carpet Ride Problem: The Maiden Voyage)

In the IOLA class, Johan reported facing three semantic demands: antonym pairs, new mathematical terms, and words or phrases with unfamiliar meanings. In terms of antonym pairs, Johan found it challenging to keep track of the meaning of the following cardinal direction terms: (a) north versus south, and (b) east versus west. For example, in Task 1 (The Carpet Ride Problem: The Maiden Voyage) (see Appendix A, Figure A.1), Johan confounded the terms "east" and "west," not being able to keep track of which one was to the left and which one was to the right:

> Interviewer: *((Shows Task 1)) Was there anything that was challenging in this, in this scenario? Do you remember the scenario from class?*
> Johan: *Oh, yeah. I think if there's any north, west, east, and south, I cannot, I need to translate it into my language, so I know which one.*
> Interviewer: *Can you tell me how?*
> Johan: *'Cause it just gets me confused. Even in my own language, sometimes the west and east, I switch it up. So I need to use like ((inaudible)) ((gestures)) to know the right order, like T ((inaudible)) like West East, so Weee. So, W first and E. S n Ssn s ((inaudible)). So Weee. So, W first and E.*

Similarly, Johan confused north with south: "At first, I thought north [was] going south." This linguistic issue was so impactful for him that he cited it as the main reason for ranking this problem as the third-most linguistically difficult in the first 12 tasks of the class.

In addition, he faced new mathematical terms, such as "span," that initially confused him but later became easier as they were explored in class:

> *[In] this one [Task 3a (Span Worksheet)], the only problem is the word Span. At that time, I'm not familiar with the word. But after taking multiple class, then span is very simple term. So I think this is easy. But at day three, I find it quite confusing.*

Johan also noted he was unfamiliar with the meaning of the term "vector notation" in Task 1 (The Carpet Ride Problem: The Maiden Voyage). Referring to term "vector notation" in the problem's instructions ["Use the vector notation for each mode of transportation as part of your explanation…,"], Johan said:

> *Oh, one thing is because at that time, we do not know what vector notation looks like because it was never explained. So I was like, "What is vector notation?"*

While the term was not central to the mathematics of the task, it was important for executing the task instructions.

A second phrase that Johan was unfamiliar with was "up to twice." In Task 7 (Meal Plans: Constraining the Number of Meals), he wasn't sure whether the phrase "up to twice" included numbers less than and equal to 2, less than 2, or greater than 2:

> Interviewer: *((Shows Task 7)) How did you feel about this problem when it was introduced, at first?*
> Johan: *The thing that make[s] it hard to think [is]..."Can you get more than two meals per day?"...I was thinking, "What do you mean by 'up to twice?' Can I go put that in twice? Can I just choose one...".*
> ...
> Interviewer: *The confusion or the question was whether it had to be exactly two meals?*
> Johan: *Yeah, or less than, or you cannot go beyond.*
> Interviewer: *... And what is your understanding of it now?*
> Johan: *I just think it, uh, at least twice,*
> Interviewer: *at least twice?*
> Johan: *Yeah and above.*
> Interviewer: *What are some examples of at least twice?*
> Johan: *At least twice three, two. It's like greater and equal.*

This challenge significantly slowed the mathematical progress of the group in class, and it remained a challenge even after the lesson was completed. As noted above, at the time of this interview, it appeared that Johan still interpreted "up to twice" to mean "at least twice."

Finally, Johan reported experiencing one syntactic demand: the column form of a vector with square brackets. Johan was unfamiliar with the square-bracket notation and column form of a vector included in Task 1 (The Carpet Ride Problem: The Maiden Voyage). Instead, he was used to seeing vectors in the row form with angled bracket notation, which is typically used in Calculus courses:

> Interviewer: *Oh, okay. Got it. Yeah. And is there vector notation in the problem that you can [point to]?*
> Johan: *Oh, I think this one is a vector notation.*

Interviewer: *Yeah. So that kind of notation there with the 3-1 and the 1-2, that's something that you were not familiar with at the beginning of the course, right?*
Johan: *Yeah. This was the first time. Usually, you use what they call the sharp bracket ((gestures the symbols "<" and >")), whatever that thing is called, to write the vector from Calculus.*
Interviewer: *Like this with the comma in between like that and, 1 comma 2 ((gestures "<1,2>")), right?*
Johan: *Yep.*

In the next subsection, I report the language demands that Johan experienced in the IOLA course in terms of the situational dimension.

Situational Demands. Recall that the situational dimension consists of four components: material, activity, semiotic, and sociocultural. The material attended to physical aspects of the classroom, such as seating arrangement, student grouping, technology, and mathematical tasks. The activity component involves the specific social engagements or interconnected chains of events in which the instructor and students partake, such as interpreting the communication of students and the instructor. The semiotic component considers aspects of various sign systems in the classroom, such as communication mediums, mathematical language systems, and linguistic repertoire. Finally, the sociocultural component draws attention to participant structures and communication modes, which reflect or induce certain ways of knowing, feeling, and being. Below I describe the material, activity, semiotic, and sociocultural aspects of the classroom that Johan experienced as language demands. Tables 4.2 and 4.3 provides a brief overview of these situational demands. Table 4.2 focuses on the situational demands induced by the communication on the tasks, whereas Table 4.3 captures the language demands from all other sources of communication (e.g., teacher and student talk) in the IOLA classroom.

Table 4.2:

Situational Demands Experienced by Johan on the IOLA Tasks

Task Feature as (Material) Situational Demand	Task Example
Too many words	Task 4 (The Carpet Ride Problem: Getting Back Home)
	Task 7 (Meal Plans: Constraining the Number of Meals)
	Task 8 (Meal Plans: Constraining the Cost)
	Task 9 (The Car Rental Problem)
Unfamiliar problem contexts	Task 7 (Meal Plans: Constraining the Number of Meals)
	Task 8 (Meal Plans: Constraining the Cost)
Instructions were too general	Task 9 (The Car Rental Problem)

Table 4.3:

Situational Demands Experienced by Johan in the IOLA Classroom's General Communication

Component	Classroom Communication Tool	Feature of Tool as Situational Demand
Material	Google Translate	Literal in translating
Activity	Class discussions	Wordy
Semiotic	Speech communication	Impermanent
		Informal
	Verbal representations	Unimaginable
	English pronunciation	Diverse pronunciation
Sociocultural	Peer interpersonal communication	Incomprehensible
	Whole-class discussions	Less comfortable speaking to a wide audience
		Need to convey information loudly
		Need to convey information succinctly

In terms of the material component, Johan highlighted various aspects of the mathematical tasks and technology that induced language challenges for him. Features of the mathematical tasks that Johan found challenging were too many words, unfamiliar problem contexts, and instructions that were too general. For example, in response to the question about whether the problem contexts in the tasks presented in class were easy or hard to access linguistically, Johan noted the tasks were challenging because they had too many words:

> *Because there's too many words, that's why I said (this). So I have to find—plus, when I do [the] problem—it used to be, but I don't know why I don't do it anymore—I underline the important aspect. The number, the task, and the important information to solve the task. Maybe I did that, but I don't think I did it for this class.*

This language demand was salient in Johan's experience with the mathematical tasks of the class. For example, when I asked Johan to rank the tasks based on linguistic complexity, he ranked Task 7 (Meal Plans: Constraining the Number of Meals) and Task 8 (Meal Plans: Constraining the Cost) as the two most challenging, and the first reason he gave for this ranking was: "Too many things to read, which I hate that the most." Other tasks for which he explicitly noted this linguistic complexity were Task 9 (The Car Rental Problem) and Task 4 (The Carpet Ride Problem: Getting Back Home).

A second challenge Johan faced on some mathematical tasks was unfamiliar contexts where it was difficult to interpret the constraints of the context. For example, while engaging with Task 7 (Meal Plans: Constraining the Number of Meals) and Task 8 (Meal Plans: Constraining the Cost), Johan incorrectly understood that it was mandatory to have two meals per day. Given this understanding, he did not attend to the part of the problem that said you could repeat the same meal type in a given day. Similarly, it was difficult for him to realize that you could choose to have no meals.

> *One thing that make it ((the Meal Plan Problems)) even hard[er]...[is that]...you can buy the same meal, which at first I didn't realize that it says you could just do breakfast, for which that implied you can have same meal plan in the same day that you—oh, another thing is, actually, in this problem, you can skip one meal. You could choose to not take anything at all for a day. So I didn't know that you can do that in the calculation, so I assumed one day you must buy two meal[s]. Yeah, I think that's the biggest problem for this problem because they never clarify in the problem.*

A third challenge Johan reported about the language on some of the mathematical tasks was that instructions were too general. For example, this was a main reason he gave for ranking Task 9 (The Car Rental Problem) as the second most challenging linguistically. He elaborated on this reason later in the interview, when he compared the linguistic complexity of Task 9 (The Car Rental Problem) and Task 10 (The Car Rental Problem: Follow-Up) in terms of generality versus specificity of the instructions:

> *Because if you see…this question ((in Task 9)) is very general, that if we stop reshuffling the car, how many would eventually end up in each location. So reshuffling the car after how many weeks. You have to state how many weeks in each location. Because it differ[s] for each week. In here ((Task 10)), it states after week 5 and above. Week 10, 50, 100, is very specific. What['s] the long-term result? This one ((Task 9)) is very general. So it ((Task 9)) needs more clarification. That's why I think this ((Task 10)) is easy to understand.*

Johan thought Task 9 (The Car Rental Problem) asked him to start with generalizing a formula that captured the relationship between the quantities in the task across different weeks, but later he realized he was being asked to start simply by capturing the relationship for specific weeks.

Finally, in terms of technology, Johan reported using Google Translate to convert some of the tasks to other languages he knew. However, he noted this tool translated literally (without accounting for the context of the original message in the target language), so he had to contextualize the translation:

> *Sometime[s] Google Translate, if there's too many words...it's direct translating, right? So you have to think a bit to make it sense. You try to convert to English*

((inaudible)), and then you can make the sentence.

In relation to the activity component, Johan experienced one language demand during class discussion: too many words. For example, he noted that sometimes he found it difficult to see the relevance in other students' thinking:

> *I think that Kaden sometime[s] find[s] out something that, for me at least, it's not relevant. Because I find it—maybe because too much words for me, I think that's too much words. Maybe it's relevant, but I cannot make it relevant in my sense.*

In terms of the semiotic component, Johan highlighted ways in which the following language systems induced language demands for him: speech medium, verbal representations, and pronunciation in the English language. Johan estimated the following percentage distribution for the IOLA teacher's use of communication mediums: 60% speaking, 30% writing, and 10% gesturing. He reported experiencing a challenge with the teacher's speech communication due to not being able to see it for later reference:

> *Yeah, because I can see it ((the writing)). Speaking, I need to remember on the spot to find the key point. But writing, I can just see it. If I forget it, it's there. It's there. The theorem is there, yeah. So I really prefer writing more.*

In addition to the challenge with the teacher's speech, Johan experienced a challenge with his classmates' speech communication when they spoke fast and used slang (in English):

> *I find it harder to listen to English, that reading is easier than listening. So sometimes, I cannot understand what people are saying because they talk fast. So in a way, I prefer a certain—for [the] professor, verbal ((speech)) is good because I can understand whatever that she's saying. But for my classmates, sometimes, they have like slang, I think, that I cannot understand.*

Whereas the issue with the teacher's communication was about recalling and connecting information, the challenge he faced with his classmates' speech was about comprehension.

Another aspect of semiotics to which Johan attended was types of representations. Johan estimated the following balance for the teacher's use of types of representations in the IOLA classroom: verbal 40%, graphical, 30%, symbolic 15%, tabular 5%. (Note: The fact that these percentages don't add up to 100% could be a result of doing math live in speech form during the interview.) Johan provided an example of how verbal representations are difficult to imagine:

> *I think I'm more to visuals. So it's like when verbal is hard for me to imagine, especially in 3D space. So having a graphical representation of the equation helps me a lot, especially for the meal plan ((Task 7)). If you remember what I said, this was ((inaudible)) calculus way to create a plane. So anything that falls off on the plane is the answer because I imagine it in plane. So I'm really leaning towards the graph, towards visual. I cannot think it in—without visual, it's hard for me. So when she pull out the graph, I'm like, "Oh, yeah. This is what I want."*

Additionally, Johan also encountered a challenge with respect to students' pronunciation of English in the IOLA class. For example, he found it challenging to understand his small-group peer Nam, for whom English was not a primary language, and he thought others found the same challenge about his own pronunciation:

> *For me to [understand] Kaden and Luis ((his small-group peers who were domestic English speakers)), it's okay. But for me to [understand] Nam, sometime[s] I need to ask him again for his pronunciation. Just like what I [have] trouble with, I think he has trouble for that too… For Nam, his pronunciation in English… because we have language barrier between each other…I think Nam felt the same way. I think all three ((of his small-group peers)) feel the same way towards me with my pronunciation.*

Finally, within the sociocultural component, Johan reported that peer interpersonal communication and whole-class discussions in the IOLA class induced language demands. Johan found peer interpersonal communication to be challenging because it was difficult for peers to understand each other:

> *That's one thing I hate talking…. And even if I force myself to talk, sometimes, too many times, that happens that we're not on the same page. The comprehension, for me, communication the most important part is comprehension, not what you can talk, but what you can understand. And most people struggle for that, including myself.*

In addition, relative to small-group discussions, whole-class discussions were challenging for Johan for two main reasons. In the whole-class format, he found himself worrying about his English pronunciation with a larger audience:

> *I don't like it. Because every time, I'm like, "Oh, did I pronounce it right? Is my English is okay?"*

In addition, he felt he needed to speak loudly, which he didn't like to do because he wasn't used to doing it. In contrast, he reported small-group discussions involved a smaller audience hearing his pronunciation and didn't require him to speak loudly.

Next, I discuss Johan's experiences with language demands in the normative dimension.

Normative Demands. Due to his cumulative experiences across two different national-cultural settings, Johan described many of the language demands he experienced in terms of conflicts in norms between Malaysia and the U.S. These conflicts were primarily shaped by the disparity he experienced between the lecture and "drill and practice" approaches in Malaysia and the IOLA class in the U.S. As indicated in Table 4.4, Johan experienced various normative language demands in the IOLA class that spanned all three types of norm tensions defined in the introduction to this chapter: class-centered norm incompatibilities, norm violations, and student-centered norm incompatibilities. These normative language demands reported by Johan reflected tensions regarding expectations for overall engagement (among students and the instructor), engagement between students and the instructor, and engagement between students in the IOLA class. All of these normative demands, except for one, related to social norms. The only relevant sociomathematical norm identified in the data was related to what constituted a mathematically productive solution.

Table 4.4:

Normative Demands Experienced by Johan in the IOLA Classroom

Audience	Norm Trait	Norm Tension	Norms or Situational Aspects	
			Johan or His Community Outside IOLA Classroom	IOLA Classroom Community
Everyone in class	Permissible language	Class-centered incompatibility	Malaysian as a primary language	English only
	Mathematically productive solution method	Violation	Unique and general way to solve a problem	Any way that solved the problem
Instructor-student	Respectful behavior	Violation	Not interrupting the instructor	Interrupting the instructor
	Effective communication	Violation	Listening quietly to teacher talk and only asking questions as a last resort after teacher talk	Asking questions during teacher talk
	Competent participation	Student-centered incompatibility	Identifying important information and writing notes	Not clear which information was important

Table 4.4:

Normative Demands Experienced by Johan in the IOLA Classroom, Continued

Audience	Norm Trait	Norm Tension	Norms or Situational Aspects	
			Johan or His Community Outside IOLA Classroom	IOLA Classroom Community
Student-student	Respectful peer communication	Violation	Appreciating being told when they have the wrong answer or method	Feeling offended when told they have the wrong answer or method
	Ethical peer behavior	Student-centered norm	Sharing only correct solutions	Not clear which solution method was correct
	Transparent peer communication	Violation	Going straight to the point	Indirect speech
	Effective peer collaboration	Violation	Working individually without talking and then comparing written solutions	Co-constructing solutions primarily through talking
	Dominant mode of peer communication	Class-centered incompatibility	Written ideas	Spoken ideas

Overall Engagement. Johan encountered two normative language demands in the IOLA classroom in relation to norms about overall engagement among the instructor and students. The first language demand was a class-centered norm incompatibility in relation to the permissible language of teaching and learning, while the second was a norm violation with respect to what constituted a mathematically productive solution method.

Johan experienced a class-centered norm incompatibility in relation to what constituted the permissible language of teaching and learning. Johan perceived that all communication in the

IOLA class had to be conducted in English, but he found this norm to be incompatible with his linguistic repertoire. Evidence that Johan perceived this norm in the IOLA class is illustrated below:

> Interviewer: *What do you feel is the language that you should speak in the class?*
> Johan: *English.*
> Interviewer: *English…every time? Or are there instances where it's okay for you to speak your language?*
> Johan: *Ah, no, I think in this class, I'm in most majority American school.*

The incompatibility of this norm (using English only) with Johan's language repertoire induced a language demand. For example, explaining why the IOLA class was harder for him to participate in than in a lecture-based class, he noted that he couldn't express himself well in English:

> *Because first I need to communicate with other people which I don't like, unless it is in my language, because I'm all fluent and something, you know, just like you say, when you want to talk, you want to express this thing, but you can only express it in your language and the meaning won't get delivered if you say it in English.*

A second linguistic challenge that resulted from Johan engaging with this norm was that it led him to question his English pronunciation:

> *I get like, I'm not good at speaking in English. So when I speak in Malay I feel more comfortable, and I don't feel, like, insecure with my English. Because sometimes my pronunciation is off, so people go like "Huh? What do you mean? Can you repeat that?" Then I'm like, "Oh, never mind, I don't want to repeat that. It's okay I got it though."*

Unfortunately, focusing on pronunciation led him to give up these conversations with his peers, thereby negatively impacting his participation.

Finally, a third linguistic issue associated with the norm of using English in class was Johan's observation that more words were required to say the same thing in English than in his home language, Malay: "Something that is expressed in English in 15 words can be expressed in two words in my language." Thus, the linguistic effort required to speak in English was exacerbated by having to use more words to express himself in a language he was already uncomfortable using. In all, the linguistic complexity of using English in all classroom communication constituted a multifaceted language demand for Johan.

Another norm violation Johan experienced was regarding what constituted a mathematically productive solution method to a given mathematical problem. According to the norms he had observed in Malaysia, a productive solution approach was the unique and general way to solve all examples related to a problem. For instance, Johan explained being unsatisfied when he couldn't identify "the" correct way to solve a problem:

> *It ((the IOLA class)) is difficult because the instructor never tell which one is the right way to get the right answer. So in Malaysia, you like, after all the discussion we have, like the teacher, show the arbitrary way of finding the answer, which usually is the shortest way. But in the class I'm like this is too long to find the answer. I don't think this is practical, so I don't like it. I need to know which one is the right way, and so I find it very difficult.*

However, his expectations contrasted with the norm he perceived from the inquiry classroom, where any solution that solved the particular problem was deemed productive:

> *The student's idea is more appreciated and there's not one right way to do solution. Just my preference is to have one right way to do solution.*

As illustrated in the excerpts above, this norm violation was a language demand for Johan because it led him to misaligned expectations about what was being communicated in class. He expected the IOLA instructor to provide the single, general way to solve a problem, so he focused on trying to locate that answer in the teacher's communication and couldn't discern it. However, the teacher may have been operating from a perspective that there is not a single way to solve a problem.

Instructor-Student Engagement. Johan faced three normative language demands in relation to norms that regulated how students should engage with the instructor in the IOLA class. These language demands were norm tensions about (a) respectful behavior, (b) effective communication, and (c) competent participation.

Johan encountered a norm violation of what constituted respectful behavior toward the teacher. Johan noted that, according to the norms in his Malaysian classrooms, students demonstrated high regard for the teacher by not raising their voice, not interrupting, and not using their phone in class:

> *So, in Malaysia, we are like mostly conservatist country. We are majority Islamic country. I'm a Muslim. So, in Islam, we prioritize our manner with the teacher very, very high. Like, you are not supposed to raise your voice, or don't interrupt when they are speaking, or don't wear your phone when they are teaching. Things like that. So in Malaysia, if you do that, you're dead!*

However, as he subsequently noted, the IOLA class violated this norm:

> *But I saw like [a] few students did that ((that is, interrupted the teacher)) in the class. So I'm like "Oh, oh, the lecturer is cool with that. So I'm like "Hoo! That's different from Malaysia!" So I'm like, "Oh, maybe the culture is different, maybe for them it's not rude and for me it's rude." They are just cultural differences.*

This norm violation in the IOLA class took Johan by surprise and reduced his opportunities to engage in face-to-face, two-way communication with the instructor. Therefore, this norm violation constituted a language demand for Johan. Johan experienced intense feelings of discomfort when other students engaged in this norm violation. It took him at least one fourth of the academic term to discover that two-way communication during the teacher's explanation was considered appropriate. As a result, his negative feelings and limited opportunities to engage with the instructor went on for relatively long in this one-semester class.

Johan also noted that the norms of the IOLA classroom violated his expectations of what constituted effective communication with the instructor. Johan prioritized listening quietly and only asking questions as a last resort:

> *Sometimes, when the teacher is talking like [an] important point, like you can say that his question will be answered if he ((a student)) let the professor talk more. I feel like, "Just wait, let the professor finish, then see if they answer your question or not… If it doesn't, then ask." That's how I think.*

By describing how students ought to engage with the instructor, he implied that students in the IOLA classroom were violating his expectations of what made engagement effective for students. He also felt that this violation might cause the instructor to be less effective, saying that "instructors are humans too, just like the students" and that "when you are not respecting the lecturer, you['re] kind of disturbing her motivation." He confirmed it made him feel uncomfortable when students interrupted the teacher. As before, this misaligned expectation constituted a language demand because it limited his opportunities to engage more interpersonally with the instructor. Indeed, he preferred to follow his previously internalized norm: "The whole class I just wanna like, don't talk [not talk] and just see the lecture."

Finally, Johan encountered a student-centered norm incompatibility of what it meant to be a competent participant. In alignment with the lecture-based math courses he had previously experienced, he had internalized that participating competently in class meant identifying important information and writing notes. Unfortunately, he felt the IOLA class made it difficult for him to engage in this norm:

> *It's like it makes me feel like I'm not interested in the class, honestly. It's like I prefer for me to take notes. At first, if you notice in the video, at first I was like, taking notes, but then throughout the progress, I am like, where is, where is the part that I should take notes, I cannot find it. So I stopped taking. So I'm like if it's a lecture-based course, I can find like "Oh, this is an important. It is this theorem, it's for this example, this theorem, for this example." But in the class, she did show the theorem for the linear independency and linear dependency, which that I write. And the one that, which one to decide for point, line, and plane. I remember all those because I think ((inaudible)), "this one is very clear," but after that, it's not clear anymore. I don't know which one she's teaching, which topic.*

In this excerpt, we can see that, while the instructor marked some important information (e.g., linear independence and dependence), it was usually not clear to Johan which topic she was teaching. This norm violation constituted a language demand for Johan because it made it difficult for him to identify the most important information in the instructor's communication. In turn, this limited his opportunities to engage in writing notes for the remainder of the course: "I think I stopped taking [notes] after like 4 classes, which, or 5, after the linear independency and linear dependency." Thus, he was not able to participate in the communicative way he knew and preferred to use. In the end, Johan said this made him feel "not interested in the class."

Student-Student Engagement. Johan faced five normative language demands in relation to norms that regulated how students should engage with one another in the IOLA classroom. These language demands were norm tensions about (a) respectful communication, (b) ethical behavior, (c) transparent communication, (d) effective peer collaboration, and (e) dominant communication.

Regarding respectful communication, Johan noted he found it difficult to engage in the IOLA activities because he had to do them with peers. He linked this reason to a norm violation he faced in the IOLA classroom regarding the meaning of respect among peers. For him, telling someone that they have the wrong answer or method is not disrespectful, but he felt that students in the IOLA classroom in general viewed this behavior as offensive.

> *From my personal experience, I'm finding that people here get easily offended by calling that this answer is wrong. For me, I'm like, "Oh, this is wrong. So I need to fix it." But for them, it's like, "Hmm, going to fight?" Joking. Yeah, it's something like that. So I think maybe they['re] used to that…thought process…, "Should I call this is wrong?" I don't mind they just say, "Hey, this is wrong." I'm like, "Oh, cool," because I think it's just a learning process for me, as long as it's done in a good manner.*

Thus, he found it challenging to successfully communicate with his peers about mathematics due to conflicting beliefs about what counted as disrespectful communication.

Johan also experienced a student-centered norm incompatibility related to what constituted ethical behavior when sharing mathematical solution methods with his peers. He was accustomed from previous mathematics classes to contributing only correct solution methods and answers during peer conversations. For example, he described how not knowing the correct method and answer from the teacher impacted his ability to communicate with peers:

> Johan: *It's like, if people ((his peers)) ask me how to do this, I'm gonna teach (them) my way. So I don't think my way is…the right answer. I need…the right, right answer to teach myself, so that I'm not teaching the wrong stuff to you. Cause it might be right only for this particular example, but wrong for others, particular example[s].*
> Interviewer: *So would you say that you want to communicate only when you have the right answer?*
> Johan: *I think so, for teaching, only when I have the right answer. For finding the answer, I'm okay any time.*

Thus, he believed he needed to have the right method and answer (validated by the instructor) as a precondition to helping his peers in class. However, realizing this normative precondition for being able to help others was challenging in this IOLA environment, since the instructor did not validate the correct solution approach or final answer. This might help explain why he rarely found himself teaching his peers during class:

> Interviewer: *Do you find yourself like teaching other students in the class?*
> Johan: *Not really, I don't think so.*

Johan additionally faced a norm violation related to what made for transparent peer communication. One norm Johan had internalized from his classes in Malaysia is that students go straight to the point when they answer peers' questions. In contrast, he felt that students in the U.S. had a less direct way of communicating:

> *But mostly I find it like American[s], their comprehension is kind of hard to see, because when you ask them, their answer doesn't relate to the question I asked…I'm like, I ask this, then suddenly they use the word "statistically speaking," "mathematically speaking," they don't go right to the answer, they like to put flowers in their way of talking, which for me, is annoying. I just wanna know which part I need to see because when they do that, I need to filter which one is the…important point.*

This norm violation constituted a language demand for Johan because he had to "filter" the important information he was receiving from his peers. This norm violation was related to *interpreting* peer communication (as opposed to *producing* peer communication).

A second norm to which he was accustomed in Malaysia was that effective peer collaboration involved first working individually on a math problem without talking and then comparing his written solutions with a classmate:

> *In Malaysia, the way we do, whenever we are discussing among our friends, it's more like, when I ask for my friend's help, he just do the work, then he show it to me, then he ask me, "which one you don't understand." So I just arrive at the answer. So that, no speaking, just write. Yeah, that's the way.*

In contrast, in the IOLA class, he was expected to co-construct mathematical ideas through speaking with his peers a great deal. For example, when I asked Johan to tell me why he had rated the difficulty to participate in the IOLA class as an eight out of 10, he implicitly referenced the norm of talking to peers as one reason:

> *I really don't, like, talk to people in ((the IOLA)) class. If you see my video ((of the classroom observations)), I rarely communicate with my neighbor or at the back. It's like, I just write the answer and I show it to them. So if they agree, they should, so we write the answer. If they disagree, I just ask them. Okay, which one is ((inaudible)). Then I can evaluate "Is it faster or not?" So, then I will (say) "Ok I agree." It's more like that.*

Therefore, in the IOLA class, he was expected to shift from his primary communication medium (writing) to a different medium (speaking), which induced a language demand for Johan. In contrast to the norm violation about transparent peer communication, this norm violation was related to *producing* peer communication.

Finally, Johan experienced a class-centered norm incompatibility regarding what constituted the dominant form of communication. As one can deduce from the previous section, Johan reported that the mathematical ideas spoken aloud in the IOLA class took precedence over the ideas that were expressed in written form. For example, during the interview, when I asked Johan what types of representations (verbal, graphical, tabular, or symbolic) he and his classmates used in class and how it may have impacted his participation or understanding, he responded:

> *I think it becomes like Kaden becomes dominant because verbal. And Nam and me, it's like we get what they call overshadowed…Overshadowed, if that's the right term, it's like when somebody is talking so our ideas become hidden. ((Note that although I asked the question with "verbal" as a representation, Johan used the word "verbal" to specifically refer to speech communication.))*

Thus, Johan felt this class-centered norm caused his written communication to be undervalued, thereby constituting a language demand. Taken as a whole, the focus and power that the IOLA class gave to speech communication (from Johan's perspective) in each of these examples may explain why he sought to adapt his forms of communication to fit the class norms:

> *Ah, "I need to learn how to speak. I have to. I have to," so I start telling myself. Then I start to talk. Not a lot, but better than before.*

Instructional Language Resources

This subsection describes the instructional aspects (e.g., the IOLA tasks and the instructor's communication) that Johan experienced as language resources in the situational dimension of the L-S-N framework. (No instructional language resources were identified in the lexico-grammatical dimension and normative dimensions during the analysis of Johan's interview data.) Hence, I report the material, activity, semiotic, and sociocultural components of the instructional aspect of the IOLA classroom that functioned as language resources. Tables 4.5 and 4.6 provide an overview of these findings based on two main sources of communication: tasks and general classroom communication.

Table 4.5:

Instructional Situational Resources Experienced by Johan in the IOLA Tasks

Component	Task Feature as Situational (Material) Resource	Task Example
Material	Fewer words (than in other tasks)	Task 11 (Geometric Interpretation of a Matrix Times a Vector)
		Task 6 (Practice for Individual Quiz)
	Familiar problem contexts	Task 6 (Practice for Individual Quiz)
		Task 11 (Geometric Interpretation of a Matrix Times a Vector)
	Instructions that were more specific (than in other tasks)	Task 10 (The Car Rental Problem: Follow-Up)
	Self-contained information	Task 3a (Span Worksheet)
Semiotic	Primacy of symbolic mathematical language	Task 3b (Group Quiz)
	Essential visual representation	Task 9 (The Car Rental Problem)

Table 4.6:

Instructional Situational Resources Experienced by Johan in the IOLA Classroom's General Communication

Component	Classroom Communication Tool as a Situational Resource	Resourceful Feature of Tool
Activity	Teacher's communication	Linguistic marker
Sociocultural	Small-group discussions	More private with the opportunity to provide or obtain clarification

In terms of the material component, Johan reported benefitting from four features of the language of some tasks, relative to other tasks: (a) fewer words, (b) familiar problem contexts, (c) instructions that were more specific, and (d) self-contained information. Johan found it helpful when tasks had fewer words. For example, describing what he found helpful in Task 11 (Geometric Interpretation of a Matrix Times a Vector), he said: "I think less word[s]." For the same reason, he found the Task 6 (Practice for Individual Quiz) easy to understand.

Another feature of some tasks that Johan found helpful was having a context or topic with which he was familiar. As I will highlight in Student Situational Resources—Material, Johan was familiar with the topic of simultaneous equations and so considered Task 6 (Practice for Individual Quiz) easy to understand. Similarly, Johan was familiar with reflections and rotations, 2-dimensional graphs, and 2x2 matrices in Task 11 (Geometric Interpretation of a Matrix Times a Vector). Therefore, he also found the context in that problem to be linguistically straightforward.

Johan also found it helpful when the task instructions on the sheets were more specific compared to the instructions for other tasks. For example, this was a main reason for assigning Task 10 (The Car Rental Problem: Follow-Up) a much lower ranking than Task 9 (The Car Rental Problem) in terms of linguistic complexity. Whereas he ranked Task 9 (The Car Rental Problem) as the second most challenging, he ranked Task 10 (The Car Rental Problem: Follow-Up) in 8th place out of 10. (Recall that students were allowed to assign the same ranking to more than one task.) As noted in Situational Demands., he felt the instructions in Task 10 (The Car Rental Problem: Follow-Up) were more specific, in comparison to those in Task 9 (The Car Rental Problem).

A final task aspect Johan found helpful was when he felt the tasks were self-contained in the information they included. For example, Johan appreciated that the Task 3a (Span Worksheet) included the definition of span, so he didn't have to search for it outside the task:

> *I guess this by putting [the] definition here ((pointing to the top of the Span Worksheet)) so I can understand what Span means. Like you put so I don't have to look up. I have it here, which is related to this question.*

In relation to the activity component, Johan found it helpful when the instructor explicitly marked the mathematical topics being covered in a given lesson, so he knew what to write in his notes:

> *In the class, she did show the theorem for the linear independency and linear dependency, which that I write. And the one that, which one to decide for point, line, and plane. I remember all those because I think ((inaudible)), "this one is very clear," but after that, it's not clear anymore.*

He felt the instructor marked such information only rarely and at the beginning of the course, so he wished the instructor did it more often.

In terms of the semiotic component, there were two language systems from the language on the IOLA tasks that Johan reported as helpful: symbolic language and visual representation. Johan found it helpful when the IOLA tasks were primarily given in symbolic mathematical language rather than with a story context in natural language. In fact, this was the primary reason he gave for ranking Task 3b (Group Quiz) as the least linguistically challenging among all the tasks collected during the classroom observations:

> *The easiest is this one because everything is very straightforward. Like the vectors in this set are R power of N, which N represents the dimension or number of entries. What this one is representing of what thing. And yeah. Here, everything is easy to understand. Everything is straightforward… Straightforward is…saying two plus two is equal to question mark. It's not like Abu buy two apple, and I buy two. How many apple in the basket? That's like you're beating behind the bush. Yeah. So that's why I mean very straightforward.*

It appears he found it helpful that he did not have to convert a story context into mathematical symbols, and he appreciated the clarity of what each symbol represented.

Johan also found it helpful when some of the IOLA tasks were represented visually. In fact, he reported that the visual representation (see Figure 4.2) given in Task 9 (The Car Rental Problem) was so central to his understanding that he couldn't have understood the problem without it:

> *If this picture doesn't exist, I think it's ((Task 9)) going to be super hard because this picture is helping a lot. And I would say 70% of my understanding come from this picture. If they turn it into sentence, I don't think I can understand this ((problem)).*

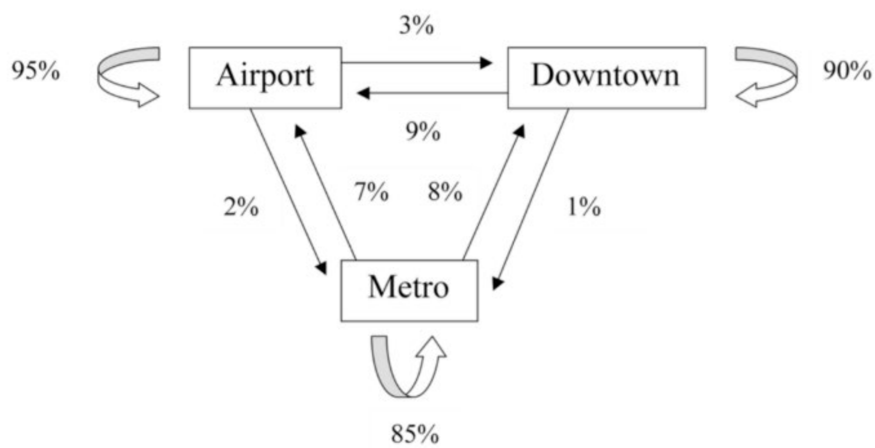


Figure 4.2:

Visual Representation from Task 9 (The Car Rental Problem)

Finally, in relation to the sociocultural component, Johan found the small-group discussions to be more comfortable for communication in comparison to whole-class discussions. As noted in the excerpt below, in a small group, he was concerned with fewer people hearing his English pronunciation and he didn't have to speak loudly:

> Interviewer: *Can you tell me what makes you more comfortable than in whole-class ((discussions))?*
> Johan: *At least if I'm going to be embarrassed of my pronunciation, there's only three that hear it instead of everyone.*

Interviewer: *Are there any other aspects that make you more comfortable in small-group ((discussions))?*
Johan: *I don't have speak loud.*
Interviewer: *Is that something that you don't like to do, to speak loud?*
Johan: *Yeah.*
Interviewer: *Can you tell me more why that is?*
Johan: *Just not used to.*

In addition, small-group discussions allowed Johan's group the opportunity to seek clarification from the IOLA instructor on some linguistic issues he experienced in the classroom. As noted in Lexico-Grammatical Demands., Johan reported that the context of Task 7 (Meal Plans: Constraining the Number of Meals) and Task 8 (Meal Plans: Constraining the Cost) were unclear. In particular, he had difficulty understanding the meaning of "up to twice" in the problem context. When I asked Johan what helped him clarify the context and associated meanings, he said his group peer Kaden was able to call the teacher for support: "Kaden asked the professor, but then professor clarified in the class."

Student Language Resources

This subsection describes the language resources that Johan drew on across the three L-S-N dimensions: lexico-grammatical, situational, and normative.

Student Lexico-Grammatical Resources. This subsection highlights the student language resources that Johan utilized in the IOLA course within the lexico-grammatical dimension. He reported drawing on one semantic resource and one syntactic resource (see Table 4.7 for an overview).

Table 4.7:

Student Lexico-Grammatical Resources Utilized by Johan in the IOLA Classroom

Component	Student Lexico-Grammatical Resources	Example	Context of Example
Semantic	Primary language translations of antonym pairs	Malay translations of "North, east, west, south"	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
Syntactic	Prior knowledge of mathematical notation	Row form of a vector with sharp brackets	Task 1 (The Carpet Ride Problem: The Maiden Voyage)

In terms of semantics, Johan also found it helpful to access the translations of certain words, such as cardinal direction antonym pairs. When I asked him if there were any particular words he remembered translating, he referenced Task 1 (The Carpet Ride Problem: The Maiden Voyage) and said "North, east, west, south."

On the syntactic end, as noted in Lexico-Grammatical Demands, Johan was not familiar with the column-form vector notation used in the IOLA course. However, he was familiar with the angled bracket and row form of a vector from his Calculus classes. He made this evident when he referenced the vector notation in Task 1 (The Carpet Ride Problem: The Maiden Voyage):

> Interviewer: *So that kind of notation there with the 3-1 and the 1-2, that's something that you were not familiar with at the beginning of the course?*
> Johan: *This is the first time. Usually, you use what they call the sharp bracket, whatever that thing is called, to write vector from Calculus 3.*
> Interviewer: *Like this with the comma in between like that ((gestures sharp brackets and horizontal form of vector)) and, 1 comma 2, right?*
> Johan: *Yep.*

Although at the beginning of the course Johan did not know how to interpret the column-form vector notation, by the time of the interview (as demonstrated in the above quote), he was able to connect it to the row-form vector notation he had learned in a prior mathematics class.

Student Situational Resources. In this subsection, I will delineate the student language resources that Johan encountered during the IOLA course in the context of the situational dimension. I report the material, activity, and semiotic aspects of the classroom that Johan utilized as resources. (I found no evidence in Johan's interview data of him drawing on a sociocultural aspect of communication as a language resource.) For a brief overview of these student situational resources, refer to Table 4.8.

Table 4.8:

Student Situational Resources Utilized by Johan in the IOLA Classroom

Component	Student Communication Tool as Situational Resource	Resourceful Feature of Tool	Example Context Where Tool Was Used
Material	Google Translate	Translating tasks to Johan's various language(s)	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
	Google search	Accessing images to clarify language on tasks	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
	Prior knowledge of certain problem contexts or topics	Interpreting problem contexts in tasks	Task 11 (Geometric Interpretation of a Matrix Times a Vector)
			Task 6 (Practice for Individual Quiz)
Activity	Metalinguistic strategies	Identifying a topic from class to revisit later	General Classroom Communication
Semiotic	Multiple languages(s)	Contextualizing literal translations of tasks	Task 1 (The Carpet Ride Problem: The Maiden Voyage)

In terms of the material component, Johan took the initiative to utilize a variety of resources to engage with the language and communication in the IOLA classroom, including: (a) translation tools, (b) web search, and (c) his prior knowledge of certain contexts or topics. For example, Johan used Google Translate to convert entire tasks into their translated version in a different language:

> *Google Translate is something I use a lot. Even in my phone, whenever professor writes something, I can understand because Google Translate, there's apps that you can just take picture and they will translate everything in the picture.*

For example, he used this tool for accessing the language on Task 1 (The Carpet Ride Problem: The Maiden Voyage):

> *So [for] this one ((problem)), what helped me is Google Translate. I just Google Translate it to my own language.*

Additionally, when Johan encountered words or notation he did not understand, he sometimes searched them on Google and sorted through images that might be helpful. For example, Johan reported doing this to gain an understanding of the vector notation in the IOLA course. During his interview, Johan showed a particular graphic that clarified the vertical form of vector notation for him by labeling the top entry as “across” and the bottom entry as “up/down.” It’s possible this graphic also helped him connect the column form of a vector to his prior knowledge about the meaning and row form of a vector.

To make sense of the contexts in the IOLA tasks, he drew on his prior knowledge of certain mathematical contexts or topics. For example, in engaging with the Task 11 (Geometric Interpretation of a Matrix Times a Vector), he drew on his familiarity with reflections and rotations, 2D graphs, and 2x2 matrices:

> *A stretch of 2 in the X direction and a stretch of 3 in the Y direction is easy to understand. A reflection, this is all what I call–I have (symmetrical) times, the rotation of 180. So all of this is nothing new for me. That’s why it’s easy. And even the graph is nothing new. It’s not 3D graph. It’s a 2D graph for me. And two by two matrix is something I’ve seen multiple times already, even prior to this class.*

Similarly, to engage with Task 6 (Practice for Individual Quiz), Johan drew on his familiarity with the topic and language of simultaneous equations from Calculus:

> *This one ((task)), [it’s] just [about] simultaneous equations, something I always see my whole life…. And everything is very familiar, “Find the intersection,” which means just solve the X, Y, Z, so you will get a point. Intersection is point, the way I understand it. If I find the intersection the first time, maybe I would ask, “Is it [a] point?” But in Calculus 3, [the] intersection has been ((inaudible)) multiple times, which defined to [a] point. ((Reads part of the task)) “Three*

planes by solving the following." Yeah. I think that's why I think this is easy.

In particular, his familiarity with the language of problems about simultaneous equations helped him to immediately interpret the visual of multiple equations (shown in Figure 4.3) as a request to simultaneously solve them.

$$\begin{aligned} x - 2y + 3z &= 2 \\ 2x - 3y + 6z &= 3 \\ -x + 5y - 3z &= 1 \end{aligned}$$

Figure 4.3:

An Image of Multiple Equations

In relation to the activity component, Johan found it challenging to figure out the topic being covered during a given IOLA lesson (see Normative Demands. for more details). To navigate this issue, he engaged in metalinguistic strategies, such as questioning his own understanding of the information presented in class. For example, explaining how he identified a topic from class to revisit at home, he said:

> *Usually…, in lecture ((in the IOLA class)),... I will ask myself in my head, like, this question that, "Can I answer this in this class?" Like, even* when *the professor asked the crowd, "What's the answer?" Even [if] she's not pointing to me, I will try to think the answer…. So if I cannot answer that question, I will go home and study the topic.*

In terms of the semiotic component, Johan drew on multiple languages to access the language on the mathematical tasks. For example, as noted earlier in this subsection, he translated full mathematical tasks into his primary language (Malay). An example of a task he translated from English to Malay was Task 1 (The Carpet Ride Problem: The Maiden Voyage) from English to Malay. During the interview, he also demonstrated how he translated a task from English into Japanese (one of his secondary languages). After translating a task or any other information from the IOLA class, he sometimes translated it back to English to contextualize the literal translation:

> *If there's anywhere I struggle, I tried to translate. Then once I understand, I translate back to English.*

Student Normative Resources. As implied in Normative Demands., there were various norms that Johan had internalized from his communities outside the IOLA classroom. This section highlights norms he noted were helpful for communicating or interpreting information in the course. To avoid repetition of interview excerpts, below I only summarize the relevant norms and describe how they constituted language resources for Johan. I present these norms in the context of the engagement between students and the instructor and engagement between students. See Table 4.9 for a brief overview.

Table 4.9:

Student Normative Resources Utilized by Johan in the IOLA Classroom

Audience	Norm Trait	Norm as a Resource
Instructor-student	Respectful behavior	Not interrupting the instructor
	Competent participation	Identifying important information and writing notes
Student-student	Respectful peer communication	Appreciating being told when they have the wrong answer or method
	Transparent peer communication	Going straight to the point
	Effective peer collaboration	Working alone and then comparing written solutions with peers

In the context of instructor-student engagement, Johan reported two norms that functioned as language resources for him in the context of instructor-student engagement in the IOLA classroom. One norm was reflected in Johan's conception of what constituted respectful behavior toward the teacher. In his prior Malaysian classrooms, students displayed great respect for their teacher by refraining from raising their voices, avoiding interruptions, and abstaining from using their phones during class. Therefore, he felt that this way of engaging communicated to the IOLA instructor that she was respected, which in turn kept her motivated to teach the lesson. In addition, he thought this form of student engagement made for more effective interpretation of the IOLA instructor's communication.

A second norm that Johan reported as helpful was reflected in his belief of what made him a competent participant in the IOLA classroom. Consistent with his past experience in lecture-based math courses, he had internalized the importance of recognizing key information from the instructor's communication and writing it down. So, he sought to apply this approach in the IOLA class to become a competent participant, and he linked this approach to his motivation to participate in the course.

In relation to student-student engagement, Johan reported three norms regulating student-student engagement that functioned as language resources for him in the IOLA class. One norm was reified in Johan's conception of what constituted respectful peer communication. For him, telling someone that they have the wrong answer or method fit within his definition of respect, so he was okay with this form of peer communication. Moreover, he found it helpful for learning and adjusting his solution to a math problem. Therefore, Johan implied he wanted other students in the IOLA class to tell him when he had a wrong answer.

Additionally, Johan reported on how his belief of what made for transparent peer communication was helpful with student-student engagement. Based on his past experience in his Malaysian classes, he was used to direct communication between peers. Therefore, he used this form of engagement in the IOLA class to help him identify the important points in his peers' communication.

Finally, a third norm that Johan reported as helpful was evident in his conception of what made for effective peer collaboration. Back in Malaysia, when he collaborated with peers to solve a math problem, he found it helpful to first work alone and then compare his written solutions with a classmate. As a result, they could evaluate which method was better. Thus, he preferred to use this method of peer collaboration in the IOLA class, even when he knew the class expected a different method of peer engagement.

Nam's Case Study

Nam is an international undergraduate student from Vietnam, who self-identified as male and Vietnamese. Nam had completed his entire K-12 education in Vietnam, after which he came to the U.S. to begin his college career. He started his undergraduate studies at a community college for two years before transferring to Southwestern University. At the time of the study's data collection, he was majoring in Software Engineering and beginning his first semester at the university. On the research study's screening form, his self-reported language background included native proficiency in Vietnamese (level six) and professional working proficiency (level four) in English.

The inquiry-oriented linear algebra (IOLA) course he took during this study was his first linear algebra course and his first inquiry-based mathematics course. It was also his first mathematics course at Southwestern University, overall. All the mathematics classes he took prior to the IOLA class in both Vietnam and the U.S. were taught in a format in a format where the teacher lectured the whole time and the students worked individually.

Nam's participation in the IOLA class was negatively impacted by language challenges induced by the classroom environment. When I asked him about his linguistic experience in the IOLA class, he said:

> *So at first I don't expect that we have to communicate with peers, so I was nervous. I don't know how to talk to them, how to show my work and then let them understand my work… Because for me, I am not a native speaker, so maybe I think the way I describe my ideas and then I talk to them, they may not understand what I mean. So that make me nervous.*

This quote highlights how some of the systems and norms of communication may have been experienced as challenges by Nam, thereby warranting a systematic analysis of the language demands he encountered in the IOLA class. It also motivates the need to understand the resources Nam experienced and utilized to navigate the language and communication used in the IOLA classroom.

As I will describe below, the demands and resources Nam experienced in the IOLA class spanned all three dimensions of the L-S-N framework. Yet, the most salient language demands and resources for Nam were captured in the normative dimension. In this section, I will describe the language demands, instructional language resources, and student language resources that Nam experienced.

Language Demands

This subsection describes the language demands that Nam experienced across the three L-S-N dimensions: lexico-grammatical, situational, and normative.

Lexico-Grammatical Demands. Below I report the language demands that Nam experienced in the IOLA course in terms of the lexico-grammatical dimension, which can be broken into semantic and syntactic components. In the interview data, there was no indication that Nam experienced any syntactic demands, so I solely describe the semantic demands he reported experiencing. An overview of these demands is shown in Table 4.10.

Table 4.10:

Lexico-Grammatical Demands Experienced by Nam in the IOLA Classroom

Lexico-Grammatical (Semantic) Demands	Example	Context of Example
Antonym pairs	"Numerator" versus "denominator"	General classroom communication
New mathematical terms	"Span" "Linearly independent"	
Words or phrases with unfamiliar meanings	"Matrix"	

In the IOLA class, Nam reported facing three types of semantic demands: (a) antonym pairs, (b) new mathematical terms, and (c) words or phrases with unfamiliar meanings. For example, Nam found it challenging to keep track of the meaning of the antonyms *numerator* and *denominator*:

> *Sometimes I don't remember the word like denominator and numerator…in the fraction…. So I don't know the denominator is in the bottom or the top…. So sometimes I mess up the word, and then I say it incorrect. People ((in the IOLA class)) don't understand me.*

In addition, Nam faced new mathematical terms, such as *span* and *linearly independent*. For example, he said: "At first I don't understand what span is." Similarly, he said: "Just like I experienced with span, I don't understand the linear independent."

Nam also encountered words with unfamiliar meanings. For example, Nam initially found it challenging to understand the English name of the mathematical term "matrix:"

> *Like about Matrix…So I know that in Vietnamese. I study it before. But in here ((in the IOLA class)), I never seen the word matrix before. So I thought, I'm like,*

"What is that?"

Situational Demands. Below, I describe the material, activity, semiotic, and sociocultural aspects of the classroom that Nam experienced as language challenges. An overview is shown in Table 4.11 and Table 4.12, with one table focusing on communication from the tasks and the other focused on more general classroom communication.

Table 4.11:

Situational Demands Experienced by Nam on the IOLA Tasks

Task Feature as (Material) Situational Demand	Task Example
Vague questions	Task 11 (Geometric Interpretation of a Matrix Times a Vector) Task 7 (Meal Plans: Constraining the Number of Meals) Task 5 (Linear Independence and Dependence: Creating Examples)
Misalignment between problem contexts and questions	Task 10 (The Car Rental Problem: Follow-Up)
Unfamiliar problem contexts	Task 7 (Meal Plans: Constraining the Number of Meals) Task 8 (Meal Plans: Constraining the Cost)

Table 4.12:

Situational Demands Experienced by Nam in the IOLA Classroom's General Communication

Component	Classroom Communication Tool	Feature of Tool as Situational Demand
Activity	Class discussions	Complexity of explaining
Semiotic	Speech communication	Impermanent
		Obscure
	Verbal representations	Unimaginable
	English pronunciation	Diverse pronunciation
Sociocultural	Whole-class discussions	Less comfortable speaking to a wide audience
		Need to convey information succinctly
	Small-group discussions	Need to relate with his peers and initiate conversations with them

In terms of the material component, Nam reported that the following features of tasks induced language demands for him: vague questions, misalignment between problem contexts and questions, and unfamiliar problem contexts. One example of a vague question that Nam provided was question 4 in Task 11 (Geometric Interpretation of a Matrix Times a Vector). This question prompted him to consider a 180-degree rotation, but it was not clear to him what point he was supposed to rotate about:

> *So when they say rotation of 180 degrees, so you see the F ((pointing to the big letter F in the graph of Task 1))… In the question ((number 4)), they don't say rotate around what point. So it can either rotate around the origin or any different point. So that's confusing to me.*

Thus, he described this question as "vague… they don't provide enough detail to understand." This confusion significantly affected Nam's interpretation of the question, leading him to rank Task 11 as the most linguistically challenging among all tasks during the first 11 days of the course. It is important to note that understanding the meaning of a rotation was central to solving question 4 of this task, which asked students to determine a matrix representation for a rotation of 180 degrees. It may have been implicit to rotate around the origin.

A second example Nam provided of a vague question was question 6b in Task 7 (Meal Plans: Constraining the Number of Meals). He suggested there were a lot of sub-questions within a single question 6b (see Figure 4.4), leaving him uncertain about which sub-question(s) he was supposed to answer:

> *So for this one, the question 6b ((in Task 7)), so that has a lot ((of questions)). That has a lot. Also I don't know, I should answer the first question or the second question or all of them.*

Nam may have found it challenging to discern which sub-question to prioritize due to a potential lack of clarity regarding the relationship between the sub-questions.

b. Consider other options such as a **10-dinner** option. What would that look like graphically and algebraically? What about a **20-dinner, 30-dinner, etc.** Where are these options located on the shape that you predicted for question 5?

Figure 4.4:

Task 7 Sub-Questions

A third example of a vague question for Nam was in Task 5 (Linear Independence and Dependence: Creating Examples). For instance, he noted a difficulty with understanding and executing the instructions in question 4 of this task, which asked him to choose one of the two boxes that were not possible. He implied that he could not execute the task because he did not have two boxes that were not possible based on his work:

> *So for this one ((Task 5)), so the number 4 ((question 4)): "Choose one of the two boxes that were not possible." So when I read that, I was confused, why it's not possible. In… box number 2 and… 5, … the answer should be not possible. But when I did the problem, I didn't think about that. I just put a set, which I was wrong. But I didn't know that I was wrong… So the word "not possible" is like confused. I don't know where it's from.*

To contextualize Nam's experience, notice that the task says "State if not possible," but it does not say there are two boxes that are not possible. So, he didn't know this information about the two boxes when he was filling in all the boxes in this task. This may explain why when he got to question 4, he "read the first sentence and…couldn't understand what they mean[t]."

Another challenging aspect of some tasks that Nam experienced was when he sensed a misalignment between a problem context and question. For example, in Task 10 (The Car Rental Problem: Follow-Up), he felt there was a contradiction between the problem context, which was about moving cars, and question 4, which was about not moving cars:

> *So this one is ((pointing to question 4 in Task 10)) confused, too. So, "Is there a number to start in this location so that we don't have to move cars?" So this is definitely the linguistic challenging. I don't understand it at all. In the problem, we have to move cars, but then number 4 ((question 4)) ask "so we don't have to move cars." It's confusing to me. Yeah. So it's number 2 ((ranked as number 2)).*

This difficulty was significant enough that Nam ranked this task as the second most linguistically challenging for that sole reason.

Yet another challenge Nam faced on some mathematical tasks was unfamiliar contexts, where it was difficult to interpret the constraints of the context. For example, while engaging with Task 7 (Meal Plans: Constraining the Number of Meals) and Task 8 (Meal Plans: Constraining the Cost), he didn't know whether it was mandatory to have two meals a day or if he could choose up to two meals a day:

> *I have a problem with the first one ((Task 7)) right here... For example, you could choose two breakfasts per day, one or five days. So I don't know that I'm allowed to choose two breakfasts every day or just up to two breakfasts a day… I don't understand that I am allowed to choose two breakfasts a day like the whole two breakfasts a day or I have to choose two breakfasts or up to two breakfasts. I can either choose one or no breakfast a day. So it's kind of confusing to me at first.*

It is worth noting that Johan also faced a similar challenge with the contexts in these tasks.

In terms of the activity component, Nam reported it was challenging for him to explain his ideas with peers during class. When I asked him if there were any challenges related to language or communication in the IOLA class, he said: "Yeah. Definitely there's some challenge. So like I said, sometimes I don't know how to explain my ideas."

In terms of the semiotic component, Nam highlighted ways in which the following language systems induced language demands for him: (a) speech medium, (b) verbal representations, and (c) pronunciation in the English language. In terms of mediums of communication, Nam reported that the instructor in the IOLA class used speech the most and then writing. Nam found it challenging that the teacher used speech the most, especially when he felt she spoke too fast. As illustrated in the following two quotes, Nam contrasted speech with graphical representations and writing, implicitly noting that it was hard for him to see the concepts in the instructor's speech communication:

> *I think the teacher need to use more graph or something like that to show the concept of the class. And she talk a lot, but I usually don't follow her. Yeah. So I try to. But I guess she need to use more graphic or some writing so I can follow her.*

In particular, in contrast to writing, Nam highlighted that the instructor's speech was impermanent, so he could not easily refer back to it to understand it:

> *I prefer more writing ((from the instructor)) because when she speaks too fast or I cannot follow her, I can go back to the writing and see.*

Similarly, in contrast to the content in graphs, Nam noted that the content in the teacher's verbal communication (speech in particular) was difficult to imagine:

> *So when she was using verbal, she talk and talk, and I couldn't imagine what happen. So if she can show the graphical, like the point on the lines ((inaudible)). I can easy imagine the problem more.*

Nam also reported experiencing a challenge with pronouncing words in English in a way that his IOLA peers would understand him:

> *The way I communicate might be confusing to everyone…. sometimes I try to explain the idea to somebody. They don't understand how I explain it… Because I have accent.*

Finally, within the sociocultural component, Nam reported that both whole-class and small-group discussions in the IOLA class induced language demands, albeit for distinct reasons. In comparison to small-group discussions, Nam indicated that whole-class discussions were more challenging because they were less comfortable and allowed fewer opportunities to both ask and give clarification. Moreover, he implicitly linked his inability to interrupt and ask for clarification in whole-class discussions to fewer opportunities to deepen his mathematical understanding:

> *Small group ((discussions)) help me understand more. I can stop wherever I want and…get more ideas. Whole class ((discussions))… just somebody talk the whole time. If I don't follow them, it's hard to interrupt them.*

Similarly, in contrast to small-group discussions, he reported having fewer opportunities to give clarification in whole-class discussions:

> *When I talk to the small group, when I say something that not clear, they ((his group peers)) can ask me what– I can repeat that or explain more. But then I talk to the whole class. Nobody ask that.*

As a result, he noted that participating in whole-class discussions required more confidence from him because he had to worry more about his mathematical understanding and English pronunciation (than in small-group discussions): "I'm not confident to talk to the whole class. But when [I] talk to [my] group, I have more confidence." He later elaborated that he needed to be confident about both his understanding of the problem and his pronunciation in English, noting that there were times when his peers didn't understand his accent. Therefore, in part by increasing the linguistic burden of focusing on his accent, whole-class discussions limited Nam's opportunities to communicate and understand mathematical ideas.

As noted above, Nam found small-group discussions to be more comfortable to participate than in whole-class discussions. However, he found that being in such an intimate setting induced a need to initiate conversations with his peers and even engage in talk about aspects from outside of class as a way of relating him with his peers. The difficulty he experienced with meeting this need is evident in the following conversation, where Nam responded to my question about whether he had experienced any challenges with being in a small group:

> Nam: *Yeah. Sometimes it's hard to start conversation. It's the first time I meet that people, right? So, I don't know what to say to…that person, and how to introduce myself. For me, it's kind of weird because I'm an introvert. So I don't talk much. I just do everything on my own. So, talking to other people is a challenge for me. So, to start a conversation, this was kind of hard for me. You have to know each other, asking about maybe something outside of classes. It's not familiar to me.*
>
> Interviewer: *Did the students also talk about things that were not related to math in the group? Or did you feel like sometimes you needed to do that?*
>
> Nam: *Yeah. I think sometimes I need to do that because we get more comfortable with each other and to know each other so we can work on class.*

In addition, he spoke specifically about the difficulty of engaging in conversations about mathematics and activities that they were doing in class:

> *So at first I don't expect that we have to communicate with peers, so I was nervous. I don't know how to talk to them, how to show my work and then let them understand my work.*

Normative Demands. Because of his accumulated experiences in two distinct national-cultural contexts, Nam articulated some of the linguistic challenges he encountered as arising from differences in norms between Vietnam and the U.S. These norm differences were chiefly influenced by the contrast he perceived between the lecture-based methods he experienced in

Vietnam and the interactive approach of the IOLA class in the U.S. As a result, Nam encountered three norm tensions in the IOLA classroom related to overall engagement among the instructor and students that induced language demands (see Table 4.13): (a) a class-centered norm incompatibility in relation to the permissible language of teaching and learning, (b) a norm violation with respect to what constituted effective student communication, and (c) a student-centered norm incompatibility of what it meant to be a competent student participant during class. All of these normative demands involved social norms, as opposed to sociomathematical norms.

Table 4.13:

Normative Demands Experienced by Nam in the IOLA Classroom

Norm Trait	Norm Tension	Norms or Situational Aspects	
		Nam or His Community Outside IOLA Classroom	IOLA Classroom Community
Permissible language	Class-centered incompatibility	Vietnamese as a primary language	English only
Effective student communication	Violation	Minimal communication and interaction among students or between students and the whole class	Intensive communication and interaction among peers or between students and the whole class
Competent student participation	Student-centered incompatibility	Identifying important information	Not clear which information was important

One class-centered norm incompatibility that Nam experienced was in relation to what constituted the permissible language of teaching and learning. Nam perceived that all communication in the IOLA class had to be conducted in English, but he found this norm to be incompatible with his linguistic repertoire. Evidence that Nam perceived this norm in the IOLA class is illustrated below:

> Interviewer: *What language did you feel like you needed to speak in when you were in the ((IOL)) class?*
> Nam: *It's English, right? Should be English because everyone speak English. But sometimes in my head, I speak Vietnamese. So I try to translate to English to talk to them ((everyone)).*

This norm was so ingrained in his way of thinking that even if all his group peers spoke Vietnamese, he would still feel he was expected to speak in English only:

> Interviewer: *What about if all your small-group peers speak Vietnamese ((in the IOLA class))?*
> Nam: *I think the same. We need to speak English in the English class ((referring to the IOLA class)).*
> Interviewer: *Thank you for sharing that. Oh, just to make sure I understand, can you tell me more about why you would speak English in class even if all Vietnamese students were in your group?*
> Nam: *I mean, I'm in the U.S., right, so I should speak English. That's it.*

The incompatibility of this norm (using English only) with Nam's language repertoire induced a language demand. For example, when I asked him whether there were any challenges related to language or communication in the IOLA class, Nam said there were:

> *Definitely there's some challenge. So like I said, sometimes I don't know how to explain my ideas. Then I don't know how to communicate with them ((his group peers)). I can just write on the ((small-group)) whiteboard or on the paper to show them how I approach the problems… So when they see the paper, they can understand me what I mean by that.*

Then he said those linguistic challenges would not be exist if he could explain his ideas in Vietnamese, which implied that English was part of the reason for the challenges he experienced:

> Interviewer: *What if all your peers in your group spoke Vietnamese ((in Vietnam)), would you have the same issues of communication?*
> Nam: *Oh. I don't know, because if they are Vietnamese I can say it in Vietnam[ese], right? So no.*

A second linguistic challenge that resulted from Nam engaging with this norm was that it led him to question his English pronunciation. This came out when I asked him about his participation in the whole-class participant structure, as noted in Language Demands—

Situational Demands. He reported that focusing on his pronunciation led him to engage less often in whole-class discussions.

Another norm tension that Nam experienced was a norm violation with respect to what constituted effective student communication. In the lecture-based classes he experienced in Vietnam, there was minimal student speech communication. In contrast, the IOLA class in the US required more student communication and interaction. Nam summarized this norm difference (which functioned as norm violation) as follows:

> *So this ((IOLA)) class…require more communication and interaction between people. In the lecture class, it's more like the teacher lecture the whole time. We barely have any communication.*

He elaborated on this difference later in the interview:

> *((In the IOLA class))... the student… communicate more… They talk more. They interact more with their peers. And also, they talk to the whole class, but that doesn't happen in the lecture-based class.*

This norm difference was a norm violation because the expectation of the IOLA class conflicted with his preference to engage individually, as he did in his prior lecture-based classes:

> *I would not say it's easier, but I prefer the lecture class...Because I usually work on my own. So I just like to research the resource and then understand work by myself. Yeah. So it's just me, not the other people.*

This norm violation induced a language demand. For example, he said:

> *So at first I don't expect that we have to communicate with peers, so I was nervous. I don't know how to talk to them, how to show my work and then let them understand my work.*

Then he linked these communication challenges to the norm violation of what constituted effective student communication in his classes in Vietnam:

> Interviewer: *Would you say that you would have the same problems ((linguistic challenges)) if you were taking the class in ((Vietnam)) or only here ((in the US))?*

Nam: *No. Because in Vietnam we don't communicate in groups like that. So, I don't know if I have any problem with that.*

The linguistic challenges induced by the norm of the IOLA class might help explain Nam's application of his previously internalized norm into the IOLA class, despite recognizing its opposition to the norm of the class:

I usually don't communicate that much. But this class require a lot of peer communication.

Thus, for Nam, what constituted effective student communication was not simply a result of the communication norms in the IOLA class but also influenced by the norms prevalent in his communities outside of the class.

Finally, Nam encountered a student-centered norm incompatibility of what it meant to be a competent student participant. In alignment with the lecture-based math courses he had previously experienced, he had internalized that participating competently in class meant identifying important information. Unfortunately, he felt the IOLA class made it difficult for him to engage in this norm, as evident in the following interview excerpt:

Nam: *So every day coming to class,...the teacher just give out the problem and then we just try to solve it. But actually, I have no idea what I'm going to learn today. So, if the teacher can give us some heads up, like today we're going to learn about this one, matrix or linear independence, so I will have more heads up and I can understand the problem more, I think.*
Interviewer: *And when you leave the class, how do you know what you learned or what was the topic of that day?*
Nam: *Yeah. Just like I said, I have no idea what I'm going to learn that day. We just come to class and solve the problem. I don't know what the problem concept is.*
Interviewer: *Yeah. And…is there a point in the class when you finally realize what the problem was about?*
Nam: *No… So I would like to know that, but that's how it is.*

In this excerpt, we see it was usually not clear to Nam which topic the instructor was teaching. This norm violation constituted a language demand for Nam because it made it difficult for him to identify the most important information in the instructor's communication. This might help explain why he didn't write notes during the teacher's communication, as illustrated in the quote below:

> Interviewer: *When the teacher is writing things on the board and talking to the whole class, how do you decide what is important information or what to write down?*
> Nam: *Actually, when the teacher write on the board, I just don't write anything on the paper when the teacher write on the whiteboard and the board.*
> Interviewer: *You don't write?*
> Nam: *No. Just listen to her. Yeah.*

Instructional Language Resources

In this section, I describe the instructional aspects (e.g., the IOLA tasks and the instructor's communication) that Nam experienced as linguistically helpful in the situational and normative dimension of the L-S-N framework. (No instructional resources were identified in the lexico-grammatical dimension during the analysis of Nam's interview data.)

Instructional Situational Resources. Nam reported various instructional situational resources he experienced in the IOLA class within the following components: material, semiotic, and sociocultural. (No instructional resources within the activity component arose during the analysis of Nam's interview data.) Table 4.14 and Table 4.15 provide a brief overview of these findings according to two sources of communication in the IOLA class: tasks and general classroom communication).

Table 4.14:

Instructional Situational Resources Experienced by Nam in the IOLA Tasks

Task Feature as Situational (Material) Resource	Task Example
Realistic problem contexts	Task 7 (Meal Plans: Constraining the Number of Meals)
Straightforward questions	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
	Task 2 (The Carpet Ride Problem: Hide and Seek)
	Task 3a (Span Worksheet)
	Task 3b (Group Quiz)
	Task 4 (The Carpet Ride Problem: Getting Back Home)
	Task 6 (Practice for Individual Quiz)
	Task 9 (The Car Rental Problem)
Marked layouts	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
	Task 9 (The Car Rental Problem)

Table 4.15:

Instructional Situational Resources Experienced by Nam in the IOLA Classroom's General Communication

Component	Classroom Communication Tool	Feature of Tool as Situational Resource
Semiotic	Supplementation of speech communication with writing	Permanent Visual
	Supplementation of verbal representations with graphical representations	Visual
Sociocultural	Small-group discussions	More private with the opportunity to provide or obtain clarification
	Whole-class discussions	Wide audience with access to more mathematical ideas

In relation to the material aspect, he attended to five features he found helpful on the language and communication in the IOLA classroom: (a) realistic problem contexts, (b) straightforward questions, and (c) explicitly marked layouts. Nam benefitted from problem contexts being realistic or imaginable. For example, he appreciated Task 7 (Meal Plans: Constraining the Number of Meals) because it had a context in which he could imagine himself:

> *So for this kind of problem, I can think of myself. Like when I go to school, I can choose how many breakfast I want. It's kind of easy to imagine. Other math problem[s] ((in other math classes)), we don't know what exactly we have to do or what we do with the real life.*

Nam also referenced multiple tasks he appreciated for having straightforward questions. For example, he summarized how Task 1 (The Carpet Ride Problem: The Maiden Voyage) and Task 3a (Span Worksheet) were straightforward in their questions:

> *So the problem ((Task 1)) asked whether we use two vector[s] to get to the point ((points first to the two column vectors in the problem context and then to the location description of the cabin in the problem's scenario)). So yeah, that's it. Everything is clear.*
>
> *So for this one ((Task 3a)), it's just asking to solve the span of these vectors, so yeah, no confusing to me.*

Similarly, when discussing Task 4 (The Carpet Ride Problem: Getting Back Home), he said:

> *This one ((Task 4)), yeah. Yeah, straight to the point. So I try to answer the question. That's it. No confusion.*

When I asked Nam what he meant by straightforward questions, he said that questions were clear and short:

> Interviewer: *So when you say straight to the point, what do you mean?*
> Nam: *So the questions. The question is clear and short. So I think that's not confusing to me.*

Two additional tasks that Nam noted had clear and short questions were Task 3b (Group Quiz) and Task 6 (Practice for Individual Quiz), both of which have no story contexts. Nam explained the simplicity of Task 3b by saying "This one [Task 3b], just answer the question. That's it." He then said that Task 6 was linguistically similar. One possible interpretation of the similarity he saw in these two tasks is that they didn't require him to make sense of a story, only the question. Altogether, he summed up Tasks 1, 2, 3a, 3b, 4, 6, 9 as easy to understand because they had straightforward questions: "The less challenging [tasks], they're clear and short, so I understand right away."

Additionally, as suggested in the above quote (which was about Task 9), Nam also appreciated the explicit layout of some task handouts that allowed him to pinpoint the scenario and question easily. Similarly, when I asked how he felt about the instructions and the communication in Task 1 (The Carpet Ride Problem: The Maiden Voyage), he referenced the clear structure of the handout to get acquainted with the task:

> *So I start to the bottom thing, that says task. So I know that will ask me to share the problem. So I just pick that one and then try to understand that. And I read the top one, and then yeah, that's all.*

In terms of the semiotic component, Nam found it helpful when the teacher and other students in whole-class discussions combined communication mediums and representation types. For example, as indicated in Situational Demands, Nam found it helpful when the teacher supplemented her speech communication with writing and graphs to convey mathematical concepts. Unlike speech alone, graphs aided him with better visualizing the concepts. Similarly, writing proved beneficial as it provided a visual and permanent form of communication that he could revisit. The utilization of graphs and writing was so beneficial for Nam that he sought more of this when he was unable to keep up with the teacher due to excessive speech or rapid pace.

During the classroom observations of this dissertation study, I saw the IOLA instructor facilitate whole-class discussions through having each small group share their whiteboards. This created opportunities for students to use speech along with what they had written on the whiteboards. So, while Nam reported that students used speech the most and writing second most during whole-class discussions, he found it helpful that students coordinated their speech with the writing on their whiteboards:

> *So I think that I can understand it. They talk about what they write on the board so there's no confusion, anything at all.*

In relation to the sociocultural component, Nam highlighted different benefits of small-group and whole-class discussions. In comparison to whole-class discussions, Nam noted that small-group discussions were helpful because they were more comfortable and allowed more opportunities to both ask and give clarification. In summary, as alluded to in Situational Demands—Sociocultural, in small-group discussions broadened the opportunities for Nam to communicate and understand mathematical ideas by reducing the burden of focusing on his accent in English.

In addition, small-group discussions allowed Nam's group the opportunity to ask the IOLA instructor for clarification on some linguistic issues he experienced in the classroom. As noted in the Material segment of Situational Demands., Nam encountered unfamiliar problems contexts in Task 7 (Meal Plans: Constraining the Number of Meals) and Task 8 (Meal Plans: Constraining the Cost): He didn't know whether it was mandatory to have two meals a day or he could choose up to two meals a day. When the teacher approached their group to clarify the task, he gained a better understanding of the problem context:

> Nam: *I think that they ((his group peers)) had the same problem understanding that ((the number of meals allowed per day)) like me. Then we asked the teacher [if] can she clarify on that.*

Interviewer: *And then now that you had clarification, how do you understand this problem?*
Nam: *So now I understand that I can choose either zero, one, or two, breakfasts, right?*

Finally, while Nam found certain aspects of small-group discussions to be more beneficial than whole-class discussions, he also reported one advantage of whole-class discussions. Specifically, Nam experienced whole-class discussions as helpful because they allowed him to access more student ideas:

> *When we work in the small group, we can see everyone['s] ideas. But the whole class, we saw more different idea[s] of how to solve the problem. So I think it's helpful to have more ideas.*

Instructional Normative Resources. Finally, Nam reported various instructional normative resources that were generally available in the IOLA classroom to support his mathematics learning. These resources included the use of realistic problem contexts on the tasks, the communication structure of each lesson, and the centrality of peer communication during class. See Table 4.16 for a brief overview.

Table 4.16:

Instructional Normative Resources Experienced by Nam in the IOLA Classroom

Tool of Communication	Typical Feature as a Normative Resource
Problem contexts of class tasks	Realistic contexts
Structure of the class lesson	Combination of different participant structures of support (e.g., individual and small group) Access to different audiences of support (e.g., peers and instructor)
Nature of overall class communication	Centrality of peer interaction

As noted in Instructional Situational Resources, Nam benefitted from the tasks being realistic. For example, he appreciated Task 7 (Meal Plans: Constraining the Number of Meals) because it had a context in which he could imagine himself. He found this feature to be typical across tasks. He implicitly made this norm evident by contrasting it with tasks in previous lecture-based mathematics classes, in which he couldn't make mathematics real for him.

Nam also found the daily structure of a lesson helpful. He reported that typically students had opportunities to individually make sense of a problem, then share their thinking with their peers, and then finally have the instructor help them solidify ideas. For example, when I asked him how easy or difficult it was to understand or follow what was going on in the IOLA class, he said:

> *I think it's easy because we have time to think about the problem, and then we can share with the teammates. And then after that, the teacher then show what is the best approach to the ideas. So I think it's kind of easy for me.*

More specifically, a norm from the IOLA class that Nam reported as helpful was the centrality of student communication and interaction, relative to his prior lecture-based mathematics classes. For example, he noted that having more student communication and interaction helped him assess his mathematical ideas and arrive at better solutions:

> *So ((in this IOLA class)) we have to work with each other and then try the best answers to the problem. So in another math class, I just work on myself and write out the solution. And I don't know if that's the best idea, but that's the best idea for me. But maybe the other one ((other students)) have a better idea, so I don't know. But in this class, I have to work as a group. So we know what the best idea is… In another math class, I just answer problem by my own. So I don't know if that'd be true or if it's wrong. So that's a problem with that.*

So, this norm from IOLA supported Nam's mathematical development. It is important to note, however, that despite this reported helpfulness, Nam still preferred to engage individually, as he had internalized from his prior lecture-based classes.

Student Language Resources

Now I describe Nam's language resources across the three L-S-N dimensions: lexico-grammatical, situational, and normative.

Student Lexico-Grammatical Resources. This subsection highlights the language resources that Nam utilized in the IOLA course within the lexico-grammatical dimension. He reported drawing on four semantic resources (see Table 4.17 for an overview): (a) Prior knowledge of mathematical terms in Vietnamese, (b) Vietnamese Translations of mathematical terms not known in English, and (c) definitions of new mathematical terms in English and Vietnamese. (There was no evidence of Nam drawing on syntactic resources of his own.)

Table 4.17:

Student Lexico-Grammatical Resources Utilized by Nam in the IOLA Classroom

Student Lexico-Grammatical (Semantic) Resources	Example	Context of Example
Prior knowledge of mathematical terms in primary language	Vietnamese term for "matrix"	General classroom communication
Primary language translations of mathematical terms not known in English	Vietnamese translation of the term "matrix" Vietnamese translation of "linear independence"	
Definitions of new mathematical terms in English and primary language	Definition of "linear independence" (in English and Vietnamese)	

Prior to entering the IOLA class, Nam had learned some linear algebra terms in Vietnamese. So, when he couldn't understand a math term in the IOLA class (in English), he often sought to connect it to the math terms he knew in Vietnamese:

> *I just test some term in math that sometimes I don't understand. But in Vietnamese I have that word, but then I try to–I don't know how to say it, but. So I'll say it. So the word in English that I have never seen before, but I have that in Vietnamese. So I try to translate that word into English that I can remember the word.*

For example, he eventually connected the meaning of term "matrix" in English to the meaning of the Vietnamese term for "matrix:"

> *Like about Matrix…So I know that in Vietnamese. I study it before. But in here ((in the IOLA class)), I never seen the word matrix before. So I thought, I'm like, "What is that?" So I see the equation with the brackets and then number in it, then I say, "Oh, that['s] the matrix…. the one I study in Vietnam before."*

Relatedly, Nam benefitted from Vietnamese translations of mathematical terms he did not know in English. For example, since he did not initially recognize the meaning of the term "matrix," he translated this term into Vietnamese using a Google search:

So if I don't know ((the term)) matrix I just type in "matrix" ((in English)), then the word ((for "what is")) in Vietnam[ese]… this mean "what is matrix?" You're going to see this word. ((He pointed to the word highlighted on the screen.)) It's matrix in Vietnamese.

Nam also benefited from accessing definitions in both English and Vietnamese of new mathematical terms he encountered in the IOLA class. For example, using Google, Nam searched for the definition of linear independence in English. In addition, he translated the term *linear independence* into Vietnamese to look up its definition (in Vietnamese). Prior to the IOLA class, Nam had not learned about linear independence in Vietnamese, but seeing the definition in Vietnamese helped him make sense of this concept. These examples are evident in the following excerpt:

Interviewer: *Based on what you saw in the class, was there anything useful from the class for understanding… of linear independence?*

…

Nam: *No. At first I was confused at what that is. And let's see. I kind of have a little ideas, but I'm not sure. So I just wrote it. And then, yeah. Then I go home and then it's just fine.*

…

Interviewer: *Can you just share screens again and… show me… how you looked up linear independence?*

…

Nam: *So you can see on the side ((right side on Google))? That's the word ((the Vietnamese translation of the English name for linear independence.))*

Interviewer: *I see. And how did you figure out its meaning based on what you saw there?*

Nam: *Oh, so I searched for the meanings in Vietnamese.... I also search in English. ((Proceeds to find and show definitions of the term "linear independence" in English by clicking on the various sites output in the Google search results.))*

In sum, Nam leveraged definitions of linear independence in both English and Vietnamese to understand this concept.

Student Situational Resources. Below, I describe the situational language resources that Nam drew from himself or his peers in the IOLA classroom across the following three components: material, semiotic, and sociocultural. (I found no evidence in Nam's interview data of him drawing on an activity aspect of communication as a language resource.) Table 4.18 provides an overview of these resources.

Table 4.18:

Student Situational Resources Utilized by Nam in the IOLA Classroom

Component	Student Communication Tool as Situational Resource	Resourceful Feature of Tool	Example Context Where Tool Was Used
Material	Mathematically specific translation tools	Translating mathematical terms like "matrix" into Vietnamese	
	Mathematical websites in multiple languages	Defining and explaining meanings of mathematical terms in Vietnamese and English	General Classroom Communication
Semiotic	Supplementation of verbal representation with graphical representation, and vice versa	Better visualizing and explaining mathematical ideas	
Sociocultural	Small-group discussions	Personalized mathematical and linguistic clarifications among peers and the instructor	Task 7 (Meal Plans: Constraining the Number of Meals) Task 8 (Meal Plans: Constraining the Cost)

In relation to the material component, Nam took the initiative to utilize mathematically specific translation tools and mathematical websites as language resources. As noted in Student Lexico-Grammatical Resources., Nam had searched online for translations and definitions of mathematical terms he did not know in English. As illustrated in that subsection, he found these resources through accessing different technologies. For example, he used Google Search as a mathematically specific translation tool to translate terms like "matrix." In addition, he used various mathematical sites on Google that defined and explained the meanings of these terms like "linear independence" in both Vietnamese and English.

Within the semiotic component, Nam reported using the following distribution of representation types: 60% verbal and 40% graphical. When engaging with his peers in the IOLA classroom, he found it helpful to supplement one representation type with another. For example, when needed, he supplemented his verbal explanations with graphs, and vice-versa:

> *If I use verbal, if they ((his group peers)) don't understand, I can use graphical ((representations)) to show what my understanding [is] about it. And if I use graphical ((representations)) and they don't understand, I can use verbal to explain what [is] going on.*

Finally, in terms of the sociocultural component, Nam noted how he and his group peers utilized the small-group discussions as a resource for solving the math problems and clarifying their mathematical and linguistic understandings. For example, when I asked Nam to describe the communication among his small-group peers and note any struggles with or ways to navigate communication, he said:

> *So I think we just try to explain our idea [of] how to solve a problem. And then if someone don't understand, they can ask again, and they can repeat what they show us*
>
> .

As shown, Nam and his group peers addressed any communication challenges in the small-group discussions by asking for clarification and repeating ideas as needed. Nam also reported leveraging his group peers as resources when he couldn't follow the instructor's talk:

> Nam: *And they ((the IOLA instructor)) talk a lot, but I usually don't follow her. Yeah. So I try to. But I guess you [she] need[s] to use more graphic[al] ((representations)) or some writing so I can follow her.*
>
> Interviewer: *And how do you normally– when you're there, how are you trying to make sense of the information in the class? What resources do you normally use?*

Nam: *I think if I don't listen, I think I just ask the teammate.*

In addition, as noted in Instructional Situational Resources, Nam and his group peers leveraged the small-group discussions to call the teacher for personalized support on linguistic and mathematical issues they were facing with the IOLA tasks. In particular, they obtained clarification of the problem context in Task 7 (Meal Plans: Constraining the Number of Meals).

Student Normative Resources. As implied in Normative Demands, there were some norms Nam had internalized from his communities outside the IOLA classroom. In this section, I highlight those norms that he framed as helpful for communicating or interpreting mathematical information. To avoid repetition of interview excerpts, I only summarize the two norms that functioned as language resources for Nam, shown in Table 4.19.

Table 4.19:

Student Normative Resources Utilized by Nam in the IOLA Classroom

Norm Trait	Norm as a Resource
Effective student communication	Minimally communicating and interacting with peers and the whole class.
Competent student participation	Identifying important information

One norm of student engagement that Nam found helpful was reflected in his preference of what constituted effective student communication in the IOLA class. Nam had internalized this norm from his prior lecture-based mathematics classes in Vietnam, where there was minimal student communication and interaction. Thus, he preferred to engage individually in the IOLA class, even when the class imposed a different norm centered on student interaction.

A second norm that Nam reported as helpful was reflected in his belief of what made him a competent participant in the IOLA classroom. Consistent with his past experience in lecture-based math courses, he had internalized the importance of recognizing key information from the instructor's communication. So, he sought to apply this approach in the IOLA class to become a competent participant. Thus, he suggested for the instructor to give students a heads up of the topic (e.g., linear independence) they would learn on a given day of instruction.

Conclusion

This chapter presented case study analyses of two international students, Johan and Nam, and addressed the research questions related to language demands, instructional language resources, and student language resources. I investigated each question through the three sociolinguistic dimensions of the L-S-N framework: lexico-grammatical, situational, and normative. The findings showed that both students experienced a diverse range of language demands in the IOLA classroom, but also leveraged a variety of student and instructional language resources.

Johan and Nam experienced similar language demands across different dimensions. For example, in the lexico-grammatical dimension, both students experienced language challenges with antonym pairs and words with unfamiliar meanings. Similarly, in the situational dimension, both encountered tasks with unfamiliar problem contexts and identified difficulties with the use of semiotic systems (e.g., speech, verbal, English communication) and sociocultural structures (e.g., a whole-class participant structure). For instance, both struggled to keep up with the pace of spoken communication in the classroom. They attributed these struggles both to the impermanence of speech along with the use of English, especially in the informal register. Finally, in the normative dimension, Nam's and Johan's experiences both spanned three types of norm tensions: class-based incompatibility, student-centered incompatibility, and violation. In addition, norm violations were the most impactful tensions for both students. Key violations that both students experienced were related to what constituted effective communication among students and the instructor. In their communities outside of the IOLA classroom, they both experienced minimal communication and interaction among students and the instructor. This contrasted with the intensive communication and interaction expected in the IOLA class. Both Nam and Johan also reported experiencing norm tensions with respect to the constitution of competent participation and the permissible language of teaching and learning in the IOLA classroom.

At the same time, each student experienced divergent language demands based on their individual and cultural-linguistic backgrounds. For example, in the situational dimension, Johan emphasized the wordiness of tasks and class discussions, while Nam focused more on the complexity of sharing his own mathematical explanations. Moreover, in the normative dimension, Johan reported encountering a significantly greater number of tensions in the IOLA classroom compared to Nam's reported experiences. The majority of the norm tensions that Johan experienced were violations, but Nam reported one norm tension of each type. One possible reason behind the differences between their experiences in the IOLA classroom may be the extent to which the IOLA class differed from their respective cultural communities. Indeed, norm violations represent the norm type with the highest degree of conflict between the IOLA class and a student's communities outside the class. This high degree of conflict was exemplified in the norm violation reported by Johan regarding what constituted respecting the teacher: "In Islam, we prioritize our manner with the teacher very, very high…. don't interrupt when they are speaking…So in Malaysia, if you do that, you're dead!" In contrast, the only violation that Nam reported facing was primarily linked to differences between the IOLA class and prior lecture-based classes, without highlighting broader cultural comparisons.

In terms of instructional language resources, Johan and Nam also reported similar experiences. For example, neither student reported experiencing instructional lexico-grammatical resources. Moreover, both students highlighted that small-group discussions served as instructional situational resources because, relative to whole-class discussions, small-group discussions provided a more comfortable space for communication and afforded more opportunities to provide and obtain clarification on communication.

However, there are also differences between the instructional resources that Johan and Nam reported experiencing. Johan did not report experiencing instructional language resources in the normative dimension, while Nam did. For example, Nam indicated benefiting from the realistic nature of most problem contexts in the IOLA tasks, the typical structure of a class lesson that combined different participant structures and afforded access to different audiences, and the centrality of peer interaction. These findings align with the interpretation that, in general, Nam viewed the norms of the class as helpful, while Johan did not.

Finally, Johan and Nam utilized similar student language resources. Within the lexico-grammatical dimension, both students translated particular words into their respective primary language and leveraged their prior knowledge of certain mathematical terms or notation. More broadly, in the context of the situational dimension, both Johan and Nam used translation tools and integrated elements from their linguistic repertoire to make sense of problem contexts and meanings of certain terms. Similarly, in terms of the normative dimension, both students applied norms of communication from their prior lecture-based courses into the IOLA class. In particular, both Johan and Nam ultimately preferred to minimize their communication and interaction with other members of the class.

Overall, this chapter illustrates the language demands that international, multilingual undergraduate students (who came to the U.S. at the start of their undergraduate studies) may face in inquiry-based mathematics courses, emphasizing their variety and significance. Additionally, it provides examples of instructional elements that can serve as language resources for these students. The chapter further highlights how students might leverage these resources to navigate language demands and engage in mathematical sense-making.

Chapter 5 Case Study Findings for Domestic Multilingual Undergraduate Students

In this chapter, I analyze the perspectives of multilingual undergraduate students from one inquiry-oriented linear algebra (IOLA) course at Southwestern University who completed at least part of their K-12 education in the U.S. I present two cases: Seok, who moved from Korea to the U.S. to complete his high school education; and Luis, who completed his entire K-12 education in the U.S. Both students were pursuing STEM degrees.

In each section of this chapter, I identify and unpack the language demands and resources identified for each case study student (Seok and Luis) primarily drawing on their interview data. I begin each section by introducing the case study and then I report the associated findings, focusing on three main subsections that correspond respectively with the three research questions: (a) language demands, (b) student language resources, and (c) instructional language resources. Each subsection is organized along the three dimensions of the L-S-N framework: lexico-grammatical, situational, and normative. I consider each dimension in relation to communication within the IOLA tasks as well as more general classroom communication (e.g., teacher and student communication) in the IOLA classroom. Descriptions of the mathematical contexts and topics embedded in the IOLA tasks can be found in Chapter 3—Data Sources, and images of these tasks are included in Appendix A. After presenting the findings, I compare Seok's and Luis' case studies by research question and dimension, highlighting the most relevant similarities and differences. Both students appeared to experience fewer or less salient language demands than Johan and Nam reported experiencing in the IOLA class. Both identified an abundance of reliable instructional resources and reported drawing on their own language resources in ways that were consistent with the norms of the IOLA course.

Seok's Case Study

At the time of this study, Seok was an international undergraduate student from South Korea who self-identified as a man. He was majoring in Automotive Systems Engineering and was in his fourth year at Southwestern University. Before beginning eighth grade, he moved to the US, where he then completed high school. In the two years before beginning as a freshman at Southwestern University, Seok completed two years of mandatory military service in South Korea. On the research study's screening form, he self-reported native proficiency (level six) in Korean and professional working proficiency (level four) in English. The IOLA course was Seok's first linear algebra course.

Much of his previous education had been largely lecture-based, but also included some courses in his major at Southwestern University that he described as project-based. The IOLA course was, however, his first inquiry-based mathematics course. Both the IOLA class and his prior project-based classes involved extensive teamwork and group presentations. Yet, one difference he noted was that the communication for group presentations during whole-class discussions in the IOLA class did not have to be as refined as in his prior project-based classes.

Unlike Johan and Nam (see Chapter 4), Seok's experience of the IOLA course was generally positive. He attributed his comfort level in the class to his prior experiences in project-based classes:

> *Maybe this kind of ((IOLA)) class are more little similar to project class…I'm taking project-based class. Basically, all we do in class is presentation, the slide presentation. So basically, it's same thing. We make the presentation, and then we explaining what we made out of and then explaining stuff. So basic concept is similar.*

This perceived connection between his prior active learning experiences and the IOLA class warrants an examination of the language demands that Seok encountered, as well as the language resources he accessed and utilized in the IOLA course.

Below, I describe the language demands, instructional language resources, and student language resources that Seok reported. The demands and resources that Seok highlighted from the IOLA class spanned all three dimensions of the L-S-N framework. Notably, he reported a few language demands, an abundance of reliable instructional resources, and a few student language resources that were consistent with the norms of the IOLA course.

Language Demands

In this subsection, I describe the language demands that Seok experienced in the IOLA class across the three L-S-N dimensions: lexico-grammatical, situational, and normative.

Lexico-Grammatical Demands. As noted in Table 5.1, Seok reported experiencing one semantic and one syntactic demand in the IOLA class. The semantic demand was new mathematical terms, such as “span,” “linearly independent,” and “linearly dependent,” that initially confused him but later became easier as they were explored in class. For example, he couldn’t understand what Task 3a (Span Worksheet) was asking him to do because he did not understand the meaning of “span.” Similarly, he reported he couldn’t complete Task 5 (Linear Independence and Dependence: Creating Examples) because he didn’t understand the meanings of “linearly independent” and “linearly dependent.”

The syntactic demand was unfamiliarity with the column form of a vector. For example, referring to the vector notation in Task 1 (The Carpet Ride Problem: The Maiden Voyage), he reported initially struggling with figuring out which entry of the vector was x and which one was *y*: "When I first saw those things ((the vector notation in Task 1)) and I didn't know, 'Oh, which one is *x* and which one is *y*?'"

Table 5.1:

Lexico-Grammatical Demands Experienced by Seok in the IOLA Classroom

Component	Lexico-Grammatical Demands	Example	Context of Example
Semantic	New mathematical terms	"Span"	Task 3a (Span Worksheet)
		"Linearly independent"	General classroom communication
		"Linearly dependent"	
Syntactic	Unfamiliar notation	Notation of a vector	Task 1 (The Carpet Ride Problem: The Maiden Voyage)

Situational Demands. Below, I report the language demands that Seok experienced in the IOLA course in terms of the situational dimension, which consists of the material, activity, semiotic, and sociocultural aspects of communication. In the interview data, I identified evidence of language demands only within the material and sociocultural components. An overview of the findings is shown in Table 5.2 and Table 5.3, with one table focusing on communication from the tasks and the other focused on more general classroom communication.

Table 5.2:

Situational Demands Experienced by Seok on the IOLA Tasks

Component	Task Feature as Situational Demand	Task Example
Material	Lots of questions	Task 7 (Meal Plans: Constraining the Number of Meals)
	Too many words	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
		Task 9 (The Car Rental Problem).
	Unfamiliar problem contexts	Task 7 (Meal Plans: Constraining the Number of Meals)
	Problem contexts that were not straightforward	None provided
Activity	Requests of mathematical explanations	Task 1 (The Carpet Ride Problem: The Maiden Voyage)

Table 5.3:

Situational Demands Experienced by Seok in the IOLA Classroom's General Communication

Component	Classroom Communication Tool	Feature of Tool as Situational Demand
Sociocultural	Whole-class discussions	Less comfortable speaking to a wide audience
		Need to convey information loudly
		Need to convey information succinctly
	Small-group discussions	Need to relate with his peers and initiate conversations with them

In terms of the material component, Seok highlighted four features of some mathematical tasks that Seok found linguistically challenging: (a) lots of questions, (b) too many words, (c) unfamiliar problem contexts, and (d) problem contexts that were not straightforward. First, an example of a task with lots of questions that he pointed to was Task 7 (Meal Plans: Constraining the Number of Meals). Second, an example of a task that he felt had too many words was Task 1 (The Carpet Ride Problem: The Maiden Voyage). Although he ultimately understood the question in the task, he felt that the wordiness initially got in the way of his understanding:

> *I think as I see this problem ((Task 1)), it's just lots of words. I mean, basically telling you, "Find this way," or, "Find direction," or how many points they went, but I don' know. Just basically this problem in the instruction there, it's basically storytelling with other words instead of math words. So I don't know how exactly I can explain, but I felt this one doesn't–I mean, this one also straightforward to telling me, "Find a solution," but it has a lot of words. So I felt kind of it's kind of wordy.*

Another task that he experienced as having too words was Task 9 (The Car Rental Problem).

A third feature that Seok found linguistically challenging in some tasks was unfamiliar problem contexts. For example, in the problem context of Task 7 (Meal Plans: Constraining the Number of Meals), he noted that his small group initially struggled to understand how many meals they were allowed per day:

> Seok: *Richard ((Seok's group peer)) asked some question about maybe that (the number of meals allowed per day)),...but because the total number is 210 and then I think he asked for should students have at least one meal…Can they skip the meal? Do they only take breakfast or do they only take lunch and dinner?...And then the professor said, "Oh, they can just have only breakfast. They can only just have breakfast only or either lunch only or either just two meals only."*
>
> Interviewer: *So would you say that was a challenge of understanding the problem of the context?*
>
> Seok: *Maybe it was some in kind of gray zone, I would say. It didn't really explain students can take only a single meal or two meals. It didn't really explain that. That's why maybe Richard asking too, I guess.*

Finally, a fourth task feature that Seok found linguistically challenging was when tasks had problem contexts that were not straightforward. His definition of "not straightforward" included contexts given in story form, where he could not easily pick out the mathematical aspects for solving it:

> *Basically the professor gave us not just simple problem. She basically gives a paper that has kind of a story to me. We read out the sentences and they have basically little stories and then we take out the numbers out of it and then make an equation out of it or the vector matrix out of it. So for me, that was pretty challenging in order to make the matrix out from the paragraph.*

In the context of the activity component, Seok reported that he found it challenging that some tasks asked students to explain in words their answers to the tasks:

> *When I see this paper ((handout of Task 1)), on the bottom, it said, "State and explain your answer," then, one of the biggest points for me to state and explain, I can kind of explain how I solve it by showing the work, but if it says, "Explain or state," then for me, it could be a little challenging to write those into the words…. That was kind of uncomfortable…I have to go into different format, not showing what I did as math work.*

Seok also reported various language demands within the sociocultural component, in relation to both whole-class and small-group discussions for distinct reasons. First, Seok noted that, in contrast to small-group discussions, whole-class discussions involved speaking more publicly with a larger audience, which induced various linguistic challenges. He felt less comfortable to speak in the whole-class format because it required explaining mathematical information more succinctly, since there were fewer opportunities to provide and request clarification. He made this feeling evident when he distinguished types of talk (explaining versus discussing) between whole-class and small-group discussions:

> *In the whole-class ((discussions)), if they're ((students)) not sure about how they solve it, or if they're not sure about their solutions, they might having challenges explaining stuff to the class….In small groups, they're not explaining to themselves. They're basically discussing about the problem. I don't think that's challenging in small groups.*

He elaborated on these two types of communication, which essentially corresponded with expository versus exploratory talk:

> *Explanation is challenging because you have to explain how you get the result, and then you have to explain the steps. But in small groups discussions, you're basically finding the ways to the result. So you can basically talk to each other*

> *and get to the results…That's why I thought it was more challenging in explanation.*

In addition, the whole-class discussion format involved communicating with students that he knew less well than his small-group peers. Ultimately, he felt that, relative to small-group format, he learned less in whole-class discussions.

Second, while Seok found certain aspects of whole-class discussions to be more challenging than small-group discussions, he also identified one challenge with small-group discussions. Specifically, he noted it was generally difficult to relate with his peers and initiate conversations with them. For example, he noted that his small group had less speech communication than in other groups as a result of not having strong relationships:

> *I felt maybe…not bad atmosphere, but since we're not that close, so we're kind of yielding someone else…to speak something first…Those kind of thing make our group less speaking compared to—((other small groups)).*

Normative Demands. Seok described four language demands in terms of conflicts in norms between the IOLA course and his prior lecture-based classes in the U.S. These four normative language demands spanned two types of norm tensions (defined in Chapter 3): class-centered norm incompatibilities and norm violations. As noted in Table 5.4, one normative demand reflected a tension regarding expectations for overall engagement throughout a whole lesson and the remaining three demands were related to engagement within a particular type of participant structure, either whole-class or small-group communication. All expectations involved in these normative demands were related to social norms, as opposed to sociomathematical norms.

Table 5.4:

Normative Demands Experienced by Seok in the IOLA Classroom

Participant Structure	Norm Trait	Norm Tension	Norms or Situational Aspects	
			Seok or His Community Outside IOLA Classroom	IOLA Classroom Community
Overall class communication	Permissible language	Class-centered incompatibility	Korean as a primary language	English only
Whole-class communication	Required genre of student communication	Violation	Answers only	Answers with explanation or justification
	Default small-group spokesperson	Class-centered incompatibility	Difficulty to break the classroom norm without explicit teacher authorization	Students who contributed the most to the small-group discussion or who seemed to enjoy talking.
Small-group communication	Logical role of communication	Violation	Share only mathematical knowledge one is confident in	Share one's mathematical thinking and struggles as a way to develop mathematical knowledge

In the context of overall communication in the IOLA course, Seok experienced one class-centered norm incompatibility related to what constituted the permissible language of teaching and learning. Like Johan and Nam (see Chapter 4), Seok perceived that all communication in the IOLA class had to be conducted in English, but there was evidence that this norm was somewhat incompatible with his linguistic repertoire, which included Korean as his primary language. Indeed, he said he would participate more if he were able to speak in Korean in the class. For example, after he noted during the interview that he spoke very little in class, I asked him if anything would be different in his participation if he could speak Korean, and he said:

I'll use more verbal stuff, but it kind of depends on–if it's really majorly speaking Korean in class, then I'll be more maybe participating, but just minor as a Korean languages in class, then it would be same because the majority is English, so it doesn't really change me.

In addition, Seok faced two norm tensions in the context of whole-class communication. The first tension was a norm violation related to what constituted the required genre of student communication during whole-class discussions. From his prior lecture-based classes, he had internalized that students can share answers to mathematical questions to the whole class without giving an explanation or justification. In contrast, he felt that the IOLA class expected students to explain or justify any answers they shared during class. As noted in the quote below, this norm violation constituted a language demand for Seok because he found it more challenging to explain than to simply give an answer:

Seok: *In this kind of type class ((the IOLA class)), students are more like, "Explain stuff, not just saying answers."*
Interviewer: *And in a lecture-based class, what do they do, normally?*
…
Seok: *One of students say whatever the answer they got…And then after that, professor would ask he or she, "How did you get?" Or something. But basically, most of lecture class just answering back to the solution, the answer that they got.*
…
Interviewer: *Are these expectations of doing more explanations than answers more difficult or the same or easier than just giving answers?*
Seok: *Answering is way easier because you can just say out the answers, whether it's right or wrong, but explanation stuff, you have to explain the board–you wrote on the board–the things on the board. And then also at the same time, you have to be more clear on your explanation, so others can understand, too.*

The second tension that Seok experienced in the context of whole-class communication was a class-centered norm incompatibility. This incompatibility was related to who constituted the default student spokesperson for a small group when the instructor asked each group to share their mathematical work. Seok perceived there was a norm that only students with one or both of the following conditions could take on the role of spokesperson: (a) they had contributed the most to the preceding small-group discussion, or (b) they seemed to enjoy talking.

> *Maybe they ((the student)) might put the most work on their whiteboard as their method, so basically, they put on the board, so they know what step they took, so they can explain easier than other classmates. That could be one case. Or otherwise, they like to talk.*

When I asked Seok if there were any challenges related to communication in the IOLA class, he pointed to this norm as a challenge:

> *In the most of the group ((whole-class discussions)), basically same people speaking every time, and then they're explaining how they solve, and then what they got on their boards. So yeah, it keeps repeating every class time.*

He subsequently implied that unless the instructor disrupted this norm, the same students would speak every time:

> *Maybe if professor…can recommends, or she can ask like, "What about other students speaking or explaining their stuff?" That could be one case. Otherwise, unless among the teammates, they're saying, "I'll explain this this time." Unless they're saying like that, I don't think. Otherwise, it would be same.*

In other words, Seok implicitly conveyed that without intentional instructional intervention, the existing norm of who is the legitimate spokesperson for a group limited other students' opportunities to speak to the whole class. In particular, he noted he wasn't comfortable with how this situation applied to his small group, where one peer, Richard (pseudonym), regularly took over the spokesperson role: "I didn't really felt any weird or bad feeling, anything bad in our group, except–I mean, only Richard is–he likes to talk."

Finally, Seok also faced a norm violation related to what constituted the logical role of communication during small-group discussions. One norm he had internalized from other classes was to produce expository talk, where he shared the mathematical knowledge of which he was confident. In contrast, he felt that in the IOLA class, he had to engage in exploratory talk, where he shared his tentative mathematical ideas and confusions. As noted in the quote below, this norm violation constituted a language demand for Seok because he initially struggled to talk about what he didn't fully understand:

> *The ((IOLA)) course is pretty different format from the other classes. So they kind of make stuff for me. So I have to participate with the team, but I'm basically pretty much no idea what this subject are. So it kind of gave me confusing for the first few times.*

Instructional Language Resources

In this subsection, I describe the instructional aspects that Seok experienced as linguistically helpful along the lexico-grammatical, situational, and normative dimensions of the L-S-N framework.

Instructional Lexico-Grammatical Resources. As indicated in Table 5.5, Seok reported experiencing one semantic resource in the IOLA class. Recall from Lexico-Grammatical Demands that Seok initially faced a challenge with making sense of the order of x and y in the column-form vector notation. Yet, while working on Task 1 (The Carpet Ride Problem: The Maiden Voyage), he was able to quickly figure out the order of x and y based on the contextual interpretation of the vector given in the task:

> *With the text on the bottom, it says "3 mile east" and then "1 mile north." So basically, north is y-axis and then east, x-axis. So, it goes 3 on the x-axis and then goes up y-axis. So, "Oh, it should be x comes first on the top and then y goes the bottom." That's how I figured out.*

He appeared to find helpful that the text in the task linked each entry in the vector with either "east" or "west," which he more readily associated with either "x" or "y."

Table 5.5:

Instructional Lexico-Grammatical Resources Experienced by Seok in the IOLA Classroom

Component	Lexico-Grammatical Resource	Example	Context of Example
Semantic	Contextual interpretation of notation	Notation of a vector	Task 1 (The Carpet Ride Problem: The Maiden Voyage)

Instructional Situational Resources. Seok reported experiencing various instructional tools from the IOLA classroom as resourceful along the following components: material, activity, semiotic, and sociocultural. Table 5.6 and Table 5.7 provide a brief overview of these findings according to two sources of communication in the IOLA class: tasks and more general classroom communication.

Table 5.6:

Instructional Situational Resources Experienced by Seok in the IOLA Tasks

Component	Task Feature as Situational Resource	Task Example
Material	Straightforward questions	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
		Task 3b (Group Quiz)
		Task 7 (Meal Plans: Constraining the Number of Meals)
		Task 6 (Practice for Individual Quiz)
		Task 10 (The Car Rental Problem: Follow-Up)
	Repeated story contexts	Task 7 (Meal Plans: Constraining the Number of Meals) & Task 8 (Meal Plans: Constraining the Cost)
	Realistic problem contexts	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
Semiotic	Essential visual representation	Task 9 (The Car Rental Problem)
		Task 11 (Geometric Interpretation of a Matrix Times a Vector)

Table 5.7:

Instructional Situational Resources Experienced by Seok in the IOLA Classroom's General Communication

Component	Classroom Communication Tool	Feature of Tool as Situational Resource
Semiotic	Supplementation of speech communication with gesturing	Visual
	Supplementation of speech communication with writing	
	Supplementation of verbal representations with graphical representations	
Sociocultural	Small-group discussions	More private with the opportunity to provide or obtain clarification
	Whole-class discussions	Wide audience with access to more mathematical ideas

In terms of the material component, Seok reported benefitting from three features of the language on some of tasks: (a) straightforward questions, (b) familiar problem contexts, and (c) realistic problem contexts. As described below, by a straightforward question, he meant that it was easy to understand what the question asked mathematically:

> *I mean, basically, for me, if it was just straightforward, "Oh, find the direction," or, "Find the numbers," then it would be easier for me, but on this one ((Task 1 (Magic Carpet Scenario One))), I just have to read out and then, oh, on this question, they want me to find this and that. So for me, personally, I would like for the– as a question, I prefer straightforward, "Find this," or, "Graph this."*

He also considered questions in a task to be straightforward when he felt they were given in symbolic language (e.g., equations) and didn't have "long sentence[s]," as with Task 3b (Group Quiz). Other examples of tasks with straightforward questions included Task 7 (Meal Plans: Constraining the Number of Meals), Task 6 (Practice for Individual Quiz), and Task 10 (The Car Rental Problem: Follow-Up).

Seok also found it helpful when tasks included story problem contexts that were repeated from previous IOLA tasks. For example, although he felt the mathematical complexity increased from Task 7 (Meal Plans: Constraining the Number of Meals) to Task 8 (Meal Plans: Constraining the Cost), he found it helpful that the story context did not change:

> *So I remember we did meal plan for two classes or three classes, and then the first classes, basically, we transformed the word into equation, and then basically stuff like how many meals that they want. And then the second classes, it's not the same problem, but the same format is the equations. The problems are the same meal plan, but it gets more difficult with the prices of each meal. And then all those things kind of makes me kind of keep connected because they use the same examples.*

Consequently, he assigned the same ranking of linguistic complexity to both Tasks 7 and 8.

In addition, Seok reported benefitting from problem contexts being realistic, or imaginable. In general, he felt this was the case with tasks that had story contexts, such as Task 1 (The Carpet Ride Problem: The Maiden Voyage):

> *It was basically storytelling. So, if those types of problems that I've explained to others, then it would be easier because while I'm reading that context, it kind of gives me imaginations of there's two however, and then magic carpet, the image's on my mind, and then they're moving some point, and then finally, they want to get some points with the fastest way.*

As noted above, he appreciated Task 1 (The Carpet Ride Problem: The Maiden Voyage) because he could imagine the situation it presented. He elaborated on this in the next quote:

> *This one ((Task 1)) also has a lot of numbers, but it basically explains pretty easy languages, not funny but pretty friendly way, just familiar. They're using familiar words, like magic carpet or hoverboard so that student can easily visualize what they're asking for.*

This linguistic feature was so salient to Seok that he ranked Tasks 1 and 2, which shared this story context, as the easiest linguistically.

In the context of the activity component, Seok highlighted an instructional strategy for removing barriers for participation during whole-class discussions for certain students. As noted in Normative Demands, Seok reported a challenge with participating in whole-class discussion due to the norm he perceived that the only spokespersons for the small groups could only be the students who contributed the most to the small-group discussion or who seemed to enjoy talking. Thus, he noticed that the spokespersons of each group tended to remain constant across lessons, thereby limiting the opportunities for other students to talk. So, he appreciated that there was a time when the IOLA instructor intentionally disrupted this norm by asking others to participate when the same student volunteered to talk for his small group:

> *Teacher asked someone else in class. Because in the class, maybe three or four students are mainly responding to a professor. So, one time, one of our classmates, she was answering back to professor, and then the professor saying like, "Oh, and is there anyone else want to answer?" And she was trying to give other students a chance. Yeah. I saw once.*

Within the semiotic component, Seok pointed to the presence of visual representations on the IOLA tasks as well to the instructional combination of communication mediums and mathematical representations as helpful. For example, he appreciated when some of the IOLA tasks were represented visually. For example, he reported that the visual representation given in Task 9 (The Car Rental Problem) was central to his understanding: "This one [Task 9], there's a lot of words, but the good thing was it has a graphic stop in the center." He noted the importance of this visual by noting that without it the task would have been placed 2 or 3 rankings higher in linguistic complexity. Another task he reported being visually represented was Task 11 (Geometric Interpretation of a Matrix Times a Vector).

Moreover, Seok found it helpful when the teacher and other students in whole-class discussions combined communication mediums and mathematical representations. In particular, Seok mentioned that the IOLA instructor frequently used hand gestures while explaining mathematical concepts in class, particularly when delving into topics like plane intersections:

"She was crossing her hands. So, there's two lines or two planes that intersects one point that makes a line." Seok especially found the instructor's gestures to be helpful because they added a visual to her speech communication:

> *For me, ((the IOLA instructor's)) speaking and gesturing pretty much is helpful, especially gesturing because it's visual stuff. So even pointing something out, that's visually. She's pointing out something on the slides or something. Basically, it's a visual stuff. For me, visual stuff helps me to learn more or even faster. So I like she's doing a lot of gesturing stuff, making graphing on her hands or something.*

He reported appreciating that the instructor often pointed to the projector and small-group whiteboards while she spoke to the whole class.

Similarly, he appreciated that when classmates shared their group's mathematical ideas to the whole class, they displayed their group's writing and different mathematical representations on their whiteboards. For example, he said:

> *Basically, as a whole class, we share small groups board. So as an individual or either our small group, we can understand either, or we can read what other students think of the same problem or how they solve it, or either maybe they use the same method that we use or either they use different method or even they both use the same method.*

Seok emphasized that he benefited from the visual aspects of graphs and equations that classmates showed:

> *Graphical and symbolic stuff are basically visual stuff, so I can see straightforward. So really basically make me to easier understanding.*

Finally, in terms of the sociocultural component, Seok highlighted features of small-group and whole-class discussions that functioned as communication resources. As alluded to in Situational Demands, Seok found that, in contrast to whole-class discussions, small-group discussions involved speaking privately with a relatively small audience, which induced various resources. He felt more comfortable to speak in the small-group format because it didn't require explaining mathematical information succinctly. In addition, the small-group discussion format involved communicating with students that he knew better than in the whole-class. Ultimately, he felt that he learned more in small-group discussions partly because the talk involved in this discussion format was more exploratory (interpersonal and discussion-based) than expository (presentational and explanation-based):

> *I'm learning more in our small groups compared to the whole class because–so, as I said, we're discussing and we're solving the problems in small groups, and then we're explaining the results in the whole group, so. And then most of the groups, they have basically similar answers, slight different steps. Yeah. That's why.*

In addition, small-group discussions allowed Seok to obtain help from both his group peers and the IOLA instructor in a more private setting. For example, he reported that his small group peers helped him clarify the meaning of new mathematical terms such as "span." Moreover, the small-group discussions allowed Seok's group the opportunity to ask the IOLA instructor for clarification on some linguistic issues they experienced in the classroom. As noted in the Material segment of Situational Demands, Seok's group encountered an unfamiliar problem context in Task 7 (Meal Plans: Constraining the Number of Meals): They didn't know how many meals were allowed per day. So, they asked the IOLA instructor for help, and they found the following response from her as helpful for understanding the context: "The professor said, 'Oh, they can just have only breakfast. They can only just have breakfast only or either lunch only or either just two meals only.'"

Yet, while Seok found certain aspects of small-group discussions to be more beneficial than whole-class discussions, he also reported one advantage of whole-class discussions. Specifically, he experienced whole-class discussions as helpful because they allowed him to access more student ideas:

Maybe in discussion, maybe in whole groups. There's basically more people than small groups, so there could be more thoughts from other students. They could maybe easier to get some ideas from students, others.

Instructional Normative Resources. Finally, Seok reported various instructional normative resources that were generally available in the IOLA classroom to support his mathematics learning. These resources included the teacher's marking of the topic of each lesson, the centrality of peer communication, and the unstructured nature of student communication. See Table 5.8 for a brief overview.

Table 5.8:

Instructional Normative Resources Experienced by Seok in the IOLA Classroom

Tool of Communication	Typical Feature as a Normative Resource
Regularity in the teacher's speech communication	Helped identify the topic(s) of each lesson
Nature of student communication	Unstructured form
Nature of overall class communication	Centrality of peer interaction

First, Seok felt that the IOLA instructor vocally marked the topic of each lesson, allowing him to see connections across lessons:

In the beginning ((of a lesson)), the professor shows, "Today's classwork will be this," and then she hands out the papers, and then basically I look through those problems over the paragraphs. Then basically that's how I see today we are learning this stuff, and then those stuff, and then basically over each class–think over each class, there's certain common stuff that goes along. So, I feel like each class is kind of connected, unless it's changed to move to different format of math stuff.

While Seok noted that the instructor vocally marked the topic of each lesson, the other multilingual students claimed they did not see the instructor do that. During my classroom observations, I did not see the instructor do that either.

Second, Seok reported that the centrality of peer interaction, relative to his prior lecture-based mathematics classes, was helpful. For example, whereas in a lecture class he had to rely more on self-learning, he found the IOLA helpful because he could learn from his small-group peers and other classmates during class discussions:

> *Since I didn't have that much knowledge on the vector and matrix system, for me, the inquiry-based class was very helpful to get to know easier and faster, I guess, because if it was lecture class, then either I had to go to office hour or get tutoring center or have to do all my homework, then I basically do self-learning. But in inquiry-based ((IOLA)) course, I can still learn from same classmate while they're solving or they're talking to each other with the one problem. So I can kind of get the idea of what they're talking or what they're putting on the boards. Those sharing ideas in the class kind of accelerate kind of the learning speed.*

Finally, Seok also appreciated the unstructured nature of student communication promoted in the IOLA class, where students shared their ideas in less generalized and final ways than in other presentation-based classes:

> *It's more kind of free to speak in this ((IOLA)) class. I mean, other class in project-based class, they're like, "We have to get ready for the presentation. It should be kind of more perfect." But in this ((IOLA)) class, we can just explain what we got, just saying, "In this problem, we got this answer," or, "How we got this," and then it shouldn't be perfect. It can be wrong. It can be a little different, and more concrete.*

He felt more liberated to speak in the IOLA environment than in his prior project-based classes. More specifically, he appreciated not having "to state the whole sentence to explain something."

Student Language Resources

This subsection describes the language resources that Seok drew from himself or his peers across the three L-S-N dimensions: lexico-grammatical, situational, and normative.

In terms of the lexico-grammatical dimension, Seok reported drawing on one semantic resource (see Table 5.9). One language resource that Seok utilized for making sense of new mathematical terms in the IOLA class was breaking down the compound structure of some of the terms into meaningful parts. For example, to make sense of the terms “linear independence” and “linear dependence,” he attended to the meanings of the embedded words “linear,” “dependent,” and “independent”:

> *When I see linear independence or dependence, those are two words combined into one term. So linear means–in math linearly something, then it's basically in the same line...And then basically dependent, independent. Dependent lean to something, and then independent, they stand alone, self.*

Table 5.9:

Student Lexico-Grammatical Resources Utilized by Seok in the IOLA Classroom

Component	Student Lexico-Grammatical Resources	Example	Context of Example
Semantic	Compound structure of new mathematical terms	Breaking down “linear independent” into “linear” and “independent” Breaking down “linear dependent” into “linear” and “dependent”	General classroom communication

In relation to the situational dimension, Seok reported three semiotic resources (see Table 5.10). The first semiotic resource was in the context of communication mediums. While Seok reported not speaking much during class, he highlighted that he relied on writing as a medium of communication: “Writing is the most. I didn't really say anything, even in my small group, more with the writings.” Similarly, he said writing constituted more than 90% of his small-group communication “because basically, we have to show the works on the [small group] board to present or show it to professor.” The second and third semiotic resources were in terms of types of mathematical representations. Specifically, Seok said he most of his participation consisted of drawing graphs and writing equations.

I'm not really talkative person, so I don't really using a lot of verbal presentations. Then, for me, I like to see things like the pictures or either graph, something drawings. So I'd rather do just drawings, graphics, and maybe just equations. Just write out interpreting the problems words into the equations and those kind of stuff.

He reported benefitting from the visual aspects of graphs and equations for his own learning.

Table 5.10:

Student Situational Resources Utilized by Seok in the IOLA Classroom

Component	Student Communication Tool as Situational Resource	Resourceful Feature of Tool	Example Context Where Tool Was Used
Semiotic	Writing Graphical representations Symbolic representations	Visual	General classroom communication

In terms of the normative dimension, Seok highlighted one norm that he framed as helpful for communicating or interpreting mathematical information. As noted in Table 5.11, this norm was reflected in his belief of what made him a competent participant in the IOLA classroom. Consistent with his past experience in lecture-based math courses, he had internalized the importance of recognizing key information from the instructor's communication and writing notes. The following quote illustrates how he learned this norm from past lecture-based courses: *Students are doing more taking notes in lecture class compared to inquiry-based ((IOLA class)). I see some people either– even I put some stuff on the notes, but most of the time, we don't really use writing something on our notes. Basically, we had a handout, basically. So the whole class is basically just solving problems, majorly.*

Even while he acknowledged notetaking was not a norm of that environment, he applied and benefited from using this norm in the IOLA class:

I'll basically take notes when the professor writing stuff on the whiteboard. That's basically what I take notes from. And then, yes, I don't really take any notes that what she's saying. Basically, I just listen to those words, and then additionally, if she's writing something on the board, I just take notes as just in case.

It is worth noting that Seok's definition of writing notes was exclusively to copy down what the instructor wrote on the board.

Table 5.11:

Student Normative Resources Utilized by Seok in the IOLA Classroom

Norm Trait	Norm as a Resource
Competent student participation	Identifying important information and writing down notes

Luis' Case Study

At the time of this study, Luis was a Latino undergraduate domestic student from the U.S. who self-identified as a man. Luis had completed his entire K-12 education in the U.S. He started his undergraduate studies in the U.S. at a community college, where he stayed for two years before transferring to Southwestern University. He was majoring in Mechanical Systems Engineering and beginning his first semester at the university. On the research study's screening form, his self-reported language background included native proficiency in Spanish (level six) and full professional proficiency (level five) in English.

Prior to the research study, Luis had already taken another linear algebra class at a local community college (online during the COVID-19 pandemic). However, he was not able to transfer the course credit to Southwestern University, so he enrolled in the IOLA course. Some of the math courses he took prior to the IOLA course were active learning courses, while others were more lecture-based.

Unlike Johan and Nam (see Chapter 4) and more like Seok, Luis' experience of the IOLA course was overwhelmingly positive: "I think it's one of the classes that has been engaging for me and yeah, had a really good experience so far." Moreover, he connected his linguistic experience of the class to his language background and other linguistic resources. For example, he said:

> *I don't feel that there is a problem ((with the language and communication in the IOLA class)), I mean, because I am bilingual, I understand both languages quite well, so it is not difficult for me, especially the teacher makes the classes very understandable. Her level of English for class is quite moderate. It's not, shall we say, extremely professional, so we students understand it quite well.*

This perceived connection between language and the IOLA class warrants a systematic analysis of language demands and resources for Luis. This analysis will help to better understand how certain language resources may have mitigated some language demands for this student.

Below, I describe the language demands, instructional language resources, and student language resources that Luis reported. The demands and resources that Luis highlighted from the IOLA class spanned all three dimensions of the L-S-N framework. Notably, he reported very few language demands, an abundance of reliable instructional resources, and a variety of student language resources that were consistent with the norms of the IOLA course.

Language Demands

Below, I describe the language demands that Luis experienced across the three L-S-N dimensions: lexico-grammatical, situational, and normative.

Lexico-Grammatical Demands. In the lexico-grammatical dimension, Luis reported experiencing two semantic demands (see Table 5.12 below). (No syntactic demands were identified.) One semantic demand Luis faced was regarding new mathematical terms, such as "span" and "linearly independent," that initially confused him but later became easier as they were explored in class. For instance, he couldn't understand what Task 3a (Span Worksheet) was asking him to do because he did not understand the meaning of "span."

A second semantic demand Luis experienced were context words, like "Cramer" and "Gauss" in Tasks 1 (The Carpet Ride Problem: The Maiden Voyage) and 2 (The Carpet Ride Problem: Hide and Seek), that he found to be misleading. Initially, he interpreted them to imply that he needed to use Cramer's Rule and Gaussian Elimination.

> *((Referring to Task 1)) But when it comes to, for example, the third paragraph from the bottom, where it says, "Your Uncle Cramer." Even though that name, Uncle Cramer, and ((referring to Task 1)) then at the very end of that paragraph, it says, "Old man Gauss." I remember that those are two techniques. Even though they're using them as names, there's rules for them in linear algebra. Yeah, it makes more sense because Gauss, it's like, "Oh, okay, this relates to row reduction or Gaussian reduction, or something." I remember that in my previous class.*

The following quote summarized his experience with these names:

> *But yeah, maybe this name right here, it may have thrown people [off] like me, where they knew exactly how that name came from.*

As he explained in these quotes, this lexico-grammatical demand was sensible for Luis because he had taken a prior linear algebra course, where he had seen the names of standard algorithms like Cramer's Rule.

Table 5.12:

Lexico-Grammatical Demands Experienced by Luis in the IOLA Classroom

Lexico-Grammatical (Semantic) Demands	Example	Context of Example
New mathematical terms	"Span"	Task 3a (Span Worksheet)
	"Linearly independent"	None provided
Misleading context words	"Cramer"	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
		Task 2 (The Carpet Ride Problem: Hide and Seek)
	"Gauss"	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
		Task 2 (The Carpet Ride Problem: Hide and Seek)

Situational Demands. Below, I report the language demands that Luis experienced in the IOLA course in terms of the situational dimension. An overview of the identified situational demands is shown in Tables 5.13 and 5.14, with one table focusing on communication within the tasks and the other focused on more general classroom communication.

Table 5.13:

Situational Demands Experienced by Luis on the IOLA Tasks

Component	Task Feature as Situational Demand	Task Example
Material	Too many words	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
		Task 7 (Meal Plans: Constraining the Number of Meals)
		Task 9 (The Car Rental Problem)
Semiotic	Disconnected visual	Task 9 (The Car Rental Problem)

Table 5.14:

Situational Demands Experienced by Luis in the IOLA Classroom's General Communication

Component	Classroom Communication Tool	Feature of Tool as Situational Demand
Sociocultural	Whole-class discussions	Need to convey information to a wide audience loudly and succinctly
	Small-group discussions	Need to relate with his peers and initiate conversations with them

In terms of the material component, Luis highlighted one aspect of some mathematical tasks that induced a communication challenge for him: too many words. Three tasks that he reported as having too many words were Task 1 (The Carpet Ride Problem: The Maiden Voyage), Task 7 (Meal Plans: Constraining the Number of Meals), and Task 9 (The Car Rental Problem).

Within the semiotic component, Luis found it challenging when tasks didn't connect visual representations with verbal descriptions. For example, he found the visual in Task 9 (The Car Rental Problem) to be confusing because the percentages in it were not described in the text:

"Even though there's not that much of notation, the fact they have too many percentages and diagrams that don't really make too much sense, it made it difficult."

Luis also reported various situational demands within the sociocultural component, in relation to both whole-class and small-group discussions for distinct reasons. Luis found that whole-class discussions involved speaking publicly with a relatively large audience, thereby making speaking in whole-class more intimidating and inducing various communication challenges for him. Specifically, he noted that, in contrast to small-group discussions, a whole-class discussion participant structure required speaking louder as well as communicating and explaining mathematical information more succinctly:

> *There's people that—not that they fear, but they don't feel comfortable talking in bigger crowds, I guess, bigger groups of people.*
>
> *Big group of people, which it may feel a little bit more intimidating or having to speak louder, having to develop your actual talking, communication skills in a big group.*
>
> *So yeah, it's a little bit easier to explain stuff to 3 or 4 guys rather than 20 or big group.*

Luis also highlighted this challenge with whole-class discussions by contrasting it with an inter-group participant structure available in the IOLA classroom. There was an individual participation assignment, which required students to gather pictures of another group's whiteboard near the end of class each day. (As part of this assignment, students were also asked to describe the mathematics on the whiteboard and to compare it to their own work.)

Relative to this inter-group participant structure, Luis felt that whole-class discussions made communication more public and limited the opportunities to speak one-on-one with other students outside of his small group. He implicitly linked these challenges to gathering less mathematical information and facing more difficulty with explaining his mathematical thinking in the whole-class format when compared to the inter-group structure: “I gain more information individually [one-on-one] rather than in a general setting because I think it’s just my personality.” He implied that this was one important reason he did not speak much in whole-class discussions, while proudly acknowledging his engagement in the inter-group conversations. He added that engaging in whole-class discussion may be more difficult for “people that they have accents or that they don’t feel comfortable speaking because they think that maybe their English is not that great.”

Finally, while Luis found certain aspects of whole-class discussions to be more challenging than small-group discussions, he also identified one challenge with small-group discussions. Specifically, he noted it was initially difficult to engage with peers because he didn’t know anyone in his group: “At first, it’s a little bit difficult [to communicate with group peers] because you don’t know anyone.”

Normative Demands. As noted in Table 5.15, Luis encountered a student-centered norm incompatibility related to what it meant to be a competent student participant in the IOLA class. In alignment with the lecture-based math courses he had previously experienced, he had internalized that participating competently in class included identifying important information and writing notes. While he was often able to engage partially in this norm through consistent notetaking, it was usually not clear to him which topic the instructor was teaching while he was in class:

> *During class, I was a little–at first, obviously, like I said, the first couple of days, you’re a little bit lost when it comes to an assignment ((IOLA task)) and stuff. But then, as soon as I went home and looked at my previous notes, I was like, “Oh, okay, yeah, I know what she’s talking about now. She’s talking about span.*

Fortunately, he was later able to identify the topic at home when he compared his notes from the IOLA class with his notes from his prior linear algebra course.

Table 5.15:

Normative Demands Experienced by Luis in the IOLA Classroom

Norm Trait	Norm Tension	Norms or Situational Aspects	
		Luis or His Community Outside IOLA Classroom	IOLA Classroom Community
Competent student participation	Student-centered incompatibility	Identifying important information and writing notes	Not clear which information was important

Instructional Language Resources

In this subsection, I describe the instructional aspects (e.g., the IOLA tasks and the instructor's communication) in the situational and normative dimension of the L-S-N framework that Luis experienced as linguistically helpful. (No instructional resources were identified in the lexico-grammatical dimension for Luis.)

Instructional Situational Resources. Luis reported various instructional situational resources he experienced in the IOLA class along the following components: material, semiotic, and sociocultural. (No instructional resources were identified within the activity component.) Tables 5.16 and 5.17 provide a brief overview of these findings according to two sources of communication in the IOLA class: communication within the tasks and more general classroom communication.

Table 5.16:

Instructional Situational Resources Experienced by Luis in the IOLA Tasks

Component	Task Feature as Situational (Material) Resource	Task Example
Material	Fewer words	Task 4 (The Carpet Ride Problem: Getting Back Home)
		Task 6 (Practice for Individual Quiz)
	Straightforward questions	Tasks 5 (Linearly Independence and Dependence: Generating Examples)
		Task 6 (Practice for Individual Quiz)
	Familiar problem contexts	Task 6 (Practice for Individual Quiz)
	Repeated story contexts	Task 1 (The Carpet Ride Problem: The Maiden Voyage) & Task 2 (The Carpet Ride Problem: Hide and Seek)
		Task 9 (The Car Rental Problem) & Task 10 (The Car Rental Problem: Follow-Up)
Semiotic	Primacy of symbolic mathematical language	Task 6 (Practice for Individual Quiz)

Table 5.17:

Instructional Situational Resources Experienced by Luis in the IOLA Classroom's General Communication

Component	Classroom Communication Tool	Feature of Tool as Situational Resource
Semiotic	Supplementation of speech communication with gesturing	Visual
	Supplementation of speech communication with writing	
	Supplementation of verbal representations with graphical representations	
Sociocultural	Small-group discussions	More private with the opportunity to provide or obtain clarification
	Whole-class discussions	Wide audience with access to more mathematical ideas

In terms of the material component, Luis identified four linguistic features of some tasks that he found helpful: (a) fewer words, (b) straightforward questions, (c) familiar problem contexts, and (d) repeated problem contexts. For example, he found that Task 4 (The Carpet Ride Problem: Getting Back Home) and Task 6 (Practice for Individual Quiz) had fewer words, which he appreciated. In addition, he ranked Task 5 (Linearly Independence and Dependence: Generating Examples) and Task 6 (Practice for Individual Quiz) among the least linguistically challenging because they had straightforward questions, where he could identify exactly what they asked him to do mathematically:

> *((Referring to Task 5)) You're able to just list a bunch of vectors that they want over here and over here to answer the questions. And this one's here ((Referring to Task 6)), yeah, also the general wording makes it a little bit nicer to get exactly what they want.*

Tasks 5 and 6 were ranked 10th and 11th, respectively, out of 12.

Additionally, Luis found it helpful when tasks included familiar problem contexts based on prior knowledge about a mathematical topic. For example, as I will highlight later in Student Situational Resources–Material, Luis was familiar with the language and topic of row reduction (e.g., "Find intersection") and so considered Task 6 (Practice for Individual Quiz) easy to understand.

Finally, Luis benefitted linguistically from repeated story contexts across tasks, provided that he understood the story context the first time it showed on a task. For example, although it took him a while to understand Task 1 (The Carpet Ride Problem: The Maiden Voyage), when he saw the same problem context about the carpet ride from that task in Task 2, Task 2 became easier to unpack linguistically than Task 1. Likewise, he reported Task 10 as being linguistically more accessible than Task 9 because it repeated the context about a car rental company. However, when Luis encountered a task with a repeated problem context from a prior task that he did not fully understand, he found the latter task more linguistically challenging than the former. This happened, for example, with Tasks 7 (Meal Plans: Constraining the Number of Meals) and 8 (Meal Plans: Constraining the Cost), where he ranked Task 8 more linguistically challenging than Task 7.

In the context of the semiotic component, Luis attended to the mathematical language system in which a task was written as well as to the use of communication mediums and types of mathematical representations by the instructor and other students during whole-class discussions. For example, Luis found it helpful when some of the IOLA tasks were primarily presented in symbolic mathematical language rather than with a story context in natural language. This was one reason he ranked Task 6 (Practice for Individual Quiz) as one of the less linguistically challenging tasks (10th out 11). He pointed to the system of equations provided in this task and noted that all he needed to do was to solve the system.

In addition, Luis found it helpful when the instructor and other students in whole-class discussions combined communication mediums and types of mathematical representation. In particular, he appreciated that the IOLA instructor regularly supplemented speech communication with both writing and gesturing. For example, she promoted this supplementation when she asked different small groups to share their mathematical ideas during whole-class discussions while simultaneously showing their whiteboards:

> *She ((the IOLA instructor)) used to display them ((whiteboards from other small groups)), and then she used to write everything that she wanted on her own board and stuff. But I think she used to write more and then refer back to the other teams more, and stuff. But she was always pointing, "Okay, you guys did this, did that," and it kind of added on to whatever she was trying to explain.*

Luis especially reported benefitting from the teacher's hand gestures because they added a visual and emphasis to the mathematical ideas she communicated, as such as when she gestured the intersections of planes:

> *Sometimes you may not really understand that something ((e.g., planes)) intersects, but then you emphasize with your hand movement that you're making a cross or something, so you know something is getting in contact or something. So that ((the IOLA instructor's gesturing)) really adds on to it. And I think it really helps out other people that maybe they're not fluent, as you were saying, in English, for example.*

Similarly, Luis found helpful that the IOLA instructor promoted the supplementation of verbal representations with graphical representations. For example, he noted that this supplementation made it easy for him to gather what other small groups shared during whole-class discussion:

> *Generally, like I said, with other students [during whole-class discussion] maybe speaking, it's the most common way. But the fact that we have graphs and also*

> *different representations of what they're doing, it adds more meaningful information to your repertoire, if that makes sense.*

He found that the graphical representations helped him "see how they're [his classmates'] thinking."

Finally, as alluded to in Situational Demands in the context of the sociocultural component, Luis found that in contrast to whole-class discussions, small-group discussions involved speaking more privately with a smaller audience, thereby making him feel more comfortable and facilitating communication in various ways. In particular, the small-group discussion participant structure did not require speaking loudly nor explaining mathematical information succinctly. In addition, by working with the same small-group peers over time, he was able to build trust ("open up") and relate to them, which further facilitated communication between them:

> *Sometimes I feel like having a small group, it's just the perfect balance for people that don't really talk too much in general in class since they get to only talk to a few members, which they already feel comfortable talking to. And then learning from them and sharing their experiences throughout the assignments ((IOLA tasks)) and all the stuff that they've been working on, it helps a lot to relate with only a few students.*

Yet, while Luis found certain aspects of small-group discussions to be more beneficial than whole-class discussions, he also reported one advantage of whole-class discussions. Specifically, Luis experienced whole-class discussions as helpful because they allowed him to access more student ideas:

> *I feel like there's benefits speaking in big groups as well, too, because you get to see how the class is standing when it comes to a topic. Because maybe since you work on small groups, you might be learning in that small group, but then other small groups are not really getting certain information, and they're not learning.*

This quote indicated that revisiting and expanding a topic from the small-group discussions in the whole-class discussions was beneficial. Conversely, he found it helpful to apply the more general ideas from whole-class discussions into his small-group conversations:

> *It also helps other group members as well through that general solution technically, which we get to discover through other group members as well, not just you're own.*

In sum, Luis reported appreciating the coordination of small-group and whole-class discussions.

Instructional Normative Resources. As indicated in Table 5.18, Luis reported four instructional aspects of the IOLA class that induced specific ways of being and communicating during class and supported his mathematics learning: (a) the nature of the instructor's communication, (b) the regularity of instructor's references to outside class materials, (c) the structure of the class lesson, and (d) the nature of interaction.

Table 5.18:

Instructional Normative Resources Experienced by Luis in the IOLA Classroom

Tool of Communication	Typical Feature as a Normative Resource
Nature of instructor's communication	Positioned students as valuable members of the IOLA classroom community
Regularity in the instructor's references to outside materials	Helped identify the topic(s) of a lesson
Structure of the class lesson	Combination of different participant structures of support (e.g., individual and small group)
	Access to different audiences of support (e.g., peers and instructor)
Nature of interaction	Centrality of peer communication and interaction

First, Luis found it helpful that the instructor communicated and behaved like a more capable peer during small-group discussions, thereby positioning students as valuable members of the IOLA classroom community. For example, when I asked him to describe the role of the instructor in the IOLA class, he said:

> *She's a professor and everything, but it feels more like another classmate since we're pretty much free of engaging with other students. She's also there giving us some input and stuff related to a problem… And they ((the IOLA instructor)) come up to you and go, "Oh, you're doing this," or, "You're setting up like this. Let me just take a look and understand a little bit of what you're doing." And then after they collect a lot of different approaches from other groups, they let other students share their own ideas. And then from there, we can make a general sense out of all the different ideas.*

Luis felt that the teacher communicating more like another classmate balanced the playing field between the students and her, which opened opportunities for students to engage with one another and contribute their own mathematical ideas during class discussions. As part of this balance, he appreciated that the instructor scaffolded the conversations in a way that simultaneously provided feedback and allowed students the freedom to explore their own ideas.

Second, as noted in Normative Demands, Luis noted he struggled to identify the topic in a given lesson during class. Yet, he said the IOLA instructor linked the class material to its associated section in the class textbook or homework, and this helped him retrospectively identify the class topic at home:

> *Whereas here in a group setting, it's a lot easier because whatever we do in class, the professor says, "Oh, this is going to be related to this section in the book or this section in the homework." And since she had already beforehand told us about it, it's a lot easier to keep track. "Oh, okay, this section belongs to this."*

Third, Luis also found the daily structure of a lesson helpful. He reported that typically students had opportunities to communicate and develop mathematics in their small groups, then share and compare their thinking with their peers in both small-group and whole-class settings, and finally have the instructor help them solidify ideas:

> *Like I mentioned before, since we're doing teamwork in this class, it is very easy to share everybody's thoughts into whatever task we're assigned. Yeah. So,*

everything flows more smoothly. And then also the fact that whatever share that you put into a teamwork assignment, other teams, not just my own, but other teams also get to share what they did. And then the professor. She's able to keep track of everybody's approach into a problem, and then she can also add on to whatever they're doing. And you start to compare other people's work with your own, and then everything clicks into place, which I feel like it has made the class a lot more smoothly compared to other classes where it's all lecture and pretty much learning on my own sometimes.

Fourth, more specifically, Luis reported that the centrality of peer interaction, relative to his prior lecture-based mathematics classes, was helpful. For example, having more peer communication in both small-group and whole-class discussions helped him clarify and build on each other's ideas:

> *But a group setting, it may be a little bit more beneficial since we're having that interaction, which may help you clarify some more things as opposed to learning on your own.*
>
> *When you want to add something that maybe one student forgot to mention or you thought it was meaningful ((during whole-class discussions)), you just can raise your hand and add whatever information you thought it was relevant to their answer, which helps out in the classroom.*

Luis added that sharing ideas within and between groups in class was beneficial because "working in teams like that, you get to learn from other approaches." Ultimately, Luis linked the typical structure of a lesson and the centrality of student communication to a positive experience in the IOLA class. He summarized his experience as follows: "I think it's one of the classes that has been engaging for me and yeah, had a really good experience so far."

Student Language Resources

In this subsection, I describe the language resources that Luis drew from himself or his peers across the three L-S-N dimensions: lexico-grammatical, situational, and normative.

Within the lexico-grammatical dimension, Luis reported utilizing one semantic resource: prior knowledge of mathematical terms (listed in Table 5.19). In the linear algebra course that

Luis took before the IOLA class, he had gained knowledge about various mathematical terms that he applied later in the IOLA class. These terms included “row reduction,” “Gaussian Elimination,” “Cramer’s Rule,” “span,” and “linearly independent.” Knowing these terms helped Luis in navigating the communication in the IOLA class because when he couldn’t understand a math term or the topic of a discussion in the IOLA class, he often attempted to connect it to the math terms and topics he had learned before:

> *I think during class. I was a little–at first, obviously, like I said, the first couple of days, you’re a little bit lost when it comes to an assignment and stuff. But then, as soon as I went home and looked at my previous notes, I was like, “Oh, okay, yeah, I know what she’s talking about now. She’s talking about span.”*

Table 5.19:

Student Lexico-Grammatical Resources Utilized by Luis in the IOLA Classroom

Student Lexico-Grammatical (Semantic) Resources	Example	Context of Example
Prior knowledge of mathematical terms	“Gaussian Elimination” “Cramer’s Rule” “row reduction” “span” “linearly independent”	General classroom communication

In the context of the situational dimension, Luis reported using various tools of communication that he drew from himself or his peers in the IOLA classroom to support his mathematics learning. All the tools he referenced fell in the material, semiotic, and sociocultural components of communication. (No situational resources were identified along the activity components.) Table 5.20 provides an overview of these resources.

Table 5.20:

Student Situational Resources Utilized by Luis in the IOLA Classroom

Component	Student Communication Tool as Situational Resource	Resourceful Feature of Tool	Example Context Where Tool Was Used
Material	Prior linear algebra notes	Interpreting IOLA classroom communication	General classroom communication
Semiotic	Supplementation of speech with gesturing	Facilitate expression and afford visualization	
	Distribution of communication mediums and mathematical representations among group members	Collectively affording various ways of communicating and interpreting information	
	Using primary language(s) for non-mathematical talk	Relating to their group peers to facilitate communication	
Sociocultural	Inter-group communication	Expanding audience of communication	

As alluded to earlier in the presentation of lexico-grammatical resources, Luis reported using one material tool of communication: His notes from his prior linear algebra class. He used his notes often to connect his new learning to his prior learning, thereby facilitating his interpretation of the mathematical communication in the IOLA class.

In terms of the semiotic component, Luis reported intentionally leveraging mediums of communication, mathematical representation types, and multiple language(s) in intentional ways. For example, Luis reported that he often gestured with his hands to learn mathematical ideas during the IOLA class: “So I tend, as I’m talking, move my hands a little bit to connect the dots.” In addition, gesturing helped him explain ideas to his peers that would otherwise “be difficult to explain…[only] through words." He felt this left his peers with "a better connection of seeing how things relate in a more detailed way."

Luis also noted that, although speech was the medium that his small group used the most, each group member took on their preferred communication mediums and types of mathematical representations to support the group in learning mathematics. As noted above, Luis often used gesturing to communicate mathematically. Luis also reported the main communication mediums and representations that other group members used (in addition to verbal communication):

> *I remember Kaden, he used to move his hands, too, a lot. I remember that when he was explaining stuff to the professor or to us, he used to move his hands a lot, which, like I said, I can relate to that since…I tend to do that as well… And like I said, it helps me out to process all that information. And yeah, Nam, for example, he didn't really move his hands too much. But I remember that he used to draw stuff on the whiteboard and stuff. And it also helped me out too because I was seeing the physical thing written on the board…And then Johan, too, I remember that…in most of the matrices problem that we got, he used to do systems of equations. And that also helped out too, just to see how things are broken down into smaller pieces and also helped me out to process all that information.*

In particular, Luis noted he found it helpful that Kaden regularly used gesturing, while Nam drew graphical representations and Johan wrote equations on their small-group whiteboards.

Finally, recall from Language Demands that Luis initially found it difficult to engage in small-group discussions because he didn't know anyone in his group, so he couldn't relate with his peers. To initiate and sustain conversations and build relationships with his small-group peers, they occasionally spoke about non-mathematical topics, including how to say certain phrases in their primary languages. For example, while acknowledging that his small group only used English for mathematical conversations, Luis noted that he and Nam spoke about the meaning of a few words or phrases in Vietnamese:

> *My group peers shared some words, but it was more like outside the topic of the class... I asked them: "How do you say this in your language?" Or something like*

a glass of coffee or I want water or basic things, but never related to class. We never shared anything ((mathematical)), since everything was shared in English.

As soon as you start introducing yourself and getting to know a little bit more, and then obviously, getting to share your part in the class at first, it's like other students feel more comfortable with you as well. So everything flows more smoothly as time keeps going.

In terms of the sociocultural component, Luis leveraged an inter-group participant structure that was induced by a participation assignment in the IOLA class. As described in Situational Demands, this assignment required gathering pictures and making sense of another group's whiteboard near the end of class each day. Although the assignment did not require speaking to others, Luis took the initiative to ask questions about their whiteboards:

I remember that I was like, well, I'm going to go take a picture of their board. And then I knew that that member specifically was in that group. So that way, when I took a picture, I was like, hey, how do you solve this?

In addition, Luis was intentional and strategic with choosing the people who communicated with for this participation assignment by focusing on those students who knew more than him or had certain types of information:

I tend to pinpoint… who's better at explaining stuff or who participates most sometimes in general in the classroom. Or sometimes there's not people that talk a lot, but…when it comes to doing group settings and stuff, they seem a little bit more reserved in the way that they act. Sometimes you can tell that those guys are more focused on what they're doing, which it also helps out to talk to them more privately in a way and explaining stuff around or just getting to know them in general when it comes to how they solve things. And yeah, sometimes they have the most valuable information.

Ultimately, he brought the mathematical knowledge he gained from his inter-group conversations back into his small group:

Since you're learning from them, you also get to share with other small team members or other members in your group, which they also have the right answer, but maybe the methods are a lot longer and sometimes you need to shorten things just for the purpose of finish things on time, which, yeah, that's–like I said, I'm more of a an easier approach type of guy for everything.

Finally, in the context of the normative dimension, Luis reported applying one norm from his prior lecture-based courses in the IOLA class: the importance of recognizing key information from the instructor's communication and writing notes. For Luis, this norm constituted competent student participation in the class and it functioned as a resource (see Table 5.21). He reported benefitted from using this norm in the IOLA class for learning mathematics and preparing for exams, even while he acknowledged notetaking was not a norm of that environment:

> Luis: *So I personally, for example, when it comes to taking notes, as you're saying, I'm a personal note taker. I just get on my notebook and as soon as the professor ((in the IOLA class)) starts talking about a particular problem in the classroom, I'm taking notes, or if she's drawing something on the board, I just automatically copy whatever she's doing. Because as she's talking and writing, I get to keep track of that as I write as well. So I tend to learn as I write stuff, too, the stuff that she drew on the board. And I'm also writing on my notes. Since she already discussed that, and then the fact that I have it physically in my notes, it's a way of like, oh, she explained this with this, and you can highlight stuff. So it's more of a natural thing. And that not only applies to that class, it applies in general to all my classes. I just take out my notes and as soon as a professor starts discussing stuff, I just take notes and gather everything. As you were saying earlier, an inquiry-based class or a lecture class, I treat them like– even though this is an inquiry class, I treat all my classes like a lecture class. I just take notes so that I can see what to expect on my exams and different other perspectives in the classroom.*
>
> Interviewer: *Do you feel like this is a norm in this class to take notes, or?*
>
> Luis: *Not really. In this class, it's not a norm. You can actually work your way without taking notes since you're getting input from other students as well. You can learn through visualizing, as I was saying, since we do a lot of graphic representations and stuff. This is, like I said, more of a personal type of thing on my own. I like to take notes because I'm more of a visual guy too. But there's people that don't really take notes, and I've seen them in the classroom. They just gather a few important points, and that's more than enough.*

As these quotes illustrated, for Luis, being a competent student included being able to write notes. Taking notes was resourceful for him because he learned through writing them and because they served as a physical reference where he could highlight important information. Ultimately, he used these notes to see what to expect on exams and capture different perspectives about approaching mathematics.

Table 5.21:

Student Normative Resources Utilized by Luis in the IOLA Classroom

Norm Trait	Norm as a Resource
Competent student participation	Identifying important information and writing notes

Conclusion

This chapter presented case study analyses of two students, Seok and Luis, and addressed the research questions. I investigated each question related to language demands, instructional language resources, and student language resources through the three sociolinguistic dimensions of the L-S-N framework: lexico-grammatical, situational, and normative. The findings showed that both students experienced a few language demands in the IOLA classroom, several reliable instructional language resources and a variety of student language resources.

There were important similarities in the demands that Seok and Luis reported experiencing. For example, both students experienced only two lexico-grammatical demands, with one being the expected language demands in any new class (new mathematical terms). Both students also experienced relatively few situational demands. More significantly, the situational demands they reported experiencing were not directly linked to any semiotic demands (issues with different language systems) in the general classroom communication. For example, none of these demands were related to uses of speech communication, verbal representations, or the English language. Instead, while both students noted a challenge with some tasks having too many words, the situational demands they experienced were more focused on difficulties with engaging in different sociocultural structures (e.g., a whole-class participant structure).

At the same time, each student experienced divergent language demands based on their individual and cultural-linguistic backgrounds. For example, out of the two lexico-grammatical demands that each experienced, Seok's second demand was about unfamiliar vector notation whereas Luis' second demand was about context words like "Cramer" and "Gauss" he found to be misleading. This difference is sensible, given that Luis had taken a prior linear algebra course, where he had seen such notation and developed predisposed meanings of those context words, while Seok had not. In addition, in the situational dimension, Seok reported experiencing issues with the problem contexts and genres of the IOLA tasks (e.g., unfamiliar problem contexts, contexts that were not straightforward, and requests of mathematical explanations), while Luis didn't report such issues.

Finally, in the normative dimension, Seok reported encountering a significantly greater number of norm tensions in the IOLA classroom compared to Luis' reported experiences, and they differed in the types of norm tensions they reported. Specifically, Luis only reported one norm tension, a student-centered incompatibility. In contrast, Seok reported four norm tensions, of which two were class-centered incompatibilities and two were violations. These differences between Seok and Luis with respect to norm tensions indicate that Seok experienced more direct conflicts with the norms of the IOLA. Indeed, student-centered incompatibilities represent the lowest degree of conflict between the IOLA class and a student's community outside the class. Two possible reasons for these differences are that Seok felt less comfortable than Luis with the English language and had fewer experiences with active learning courses. The plausibility of these explanations may be seen, for example, in the fact that one class-centered norm incompatibility that Seok experienced was with respect to the permissible language of teaching and learning. Moreover, the two norm violations that Seok experienced were related to differences between the IOLA class and prior courses with fewer elements of active learning. For example, one norm violation contrasted the complexity of explaining and justifying one's answers in the IOLA class, in contrast to simply giving answers in prior classes.

In terms of instructional resources, Seok and Luis reported similar experiences. For example, both students reported experiencing the following as reliable instructional situational resources: supplementation of the teacher's speech communication with gesturing, writing, and graphical representations. Moreover, both Seok and Luis found small-group discussions helpful because, relative to whole-class discussions, they provided a more private space for communication and afforded more opportunities to provide and obtain clarification on communication.

In addition, both Seok and Luis described a variety of instructional normative resources, with Luis identifying one more resourceful norm than Seok. Both students reported benefitting from the centrality of peer interaction and pointed to helpful regularities in the teacher's communication that helped identify the topic(s) of each lesson. However, one salient aspect Luis reported experiencing, that Seok did not, was related to the nature of the instructor's communication. Specifically, Luis reported that the instructor communicated and behaved like a more capable peer during small-group discussions, which he felt positioned students as valuable members of the IOLA classroom community. These findings align with the interpretation that, in general, both Seok and Luis viewed the norms of the class as helpful, with Luis finding them even more beneficial.

Finally, in terms of student language resources, Seok and Luis both students relied on their own normative resource of identifying important information and writing notes as a way of positioning themselves as competent participants in the IOLA classroom. It appeared that both students were able to sustain their notetaking practice throughout the IOLA lessons. In addition, both students relied on a variety of mathematical representations, including graphs and equations. However, they were stark differences between Seok and Luis in the other student language resources they reported utilizing. For example, in terms of communication mediums, Luis reported mainly using speech and secondarily gesturing for communication, while Seok relied almost exclusively on writing. Moreover, while Seok reported having trouble relating to his small-group peers and engaging in speech communication, Luis reported successfully drawing on his multilingual peers' primary language(s) to relate to his group peers and facilitate communication between them. Finally, Luis reported drawing on his prior linear algebra notes, while Seok did not have such resource available, since he had not taken a prior course on linear algebra as Luis had done. It is then plausible that these differences in student language resources between Seok and Luis may have been a result of a combination of factors, including previous mathematical experiences (e.g., taking a prior linear algebra course or not), group dynamics (relating to his group peers or not), and the extent of the student's comfort in using speech communication with their group peers.

Overall, this chapter exemplified the variety and salience of language demands that multilingual undergraduate students who have spent part of their K-12 education in the U.S. might face in inquiry-based mathematics courses. Simultaneously, this chapter provided examples of instructional aspects that might function as language resources for such students.

Most importantly, this chapter highlights the ways that such students might leverage these and their own resources to navigate language demands and engage in mathematical sense-making.

In particular, this chapter highlighted the cases of two students, Seok and Luis, illustrating how their personal and cultural-linguistic backgrounds mediated the language demands and instructional resources they reported experiencing as well as the student language resources they reported utilizing. Both students experienced a few language demands and experienced an abundance of reliable instructional resources, possibly due to having been in the U.S. for a significant number of years prior to this dissertation study. However, potentially as a byproduct of differences in previous mathematical experiences and instructional settings, there were stark differences in the normative demands that Seok and Luis experienced as well as in the student language resources they utilized.

Chapter 6 Case Study Findings from Inquiry Instructor

In this chapter, I analyze the perspective of the instructor of one inquiry-oriented linear algebra (IOLA) course at Southwestern University regarding the language demands and resources for multilingual students. She identified as white and reported being a predominantly English speaker.

At the time of the study, the IOLA instructor was a recognized teacher-researcher, who had prior experience with teaching IOLA for more than 10 years and focused on IOLA as one of her research areas. For her teaching, she reported adopting the four principles of inquiry-oriented instruction described in Kuster et al. (2018): (a) generating student ways of reasoning, (b) building on student contributions, (c) developing a shared understanding, and (d) connecting to standard mathematical language and notation. During the teacher interview, she exemplified how she implemented the first three principles. (She did not discuss how she integrated the fourth principle.) She integrated the first principle through promoting class discussions with different participant structures and using open-ended tasks that "allowed students to have a chance at thinking about something." In addition, in order to implement the second principle, she externalized students' mathematical thinking through encouraging students to write their ideas on their small-group whiteboards and present their work with the whole class. She then tried to build on students' ideas by encouraging students to connect ideas between the whiteboards. More generally, for her, "building on students' contributions is not just that you have students state their contributions, but you interweave them, and you connect them to what you know about the mathematics." Finally, to facilitate the accomplishment of the third principle, she tried to focus the class discussion on a few important mathematical ideas.

She was involved in the design of the nationally and publicly available IOLA curriculum developed by Wawro et al. (2013). She drew from this curriculum to teach the IOLA course in this dissertation study because she felt that the associated tasks were open-ended and, thus, aligned with her principles of instruction. She adapted the use of this curriculum for her class demographics. The IOLA course she taught was designated for students in STEM degree programs. Most of her students were predominantly engineering students who had previously taken Calculus 3 and were in the third and fourth years of their undergraduate studies. Many of her students were also from non-traditional backgrounds, including students in their thirties or older. This may have been a byproduct of the course being situated in a satellite campus of the university.

The overall design of her IOLA course was task-based and discussion-based. Most of the tasks she used came directly from the curriculum developed by Wawro et al. (2013). Other tasks came from an expanded version of this curriculum, which had not yet been made available to the public. The remaining tasks were group quizzes that she designed individually for her own courses. In addition to these tasks, she implemented individual quizzes, exams, and a participation assignment. The participation assignment required students to gather pictures of another group's whiteboard near the end of class each day. Then, for homework, students were asked to describe the mathematics on the whiteboard they photographed and compare it to their own work. The students were expected to discuss and solve the tasks in small groups, and then present and discuss their solutions with the whole class.

In this chapter, I report the findings addressing the three research questions, focusing on identifying the language demands, student language resources, and instructional language resources for multilingual students. The instructor often identified language demands and resources for all students, while occasionally highlighting how a demand or resource may be more pronounced for a multilingual student. Therefore, the style of presentation of this chapter preserves her approach: I present the findings she reported for all students, while highlighting the salience for multilingual students, when she described it. To further emphasize the relevance for multilingual students and leverage this chapter as context for the previous student case studies, I make explicit connections between the instructor's perspective and the findings from the student case studies as I discuss each finding from the instructor.

The instructor reported one student language resource and a variety of language demands and instructional language resources. The student language resource that the instructor highlighted was that students can leverage some everyday meanings of certain mathematical terms. For example, she noted that "span" has an everyday meaning that closely matches its mathematical meanings: "Let's think about the span one ((term))…. 'Oh, it spans the whole space' like with hand motion. 'It spans. It covers…some English language meanings of span can be relatively helpful.'" This general finding connects to Seok's specific use of the everyday meanings of "independent" and "dependent" for making sense of the terms "linearly independent" and "linearly dependent." In the two sections below, I present the language demands and instructional language resources that the instructor reported in the context of the IOLA course.

Language Demands

In this section, I describe the language demands that the instructor identified for students in the IOLA class across the three L-S-N dimensions: lexico-grammatical, situational, and normative.

Lexico-Grammatical Demands

The instructor highlighted three lexico-grammatical demands for students in the IOLA class: (a) new mathematical terms, (b) unfamiliar vector notation, and (c) terms with multiple meanings. The first two demands were similar to those that the case-study students highlighted, and the third one was an additional demand observed by the instructor. For example, the instructor noted the following about new mathematical terms: "I think any new technical term tends to be challenging, like span, linear independence. I mean, I think we're still struggling now, even though we did it the first two weeks of class."

The IOLA instructor also provided insights regarding the challenges students might face with unfamiliar notation, especially in the context of her students. She noted this vector notation is unfamiliar for many students because they are used to seeing the pointy-bracket and horizontal-form vector notation from their prior Calculus 3 classes or the i,j vector notation from their prior physics courses. In addition, she noted she has observed students mix the order of x and y in the vector notation. She believed one plausible explanation might be that students associated the meaning of vector as going over and up on a grid with the concept of slope, which involved a change in y over a change in x, leading them to conclude that y was the top entry and x was the bottom entry of a two-dimensional vector:

> *I think it ((unfamiliarity with the vector notation)) just has to do with past experience in math classes…So in our class, we use vertical vectors, which are written in this way where the x is on top and the y is on the bottom. And vertical vectors can also be… thought of as a dot…Or I could think of that as starting from the origin…And with Magic Carpet ride ((Task 1)), it could be you did something else first and you're over at some other dot, and then you're going to go over one and up two from that other dot, right?...It's partially weird [for students] because prior to this, vertical vectors were not the way they wrote vectors. So I'll get to the real reason that I think you brought it up in just a second. But first, I want to just say that in their Calc 3 class, they mostly use this vector notation, the pointy brackets. And a lot of people teach this linear algebra*

class to people who are a little bit younger and haven't had Calc 3 yet, but my students, most of them have had Calc 3.

And then the other one that they may have seen in Calc 3, but they may have also just seen in physics…The i,j thing ((vector notation))...And so then to use the vertical notation is weird for them anyway, just because it's new. But the thing that makes it more confusing, to circle back, is the fact that you're going over and up. So the fact that you're thinking this is a direction vector…But the most common time when students think about sort of–if not directing, slope, right?...And when we do tiltiness of the line–slope–...we think of change in x over change in y which is a case where the x is on top and the y is on the bottom ((Based on the context of this quote, the instructor misspoke here; she really meant to say x when she said y and vice versa)). And the first day of class, you'll sometimes see people put a little line like that ((between the two entries)) on their vector, almost thinking of it as a fraction…And so…it ((a vector)) might be thought of as more like this ((a fraction)), and that's a switch from what people are used to…. But that's something that occurs every semester.

In addition, the instructor reported a lexico-grammatical demand that students might experience in the IOLA class with terms that have multiple meanings. As noted before, the instructor noted that span is one such term, highlighting its everyday meanings as potential student language resources for starting to make sense of the mathematical meaning. On the other hand, she noted that the everyday meanings of span can also pose challenges for students because those meanings may not match exactly the mathematical meaning. She acknowledged that such lexico-grammatical demands may apply to all students, but may be even more complex for multilingual students:

There can be words that, in English, kind of casual language, the language you always used growing up, mean certain things and maybe mean a range of things…But then in math, they may get used in a very specific way that may or may not be a subset of that range of things you learned that that word means. Even span is an example of that. So that's a challenge for students. And that's normal. We know that as researchers. Now, there may be an additional level of that for someone who's very experienced in another language because that word or a cognate of that word, a close word, might have even other meanings. So maybe the word for English speakers is like, "Yeah, that's one of those meanings I know for that word, the math meaning." But maybe for other speakers, it's like, "Oh, I thought they meant this other thing."

She provided one example of a mismatch in meanings that students may experience with the term “span,” noting how an everyday meaning may not fully support students in thinking of it as something more discrete, as being generated from specific vectors:

> *Sometimes textbooks would like more using words like generating vectors or something like that. Because I think it’s span is a little bit–it makes me think of like I’m going to take butter and spread it as opposed to something a little more discrete.*

For terms like “dependent” and “solution” that have multiple meanings, students might interpret an unintended meaning in the context of linear algebra. For instance, the term “dependent” when discussing “linear independence” might be erroneously interpreted in the context of calculus, where “dependent” typically describes a characteristic of a single variable in relation to another “independent” variable. In contrast, the term “linearly dependent” in linear algebra is defined as a feature of a set of vectors, even while the way people speak about linear independence may make it sound like it’s a feature of a vector. The concept of linear dependence is meant to apply to a set with any finite number of vectors (not simply a comparison between two vectors).

Another example of a term with multiple meanings that the instructor has observed students use unintendedly is “solution,” when discussing solutions of a vector equation. She has seen students interpret “solution” as the right-hand side of a vector equation, when she really meant the vector that satisfied the vector equation. She went on to explain how this student interpretation of “solution” is sensible given students’ prior schooling experiences:

> *The other one ((linguistic challenge)) I was going to mention is “solution,” right?...So you’re like, “Oh, let’s talk about the solution to this vector equation.” So, it’s kind of natural to say, “Oh, the solution is the thing after the equals sign on the right-hand side,” right? Because that’s the most normal way or most often occurring way it is, right, in previous courses.*
>
> *It’s like you do some sort of calculation, just a simple one here, and then even from elementary school, you do a calculation, and then what you get, you put over here on the right-hand side, and that’s the solution, right? But here ((in the IOLA*

class)), we don't mean that. We mean the solution is the x's and y's that make this ((equation)) true.

Situational Demands

Below, I report the language demands that the instructor identified for students in the IOLA class within the situational dimension. I organize these demands according to the following components of this dimension: material, activity, semiotic, and sociocultural.

In terms of the material component, the instructor highlighted three features of some IOLA tasks that might induce language demands for some students: (a) too many words, (b) problem contexts that were not straightforward, and (c) unrelatable problem contexts. The first two features were also noted by the case students, and the third feature was reported solely by the instructor. For example, regarding the number of words on some tasks, the instructor acknowledged: "Just reading the task sheet for the activity. Usually, it's fairly dense. It has quite a few words on it."

In addition, the instructor reported observing students struggling with understanding the problem context in Task 7 (Meal Plans: Constraining the Number of Meals). She also shared the tensions faced with wording this task as part of the research team that designed it. On the one hand, she wanted to communicate to students the quantitative relationships and constraints embedded in the task. On the other hand, she wanted to do it in a way that still allowed students to engage in the practice of mathematizing (that is, figuring out the math in the problem context), so she didn't want to express the relationships directly in mathematical language.

So I think there has been a lot more wording issues with this one ((Task 7)).... as a team…we've gone through several iterations…As we're developing new tasks, we're like, "Does this make sense? Should we word it differently? Should we ask a little different question?"...I don't say this one is perfect, but I know it was worse…It's weird because we're trying to communicate that breakfast plus lunch plus dinner has to equal 210 without saying that directly…And then we're trying to communicate something about, "Oh, basically, the whole thing is we want them *to have the boundaries for breakfast, lunch, and dinner to be between 0 and 210." But we're not trying to say that directly in a mathematical way…And so then people ((students)) are like, "Oh, 105 days per semester and 2 breakfasts." They want to be like, "Do I have to say how many meals I'm going to eat each day? Can I have three breakfasts on one day?" And I'm like, "It doesn't matter because we just want the total number of breakfasts for the semester." And there's lots of wording issues with this one, and I am more hypersensitive about them because it's a new task.*

She highlighted that the wording of the problem context of the task was the best way they could word it so far, while acknowledging that it could induce unintended linguistic complexities for some students:

((Students could be)) like, "Why are you telling me that there's two breakfasts per day for 105 days? Am I supposed to use that somehow? And…what does that imply about the mathematical situation and what I'm being asked to do in this mathematical situation?"

Finally, the instructor identified the potential for some IOLA tasks to be unrelatable to some students, which the IOLA design team worked to address. For example, she pointed to Task 1 (The Carpet Ride Problem: The Maiden Voyage), which asked students to imagine themselves as a young traveler. She explained that this problem context might not be relatable to the majority of her students "because I often have a lot of older students [than the typical nineteen-year-olds]," who might not identify as "young" travelers. She described this issue after describing the serious consideration the IOLA design team has given toward addressing it:

This ((accessibility)] is a very challenging issue that we have discussed a lot in our research team. I think we've always had the goal, is that they were very accessible…Let's start with an example…The Magic Carpet text…it's written…like, "You are a young traveler,"...So I've always had a little bit of a problem with that because I think that was written for people who are teaching students who are sophomores at a kind of major research institution where all their students are coming straight through.

Within the activity component, the instructor acknowledged a potential challenge for students with explaining mathematical ideas, which some case-study students also reported experiencing. The instructor added that this could be a challenge even for monolingual students:

> *Well, people have different, I don't know, comfort levels or experience levels in, say, explaining their ideas to someone else. Even if you're a monolingual speaker, you might not have a lot of experience at that. Or even if you're a monolingual speaker, it might not be something that you're especially good at to just hear a bunch of words thrown at you at once.*

In terms of the semiotic component, the instructor reported two potential challenges students might face in relation to speech communication and graphical representations. For example, in alignment with the case-study students' reports, the instructor noted she may have unintentionally spoken fast on occasions, especially when she had a new idea she wanted to share with students:

> *So I try to get the double—the verbiage and the written so that they have a better chance to catch up. But I'm pretty sure that sometimes I talk pretty fast, and I have a new idea off the cuff, and I just blah, blah, blah.*

The instructor also highlighted a potential semiotic demand related to graphical representations. She noted that students may interpret a graph as a literal picture rather than as a relationship between the indicated quantities. She exemplified this with a distance-time graph:

> *Here's the graph of time versus distance from the school. And so you know this means the time business. So that the person goes kind of away from the school and they come back to the school. But it might be read in a more picture way, like, "Oh, they went up the hill and down the hill," or something that's more just like what that looks like in the picture as opposed to interpreting the rate of change of time versus distance. So, we know as researchers that there's all of these kinds of things going on, always.*

Finally, in the context of the sociocultural component, the instructor highlighted potential challenges for students with communicating within small-group and whole-class discussions, which the case-study students also reported facing. The instructor suggested whole-class discussions might be challenging to participate in due to their more public nature, in comparison to small-group discussions:

> *So I think most people–it may be easier to hear and understand things than to actually form your ideas and get them out, especially if it's not your native language, but it could be even if it's your native language…and this occurs…also for people who are shy or not as comfortable with public speaking. It may be hard to actually stand up and say, "These are my ideas."*

In addition, the instructor noted that speaking with group peers may be difficult for students, especially when they don't know each other and have not yet built trust:

> *You might be in a group…with a bunch of people that you do not know and you're not sure if you trust. I think that's another issue that can occur about communicating. How do you feel like you can trust the other people with your ideas and with your maybe not knowing it right away, but trying to kind of gradually express your ideas? I think that can be a challenge in communication.*

Normative Demands

Below, I report the language demands that the instructor identified for students in the IOLA class within the normative dimension. All but one of these normative demands involved social norms. The only relevant sociomathematical norm identified in the data was related to what constituted a mathematically productive solution. The normative demands reported by the instructor reflected tensions regarding expectations for overall communication (spanning the whole lesson), communication during the whole-class discussions, and communication during the small-group discussions. All normative demands, except one, were emphasized by the case-study students. Thus, I begin by contextualizing each demand reported by the instructor with what the students reported.

In the context of overall communication, all case-study students reported a norm of English only in the IOLA class, which for three of them induced a class-centered norm incompatibility regarding the permissible language of teaching and learning. Related to this norm, the instructor reported there was an expectation of a certain level of English proficiency in various forms of student engagement. For example, when I asked how easy or difficult it might be for students, especially multilingual students who are learning the language of instruction, to participate in the IOLA class, she noted:

> *I think it's challenging for all students…I'm guessing that it's harder for students if they don't feel as comfortable with English…Because there's quite a bit of verbal communication, both in writing and in listening-speaking, that's expected in this kind of ((inquiry)) class… So both reading, writing, and speaking, listening, I think there's a fair level of English proficiency expected. And so I'm not quite sure how that plays out for different people. And even a native English speaker might find that challenging.*

Similar to three case-study students (Johan, Nam, and Seok), the instructor highlighted a norm violation in the IOLA class regarding what constituted effective student communication. She noted the lecture-based had a norm of not having to verbally communicate, whereas the IOLA class expected active student participation:

> *Well, I assume it's harder because in a lecture-based class, you don't have to participate very much. You might not have to participate at all if you don't want to. So now, you're kind of being forced to participate. You could hide out if you try hard. If your goal is to hide out, you maybe can, right? ((laughs)) But it's pretty hard, right? So it's got to be harder, but I don't know.*

Johan reported experiencing a norm violation about what constituted a mathematically productive solution, which he defined as a unique and general way of solving a problem. In contrast, he perceived the IOLA class had a sociomathematical norm of validating multiple solution methods as productive. The instructor acknowledged this was a norm she intended in the IOLA class and provided a rationale for encouraging it. Specifically, she felt that by having students engage with a variety of ideas, they would build deeper mathematical connections:

> *The hope…is that by having to grapple with ideas and not just hear a nice presentation come by you, that you build deeper, stronger connections with the ideas as you eventually build them. And this one's a little harder, but maybe*

understand a wider variety of ways to think about it than just that slick way the nice lecture presented it.

In relation to communication during the whole-class discussions, three case-study students (Johan, Nam, and Luis) reported a class-centered norm incompatibility regarding competent student participation, namely that they didn't feel competent at figuring out the big mathematical ideas or topics in a class lesson. From her perspective, the instructor acknowledged this issue and explained it as stemming from her attempt to highlight important mathematical ideas and topics while simultaneously encouraging students to generate and connect their own ideas. She described how challenging it may be even for an inquiry instructor to identify all the students' ideas and connect them to larger themes:

There's so many ideas being generated. And as the instructor, I'm often having trouble just keeping track of all the ideas and how they're connected to each other. So, for students, it must be crazy hard to try to do that. And I think sometimes they just kind of, "I'll just pay attention to these two ideas because the rest of that, I don't know what's going on." So I think that people appreciate it when you, as an instructor, kind of bring things together and highlight things because it's, "What am I supposed to be taking from all this stuff that's going on?" "Okay, here are some things that you can take." And trying to find the right balance on that, again, with the inquiry and with wanting them to generate ideas and their ideas be appreciated and them understanding each other's ideas, but at the same time, not be freaking out because like, "What do I need to know for the test?"

Like Seok, the instructor highlighted a class-centered norm incompatibility related to who was the default spokesperson for a small group during whole-class discussions, noting that it tended to be the same person in every lesson. The instructor also reported observing this pattern and shared how challenging it was for her to disrupt this norm in the face-to-face inquiry setting:

One thing that often happens in groups is that the louder a person–or the person who's more comfortable talking will be the one who presents the group's board. And even though I know that's the case and I don't necessarily want that to happen, it's a lot of work to try to not have that happen. And I'm not successful. I was better successful in COVID because we had some sort of point system where you had to present the board…"In this quarter of the semester, you have to

> *present twice, and you get plus five points for each time you present." Oh, and so it feels like, "Oh, we're coming to this board that has these four names on it, but these guys already have their 10 points…," then "Oh, we need somebody else to explain." So we had a system we were good at. We were better about implementing that to get more people to explain their ideas in COVID with the jamboards. But I also had a dedicated graduate student on Zoom recording all of that for me. And I think going back to in-person, I kind of slipped back to a little more of my old ways of I'd like someone else to speak besides that person yet again, but not always being able to pull that off. I think I tried a few times when you were there.*

In the context of communication during the small-group discussions, Johan reported experiencing a norm violation regarding respectful peer communication. He wanted students to be able to tell each other when their answer was wrong without feeling disrespected but felt that students in the U.S. generally felt offended when they were told their answer was wrong. The instructor explained these contrasting perspectives by noting that some students, like Johan, may be focused on arriving at a right answer as efficiently as possible, whereas other students may not want to tell others their answer is wrong as an attempt to validate (or at least not invalidate) others' ideas:

> *I know just as a person trying to work with other people, that you want to feel like your idea is validated in some way, even if it's not the one that your group ends up using on the board. And I think it's always a tricky issue, especially if there's a grade involved or a sort of stress about correctness involved because you want to be like, "Oh, we want the right answer, and we want it efficiently." So if your group is putting that thought process in like, "We want the right answer, and we want it as soon as possible," then they might not want to take time to hear an idea that at least they don't immediately recognize as useful.*

The instructor reported a class-centered norm incompatibility in relation to who constituted the default spokesperson for a small group when the instructor visited the groups during small-group discussions. (None of the case-study students reported experiencing this norm tension.) Although she did not identify the criteria of what made a student the default spokesperson, she observed there was tendency for the same person in each group to be the one who took over the role of speaking with the instructor.

There's the additional things one does when one walks around to the groups to try to promote that further ((that is, promote working together and writing something on their whiteboard)). So like, "Oh, what have you guys doing?" Oh, and you ask other–you try not to have only the one spokesperson be the only one who tells you something, which is so hard. And, "Oh, but what about you? What do you think about what's on this board? Try, you try."

Instructional Language Resources

The instructor highlighted various aspects of design and instruction that she intended as helpful for students. These aspects spanned the situational and normative dimensions of communication in the IOLA class. (No instructional lexico-grammatical resources were identified in the teacher's interview data.)

Instructional Situational Resources

Below I describe the instructional resources that the instructor reported leveraging in the IOLA class along the following components: material, activity, semiotic, and sociocultural. The instructor reported one resource within the material component, highlighting that, as part of the IOLA design team, she considered ways to make the IOLA tasks relatable to students. For example, she said the team intentionally included a female character in one of the tasks to make its problem context more identifiable for female students:

There was some effort made when these were created way back in the day to have a female person on one of the things so that maybe more people can identify with that. So that's one of our older tasks.

In terms of the activity component, the instructor reported using various techniques primarily within either the whole-class or small-group discussions that she intended as helpful. The following six instructional techniques were utilized in the whole-class discussions: (a) mitigating unrelatable problem contexts, (b) implementing a quick round-robin routine, (c) selecting particular groups to share based on an observed chain of mathematical ideas, (d) promoting a comparing and contrasting technique, (e) highlighting and summarizing key mathematical ideas or topics, and (f) attempting to disrupt a norm about who constituted the default spokesperson to the whole class.

First, the instructor tried to mitigate a potentially unrelatable problem context by changing the words or de-emphasizing certain wording on some tasks when she introduced them in class. For instance, when she discussed the problem context from Task 1 (The Carpet Ride Problem: The Maiden Voyage), she tried to de-emphasize the wording of a "young traveler" because she felt that might not be relatable to her older students: "I tend to change the wording or de-emphasize the wording when I constitute the task [Task 1] with my class because I often have a lot of older students, especially on my campus."

Second, the instructor reported occasionally using a quick round-robin routine during whole-class discussions, where she had each group share their work one by one. She implemented this routine as a way of allowing as many students as possible to contribute to the discussion within the constraints of the limited class time:

> *You want as many people to participate as possible, but if every group walks up to the front, and tells you every single thing on their board, you will run out of time to do what you need to do that day. So that's one of the ways that I try to deal with presentations is like, "Okay, just tell one idea from your board." And try to go, if you can, around every single board…It won't be every person because groups have their loud-mouth person who always likes to speak and present the board, and you have to work hard to try to get someone else to present the board. But at least every board that gets some– maybe a third of the class speaking and– but to do it fast, get some sense of what's going on.*

She elaborated on how this round-robin routine developed from her experience of teaching during the COVID-19 pandemic. More specifically, the routine grew out of her attempt to promote more equitable student participation while being limited to a virtual environment:

> *My use of that, everybody just say one thing about your board, came out a little bit more through the COVID process of having people write on the jam boards, and then I would go through and be like, "Just tell one thing on your jam board." And I used to do less of that until I went through the COVID thing.*

There was a big sort of social justice thing going on in the middle of COVID. And people were thinking harder about those [social justice] issues, and "Does everyone have an equitable chance to present their ideas?" So with that in mind, but also the fact that in COVID, most people were online and they were writing on jamboards, and…I couldn't do my little trick that I used to do pre-COVID of like, "Oh, I see where the boards are. So I'll start with this one and then move to this." I mean, you can't always pull that off anyway, but if you're really on it, sometimes you can pull off a nice chain of thoughts by picking on groups. But in COVID, it was just like, "I can't manage this."

Third, as evidenced in the previous quote, she contrasted her round-robin routine with another practice of selecting different groups to share their ideas during whole-class discussions: sequencing the participation of certain groups based on particular mathematical ideas with the goal of creating a more coherent story. She reported occasionally integrating this alternative practice into her face-to-face IOLA class:

Sometimes what I do with the whiteboards is I try to tell more of a story. So this one that we're just talking about, round-robin, is just like, I'll start in here and go around. Or sometimes I'll try to mix it up. I'll start in the back and go around or start on this side. But sometimes what I do with the boards is try to tell more of a story out of the boards. If I kind of have a better sense of what's on all the boards and I want to get certain ideas out and maybe I want to get them out in a certain order to kind of build a story of different types of approaches, then I might pick the board. Like, "Okay, let me grab your board," and then I usually talk more about their board. Or sometimes I'll have them talk from their seat about the board, but I've got it held up in front. So that's another strategy. Sometimes I'll try to compare across boards.

Fourth, when the instructor used a round-robin routine, she asked each group to share something from their small-group whiteboard that was different from the other boards presented. She promoted this comparing and contrasting technique to focus the whole-class discussion on a few important mathematical ideas:

So kind of managing the issue of not overloading people cognitively but still trying to push them to learn to hear other people's ideas and interact with them…One way of doing that is…going around and saying, "Show something different." So you had to pay attention to what the other boards were to know if yours was different or not. And what I think is different might be different than *what the students think is different. They're like, "Oh, it was the same as that other board." "Well, actually, there was quite a few differences. Here's some." I might point them out myself.*

The instructor emphasized how this routine of comparing and contrasting the work from different whiteboards helped the class generate new ideas:

Trying to have them listen to each other,...[I would be] like, "Oh, when you raise up your board, especially emphasize something that hasn't been said yet that's not on one of the other boards," so we get more new ideas.

Fifth, another technique the instructor used was highlighting and summarizing key mathematical ideas or topics:

Sometimes I've been able to do it in that day of class, highlighting certain things or summarizing. Sometimes I was able to do it at the beginning of the next class. "Here's a summary of what we did last class. Let me remind you of some things, go through some notes."

She elaborated on what she meant by highlighting:

Sometimes I talk about people's boards. I point out specific things that I think are important about people's boards. So that's highlighting their ideas. And sometimes I might point out connections across boards. I might have more than one board up at a time. Like, "Okay. Notice how they did this part the same, but they did this part different." I guess those would be examples of that ((that is, highlighting)).

As can be seen, highlighting included using linguistic markers such as "Notice [this]." Highlighting or summarizing could be considered one way the instructor tried to address the class-centered norm incompatibility that the three of the case-study students reported, where they didn't feel competent at figuring out the mathematical ideas or topics in a class lesson.

Sixth, the instructor tried to break a norm about who constituted the default spokesperson for a small group during whole-class discussion by asking different students to take on the role of spokesperson: "I'd like someone else to speak besides that person yet again." The norm she tried to disrupt was the same norm involved in the class-centered norm incompatibility reported by Seok.

In addition to the instructional techniques used during the whole-class discussions, the instructor reported leveraging three techniques during the small-group discussions: (a) *visiting each group to promote communication, (b) attempting to disrupt the norm of who constituted the default spokesperson to the instructor, and (c) drawing attention to the ideas of certain students.*

First, one technique the instructor implemented during small-group discussions was to visit each group to promote peer communication (as well as collaboration and sensemaking) by drawing students' attention to certain aspects of their mathematical thinking, asking and answering different types of questions, and giving them hints, among other strategies. She exemplified the types of questions she asked in the small groups:

> *So there's things you try to do to get them ((students in small groups)) working together, or understanding like, "Oh, I see this student wrote this on the board and is explaining it to me. What about you guys? What do you think? Do you think this makes sense? Can somebody else explain this?" Like, "Oh, you don't think it makes sense. Okay, well, how about you guys make sure that that person understands?" So there's stuff you try to do to try to keep the communication going.*

Second, she attempted to break a norm, whereby the group member who spoke to her during small-group discussions tended to be the same across lessons. As noted in Normative Demands, the instructor prompted more equitable participation when she visited each small group by making statements and posing questions such as: "Oh, but what about you? What do you think about what's on this board? Try, you try."

Third, another technique the instructor leveraged during small-group discussions was to draw attention to the ideas of certain students to ensure that their ideas were not overshadowed:

> *Often what I'll do when I'm walking around is, sure, give hints and stuff and answer questions, but like, "Oh, you guys have a cool idea about this. Can you*

> *make sure that gets on your board?"... So that the ideas that people have are getting actually on the boards and we don't lose some cool ideas because another member of the group was louder and has everything. So there's a certain amount of making sure good ideas get on the board.*

This instructional technique could be seen as an attempt to disrupt the norm violation that Johan reported regarding dominant student communication, where he felt his ideas were overshadowed by more vocal students.

In relation to the semiotic component, the instructor leveraged multiple communication modes and mathematical representations. The instructor reported supplementing her speech with other modes of communication, particularly gesturing and writing:

> *I think that if I'm not writing it down, I'm probably doing one of these other things ((modes of communication)) with my speaking to have a second way of communicating. I think it's less likely that I just talk. I like gesturing.*

She explained her reasoning for integrating multiple modes of communication, noting that each may have different affordances for different students:

> *Students may get more out of one of these modes of communication than another. So you may want to use multiple modes of communication in hopes that one of them will stick. One of them will work better than others. So the example I gave you earlier was talking about how I try to write stuff down as well as say the stuff because I know that has a better chance of comprehension if people have access to both of those, but that I don't always succeed.*

Specifically, she pointed to the visual affordance of writing and gesturing:

> *I try to have visuals, either maybe something I've typed up ahead of time that I'm showing while I'm pointing at it and stuff, or they have the handout in their hand for the activity, or I try to write something on the board.*

She also exemplified how she intentionally integrated gesturing with speech to support student learning of mathematical topics, such as span and intersections of planes:

> *When we're talking about span or… intersecting our planes…, I do a lot of arm motions that–, "Oh, the planes intersect like this ((gestures)) and see that's a line,"..., or one vector pointing this way ((gestures)), "We can multiply by anything to get anything on this line," those–it could be a gesture.*

This integration of modes of communication was an instructional resource that all case-study students reported experiencing.

In addition, the instructor reported using multiple mathematical representations as a resource for student learning. For example, she said she used tabular representations, although she preferred not to call them that:

> *I think tabular is very making a table of values for a graph. And I do think that that sometimes occurs in my class, but I would rather say numeric for that one because I think it's really useful just in general, when you're first learning a math concept, if you can see specific examples as well as think about the generality. And so using numeric representations allows you to look at more specifics.*

She also noted there are different representations within a given type of representation that could be helpful for students. For example, she pointed to three different representations of matrix multiplication, representing three views of A**x**=**b**, where A is a matrix, and **x** and **b** are column vectors: (a) as a linear combination, as a system of equations, and as a transformation. She added that "it's also useful to see them [concepts] in different contexts…to get an overall understanding."

Finally, in terms of the sociocultural component, the instructor reporting leveraging different participant structures–small group-discussions, working in pairs, and whole-class discussions–to encourage student participation and generate students' mathematical ideas:

> *Generating students' ways of reasoning. So that's very important to me in the class. And generating student ways of reasoning could occur in small groups, kind of most often because people need to get started thinking about things and have a chance to do that. Sometimes, though, it could be like, "Think about this with your neighbor for one minute, or something shorter." And sometimes it could be me just kind of posting something like, "Hey, what do you guys think about this in the whole-class discussion?"*

Instructional Normative Resources

The instructor highlighted three aspects of her instructional design and teaching practice that she intended to be linguistically helpful for her students: (a) designing tasks with realistic problem contexts, (b) constituting the tasks (that is, explaining and negotiating the intended meanings and problem contexts in the tasks), (c) and encouraging the use of small-group whiteboards during small-group and whole-class discussions.

First, as part of the IOLA design team, the instructor highlighted her intentionality in designing problem contexts that were realistic, or imaginable, for students. For example, she noted how Task 1 (The Carpet Ride Problem: The Maiden Voyage) was designed to be imaginable for all students, even those who might not identify as young travelers:

> *I can still imagine a young traveler. But I don't feel like I'm a young traveler, personally. So it's just, where's the level of imagination?...That's an example of when we're ((the IOLA design team)) trying to kind of manage that issue of it being written for a 19-year-old, and maybe my student is 32 and a military veteran…I think it's [the task] still accessible.*

Likewise, the instructor described how the problem context in Task 7 (Meal Plans: Constraining the Number of Meals) might be imaginable even to students who do not buy a meal plan:

> *The scenario for the meal plan,...it's kind of based on this sort of traditional student at a college. And I think people get ((that is, imagine)) it. And even if you're not a traditional student, you might buy a meal card, if you get a discount or something.*

This instructional resource reported by the instructor connects to the report from Nam and other case-study students who benefited from the IOLA tasks having problem contexts that were imaginable to them.

Second, the instructor reported regularly implementing a practice of constituting the tasks to negotiate the intended meanings and problem contexts:

Maybe we haven't figured out the best way to say ((that is, write)) it. Maybe there is no best way to say it or format it. And so we also know that the instructor needs to talk with the students and negotiate the task. So like, "Oh, you thought it meant that." "No, actually, I meant this." And that's just a normal part of the process of teaching a task.

Any task where there's a bunch of words on a piece of paper, there's going to be some like, "Okay. What does this mean? What's she asking?"

The instructor elaborated on how she constituted the tasks by giving her own version of the text on the task handout and clarifying the problem contexts:

One thing that I do is try to constitute the task. And what I mean by that is kind of talk through what's on this page with them and maybe draw a little graph on the board or indicate some things.

She exemplified how she constituted Task 1 (The Carpet Ride Problem: The Maiden Voyage):

One of the things that comes up with this task ((Task 1)), in particular, is what does negative mean, or what does going backwards mean? Like, "Oh, you can't time travel backwards"...but yeah, that's not what we intend here. We just mean you go the opposite direction…And my mindset is to come to a joint understanding so that if something comes up and I'm like, "Oh, yeah. No, actually, I was trying to communicate this," and I'll try to explain why I was communicating and negotiate an understanding. It could happen with monolingual English speakers as well.

Finally, the instructor also noted that she constituted Task 7 (Meal Plans: Constraining the Number of Meals) to clarify wording issues:

And there's lots of wording issues with this one ((Task 7)), and I am more hypersensitive about them because it's a new task. But in the bottom line, it still comes down to like, "This is the best I can figure out to word this. I know it will still have some confusing issues, so we need to constitute the task together so we come to a shared understanding of what's being asked"

All case-study students reported benefiting from this instructional practice when they mentioned that they found it helpful that the IOLA instructor clarified the problem contexts and other aspects of communication within the tasks.

Third, the instructor consistently encouraged students to utilize their whiteboards during small-group discussions and asked them to present their whiteboard work to the entire class. She observed that establishing this expectation of presenting whiteboard work motivated students to engage in collaborative writing within their small groups. Conversely, engaging in meaningful small-group discussions and preparing their whiteboard presentations equipped them for active participation in whole-class discussion:

> *That's a norm in the class that you will put stuff on your whiteboard ((during small-group discussions)) and you may be asked to present ((during whole-class discussions)), that you are expected to work in a group on the whiteboard. So I think that it's expected or helpful if people answer questions or respond ((to questions)) to [promote] having a discussion.*
>
> *I want people to work with each other. So that's why we have whiteboards, as it promotes that. Because then you need–you have whiteboard, and you have to have something to present on your whiteboard. So that promotes working together and putting something on your whiteboard.*

Conclusion

This chapter presented a case study analysis of one IOLA instructor and addressed the research questions related to language demands, instructional language resources, and student language resources. There was much overlap between what the instructor identified and what the case study students reported experiencing. The instructor primarily identified language demands and instructional language resources, adding important contextualization and explanations of the findings from the case study students as well as a handful of new language demands and resources. Below, I describe three new language demands and several instructional resources that the IOLA instructor added. Then I describe how the instructor contextualized the findings from the case study students, which included navigating three tensions as an inquiry mathematics instructor.

The IOLA instructor added three new language demands. The first was a lexico-grammatical demand related to terms with multiple meanings, highlighting how an unintended meaning of a certain mathematical term may negatively influence a student's mathematical interpretation of the term. The second was a situational demand regarding the potential for some problem contexts on the IOLA tasks to be unrelatable to certain students. The third demand was a class-centered norm incompatibility about who tended to speak with the instructor on behalf of a small group during small-group discussions.

In addition, the IOLA instructor added several instructional language resources in the situational and normative dimensions. Two situational resources were using problem contexts that were relatable to students or mitigating potentially unrelatable problem contexts. Other situational resources included: (a) implementing a quick round-robin routine, (b) sequencing group sharing based on an observed chain of mathematical ideas, (c) promoting a comparing and contrasting technique, (d) highlighting and summarizing key mathematical ideas or topics, and (e) attempting to disrupt a norm about who constituted the default spokesperson to the whole class. In the normative dimension, one teaching practice she implemented across lessons was constituting the tasks, which included clarifying wording issues.

Furthermore, the IOLA instructor contextualized some of the findings in terms of contrasting student perspectives and instructional tensions that she faced as an inquiry instructor. For example, she contextualized the normative demand that Johan faced regarding what constituted respectful peer communication as stemming from contrasting priorities between different students. She observed that certain students, such as Johan, might prioritize reaching the correct answer as quickly as they can, while others may refrain from pointing out incorrect answers to affirm or at least not discredit their peers' ideas.

The instructor also reported having to navigate three tensions in the IOLA class between facilitating linguistic access and remaining true to an inquiry-based approach. For example, she discussed a challenge she encountered while writing Task 7 (Meal Plans: Constraining the Number of Meals) as a part of the IOLA design team. On one hand, she aimed to convey to students the quantitative relationships and constraints inherent in the task. On the other hand, she sought to achieve this without directly presenting the relationships in mathematical language to allow students to actively engage in the valued practice of mathematizing a problem context. She observed this tension may have given way to situational demands for students related to not understanding the problem context.

Another tension that the IOLA instructor faced lay in her effort to emphasize important mathematical ideas and topics while also fostering students' ability to develop and link their own ideas. Furthermore, she explained the difficulty that an inquiry instructor might face in recognizing all of the students' ideas and integrating them into broader themes. She noted how trying to balance this tension and navigating this instructional challenge may have contributed to a normative demand where students struggled to identify the mathematical ideas or topic(s) in a class lesson.

Finally, the instructor reported trying to balance a tension between promoting more equitable student participation and pulling a story of connected themes out of students' mathematical ideas. More specifically, she noted it was difficult to highlight a coherent chain of connected ideas while simultaneously giving all groups an equal voice during whole-class discussions (as in a quick round robin routine). On the other hand, selecting particular groups to contribute to whole-class discussions based on particular mathematical ideas promoted more coherence between ideas but might not guarantee an equal chance for each group to participate in the discussion.

Overall, this chapter further confirmed and expanded the variety and salience of language demands and instructional resources that students, especially multilingual learners, might experience in inquiry-based mathematics courses. In addition, this chapter provided new insights that contextualized the identified demands and resources through the perspective of a central member of the inquiry-classroom: the instructor. The instructor's perspective elucidated instructional tensions that evidence how difficult it is to balance linguistic access (or more broadly, equity) with aspects of inquiry.

Chapter 7 Discussion

In this dissertation study, I utilized a case study approach to investigate the linguistic experiences of four undergraduate multilingual students – Johan, Nam, Seok, and Luis – in one undergraduate inquiry-oriented linear algebra (IOLA) course at Southwestern University. To capture their linguistic experiences, I addressed three research questions targeting language demands, instructional language resources, and student language resources. I also analyzed the perspectives of the instructor of the IOLA course in relation to the three research questions.

Reflecting on the case studies of the four multilingual students and the instructor, I identify five themes related to the research questions. The first three themes focus on students' linguistic experiences, indicating that, while they are not monolithic and may be mediated by the students' backgrounds, there are still common elements across students. The last two themes situate these common elements in a systemic way regarding inquiry-based mathematics education. Below, I start with a section that describes these themes. Then I discuss implications for practice and research as well as the limitations of the study and future research. Finally, I highlight the significance of the study.

Themes

In this section, I illustrate five themes related to the study's research questions focused on the language demands, instructional language resources, and student language resources for multilingual students in one inquiry-based mathematics course. The five themes are:

(a) The linguistic experiences of multilingual students in inquiry-based mathematics courses are not monolithic.

(b) The linguistic experiences of multilingual students may be strongly mediated by the students' backgrounds.

(c) There are important similarities in the linguistic experiences of various multilingual students in inquiry-based mathematics courses.

(d) Inquiry-based mathematics education can induce language demands for multilingual students and unintentionally undervalue their language resources.

(e) Balancing inquiry and linguistic access is a complex process.

Theme 1: Diversity in Linguistic Experiences with Inquiry Mathematics

The experiences of multilingual students with respect to language demands, instructional language resources, and student language resources in an inquiry-based classroom are not monolithic. This diversity of linguistic experiences can be seen in Table 7.1, where I summarize the four unique case studies according to various aspects (demands, instructional resources, and student resources). To capture and compare between the students' linguistic experiences, I broke down each aspect of their experience by dimension and inferred a corresponding measure of salience for each student case study. For example, I identified how salient were normative demands for Johan's case study. For each student case study, I obtained a broad measure of salience for a given dimension within an aspect of experience by considering various conditions: (a) the number of codes lying in that dimension, (b) the number of episodes with instances labeled with codes from that dimension, (c) the intensities with which the student described the instances labeled with codes from that dimension, and (d) the degree of conflict between the student's situational or normative aspects and those of the IOLA class. Below, I illustrate these measures for each case student and summarize the student's overall attitudes toward the course.

Table 7.1:

Diversity of the Four Case Study Students' Linguistic Experiences in the IOLA Course

Linguistic Experience	Broken by dimension	Johan	Nam	Seok	Luis
Demands	Lexico-grammatical	High	Moderate	Low	Moderate
	Situational	High	High	Moderate	Low
	Normative	High	Moderate	Moderate	Low
Instructional Resources	Lexico-grammatical	Low	Low	Low	Low
	Situational	Moderate	Moderate	High	High
	Normative	Low	Moderate	Moderate	High
Student Resources	Lexico-grammatical	Moderate	Moderate	Low	Low*
	Situational	High	High	Moderate	High
	Normative	High*	Moderate*	Low	Low
Overall	Linguistic complexity of the class	High	Moderately high	Moderately low	Low

Note. *Student resources that were perceived to be in tension with aspects of the IOLA course.

For Johan, language demands played a very salient role in his experience of the IOLA class. For example, he noted a significant number of normative demands, of which most were norm violations (the most conflicting type of norm tensions). In addition, he did not experience any normative language resources. Moreover, he had plenty of language resources on his own, but most were in conflict with the norms or situational aspects of the IOLA class. For example, he couldn't sustain his practice of notetaking during class. Overall, he found the communication in the IOLA class very difficult to navigate and did not find the class's ways of operating as helpful for his learning. Ultimately, he preferred his own way of communicating and learning mathematics, which aligned with instructional approaches with lower degrees of active learning.

For Nam, as with Johan, language demands played a salient role in his experience. However, the degree of salience for Nam was a bit lower. For example, he only reported a moderate number of normative demands, only one of which was a norm violation. In addition, unlike Johan, Nam reported a moderate number of instructional normative resources. Moreover, like Johan, Nam had some normative resources on his own, with most being in conflict with the class. Overall, he found the communication in the IOLA class somewhat difficult to navigate and the class's ways of operating as somewhat helpful for his learning. Ultimately, he preferred his own way of communicating and learning mathematics, which aligned with more traditional lecture-based approaches.

For Seok, as with Nam, normative language demands played a moderate role in his experience. However, situational demands were much less salient. For example, unlike Johan and Nam, Seok did not mention finding the teacher's or other students' speech communication difficult to follow. Like Nam, however, Seok experienced a moderate number of instructional normative resources, and even more instructional situational resources. Finally, in contrast to Johan and Nam, Seok relied less on his own language resources to navigate the class; yet his use of his own language resources were not in conflict with the IOLA class. Overall, Seok found the communication in the IOLA class just a little difficult to navigate and generally found the class's ways of operating as helpful for his learning. Ultimately, he stated no preference in ways of communicating and learning mathematics between inquiry-based approaches and other approaches with lower degrees of active learning.

Unlike for Seok, language demands did not play a salient role in Luis' experience of the class. However, like Seok, Luis experienced a high number of instructional situational resources, while reporting a higher number of instructional normative resources than Seok. In addition, like Seok, Luis reported a low number of student normative resources, with those resources not conflicting with the IOLA class. Overall, Luis found the communication in the IOLA class very easy to navigate and very helpful for his learning. Ultimately, he stated no preference in ways of communicating and learning mathematics between inquiry-based approaches and other approaches with lower degrees of active learning.

In sum, as depicted in the last row of Table 7.1, the linguistic experiences across the four case study students can be arranged in decreasing order of linguistic complexity–starting with Johan, who reporting struggling the most with the communication in the IOLA class and ending with Luis, who reported struggling the least with the class communication. In addition, as illustrated in Figure 7.1, the students who reported experiencing less linguistic complexity in the IOLA class found the class more helpful for their overall learning of mathematics, whereas the students who reported experiencing more linguistic complexity found the class less helpful for their learning. Next, Theme 2 will examine the cultural- linguistic backgrounds that may have influenced this diversity of linguistic and learning experiences.

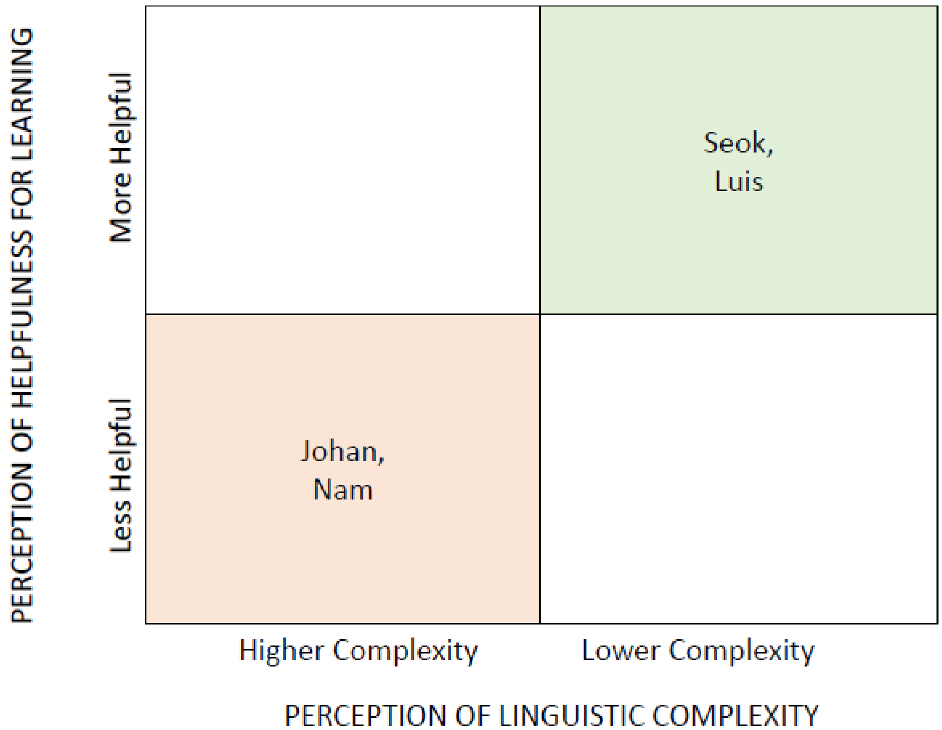


Figure 7.1:

Diversity of the Four Case Study Students' Linguistic and Learning Experiences of the IOLA Course

Theme 2: Relationship Between Background and Linguistic Experiences with Inquiry Mathematics

The experiences of multilingual students in an inquiry-based classroom may be strongly mediated by the students' backgrounds (comfort level with English, prior exposure to U.S. culture, and prior experience with inquiry-based culture). Table 7.2 provides a summary of the case study students' backgrounds. Below, I illustrate how these factors may have influenced the diversity of the students' linguistic and learning experiences identified in Theme 1.

Table 7.2:

Potential Factors Influencing Case Study Students' Experiences of IOLA Course

Factor	Measure	Source	Johan	Nam	Seok	Luis
Comfort level with English	Self-reported proficiency in English (out of 6)	Screening survey	4	4	4	6
	Confidence level with English communication in class	Inferred from interview	Low	Low	Moderate	High
Prior exposure to U.S. culture	Total time in the U.S. (years)	Inferred from interview	2	3	8	> 20
	Salience of cultural background	Inferred from interview	High	Moderate	Low	None
Prior exposure to inquiry-based culture	Prior active learning courses	Inferred from interview	Low	Low	Moderate	High
Prior exposure to linear algebra	Previous linear algebra course taken	Inferred from interview	Low	Low	Low	High

In terms of comfort level with English, the three students who self-reported a lower proficiency with English (Johan, Nam, and Seok) experienced a normative demand related to the norm of English only in the IOLA class. In addition, Seok, who reported a higher level of confidence with English than Johan and Nam, experienced less salient situational demands related to pronunciations in English and the use of speech communication. In contrast, Luis, who self-reported a high proficiency with English and a high level of confidence with English in class, did not experience any situational or normative demand related to the English language or speech communication.

In terms of prior exposure to U.S. culture, the two students who had the least number of years in the U.S. also reported the most culture shock in the IOLA class, as evidenced by the normative demands they experienced. For example, Johan described how his expectations of how students should communicate and interact in class were related to his Muslim identity, which he considered more conservative than the culture in the U.S. Similarly, Nam described norm tensions in the IOLA class in relation to norms in Vietnam, but those norms were limited to classroom culture. This may explain why Nam may have found the IOLA class somewhat easier to navigate than Johan did, as well as why he may have found the class to be somewhat more helpful for learning overall.

Finally, in relation to their exposure to inquiry-based culture, the two students who reported the least prior experience with active learning courses (Johan and Nam) experienced more norm tensions between an inquiry approach and more traditional approaches with lower degrees of active learning. On the other hand, Luis, who reported having more experience than Seok with active learning courses, experienced fewer such norm tensions than Seok did. Thus, students with greater prior exposure to inquiry-based culture generally experienced less salient normative demands in the IOLA class.

Theme 3: Similarities in Linguistic Experiences with Inquiry Mathematics

Although the linguistic experiences of multilingual students in an inquiry-based mathematics classroom are not monolithic, there may be some key aspects of their experience that are shared across multiple students. In Table 7.3, I provide a list of the most relevant similarities in language demands, instructional language resources, and student language resources across the four case study students. A similarity is defined as a code (or combination of similar codes) that arose for at least two students. Below, I unpack six key similarities in language demands: one lexico-grammatical, three situational, and two normative.

One lexico-grammatical demand that both Johan and Nam experienced was antonym pairs. For Johan, it was difficult to distinguish between North and South as well as between East and West. He acknowledged that this was a personal challenge he faced even in his primary language (Malay), but the challenge worsened when he encountered these words in English. Similarly, Nam experienced a challenge with distinguishing between numerator and denominator.

One situational demand that both Johan and Nam experienced was related to the use of speech communication. For example, Johan often struggled to understand his peers' speech and make himself understood. In addition, both students often found it difficult to follow the teacher's speech, especially when they perceived her as speaking too fast. They attributed these challenges to speech being impermanent, informal, or obscure. In other words, speech disappeared (so could not be revisited), involved slang words, and was not easy to visualize (so it hid its meanings).

A second situational demand that Johan, Nam, and Seok experienced was tasks with unfamiliar problem contexts. One example of such a task was Task 7 (Meal Plans: Constraining the Number of Meals), where they could not figure out how many meals they were allowed per day in the context. A third situational demand that all case study students experienced was that some IOLA tasks had too many words.

Table 7.3:

Key Similarities Among the Four Case Study Students' Linguistic Experiences in the IOLA Course

Linguistic Experience	Broken by dimension	Key Examples
Demands	Lexico-grammatical	Antonym pairs, unfamiliar notation, and words or phrases with unfamiliar meanings
	Situational	Speech communication, English pronunciation, whole-class and small-group discussion structures, tasks with too many words, and tasks with unfamiliar problem contexts
	Normative	Tensions related to what constituted the permissible language, dominant student communication, effective student communication, and competent student participation
Instructional Resources	Lexico-grammatical	None
	Situational	Tasks with fewer words, familiar problem contexts, repeated story contexts, primacy of symbolic mathematical language, visual representations, supplementation of speech with other mediums (gesturing, writing, and graphical representations), small-group and whole-class discussion structures, and linguistic markers of important information
	Normative	Tasks with realistic problem contexts*, centrality of peer communication, and structure of the class lesson

Table 7.3:

Key Similarities Among the Four Case Study Students' Linguistic Experiences in the IOLA Course, Continued

Linguistic Experience	Broken by dimension	Key Examples
Student Resources	Lexico-grammatical	Primary language translations of certain words and prior knowledge of mathematical terms
	Situational	Translating tools, prior knowledge of certain problem contexts or topics, multiple language(s), writing*, graphical representations, and symbolic representations
	Normative	Identifying important information and writing notes, and minimally communicating through speech

Note. *Examples that came up for multiple students across different dimensions.

There was also a normative demand that both Johan and Nam experienced in relation to what constituted dominant student communication. While Nam did not report it in his interview, Johan identified this demand as applying to both him and Nam. He found that their ideas tended to be overshadowed in their small group discussions because their ideas were mostly shared in written form, whereas the most valued form in the group was speech. Seok did not report facing this issue in the small-group discussions, but he identified a similar theme about dominance in the whole-class discussion. He found that there was a tendency for the same person to take over the role of spokesperson for a small group when the groups shared their work with the whole class. It is worth noting that the instructor of the IOLA course also identified these language demands and reported observing an additional form of speech dominance: When she visited the small groups during small-group discussions, the same student tended to speak with her on behalf of the group.

Finally, one normative demand that all four students experienced (though with different degrees of salience) was related to what constituted competent student participation. All students struggled to figure out the mathematical topic(s) and major takeaways from each lesson during class. This demand was especially impactful for Johan, who as a result could not sustain his note-taking practice and subsequently reported losing some interest in the class. In contrast, this demand was not as impactful for Luis because he had taken a prior linear algebra class, so after an IOLA class lesson, he would go home and figure out the topic(s) covered based on his prior class notes.

Theme 4: Problematizing Inquiry-Based Mathematics Education (IBME) for Multilingual Students

As summarized in Chapter 2: Literature Review, four typical features of inquiry-based mathematics education (IBME) include: (a) using tasks with authentic problem contexts, (b) building on students' everyday resources, (c) relying on small-group and whole-class discussions, and (d) establishing norms of participation for the particular classroom community. Below, I draw on the findings from this study to argue and illustrate that, without a language-focused intervention, these well-intended instructional features can unintentionally undervalue student language resources or induce language demands that might negatively impact multilingual students. Table 7.4 provides an overview of this argument.

Table 7.4:

Ways that IBME Can Unintentionally Induce Challenges for Multilingual Students

Typical Features of IBME	Implications for Multilingual Students
Using tasks with authentic problem contexts	Authentic problem contexts use complex language, especially for multilingual students.
Building on students' everyday resources	"Everyday" language resources for students from the dominant community might not function as such for multilingual students.
Relying on small-group and whole-class discussions	An emphasis on verbal participation can obscure multilingual students' communication resources.
Establishing certain norms of participation for the classroom community	Inquiry classrooms can induce norm tensions about communication for multilingual students when the norms of their communities outside the inquiry classroom are not explicitly considered.

First, using tasks with authentic problem contexts can induce challenges for multilingual students because authentic problem contexts can come with complex language. For example, all case study students found many of the IOLA tasks challenging because they had too many words. In addition, three of the case study students found that some tasks had unfamiliar problem contexts.

Second, building on students' everyday resources often involves using informal or everyday language. However, what might constitute an "everyday" language resource for students from the dominant community might not function as such for a multilingual student. This issue might arise due to limiting communication to English as the legitimized language of teaching and learning or to the limited focus on U.S. culture. For example, Johan experienced the phrase "up to twice" in Task 7 (Meal Plans: Constraining the Number of Meals) as a barrier to understanding the problem context in the task. Johan interpreted "up to twice" to mean "at least twice," which slowed his mathematical progress during class.

Third, without systematic intervention, relying on small-group and whole-class discussions can over-emphasize speech communication at the detriment of other legitimate forms of communication. For example, Johan reported that his and Nam's written ideas were overshadowed in his small-group's discussions by the spoken ideas from other group members.

Fourth, norms of participation in the inquiry classroom are often promoted by the instructor without explicitly attending to the norms of the students' communities outside the particular classroom. This can be problematic because, as this study indicated, inquiry-based mathematics education can induce various norm tensions (class-centered norm incompatibilities, norm violations, and student-centered norm incompatibilities) for students. In turn, many such tensions can be experienced as language demands by multilingual students. For example, while most students in the IOLA class asked questions and made comments during teacher talk in the form of interpersonal communication, Johan felt this form of student participation was disrespectful to the instructor. His perception of disrespect was based on the norms of his Malaysian and Muslim community, where they regarded the teacher very highly. He was used to letting the teacher finish an entire explanation before asking questions or contributing any ideas. An important factor that further problematizes such norm tensions is that norms are often negotiated implicitly through classroom discourse, so it takes time for students to figure out the intended norms. Indeed, it took Johan between two and three weeks of the course to realize this was not a class where he could take notes, as he was used to doing in a lecture-based class.

Theme 5: Balancing Inquiry-Based Mathematics Education and Linguistic Access

Balancing IBME and linguistic access is complex. There are various tensions that must be considered from an instructional perspective. At the same time, there are various instructional tools that can be used to promote linguistic access for all students, especially multilingual students, while keeping an inquiry-based spirit for the class. For example, during her interview, the IOLA instructor reported attempting to balance four tensions in her instructional approach (three of which are discussed in Chapter 6). On the other hand, the instructor also reported applying various instructional resources, which can be cast as an effort to balance inquiry-based mathematics education and linguistic access. Below, I briefly summarize four tensions the IOLA instructor faced during her teaching and how she may have attempted to balance some of them.

One tension related to designing tasks in linguistically accessible ways, while still allowing students opportunities to mathematize authentic problem contexts. More specifically, she strived to communicate the quantitative relationships and constraints of the task to students, but she endeavored to do so without directly employing mathematical language. She noted that this balancing act might have led to situational challenges for students, particularly in grasping the problem context.

The remaining tensions the IOLA instructor faced were related to class discussions. A second tension she reported facing as an experienced inquiry instructor was highlighting essential mathematical concepts while encouraging student autonomy. By highlighting essential mathematical concepts, she supported students in accessing the big picture of the communication in a given lesson. This support was important to her because she acknowledged the difficulty for students to capture and synthesize all the student mathematical ideas shared in class. In fact, she admitted it was difficult to do even as an instructor. However, seeking to remain true to an inquiry approach, she had to be cautious not to do much of the highlighting of ideas herself to allow students to connect many of the mathematical ideas themselves. To attempt to balance this tension, she first allowed as many opportunities as possible for students to make and report connections themselves, and then she highlighted other specific connections the students did not highlight. For example, during the whole-class discussions, she often used a quick round robin routine where she had each group share something about their small-group whiteboard that was different from the previous boards. When she noticed a connection the students did not share with the class, she would sometimes point it out herself.

A third tension the instructor considered was promoting equitable participation while maintaining coherence in class discussions. In particular, she remarked on the challenge of maintaining a cohesive flow of interconnected ideas while ensuring equal participation from all groups in whole-class discussions through a quick round-robin routine. Conversely, prioritizing certain groups to contribute based on specific mathematical concepts enhanced coherence among ideas but potentially compromised equal participation opportunities for all groups. The instructor might have attempted to address this tension by alternating between these two instructional approaches of sequencing group sharing.

Finally, a fourth tension the IOLA instructor faced was allowing students to access their own digital resources while encouraging an inquiry approach to learning during class. She ultimately allowed students to use their own technology during class because she was fine with them using it as a source to help them think about a topic or problem. At the same time, she feared some students might just try to find similar problems and try to mimic the solution approaches shown online. To her point, during the first week of my classroom observations, I did notice a few students in her class skimming through the article by Wawro et al. (2013), which discussed solutions to Tasks 1-4 of the IOLA class. However, as exemplified by Johan's case study (see Chapter 4), accessing their own digital technology can provide critical language resources, such as translating tools, for multilingual students. Thus, it is important to allow multilingual students to use their own digital resources, especially when a course privileges English only during official class communication.

Implications for Practice

Collectively, the five themes discussed above highlight important practical implications about inquiry-based mathematics education in light of the linguistic experiences of the four case study students and the tensions faced by the IOLA instructor. Below, I describe the following implications for practice: (a) inquiry does not guarantee positive linguistic and learning experiences for multilingual students, (b) any language-focused intervention must consider the diversity of backgrounds and linguistic experiences of multilingual students, (c) there are instructional tools that could be considered as starting points for promoting linguistic access for multilingual students, (d) any language-focused intervention must consider the tensions experienced by inquiry instructors in balancing aspects of inquiry with linguistic accessibility.

First and foremost, one implication is that inquiry does not guarantee positive linguistic and learning experiences for multilingual students. For example, Theme 4 highlighted general ways in which features of inquiry-based mathematics education may unintentionally create challenges for multilingual students. More specifically, Theme 3 showcased common language demands that various multilingual students experienced in the IOLA class. Similar findings were recently documented by Rios (2024) in undergraduate mathematics courses that used groupwork. For example, Rios identified the following discourse described by multilingual students about participation during groupwork: "Participation in groupwork meant vocal communication" (p. 8). This discourse connects to my finding regarding speech dominance, where, for example, the ideas of two multilingual students who expressed in written form were overshadowed by the spoken ideas of more vocal group peers.

Rios (2024) also identified the following discourses about language: (a) “Students should only speak English in the classroom” (p. 8), and (b) “Students should sound like native speakers (i.e. speaking without an accent or grammatical errors) (p. 8).” These two discourses connect to my findings, where three of the four case study students experienced a normative demand regarding what constituted the permissible language of teaching and learning in the IOLA class. One way that this normative demand played out is that two of the multilingual students were worried about speaking with an accent and subsequently decreased their participation in class. In particular, they found whole-class discussions harder for participating than small-group discussions because they worried more about their accent in the larger discussion setting. They felt that participation in whole-class discussions required more succinct communication, with fewer errors. Given these induced challenges for multilingual students, it is critical to systematically design language-focused interventions in inquiry-based undergraduate mathematics classrooms.

A second implication from this study is that any intervention designed to improve the learning experiences of multilingual students must consider the diversity in their linguistic experiences, exemplified in Theme 1. Moreover, as suggested by Theme 2, one window for contextualizing the diversity of linguistic experiences is attending to the students’ backgrounds (factors such as comfort level with English, prior exposure to U.S. culture, and prior experience with inquiry-based culture). One takeaway is that when attempting to design more linguistically accessible learning environments, it might not suffice to consider only the experiences of multilingual students from certain backgrounds. For example, if we only considered students from backgrounds like Seok who have had more prior experiences with active learning courses and longer time in the U.S., we might miss the need to account for the culture shock that other students like Johan experienced in the IOLA class. Collectively, the student case studies described in Chapters 4 and 5 highlighted a diversity of instructional tools that were found to be helpful for individual multilingual students from certain backgrounds.

Yet, while there may be no one-size-fits-all solution to address all systemic challenges with respect to linguistic access, a third implication is that there may be instructional tools that might be experienced as language resources for multilingual students from various backgrounds. Table 7.3 in Theme 3 highlighted some examples of such instructional resources based on the experiences of the case study students. For instance, various multilingual students appreciated the instructional supplementation of speech with other mediums (e.g., writing and gesturing) and mathematical representations (e.g., equations and graphs) that made mathematical communication more visual, retraceable, concrete, or formal.

Furthermore, the IOLA instructor's case study (see Chapter 6) highlighted various instructional tools she occasionally integrated in her class that could serve as a starting point for other instructors to expand linguistic access in their classes. For example, in addition to designing for accessibility in the wording of the IOLA tasks, the instructor occasionally implemented the following instructional techniques during class: (a) mitigating potentially unrelatable problem contexts, (b) implementing a quick round-robin routine, (c) sequencing group sharing based on an observed chain of mathematical ideas, (d) promoting a comparing and contrasting technique, (e) highlighting and summarizing key mathematical ideas or topics, and (f) attempting to disrupt a norm about who constituted the default spokesperson to the whole class. Moreover, she also implemented a practice of constituting the tasks, which she leveraged throughout each lesson to promote linguistic access to the tasks for all students. To lend insight into this practice, she referred me to Jackson et al. (2012), which discusses four important elements of launching complex tasks: (a) Discuss the key conceptual features, (b) discuss the key mathematical ideas, (c) develop common language to describe the key features, and (d) maintain the cognitive demand. As noted in Chapters 4 and 5, various case study students found this instructional practice helpful for clarifying linguistic issues with some problem contexts in the IOLA tasks, including Task 7 (Meal Plans: Constraining the Number of Meals). Thus, this practice of constituting a task may be worth integrating as part of any language-focused intervention in inquiry-based mathematics classrooms.

Some of the instructional tools that the IOLA instructor integrated into her IOLA class aligned with the principles for curriculum design and pedagogy in multilingual secondary mathematics classrooms described in Zahner, Calleros et al. (2021). One principle is to "align the conceptual focus across the curriculum and carefully choose problem contexts" (p. 238). The IOLA tasks engender this principle in that they are organized as instructional units based on certain concepts. For example, the first six tasks (Tasks 1, 2, 3a, 3b, 4, and 5) made up a unit focused on span and linear independence. Various case study students described this feature of the tasks as helpful Another way the IOLA curriculum aligns with the first principle is that some tasks repeated story problem contexts. The student case studies provided evidence of how this task feature served as an instructional language resource for some of them (see, for example, Chapter 5–Luis' Case Study).

Another principle described by Zahner et al. (2021) is to incorporate structures that enable the widest possible participation in classroom discourse. Two language structures that are provided as examples of implementing this principle are: (a) alternating whole-class and small-group discussions, and (b) implementing mathematical language routines. The IOLA instructor regularly applied the first structure in her IOLA class. In addition, she also applied instructional techniques that can be considered MLRs, which are adaptable language mechanisms that can be used to amplify, evaluate, and develop students' language (Zwiers et al., 2017). For example, the comparing and contrasting routine she promoted in her class resembles the "Compare and Connect" MLR described in Zwiers et al. (2017, p. 16). Another technique she used to enable the widest possible participation during group sharing in whole-class discussions was a quick round robin. These design and teaching principles could form the foundation for enhancing linguistic accessibility in undergraduate mathematics classrooms that utilize inquiry-based approaches.

Finally, a fourth implication for practice from this study is that any language-focused intervention must consider the tensions experienced by inquiry instructors in balancing aspects of inquiry with linguistic accessibility. Theme 5 highlighted important tensions she experienced as the IOLA instructor. This theme also described ways that the IOLA instructor attempted to address some of these tensions.

These implications highlight the need to continue improving inquiry-based mathematics education to make it linguistically accessible to all students, including multilingual students. This highlighted area of improvement aligns with Laursen and Rasmussen's (2019) vision of adding a fourth pillar to inquiry-based mathematics education focusing on equitable instructional practice. While the IOLA instructor in my study attended to equitable student participation, she may not have specifically focused on all facets of linguistic access for multilingual students.

It is worth noting that none of the implications described in this section imply that mathematics education should move away from inquiry-based approaches into more traditional approaches. Traditional lecturing approaches have their own limitations with respect to linguistic access (e.g., Lew et al., 2016). Despite the need to continue improving inquiry approaches, these instructional methods have been shown to be a better alternative to lecturing (e.g., Freeman et al., 2014; Laursen et al., 2014; Rasmussen et al., 2006).

Practical Recommendations

Based on the findings and reflections from this study, I offer the following practical recommendations for inquiry instructors: (a) consider the perspectives of international students and other multilingual students, (b) promote a translanguaging space among students and the instructor, (c) intentionally support language demands, and (d) make classroom norms explicit.

First, I recommend that instructors consider the prior experiences and expectations of international students and other multilingual students. For example, instructors could consider ways in which their instruction might induce norm tensions, such as norm violations, student-centered norm incompatibilities, and class-centered norm incompatibilities. This recommendation aligns with prior research in K-12 mathematics settings, which shows that classrooms are places where individuals from different cultural norms of communication meet (e.g., Cazden, 2001; Seifert & Sutton, 2020).

To account for the clashes in expectations between students and inquiry approaches, I suggest that instructors make classroom norms and their negotiation explicit. When students join classroom environments that are guided by unfamiliar norms of interaction, students may be uncertain, perplexed, or even resistant to new methods of working (Sfard, 2007; Siegel & Borasi, 1994). So, explicitness can serve as a tool to address students' unfamiliarity with certain classrooms (Selling, 2016).

Additionally, I encourage instructors to intentionally identify and address the language demands for multilingual students in their classroom. For example, instructors could use the L-S-N framework (see Chapter 2) as a lens to begin identifying language demands as well as language resources that could balance those demands. This would allow instructors to recognize resources that may already be available in their instructional approach and instructional materials and ensure that they indeed function as resources for their multilingual students.

Finally, I invite instructors to adopt a translanguaging stance (García, 2009; García et al., 2017) that encourages students to draw on their full linguistic repertoire to communicate and learn mathematics among peers and the instructor. By allowing students to draw on their own language resources (e.g., their home language and translanguaging practices), instructors would increase multilingual students' opportunities to engage in discussions. In turn, multilingual students can develop multiple dimensions of mathematical literacy through increased participation in discussions (Zahner et al., 2021).

Implications for Research

This dissertation study contributed various conceptual and analytical tools to the field of mathematics education for investigating issues related to multilingual students. Three important research contributions were: (a) a conceptual framework for analyzing language demands and resources for students in a mathematics classroom, (b) a framework for analyzing norm tensions for students in a mathematics classroom, and (c) an application of a sociocultural theory of learning to the study of social and sociomathematical norms in the mathematics classroom.

First, the L-S-N framework was created with a sociocultural perspective as part of this study to analyze language demands and resources for multilingual students in one inquiry-oriented linear algebra class. As evidenced in the rich findings and themes generated from this study, the L-S-N framework revealed important insights about multilingual students regarding language demands, instructional resources, and student language resources. In addition, it highlighted the complex interrelation between these different aspects of linguistic experience and across different dimensions. While this framework was applied in one IOLA context with multilingual students as the focus, it also has the potential to be applied in different teaching and learning settings with students from other demographic groups.

Second, the Norm Tensions framework was created inductively from the initial stages of data analysis in this dissertation study and was then used deductively in subsequent stages to analyze norm tensions that might induce language challenges for multilingual students. The framework highlighted the importance of the source of the norms involved – that is, whether the norms came from the IOLA class or from a student's communities outside the class. In addition, the use of the framework afforded important insights about the role that norms played in mediating the students' linguistic experiences of the IOLA class. Such roles included violating or being incompatible with certain aspects of the IOLA class or a particular student. It is worth noting that while this framework was used to identify norm tensions that induced language challenges, it could also be used to identify norm tensions that induce other types of social challenges (not just linguistic).

Third, the application of a sociocultural lens to the investigation of norm-related issues afforded a new way of conceptualizing the role of students in the mathematics classroom that has not yet been explicitly leveraged in the mathematics education literature. Most, if not all, research articles on the identification and establishment of social and sociomathematical norms in the mathematics classroom appear to have drawn on an emergent perspective of learning (Cobb & Yackel, 1996). One example is the article by Yackel et al. (2000), where the authors showcase how social norms and sociomathematical norms regarding explanation were constituted in two different differential equations undergraduate classrooms. In all such articles, the instructor's role is positioned (if implicitly) as the broker between the local classroom community and the broader mathematical community. Zandieh et al. (2017) define a broker as "someone who can facilitate and fluidity of practices between different communities and who has membership status in all the different communities" (p. 97). These authors further demonstrate two distinct types of instructor brokering practices: (a) creating a boundary encounter, and (b) interpreting between communities. They define the first practice as "when a broker (i.e., the instructor) sets up an indirect interface between the classroom community and the broader mathematical community," (p. 98) – where a boundary encounter encompasses a boundary object that offers students a chance to participate in one or practices of the broader mathematical community. I argue that, by taking a sociocultural approach, my study expanded the notion of who can act as a broker and which communities are involved in the brokering process.

Specifically, my study positioned students themselves as brokers who could create their own boundary encounters and interpret between their local classroom community and the broader socio-linguistic and mathematical communities. For example, when the case study students used translating tools or mathematical websites, they were leveraging those tools as boundary objects that allowed them to interpret between communities with different language practices and mathematical approaches. It is worth noting that my Norm Tensions framework explicitly engendered the brokering role of students in relating norms or situational aspects between the IOLA classroom community and their other communities outside the class. This view on the role of students was critical for my study because it focused on multilingual students, whose identities lied at the intersection of various socio-linguistic communities.

In sum, a sociocultural lens afforded me the ability to situate the findings about the case study students' linguistic experiences in the context of their multilingual identities. This lens was also leveraged in creating and interpreting the L-S-N framework and the Norms Tension framework. Through these frameworks, I expanded the role of broker to all students, and in particular to multilingual students. This expanded view of students can allow researchers to shed light on how multilingual students leverage and navigate between their different communities, and how that process might mediate their linguistic experiences in the mathematics classroom, especially in the inquiry-based context.

Limitations and Future Research

The study was bounded by certain limitations that present opportunities for future research. Below, I present four limitations regarding the following aspects: the case selected for inquiry-based mathematics education, the backgrounds of the selected participants, the primary data source that was drawn for analysis, and the scope of the study. As I present each limitation, I discuss ways to address these limitations in future research studies. I end by providing a holistic plan for next steps to build on the research from this study.

First, the study was situated in one particular case of inquiry-based mathematics education: one inquiry-oriented linear algebra class. While this case successfully provided insights about the linguistic experiences of multilingual students in inquiry-based settings, it might also have presented some limitations. For example, this might explain why only one sociomathematical norm was identified as a language demand. Indeed, the IOLA instructor noted that she did not see sociomathematical norms as relevant for this class. In contrast, she could see sociomathematical norms as relevant in other classes, including proof-based courses where it may be important to structure or compose an argument in specifically mathematical ways. Thus, to gain more insights about the potential relationship between sociomathematical norms and language, a follow-up study could analyze language demands and resources in a proof-based undergraduate mathematics course.

Second, the participants selected as case study students were all male and mostly Asian. Moreover, the one Latino student in the case studies was born in the U.S. and was thus very comfortable with English. As suggested in Theme 2, the backgrounds of the case study students appeared to be critical in mediating the students' linguistic experiences. Thus, a follow-up study that investigates the linguistic experiences of students from additional backgrounds, including women and Latinx students who may not be comfortable with English, is warranted to understand the diversity of linguistic experiences more comprehensively in inquiry-based undergraduate mathematics classrooms and its relationship to students' backgrounds.

Third, the data that comprised the main source for analysis was the first interview of each case study participant. While focusing on Interview 1 provided a detailed view of the linguistic experiences of multilingual students, a systematic analysis of the classroom data in conjunction with Interview 2 (which included a video-recall component) could produce a more process-focused view of multilingual students' experiences. For example, this view might yield insights regarding how certain normative demands (norm tensions inducing language challenges) played out in the classroom interaction.

Fourth, to address the research questions, this study adopted a narrow focus of capturing multilingual students' experiences within the dimensions of the L-S-N framework. However, this focus is only one framing (among many possible framings) of how multilingual students may experience inquiry-based mathematics education. For example, this study did not focus on how the IOLA class may have met students' other goals – such as forming new intercultural friendships and developing conceptual understanding – that seem much more likely to develop in an inquiry class compared to a lecture. Other studies are needed that shed light on different aspects of the experiences of multilingual students in inquiry-based mathematics settings.

Overall, I plan to continue to refine the L-S-N framework by testing it in different inquiry-based undergraduate mathematics classrooms. I then plan to use the refined framework to develop principles for addressing language demands and leveraging language resources. Ultimately, I plan to design, test, and refine a language-focused intervention to improve the experiences of multilingual students in inquiry-based mathematics classes.

Significance of This Study

The findings from this dissertation study illuminated ways in which inquiry-based mathematics education can induce linguistic challenges for multilingual students. More generally, the interpretations of these findings suggested that inquiry does not guarantee positive linguistic and learning experiences for multilingual students, which aligns with the recent and more general argument that inquiry does not guarantee equity (Johnson et al., 2020). On the other hand, this study highlighted opportunities and tools for improving inquiry-based undergraduate mathematics classrooms for multilingual students.

This study also introduced the L-S-N framework – a framework rooted in a sociocultural theory of learning that could be used to analyze language demands, instructional resources, and student language resources for multilingual students in inquiry-based mathematics courses. The framework consists of three socio-linguistic dimensions–lexico-grammatical, situational, and normative–that capture different, but related conceptualizations of communication in mathematics classrooms. Moreover, through the initial stages of data analysis, an additional framework was developed for analyzing norm tensions between members of different communities. This study highlighted the affordances of applying a sociocultural lens for investigating issues and resources related to multilingual students.

Chapter 8 Conclusion

In this dissertation, I aimed to investigate the linguistic experiences of multilingual undergraduate students in inquiry-based mathematics education. To do so, I used a case study approach to identify the language demands, instructional language resources, and student language resources for four multilingual students in one inquiry-oriented linear algebra (IOLA) class. I also contextualized these findings through the perspective of the IOLA instructor. Below, I provide a brief overview of the chapters in this dissertation. Then I briefly address the research questions. Finally, I highlight the significance of this dissertation study.

Chapter Overview

In Chapter 1, I motivated this study. Prior research has shown that inquiry-based mathematics education (IBME) is more effective overall than traditional lecture. However, IBME may not be equitable for students from certain backgrounds (e.g., women). These findings, along with the demographic trend of increasing linguistic diversity in the undergraduate student population and the perceived centrality of communication in IBME approaches, raised the question about how multilingual students might experience such instructional approaches.

In Chapter 2, I contextualized inquiry-based mathematics education, introduced the study's overarching and conceptual frameworks, and reviewed the literature in relation to the conceptual framework and multilingual students. The overarching theory of learning in this study was a sociocultural perspective, where learning happens through socialization. The framework consisted of three dimensions that allowed me to conceptualize language demands and resources from different, but interrelated views of communication in the mathematics classroom: (a) lexico-grammatical, (b) situational, and (c) normative. Each dimension was operationalized by adapting a framework from the literature through a sociocultural perspective. The lexico-grammatical dimension focused on semantic and syntactic aspects of words and phrases. The situational dimension focused on daily uses of materials, activities, sign systems of communication, and structures of communication that either reflect or contradict beliefs about learning from a sociocultural perspective.

Through the literature review, I identified various gaps in the existing literature that I aimed to address through this dissertation study. Firstly, there was a scarcity of research rooted in the classroom environment regarding the lexico-grammatical dimension. Secondly, there was a dearth of research explicitly dedicated to exploring language resources within the lexico-grammatical dimension. Thirdly, substantive research within the situational dimension was notably lacking at the undergraduate level. Fourthly, while there existed substantive research within the normative dimension at the undergraduate level, it often failed to explicitly analyze and relate its findings with language demands and resources. Lastly, there was a noticeable absence of research in mathematics education concerning multilingual undergraduate students, with existing studies primarily originating outside the U.S. Furthermore, such research tended to adopt a cognitive perspective, potentially fostering a deficit view of multilingual students' abilities.

In Chapter 3, I discussed the framing and methods used in my study. I used a case study approach that allowed me to capture the multilingual students' linguistic experiences in the context of their unique backgrounds. Through convenience sampling, I selected an expert inquiry-based instructor with extensive teaching experience and background as a researcher of IOLA. She identified as White and reported being a predominant English speaker. Then, using a criterion sampling method, I selected four multilingual students to participate in the study. Three of the participants (Johan, Nam, and Seok) were international students who self-rated their

English proficiency as a four out of six and self-reported native proficiency in Malaysian, Vietnamese, and Korean, respectively. The remaining multilingual participant, Luis, was a U.S. domestic, Latino student who self-rated his English proficiency as a five and self-reported native proficiency in Spanish. For the data sources, this study combined the use of ethnographic classroom observations with two semi-structured interviews with the teacher and each multilingual student. I leveraged the first interview from each participant (which took an average of 2.5 hours) as my main source for analysis. For each participant case study, I analyzed the data using two-cycle coding, where the first cycle consisted of assigning initial codes to the interview data of each student, and the second cycle focused on compiling and grouping those codes into more general categories. Additionally, I applied a cross-case clustering technique to inductively group similar codes together. Finally, I conducted a cross-case comparison using various approaches, which included capturing and comparing the prominence of certain codes across the case study students.

In Chapter 4, I presented two case studies of international multilingual undergraduate students who had moved to the U.S. at the beginning of their undergraduate studies: Johan (from Malaysia) and Nam (from Vietnam). Both students experienced many language demands, few instructional language resources, and several student language resources. However, Johan faced significantly more normative demands than Nam, possibly due to a greater culture shock resulting from perceived differences between their home cultures and the U.S. culture.

In Chapter 5, I presented two case studies of multilingual undergraduate students who completed at least part of their K-12 education in the U.S. prior to their undergraduate studies: Seok (from Korea) and Luis (Latino student from the U.S.). Both students experienced few language demands in the IOLA classroom, many reliable instructional language resources, and several student language resources. However, Luis faced significantly fewer language demands, possibly due to greater confidence in English and prior experience with a linear algebra course and classes with high degrees of active learning.

In Chapter 6, I presented a case study of the IOLA instructor in relation to language demands, instructional language resources, and student language resources. She mainly highlighted language demands and instructional language resources. There was close alignment between the instructor's observations and the experiences reported by the case study students. The instructor added three new language demands and a significant amount of instructional language resources in the situational and normative dimensions. Additionally, the instructor provided valuable contextualization and insights into the findings reported by the case study participants. Specifically, she highlighted several tensions that she faced as an inquiry mathematics instructor regarding balancing aspects of inquiry with language accessibility: (a) designing tasks in linguistically accessible ways, while still allowing students opportunities to mathematize authentic problem contexts, (b) highlighting essential mathematical concepts while encouraging student autonomy, and (c) promoting equitable participation while maintaining coherence in class discussions.

In Chapter 7, I delved into five themes derived from reflecting on the findings across the case studies. The initial three themes captured the diverse linguistic experiences of multilingual students, emphasizing that these experiences, though influenced by individual backgrounds, may share certain consistent traits. The final two themes contextualized these shared traits within the broader inquiry-based mathematics education system. The identified themes were:

(a) The linguistic experiences of multilingual students in inquiry-based mathematics courses are not monolithic.

(b) The linguistic experiences of multilingual students may be strongly mediated by the students' backgrounds.

(c) There are important similarities in the linguistic experiences of various multilingual students in inquiry-based mathematics courses.

(d) Inquiry-based mathematics education can induce language demands for multilingual students and unintentionally undervalue their language resources.

(e) Balancing inquiry and linguistic access is a complex process.

In addition, I built on these themes to derive implications for practice and research. Practical implications included that: (a) inquiry does not guarantee positive linguistic and learning experiences for multilingual students, (b) any language-focused intervention must consider the diversity of backgrounds and linguistic experiences of multilingual students, (c) there are instructional tools that could be considered as starting points for promoting linguistic access for multilingual students, (d) any language-focused intervention must consider the tensions experienced by inquiry instructors in balancing aspects of inquiry with linguistic accessibility. One instructional tension highlighted in the last theme was allowing students to access their own digital resources while encouraging an inquiry approach to learning during class.

Furthermore, three important research contributions were: (a) a conceptual framework for analyzing language demands and resources for students in a mathematics classroom, (b) a framework for analyzing norm tensions for students in a mathematics classroom, and (c) an expanded view on the role of students in the mathematics classroom afforded through a sociocultural lens.

Research Questions

The research questions I investigated were:

(RQ1) What language demands do multilingual students experience in an undergraduate IOLA course?

(RQ2) What instructional language resources do multilingual students experience in an undergraduate IOLA course?

(RQ3) What language resources do multilingual students use to interpret and communicate mathematical information in an IO linear algebra course?

The linguistic experiences of the case study multilingual students in an inquiry-based mathematics classroom were not monolithic, but there were some key aspects of their experience that were shared across multiple students. Table 8.1 provides a list of the most relevant similarities in language demands, instructional language resources, and student language resources across the four case study students. A similarity was defined as a code (or combination of similar codes) that arose for at least two students.

Table 8.1:

Key Similarities of the Four Case Study Students' Linguistic Experiences in the IOLA Course

Linguistic Experience	Broken by dimension	Key Examples
Demands	Lexico-grammatical	Antonym pairs, unfamiliar notation, and words or phrases with unfamiliar meanings
	Situational	Speech communication, English pronunciation, whole-class and small-group discussion structures, tasks with too many words, and tasks with unfamiliar problem contexts
	Normative	Tensions related to what constituted the permissible language, dominant student communication, effective student communication, and competent student participation
Instructional Resources	Lexico-grammatical	None
	Situational	Tasks with fewer words, familiar problem contexts, repeated story contexts, primacy of symbolic mathematical language, visual representations, supplementation of speech with other mediums (gesturing, writing, and graphical representations), small-group and whole-class discussion structures, and linguistic markers of important information
	Normative	Tasks with realistic problem contexts*, centrality of peer communication, and structure of the class lesson

Table 8.1

Key Similarities of the Four Case Study Students' Linguistic Experiences in the IOLA Course, Continued

Linguistic Experience	Broken by dimension	Key Examples
Student Resources	Lexico-grammatical	Primary language translations of certain words and prior knowledge of mathematical terms
	Situational	Translating tools, prior knowledge of certain problem contexts or topics, multiple language(s), writing*, graphical representations, and symbolic representations
	Normative	Identifying important information and writing notes, and minimally communicating through speech

Note. *Examples that came up for multiple students across different dimensions.

Significance

The results of this dissertation shed light on the linguistic challenges that inquiry-based mathematics education can induce for multilingual students. Specifically, these results suggested that inquiry-based approaches may not always ensure positive linguistic and learning experiences for these students, echoing broader discussions that inquiry doesn't inherently promote equity. However, the study also identified avenues and resources to enhance the effectiveness of inquiry-based mathematics instruction for multilingual learners.

Additionally, this dissertation introduced the L-S-N framework–grounded in a sociocultural learning theory–which offers a method to examine language challenges, instructional supports, and student language resources in inquiry-based mathematics classes. This framework comprises three socio-linguistic dimensions—lexico-grammatical, situational, and normative—that provide varied yet interconnected perspectives on communication within math classrooms. Furthermore, the study presented a supplementary framework for analyzing norm conflicts among diverse community members, which emerged during the initial data analysis stages. This study underscored the benefits of adopting a sociocultural perspective to explore challenges and resources for multilingual students.

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Appendix A

This section contains the mathematical tasks covered in the IOLA class during the classroom observations. There were 12 mathematical tasks in total, each covering about one class day. The mathematical contexts and topics embedded in the tasks are described in Chapter 3–Data Sources–Classroom Observations–Mathematical Tasks.

I assigned a number to each task based on the first day of observation. For example, Task 2 was introduced on the second day of the course. On the third day, two separate tasks were introduced, so they are labeled 3a and 3b. I named the tasks to closely mirror the titles printed on the worksheets or their content. Some tasks consisted of two pages.

THE CARPET RIDE PROBLEM

Name ________________________ Group Members ______________________________

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the *hover board's* movement by the vector $\begin{bmatrix}3\\1\end{bmatrix}$. By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 miles East and 1 mile North of its starting location.

We denote the restriction on the *magic carpet's* movement by the vector $\begin{bmatrix}1\\2\end{bmatrix}$. By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 mile East and 2 miles North of its starting location.

SCENARIO ONE: THE MAIDEN VOYAGE

Your Uncle Cramer suggests that your first adventure should be to go visit the wise man, Old Man Gauss. Uncle Cramer tells you that Old Man Gauss lives in a cabin that is 107 miles East and 64 miles North of your home.

TASK:

Investigate whether or not you can use the hover board and the magic carpet to get to Gauss's cabin. If so, how? If it is not possible to get to the cabin with these modes of transportation, why is that the case?

As a group, state and explain your answer(s) on the group whiteboard. Use the vector notation for each mode of transportation as part of your explanation and use a diagram or graphic to help illustrate your point(s).

Figure A.1:

Task 1 (The Carpet Ride Problem: The Maiden Voyage)

The Carpet Ride Problem: Hide and Seek

Name ______________________ Group Members ___________________________

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the *hover board's* movement by the vector $\begin{bmatrix}3\\1\end{bmatrix}$. By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 units East and 1 unit North of its starting location.

We denote the restriction on the *magic carpet's* movement by the vector $\begin{bmatrix}1\\2\end{bmatrix}$. By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 unit East and 2 units North of its starting location.

Scenario Two: Hide-and-Seek

Old Man Gauss wants to move to a cabin in a different location. You are not sure whether Gauss is just trying to test your wits at finding him or if he actually wants to hide somewhere that you can't visit him.

Are there some locations that he can hide and you cannot reach him with these two modes of transportation? Describe the places that you can reach using a combination of the hover board and the magic carpet and those you cannot. Specify these geometrically and algebraically. Include a symbolic representation using vector notation. Also, include a convincing argument supporting your answer.

Use your group's whiteboard as a space to write out our work as you work together on this problem.

Figure A.2:

Task 2 (The Carpet Ride Problem: Hide and Seek)

The definition of the span of a set of vectors:

DEFINITION If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or **generated**) **by** $\mathbf{v}_1, \ldots, \mathbf{v}_p$. That is, Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with $c_1, \ldots, c_p$ scalars.

Determine or describe each of the following:

(a) $\text{Span}\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$

(b) $\text{Span}\left\{\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix}\right\}$

(c) $\text{Span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \end{bmatrix}\right\}$

(d) $\text{Span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \end{bmatrix}\right\}$

(e) $\text{Span}\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$

(f) $\text{Span}\left\{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}\right\}$

(g) $\text{Span}\left\{\begin{bmatrix} -3 \\ 0 \\ -2 \end{bmatrix}\right\}$

In other words, tell whether each is a point, line, plane or other shape. Also, describe – using an equation, graph or other means – which point, line plane or other shape it is.

Figure A.3:

Task 3a (Span Worksheet)

QUIZ 1, GROUP *Names*

1. (3 points) Answer the following questions regarding span{[4 2], [1 0] }.

   (a) The vectors in this set are in R^n, where n = _________.

   (b) This set spans a (circle one): point / line / plane / 3 dimensional space / 4 dimensional space

2. (2 points) The line indicated by $[x\ y\ ] = t[4\ 2\ ]$ can be rewritten in $y = mx + b$ format as:

3. (1 points) The line indicated by $[x\ y\ ] = t[4\ 2\ ] + [1\ 5\ ]$ can be rewritten in $y = mx + b$ format as:

4. (4 points) Answer the following questions regarding the set of vectors $\{[1\ 0\ 2\ ], [0\ 2\ 4\ ], [-3\ 0\ -6\ ]\}$.

   (a) The vectors in this set are in R^n, where n = _________.

   (b) This set spans a (circle one): point / line / plane / 3 dimensional space

   (c) List one vector in the span, indicating the coefficients (i.e., scalars) that show this vector is in the span.

   (d) List one vector not in the span, explaining how you know it is not in the span OR explain why it is impossible to list such a vector.

Figure A.4:

Task 3b (Group Quiz)

THE CARPET RIDE PROBLEM: GETTING BACK HOME

Name ________________________ Group Members ______________________________

Suppose you are now in a three-dimensional world for the carpet ride problem, and you have three modes of transportation: $\boldsymbol{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix}6\\3\\8\end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix}4\\1\\6\end{bmatrix}$.

You are only allowed to use each mode of transportation **once** (in the forward or backward direction) for a fixed amount of time (c_1 on $\mathbf{v}_1$, c_2 on $\mathbf{v}_2$, c_3 on $\mathbf{v}_3$). Find the amounts of time on each mode of transportation (c_1, c_2, and c_3, respectively) needed to go on a journey that starts and ends at home OR explain why it is not possible to do so.

As a group, state and explain your answer(s) on the group whiteboard. Use the vector notation for each mode of transportation as part of your explanation and use a diagram or graphic to help illustrate your point(s).

After you have completed the part above, answer the questions on the back of the page:

Figure A.5:

Task 4 (The Carpet Ride Problem: Getting Back Home)

1. Is there more than one way to make a journey that meets the requirements described above? (In other words, are there different combinations of times you can spend on the modes of transportation so that you can get back home?) If so, how?

2. Is there anywhere in this 3D world that Gauss could hide from you? If so, where? If not, why not?

3. What is $span\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}6\\3\\8\end{bmatrix}, \begin{bmatrix}4\\1\\6\end{bmatrix}\right\}$?

Figure A.5:

Task 4 (The Carpet Ride Problem: Getting Back Home, Continued)

Linear Independence and Dependence: Creating Examples

1. Fill in the following chart with the requested sets of vectors. Keep track of the strategies you use to generate the examples. Use set notation. State if not possible.

	Linearly dependent set	**Linearly independent set**
A set of 2 vectors in $\mathbb{R}^2$		
A set of 3 vectors in $\mathbb{R}^2$		
A set of 2 vectors in $\mathbb{R}^3$		
A set of 3 vectors in $\mathbb{R}^3$		
A set of 4 vectors in $\mathbb{R}^3$		

On the back of this page, you will be asked to show your work for three of the above.

Figure A.6:

Task 5 (Linear Independence and Dependence: Creating Examples)

2. Choose one set that you said was linearly dependent. Show a relationship between the vectors that proves the set is linearly dependent. [Optional, if time permit, use row reduction by hand to find all scalar solutions that show the linear dependence relationship.]

3. Choose one set that you said was linearly independent. Use row reduction, by hand, to show that the only solution to the appropriate vector equation is the trivial solution.

4. Choose one of the two boxes above that were not possible. Explain why it is not possible to create the vector set requested in that box. You may explain in geometric or intuitive terms. [Optional, if time permits, explain what happens in the row reduction process that explains why this set is not possible to construct.]

Figure A.6:

Task 5 (Linear Independence and Dependence: Creating Examples, Continued)

QUIZ 3 *Name* *Class Time*

1. (10 points) Find the intersection of the following three planes by solving the following system using row reduction *by hand*. Show all your work.

$$\begin{aligned} x - 2y + 3z &= 2 \\ 2x - 3y + 6z &= 3 \\ -x + 5y - 3z &= 1 \end{aligned}$$

After row reducing, state whether your solution indicates that the intersection is a point, line, plane or there is no intersection.

Figure A.7:

Task 6 (Practice for Individual Quiz)

Task 1: Meal Plans - Constraining the Number of Meals

You are enrolled in a semester long meal plan, which includes 210 meals. Please enter your desired number of each type of meal according to the following rules:

- There are 105 days per semester.
- Your total number of meals must equal 210.
- Each meal (breakfast, lunch, dinner) can be purchased up to twice per day. For example, you could choose 2 breakfasts per day for the 105 days (with no lunches or dinners at all) to cover the 210 meals.

Breakfast ☐

Lunch ☐

Dinner ☐

When the student signs up for the plan online, the student must type in how many of each meal type (breakfasts, lunches, and dinners) they want to purchase, but they do not have to decide on which days they will eat which meals.

Questions for whole class discussion:

1. List examples of meal plan options for the 210-meal plan. That is, how many breakfasts, lunches, and dinners could someone type in that would satisfy the 210-meal plan constraints?

2. Write an equation that captures the relationship between the number of breakfasts, lunches, and dinners that a student can choose for the meal plan.

3. In addition to the equation in 2, what other restrictions do we have on the numbers we can use for breakfast, lunches and dinners given the description of the scenario above.

4. The total number of possible choices for the 210-meal plan is closest to (circle one):

   1   20   2000   20,000   200,000

   Create a systematic list of solutions to guide your estimate.

Figure A.8:

Task 7 (Meal Plans: Constraining the Number of Meals)

In your small groups, consider the following problems:

5. **If we plotted all points, (B, L, D) that satisfy the conditions, what shape would that form on a 3D graph?** For example, would the points all lie along a line or perhaps 2 or 3 lines in different directions? Would the points instead all lie on the same plane or 2 or 3 specific planes? Or would the points be scattered (randomly or in some organized manner) throughout 3D space (beyond a few lines or planes)?

    a. Identify 8-10 (B, L, D) points that you think will help you determine the overall shape of the 3D graph, and plot them using GeoGebra 3D https://www.geogebra.org/3d?lang=en

    b. On your group's whiteboard or Jamboard, state a best guess for the shape that all such points would form. Write an argument as to why this makes sense and fits the points you have plotted. Write any questions you have or things you are wondering about this graph, and be prepared to share screenshots from Geogebra.

6. **No dinners**. Suppose in a particular semester you have evening obligations that prevent you from visiting the cafeteria, so you NEVER eat dinner as part of your meal plan.

    a. List at least 5 **no dinner** choices for your semester-long meal plan. How could you algebraically represent all of the "no dinner" choices? How could you geometrically represent all of the "no dinner" choices?

    b. Consider other options such as a **10-dinner** option. What would that look like graphically and algebraically? What about a **20-dinner, 30-dinner, etc.** Where are these options located on the shape that you predicted for question 5?

    c. *If time permits:* What can you say about all of the "no breakfast" options? All of the "no lunch" options?

Figure A.8:

Task 7 (Meal Plans: Constraining the Number of Meals, Continued)

Task 2: Meal Plans - Constraining the Cost

You are enrolled in a semester long meal plan, which includes 210 meals. Please enter your desired number of each type of meal according to the following rules:

- There are 105 days per semester.
- Your total number of meals must equal 210.
- Each meal (breakfast, lunch, dinner) can be purchased up to twice per day. For example, you could choose 2 breakfasts per day for the 105 days (with no lunches or dinners at all) to cover the 210 meals.

Breakfast $5 [ ]

Lunch $7 [ ]

Dinner $10 [ ]

New constraint: Cost of Meals

You have already found many different options for the 210 meal plan. As you explore the website more carefully, you notice that the cost of your 210 meals must add up to exactly $1500. Note that each meal is a different price (breakfasts cost $5 each, lunches cost $7 each, and dinners cost $10 each).

Questions for whole-class discussion (with think-pair-share as needed):

1. List some options that satisfy the *cost constraint* but not the *number of meals constraint* (in other words, the number of meals doesn't add to 210).

2. List some options that satisfy the constraints for both the *cost* and the *number* of meals.

3. In Task 1, we expressed the constraint on the number of meals as B+L+D=210.
   a. Write an equation for the *cost constraint.*

   b. If all of the solutions to the equation for the *cost constraint only* were plotted, what shape do you think would they form on a 3D graph and why?

Figure A.9:

Task 8 (Meal Plans: Constraining the Cost)

In your small groups, consider the following problems:

4. **If we** plotted **all points, (B, L, D) that satisfy the constraints for BOTH the cost and number of meals, what shape would that form on a 3D graph?**

    a. Identify 4-6 (B, L, D) points that you think will help you determine the overall shape of the 3D graph, and plot them using GeoGebra 3D https://www.geogebra.org/3d?lang=en

    b. On your group's whiteboard, state a best guess for the shape that all such points would form. Explain your reasoning.

    c. Find a way to write one or more algebraic expressions that describe these points. (You may use methods such as substitution, elimination, and/or row reduction to manipulate your algebraic expressions if you think it is helpful.)

5. **If time permits:** Write another (3rd) equation for the meal plan context. How many solutions do you think there are for the system consisting of this equation, as well as the equations for the cost and number of meals? How could you check?

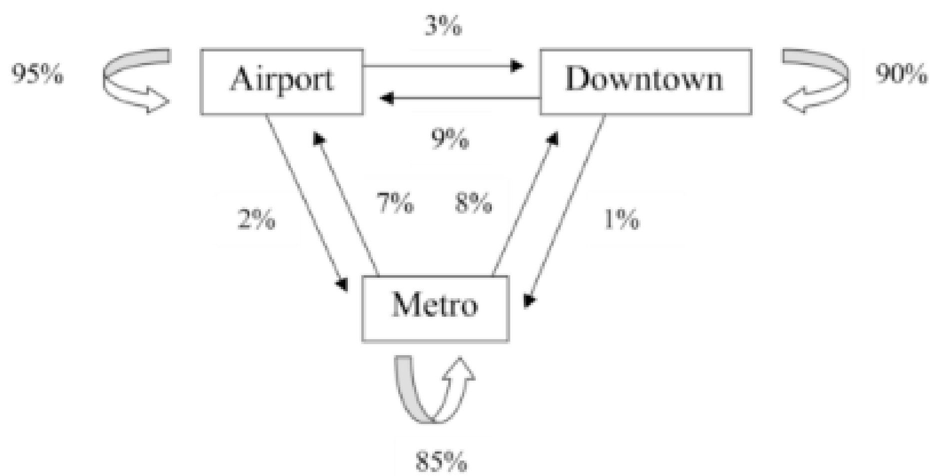
Figure A.9:

Task 8 (Meal Plans: Constraining the Cost, Continued)

The Car Rental Problem

The Mertz car rental company has three locations in a large west coast city: at the Airport, Downtown, and off the Metro. The company has been doing exceptionally well during the last year, and the management believes that this success has been in part due to their policy of allowing customers to return a rental to any of the three locations, regardless of which location the vehicle was rented from. Unfortunately, this policy has created something of a logistical nightmare for the company, as they have started to have problems with too many vehicles at the Airport and not enough at Metro. Currently, the company reshuffles the cars at the end of each week so that there are 500 cars at the Airport, 250 Downtown, and 200 at Metro. There is always more demand at each location than can currently be met.

Each week, about 95% of the vehicles rented from the Airport location are returned at the Airport location, about 3% rented at the Airport are returned Downtown, and about 2% of the cars rented from the Airport location are returned at the Metro location. The diagram below indicates the analogous statistics for the Downtown and Metro locations.



The management of Mertz has hired your team of consultants to help them build a better understanding of the distribution of cars, how that distribution changes over time, and how the company might most efficiently manage its resources to meet demand and optimize profit. In order to do this, the management has given you two tasks:

Eventually (big picture) we would like to determine how the company might most efficiently manage its resources to meet demand and optimize profit. In other words we would like to know (1) If we stopped reshuffling the cars, how many would eventually end up in each location and does the answer to that depend on the number of cars we start with at each location? (2) Is there a number of cars we could start with at each location that would give a relatively stable distribution?

Start with the simpler task of finding a way to use some methods of linear algebra to model this scenario. Use your model(s) to determine how many cars will be at each location at the end of week 1, 2 and 3, if there are initially 500 cars at the Airport, 250 Downtown, and 200 at Metro.

Figure A.10:

Task 9 (The Car Rental Problem)

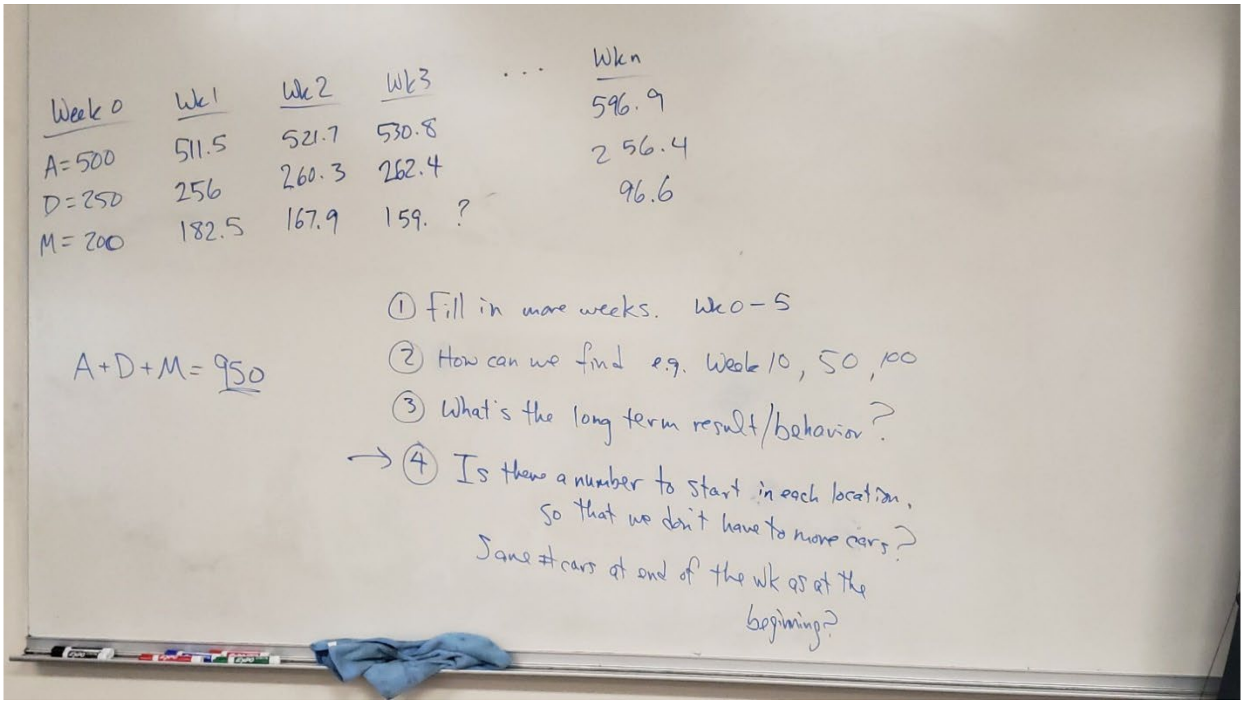
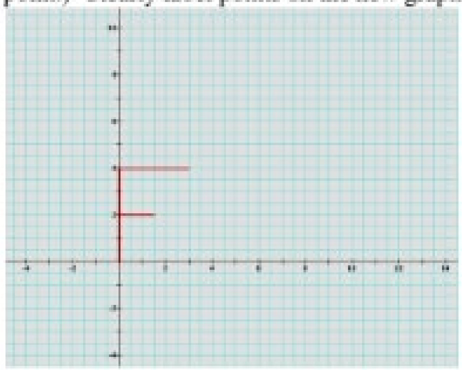


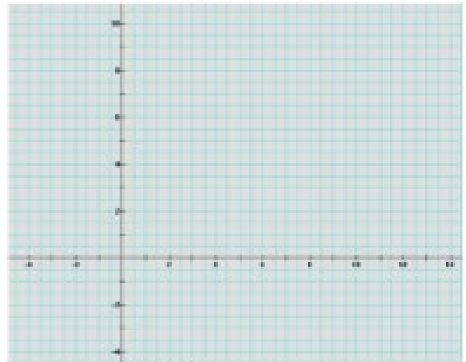
Figure A.11:

Task 10 (The Car Rental Problem: Follow-Up)

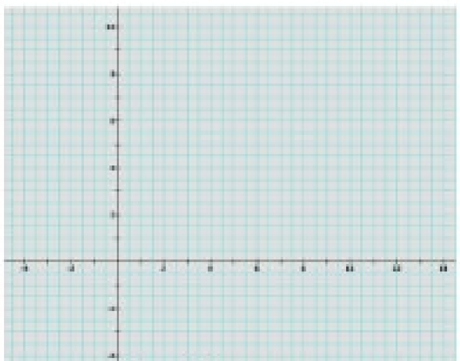
Consider the capital letter F as drawn on the upper left graph below. Some important points on the F occur at (0,0), (0,2), (0,4), (3,4) and (1.5,2). For each of the matrices listed, draw the image of the F when each point of the original F is multiplied by that matrix (e.g., for matrix **A** and point **x**, multiply **Ax** to find the new point.) Clearly label points on the new graphs.



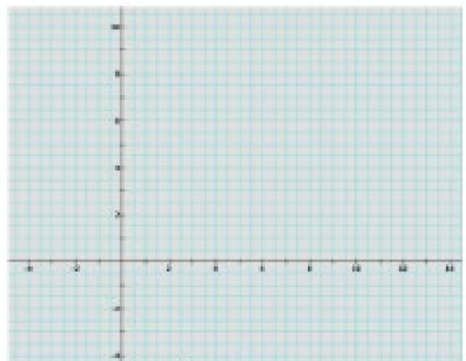
Original



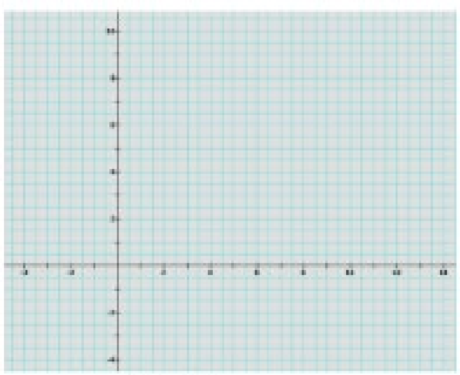
$$\mathbf{A} = \begin{bmatrix} \frac{5}{3} & 0 \\ 0 & 2 \end{bmatrix}$$



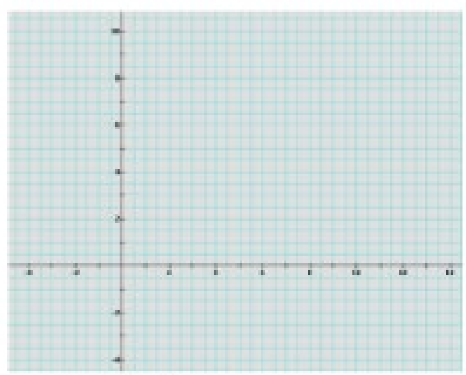
$$\mathbf{B} = \begin{bmatrix} \frac{5}{3} & 1 \\ 0 & 2 \end{bmatrix}$$



$$\mathbf{C} = \begin{bmatrix} \frac{5}{3} & 0 \\ 1 & 2 \end{bmatrix}$$



$$\mathbf{D} = \begin{bmatrix} \frac{5}{3} & 0 \\ 0 & -1 \end{bmatrix}$$



$$\mathbf{E} = \begin{bmatrix} \frac{5}{3} & 0 \\ 0 & 0 \end{bmatrix}$$

Figure A.12:

Task 11 (Geometric Interpretation of a Matrix Times a Vector)

Generalizations.

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Determine numeric values for *a*, *b*, *c* and *d* so that when matrix M is multiplied by the points in the letter F, each of the following transformations will occur:

1. A stretch of 2 in the x direction and a stretch of 3 in the y direction.
2. A reflection across the x-axis.
3. A reflection across the line y = x.
4. A collapse on the y-axis.
5. A rotation of 180 degrees.
6. A rotation counterclockwise of 90 degrees.

Figure A.12:

Task 11 (Geometric Interpretation of a Matrix Times a Vector, Continued)

Appendix B

This section contains the protocols for the semi-structured interviews and the contact summary form. There were two interviews for each participant (student or teacher). The contact summary template was used after each classroom observation or semi-structured interview event as a form of preliminary analysis.

Student Interview 1: Language Demands and Resources

Before we start, I would like to ask your permission to record this interview. The recordings will primarily be used by me to recall what we discussed. Otherwise, members of my dissertation committee may ask to see the recordings. I will not post these recordings on the internet, or in any way make the video clips available to the general public. I will also use a pseudonym for both you and the school in any publications or presentations. Do you have any questions for me? Do you mind if I record this interview?

(START RECORDING!)

Thank you for agreeing to participate in my dissertation study. My name is Ernesto Calleros and I am a PhD student in the Mathematics and Science Education joint doctoral program at San Diego State University and UC San Diego. Dr. Bill Zahner is my graduate advisor at SDSU. You are welcome to contact him should you have any questions or concerns about this study. Both of our contact information is on the assent form.

During this interview, I will ask you some questions about your experiences in your math classes.

Component 1: General Questions

1. Are you aware that the course you are taking may be classified as an inquiry-based course, as opposed to a lecture-based course?
    a. What does "inquiry-based" mean to you?
2. Is this your first time taking an inquiry-based course?
    a. If not, when did you take your first inquiry-based course?
    b. Have you taken math courses conducted as lectures instead?
3. How easy or difficult is it to participate in this inquiry-based course?
    a. In what way is it difficult to participate?
    b. In what way is it easy to participate?
4. How easy or difficult is it to understand or follow what is going on in this inquiry-based course?
    a. In what way it is difficult to understand or follow what is going on in class?
    b. In what ways is it easy to understand or follow what is going on in class?
5. Are there any challenges related to language or communication in this class?
    a. If so, how do they impact you or your peers?
6. Is it easier or harder to participate in an inquiry-based class compared to in a lecture-based class?

    a. How so?

7. Is it easier or harder to understand what is going on in this inquiry-based class compared to a lecture-based class?
    a. How so?

Component 2: Lexico-Grammatical

8. Are there any words or phrases that have been difficult or confusing during class?
    a. Can you give some examples?
9. Are there any particular mathematical terms, symbols, or visuals that may be difficult or confusing for you during class?
    a. Can you give some examples?
10. Are there words or sentence structures for expressing mathematical relationships (e.g., the words "greater than" for comparison and "if-then" structure for logical statements) that were difficult to understand?
11. What do you think the terms linear independence and span mean?
12. Was it easy or hard to understand the meaning of linear independence and span during class?
    a. How so?

Component 3: Situational

13. How would you describe the seating arrangement and technology used (or banned) in your class (e.g., table groups, lined up individual desks)?
    a. Does that arrangement make it easier or harder to understand or participate in class?
    b. Does that technology use (or ban) make it easier or harder to understand or participate in class?
14. How would you describe the types of math problems and activities presented or promoted in your class?
    a. Were the math problems easy or hard to access linguistically?
        i. How so?
    b. Were the contexts in the math problems easy or hard to understand?
        i. Can you give some examples?
    c. Were the instructions or expectations in the problem clear or unclear?
    d. What type of writing or communication did the problems or tasks request?
    e. What types of mathematical activity did you see the teacher doing through these tasks or problems?
    f. What types of mathematical activity did you find yourself doing through these tasks or problems?
        i. Were these activities difficult to engage in?
15. What modes of communication (e.g., writing, speech, gesturing, (en)acting) does the teacher use in this class?
    a. How does this make it easier or harder to understand?
    b. What modes are prioritized?
    c. How does this impact your understanding?
16. What modes of communication (e.g., writing, speech, gesturing, (en)acting) do you and

your classmates use in this class?
    a. How does this make it easier or harder to participate?
    b. What modes are prioritized?
    c. How does this impact your understanding or participation?
17. What types of representations (e.g., verbal, graphical, tabular, or symbolic) does the teacher use?
    a. How does this make it easier or harder to participate?
    b. What representations are prioritized?
    c. How does this impact your understanding or participation?
18. What types of representations (e.g., verbal, graphical, tabular, or symbolic) do you and your classmates use in this class?
    a. How does this make it easier or harder to participate?
    b. What modes are prioritized?
    c. How does this impact your understanding or participation?
19. What language(s) does the teacher use in this class?
    a. How does this make it easier or harder to understand?
    b. What modes are prioritized?
    c. How does this impact your understanding?
20. What language(s) do you and your classmates use in this class?
    a. How does this make it easier or harder to understand?
    b. What modes are prioritized?
    c. How does this impact your understanding?
21. How would you say the discussion is organized?
    a. If you had to describe how the discussion is organized into various segments (or episodes), what would each segment be?
    b. How does this impact your experience of the class?
22. Your class switched between whole-class and small-group discussions.
    a. How do you feel about communicating in whole-class discussions?
        i. How so?
    b. How do you feel about communicating in small-group discussions?
        i. How so?
    c. Do you find it easier or harder to communicate in small-group discussions than in whole-class discussions?
        i. How so?
    d. Do you find it helpful or unhelpful that the class included both whole-class and small-group discussions?
        i. How so?
    e. Do you find it helpful or unhelpful that the class switched back and forth between whole-class and small-group discussions?
        i. How so?

Component 4: Normative

23. How are you expected to participate in this class?
    a. Are these expectations easy or hard to meet?
24. Are there particular ways students should say certain things in this class due to it being an inquiry-based class?

    a. For example, if you want to provide an explanation, does your explanation need to be said in a certain way or include certain things?
    b. Can you give some examples?
    c. Are these expectations easy or hard to meet?
25. Are there particular ways in which students should act or participate in this class due to it being a math class versus a class in a different subject?
    a. Can you give some examples?
    b. Are these expectations easy or hard to meet?
        i. How so?
26. How do you know when it's a good time to participate?
    a. When you want to ask or say something, what tells you when it's an appropriate time?
27. How easy or difficult is it to figure out how you are expected to say things in this class?
    a. Can you give some examples?
28. How easy or difficult is it to figure out when you are allowed or not to participate in this class in a given moment?
    a. Can you give some examples?

Component 5: Mathematical Tasks—Sorting Activity

Here are the first 12 mathematical tasks you saw in your linear algebra class. Please sort these tasks by how linguistically challenging they are (with 1 being the most challenging) and explain your reasoning.

Student Interview 2: Video-Stimulated Recall Dialogue (VSRD)

Hello! It's great to see you again! Before we start, I would like to ask your permission to record this interview. The recordings will primarily be used by me to recall what we discussed. Otherwise, members of my dissertation committee may ask to see the recordings. I will not post these recordings on the internet, or in any way make the video clips available to the general public. I will also use a pseudonym for both you and the school in any publications or presentations. Do you have any questions for me? Do you mind if I record this interview?

(START RECORDING!)

Last time I learned so much from and with you! In particular, you shared several ways that certain words and terms, participation expectations, and arrangements of communication might make inquiry-based classes harder or easier to participate in and/or to understand what was going on in class.

To help us reflect on your experiences, I'll show you several video clips of moments that happened in your class, and I'll ask some questions regarding any linguistic challenges you faced, as well as any resources you experienced or utilized.

Here is the video clip. As you watch it, feel free to pause the video when you identify a linguistic challenge or support. [Start playing video clip]

[If the student participant pauses the video at any moment, ask them to share why they paused it. If they share a linguistic challenge, ask them if the challenge was addressed. If so, ask what or who helped address it and how. Let them continue watching the video and repeat the procedure at any subsequent pauses.]

Teacher Interview 1: Language Demands and Resources

Before we start, I would like to ask your permission to record this interview. The recordings will primarily be used by me to recall what we discussed. Otherwise, members of my dissertation committee may ask to see the recordings. I will not post these recordings on the internet, or in any way make the video clips available to the general public. I will also use a pseudonym for both you and the school in any publications or presentations. Do you have any questions for me? Do you mind if I record this interview?

(START RECORDING!)

Thank you for agreeing to participate in my dissertation study. My name is Ernesto Calleros and I am a PhD student in the Mathematics and Science Education joint doctoral program at San Diego State University and UC San Diego. Dr. Bill Zahner is my graduate advisor at SDSU. You are welcome to contact him should you have any questions or concerns about this study. Both of our contact information is on the assent form.

During this interview, I will ask you some questions about your experiences teaching this linear algebra course to multilingual students. When I say students in the interview, I mean multilingual students. If you can't specify for multilingual students, you can share your experiences with all students.

Component 1: General Questions

1. How would you classify this course as an inquiry-based course, as opposed to a lecture-based course?
    a. What does "inquiry-based" mean to you?
2. How long have you been teaching an inquiry-based course?
    a. Have you also taught math courses conducted as lectures instead?
3. How easy or difficult is it for students to participate in this inquiry-based course?
    a. In what way is it difficult for students to participate?
    b. In what way is it easy for students to participate?
4. How easy or difficult is it for students to understand or follow what is going on in this inquiry-based course?
    a. In what way is it difficult to understand or follow what is going on in class?
    b. In what ways is it easy to understand or follow what is going on in class?
5. Are there any challenges related to language or communication in this class for students?
    a. If so, how do they impact the students?
6. Is it easier or harder for students to participate in an inquiry-based class compared to in a lecture-based class?
    a. How so?
7. Is it easier or harder for students to understand what is going on in this inquiry-based class compared to a lecture-based class?
    a. How so?

Component 2: Lexico-Grammatical

8. Are there any words or phrases that have been difficult or confusing for students during

class?
    a. Can you give some examples?
9. Are there any particular mathematical terms, symbols, or visuals that may be difficult or confusing for your students during class?
    a. Can you give some examples?
10. Are there words or sentence structures for expressing mathematical relationships (e.g., the words "greater than" for comparison and "if-then" structure for logical statements) that have been difficult for students to understand or use appropriately?
    a. Can you give some examples?
11. What do you think the terms linear independence and span mean to students?
12. Was it easy or hard for students to understand the meaning of linear independence and span during class?
    a. How so?

Component 3: Situational

13. How would you describe the seating arrangement and technology used (or banned) in your class (e.g., table groups, lined up individual desks)?
    a. Does that arrangement make it easier or harder for students to understand or participate in class?
    b. Does that technology use (or ban) make it easier or harder for students to understand or participate in class?
14. How would you describe the types of math problems and activities presented or promoted in your class?
    a. Were the math problems easy or hard for students to access linguistically?
        i. How so?
    b. Were the contexts in the math problems easy or hard for students to understand?
        i. Can you give some examples?
    c. Were the instructions or expectations in the problem clear or unclear for your students?
    d. What type of writing or communication did the problems or tasks request?
    e. What types of mathematical activity did you see yourself as the teacher doing through these tasks or problems?
    f. What types of mathematical activity did you see your students doing through these tasks or problems?
        i. Were these activities difficult for students to engage in?
15. What modes of communication (e.g., writing, speech, gesturing, (en)acting) do you use in this class?
    a. How does this make it easier or harder for your students to understand?
    b. What modes are prioritized?
    c. How does this impact your students' understanding?
16. What modes of communication (e.g., writing, speech, gesturing, (en)acting) do students use in this class?
    a. How does this make it easier or harder to participate?
    b. What modes are prioritized?
    c. How does this impact your students' understanding or participation?
17. What types of representations (e.g., verbal, graphical, tabular, or symbolic) did you use?

    a. How does this make it easier or harder for your students to participate?
    b. What representations are prioritized?
    c. How does this impact your students’ understanding or participation?
18. What types of representations (e.g., verbal, graphical, tabular, or symbolic) do students use in this class?
    a. How does this make it easier or harder to participate?
    b. What modes are prioritized?
    c. How does this impact your students’ understanding or participation?
19. What language(s) do you use in this class?
    a. How does this make it easier or harder for your students to understand?
    b. What modes are prioritized?
    c. How does this impact your students’ understanding?
20. What language(s) do your students use in this class?
    a. How does this make it easier or harder to understand?
    b. What modes are prioritized?
    c. How does this impact your students’ understanding?
21. How would you say the discussion is organized?
    a. If you had to describe how the discussion is organized into various segments (or episodes), what would each segment be?
    b. How does this impact your students’ experiences of the class?
22. Your class switched between whole-class and small-group discussions.
    a. How do your students feel about communicating in whole-class discussions?
        i. How so?
    b. How do your students feel about communicating in small-group discussions?
        i. How so?
    c. Do you find it easier or harder for your students to communicate in small-group discussions than in whole-class discussions?
        i. How so?
    d. Do you find it helpful or unhelpful for your students that the class included both whole-class and small-group discussions?
        i. How so?
    e. Do you find it helpful or unhelpful for your students that the class switched back and forth between whole-class and small-group discussions?
        i. How so?

Component 4: Normative

21. How are students expected to participate in this class?
    a. Are these expectations easy or hard for students to meet?
22. Are there particular ways students should say certain things in this class due to it being an inquiry-based class?
    a. For example, if a student wants to provide an explanation, does their explanation need to be said in a certain way or include certain things?
    b. Can you give some examples?
    c. Are these expectations easy or hard for students to meet?
        i. How so?
23. Are there particular ways in which students should act or participate in this class due to it

being a math class versus a class in a different subject?

    a. Can you give some examples?
    b. Are these expectations easy or hard for students to meet?
        i. How so?

24. How would your students know when it's a good time to participate?
    a. When students want to ask or say something, what tells them when it's an appropriate time?
25. How easy or difficult is it for students to figure out how they are expected to say things in this class?
    a. Can you give some examples?
26. How easy or difficult is it for students to figure out when they should or should not participate in this class in a given moment?
    a. Can you give some examples?

Teacher Interview 2: Video-Stimulated Recall Dialogue (VSRD)

Hello! It's great to see you again! Before we start, I would like to ask your permission to record this interview. The recordings will primarily be used by me to recall what we discussed. Otherwise, members of my dissertation committee may ask to see the recordings. I will not post these recordings on the internet, or in any way make the video clips available to the general public. I will also use a pseudonym for both you and the school in any publications or presentations. Do you have any questions for me? Do you mind if I record this interview?

(START RECORDING!)

Last time I learned so much from and with you! In particular, you shared several ways that certain words and terms, participation expectations, and arrangements of communication might make inquiry-based classes harder or easier for students to participate in and/or to understand what was going on.

To help us reflect on your students' experiences, I'll show you several video clips of moments that happened in your class, and I'll ask some questions regarding any linguistic challenges your students may have faced, as well as any resources they may have experienced or utilized.

Here is the video clip. As you watch it, feel free to pause the video when you identify a linguistic challenge or support for your students. [Start playing video clip]

[If the teacher pauses the video at any moment, ask them to share why they paused it. If she shares a linguistic challenge, ask her if the challenge was addressed. If so, ask what or who helped address it and how. Let her continue watching the video and repeat the procedure at any subsequent pauses.]

Contact Summary Form

Data source:____________________

Event date:____________________

Event topic or interviewee:____________________

1. What salient categories did I encounter in this data collection event? (Include here any salient information regarding language demands, instructional language resources, or student language resources. Tag any potential categories with their primary dimensions from the L-S-N conceptual framework.)

2. Summarize the information I got (or failed to get) in relation to each dimension and component from the L-S-N conceptual framework. (As needed, add rows to the tables.)

Table B.1:

Lexico-Grammatical Dimension

Component	Language Demands	Instructional Language Resources	Student Language Resources
Semantic			
Syntactic			

Table B.2:

Situational Dimension

Component	Language Demands	Instructional Language Resources	Student Language Resources
Material			
Activity			
Semiotic			
Sociocultural			

Table B.3:

Normative Dimension

Component	Language Demands	Instructional Language Resources	Student Language Resources
Social			
Socio-mathematical			

3. What made certain categories more salient, interesting, or important than others during this event? (Include any relevant timestamps from the fieldnotes.)

4. What new (or remaining) questions do I have in considering the next data collection event? (Include here any difficulties with assigning categories or operationalizing a construct from the framework, emergent ideas of new constructs, follow-up questions for a particular participant, or wonderings about an aspect of the IOLA classroom environment.)

Appendix C

This section contains the lists of finalized codes that spanned all student case study interview data. There is one table for each research question and each dimension of the L-S-N framework, except for the situational dimension. For each research question, the situational dimension has two tables, one for the communication within the tasks and one for general classroom communication.

Table C.1:

Lexico-Grammatical Demands Experienced in the IOLA Classroom

Component	Lexico-Grammatical Demands	Example	Context of Example
Semantic	Antonym pairs	"East" versus "West"	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
		"North" versus "South"	
		"Numerator" versus "denominator"	General classroom communication
	New mathematical terms	"Vector notation"	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
		"Span"	Task 3a (Span Worksheet)
			General classroom communication
		"Linearly independent"	General classroom communication
			None provided
		"Linearly dependent"	General classroom communication
	Words or phrases with unfamiliar meanings	"Up to twice"	Task 7 (Meal Plans: Constraining the Number of Meals)
		"Matrix"	General classroom communication

Table C.1:

Lexico-Grammatical Demands Experienced in the IOLA Classroom, Continued

Component	Lexico-Grammatical Demands	Example	Context of Example
Syntactic	Unfamiliar notation	Notation of a vector	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
	Misleading context words	“Cramer”	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
			Task 2 (The Carpet Ride Problem: Hide and Seek)
		“Gauss”	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
			Task 2 (The Carpet Ride Problem: Hide and Seek)

Table C.2:

Situational Demands Experienced on the IOLA Tasks

Component	Task Feature as Situational Demand	Task Example
Material	Too many words	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
		Task 4 (The Carpet Ride Problem: Getting Back Home)
		Task 7 (Meal Plans: Constraining the Number of Meals)
		Task 8 (Meal Plans: Constraining the Cost)
		Task 9 (The Car Rental Problem)
	Unfamiliar problem contexts	Task 7 (Meal Plans: Constraining the Number of Meals)
		Task 8 (Meal Plans: Constraining the Cost)
	Instructions were too general	Task 9 (The Car Rental Problem)
	Vague questions	Task 11 (Geometric Interpretation of a Matrix Times a Vector)
		Task 7 (Meal Plans: Constraining the Number of Meals)
		Task 5 (Linear Independence and Dependence: Creating Examples)
	Misalignment between problem contexts and questions	Task 10 (The Car Rental Problem: Follow-Up)
	Lots of questions	Task 7 (Meal Plans: Constraining the Number of Meals)
	Problem contexts that were not straightforward	None provided

Table C.2:

Situational Demands Experienced on the IOLA Tasks, Continued

Component	Task Feature as Situational Demand	Task Example
Activity	Requests of mathematical explanations	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
Semiotic	Disconnected visual	Task 9 (The Car Rental Problem)

Table C.3:

Situational Demands Experienced in the IOLA Classroom's General Communication

Component	Classroom Communication Tool	Feature of Tool as Situational Demand
Material	Google Translate	Literal in translating
Activity	Class discussions	Wordy
		Complexity of explaining
Semiotic	Speech communication	Impermanent
		Informal
		Obscure
	Verbal representations	Unimaginable
	English pronunciation	Diverse pronunciation
Sociocultural	Peer interpersonal communication	Incomprehensible
	Whole-class discussions	Need to convey information loudly
		Need to convey information succinctly
		Less comfortable speaking to a wide audience
	Small-group discussions	Need to relate with his peers and initiate conversations with them

Table C.4:

Normative Demands Experienced in the IOLA Classroom

Audience	Norm Trait	Norm Tension	Norms or Situational Aspects	
			Community Outside IOLA Classroom	IOLA Classroom Community
Overall class communication	Permissible language	Class-centered incompatibility	Malaysian as a primary language	English only
			Vietnamese as a primary language	
			Korean as a primary language	
	Mathematically productive solution method	Violation	Unique and general way to solve a problem	Any way that solved the problem
	Required genre of student communication		Answers only	Answers with explanation or justification
	Default small-group spokesperson	Class-centered incompatibility	Difficulty to break the classroom norm without explicit teacher authorization	Students who contributed the most to the small-group discussion or who seemed to enjoy talking.
	Effective student communication	Violation	Minimal communication and interaction among students or between students and the whole class	Intensive communication and interaction among peers or between students and the whole class

Table C.4:

Normative Demands Experienced in the IOLA Classroom, Continued

Audience	Norm Trait	Norm Tension	Norms or Situational Aspects	
			Community Outside IOLA Classroom	IOLA Classroom Community
Instructor-student	Respectful behavior	Violation	Not interrupting the instructor	Interrupting the instructor
	Effective communication		Listening quietly to teacher talk and only asking questions as a last resort after teacher talk	Asking questions during teacher talk
	Competent participation	Student-centered incompatibility	Identifying important information and writing notes	Not clear which information was important
Student-student	Respectful peer communication	Violation	Appreciating being told when they have the wrong answer or method	Feeling offended when told they have the wrong answer or method
	Ethical peer behavior	Student-centered norm	Sharing only correct solutions	Not clear which solution method was correct
	Transparent peer communication	Violation	Going straight to the point	Indirect speech
	Effective peer collaboration	Violation	Working individually without talking and then comparing written solutions	Co-constructing solutions primarily through talking
	Dominant mode of peer communication	Class-centered incompatibility	Written ideas	Spoken ideas

Table C.4

Normative Demands Experienced in the IOLA Classroom, Continued

Audience	Norm Trait	Norm Tension	Norms or Situational Aspects	
			Community Outside IOLA Classroom	IOLA Classroom Community
Small-group communication	Logical role of communication	Violation	Share only mathematical knowledge one is confident in	Share one's mathematical thinking and struggles as a way to develop mathematical knowledge

Table C.5:

Instructional Lexico-Grammatical Resources Experienced in the IOLA Classroom

Component	Lexico-Grammatical Resource	Example	Context of Example
Semantic	Contextual interpretation of notation	Notation of a vector	Task 1 (The Carpet Ride Problem: The Maiden Voyage)

Table C.6:

Instructional Situational Resources Experienced in the IOLA Tasks

Component	Task Feature as Situational Resource	Task Example
Material	Fewer words (than in other tasks)	Task 11 (Geometric Interpretation of a Matrix Times a Vector)
		Task 6 (Practice for Individual Quiz)
		Task 4 (The Carpet Ride Problem: Getting Back Home)
	Familiar problem contexts	Task 6 (Practice for Individual Quiz)
		Task 11 (Geometric Interpretation of a Matrix Times a Vector)
	Realistic problem contexts	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
		Task 7 (Meal Plans: Constraining the Number of Meals)
	Repeated story contexts	Task 1 (The Carpet Ride Problem: The Maiden Voyage) & Task 2 (The Carpet Ride Problem: Hide and Seek)
		Task 7 (Meal Plans: Constraining the Number of Meals) & Task 8 (Meal Plans: Constraining the Cost)
		Task 9 (The Car Rental Problem) & Task 10 (The Car Rental Problem: Follow-Up)
	Instructions that were more specific (than in other tasks)	Task 10 (The Car Rental Problem: Follow-Up)
	Self-contained information	Task 3a (Span Worksheet)
	Marked layouts	Task 1 (The Carpet Ride Problem: The Maiden Voyage)

Table C.6:

Instructional Situational Resources Experienced in the IOLA Tasks, Continued

Component	Task Feature as Situational Resource	Task Example
Material (continued)	Straightforward questions	Task 9 (The Car Rental Problem)
		Task 2 (The Carpet Ride Problem: Hide and Seek)
		Task 3a (Span Worksheet)
		Task 3b (Group Quiz)
		Task 4 (The Carpet Ride Problem: Getting Back Home)
		Tasks 5 (Linearly Independence and Dependence: Generating Examples)
		Task 6 (Practice for Individual Quiz)
		Task 7 (Meal Plans: Constraining the Number of Meals)
		Task 9 (The Car Rental Problem)
		Task 10 (The Car Rental Problem: Follow-Up)
Semiotic	Primacy of symbolic mathematical language	Task 3b (Group Quiz)
		Task 6 (Practice for Individual Quiz)
	Essential visual representation	Task 9 (The Car Rental Problem)
		Task 11 (Geometric Interpretation of a Matrix Times a Vector)

Table C.7:

Instructional Situational Resources Experienced in the IOLA Classroom's General Communication

Component	Classroom Communication Tool as a Situational Resource	Resourceful Feature of Tool
Activity	Teacher's communication	Linguistic marker
Sociocultural	Small-group discussions	More private with the opportunity to provide or obtain clarification
	Whole-class discussions	Wide audience with access to more mathematical ideas
Semiotic	Supplementation of speech communication with writing	Permanent Visual
	Supplementation of verbal representations with graphical representations	Visual
	Supplementation of speech communication with gesturing	

Table C.8:

Instructional Normative Resources Experienced in the IOLA Classroom

Tool of Communication	Typical Feature as a Normative Resource
Problem contexts of class tasks	Realistic contexts
Structure of the class lesson	Combination of different participant structures of support (e.g., individual and small group) Access to different audiences of support (e.g., peers and instructor)
Nature of overall class communication	Centrality of peer interaction
Regularity in the teacher's speech communication	Helped identify the topic(s) of each lesson
Nature of student communication	Unstructured form
Nature of interaction	Centrality of peer communication and interaction

Table C.9:

Student Lexico-Grammatical Resources Utilized in the IOLA Classroom

Component	Student Lexico-Grammatical Resources	Example	Context of Example
Semantic	Primary language translations of antonym pairs	Malay translations of "North, east, west, south"	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
	Prior knowledge of mathematical terms in primary language	Vietnamese term for "matrix"	General classroom communication
	Primary language translations of mathematical terms not known in English	Vietnamese translation of the term "matrix"	
		Vietnamese translation of "linear independence"	
	Definitions of new mathematical terms in English and primary language	Definition of "linear independence" (in English and Vietnamese)	
	Compound structure of new mathematical terms	Breaking down "linear independent" into "linear" and "independent"	
	Prior knowledge of mathematical terms	"Gaussian Elimination"	
		"Cramer's Rule"	
		"row reduction"	
		"span"	
		"linearly independent"	

Table C.9:

Student Lexico-Grammatical Resources Utilized in the IOLA Classroom, Continued

Component	Student Lexico-Grammatical Resources	Example	Context of Example
Syntactic	Prior knowledge of mathematical notation	Row form of a vector with sharp brackets	Task 1 (The Carpet Ride Problem: The Maiden Voyage)

Table C.10:

Student Situational Resources Utilized in the IOLA Classroom

Component	Student Communication Tool as Situational Resource	Resourceful Feature of Tool	Example Context Where Tool Was Used
Material	Google Translate	Translating tasks to Johan's various language(s)	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
	Google search	Accessing images to clarify language on tasks	
	Prior knowledge of certain problem contexts or topics	Interpreting problem contexts in tasks	Task 11 (Geometric Interpretation of a Matrix Times a Vector)
			Task 6 (Practice for Individual Quiz)
	Mathematically specific translation tools	Translating mathematical terms like "matrix" into Vietnamese	General Classroom Communication
	Mathematical websites in multiple languages	Defining and explaining meanings of mathematical terms in Vietnamese and English	
	Prior linear algebra notes	Interpreting IOLA classroom communication	
Activity	Metalinguistic strategies	Identifying a topic from class to revisit later	General Classroom Communication

Table C.10:

Student Situational Resources Utilized in the IOLA Classroom, Continued

Component	Student Communication Tool as Situational Resource	Resourceful Feature of Tool	Example Context Where Tool Was Used
Semiotic	Multiple languages(s)	Contextualizing literal translations of tasks	Task 1 (The Carpet Ride Problem: The Maiden Voyage)
	Writing	Visual	General classroom communication
	Graphical representations		
	Symbolic representations		
	Supplementation of verbal representation with graphical representation, and vice versa	Better visualizing and explaining mathematical ideas	
	Supplementation of speech with gesturing	Facilitate expression and afford visualization	
	Distribution of communication mediums and mathematical representations among group members	Collectively affording various ways of communicating and interpreting information	
	Using primary language(s) for non-mathematical talk	Relating to their group peers to facilitate communication	

Table C.10:

Student Situational Resources Utilized in the IOLA Classroom, Continued

Component	Student Communication Tool as Situational Resource	Resourceful Feature of Tool	Example Context Where Tool Was Used
Sociocultural	Small-group discussions	Personalized mathematical and linguistic clarifications among peers and the instructor	Task 7 (Meal Plans: Constraining the Number of Meals) Task 8 (Meal Plans: Constraining the Cost)
	Inter-group communication	Expanding audience of communication	General classroom communication

Table C.11:

Student Normative Resources Utilized in the IOLA Classroom

Audience	Norm Trait	Norm as a Resource
Instructor-student	Respectful behavior	Not interrupting the instructor
	Competent participation	Identifying important information and writing notes
Student-student	Respectful peer communication	Appreciating being told when they have the wrong answer or method
	Transparent peer communication	Going straight to the point
	Effective peer collaboration	Working alone and then comparing written solutions with peers
	Effective student communication	Minimally communicating and interacting with peers and the whole class.