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ANALYSIS OF STRESS CONCENTRATIONS IN THIN SPHERICAL SHELLS

BY

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FORWARD

This report is submitted in fulfillment of contract No. UCX 2271 with the Lawrence Radiation Laboratory, Livermore, California.

The investigation was conducted by M. K. S. Rajan, Graduate Research Engineer, under the general supervision and technical responsibility of J. Penzien, Professor of Civil Engineering, and E. P. Popov, Professor of Civil Engineering, Department of Civil Engineering, College of Engineering, University of California, Berkeley, California.

NOTATION

M	Moment per linear unit of middle surface.
N	Direct force per linear unit of middle surface.
Q	Shear force per linear unit of middle surface.
σ	Stress.
ϵ	Strain.
w	Raidal displacements.
F	Auxiliary function.
R, ϕ, θ	Polar coordinate, Fig. 3.
x, ϕ_e, θ	Conical coordinates, Figs. 4 or 5
Φ	$\equiv w + \lambda F$, a complex potential.
λ	Imaginary constant.
R	Radius of spherical shell.
t	Thickness of shell.
E	Elastic modulus.
ν	= Poisson's ratio.

Other symbols, used less frequently than these, are defined as they appear.

INTRODUCTION

This report illustrates practical applications of the general elastic theory for spherical shells developed in Ref. 1. Special design charts and tables applicable to specific problems which frequently arise in practice have been developed to facilitate the solution of such problems.

In this report, the flexural behavior of thin spherical shells in the elastic range of stresses is considered in some detail. Attention is restricted to cases in which the displacements under loading are of smaller order than the thickness of the shell. The class of problems considered is that of a loading which produces stress concentrations in the proximity of the loaded zone and in which deformations are small.

The use of the conical co-ordinate system, which is tangent to a sphere at a selected location, leads to solutions which can be termed exact at the point of tangency corresponding to the meridional angle ϕ_e, see Figure 1. Within the limitations of a shallow shell theory, the solutions are valid within a shallow zone ($|\phi - \phi_e| < \frac{\pi}{6}$) surrounding the point of tangency and apply with increasing accuracy as $\phi \rightarrow \phi_e$. No restriction on ϕ_e need be placed except that the origin of the conical co-ordinates must be properly selected. This choice depends on whether $0 \leq \phi_e < \frac{\pi}{2}$, $\phi_e = \frac{\pi}{2}$, or $\frac{\pi}{2} < \phi_e \leq \pi$ and the proper co-ordinates are shown in Fig. 2. In general, it is desirable to select the point of tangency of the cone on this basis at the point of highest stress concentration. On this basis the solutions are more accurate in the zone considered.

In this report the solutions are presented, where possible, in terms of functions readily tabulated (2), (3), (4). In some instances, additional approximations are introduced to simplify direct use in design. The restrictions that arise due to such simplifications are discussed in detail wherever they occur.

Specifically the cases considered in this report are:

1) Internally pressurized spherical shell with a rigid insert.
2) Axially loaded insert
3) Ring Load
4) Uniformly loaded spherical cap with fixed ends.
5) External moment on an insert
6) Tangentially loaded insert.

Examples are also presented for the specific cases to illustrate the use of the tables and charts.

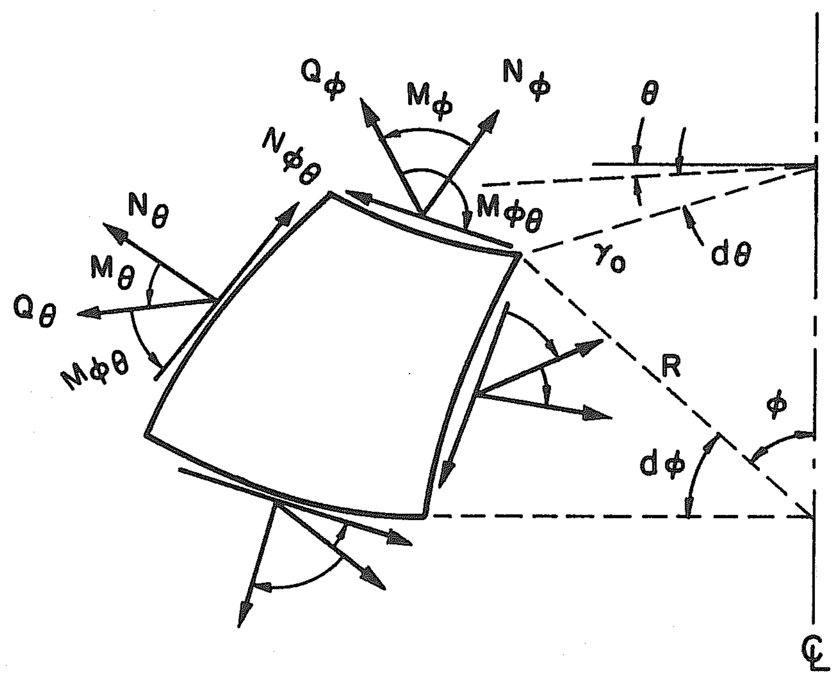


FIG. I — SIGN CONVENTION AND NOTATION
SPHERICAL SHELL

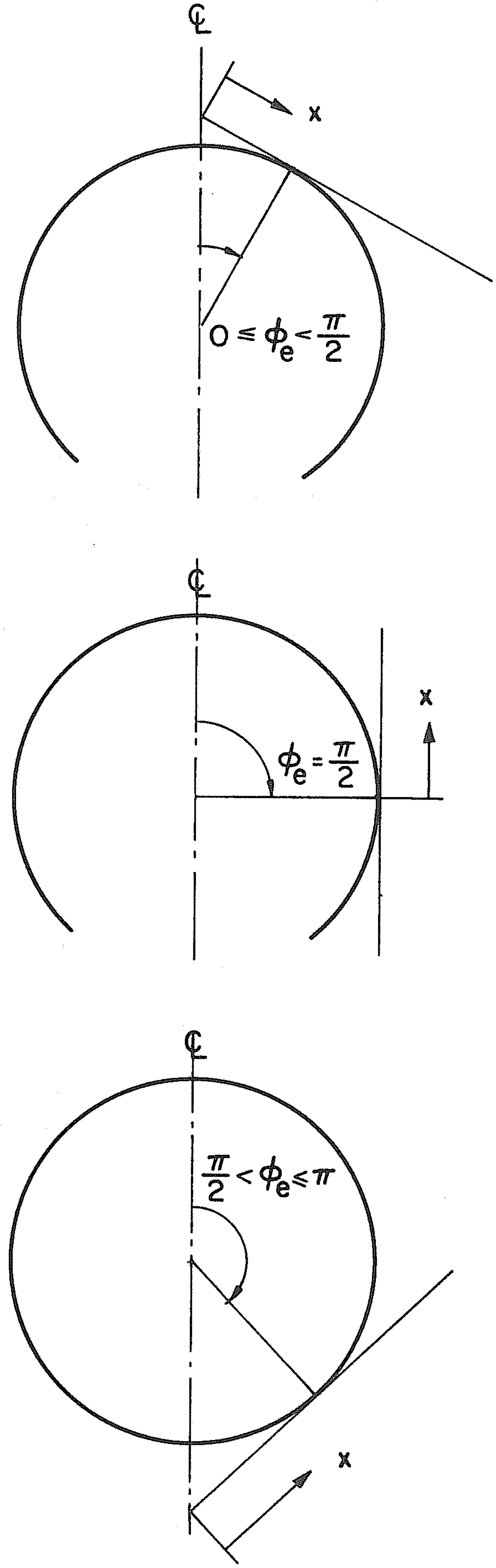


FIG. 2 — CHOICE OF ORIGIN OF CONICAL CO-ORDINATE

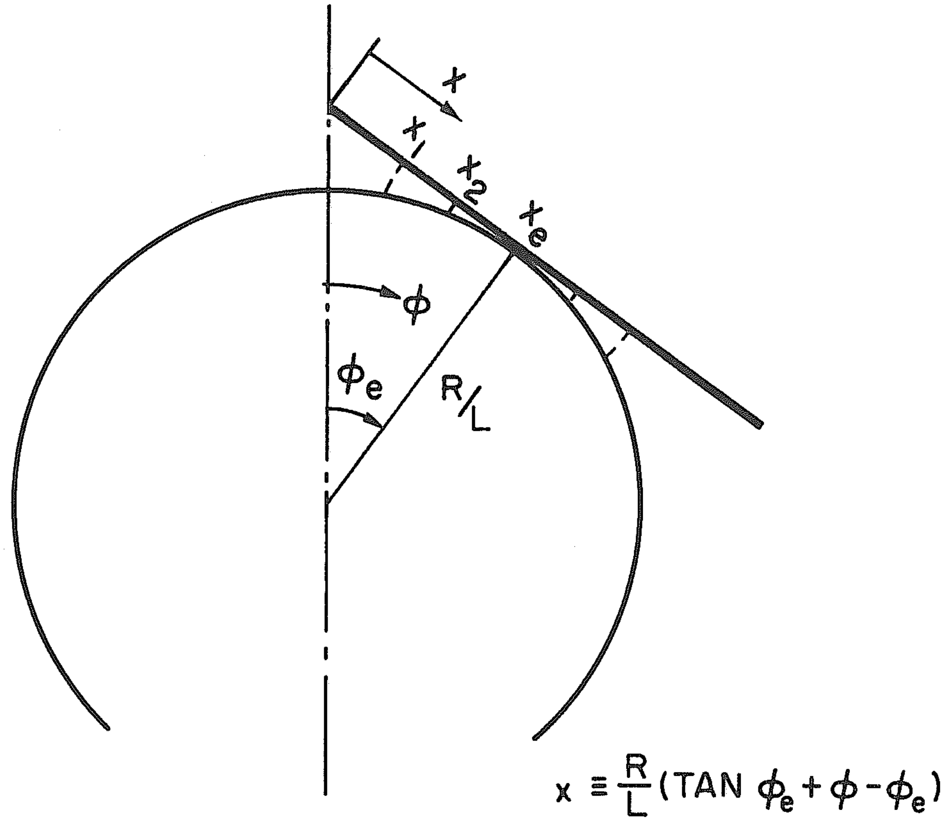


FIG. 3a — PROJECTION OF ZONE OF SHELL ONTO CONICAL COORDINATE SURFACE — LINEAR PROJECTION OF MERIDIONAL ORDINATES

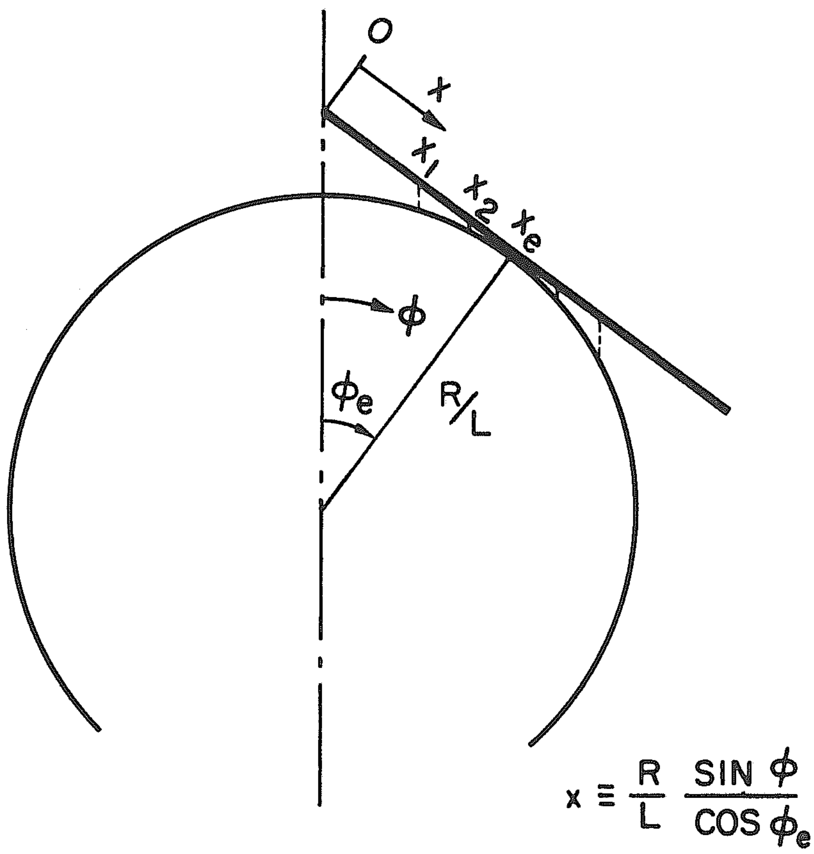


FIG. 3b — VERTICAL PROJECTION ONTO CONICAL COORDINATE SURFACE

Section I

AXI-SYMMETRICAL BENDING

The equations governing the radial displacements w and the stress function F, as shown in Eq. 25 and Eq. 23b of Ref. 1, are given by

$$\nabla^4 w + w = \frac{qR^2}{Et} \qquad (1)$$

and $$\nabla^6 F + \nabla^2 F = qR \qquad (2)$$

where $$\nabla^6 = \nabla^4 \cdot \nabla^2 \; ; \quad \nabla^4 = \nabla^2 \cdot \nabla^2 \; ; \quad \nabla^2 = \frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} \qquad (3)$$

and $$x = \frac{R}{L}(\tan \phi_e + \phi - \phi_e) \qquad (4a)$$

$$\text{or} \simeq \frac{R}{L}\,\frac{\sin \phi}{\cos \phi_e} \quad (\text{if } \phi_e \text{ is restricted to a shallow zone}) \qquad (4b)$$

The conical co-ordinate systems as given by Eqs. 4a and 4b are shown in Figs. 3a and 3b respectively.

Proceeding as in Eq. 56 and 57 of Ref. 1, the solution can be written as

$$w = \text{Re}\left[A_1 \, Jo(x\sqrt{i}) + A_2 \, Ho^{(1)}(x\sqrt{i})\right] + \frac{qR^2}{Et} \qquad (5a)$$

or alternatively

$$w = C_1 \text{ber } x + C_2 \text{bei } x + C_3 \text{ker } x + C_4 \text{kei } x + \frac{qR^2}{Et} \qquad (5b)$$

where A_1 and A_2 are complex constants, whereas C_1 to C_4 are real.

Similarly, from Eqs. 66 of Ref. 1, we have

$$F = \mathrm{Re}\left[B_1\, Jo(x\sqrt{i}) + B_2\, Ho^{(1)}(x\sqrt{i})\right] + C_5 \log x + C_6 = \frac{qRx^2}{4} \qquad (6)$$

The complex constants Be's are related to Ae's by the relationship

$$\overline{B_e} = \frac{1}{\lambda}\overline{A_e} \quad ; \text{ where } \quad \lambda = \frac{iR}{Et} \qquad (7)$$

as shown in Eq. 67 of Ref. 1. $\overline{A_e}$ and $\overline{B_e}$ are complex conjugates of A_e and B_e. This also can be shown to be true from the relation

$$\nabla^2 F - \frac{Et}{R} w = 0 \qquad (8)$$

which has been used in deriving the governing differential equations. Hence we can write

$$F = \frac{Et}{R}\left[C_1 \mathrm{bei}\, x - C_2 \mathrm{ber}\, x + C_3 \mathrm{kei}\, x - C_4 \mathrm{ker}\, x\right] + C_5 \log x + C_6 + \frac{qRx^2}{4} \qquad (9)$$

The functional values of zero order Bessel functions of the first kind $Jo(x\sqrt{i})$ and the third kind $Ho^{(1)}(x\sqrt{i})$ and their derivatives are tabulated in Ref. 2. Most comprehensive values of ber x, bei x, ker x and kei x can be found in Ref. 3.

From Eqs. 29 and 68 of Ref. 1, we can write

$$N_\phi = \frac{Et}{R}\left[-C_1 \frac{\mathrm{bei}'x}{x} + C_2 \frac{\mathrm{ber}'x}{x} - C_3 \frac{\mathrm{kei}'x}{x} + C_4 \frac{\mathrm{ker}'x}{x}\right] - C_5 \cdot \frac{1}{x^2} - \frac{qR}{2} \qquad (10a)$$

$$N_\theta = \frac{Et}{R}\left[C_1\left(\frac{\mathrm{bei}'x}{x} - \mathrm{ber}\, x\right) - C_2\left(\frac{\mathrm{ber}'x}{x} + \mathrm{bei}\, x\right) + C_3\left(\frac{\mathrm{kei}'x}{x} - \mathrm{ker}\, x\right) - C_4\left(\frac{\mathrm{ker}'x}{x} + \mathrm{kei}\, x\right)\right] + C_5 \cdot \frac{1}{x^2} - \frac{qR}{2} \qquad (10b)$$

$$M_\phi = \frac{D}{L^2}\left[C_1\left\{(1-\nu)\frac{\mathrm{ber}'x}{x} + \mathrm{bei}\, x\right\} + C_2\left\{(1-\nu)\frac{\mathrm{bei}'x}{x} - \mathrm{ber}\, x\right\} + C_3\left\{(1-\nu)\frac{\mathrm{ker}'x}{x} + \mathrm{kei}\, x\right\} + C_4\left\{(1-\nu)\frac{\mathrm{kei}'x}{x} - \mathrm{ker}\, x\right\}\right] \qquad (10c)$$

$$M_\theta = \frac{D}{L^2}\left[C_1\left\{\nu\,\mathrm{bei}\,x - (1-\nu)\frac{\mathrm{ber}'x}{x}\right\} - C_2\left\{\nu\,\mathrm{ber}\,x + (1-\nu)\frac{\mathrm{bei}'x}{x}\right\}\right.$$

$$\left. + C_3\left\{\nu\,\mathrm{kei}\,x - (1-\nu)\frac{\mathrm{ker}'x}{x}\right\} - C_4\left\{\nu\,\mathrm{ker}\,x + (1-\nu)\frac{\mathrm{kei}'x}{x}\right\}\right] \qquad (10d)$$

$$Q_\phi = \frac{D}{L^3}\left[C_1\mathrm{bei}'x - C_2\mathrm{ber}'x + C_3\mathrm{kei}'x - C_4\mathrm{ker}'x\right] \qquad (10e)$$

Case (1):

Rigid Cylindrical Inserts

In the type of stress concentration problems that follow, it has been assumed that the external boundaries are far off from the stress concentration zone, such that their effect can be neglected. Therefore, the radial displacement w and $\frac{dw}{dx}$ are expected to dampen out as x increases. Hence $A_1 = 0$ or $C_1 = C_2 = 0$ in Eqs. 5 and 10.

The boundary conditions at the insert edge are, at $x = x_e$

$$\frac{dw}{dx} = 0 \quad \text{and} \quad \epsilon_\theta = \frac{1}{Et}(N_\theta - \nu N_\phi) = 0 \tag{11}$$

Making this substitution, we get

$$C_3 = \frac{-R}{Et}\left[(1+\nu)C_5\frac{1}{x_e^2} - \frac{qR}{2}(1-\nu)\right]\frac{\text{kei}'x_e}{\text{ker}'x_e}\cdot F(x_e) \tag{12a}$$

$$C_4 = \frac{R}{Et}\left[(1+\nu)C_5\frac{1}{x_e^2} - \frac{qR}{2}(1-\nu)\right]F(x_e) \tag{12b}$$

$$\text{where } F(x_e) = \frac{x_e \ \text{ker}' x_e}{(1+\nu)\left[\text{ker}'^2 x_e + \text{kei}'^2 x_e\right] - x_e\left[\text{ker}\, x_e\, \text{kei}' x_e - \text{ker}' x_e\, \text{kei}\, x_e\right]} \tag{12c}$$

For small argument values, we have (Ref. 4)

$$\text{ker}\, x = -\ln x + (\ln 2 - \gamma) + \frac{\pi x^2}{16} + \ldots\ldots\ldots \tag{13a}$$

$$\text{kei}\, x = -\left(\frac{x^2}{4}\right)\ln x - \frac{\pi}{4} + (1 + \ln 2 - \gamma)\frac{x^2}{4} + \ldots\ldots \tag{13b}$$

$$\text{ker}'x = \frac{-1}{x} + \frac{\pi x}{8} + \left(\frac{x^3}{16}\right)\ln x + \ldots\ldots\ldots \tag{13c}$$

$$\text{kei}'x = -\left(\frac{x}{2}\right)\ln x + (1 + \ln 4 - 2\gamma)\frac{x}{4} + \ldots\ldots \tag{13d}$$

Hence as $x_e \rightarrow 0$, the explicit behavior of the coefficients C_3 and C_4 is given by

$$C_3 = 0 \quad \text{and} \quad C_4 = -\frac{R}{Et}\left[C_5 - \frac{qR}{2}\frac{(1-\nu)}{(1+\nu)} x_e^2 \right] \tag{14}$$

On the other hand, for large argument values

$$\text{ker } x \simeq \frac{\exp(-x/\sqrt{2})}{\sqrt{2x/\pi}} \cos\left(\frac{x}{\sqrt{2}} + \frac{\pi}{8}\right) \tag{15a}$$

$$\text{kei } x \simeq \frac{-\exp(-x/\sqrt{2})}{\sqrt{2x/\pi}} \sin\left(\frac{x}{\sqrt{2}} + \frac{\pi}{8}\right) \tag{15b}$$

$$\text{ker}' x \simeq \frac{-\exp(-x/\sqrt{2})}{\sqrt{2x/\pi}} \cos\left(\frac{x}{\sqrt{2}} - \frac{\pi}{8}\right) \tag{15c}$$

$$\text{kei}' x \simeq \frac{\exp(-x/\sqrt{2})}{\sqrt{2x/\pi}} \sin\left(\frac{x}{\sqrt{2}} - \frac{\pi}{8}\right) \tag{15d}$$

Such that as $x_e \rightarrow \infty$

$$C_3 = \frac{2R}{Et}\left[\frac{qR}{2}(1-\nu) - (1+\nu)C_5 \cdot \frac{1}{x_e^2} \right] \sqrt{\frac{x_e}{\pi}} \exp\left(\frac{x_e}{\sqrt{2}}\right) \sin\left(\frac{x_e}{\sqrt{2}} - \frac{\pi}{8}\right) \tag{16a}$$

$$C_4 = \frac{2R}{Et}\left[\frac{qR}{2}(1-\nu) - (1+\nu)C_5 \cdot \frac{1}{x_e^2} \right] \sqrt{\frac{x_e}{\pi}} \exp\left(\frac{x_e}{\sqrt{2}}\right) \cos\left(\frac{x_e}{\sqrt{2}} - \frac{\pi}{8}\right) \tag{16b}$$

The coefficient C_5 is dependent on the type of loading.

(a) Internally Pressurized Shell:

Let the internal pressure be $q = -p$. (Fig. 4) The arbitrary constant C_5 is determined from the condition.

$$\text{At } x \geqslant x_e \;;\quad R_v = N_\phi \operatorname{Sin} \phi + Q_\phi \cos \phi = \frac{pLx\cos \phi_e}{2} \tag{17}$$

This condition is applied at the insert edge $x = x_e$, where the stress concentration occurs, so that it may be more accurate in the zone considered. This leads to $C_5 = 0$.

$$\text{Hence,}\quad C_3 = \frac{-pR^2}{2Et} (1 - \nu) \frac{\operatorname{kei}' x_e}{\operatorname{ker}' x_e} F(x_e) \tag{18a}$$

$$C_4 = \frac{pR^2}{2Et} (1 - \nu) F(x_e) \tag{18b}$$

The quantities of interest follow as

$$\begin{aligned}
w &= a_1 \frac{pR^2}{Et} & N_\Theta &= a_6 \, pR \\
M_\phi &= a_2 \, pRt & \sigma_{\phi_i} &= a_7 \frac{pR}{t} \\
M_\Theta &= a_3 \, pRt & \sigma_{\phi_e} &= a_8 \frac{pR}{t} \\
Q_\phi &= a_4 \, p \sqrt{Rt} & \sigma_{\Theta_i} &= a_9 \frac{pR}{t} \\
N_\phi &= a_5 \, pR & \sigma_{\Theta_e} &= a_{10} \frac{pR}{t}
\end{aligned} \tag{19}$$

The variation of the coefficients a_1 to a_{10} with distance from the edge of the insert $x = x_e$ is tabulated in Table I, for various sizes of insert. Graphs of w, M_ϕ, M_Θ, Q_ϕ, N_ϕ, N_Θ are given in Figs. 5 - 9. The dot-dashed curves give the values at the circumference of the rigid attachment, for varying values of x_e. The solid curves give the values away from the attachment for given values of x_e and varying values of x. Similarly graphs of σ_ϕ and σ_θ are given in Figs. 10 - 11, where the solid lines refer to the stresses on the interior face of the shell and dashed curves indicate the stresses on the exterior face of the shell.

In the limiting case, as the diameter of the insert approaches zero, ($x_e \rightarrow 0$), the bending moments and shear approach zero, whereas the direct inplane forces N_ϕ and N_θ approach $pR/(1+\nu)$ and $\nu pR/(1+\nu)$ respectively. When $x_e = 0$, $C_3 = C_4 = 0$ and hence $M_\phi = M_\theta = Q_\phi = 0$ and $N_\phi = N_\theta = pR/2$. Thus the limit $x_e = 0$, can be considered as defining the stress concentration factor for the direct inplane forces.

As the diameter of the insert increases, the meridional bending moment M_ϕ and circumferential bending moment M_θ at the insert edge increase, developing an appreciable negative valued region. The direct inplane force N_ϕ approaches its membrane limit, whereas N_θ deviates slightly from its membrane limit pR/2.

Using the expressions (15) for large values of x, we have

$$\frac{M_\phi}{pRt} \simeq \frac{1}{2}\sqrt{\frac{(1-\nu)}{3(1+\nu)}}\sqrt{\frac{x_e}{2x}}\left\{\frac{x_e}{x_e+\sqrt{2}\,(1+\nu)}\right\}\exp\left\{-\frac{x-x_e}{\sqrt{2}}\right\}\left\{\cos\left(\frac{x-x_e}{\sqrt{2}}+\frac{\pi}{4}\right)-\frac{(1-\nu)}{x}\sin\left(\frac{x-x_e}{\sqrt{2}}\right)\right\} \tag{20a}$$

$$\frac{M_\theta}{pRt} \simeq \sqrt{\frac{1}{2}\frac{(1-\nu)}{3(1+\nu)}}\sqrt{\frac{x_e}{2x}}\left\{\frac{x_e}{x_e+\sqrt{2}\,(1+\nu)}\right\}\exp\left(-\frac{x-x_e}{\sqrt{2}}\right)\left\{\nu\cos\left(\frac{x-x_e}{\sqrt{2}}+\frac{\pi}{4}\right)+\frac{(1-\nu)}{x}\sin\left(\frac{x-x_e}{\sqrt{2}}\right)\right\} \tag{20b}$$

$$\frac{Q_\phi}{p\sqrt{Rt}} \simeq -\frac{(1-\nu)}{\left[12(1-\nu^2)\right]^{1/4}} \sqrt{\frac{x_e}{2x}} \left\{ \frac{x_e}{x_e + \sqrt{2}(1+\nu)} \right\} \exp\left(-\frac{x - x_e}{\sqrt{2}}\right) \cos\left(\frac{x - x_e}{\sqrt{2}}\right) \quad (20c)$$

$$\frac{N_\phi}{pR} \simeq \frac{1}{2} + \frac{(1-\nu)}{x} \sqrt{\frac{x_e}{2x}} \left\{ \frac{x_e}{x_e + \sqrt{2}(1+\nu)} \right\} \exp\left(-\frac{x - x_e}{\sqrt{2}}\right) \cos\left(\frac{x - x_e}{\sqrt{2}}\right) \quad (20d)$$

$$\frac{N_\theta}{pR} \simeq \frac{1}{2} - \frac{(1-\nu)}{x} \sqrt{\frac{x_e}{2x}} \left\{ \frac{x_e}{x_e + \sqrt{2}(1+\nu)} \right\} \exp\left(-\frac{x - x_e}{\sqrt{2}}\right)$$

$$\left\{ \cos\left(\frac{x - x_e}{\sqrt{2}}\right) + x \sin\left(\frac{x - x_e}{\sqrt{2}} + \frac{\pi}{4}\right) \right\} \quad (20e)$$

As $x_e \rightarrow \infty$, these expressions (20) can be further simplified to:

$$\frac{M_\phi}{pRt} \simeq \frac{1}{2} \sqrt{\frac{(1-\nu)}{6(1+\nu)}} \exp\left(-\frac{x - x_e}{\sqrt{2}}\right) \cos\left(\frac{x - x_e}{\sqrt{2}} + \frac{\pi}{4}\right) \quad (21a)$$

$$M_\theta \simeq \nu M_\phi \quad (21b)$$

$$\frac{Q_\phi}{p\sqrt{Rt}} \simeq -\frac{(1-\nu)}{2\left[3(1-\nu^2)\right]^{1/4}} \exp\left(-\frac{x - x_e}{\sqrt{2}}\right) \cos\left(\frac{x - xe}{\sqrt{2}}\right) \quad (21c)$$

$$\frac{N_\phi}{pR} \simeq \frac{1}{2} \quad (21d)$$

$$\frac{N_\theta}{pR} \simeq \frac{1}{2} - \frac{(1-\nu)}{\sqrt{2}} \exp\left(-\frac{x - x_e}{\sqrt{2}}\right) \sin\left(\frac{x - x_e}{\sqrt{2}} + \frac{\pi}{4}\right) \quad (21e)$$

From these expressions (21) we can determine the assymptotic maximum and minimum values as indicated in Figs. 5 - 9.

Similarly the limiting values of the meridional stress σ_ϕ and circumferential stress σ_θ in the extreme fibers (interior and exterior) of the shell can be determined and are shown in Figs. 10 - 11. It can be observed that the greatest values of the meridional stress occurs on the inner surface of the shell, at the insert edge. In the limiting case when $x_e \rightarrow \infty$, this value approaches

$$\sigma_\phi \left(x_e + \frac{t}{2}\right) \simeq \frac{pR}{2t} \left[1 + \sqrt{\frac{3(1 - \quad)}{(1 + \quad)}}\right] \tag{22}$$

Equations (20) and (21) give solutions with sufficient accuracy for values of x_e greater than 15 and 20 respectively.

(b) Axially Loaded Insert

Let the radial load acting on the insert be P (Fig. 12). With $q = 0$, the arbitrary constant C_5 is determined from the condition that

$$\text{At } x \geq x_e \; ; \; Rv = N_\phi \sin\phi + Q_\phi \cos\phi = - \frac{P}{2\pi L x \cos\phi_e} \qquad (23)$$

This condition is applied at the insert edge $x = x_e$ (i.e., $\phi = \phi_e$) so that the resulting solution is more accurate in the region of high stress concentration. On this basis, we have

$$C_5 = \frac{PR}{2\pi L^2 \cos^2\phi_e} = \frac{PR}{2\pi L^2}\left(1 + \frac{L^2 x_e^2}{R^2}\right) \qquad (24)$$

When ϕ_e is small, in order to reduce the computations due to introduction of more general parameters that will occur, we can adopt

$$C_5 = \frac{PR}{2\pi L^2} \qquad (25)$$

This restricts the solutions to stress concentration for small angles of ϕ_e.

Imposing the boundary conditions given by Eq. 11 and noting that $q = 0$, we get from Eqs. 12 and 25 that

$$C_3 = -(1+\nu)\frac{PL^2}{2\pi D}\frac{\text{kei}' x_e}{\text{ker}' x_e}\frac{F(x_e)}{x_e^2} \qquad (26a)$$

$$C_4 = (1+\nu)\frac{PL^2}{2\pi D}\frac{F(x_e)}{x_e^2} \qquad (26b)$$

where $F(x_e)$ is given by Eq. 12c as before.

As $x_e \rightarrow 0$, the behavior of the coefficients C_3 and C_4 is given from Eq. 14 as

$$C_3 = 0 \text{ and } C_4 = -\frac{PL^2}{2\pi D} = -\frac{R}{Et} C_5 \qquad (27)$$

Using Eqs. 26, quantities of interest can be represented as

$$w = b_1 \frac{PR}{Et^2} \qquad N_\Theta = b_6 \frac{P}{t}$$

$$M_\phi = b_2 P \qquad \sigma_{\phi_i} = b_7 \frac{P}{t^2}$$

$$M_\Theta = b_3 P \qquad \sigma_{\phi_e} = b_8 \frac{P}{t^2}$$

$$Q_\phi = b_4 \frac{P}{\sqrt{Rt}} \qquad \sigma_{\Theta_i} = b_9 \frac{P}{t^2}$$

$$N_\phi = b_5 \frac{P}{t} \qquad \sigma_{\Theta_e} = b_{10} \frac{P}{t^2}$$

The variation of the coefficients b_1 to b_{10} for each insert size is tabulated in Table II, for various values of x_e as before, and are also graphically represented in Figs. 13 - 19.

As the diameter of the insert reduces to zero, in the limiting case (i.e., $x_e \rightarrow 0$,) the problem reduces to the case of a concentrated load. In this case, at the insert edge, the bending moments M_ϕ and M_Θ and shear Q_ϕ approach infinity, whereas the direct inplane forces N_ϕ and N_Θ are finite and both tend to $-\frac{P}{8t}\sqrt{3(1-\nu^2)}$. At any distance x from the concentrated load, we have

$$w = -\frac{PR}{Et^2} \frac{\xi^2}{2} \text{ kei } x \qquad (29a)$$

$$M_\phi = \frac{P}{2\pi} \left\{ \text{ker } x - (1-\nu) \frac{\text{kei}' x}{x} \right\} \qquad (29b)$$

$$M_\theta = \frac{P}{2\pi}\left\{\ker x + (1-\nu)\frac{\text{kei}' x}{x}\right\} \qquad (29c)$$

$$Q_\phi = \frac{P}{Rt} \cdot \frac{\xi^2}{2\pi} \ker' x \qquad (29d)$$

$$N_\phi = -\frac{P}{t} \cdot \frac{\xi^2}{2\pi}\left\{\frac{\ker' x}{x} + \frac{1}{x^2}\right\} \qquad (29e)$$

$$N_\theta = \frac{P}{t} \cdot \frac{\xi^2}{2\pi}\left\{\text{kei}\, x + \frac{\ker' x}{x} + \frac{1}{x^2}\right\} \qquad (29f)$$

where $$\xi^2 = \left[12(1-\nu^2)\right]^{1/4} \qquad (29g)$$

The error involved in the above calculations, at the edge of the insert, is due to the approximation made in Eq. 25 by assuming $\cos^2 \phi_e \simeq 1$. In a given problem where the angle ϕ_e is known, we can get the exact values by multiplying with $\sec^2 \phi_e$ the values in the table for the corresponding x_e.

It can also be shown that the results for an axially loaded insert can be deduced from the results of the corresponding pressurized shell with an insert. By comparison, it can be noted that

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \\ b_8 \\ b_9 \\ b_{10} \end{bmatrix} = \frac{1+\nu}{1-\nu}\,\frac{t}{R}\,\frac{\text{cosec}^2 \phi_e}{\pi} \begin{bmatrix} (1 + a_1) \\ a_2 \\ a_3 \\ a_4 \\ \left\{a_5 - \frac{1}{1+\nu}\right\} \\ \left\{a_6 - \frac{\nu}{1+\nu}\right\} \\ \left\{a_7 - \frac{1}{1+\nu}\right\} \\ \left\{a_8 - \frac{1}{1+\nu}\right\} \\ \left\{a_9 - \frac{\nu}{1+\nu}\right\} \\ \left\{a_{10} - \frac{\nu}{1+\nu}\right\} \end{bmatrix} \qquad (30)$$

The coefficients b_1 to b_{10} that can be obtained as shown above are not restricted to shallow angles ϕ_e, but can tend to $\pi/2$ if desired.

Case 2:

Ring Load

A ring load of total magnitude P (Fig. 20) acting at a distance x_e in the conical co-ordinate system, corresponding to an ordinate ϕ_e is considered. The boundary and transition conditions are

$$\text{At } x = 0, \quad w,\ N_\phi \text{ and } N_\theta \text{ are finite and } \frac{dw}{dx} = 0 \tag{31}$$

$$x = x_e \quad w,\ \frac{dw}{dx},\ M_\phi \text{ and } N_\phi \text{ are continuous} \tag{32}$$

$$x = \infty \quad w = \frac{dw}{dx} = 0 \tag{33}$$

Furthermore, outside the ring load region for $x \geq x_e$

$$R_v = \frac{P}{2\pi L x \cos \phi_e} \tag{34}$$

The ring load divides the shell into two regions: a) the portion corresponding to $x \leq x_e$, the solutions to which are designated with a subscript 1; b) the portion $x \geq x_e$, designated with a subscript 2.

Boundary condition Eq. 31, with Eqs. 5b and 10 gives $C_3 = C_4 = C_5 = 0$, and reduces the general solution in region 1 for $x \leq x_e$ to:

$$w_1 = C_1 \operatorname{ber} x + C_2 \operatorname{bei} x \tag{35a}$$

$$M_{\phi_1} = \frac{D}{L^2}\left[C_1 \left\{ \operatorname{bei} x + (1-\nu)\frac{\operatorname{ber}' x}{x} \right\} - C_2 \left\{ \operatorname{ber} x - (1-\nu)\frac{\operatorname{bei}' x}{x} \right\} \right] \tag{35b}$$

$$M_{\theta_1} = \frac{D}{L^2}\left[C_1 \left\{ \nu \operatorname{bei} x - (1-\nu)\frac{\operatorname{ber}' x}{x} \right\} - C_2 \left\{ \nu \operatorname{ber} x + (1-\nu)\frac{\operatorname{bei}' x}{x} \right\} \right] \tag{35c}$$

$$Q_{\phi_1} = \frac{D}{L^3}\left[C_1 \operatorname{bei}' x - C_2 \operatorname{ber}' x \right] \tag{35d}$$

$$N_{\phi_1} = -\frac{Et}{R}\left[C_1 \frac{\operatorname{bei}' x}{x} - C_2 \frac{\operatorname{ber}' x}{x} \right] \tag{35e}$$

$$N_{\theta_1} = -\frac{Et}{R}\left[C_1 \left(\operatorname{ber} x - \frac{\operatorname{bei}' x}{x} \right) + C_2 \left(\operatorname{bei} x + \frac{\operatorname{ber}' x}{x} \right) \right] \tag{35f}$$

Eqs. 33 as before lead to $C_1 = C_2 = 0$ in region 2 for $x \geq x_e$ The equilibrium condition 34 as in Eq 24 in order to simplify calculations leads to

$$C_5 = \frac{PR}{2\pi L^2 \cos^2 \phi_e} \simeq \frac{PR}{2\pi L^2} \tag{36}$$

Hence the general solutions in region 2 reduce to:

$$w_2 = C_3 \ker x + C_4 \operatorname{kei} x \tag{37a}$$

$$M_{\phi_2} = \frac{D}{L^2}\left[C_3 \left\{ \operatorname{kei} x + (1-\nu)\frac{\ker' x}{x} \right\} - C_4 \left\{ \ker x - (1-\nu)\frac{\operatorname{kei}' x}{x} \right\} \right] \tag{37b}$$

$$M_{\theta_2} = \frac{D}{L^2}\left[C_3 \left\{ \nu \operatorname{kei} x - (1-\nu)\frac{\ker' x}{x} \right\} - C_4 \left\{ \nu \ker x + (1-\nu)\frac{\operatorname{kei}' x}{x} \right\} \right] \tag{37c}$$

$$Q_{\phi_2} = \frac{D}{L^3}\left[C_3 \operatorname{kei}' x - C_4 \ker' x \right] \tag{37d}$$

$$N_{\phi_2} = -\frac{Et}{R}\left[C_3 \frac{\operatorname{kei}' x}{x} - C_4 \frac{\ker' x}{x} + \frac{PL^2}{2\pi D}\frac{1}{x^2} \right] \tag{37e}$$

$$N_{\theta_2} = -\frac{Et}{R}\left[C_3 \left(\ker x - \frac{\operatorname{kei}' x}{x}\right) + C_4 \left(\operatorname{kei} x + \frac{\ker' x}{x}\right) - \frac{PL^2}{2\pi D}\frac{1}{x^2} \right] \tag{37f}$$

The continuity conditions (32) for w, $\frac{dw}{dx}$, M_ϕ, and N_ϕ are of the form

$$C_1 \operatorname{ber} x_e + C_2 \operatorname{bei} x_e - C_3 \ker x_e - C_4 \operatorname{kei} x_e = 0 \tag{38a}$$

$$C_1 \operatorname{ber}' x_e + C_2 \operatorname{bei}' x_e - C_3 \ker' x_e - C_4 \operatorname{kei}' x_e = 0 \tag{38b}$$

$$C_1 \operatorname{bei} x_e + C_2 \operatorname{ber} x_e - C_3 \operatorname{kei} x_e - C_4 \ker x_e = 0. \tag{38c}$$

$$C_1 \operatorname{bei}' x_e - C_2 \operatorname{ber}' x_e - C_3 \operatorname{kei}' x_e + C_4 \ker' x_e = \frac{PL^2}{2\pi D}\frac{1}{x_e} \tag{38d}$$

Using the relations (Ref. 5)

$$\text{ber } x \text{ ker}' x + \text{bei}' x \text{ kei } x - \text{ber}' x \text{ ker } x - \text{bei } x \text{ kei}' x = -\frac{1}{x} \tag{39a}$$

$$\text{ber } x \text{ kei}' x + \text{bei } x \text{ ker}' x - \text{ber}' x \text{ kei } x - \text{bei}' x \text{ ker } x = 0 \tag{39b}$$

which can be obtained by putting $x\sqrt{i}$ for x and equating real and imaginary parts in

$$I_o(x)\, K_o^{1}(x) - I_o^{1}(x)\, K_o(x) = -\frac{1}{x} \tag{39c}$$

The system 38 of four simultaneous equations can be solved in closed form as follows:

$$C_1 = -\frac{PL^2}{2\pi D} \text{ kei } x_e \tag{40a}$$

$$C_2 = -\frac{PL^2}{2\pi D} \text{ ker } x_e \tag{40b}$$

$$C_3 = -\frac{PL^2}{2\pi D} \text{ bei } x_e \tag{40c}$$

$$C_4 = -\frac{PL^2}{2\pi D} \text{ ber } x_e \tag{40d}$$

With these we can write:

For $x \leq x_e$ (41)

$$w_1 = f_1 \cdot \frac{PR}{Et^2} \qquad N_{\theta_1} = f_6 \cdot \frac{P}{t}$$

$$M_{\phi_1} = f_2 \cdot P \qquad \sigma_{\phi_{1_i}} = f_7 \cdot \frac{P}{t^2}$$

$$M_{\theta_1} = f_3 \cdot P \qquad \sigma_{\phi_{1_e}} = f_8 \cdot \frac{P}{t^2}$$

$$Q_{\phi_1} = f_4 \cdot \frac{P}{\sqrt{Rt}} \qquad \sigma_{\theta_{1_i}} = f_9 \cdot \frac{P}{t^2}$$

$$N_{\phi_1} = f_5 \cdot \frac{P}{t} \qquad \sigma_{\theta_{1_e}} = f_{10} \cdot \frac{P}{t^2}$$

For $x \geq x_e$ (42)

$$w_2 = g_1 \cdot \frac{PR}{Et^2} \qquad N_{\theta_2} = g_6 \cdot \frac{P}{t}$$

$$M_{\phi_2} = g_2 \cdot P \qquad \sigma_{\phi_{2_i}} = g_7 \cdot \frac{P}{t^2}$$

$$M_{\theta_2} = g_3 \cdot P \qquad \sigma_{\phi_{2_e}} = g_8 \cdot \frac{P}{t^2}$$

$$Q_{\phi_2} = g_4 \cdot \frac{P}{\sqrt{Rt}} \qquad \sigma_{\theta_{2_i}} = g_9 \cdot \frac{P}{t^2}$$

$$N_{\phi_2} = g_5 \cdot \frac{P}{t} \qquad \sigma_{\theta_{2_e}} = g_{10} \cdot \frac{P}{t^2}$$

As before, the variation of the coefficients f_1 to f_{10} and g_1 to g_{10} are tabulated in Table III and are graphically represented in Figs. 21 - 27.

In the limiting case, as the diameter of the ring load reduces to zero, i.e., $x_e \longrightarrow 0$, the general solutions reduce to that of a concentrated load P at the apex as given by Eqs. 29.

As in the case of an axially loaded insert, the error involved in the above calculations at the edge of the ring load is due to the approximation of letting $\cos^2 \phi_e \simeq 1$ in Eq. 36. Hence, in a particular problem where ϕ_e is known, the exact solution at the edge of the ring load can be obtained by multiplying the results in the table for the corresponding x_e by $\sec^2 \phi_e$.

Case (3):

Uniformly loaded spherical cap

A spherical shell with edges restrained against rotation and acted upon by a uniform hydrostatic pressure of intensity p (Fig. 28) over its surface is considered. Let the ordinate to the fixed edge of the shell, in conical co-ordinates, be x_e , corresponding to the meridional angle ϕ_e. The boundary conditions are:

$$\text{At } x = 0; \qquad w,\ N_\phi\ ,\ \text{and } N_\theta \text{ are finite and } \frac{dw}{dx} = 0 \tag{43}$$

$$x = x_e\ ; \qquad \frac{dw}{dx} = 0 \text{ and } \epsilon_\theta = \frac{1}{Et}\left(N_\theta - \nu N_\phi\right) = 0 \tag{44}$$

Furthermore, for $x \leq x_e$

$$R_v = \frac{p\,L\,x\cos\phi_e}{2} \tag{45}$$

Boundary condition Eq. 43 with general solutions 5b and 10 gives $C_3 = C_4 = C_5 = 0$. The condition 44 at the fixed edge of the shell leads to:

$$C_1 = -\frac{(1-\nu)\,pR^2}{2Et}\ \frac{\text{bei}'\,x_e}{\text{ber}'\,x_e}\ \cdot\ F(x_e) \tag{46a}$$

$$C_2 = \frac{(1-\nu)\,pR^2}{2Et}\ \cdot\ F(x_e) \tag{46b}$$

$$\text{where } F(x_e) = \frac{x_e\ \text{ber}'\,x_e}{(1+\nu)\left[\text{ber}'^2\,x_e + \text{bei}'^2\,x_e\right] - x_e\left[\text{ber}\,x_e\ \text{bei}'\,x_e - \text{ber}'\,x_e\ \text{bei}\,x_e\right]} \tag{46c}$$

For large argument values, we have

$$\text{ber } x \simeq \frac{\exp(x/\sqrt{2})}{\sqrt{2\pi x}} \cos\left(\frac{x}{\sqrt{2}} - \frac{\pi}{8}\right) \qquad (47a)$$

$$\text{bei } x \simeq \frac{\exp(x/\sqrt{2})}{\sqrt{2\pi x}} \sin\left(\frac{x}{\sqrt{2}} - \frac{\pi}{8}\right) \qquad (47b)$$

$$\text{ber}' x \simeq \frac{\exp(x/\sqrt{2})}{\sqrt{2\pi x}} \cos\left(\frac{x}{\sqrt{2}} + \frac{\pi}{8}\right) \qquad (47c)$$

$$\text{bei}' x \simeq \frac{\exp(x/\sqrt{2})}{\sqrt{2\pi x}} \sin\left(\frac{x}{\sqrt{2}} + \frac{\pi}{8}\right) \qquad (47d)$$

Hence as $x_e \longrightarrow \infty$, the behavior of the coefficients is given by

$$C_1 = (1 - \nu) \frac{pR^2}{Et} \sqrt{\pi x_e} \exp\left(-\frac{x_e}{\sqrt{2}}\right) \sin\left(\frac{x_e}{\sqrt{2}} + \frac{\pi}{8}\right) \qquad (48a)$$

$$C_2 = -(1 - \nu) \frac{pR^2}{Et} \sqrt{\pi x_e} \exp\left(-\frac{x_e}{\sqrt{2}}\right) \cos\left(\frac{x_e}{\sqrt{2}} + \frac{\pi}{8}\right) \qquad (48b)$$

The condition 45 is automatically satisfied at the fixed edge of the shell, where the stress concentration occurs.

With the coefficients defined as in Eqs. 46, we can write the general solutions in the same manner as Eqs. 19 for a pressurized shell with an insert, replacing coefficients a_1 to a_{10} by h_1 to h_{10} respectively.

The variation of the coefficients h_1 to h_{10} are given in Table IV. for a series of values of x_e corresponding to the fixed edge of the shell. Also they are graphically represented in Figs. 29 - 35.

In the limiting case, when $x_e = 0$, all the quantities are zero, as there doesn't exist any problem.

For large values of x, using expressions (47), we can write

$$\frac{Et}{pR^2}\, w \simeq (1-\nu)\sqrt{\frac{x_e}{2x}}\left\{\frac{x_e}{x_e-(1+\nu)\sqrt{2}}\right\}\exp\left\{-\frac{x_e-x}{\sqrt{2}}\right\}\sin\left(\frac{x_e-x}{\sqrt{2}}+\frac{\pi}{4}\right)-1 \qquad (48c)$$

$$\frac{M_\phi}{pRt} \simeq \frac{1}{2}\sqrt{\frac{(1-\nu)}{3(1+\nu)}}\sqrt{\frac{x_e}{2x}}\left\{\frac{x_e}{x_e-(1+\nu)\sqrt{2}}\right\}\exp\left\{-\frac{x_e-x}{\sqrt{2}}\right\}\left[\cos\left\{\frac{x_e-x}{\sqrt{2}}+\frac{\pi}{4}\right\}+\frac{(1-\nu)}{x}\sin\left(\frac{x_e-x}{\sqrt{2}}\right)\right] \qquad (48d)$$

$$\frac{M_\theta}{pRt} \simeq \frac{1}{2}\sqrt{\frac{(1-\nu)}{3(1+\nu)}}\sqrt{\frac{x_e}{2x}}\left\{\frac{x_e}{x_e-(1+\nu)\sqrt{2}}\right\}\exp\left\{-\frac{x_e-x}{\sqrt{2}}\right\}\left[\nu\cos\left\{\frac{x_e-x}{\sqrt{2}}+\frac{\pi}{4}\right\}-\frac{(1-\nu)}{x}\sin\left\{\frac{x_e-x}{\sqrt{2}}\right\}\right] \qquad (48e)$$

$$\frac{Q_\phi}{p\sqrt{Rt}} \simeq \frac{(1-\nu)}{\left[12(1-\nu^2)\right]^{1/4}}\sqrt{\frac{x_e}{2x}}\left\{\frac{x_e}{x_e-(1+\nu)\sqrt{2}}\right\}\exp\left\{-\frac{x_e-x}{\sqrt{2}}\right\}\cos\left\{\frac{x_e-x}{\sqrt{2}}\right\} \qquad (48f)$$

$$\frac{N_\phi}{pR} \simeq \frac{1}{2}-\frac{(1-\nu)}{x}\sqrt{\frac{x_e}{2x}}\left\{\frac{x_e}{x_e-(1+\nu)\sqrt{2}}\right\}\exp\left\{-\frac{x_e-x}{\sqrt{2}}\right\}\cos\left\{\frac{x_e-x}{\sqrt{2}}\right\} \qquad (48g)$$

$$\frac{N_\theta}{pR} \simeq \frac{1}{2}+\frac{(1-\nu)}{x}\sqrt{\frac{x_e}{2x}}\left\{\frac{x_e}{x_e-(1+\nu)\sqrt{2}}\right\}\exp\left\{-\frac{x_e-x}{\sqrt{2}}\right\}\left[\cos\left\{\frac{x_e-x}{\sqrt{2}}\right\}-x\sin\left\{\frac{x_e-x}{\sqrt{2}}+\frac{\pi}{4}\right\}\right] \qquad (48h)$$

As before Eqs. 48 are sufficiently accurate for x_e greater than 15. The radial deflections w tabulated are only relative values. Hence, the relative radial deflection for no deflection at the fixed edges of the shell can be easily worked out.

SECTION II

UNSYMMETRICAL BENDING

In the elastic range from Ref. 1, the governing differential equation for the complex potential $\bar{\phi}$ is given by

$$\nabla^4 \bar{\phi} - i \nabla^2 \bar{\phi} = \frac{qR^2}{Et} \tag{49a}$$

where $\bar{\phi} = W + \lambda F$ with $\lambda = \frac{iR}{Et}$ (49b)

and $$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x} + \frac{1}{x^2 \cos^2 \phi_e}\frac{\partial^2}{\partial \theta^2} \tag{49c}$$

where x is given by Eqs. 4 as before.

Proceeding as in Eqs. 103 and 104 as in Ref. 1, the solutions can be written as

$$W = \mathrm{Re}\left[\bar{\phi}\right] = \mathrm{Re} \sum_{n=0}^{\infty} \left[A_v J_v(xi^{3/2}) + B_v H_v^{(2)}(xi^{3/2})\right] \begin{matrix}\cos n\theta \\ \sin n\theta\end{matrix} + \frac{qR^2}{Et} \tag{50}$$

$$\text{and } F = \mathrm{Re}\left[\bar{\phi}\frac{1}{\lambda}\right] = \frac{Et}{R}\,\mathrm{Im} \sum_{n=0}^{\infty} \left[A_v J_v(xi^{3/2}) + B_v H_v^{(2)}(xi^{3/2})\right] \begin{matrix}\cos n\theta \\ \sin n\theta\end{matrix}$$

$$+ c_0 \log x + d_o + \sum_{n=1}^{\infty} \left[c_v x^{-v} + d_v x^v\right] \begin{matrix}\text{cox } n\theta \\ \sin n\theta\end{matrix} + \frac{qR}{4} x^2 \tag{51}$$

where A_v and B_v are complex constants, whereas c_v and d_v are real constants.

Now $v = \frac{n}{\cos \phi_e}$ where ϕ_e corresponds to the ordinate to the zone of stress concentration. Hence v may not be an integer depending on the value of ϕ_e. This will involve the use of Bessel functions of non-integer orders with imaginary arguments which may not be available readily and also may result tedious calculations. In order to use the readily available Bessel functions of integer order which are tabulated and also as the ordinate ϕ_e, in the case of unsymmetrical bending, tends to have less significance since the concentration of stress considered need not occur at a constant ϕ around the shell, we assume that $\phi_e = 0$, so that v = n. However, choosing a value for ϕ_e in the region of maximum stress leads to greater accuracy than that given by the value $\phi_e = 0$. This approximation of letting $\phi_e = 0$ restricts the solutions to stress concentrations at shallow angles only, i.e., $\phi_e < \pi/6$.

With $\phi_e = 0$, i.e., v = n. and noting that

$$\int_0^{2\pi} \cos\theta \cdot \cos n\theta = \int_0^{2\pi} \sin\theta \cdot \sin n\theta = \begin{cases} 0 & n \neq 1 \\ \pi & n = 1 \end{cases} \tag{52a}$$

$$\int_0^{2\pi} \cos\theta \cdot \sin n\theta = \int_0^{2\pi} \sin\theta \cdot \cos n\theta = 0 \tag{52b}$$

it can be shown that for equilibrium for the type unsymmetrical loading that will be considered we need only the solutions corresponding to n = 1. Also, where symmetry occurs about a plane through the poles ($\theta = o, \pi$), only the cosine series is required. Hence, from Eqs. 50 and 51 we have

$$W = \mathrm{Re}\left[A_1 J_1(xi^{3/2}) + B_1 H_1^{(2)}(xi^{3/2})\right] \cos\theta + \frac{qR^2}{Et} \tag{53}$$

$$F = \frac{Et}{R} \ \mathrm{Im}\left[A_1 J_1(xi^{3/2}) + B_1 H_1^{(2)}(xi^{3/2})\right] \cos\theta$$

$$+ \left[c_1 \frac{1}{x} + d_1 x\right] \cos\theta + \frac{qRx^2}{4} \tag{54}$$

Using the relationships

$$J_1(xi^{3/2}) = ber_1 x + i\, bei_1 x \tag{55a}$$

$$H_1^{(2)}(xi^{3/2}) = -\frac{2}{\pi}(Kei_1 x - i\, Ker_1 x) = \frac{2i}{\pi}(Ker_1 x + iKei_1 x) \tag{55b}$$

and the recurrence formulas

$$ber_1 x = \frac{1}{\sqrt{2}}(ber' x - bei' x) \tag{56a}$$

$$bei_1 x = \frac{1}{\sqrt{2}}(ber' x + bei' x) \tag{56b}$$

and similar relations for $Ker_1 x$ and $Kei_1 x$ we can write

$$W = C_1 ber' x + C_2 bei' x + C_3 Ker' x + C_4 Kei' x + \frac{qR^2}{Et} \tag{57}$$

$$F = \frac{Et}{R}\left[C_1 bei_, x - C_2 ber' x + C_3 Kei' x - C_4 Ker' x + C_5 \cdot \frac{1}{x} + C_6 \cdot x\right] \cos\theta + \frac{qRx^2}{4} \tag{58}$$

Rigid Cylindrical Inserts

As before, if the radial deflections must dampen out for large values of x, we must have $C_1 = C_2 = 0$. Hence from Eqs. 82 and 108 and the recurrence relations

$$\text{Ker}''x = -\text{Kei}\,x - \frac{\text{Ker}'x}{x} \tag{59a}$$

$$\text{Kei}''x = \text{Ker}\,x - \frac{\text{Kei}'x}{x} \tag{59b}$$

$$\text{Ker}'''x = -\text{Kei}'x + \frac{\text{Kei}\,x}{x} + \frac{2\,\text{Ker}'x}{x^2} \tag{59c}$$

$$\text{Kei}'''x = \text{Ker}'x - \frac{\text{Ker}\,x}{x} + \frac{2\,\text{Kei}'x}{x^2} \tag{59d}$$

We have for $q = 0$

$$N_\phi = \frac{Et}{R}\left[C_3 \cdot \frac{1}{x}\left(2\frac{\text{Kei}'x}{x} - \text{Ker}\,x\right) - C_4 \cdot \frac{1}{x}\left(2\frac{\text{Ker}'x}{x} + \text{Kei}\,x\right) + 2\,C_5 \cdot \frac{1}{x^3}\right]\cos\theta \tag{60a}$$

$$N_\theta = \frac{Et}{R}\left[C_3\left\{-\text{Ker}'x - \frac{1}{x}\left(2\frac{\text{Kei}'x}{x} - \text{Ker}\,x\right)\right\} - C_4\left\{\text{Kei}'x - \frac{1}{x}\left(2\frac{\text{Ker}'x}{x} + \text{Kei}\,x\right)\right\} - 2\,C_5 \cdot \frac{1}{x^3}\right]\cos\theta \tag{60b}$$

$$N_{\phi\theta} = \frac{Et}{R}\left[C_3\frac{1}{x}\left(2\frac{\text{Kei}'x}{x} - \text{Ker}\,x\right) - C_4\frac{1}{x}\left(2\frac{\text{Ker}'x}{x} + \text{Kei}\,x\right) + 2\,C_5 \cdot \frac{1}{x^3}\right]\sin\theta \tag{60c}$$

$$M_\phi = \frac{D}{L^2}\left[C_3\left\{\text{Kei}'x - \left(\frac{1-\nu}{x}\right)\left(2\frac{\text{Ker}'x}{x} + \text{Kei}\,x\right)\right\} - C_4\left\{\text{Ker}'x + \left(\frac{1-\nu}{x}\right)\left(2\frac{\text{Kei}'x}{x} - \text{Ker}\,x\right)\right\}\right]\cos\theta \tag{60d}$$

$$M_\theta = \frac{D}{L^2}\left[C_3\left\{\nu \text{Kei}'x + \left(\frac{1-\nu}{x}\right)\left(2\frac{\text{Ker}'x}{x} + \text{Kei } x\right)\right\}\right.$$

$$\left. - C_4\left\{\nu \text{Ker}'x - \left(\frac{1-\nu}{x}\right)\left(2\frac{\text{Kei}'x}{x} - \text{Ker } x\right)\right\}\right]\cos\theta \qquad (60e)$$

$$M_{\phi\theta} = -(1-\nu)\frac{D}{L^2}\left[C_3 \cdot \frac{1}{x}\left(2\frac{\text{Ker}'x}{x} + \text{Kei } x\right) + C_4 \cdot \frac{1}{4}\left(2\frac{\text{Kei}'x}{x} - \text{Ker } x\right)\right]\sin\theta \qquad (60f)$$

$$Q_\phi = \frac{D}{L^3}\left[C_3\left(\text{Ker } x - \frac{\text{Kei}'x}{x}\right) + C_4\left(\text{Kei } x + \frac{\text{Ker}'x}{x}\right)\right]\cos\theta \qquad (60g)$$

$$Q_\theta = \frac{D}{L^3}\left[-C_3\frac{\text{Kei}'x}{x} + C_4\frac{\text{Ker}'x}{x}\right]\sin\theta \qquad (60h)$$

Introducing the boundary conditions that at $x = x_o$

$$\frac{dW}{dx} = \frac{W}{x} \quad \text{and} \quad \epsilon_\theta = \frac{1}{Et}\left(N_\theta - \nu N_\phi\right) = 0\,, \qquad (61)$$

where x_o corresponds to the insert edge in the conical co-ordinate system with $x_e = 0$ we get

$$C_3 = C_5 \cdot F(x_o) \qquad (62a)$$

$$C_4 = -C_3\left\{\frac{2\dfrac{\text{Ker}'x_o}{x_o} + \text{Kei } x_o}{2\dfrac{\text{Kei}'x_o}{x_o} - \text{Ker } x_o}\right\} \qquad (62b)$$

$$F(x_o) = \frac{-\dfrac{2(1+\nu)}{x_o^3}\left(2\dfrac{\text{Kei}'x_o}{x_o} - \text{Ker } x_o\right)}{\left[\left(2\dfrac{\text{Kei}'x_o}{x_o} - \text{Ker } x_o\right)\left\{\text{Ker}'x_o + \dfrac{(1+\nu)}{x_o}\left(2\dfrac{\text{Kei}'x_o}{x_o} - \text{Ker } x_o\right)\right\} - \left(2\dfrac{\text{Ker}'x_o}{x_o} + \text{Kei } x_o\right)\left\{\text{Kei}'x_o - \dfrac{(1+\nu)}{x_o}\left(2\dfrac{\text{Ker}'x_o}{x_o} + \text{Kei } x_o\right)\right\}\right]} \qquad (62c)$$

(a) External Moment Acting Upon a Rigid Cylindrical Insert in a Spherical Shell

An external moment of magnitude M as shown in Fig. 36 is considered. For equilibrium about a horizontal section at any ordinate ϕ, we have

$$M = \int_0^{2\pi} \left[-(N_\phi \sin\phi + Q_\phi \cos\phi) R \sin\phi \,.\, \cos\theta - M_\phi \cos\theta + M_{\phi\theta} \sin\theta \right] . \; . \, R \sin\phi \,.\, d\theta \qquad (63)$$

Now considering the symmetry of the loading about a plane through the poles, $\theta = 0, \pi$, we can express the stress resultants and couples as

$$\begin{aligned} N_\phi &= \sum_n N_{\phi_n}(x) \,.\, \cos n\theta \\ N_{\phi\theta} &= \sum_n N_{\phi\theta_n}(x) \,.\, \sin n\theta \\ Q_\phi &= \sum_n Q_{\phi_n}(x) \,.\, \cos n\theta \\ M_\phi &= \sum_n M_{\phi_n}(x) \,.\, \cos n\theta \\ M_{\phi\theta} &= {}_n \; M_{\phi\theta_n}(x) \quad \sin n\theta \end{aligned} \qquad (63a)$$

Making use of Eqs. 52, on integration of Eq. (63) we notice that, all terms except that for n = 1 vanish. Dropping the subscript 1, Eq. 63 leads to

$$M = -\pi R^2 \sin\phi \left[\left\{ N_\phi(x) \sin\phi + Q_\phi(x) \cos\phi \right\} \sin\phi - \frac{1}{R} \left\{ M_\phi(x) - M_{\phi\theta}(x) \cos\phi \right\} \right] \qquad (64)$$

x in parenthesis indicates that there is no θ dependence of the stress resultants and couples after integration. Now expressing $\cos\phi$ in terms of $\sin\phi$ and noting that $\sin\phi = \frac{L}{R} x$, we have

$$\cos\phi = 1 - \left(\frac{L^2 x^2}{2R^2}\right) + \frac{1}{2!}\left(\frac{L^2 x^2}{2R^2}\right)^2 - \frac{1.3}{3!}\left(\frac{L^2 x^2}{2R^2}\right)^3 + - \ldots\ldots$$

Substituting the above expressions for $\sin\phi$ and $\cos\phi$ and Eqs. (60) in Eq. (64), and considering only terms containing equal and lowest order of $\frac{L^2}{R^2}$ and neglecting higher order terms of $\frac{L^2}{R^2}$ as $\frac{L^2}{R^2} << 1$ for thin shells under consideration we get

$$C_5 = - \frac{ML}{2\pi D} = - \frac{MR^2}{2\pi \; EtL^3} \qquad (65)$$

Using Eqs. 60, 62 and 65, the quantities of interest can be written as (66)

$$W = d_1 \cdot \frac{M}{Et^2} \sqrt{\frac{R}{t}} \cdot \cos\theta \qquad N_\theta = d_8 \cdot \frac{M}{Rt}\sqrt{\frac{R}{t}} \cdot \cos\theta$$

$$M_\phi = d_2 \cdot \frac{M}{R} \sqrt{\frac{R}{t}} \cdot \cos\theta \qquad N_{\phi\theta} = d_7 \cdot \frac{M}{Rt}\sqrt{\frac{R}{t}} \cdot \sin\theta$$

$$M_\theta = d_3 \cdot \frac{M}{R} \sqrt{\frac{R}{t}} \cdot \cos\theta \qquad \sigma_{\phi_i} = d_9 \cdot \frac{M}{Rt^2}\sqrt{\frac{R}{t}} \cdot \cos\theta$$

$$M_{\phi\theta} = d_4 \cdot \frac{M}{R} \sqrt{\frac{R}{t}} \cdot \sin\theta \qquad \sigma_{\phi_e} = d_{10} \cdot \frac{M}{Rt^2}\sqrt{\frac{R}{t}} \cdot \cos\theta$$

$$Q_\phi = d_5 \cdot \frac{M}{Rt} \cdot \cos\theta \qquad \sigma_{\theta_i} = d_{11} \cdot \frac{M}{Rt^2}\sqrt{\frac{R}{t}} \cdot \cos\theta$$

$$Q_\theta = d_6 \cdot \frac{M}{Rt} \cdot \sin\theta \qquad \sigma_{\theta_e} = d_{12} \cdot \frac{M}{Rt^2}\sqrt{\frac{R}{t}} \cdot \cos\theta$$

$$N_\phi = d_7 \cdot \frac{M}{Rt}\sqrt{\frac{R}{t}} \cdot \cos\theta$$

The variation of the coefficients d_1 to d_{10} for several insert sizes is given in Table V for various values of insert size x_o. The results are also graphically represented in Figs. 37 - 46.

As the diameter of the insert reduces to zero in the limiting case, i.e., $x_o = 0$, the problem reduces to the case of a concentrated moment at the apex of the shell. In this case at any distance x from the apex, we have

$$W = \frac{M}{Et^2} \cdot \sqrt{\frac{R}{t}} \frac{\xi^3}{2\pi} \text{Kei}' x \cos\theta \qquad (67a)$$

$$M_\phi = -\frac{M}{R}\sqrt{\frac{R}{t}} \frac{\xi}{2\pi}\left[\text{Ker}' x + \frac{(1-\nu)}{x}\left(2\frac{\text{Kei}' x}{x} - \text{Ker } x\right)\right]\cos\theta \qquad (67b)$$

$$M_\theta = -\frac{M}{R}\sqrt{\frac{R}{t}} \frac{\xi}{2\pi}\left[\nu\text{Ker}' x - \frac{(1-\nu)}{x}\left(2\frac{\text{Kei}' x}{x} - \text{Ker } x\right)\right]\cos\theta \qquad (67c)$$

$$M_{\phi\theta} = -\frac{M}{R}\sqrt{\frac{R}{t}} \frac{\xi}{2\pi}\frac{(1-\nu)}{x}\left(2\frac{\text{Kei}' x}{x} - \text{Ker } x\right)\sin\theta \qquad (67d)$$

$$Q_\phi = \frac{M}{Rt}\frac{\xi^2}{2\pi}\left(\text{Kei } x + \frac{\text{Ker}' x}{x}\right)\cos\theta \qquad (67e)$$

$$Q_\theta = \frac{M}{Rt}\frac{\xi^2}{2\pi}\frac{\text{Ker}' x}{x}\sin\theta \qquad (67f)$$

$$N_\phi = -\frac{M}{Rt}\sqrt{\frac{R}{t}}\frac{\xi^3}{2\pi}\left[\frac{1}{x}\left(2\frac{\text{Ker}' x}{x} + \text{Kei } x\right) + \frac{2}{x^3}\right]\cos\theta \qquad (67g)$$

$$N_\theta = -\frac{M}{Rt}\sqrt{\frac{R}{t}}\frac{\xi^3}{2\pi}\left[\text{Kei}' x - \frac{1}{x}\left(2\frac{\text{Ker}' x}{x} + \text{Kei } x\right) - \frac{2}{x^3}\right]\cos\theta \qquad (67h)$$

$$N_{\phi\theta} = -\frac{M}{Rt}\sqrt{\frac{R}{t}}\frac{\xi^3}{2\pi}\left[\frac{1}{x}\left(2\frac{\text{Ker}' x}{x} + \text{Kei } x\right) + \frac{2}{x^3}\right]\sin\theta \qquad (67i)$$

where $$\xi = \left[12(1-\nu^2)\right]^{1/4} \qquad (67j)$$

As $x \to 0$, W, N_ϕ, and $N_{\phi\theta}$ tend to zero, whereas M_ϕ, M_θ, $M_{\phi\theta}$, Q_ϕ, and Q_θ tend to infinity.

(b) Tangentially Loaded (Shear Load) Insert

A force of magnitude P (Fig. 47) is considered acting on the insert, tangentially to the surface of the shell at the apex. For equilibrium at any ordinate ϕ, we have

$$P = \int_0^{2\pi} \left[N_{\phi\theta} \sin\theta + (Q_\phi \sin\phi - N_\phi \cos\phi) \cos\theta \right] R \sin\phi \, d\theta \qquad (68)$$

As in the case of an external moment this reduces to

$$P = \pi R \sin\phi \left[N_{\phi\theta}(x) + Q_\phi(x) \sin\phi - N_\phi(x) \cos\phi \right] \qquad (69)$$

As before substituting the expressions for sin ϕ and cos ϕ and Eqs. (60) in (69) and neglecting higher order terms of $\frac{L^2}{R^2}$ and noting that $N_\phi(x) = N_{\phi\theta}(x)$, we notice that the lower order terms cancel out. Hence in order to take care of the external tangential loading P at the apex, we introduce a membrane solution, satisfying Eq. (69), given by

$$N_\phi(x) = - N_\theta(x) = - N_{\phi\theta}(x) = - \frac{P}{2\pi L} \; \frac{1}{x} \qquad (70)$$

Incorporating this correction in the expressions for N_ϕ, N_θ and $N_{\phi\theta}$, and applying the boundary conditions Eq. (61) as before, with $C_5 = 0$, we can solve for C_3 and C_4 explicitly. The quantities of interest follow as:

$$\begin{aligned}
W &= e_1 \cdot \frac{P}{Et} \sqrt{\frac{R}{t}} \cos\theta & N_\phi &= e_7 \cdot \frac{P}{t} \sqrt{\frac{t}{R}} \cos\theta \\
M_\phi &= e_2 \cdot P \sqrt{\frac{t}{R}} \cos\theta & N_\theta &= e_8 \cdot \frac{P}{t} \sqrt{\frac{t}{R}} \cos\theta \\
M_\theta &= e_3 \cdot P \sqrt{\frac{t}{R}} \cos\theta & N_{\phi\theta} &= e_9 \cdot \frac{P}{t} \sqrt{\frac{t}{R}} \sin\theta \\
M_{\phi\theta} &= e_4 \cdot P \sqrt{\frac{t}{R}} \sin\theta & \sigma_{\phi_i} &= e_{10} \cdot \frac{P}{t^2} \sqrt{\frac{t}{R}} \cos\theta \\
Q_\phi &= e_5 \cdot \frac{P}{R} \cos\theta & \sigma_{\phi_e} &= e_{11} \cdot \frac{P}{t^2} \sqrt{\frac{t}{R}} \cos\theta \\
Q_\theta &= e_6 \cdot \frac{P}{R} \sin\theta & \sigma_{\theta_i} &= e_{12} \cdot \frac{P}{t^2} \sqrt{\frac{t}{R}} \cos\theta
\end{aligned} \qquad (71)$$

$$\sigma_{\theta_e} = e_{13} \cdot \frac{P}{t^2} \sqrt{\frac{t}{R}} \cos\theta$$

As before, the variation of the coefficients e_1 to e_{13} are given in Table VI and are also graphically represented in Figs. 48-58.

DESIGN CHARTS AND TABLES

Table I

Internally pressurized spherical shell with a rigid insert
$X_e = 0.2$

x	a_1	a_2	a_3	a_4	a_5
0.2	9927354050-	5717052348	1715115648	2900614149-	7636426350
0.4	9929915050-	2391125748	2211187848	1391251049-	5632267950
0.6	9935219050-	1270366748	1765563448	8657105248-	5262287250
0.8	9941654050-	6581998447	1353999648	5901099648-	5134090750
1.0	9948479050-	2732579847	1026586548	4178072848-	5075950750
1.2	9955266050-	2005434046	7715042247	2997544948-	5045408850
2.0	9978164050-	3485026547-	2118446247	6853374347-	5006229250
3.0	9994211050-	2795264947-	7405178645	1017716547	4999383350
4.0	1000006051-	1316873547-	2086217146-	1787260147	4999187750
5.0	1000114051-	3885566346-	1168107546-	1013149947	4999631650
6.0	1000076851-	1294227445-	2825908545-	3450522146	4999895550
7.0	1000029851-	6934275945	6335773344	3393727545	4999991250
8.0	1000004751-	5141418545	1036092345	4816938145-	5000011050
9.0	9999969050-	2168163045	5718627044	4126612445-	5000008350
10.0	9999968050-	4351433044	1678141244	1893220845-	5000003550

x	a_6	a_7	a_8	a_9	a_{10}
0.2	2290927750	7979449450	7293403250	2393834650	2188020850
0.4	4297647450	5775735450	5488800450	4430318750	4164976150
0.6	4672931750	5338509250	5186065250	4778865550	4566997950
0.8	4807563250	5173582750	5094598750	4888803250	4726323250
1.0	4872527950	5092346250	5059555250	4934123150	4810932750
1.2	4909856950	5046612150	5044205550	4956147250	4863566650
2.0	4971935050	4985319050	5027139450	4984645750	4959224350
3.0	4994827950	4982611750	5016154950	4995272250	4994383650
4.0	5000872250	4991286550	5007088950	4999620550	5002123950
5.0	5001508550	4997300350	5001962950	5000807650	5002209450
6.0	5000872850	4999817850	4999973250	5000703250	5001042450
7.0	5000306850	5000407350	4999575150	5000344850	5000268850
8.0	5000035850	5000319550	4999702550	5000098050	4999973650
9.0	4999960650	5000138450	4999878250	4999994950	4999926350
10.0	4999964450	5000029650	4999977450	4999974550	4999954350

$X_e = 0.4$

x	a_1	a_2	a_3	a_4	a_5
0.4	9763339050-	1467009349	4401028048	5523612149-	7510260850
0.6	9770778050-	7548473948	5278607748	3468081049-	6050736150
0.8	9787340050-	4151659648	4555054348	2388261649-	5542684850
1.0	9808115050-	2148220848	3660148848	1710485049-	5310938850
1.2	9830496050-	8620611047	2852744648	1243453249-	5188366650
1.4	9852933050-	1264176046	2178322248	9044467048-	5117438550
2.0	9912765050-	1105468148-	8573431947	3128523948-	5028435850
3.0	9974846050-	1046861448-	5137357046	2161625647	4998690150
4.0	9998925050-	5254416447-	7658846646-	6540255447	4997027750
5.0	1000405951-	1674443447-	4689934946-	3993886247	4998547950
6.0	1000299251-	1371055246-	1274700046-	1458729147	4999558050
7.0	1000123951-	2421770346	1645571145	2021618346	4999947550
8.0	1000023751-	1986411346	3861731245	1624848346-	5000036950
9.0	9999901050-	8922468945	2289399045	1578619146-	5000031950
10.0	9999878050-	2069603845	7357278844	7713395045-	5000014050

x	a_6	a_7	a_8	a_9	a_{10}
0.4	2253078350	8390466450	6630055250	2517140050	1989016650
0.6	3720042050	6503644550	5597827750	4036758550	3403325550
0.8	4244655650	5791784450	5293585250	4517958950	3971352350
1.0	4497175950	5439832150	5182045650	4716784850	4277567050
1.2	4642129250	5240090350	5136642950	4813293950	4470964550
1.4	4735494750	5118197050	5116680050	4866194050	4604795450
2.0	4884329250	4962107750	5094763950	4935769850	4832888650
3.0	4976155950	4935878450	5061501850	4979238350	4973073550
4.0	5001897050	4965501250	5028554250	4997301750	5006492350
5.0	5005511150	4988501250	5008594650	5002697150	5008325150
6.0	5003434250	4998735450	5000380650	5002669450	5004199050
7.0	5001291850	5001400650	4998494450	5001390550	5001193150
8.0	5000200250	5001228750	4998845150	5000431950	4999968550
9.0	4999869450	5000567250	4999496650	5000006850	4999732050
10.0	4999864150	5000138250	4999889850	4999908250	4999820050

$X_e = 0.6$

x	a_1	a_2	a_3	a_4	a_5
0.6	9566380050-	2351685449	7055056048	7785350249-	7358753650
0.8	9578932050-	1336782449	7903009048	5426109549-	6232974950
1.0	9608023050-	7637625248	7007620848	3937936949-	5715853950
1.2	9645416050-	4024062248	5768199648	2905265549-	5440109150
1.4	9686283050-	1633070148	4570912248	2149242349-	5279069950
1.6	9727620050-	3037270046	3525967448	1578853449-	5179381450
2.0	9804813050-	1704962848-	1947815248	8046587248-	5073137150
3.0	9938285050-	2155161448-	1733329247	2656370046-	5000160950
4.0	9994072050-	1173918048-	1532615647-	1302629748	4994080050
5.0	1000775451-	4073839347-	1055612547-	8792435647	4996803450
6.0	1000648951-	5356024746-	3242636146-	3478876147	4998946050
7.0	1000290451-	4484891246	1309011745	6307387646	4999836250
8.0	1000066651-	4263432046	7897041745	2832979346-	5000064450
9.0	9999847050-	2066215746	5135334545	3344641646-	5000067550
10.0	9999746050-	5518140445	1821166545	1766032846-	5000032150

x	a_6	a_7	a_8	a_9	a_{10}
0.6	2207626150	8769764850	5947742450	2630929550	1784322750
0.8	3345956950	7035044350	5430905550	3820137450	2871776450
1.0	3892169550	6174111450	5257596450	4312626850	3471712350
1.2	4205306850	5681552850	5198665450	4551398850	3859214850
1.4	4407213450	5377054150	5181085750	4681468150	4132958750
1.6	4548238450	5181203850	5177559050	4759796450	4336680450
2.0	4731675750	4970839350	5175434950	4848544650	4614806850
3.0	4938124250	4870851250	5129470650	4948524250	4927724250
4.0	4999991950	4923644950	5064515150	4990796250	5009187650
5.0	5010950350	4972360450	5021246450	5004616650	5017284050
6.0	5007543550	4995732450	5002159650	5005597950	5009489150
7.0	5003068150	5002527150	4997145350	5003146650	5002989650
8.0	5000601750	5002622550	4997506350	5001075550	5000127950
9.0	4999779150	5001307250	4998827850	5000087250	4999471050
10.0	4999713650	5000363250	4999701050	4999822950	4999604350

$X_e = 0.8$

x	a_1	a_2	a_3	a_4	a_5
0.8	9366229050-	3145745049	9437234048	9702906849-	7204791650
1.0	9383496050-	1894898949	1002002149	7141061749-	6298130750
1.2	9424531050-	1116016949	8945714048	5349000749-	5810302450
1.4	9478204050-	5999569148	7447715948	4024337549-	5522543050
1.6	9537544050-	2490400848	5954766448	3014113649-	5342448650
1.8	9598035050-	9924430046	4618372848	2230773149-	5225288350
2.0	9656746050-	1500536148-	3482943648	1619034849-	5147157350
3.0	9880880050-	3425242948-	4186122747	1041124348-	5006308750
4.0	9982710050-	2061028148-	2342399047-	1977507348	4991013050
5.0	1001104751-	7802097247-	1868562247-	1518196448	4994480350
6.0	1001101251-	1387888147-	6463560746-	6536922147	4998019450
7.0	1000536351-	6067163046	2322747045-	1456914647	4999621650
8.0	1000143651-	7144598246	1243854246	3417814446-	5000077750
9.0	9999862050-	3769712846	9055960845	5516425746-	5000111450
10.0	9999589050-	1143371446	3543724945	3184018946-	5000057950

x	a_6	a_7	a_8	a_9	a_{10}
0.8	2161437450	9092238650	5317344650	2727671450	1595203450
1.0	3085364850	7435070050	5161191450	3686566150	2484163550
1.2	3614228350	6479912550	5140692350	4150971150	3077485550
1.4	3955661550	5882517250	5162568950	4402524550	3508798650
1.6	4195095950	5491872750	5193024650	4552381950	3837809950
1.8	4372747050	5231243050	5219333650	4649849450	4095644650
2.0	4509588250	5057125150	5237189550	4718564850	4300611650
3.0	4874571650	4800794150	5211823350	4899688350	4849454950
4.0	4991697450	4867351350	5114674750	4977643050	5005751850
5.0	5016566550	4947667750	5041292950	5005355150	5027777950
6.0	5012992150	4989692150	5006346750	5009114050	5016870250
7.0	5005741850	5003261950	4995981350	5005602450	5005881250
8.0	5001358050	5004364550	4995790950	5002104350	5000611750
9.0	4999750650	5002373250	4997849650	5000294050	4999207250
10.0	4999531250	5000743950	4999371950	4999743850	4999318650

$X_e = 1.0$

x	a_1	a_2	a_3	a_4	a_5
1.0	9176100050-	3836077849	1150823449	1132409050-	7058538350
1.2	9197529050-	2399465649	1171723049	8613660849-	6304854850
1.4	9249330050-	1446661749	1043911049	6588541549-	5855493950
1.6	9317967050-	7919164048	8726884048	5025877949-	5571015150
1.8	9394543050-	3364992048	7004675848	3798929649-	5383658150
2.0	9473117050-	2144751047	5442902148	2828024049-	5257044850
3.0	9799356050-	4644398448-	8338324447	3269655648-	5019812350
4.0	9961922050-	3157864548-	3002376747-	2508779148	4988598650
5.0	1001260551-	1304818748-	2888634247-	2283252748	4991698850
6.0	1001622951-	2880038147-	1120069447-	1073826348	4996746550
7.0	1000866351-	6253643046	1133939646-	2820787147	4999267550
8.0	1000264851-	1037325747	1665677746	2616791046-	5000059550
9.0	1000001051-	6015123946	1394260546	7853234346-	5000158750
10.0	9999430050-	2045766646	6012760445	5018954146-	5000091250

x	a_6	a_7	a_8	a_9	a_{10}
1.0	2117561550	9360185050	4756891650	2808055550	1427067550
1.2	2892673950	7744534250	4865175450	3595707750	2189640150
1.4	3393836550	6723490950	4987496950	4020183150	2767489950
1.6	3746952250	6046164950	5095865350	4270565250	3223339250
1.8	4010885150	5585557650	5181758650	4431165750	3590604650
2.0	4216072650	5269913350	5244176350	4542646750	3889498550
3.0	4779543950	4741148450	5298476250	4829573850	4729514050
4.0	4973323650	4799126750	5178070550	4955309350	4991337950
5.0	5020906050	4913409750	5069987950	5003574250	5038237850
6.0	5019482150	4979466350	5014026750	5012761750	5026202550
7.0	5009395550	5003019750	4995515350	5008715150	5010075950
8.0	5002588350	5006283550	4993835550	5003587750	5001588950
9.0	4999851050	5003767850	4996549650	5000687650	4999014450
10.0	4999338850	5001318750	4998863750	4999699650	4998978050

$X_e = 3.0$

x	a_1	a_2	a_3	a_4	a_5
3.0	8015047050-	7277900949	2183370249	1923306450-	6165421150
3.2	8057745050-	5080751749	1869774249	1593125050-	5905014550
3.4	8166652050-	3343683849	1543930349	1297409850-	5693671750
3.6	8317863050-	1990839649	1231495049	1037106450-	5523692750
3.8	8493001050-	9571499048	9471598448	8116416149-	5388272250
4.0	8678220050-	1864907048	6982672148	6193840149-	5281485250
5.0	9491267050-	1192107649-	1271248047-	6324975148-	5022995650
6.0	9915818050-	8572091248-	1438973148-	7748163048	4976525150
7.0	1004654151-	3566471448-	8959540147-	6661220948	4982701350
8.0	1005041951-	7266564047-	2884210547-	3054919948	4993058350
9.0	1002563651-	2420384947	4320250044-	7412074047	4998502950
10.0	1000724951-	3289029147	6410134846	1333208347-	5000242350

x	a_6	a_7	a_8	a_9	a_{10}
3.0	1849626350	1053216251	1798680650	3159648450	5396042049
3.2	2152730450	8953465550	2856563550	3274594950	1030865950
3.4	2472980950	7699882050	3687461450	3399339150	1546622750
3.6	2794170450	6718196550	4329188950	3533067450	2055273450
3.8	3104728750	5962562150	4813982350	3673024650	2536432850
4.0	3396734550	5393379650	5169590850	3815694850	2977774250
5.0	4468271750	4307731050	5738260250	4460644250	4475899250
6.0	4939292750	4462199650	5490850650	4852954350	5025631150
7.0	5063839750	4768713050	5196689650	5010082550	5117596950
8.0	5057360750	4949458950	5036657750	5040055450	5074666050
9.0	5027132850	5013025250	4983980650	5027106950	5027158750
10.0	5007006350	5019976550	4980508150	5010852450	5003160250

$X_e = 5.0$

x	a_1	a_2	a_3	a_4	a_5
5.0	7537208050-	8438200649	2531460149	2194506650-	5797852550
5.2	7587201050-	6024968349	2057685949	1846222150-	5645410850
5.4	7716264050-	4054025749	1622094549	1523732950-	5512945050
5.6	7897605050-	2475078749	1234158049	1232033750-	5399935850
5.8	8109910050-	1237816349	8981476748	9735778249-	5305139450
6.0	8336619050-	2935844748	6145724248	7489077949-	5226899150
7.0	9353655050-	1497181649-	1359227948-	7456154548-	5019362950
8.0	9898038050-	1113022149-	2240773948-	1039142249	4976387650
9.0	1006684851-	4702072248-	1239208348-	8930934348	4981961150
10.0	1006976151-	9479838247-	3646315147-	4104613148	4992538450

x	a_6	a_7	a_8	a_9	a_{10}
5.0	1739355950	1086077351	7349321049	3258232050	2204798049
5.2	1941790650	9260391850	2030429850	3176402150	7071791049
5.4	2203319350	7945360450	3080529650	3176576050	1230062650
5.6	2497668850	6884983050	3914888650	3238163650	1757174050
5.8	2804770350	6047829250	4562449650	3343658950	2265881750
6.0	3109719850	5403049850	5050748450	3478463350	2740976450
7.0	4334292450	4121053950	5917671950	4252738750	4415846150
8.0	4921650950	4308574350	5644200950	4787204550	5056097350
9.0	5084887150	4699836850	5264085450	5010534650	5159239650
10.0	5077222850	4935659450	5049417450	5055344950	5099100750

$X_e = 7.0$

x	a_1	a_2	a_3	a_4	a_5
7.0	7286259050-	9002192049	2700657549	2328974950-	5604814950
7.2	7339815050-	6488701949	2140583849	1971863250-	5497851750
7.4	7478796050-	4409176049	1642524849	1636789550-	5402083950
7.6	7675117050-	2723349649	1210177549	1330278450-	5318188650
7.8	7906123050-	1387487949	8434835548	1056036550-	5246116050
8.0	8153953050-	3568755048	5396849448	8155809549-	5185324550
9.0	9278384050-	1656427449-	2251526548-	8068864048-	5016297750
10.0	9888947050-	1255964549-	2776684848-	1189327349	4978379950

x	a_6	a_7	a_8	a_9	a_{10}
7.0	1681444550	1100613051	2034997049	3301839050	6105000048
7.2	1841962950	9391072850	1604630650	3126313250	5576126049
7.4	2076711650	8047589550	2756578350	3062226550	1091196750
7.6	2356928650	6952198450	3684178850	3083035150	1630822150
7.8	2660006650	6078608750	4413623350	3166096750	2153916550
8.0	2968628750	5399449850	4971199250	3292439750	2644817750
9.0	4262086050	4022441350	6010154150	4126994450	4397177650
10.0	4910567450	4224801250	5731958650	4743966350	5077168550

$X_e = 10.0$

x	a_1	a_2	a_3	a_4	a_5
10.0	7076147050-	9452378949	2835713749	2438002750-	5443189850

x	a_6	a_7	a_8	a_9	a_{10}
10.0	1632957150	1111461751	2282375049-	3334385350	6847110048-

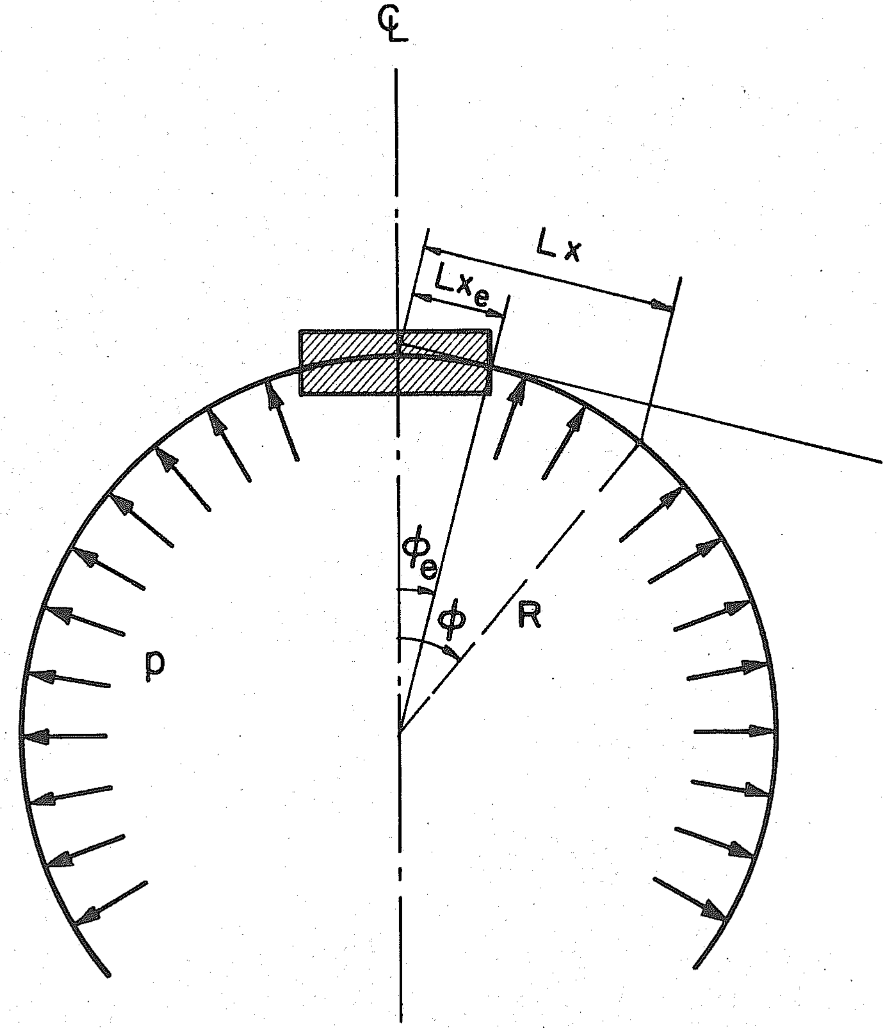
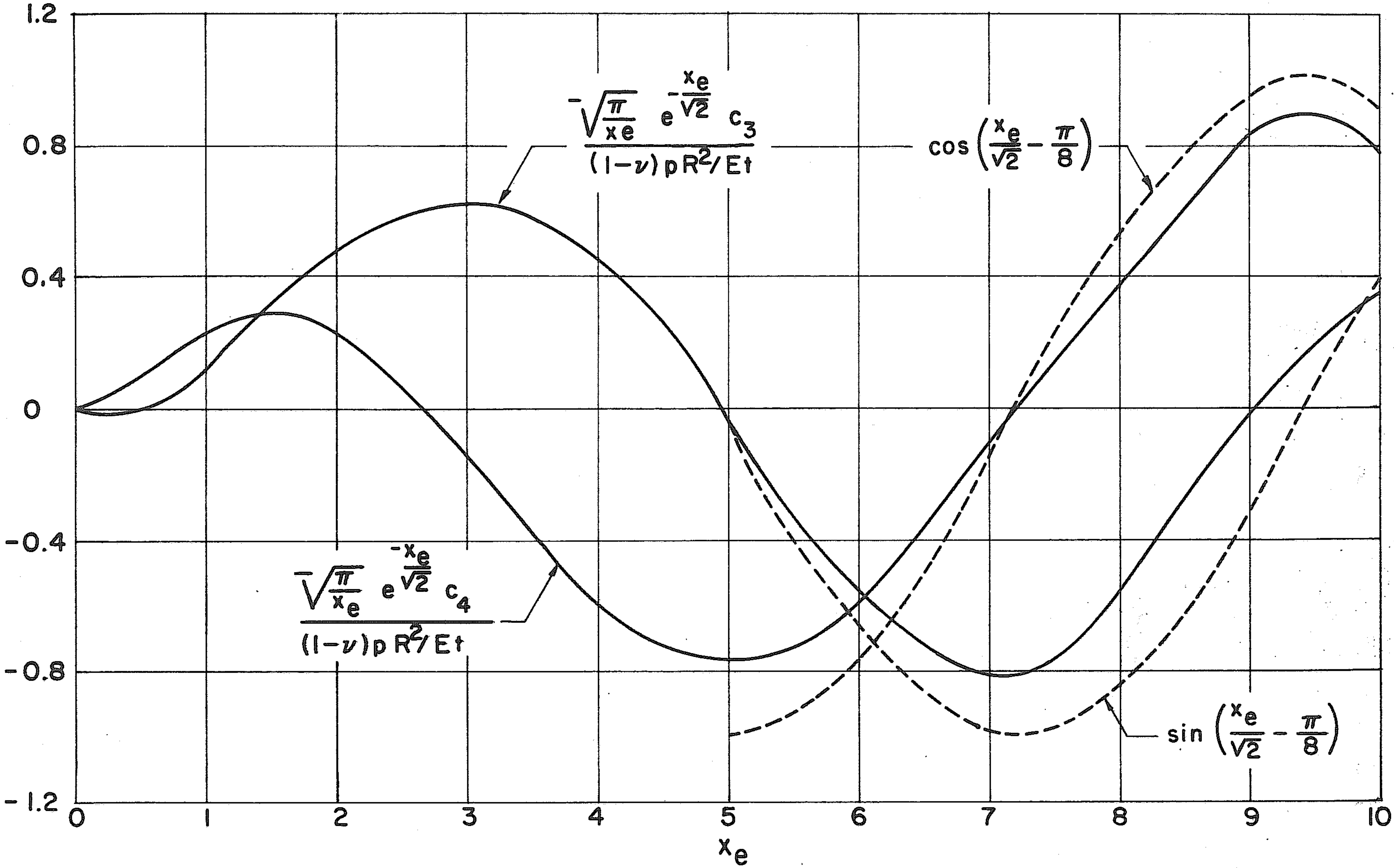
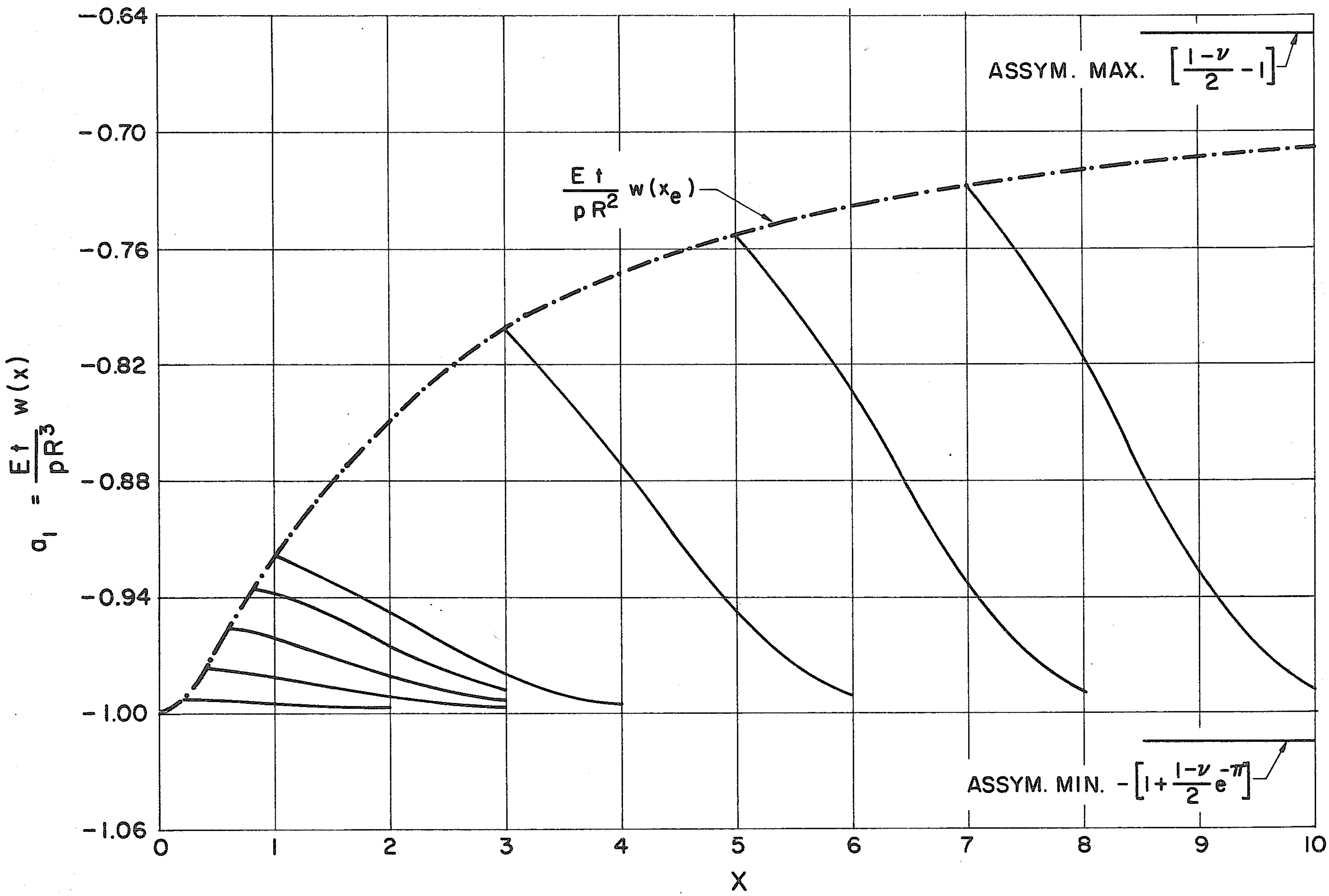


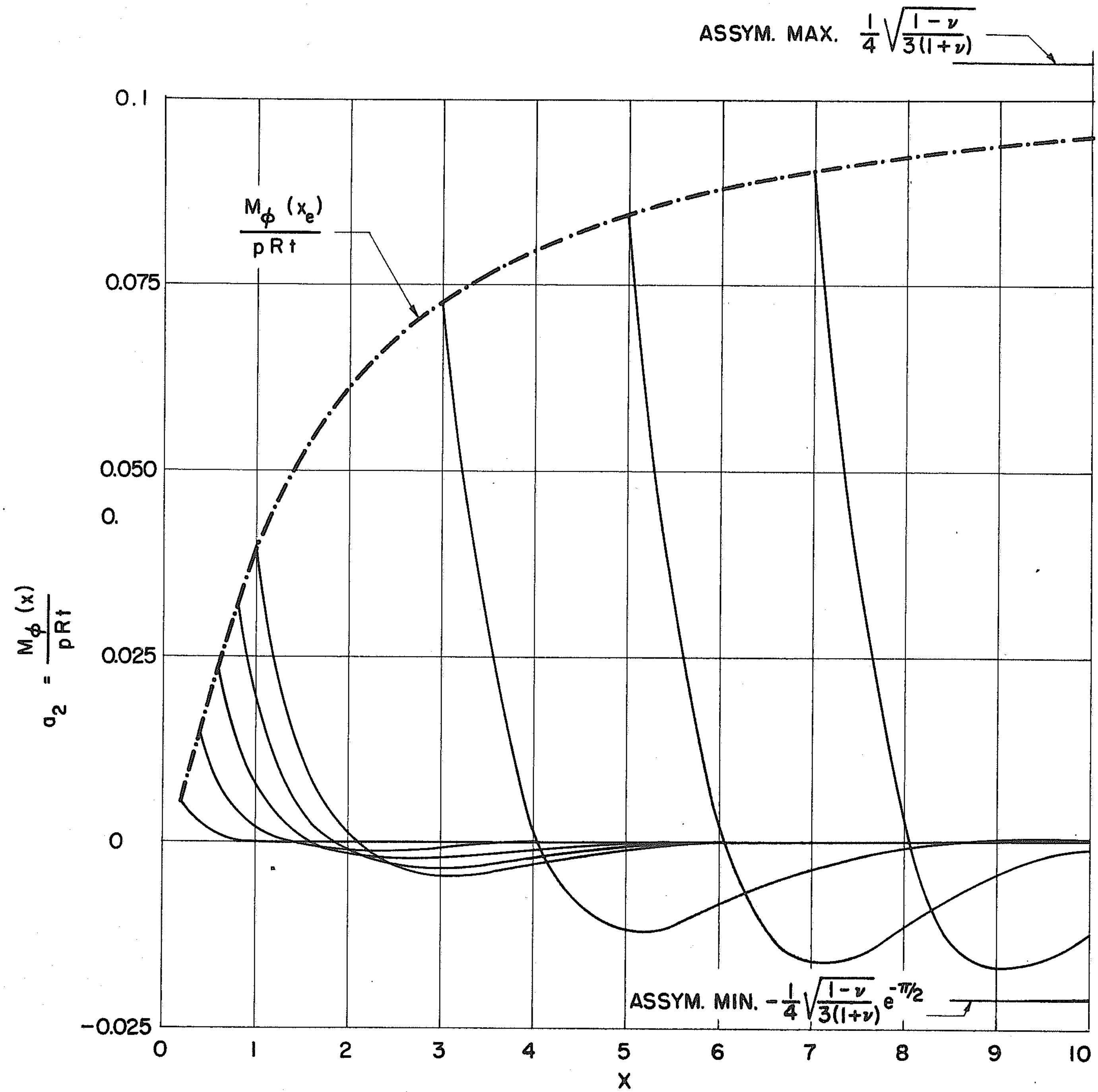
FIG. 4 — INTERNALLY PRESSURIZED SPHERICAL SHELL



CONSTANTS OF INTEGRATION FOR A PRESSURIZED SPHERICAL
SHELL WITH AN INSERT
FIG. 4a.

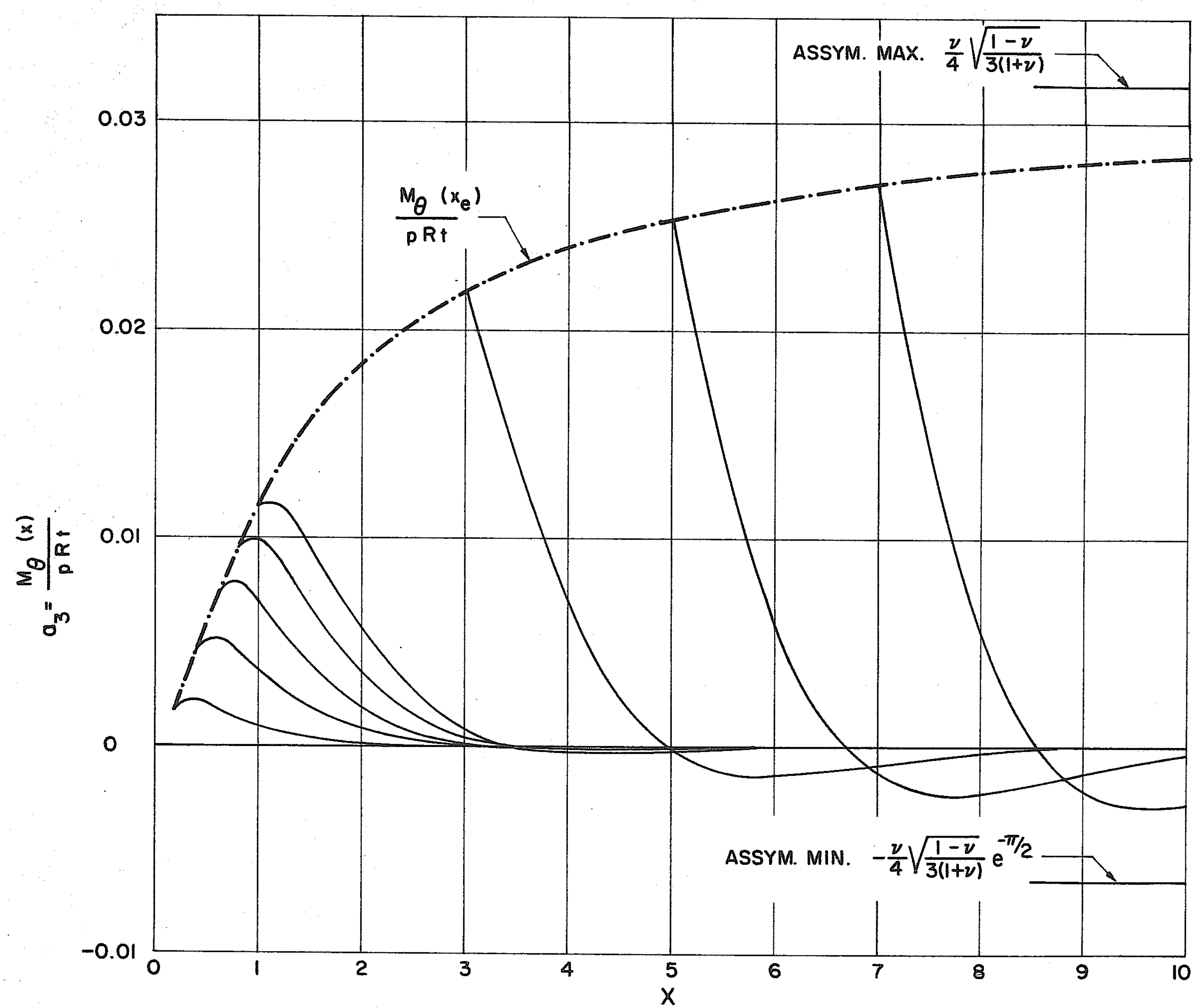


RADIAL DEFLECTION W FOR A PRESSURIZED SPHERICAL SHELL
WITH AN INSERT
FIG. 5



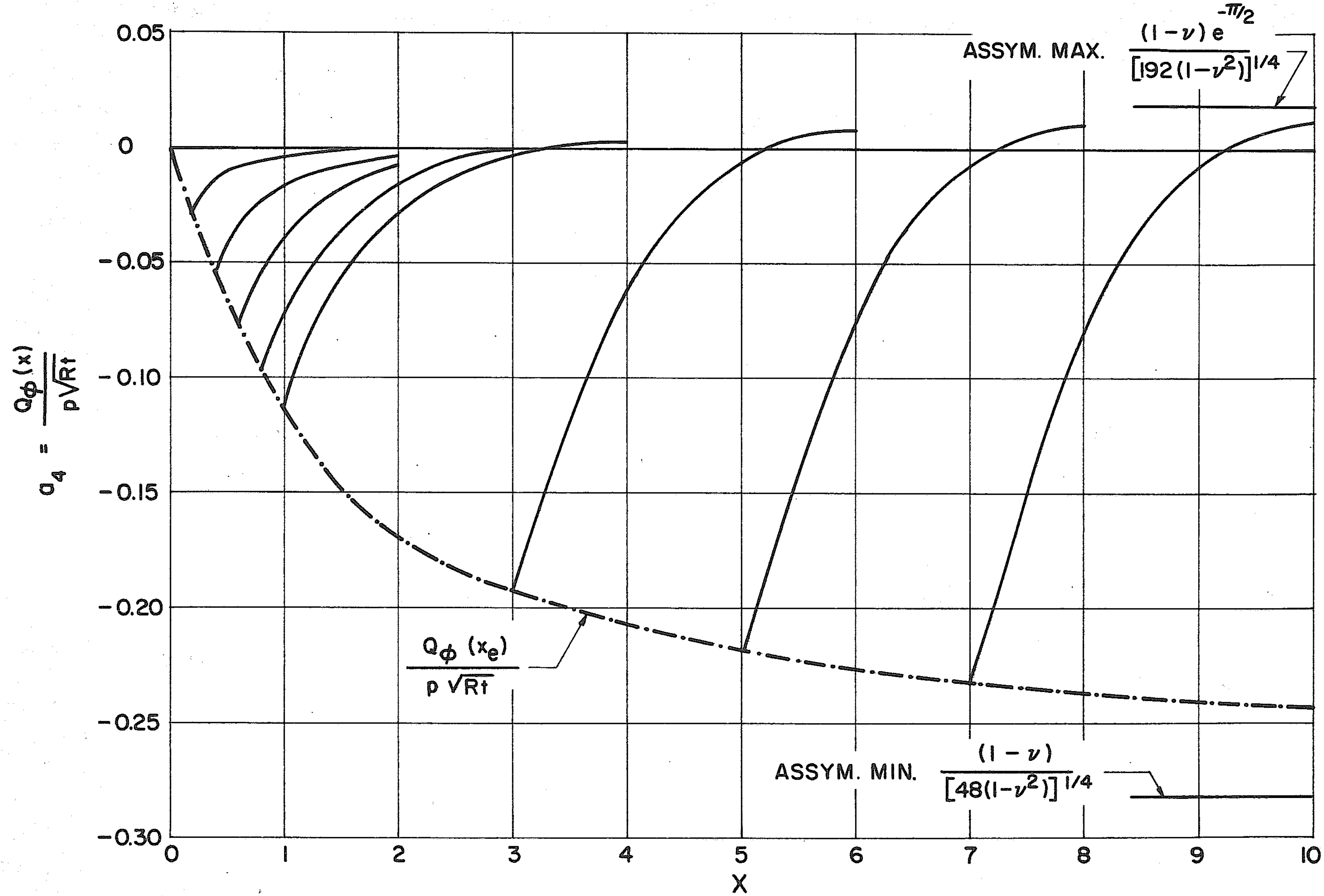
MERIDIONAL MOMENT M_ϕ FOR A PRESSURIZED SPHERICAL SHELL WITH AN INSERT

FIG. 6



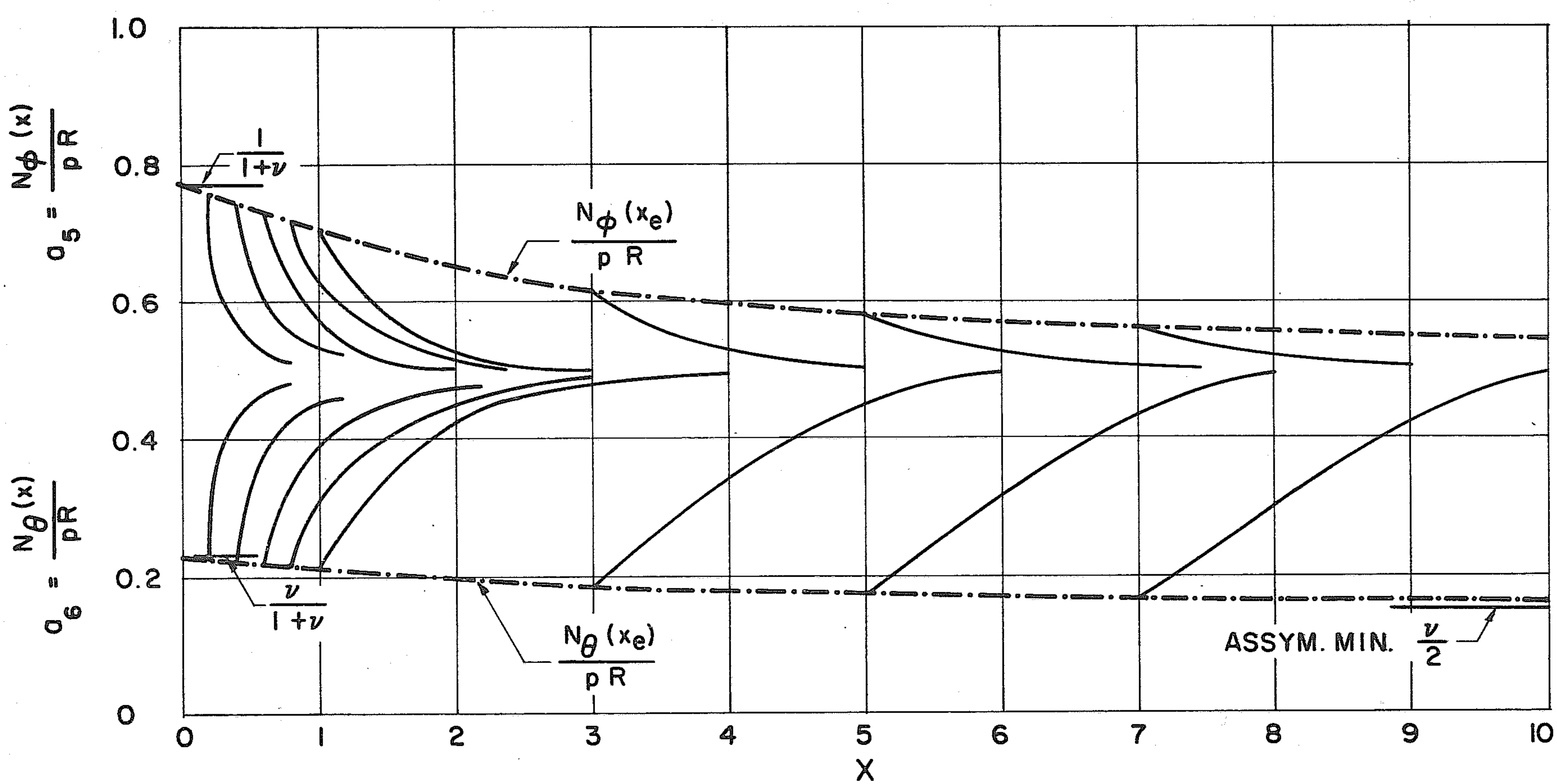
CIRCUMFERENTIAL MOMENT M_Θ FOR A PRESSURIZED SPHERICAL WITH AN INSERT

FIG. 7



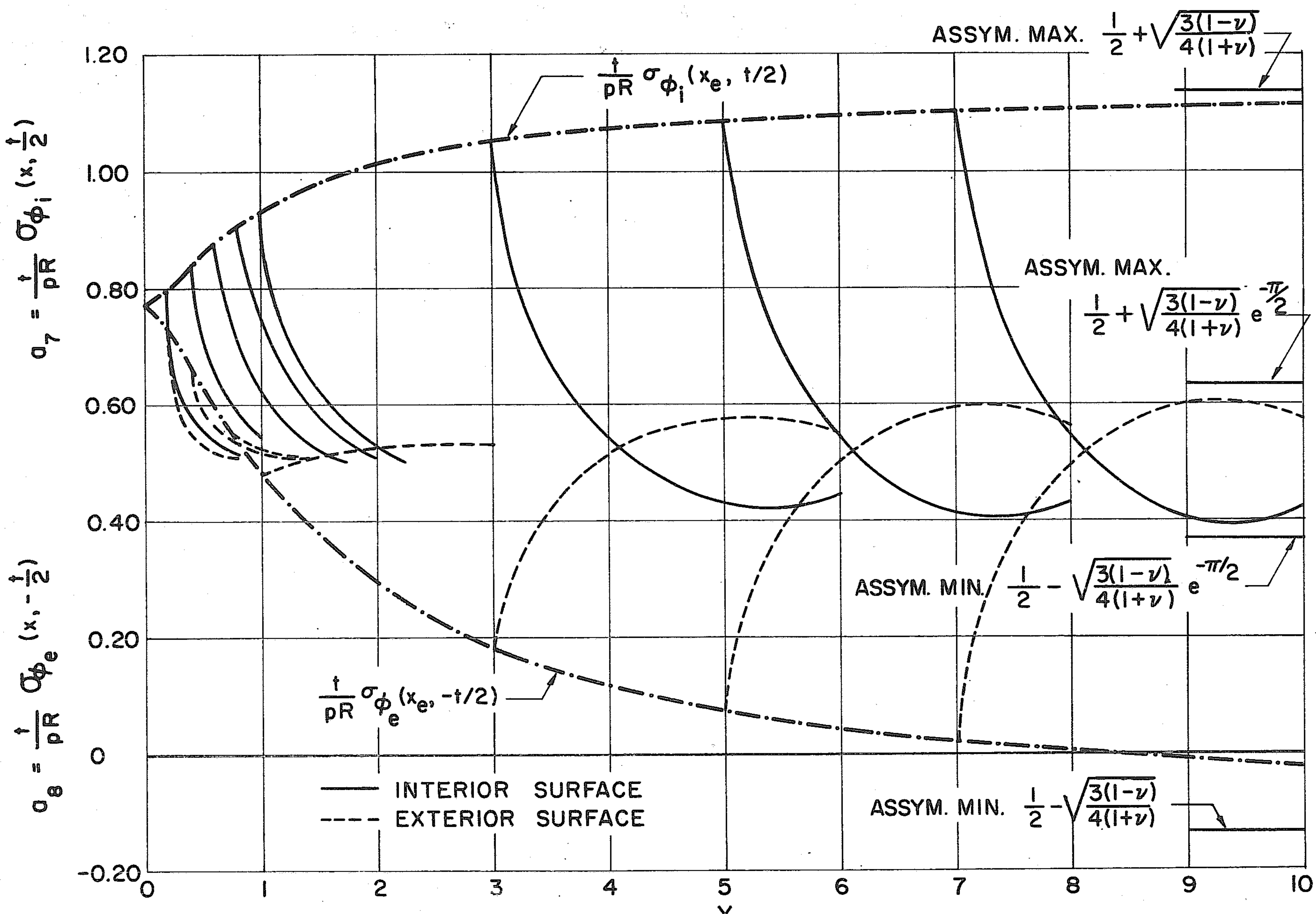
SHEAR RESULTANT Q_ϕ FOR A PRESSURIZED SPHERICAL SHELL WITH AN INSERT

Fig. 8



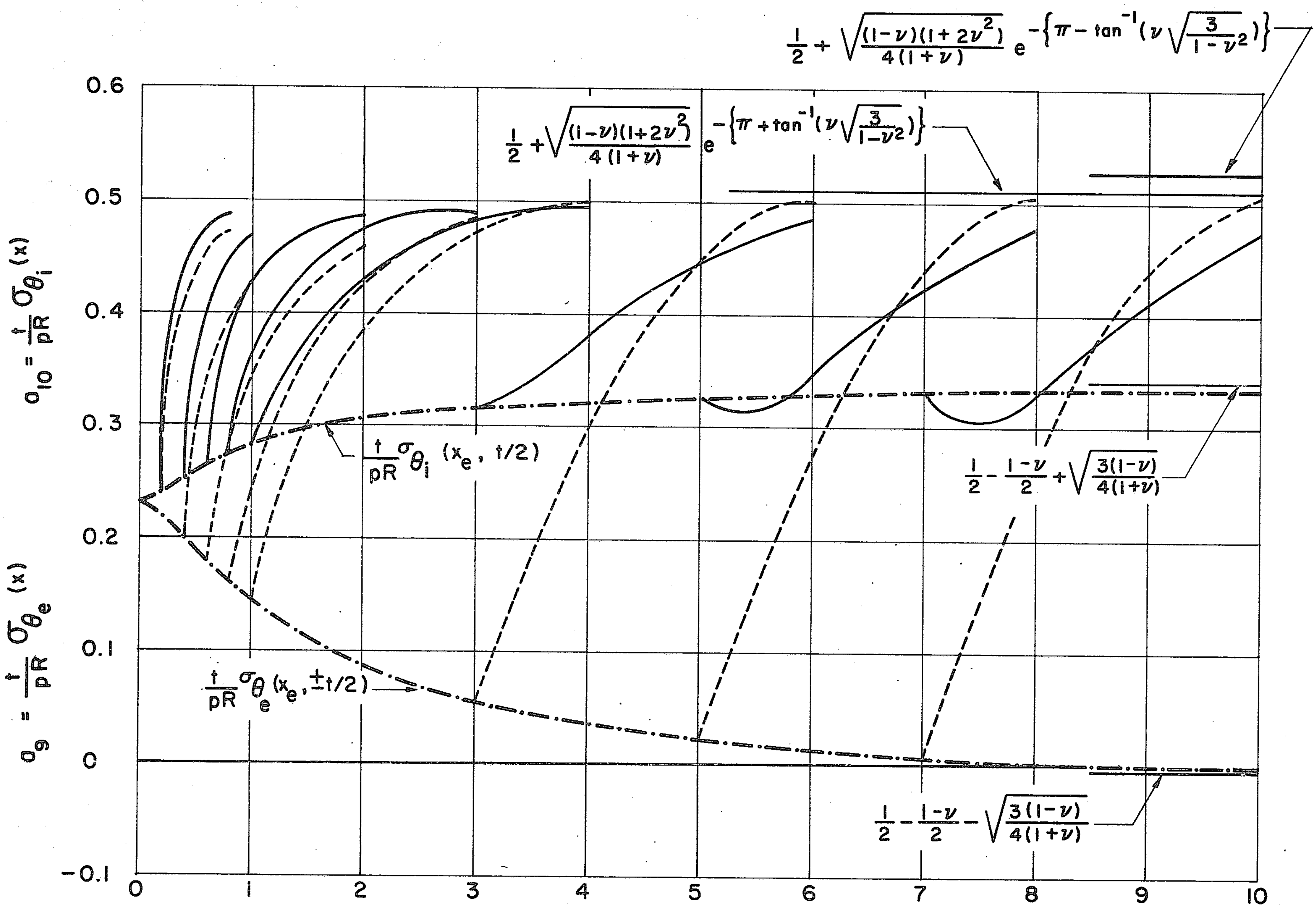
DIRECT STRESS RESULTANTS N_ϕ AND N_θ FOR A PRESSURIZED SPHERICAL SHELL WITH AN INSERT

FIG. 9



MERIDIONAL STRESS σ_ϕ FOR A PRESSURIZED SPHERICAL SHELL WITH AN INSERT

FIG. 10



CIRCUMFERENTIAL STRESS σ_θ FOR A PRESSURIZED SPHERICAL SHELL WITH AN INSERT

FIG. 11

TABLE II

Axially loaded insert

$X_e = 0.2$

x	b_1	b_2	b_3	b_4	b_5
0.2	3547791650	2792022750	8376067049	1416565751-	2729060650-
0.4	3422710750	1167748150	1079872350	6794418150-	1992978150-
0.6	3163698450	6204058349	8622438549	4227849050-	1800029750-
0.8	2849433150	3214434349	6612494249	2881905450-	1669168750-
1.0	2516141050	1334503349	5013515549	2040435150-	1550158050-
1.2	2184673950	9793889047	3767776149	1463903650-	1434702450-
2.0	1066394250	1701973749-	1034580349	3346965449-	1010623250-
3.0	2827050949	1365116649-	3616448847	4970196848	6144881449-
4.0	2923802047-	6431182848-	1018840748-	8728397948	3683759949-
5.0	5567898948-	1897584448-	5704657947-	4947895048	2283626649-

x	b_6	b_7	b_8	b_9	b_{10}
0.2	8187356649-	1402307551	1948119751-	4206904550	5844375940-
0.4	1429733750-	5013510550	8999466750-	5049500150	7908967550-
0.6	1363669150-	1922405350	5522464750-	3809794050	6537132250-
0.8	1180264450-	2594919049	3597829350-	2787232150	5147760950-
1.0	9659831249-	7494560049-	2350860050-	2042126250	3974092450-
1.2	7499715549-	1375939150-	1493465750-	1510694250	3010637350-
2.0	5577095848-	2031807450-	1056100048	5649772249	6765191449-
3.0	3317830549	1433558150-	2045818249	3534817449	3100843649
4.0	3712997949	7542469649-	1749498048	3101693549	4324302349
5.0	2840416449	3422177249-	1145076049-	2498136949	3182695949

$X_e = 0.4$

x	b_1	b_2	b_3	b_4	b_5
0.4	2889436450	1791099250	5373298049	6743881850-	2222645450-
0.6	2798613850	9216073549	6444751349	4234245250-	1780650750-
0.8	2596401150	5068839249	5561351449	2915873550-	1591983150-
1.0	2342763850	2622803049	4468744549	2088363350-	1463032150-
1.2	2069508550	1052506649	3482969549	1518155450-	1352518050-
1.4	1795565850	1543460047	2659554649	1104256050-	1249508950-
2.0	1065069450	1349686849-	1046746449	3819673749-	9676578049-
3.0	3071097249	1278132849-	6272295747	2639169448	6003633549-
4.0	1312850648	6415215848-	9350830047-	7985120648	3649980949-
5.0	4955741548-	2044359548-	5726029747-	4876210548	2281020449-

x	b_6	b_7	b_8	b_9	b_{10}
0.4	6667925349-	8523950050	1296924051-	2557186350	3890771350-
0.6	1017963650-	3748993450	7310294850-	2848887250	4884814450-
0.8	1004418250-	1449320450	4633286650-	2332392650	4341229050-
1.0	8797317349-	1106497049	3036713950-	1801515050	3560978450-
1.2	7169903749-	7210140049-	1984022050-	1372791350	2806772150-
1.4	5460568349-	1240248150-	1258769750-	1049676050	2141789650-
2.0	9741160048-	1777469950-	1578457249-	5306362449	7254594449-
3.0	2932536449	1367243050-	1665163349	3308874149	2556198749
4.0	3518695849	7499110449-	1991486048	2957646049	4079745649
5.0	2776594549	3507636149-	1054404749-	2433032749	3120156349

$X_e = 0.6$

x	b_1	b_2	b_3	b_4	b_5
0.6	2352956350	1276096450	3828289249	4224569150-	1809967150-
0.8	2284845150	7253789749	4288414549	2944373050-	1527223750-
1.0	2126984850	4144408949	3802549349	2136845050-	1374900350-
1.2	1924082450	2183579249	3130001549	1576485850-	1264155250-
1.4	1702322750	8861537048	2480316749	1166244650-	1169019450-
1.6	1478018050	1648110047	1913297749	8567340949-	1081052250-
2.0	1059146150	9251650048-	1056944149	4366324249-	9179716049-
3.0	3348831449	1169456549-	9405573747	1441423047-	5834979749-
4.0	3216763448	6370038048-	8316441047-	7068467148	3608322749-
5.0	4207408748-	2210589848-	5728076247-	4771044348	2277196849-

x	b_6	b_7	b_8	b_9	b_{10}
0.6	5429898949-	5846611350	9466545550-	1753983650	2839963450-
0.8	7576213849-	2825050150	5879497550-	1815427350	3330670150-
1.0	7520845449-	1111745050	3861545650-	1529445150	3033614150-
1.2	6599272749-	4599230048	2574302750-	1218073650	2537928250-
1.4	5333033549-	6373272049-	1700711650-	9548867049	2021493450-
1.6	3969657449-	1071163550-	1090940950-	7510129049	1544944350-
2.0	1411745349-	1473070650-	3628726049-	4929919349	7753409949-
3.0	2486148349	1285171950-	1181759349	3050482749	1921813949
4.0	3286646449	7430345549-	2137001048	2787659949	3785632949
5.0	2697937649	3603550749-	9508429048-	2354253049	3041622249

$X_e = 0.8$

x	b_1	b_2	b_3	b_4	b_5
0.8	1934457550	9601747649	2880524349	2961615250-	1488044150-
1.0	1881754550	5783793949	3058408149	2179664150-	1297062050-
1.2	1756503450	3406415149	2730497649	1632673850-	1179037950-
1.4	1592674950	1831246549	2273264149	1228347350-	1088381750-
1.6	1411551550	7601442048	1817571549	9199969749-	1009176050-
1.8	1226915750	3029230047	1409664549	6808981749-	9356070049-
2.0	1047714550	4580082048-	1063097849	4941774749-	8656678049-
3.0	3635888949	1045485849-	1277728948	3177820348-	5651155049-
4.0	5277294448	6290869648-	7149697447-	6035939548	3561398549-
5.0	3371819948-	2381431748-	5703406447-	4633986248	2272213949-

x	b_6	b_7	b_8	b_9	b_{10}
0.8	4464134749-	4273004550	7249092750-	1281901150	2174728150-
1.0	5846925649-	2173214350	4767338350-	1250352350	2419737550-
1.2	5774654449-	8648112049	3222887050-	1060833250	2215764050-
1.4	5042931649-	1036620048	2187129650-	8596653049	1868251750-
1.6	4023755249-	5530895049-	1465262550-	6881674049	1492918450-
1.8	2913085749-	9174316249-	9537823849-	5544901349	1137107350-
2.0	1820467449-	1140472750-	5908628849-	4558119449	8199054249-
3.0	2015266149	1192407050-	6217598048	2781903449	1248628849
4.0	3033669149	7335920349-	2131233048	2604687349	3462650949
5.0	2609395849	3701072949-	8433549048-	2267191449	2951600249

$X_e = 1.0$

x	b_1	b_2	b_3	b_4	b_5
1.0	1609464450	7493662949	2248098949	2212127050-	1238049550-
1.2	1567603550	4687284049	2288925649	1682652850-	1103326450-
1.4	1466410750	2826010249	2039248849	1287051850-	1012156450-
1.6	1332330450	1546985149	1704770649	9817901649-	9389696049-
1.8	1182741750	6573410048	1368342449	7421095149-	8737891049-
2.0	1029249450	4189710047	1063254649	5524460049-	8127064049-
3.0	3919516249	9072693048-	1628866748	6387174048-	5456685249-
4.0	7438395348	6168793548-	5865052947-	4900824848	3509811949-
5.0	2462307348-	2548924248-	5642860247-	4460265648	2265897949-

x	b_6	b_7	b_8	b_9	b_{10}
1.0	3714149149-	3258148250	5734247250-	9774444049	1720274250-
1.2	4642769949-	1709044050	3915696850-	9090784049	1837632450-
1.4	4542543349-	6834497049	2707762550-	7692950049	1677803650-
1.6	3933607549-	1077854048-	1867160750-	6295017049	1416223250-
1.8	3089525549-	4793845049-	1268193750-	5120528949	1129958050-
2.0	2165431049-	7875681449-	8378446649-	4214096649	8544958649-
3.0	1537169049	1090030150-	1306940047-	2514489049	5598490048
4.0	2765972549	7211088049-	1914642048	2414069349	3117875749
5.0	2512128649	3795252449-	7365434048-	2173557049	2850700249

$X_e = 3.0$

x	b_1	b_2	b_3	b_4	b_5
3.0	4308384049	1579684649	4739053848	4174579449-	3314141549-
3.2	4215708249	1102788549	4058386548	3457913449-	3171722749-
3.4	3979321849	7257540348	3351135148	2816057049-	3043974549-
3.6	3651115949	4321161848	2672987448	2251062649-	2921448149-
3.8	3270975349	2077515348	2055831648	1761686249-	2799450349-
4.0	2868953749	4047822047	1515604548	1344386849-	2676119149-
5.0	1104215849	2587496448-	2759273046-	1372849948-	2053824449-

x	b_6	b_7	b_8	b_9	b_{10}
3.0	9942426048-	6163966149	1279224950-	1849189749	3837674949-
3.2	10439856 49-	3445008349	9788453749-	1391046349	3479017549-
3.4	9353473048-	1310549749	7398498749-	1075333849	2946028449-
3.6	7296677048-	3287510048-	5514145249-	8741247048	2333460149-
3.8	4715249048-	1552941149-	4045959549-	7619741048	1705023949-
4.0	1928345048-	2433249849-	2918988449-	7165282048	1102197249-
5.0	9496087048	3606322249-	5013266048-	9330530648	9661643448

$X_e = 5.0$

x	b_1	b_2	b_3	b_4	b_5
5.0	1924396049	6593509148	1978052748	1714761249-	1480304549-

x	b_6	b_7	b_8	b_9	b_{10}
5.0	4440914748-	2475801049	5436410049-	7427401048	1630923149-

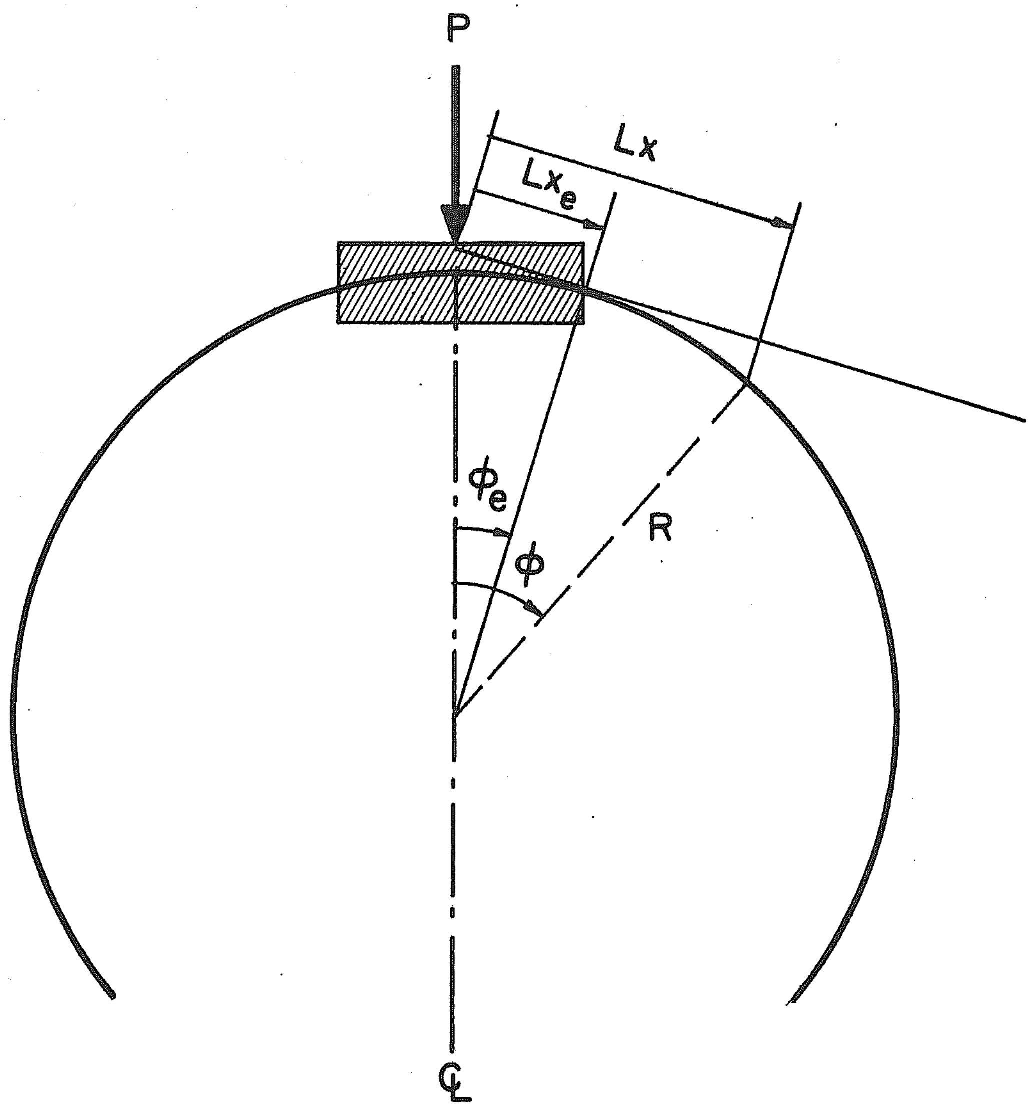
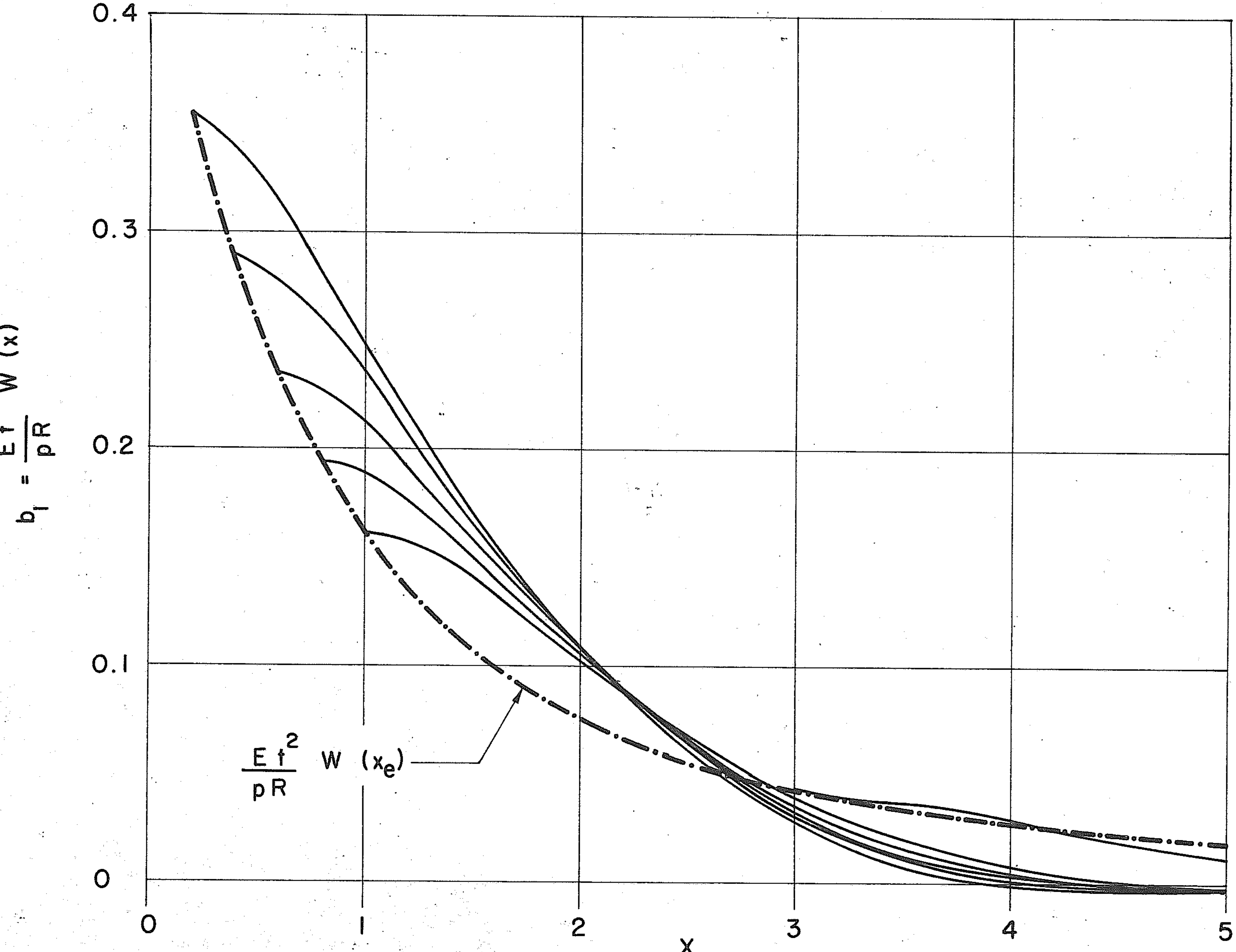
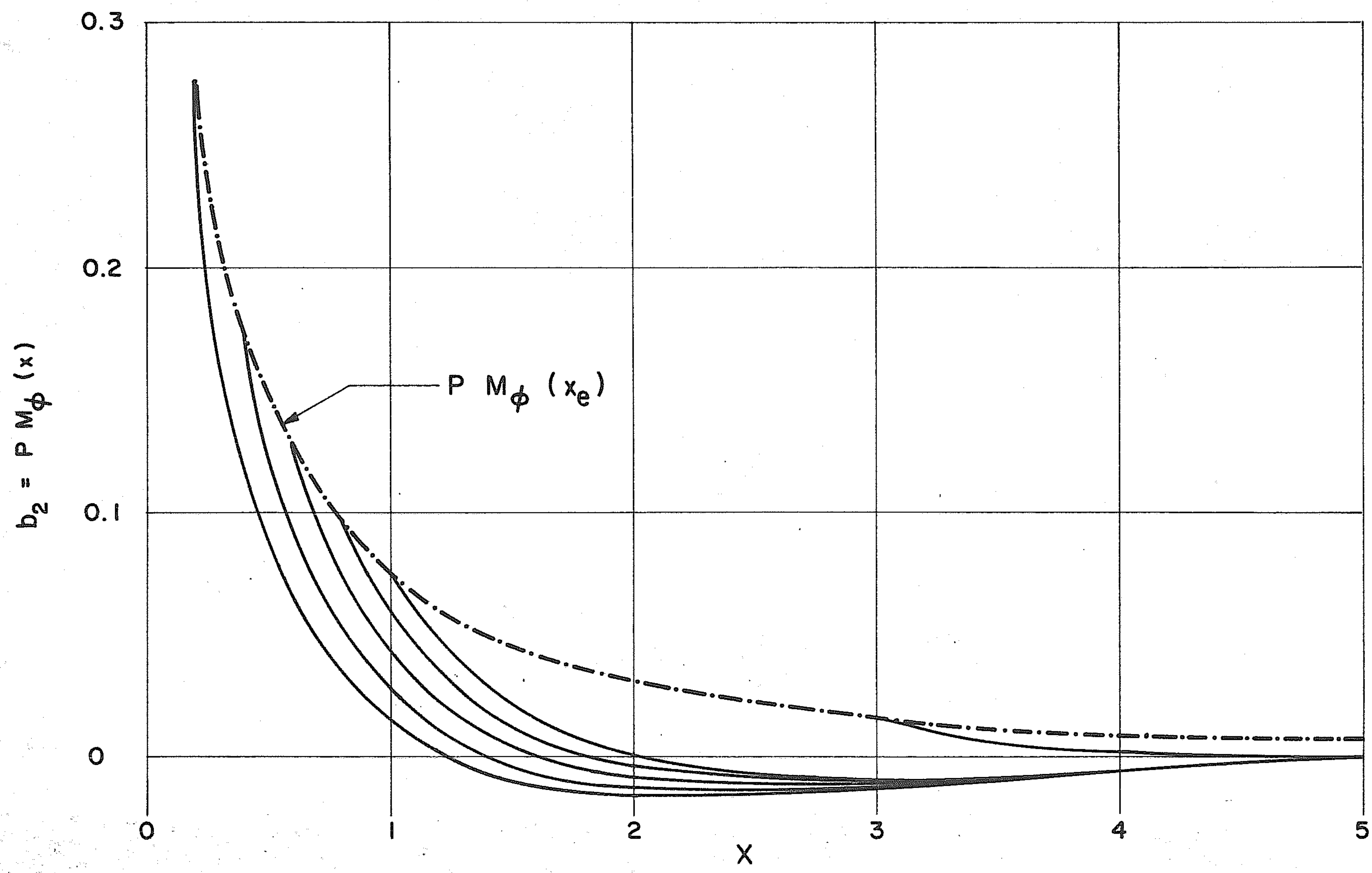


FIG. 12 — AXIALLY LOADED INSERT



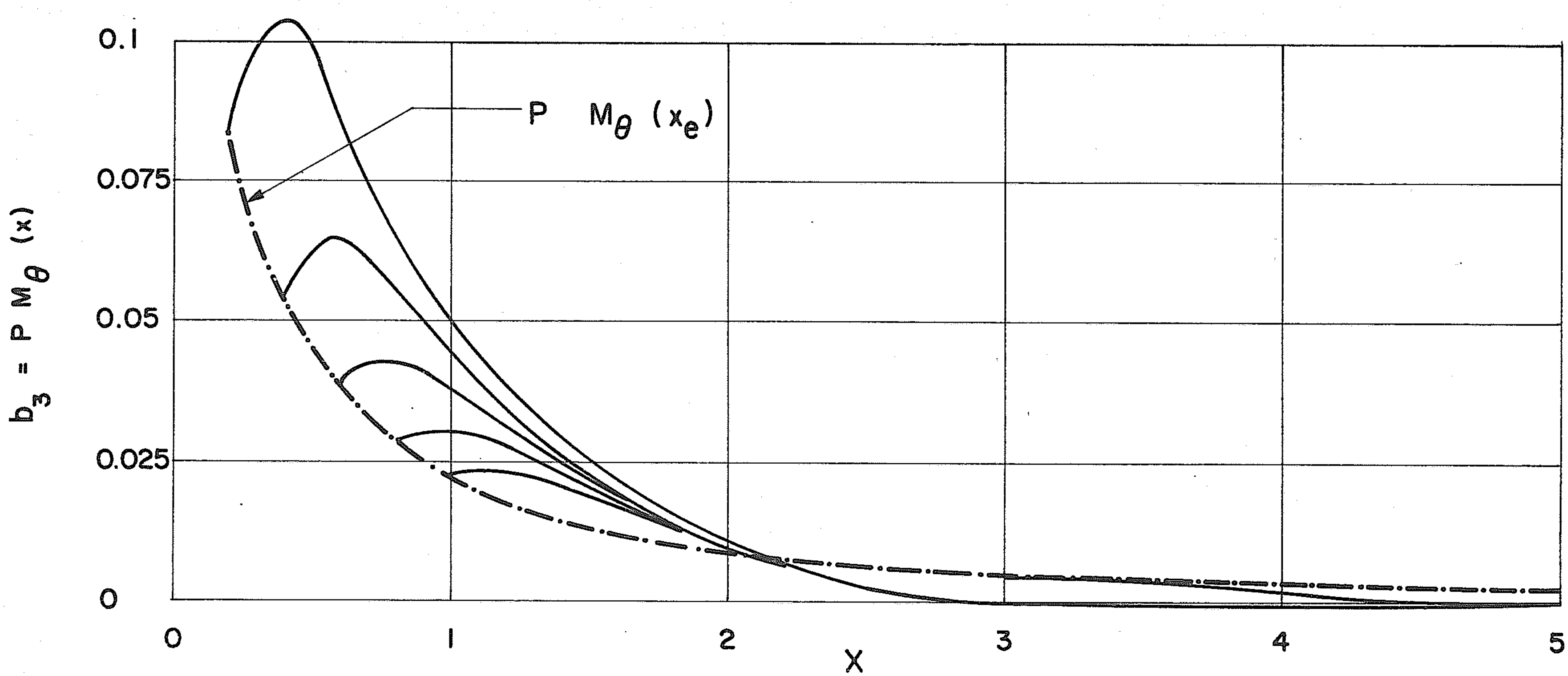
RADIAL DEFLECTION W FOR AN AXIALLY LOADED INSERT IN A SPHERICAL SHELL

FIG. 13



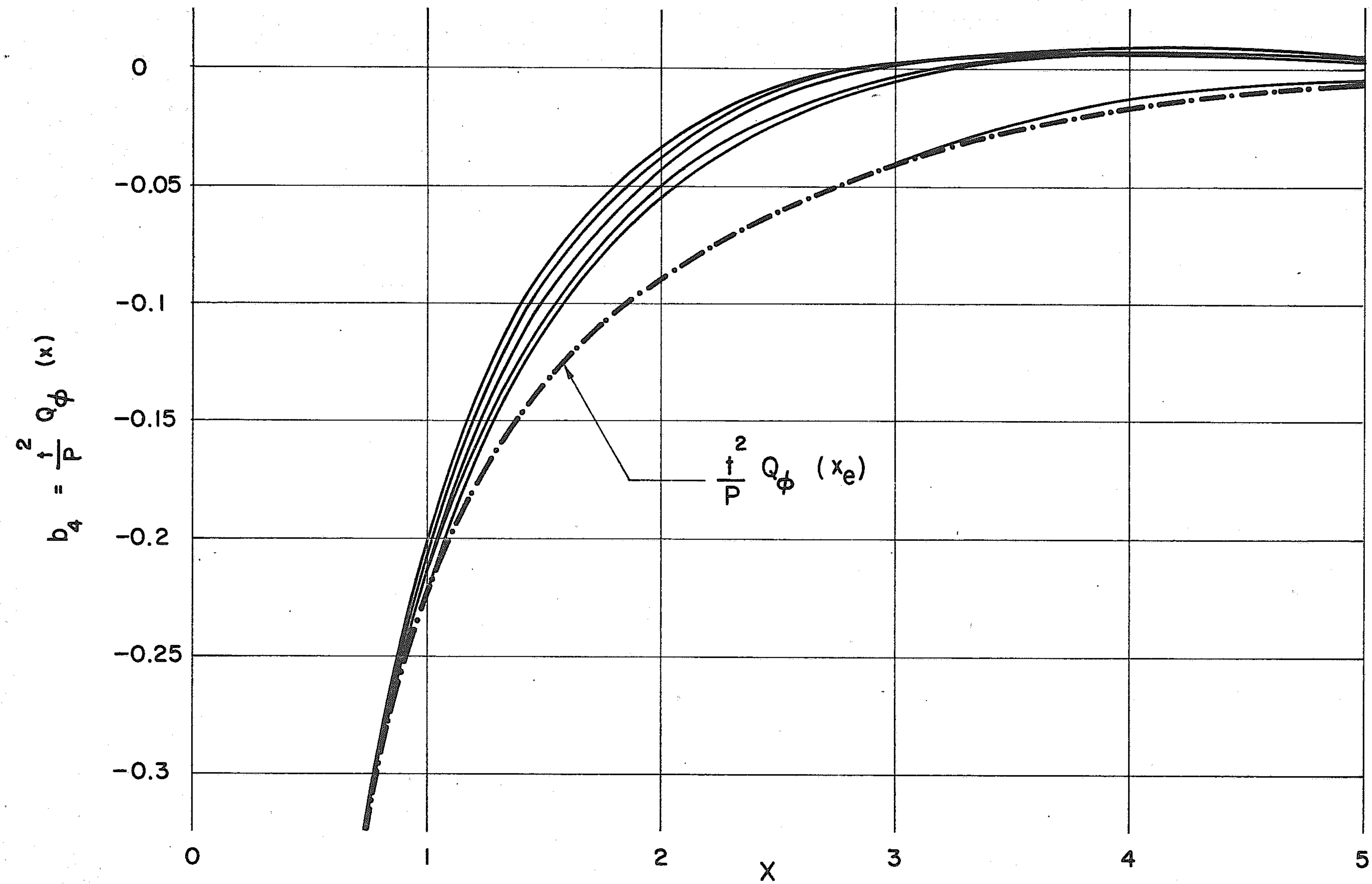
MERIDIONAL MOMENT M_ϕ FOR AN AXIALLY LOADED INSERT IN A SPHERICAL SHELL

FIG. 14



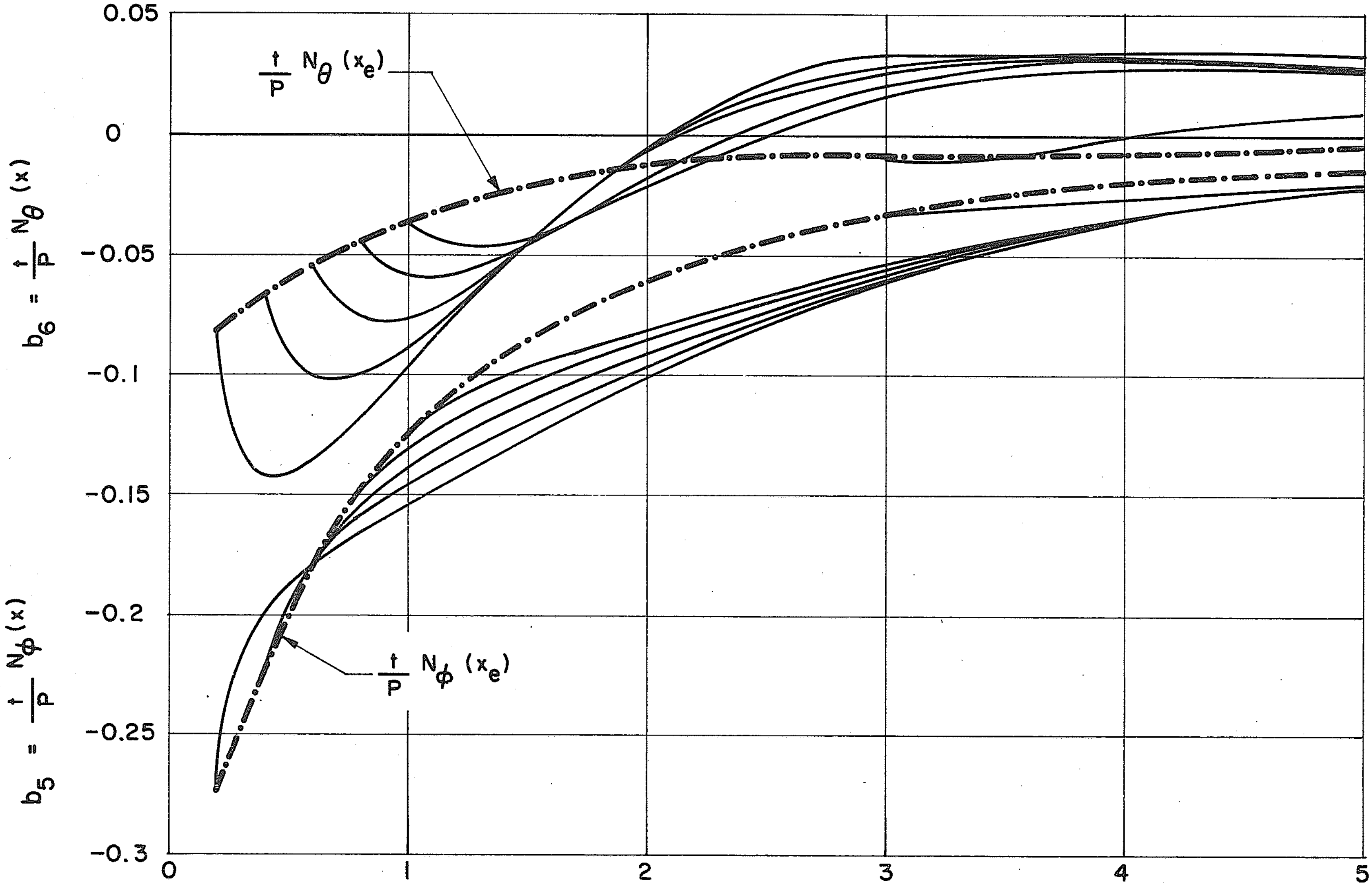
CIRCUMFERENTIAL MOMENT M_θ FOR AN AXIALLY LOADED INSERT IN A SPHERICAL SHELL

FIG. 15



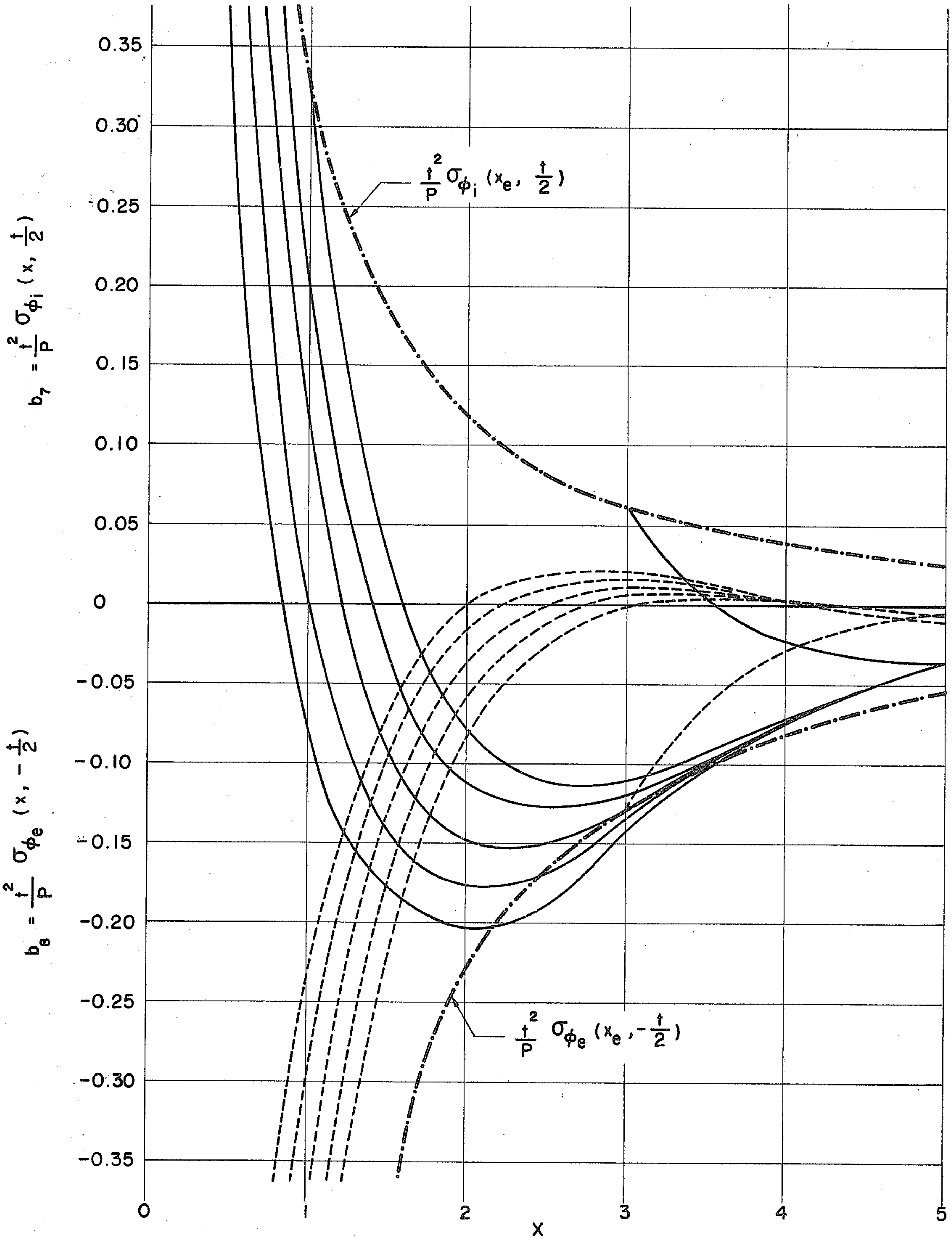
SHEAR RESULTANT Q_ϕ FOR AN AXIALLY LOADED INSERT IN A SPHERICAL SHELL

FIG. 16



DIRECT STRESS RESULTANTS N_ϕ AND N_θ FOR AN AXIALLY LOADED INSERT IN A SPHERICAL SHELL

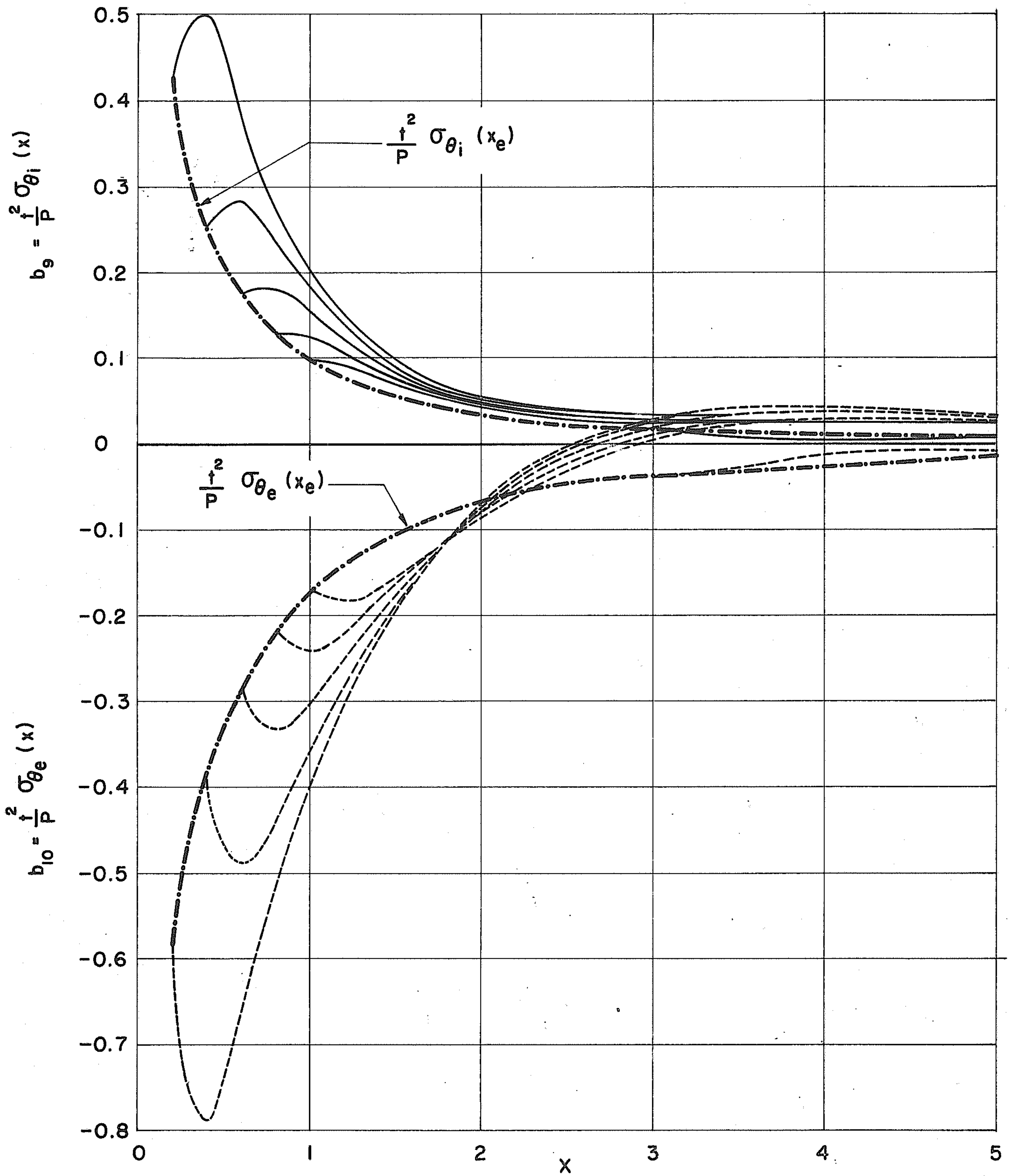
FIG. 17



MERIDIONAL STRESS σ_ϕ FOR AN AXIALLY LOADED INSERT

IN A SPHERICAL SHELL

FIG. 18



CIRCUMFERENTIAL STRESS σ_θ FOR AN AXIALLY LOADED INSERT IN A SPHERICAL SHELL

FIG. 19

TABLE I

$X_e = 0.2$

Ring Load

x	f_1	f_2	f_3	f_4	f_5
0.2	3895987050	1802841950	1798651050	2168303849	1970814750-

x	g_1	g_2	g_3	g_4	g_5
0.2	3895987050	1802841950	1798651050	1424908051-	1970818850-
0.4	3645546750	9058470049	1307243750	6813751350-	1905116550-
0.6	3316027750	4938404549	9533891849	4223999250-	1811693350-
0.8	2954828150	2435377149	7051765649	2866856550-	1703364250-
1.0	2588202150	7935213048	5240737349	2019760150-	1587741650-
1.2	2232398750	3026568548-	3886251749	1440734050-	1469801150-
2.0	1066656150	1849336049-	1029189749	3147678249-	1028736850-
3.0	2723861449	1401241549-	2500968847	5947003748	6204070449-
4.0	9659084847-	6436109748-	1053707348-	9037895148	3697825349-
5.0	5823245548-	1835469348-	5694116547-	4976611848	2284670649-

x	f_6	f_7	f_8	f_9	f_{10}
0.2	1925172450-	8846236050	1278786651-	8866734050	1271707851-

x	g_6	g_7	g_8	g_9	g_{10}
0.2	1925171950-	8846232050	1278787051-	8866734050	1271707851-
0.4	1740431250-	3529965550	7340198550-	6103031050	9583893450-
0.6	1504335650-	1151349450	4774736050-	4215999550	7224670750-
0.8	1251464150-	2421379049-	3164590550-	2979595350	5482523550-
1.0	1000460550-	1111628850-	2063854450-	2143981950	4144902950-
1.2	7625975849-	1651395250-	1288207050-	1569153450	3094348650-
2.0	3791923748-	2138338450-	8086480048	5795945849	6554330649-
3.0	3480209149	1461151950-	2203378649	3630267249	3330151049
4.0	3794416149	7559491149-	1638405048	3162191749	4426640549
5.0	2866995149	3385952249-	1183389049-	2525348149	3208642149

$X_e = 0.4$

x	f_1	f_2	f_3	f_4	f_5
0.2	3645546750	1108495750	1104595050	2020833449	1836776050-
0.4	3476507350	1135657050	1120291950	3948929549	1794630550-

x	g_1	g_2	g_3	g_4	g_5
0.4	3476507350	1135657050	1120291950	6838062250-	1794632750-
0.6	3205416150	6052340249	8810222749	4252636250-	1724930650-
0.8	2882259450	3117420549	6717734749	2896946850-	1634989950-
1.0	2541992050	1261751749	5077509749	2049588750-	1533517750-
1.2	2204903650	3872612147	3808063849	1469226550-	1426638950-
1.4	1883450550	7668114148-	2826678549	1051092550-	1318539550-
2.0	1072830350	1734181549-	1039981849	3337293849-	1011502350-
3.0	2828605849	1378698349-	3471625947	5145827948	6155523449-
4.0	3946773447-	6470497648-	1030157348-	8826958848	3688239149-
5.0	5639417248-	1899695048-	5737305347-	4981863048	2284861649-

x	f_6	f_7	f_8	f_9	f_{10}
0.2	1808770750-	4814198250	8487750250-	4818799350	8436340750-
0.4	1681876850-	5019311550	8608572550-	5039874650	8403628250-

x	g_6	g_7	g_8	g_9	g_{10}
0.4	1681875450-	5019309350	8608574750-	5039876050	8403626850-
0.6	1480486950-	1906473550	5356334750-	3805646750	6766620550-
0.8	1247269350-	2354624049	3505442250-	2783371550	5277910150-
1.0	1008474250-	7764667049-	2290568750-	2038031650	4054980050-
1.2	7782646549-	1403403250-	1449874650-	1506573750	3063103050-
1.4	5649111849-	1778626450-	8584527049-	1131095950	2260918350-
2.0	6132803748-	2052011250-	2900660048	5626610449	6853171249-
3.0	3326917749	1442771350-	2116666449	3535215349	3118620249
4.0	3727706849	7570537749-	1940595048	3109612449	4345801249
5.0	2848803249	3424678649-	1145044649-	2504564949	3193041549

$X_e = 0.6$

x	f_1	f_2	f_3	f_4	f_5
0.2	3316027950	7253837849	7218458349	1834215649	1667155350-
0.4	3205416350	7501264449	7361298449	3607812049	1639606250-
0.6	3017768250	7903754649	7594708549	5258349749	1593139750-

x	g_1	g_2	g_3	g_4	g_5
0.6	3017768050	7903755149	7594708149	4296135350-	1593139550-
0.8	2758370150	4251597949	6154026749	2944224750-	1527560050-
1.0	2462395350	2041255549	4800253449	2097278150-	1446826350-
1.2	2156850950	6079371848	3673879549	1515267950-	1356892250-
1.4	1857982950	3376829048-	2768694749	1094218050-	1262542750-
1.6	1575755150	9516731748-	2052429349	7808525749-	1167264950-
2.0	1082053050	1540429349-	1056937549	3650137149-	9830673449-
3.0	3000436749	1339729249-	5086707147	3804736548	6074260549-
4.0	5582346847	6521367948-	9898579747	8466399348	3671853149-
5.0	5327247548-	2004890748-	5803584447-	4985637548	2284998849-

x	f_6	f_7	f_8	f_9	f_{10}
0.2	1648872650-	2685147450	6019458050-	2682202450	5979947650-
0.4	1565809950-	2861152450	6140364850-	2850969150	5982588950-
0.6	1424628350-	3149113150	6335392550-	3132196850	5981453450-

x	g_6	g_7	g_8	g_9	g_{10}
0.6	1424629250-	3149113650	6335392650-	3132195750	5981454150-
0.8	1230810050-	1023398750	4078518750-	2461606050	4923226050-
1.0	1015569150-	2220730049-	2671579650-	1864582950	3895721150-
1.2	7999586949-	9921299049-	1721654550-	1404369050	3004286450-
1.4	5954400949-	1465152450-	1059933050-	1065776750	2256656950-
1.6	4084901149-	1738268850-	5962610049-	8229675049	1639947750-
2.0	9898564948-	1907324950-	5880976048-	5351768549	7331481549-
3.0	3073823849	1411263650-	1964114749	3379026249	2768621449
4.0	3616029749	7584673849-	2409676048	3022114949	4209944549
5.0	2817723549	3487933249-	1082064449-	2469508449	3165938649

$X_e = 0.8$

x	f_1	f_2	f_3	f_4	f_5
0.2	2954828150	4759303849	4727839049	1632039749	1483393550-
0.4	2882259450	4979998849	4855156249	3224362649	1465343850-
0.6	2758370350	5341329749	5064294849	4735629149	1434769450-
0.8	2578833850	5833271749	5350547849	6121299849	1390943050-

x	g_1	g_2	g_3	g_4	g_5
0.8	2578833850	5833271449	5350548049	3004347450-	1390943450-
1.0	2345597550	3130166349	4401343949	2159755850-	1333252150-
1.2	2084932450	1404937049	3477982749	1576656450-	1263896850-
1.4	1818365450	2648470048	2681564049	1152401250-	1186994450-
1.6	1559224650	4889845348-	2025008249	8345641349-	1106240550-
1.8	1315734050	9736349648-	1496605149	5928773349-	1024500250-
2.0	1092715450	1265472449-	1078497549	4081213349-	9438859749-
3.0	3235117349	1282265349-	7341059647	1915967048	5959811249-
4.0	1893451248	6579018848-	9312630147-	7942966648	3648065249-
5.0	4878268748-	2148210748-	5884351647-	4980464048	2284810749-

x	f_6	f_7	f_8	f_9	f_{10}
0.2	1471434650-	1372188850	4338974850-	1365268850	4308138050-
0.4	1416915450-	1522655550	4453343150-	1496178350	4330009150-
0.6	1323600850-	1770028450	4639567250-	1714976150	4362177750-
0.8	1187890850-	2109020050	4890906050-	2022437950	4398219550-

x	g_6	g_7	g_8	g_9	g_{10}
0.8	1187890250-	2109019450	4890906250-	2022438650	4398219050-
1.0	1012345250-	5448477049	3211351950-	1628461150	3653151550-
1.2	8210356949-	4209346049-	2106859050-	1265753950	2907825350-
1.4	6313710249-	1028086250-	1345902650-	9775674049	2240309450-
1.6	4529841349-	1399631250-	8128498049-	7620208049	1667989050-
1.8	2912336049-	1608681250-	4403192049-	6067294649	1189196750-
2.0	1488294249-	1703169450-	1846025349-	4982690849	7959279249-
3.0	2724693949	1365340350-	1733780649	3165157549	2284230349
4.0	3458720149	7595476549-	2993461048	2899962349	4017477949
5.0	2772637649	3573737149-	9958843048-	2419576549	3125698749

$X_e = 1.0$

x	f_1	f_2	f_3	f_4	f_5
0.2	2588202150	3030889249	3003369449	1427950249	1297892550-
0.4	2541992050	3224347849	3114913649	2830658149	1286420950-
0.6	2462395350	3542636549	3298872449	4181459649	1266870850-
0.8	2345597550	3979293549	3552216849	5450875349	1238602450-
1.0	2186369350	4524779949	3870497649	6605279749	1200734250-

x	g_1	g_2	g_3	g_4	g_5
1.0	2186369350	4524780049	3870497449	2232654350-	1200734350-
1.2	1984584350	2428874249	3212560649	1650245350-	1152419450-
1.4	1760678950	1041807249	2559430849	1223377150-	1094835350-
1.6	1532276750	1101401548	1982325149	9009094149-	1030862350-
1.8	1310571850	5090873248-	1497651249	6534298849-	9633476649-
2.0	1102552850	9059322048-	1102455849	4623093149-	8946334449-
3.0	3526477349	1203464949-	1022520948	5294674247-	5811630849-
4.0	3610510848	6629797548-	8522643447-	7238618348	3616055349-
5.0	4281064248-	2325524248-	5967472847-	4955878948	2283916849-

x	f_6	f_7	f_8	f_9	f_{10}
0.2	1290309750-	5206410049	3116426050-	5117119049	3092331350-
0.4	1255571050-	6481878049	3221029650-	6133772049	3124519250-
0.6	1195524650-	8587111049	3392452750-	7837988049	3174848050-
0.8	1106994950-	1148973750	3626178550-	1024335250	3238325050-
1.0	9856350849-	1514133750	3915602150-	1336663550	3307933750-

x	g_6	g_7	g_8	g_9	g_{10}
1.0	9856350049-	1514133750	3915602350-	1336663450	3307933450-
1.2	8321648749-	3049051049	2609743950-	1095371550	2759701350-
1.4	6658435049-	4697510049-	1719919650-	8698150049	2201502050-
1.6	5014143449-	9647782049-	1096946450-	6879808049	1690809450-
1.8	3472241649-	1268800150-	6578952749-	5513665649	1245814950-
2.0	2079193849-	1438192850-	3510741249-	4535541049	8693928649-
3.0	2285153649	1303242050-	1409158649	2898666149	1671641149
4.0	3255004449	7593933849-	3618232048	2743645849	3766363049
5.0	2712023349	3679231349-	8886023048-	2353974949	3070071749

$X_e = 3.0$

x	f_1	f_2	f_3	f_4	f_5
1.0	3526477249	5112154148-	5899974148-	8567737648	1557477649-
2.0	5448784349	1911182048	2052443048-	2312502349	2101879749-
2.2	5807304349	4236219448	8001853647-	2694071149	2226086649-
2.4	6100427549	6930407448	6456502147	3098255849	2346722249-
2.6	6295303649	1001021049	2295369248	3516756949	2458808249-
2.8	6354380549	1348187149	4155365348	3937990449	2556656049-
3.0	6235345249	1733787849	6226550148	4346644349	2633834749-

x	g_1	g_2	g_3	g_4	g_5
3.0	6235345249	1733787949	6226548748	5297297749-	2633833849-
3.2	5913871949	1150382949	5198995148	4317329749-	2683509749-
3.4	5446402549	6967768448	4190108648	3456073749-	2701783749-
3.6	4894707449	3502965348	3258410348	2710766249-	2689318149-
3.8	4305307349	9167439147	2434431148	2075432649-	2649360649-
4.0	3712377349	9545519747-	1730368648	1541997149-	2586313049-
5.0	1303353749	3790718648-	1669860847-	6415621947-	2080411849-

x	f_6	f_7	f_8	f_9	f_{10}
1.0	1968999749-	4624770149-	1509814949	5508984249-	1570984849
2.0	3346904649-	9551705048-	3248588949-	4578370449-	2115438849-
2.2	3581218049-	3156450048	4767818249-	4061329249-	3101106849-
2.4	3753705249-	1811522249	6504966649-	3366315149-	4141095349-
2.6	3836495649-	3547317849	8464934249-	2459268149-	5213723149-
2.8	3797724849-	5532466649	1064577940-	1304505649-	6290944049-
3.0	3601510549-	7768892049	1303656250-	1344196048	7337440649-

x	g_6	g_7	g_8	g_9	g_{10}
3.0	3601511449-	7768893049	1303656150-	1344178048	7337440649-
3.2	3230362249-	4218787749	9585807149-	1109651048-	6349759349-
3.4	2744618849-	1478877349	6882444749-	23055360448-	5258684049-
3.6	2205389249-	5875989048-	4791037349-	2503430048-	4160435449-
3.8	1655946449-	2099314349-	3199407049-	1952877048-	3116605149-
4.0	1126064249-	3159044249-	2013581849-	8784300047-	2164285449-
5.0	7770580848	4354843049-	1940194048	6768664348	8772497348

$X_e = 5.0$

x	f_1	f_2	f_3	f_4	f_5
1.0	4281064248-	1532156248-	1390115348-	1401980948-	2548577048
2.0	1463498648	2213198348-	1827868148-	1325979248-	1205208948
3.0	1303353749	2015766148-	1941938748-	2378535648	1441265948-
4.0	2896387449	1433567648	6349943147-	1202983249	5467077548-
4.2	3191278449	2784124348	7452525646-	1473023349	6375525748-
4.4	3445118049	4408820448	6131179947	1763401449	7285413248-
4.6	3637287549	6323565448	1437410548	2068600149	8174748248-
4.8	3743557849	8537299448	2405375148	2380903849	9016879448-
5.0	3735992949	1104955149	3520629848	2690047549	9780151848-

x	g_1	g_2	g_3	g_4	g_5
5.0	3735992949	1104955249	3520628948	3096316949-	9780150648-

x	f_6	f_7	f_8	f_9	f_{10}
1.0	1732486848	6644360248-	1174151449	6608205048-	1007317949
2.0	2668707648-	1207398149-	1448439949	1363591749-	8298501048
3.0	1159227149-	1353586349-	1065333149	2324390349-	5936100046
4.0	2349679749-	3134328148	1406848349	2730676349-	1968683149-
4.2	2553725849-	1032922049	2308027249-	2598441049-	2509010649-
4.4	2716576849-	1916750949	3373833549-	2348706049-	3084447649-
4.6	2819812849-	2976664449	4611614049-	1957366549-	3682259149-
4.8	2841869849-	4220691749	6024067549-	1398644749-	4285094949-
5.0	2757977849-	5651715449	7607745849-	6455999048-	4870355749-

x	g_6	g_7	g_8	g_9	g_{10}
5.0	2757977849-	5651716149	7607746349-	6456005048-	4870355149-

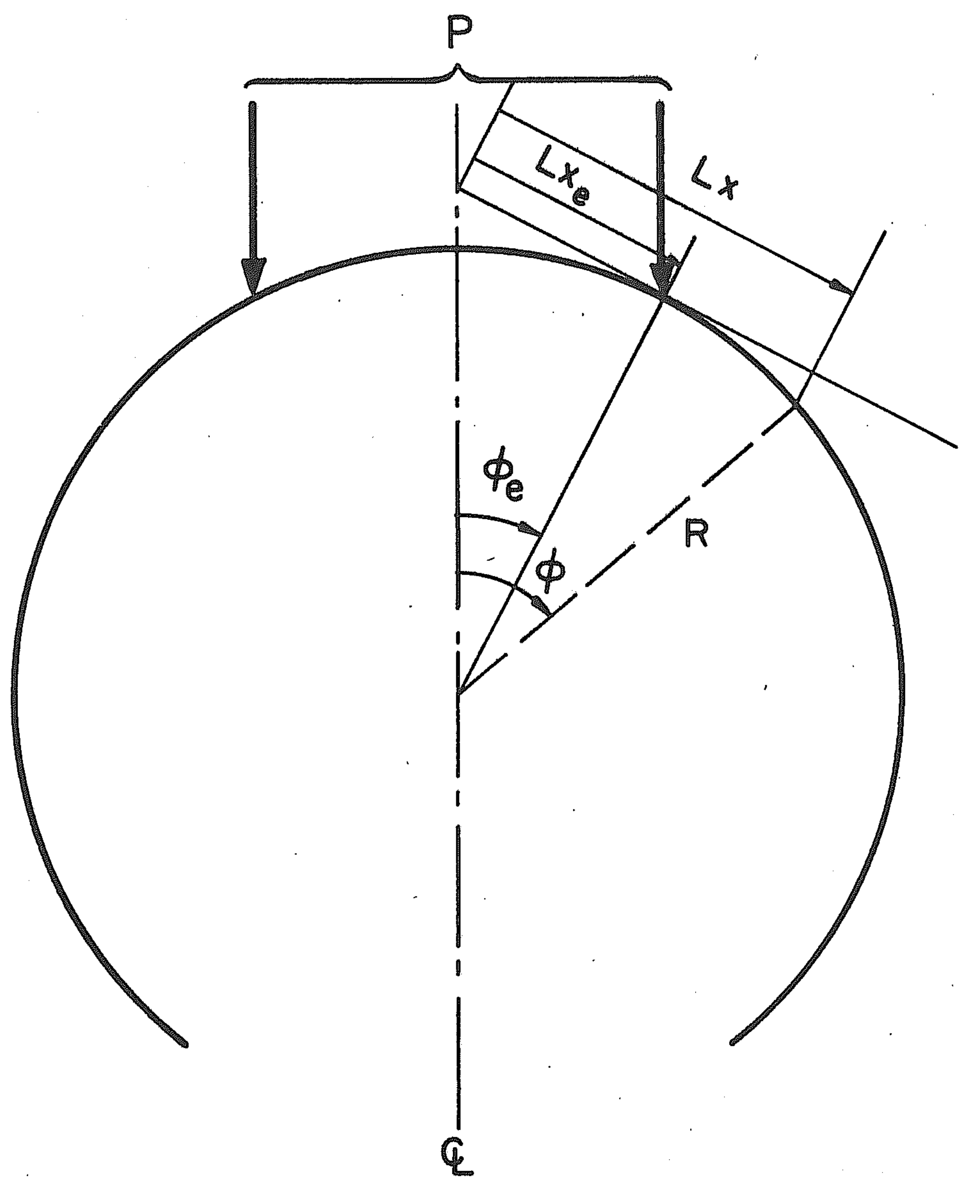
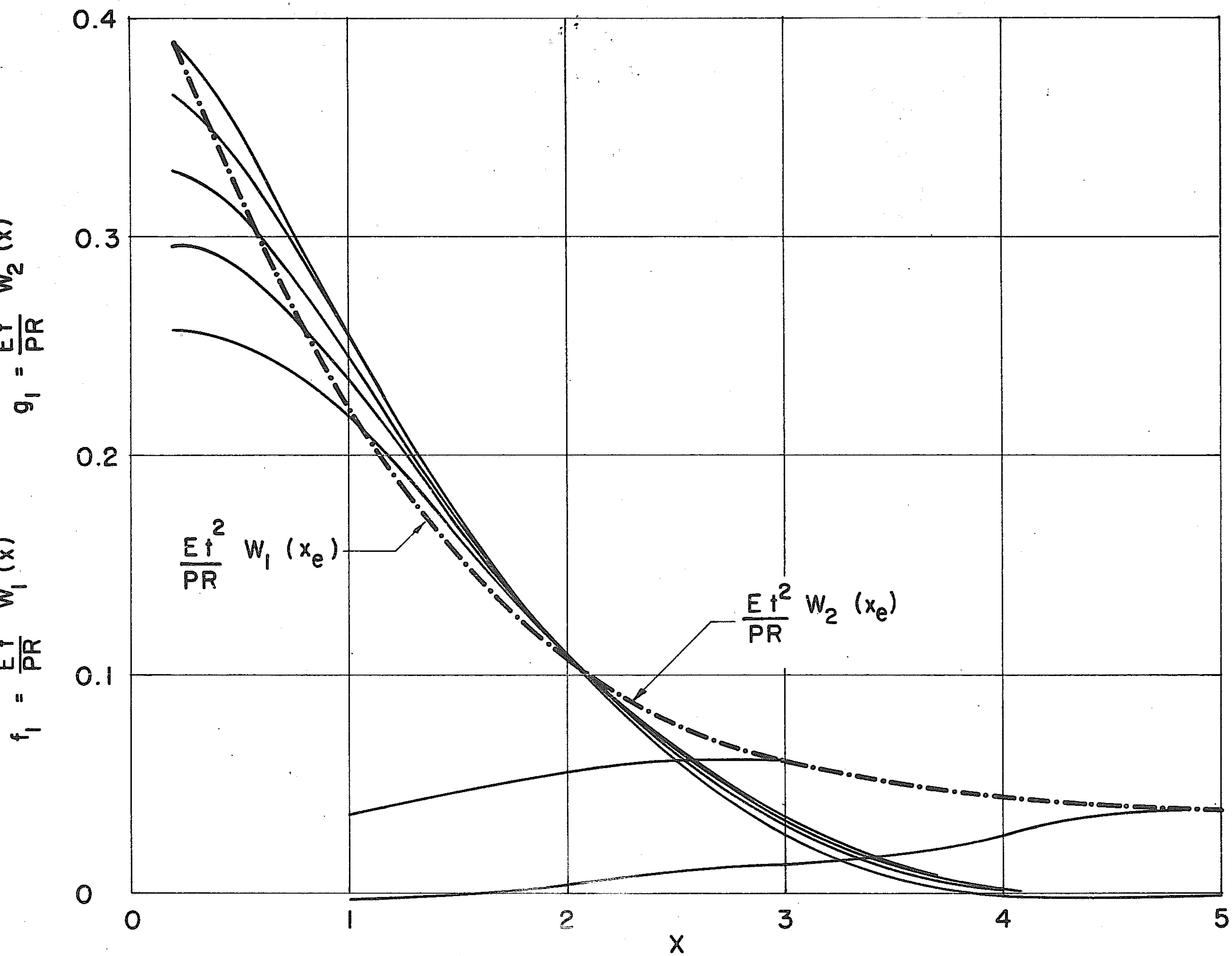
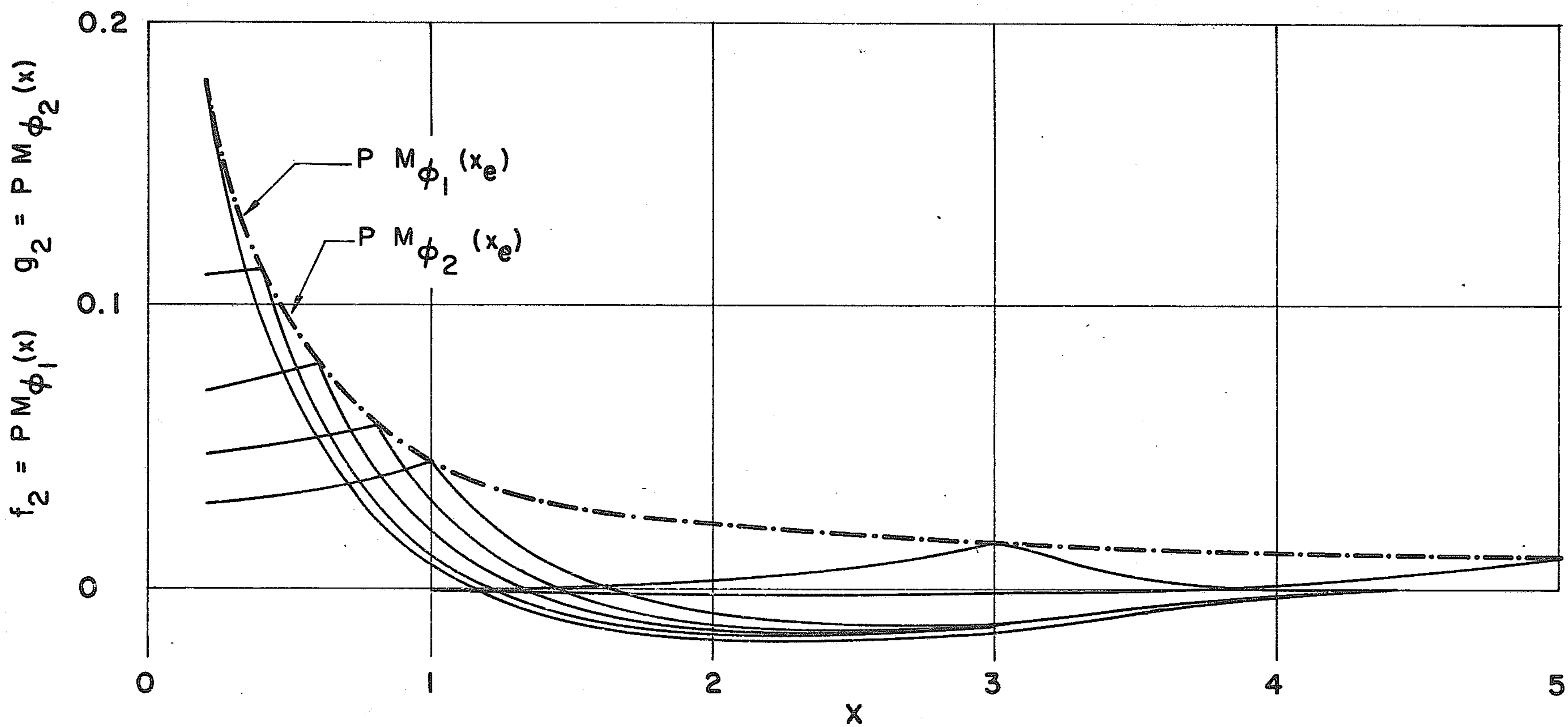


FIG. 20 — RING LOAD ON A SPHERICAL SHELL



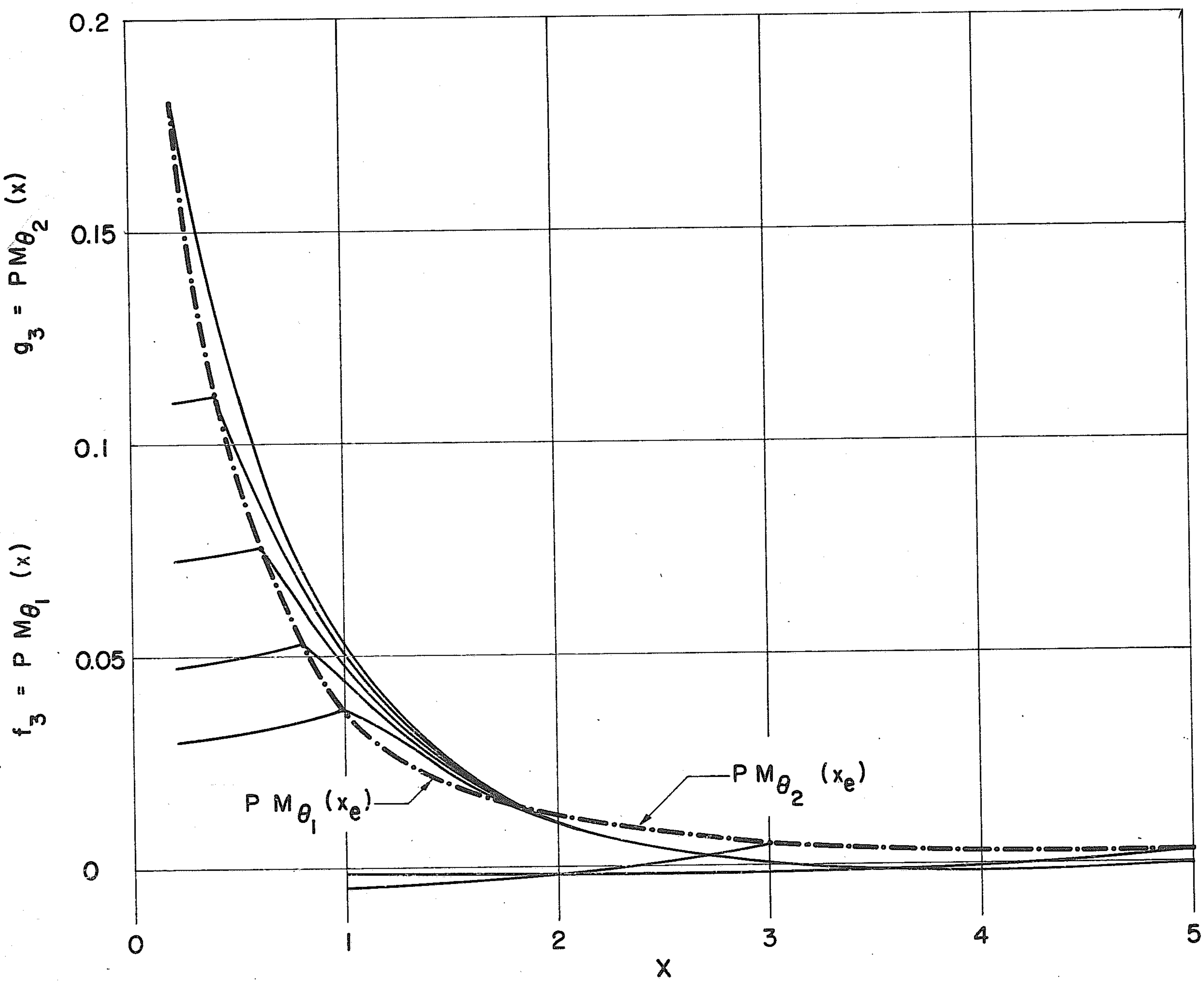
RADIAL DEFLECTION W FOR A RING LOAD ON A SPHERICAL SHELL

FIG. 21



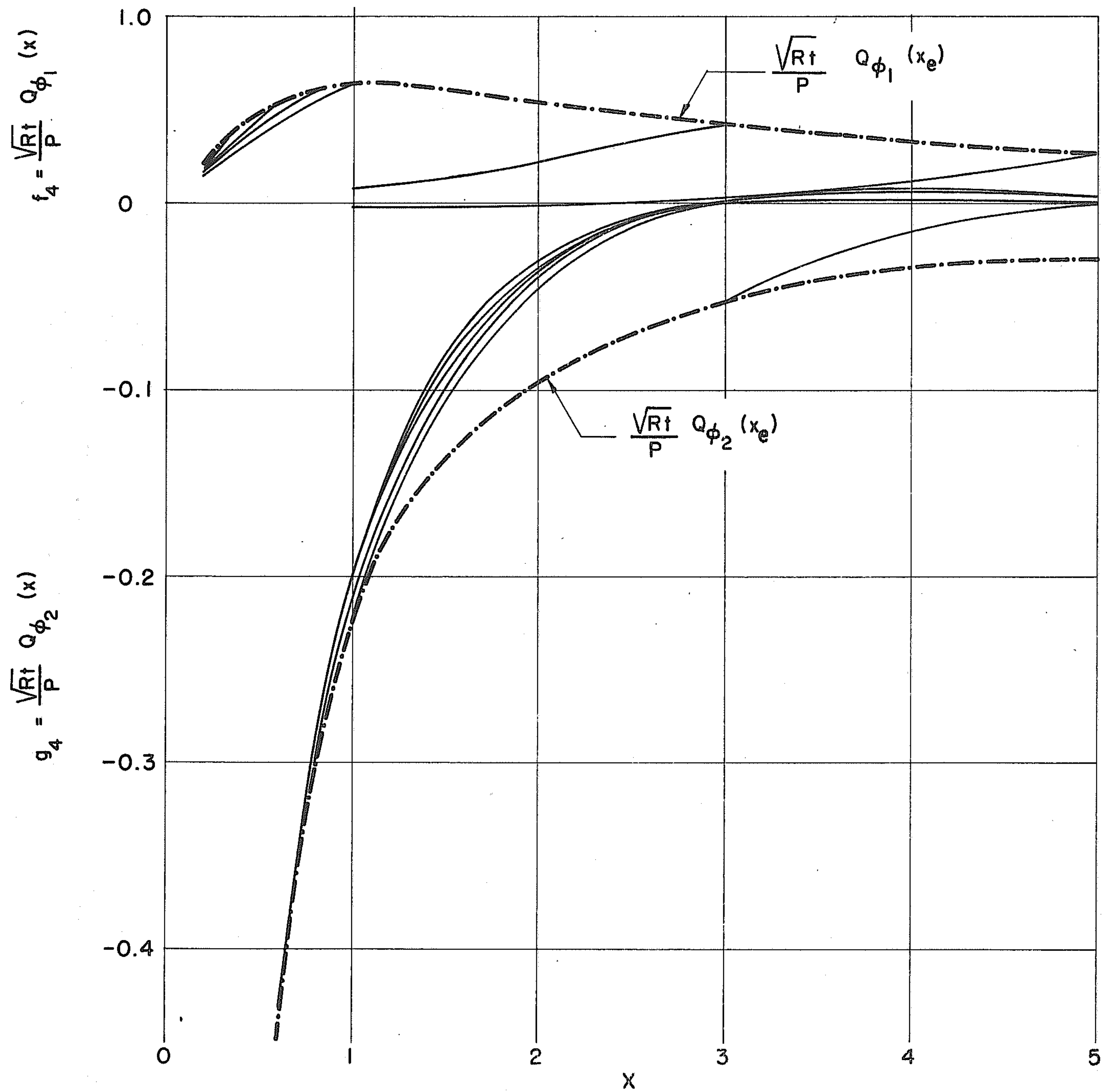
MERIDIONAL MOMENT M_ϕ FOR A RING LOAD ON A SPHERICAL SHELL

FIG. 22



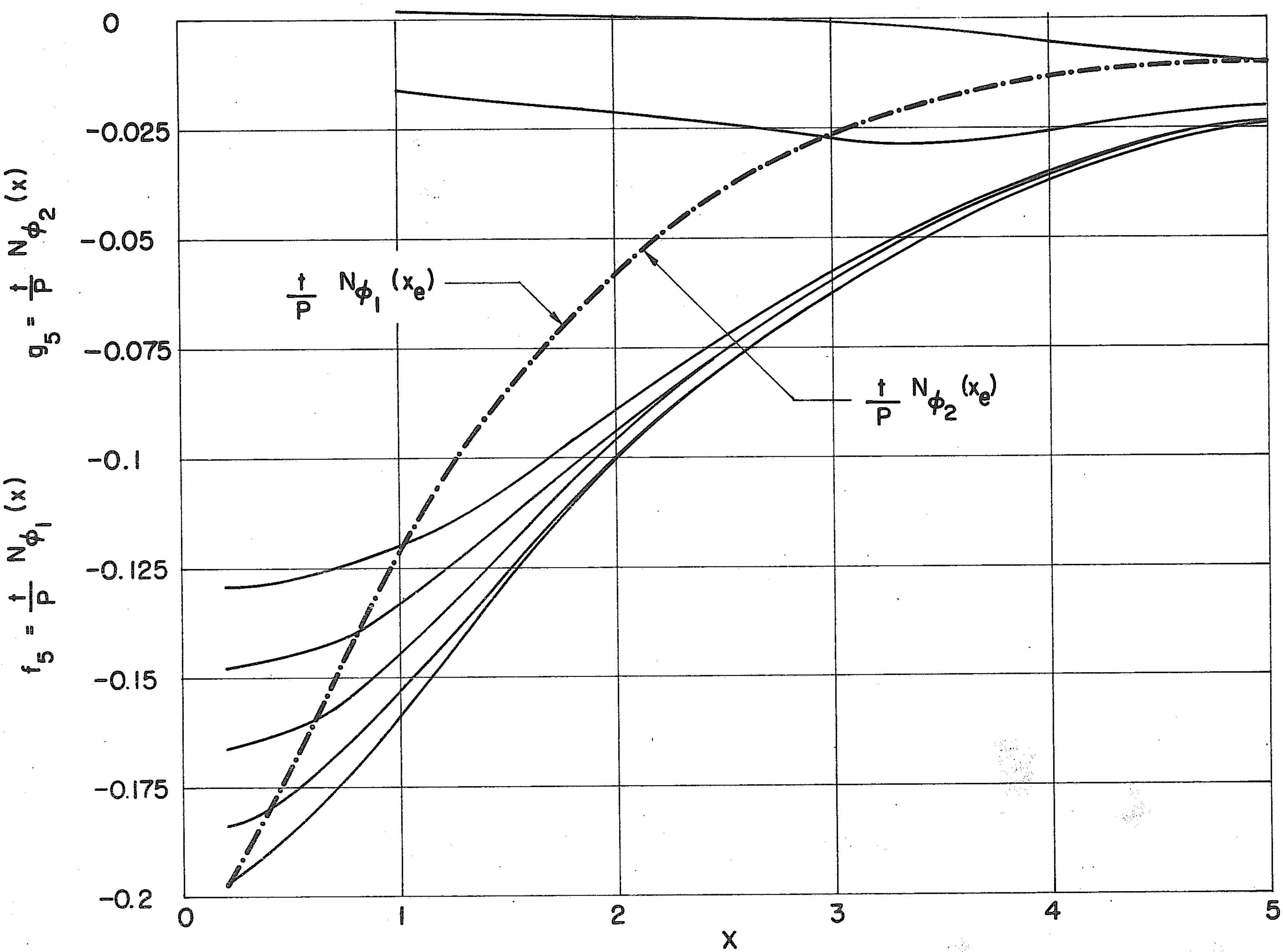
CIRCUMFERENTIAL MOMENT M_Θ FOR A RING LOAD ON A SPHERICAL SHELL

FIG. 23



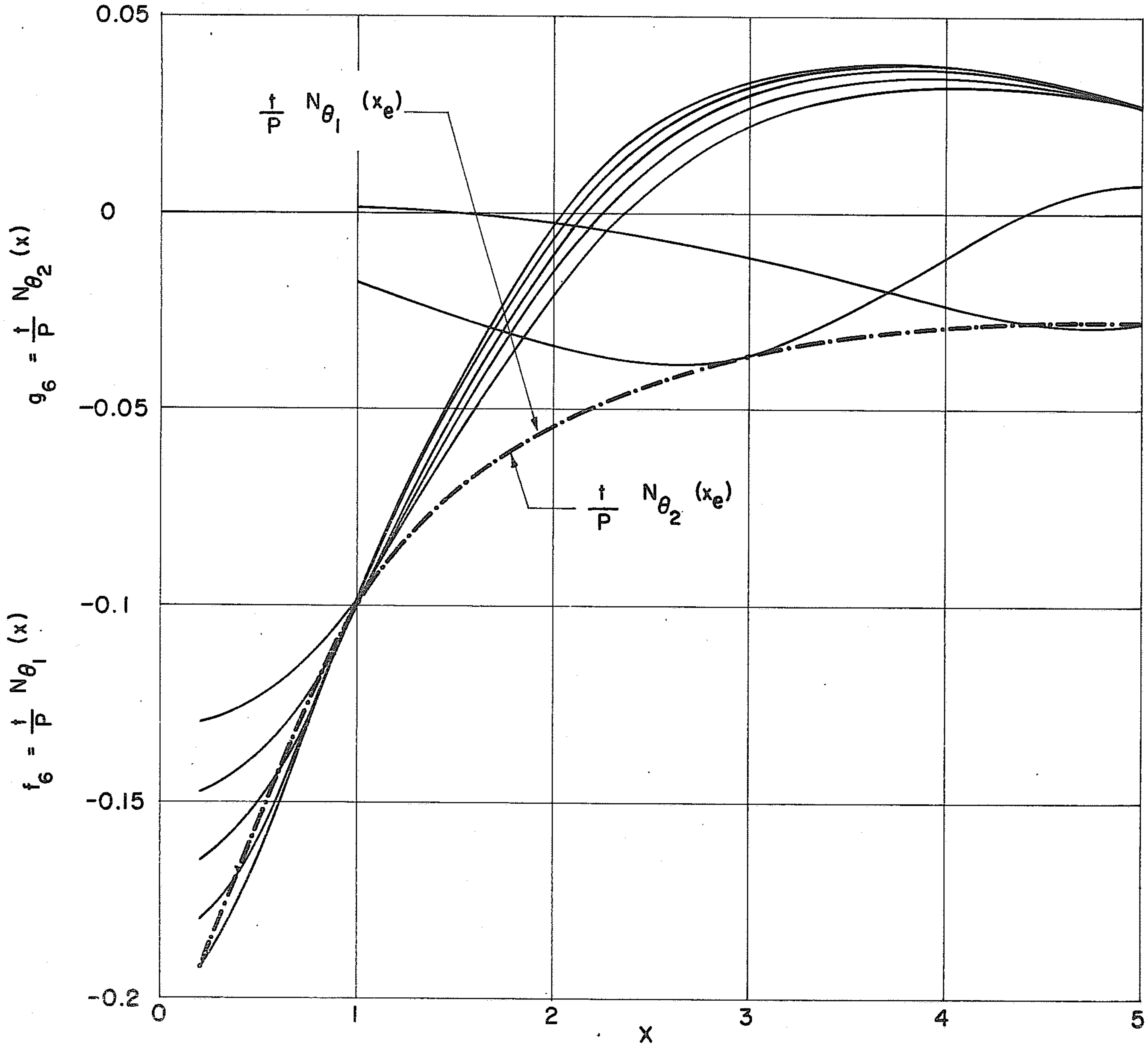
SHEAR RESULTANT Q_ϕ FOR A RING LOAD ON A SPHERICAL SHELL

FIG. 24



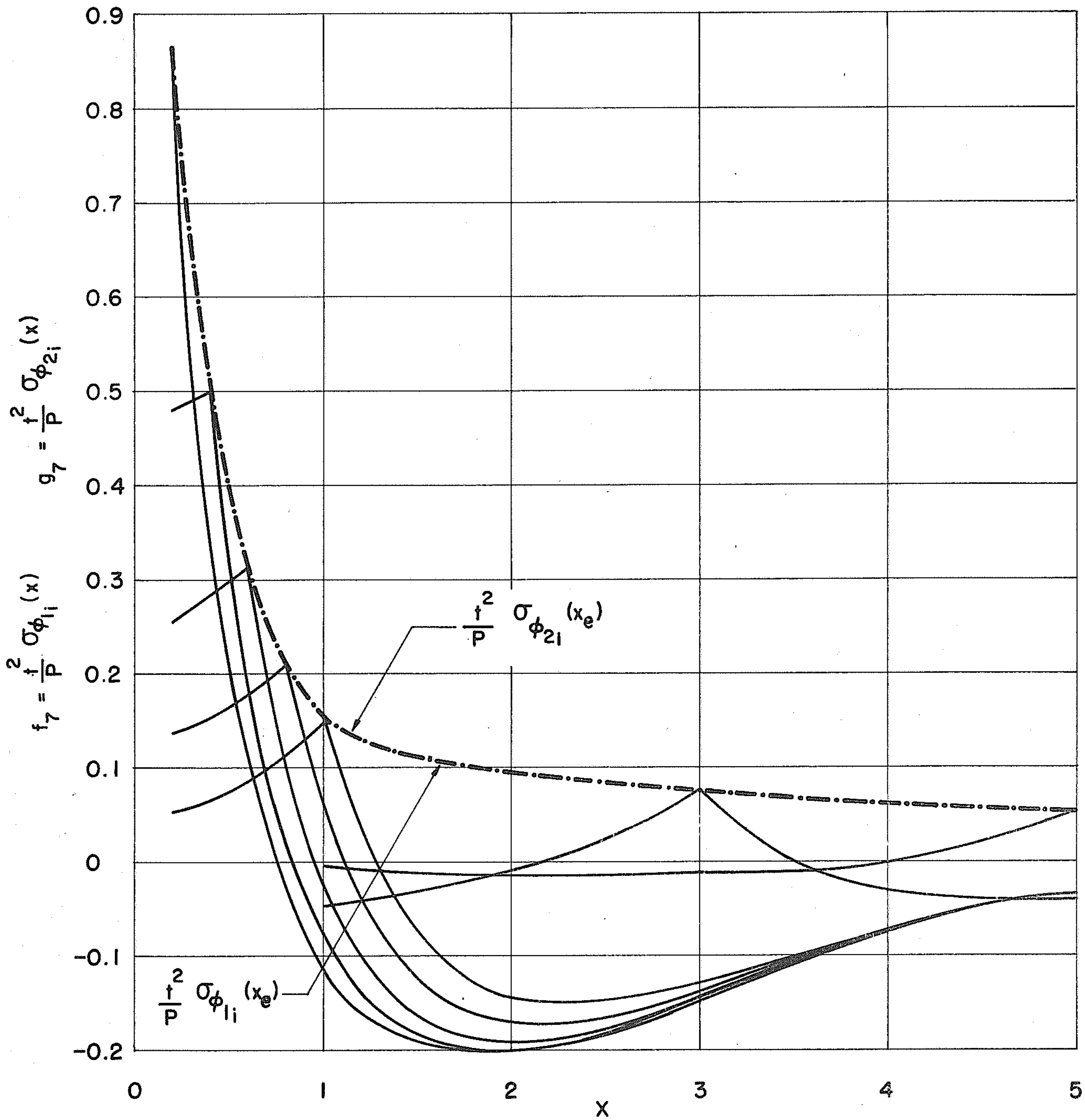
MERIDIONAL DIRECT FORCE N_ϕ FOR A RING LOAD ON A SPHERICAL SHELL

FIG. 25



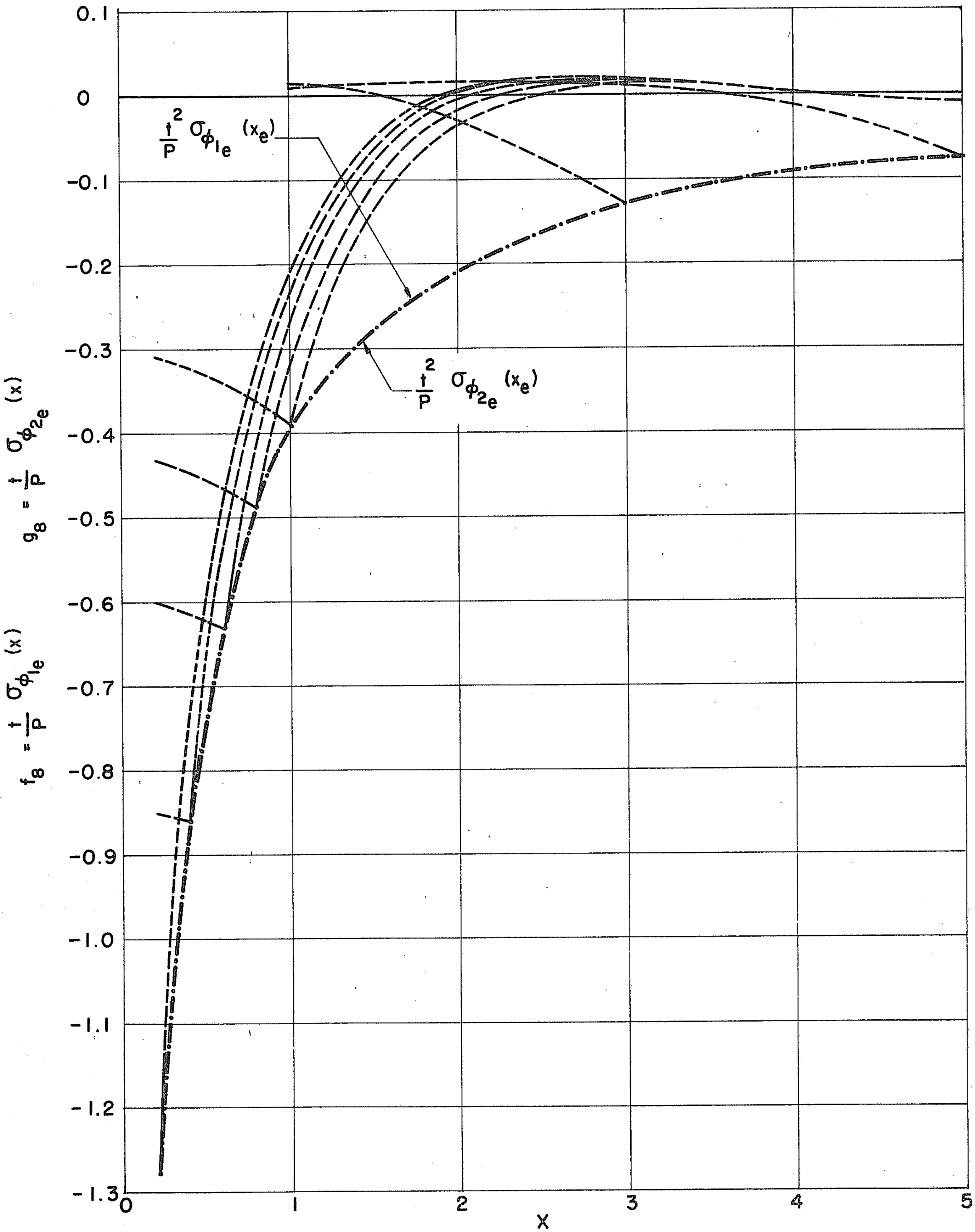
CIRCUMFERENTIAL DIRECT FORCE N_θ FOR A RING LOAD ON A SPHERICAL SHELL

FIG. 25a



MERIDIONAL STRESS σ_{ϕ_i} AT THE INTERIOR FACE FOR A RING LOAD ON A SPHERICAL SHELL

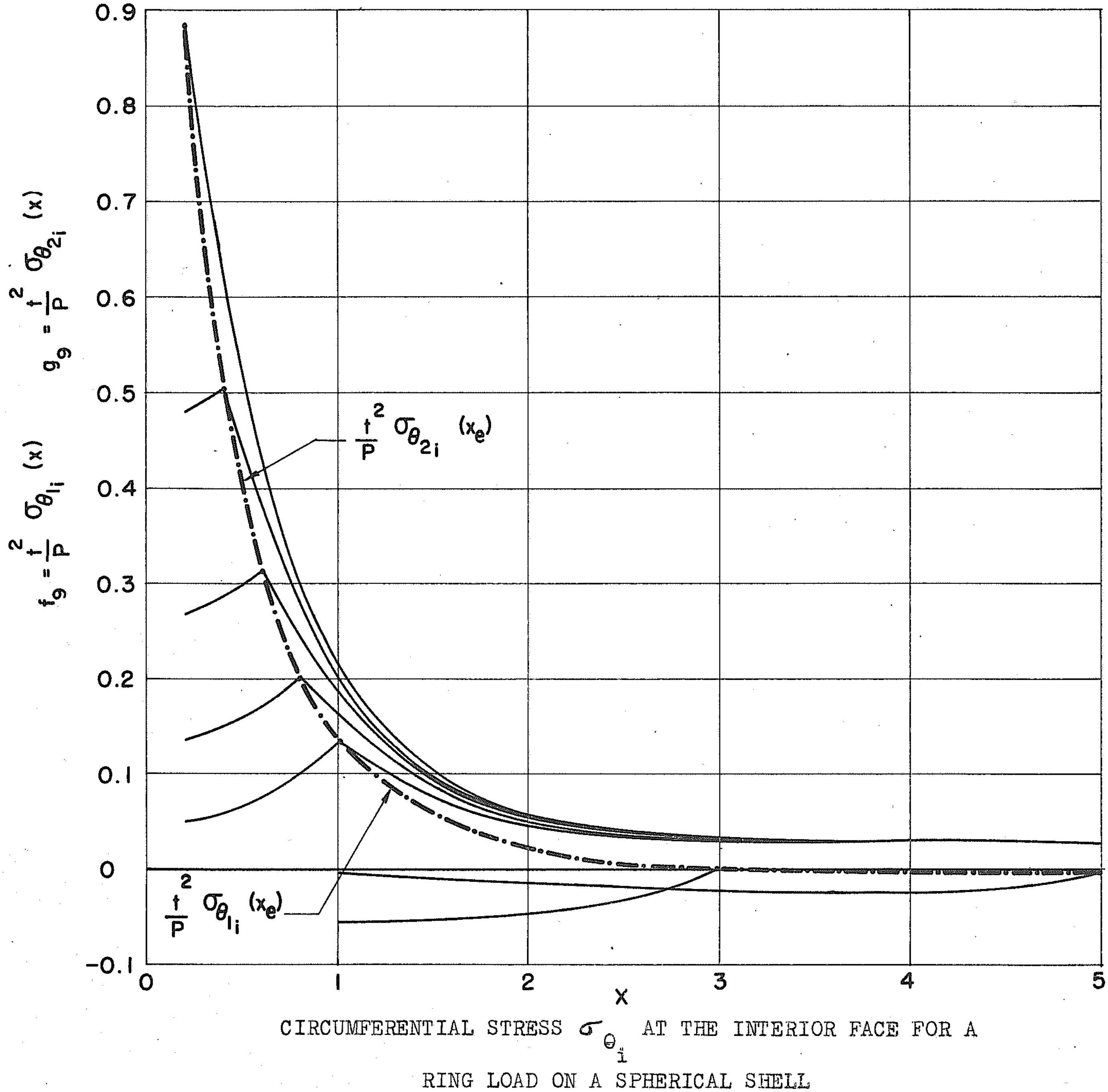
FIG. 26



MERIDIONAL STRESS σ_{ϕ_e} AT THE EXTERIOR FACE FOR A RING

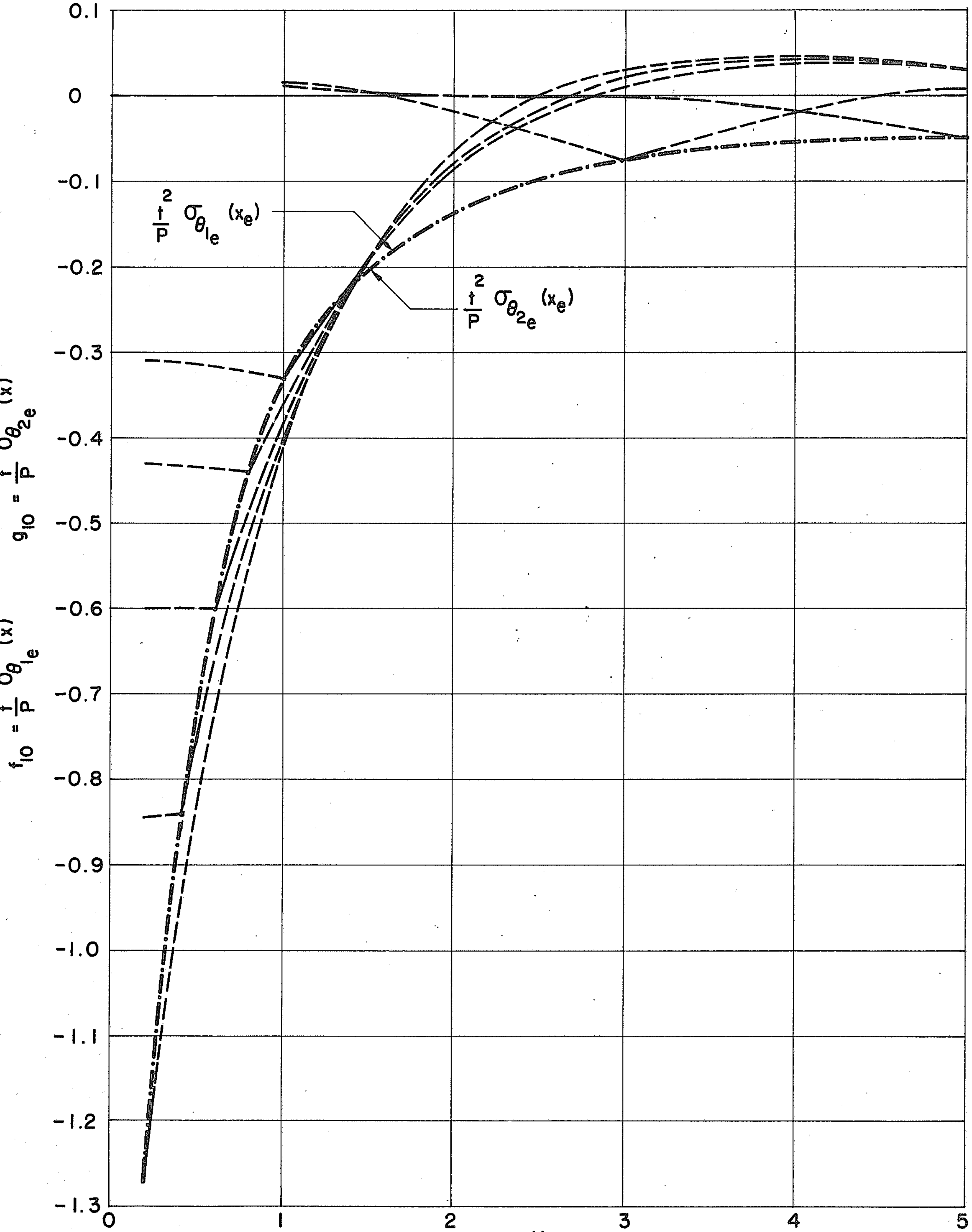
LOAD ON A SPHERICAL SHELL

FIG. 26a



CIRCUMFERENTIAL STRESS σ_{θ_i} AT THE INTERIOR FACE FOR A RING LOAD ON A SPHERICAL SHELL

FIG. 27



CIRCUMFERENTIAL STRESS σ_{θ_e} AT THE EXTERIOR FACE FOR A RING LOAD ON A SPHERICAL SHELL

FIG. 27a

Table IV

Uniformly loaded spherical cap with fixed ends

$X_e = 2.0$

x	h_1	h_2	h_3	h_4	h_5
0.5	2859300050-	6225011049-	6687821049-	9666731649	1485485750
1.0	2264981050-	2850653749-	4758914149-	2019829450	1328273350
1.5	1557601050-	3087442749-	1392352449-	3207436750	1112928950
2.0	1186550050-	1188793450	3566380349	4496841450	9127309049

x	h_6	h_7	h_8	h_9	h_{10}
0.5	1373813850	2249520950-	5220492350	2638878850-	5386506450
1.0	9367079049	3821189049-	3038665550	1918640650-	3792056450
1.5	4446722049	2965394550	7395367049-	3907392449-	1280083650
2.0	2738193049	8045491350	6220029550-	2413647550	1866008950-

$X_e = 4.0$

x	h_1	h_2	h_3	h_4	h_5
1.0	9797846050-	4289439949-	4248461949-	1806468148-	5032838750
2.0	8188012050-	3568317649-	3973457749-	4075607249	4629559950
2.5	7042932050-	1972363149-	3268745549-	9182345049	4332318650
3.0	5817797050-	1159131249	1844849949-	1670294250	3987890850
3.5	4761809050-	6392029049	5807035048	2645441550	3626003150
4.0	4286683050-	1414416850	4243250649	3746318150	3297448350

x	h_6	h_7	h_8	h_9	h_{10}
1.0	4765007450	2459174850	7606502650	2215930350	7314084550
2.0	3558452350	2488569350	6770550550	1174377750	5942526950
2.5	2710613350	3148900750	5515736550	7493660049	4671860650
3.0	1829906550	4683369550	3292412150	7229966049	2936816450
3.5	1135806250	7461220550	2092143049-	1484228350	7873841049
4.0	9892345049	1178394951	5189052550-	3535184950	1556715950-

$X_e = 6.0$

x	h_1	h_2	h_3	h_4	h_5
1.0	1054241051-	3744820048-	2282868748-	15129368849-	5275027750
2.0	1040249851-	1362231549-	8042640248-	2758121849-	5250691250
3.0	9848126050-	2618754649-	1597237749-	2557822949-	5154990450
4.0	8543402050-	2815906449-	2037019949-	1757501949	4920128650
4.5	7589518050-	1669838149-	1756574749-	6574345449	4734419850
5.0	6541707050-	9640320848	8498559848-	1360625150	4505320250
5.5	5617928050-	5622922649	9346749548	2274236150	4248327850
6 0	5195220050-	1271656650	3814969749	3312755550	3996323550

x	h_6	h_7	h_8	h_9	h_{10}
1.0	5267382350	5050338550	5499716950	5130410250	5404354450
2.0	5151806950	4433352350	6068030150	4669248550	5634365350
3.0	4693136050	3583737650	6726243250	3734793450	5651478650
4.0	3623273450	3230584850	6609672450	2401061550	4845485350
4.5	2855098650	3732516850	5736322750	1801153850	3909043450
5.0	2036386450	5083739550	3926901050	1526472850	2546300050
5.5	1369600250	7622081450	8745742049	1930405250	8087952049
6.0	1198896950	1162626351	3633616150-	3487878750	1090084950-

$X_e = 8.0$

x	h_1	h_2	h_3	h_4	h_5
1.0	1006416951-	2588425648	2696038348	1280270648-	5023273350
2.0	1016137251-	1178278248	1973338448	5348936248-	5048617650
3.0	1027215951-	3223419348-	2194271047-	1380589549-	5083656450
4.0	1025588651-	1229401949-	4794540948-	2401095249-	5109120250
5.0	9837606050-	2408178349-	1128043949-	2352263549-	5085520850
6.0	8691033050-	2661950849-	1518029149-	1483232749	4955062050
6.5	7817981050-	1620854449-	1289290349-	5964431049	4833194150
7.0	6842473050-	8538907048	5066410048-	1263010750	4672007050
7.5	5971493050-	5307801749	1069257549	2141107350	4481041250
8.0	5569307050-	1216945250	3650835749	3150628450	4284082750

x	h_6	h_7	h_8	h_9	h_{10}
1.0	5040895650	5178578850	4867967850	5202657950	4879133350
2.0	5112754050	5119314350	4977920950	5231154350	4994353750
3.0	5188503050	4890251250	5277061650	5175337450	5201668650
4.0	5146766350	4371479150	5846761350	4859093950	5434438850
5.0	4752085150	3640613850	6530427850	4075258850	5428911450
6.0	3735970750	3357891550	6552232550	2825153250	4646788250
6.5	2984786850	3860681550	5805706750	2211212650	3758361050
7.0	2170465950	5184341450	4159672650	1866481350	2474450550
7.5	1490451750	7665722250	1296360250	2132006250	8488972049
8.0	1285224850	1158575451	3017588550-	3475726250	9052766049-

$X_e = 10.0$

x	h_1	h_2	h_3	h_4	h_5
1.0	9968022050-	4857809647	3938796447	9364987747	4982976050
2.0	9986215050-	1046987648	7298365047	1443696348	4986878050
3.0	1003431651-	1532928848	1087654248	5655998047	4996572850
4.0	1012330751-	8766097047	9924595047	3316898248-	5015074050
5.0	1022545351-	2773240248-	3971966047-	1148502649-	5041755950
6.0	1021852851-	1113922649-	3970695648-	2165974649-	5065623350
7.0	9841583050-	2254344549-	9426065048-	2215700449-	5057539850
8.0	8769909050-	2552158049-	1275413449-	1352965249	4969256650
8.5	7940965050-	1578996249-	1059325949-	5638010349	4879423550
9.0	7006947050-	7952865048	3316172048-	1208393850	4755925950
9.5	6167285050-	5122554949	1137289649	2064812050	4604895050
10.0	5777780050-	1184603150	3553809449	3056120250	4444446250

x	h_6	h_7	h_8	h_9	h_{10}
1.0	4985046350	5012122950	4953829150	5008679150	4961413550
2.0	4999337150	5049697350	4924058750	5043127350	4955546950
3.0	5037743450	5088548550	4904597150	5103002750	4972484150
4.0	5108233050	5067670650	4962477450	5167780650	5048685450
5.0	5183696850	4875361550	5208150350	5159865050	5207528650
6.0	5152905050	4397269750	5733976950	4914663350	5391146750
7.0	4784043650	3704933150	6410146550	4218479750	5349607550
8.0	3800652650	3437961850	6500551450	3035404650	4565900650
8.5	3061541050	3932025850	5826821250	2425945550	3697136550
9.0	2251021350	5233097850	4278754050	2052051050	2449991650
9.5	1562390350	7678427950	1531362150	2244764150	8800165049
10.0	1333334150	1155206551	2663172450-	3465619750	7989515049-

X_e = 20.0

x	h_1	h_2	h_3	h_4	h_5
8.0	9999776050-	4896918446-	1942913546-	5278583646-	5000119950
10.0	9993566050-	4177397045	1069187446-	2444434147	4999555650
11.0	9990349050-	2388827647	6959282846	6609541547	4998907750
12.0	9995665050-	6937639247	2400717247	1036945948	4998429250
13.0	1002363951-	1156472348	4412827147	5724550247	4999199550
14.0	1008666151-	7956192047	3951878247	2256131548-	5002929550
15.0	1016751851-	2069232748-	4984589047-	8937934548-	5010831850
16.0	1016663751-	9271896348-	2966642748-	1804585049-	5020502850
17.0	9851422050-	1979631049-	6878972648-	1948530149-	5020836050
18.0	8909205050-	2341362049-	9050375548-	1162141149	4988263450
18.5	8161290050-	1496301249-	6935705648-	5073331149	4950148650
19.0	7305478050-	6792594048	4548175047-	1108824250	4893912450
19.5	6525498050-	4752521549	1243722249	1923187250	4820715650
20.0	6159313050-	1120415750	3361247249	2883283150	4737932650

x	h_6	h_7	h_8	h_9	h_{10}
8.0	4999655850	4997181750	5003058150	4998490150	5000821550
10.0	4994010850	4999806250	4999305050	4993369350	4994652350
11.0	4991441050	5013240750	4984574750	4995616650	4987265450
12.0	4997235750	5040055050	4956803450	5011640050	4982831450
13.0	5024439650	5068587850	4929811250	5050916650	4997962650
14.0	5083732050	5050666750	4955192350	5107443350	5060020750
15.0	5156686550	4886677850	5134985850	5126779050	5186594050
16.0	5146134750	4464189050	5576816650	4968136150	5324133350
17.0	4830586550	3833057450	6208614650	4417848150	5243324950
18.0	3920941750	3583446250	6393080650	3377919250	4463964250
18.5	3211141150	4052367950	5847929350	2794998850	3627283450
19.0	2411565650	5301468050	4486356850	2384276650	2438854750
19.5	1704782350	7672168550	1969262750	2451015650	9585490049
20.0	1421380050	1146042751	1984561650-	3438128350	5953683049-

$X_e = 40.0$

x	h_1	h_2	h_3	h_4	h_5
26.0	1000030651-	2985151045-	7436923044-	1548889346-	5000010850
28.0	9999818050-	3724230146-	1216986746-	4052460346-	5000026350
30.0	9994958050-	6194050044	3049344045-	1938094047	4999882650
31.0	9992274050-	1954645847	5794417946	5454007447	4999680250
32.0	9996630050-	5912588247	1874186447	8839330847	4999497950
33.0	1002056551-	1016860948	3372447147	5100155947	4999719150
34.0	1007591451-	7301753447	2762741447	1978899448-	5001058050
35.0	1014857151-	1843604048-	5053245847-	8055166848-	5004183750
36.0	1014827351-	8529853148-	2633847348-	1658863949-	5008376550
37.0	9856674050-	1858064449-	5976726448-	1823978049-	5008961450
38.0	8970389050-	2239900549-	7628602348-	1091810349	4994777050
38.5	8259494050-	1453124349-	5480734948-	4828621449	4977200950
39.0	7440445050-	6268192048	7144076047	1063490250	4950429450
39.5	6689158050-	4574943449	1285649549	1857025350	4914537350
40.0	6334491050-	1089450350	3268350849	2801454250	4872685150

x	h_6	h_7	h_8	h_9	h_{10}
26.0	5000295150	4999831750	5000189950	5000250550	5000339750
28.0	4999791450	4997791850	5002260850	4999061250	5000521650
30.0	4995075450	4999919850	4999845450	4994892450	4995258450
31.0	4992593450	5011408150	4987952350	4996070150	4989116750
32.0	4997132150	5034973450	4964022450	5008377250	4985887050
33.0	5020846250	5060730850	4938706450	5041080950	5000611550
34.0	5074855950	5044868550	4957247550	5091432350	5058279550
35.0	5144387550	4893567550	5114799950	5114068050	5174707050
36.0	5139896650	4496585350	5520167750	4981865850	5297927450
37.0	4847712450	3894122850	6123800050	4489108850	5206316050
38.0	3975612450	3650836750	6338717350	3517896350	4433328550
38.5	3282292850	4105326350	5849065550	2953448750	3611136950
39.0	2490015050	5326520950	4574337950	2532879550	2447150550
39.5	1774620150	7659503350	2169571350	2546009850	1003230450
40.0	1461805550	1140938751	1664016750-	3422816050	4992050049-

$X_e = 60.0$

x	h_1	h_2	h_3	h_4	h_5
44.0	1000003251-	1928371245	6212016844	1228834445	4999999550
46.0	1000027851-	2699188345-	7309392044-	1414103646-	5000005650
48.0	9999827050-	3466023446-	1093724746-	3770798946-	5000014350
50.0	9995278050-	1565920044	1782026045-	1821078647	4999933850
51.0	9992737050-	1848244647	5504110146	5163889847	4999815950
52.0	9996867050-	5639721247	1750694547	8426570447	4999705450
53.0	1001973251-	9771517047	3123413447	4903745247	4999831850
54.0	1007292551-	7089674547	2474078347	1903543748-	5000640850
55.0	1014314551-	1779787948-	5044692247-	7797784948-	5002577350
56.0	1014290151-	8304903448-	2538255948-	1614445949-	5005250750
57.0	9858428050-	1819745049-	5714651948-	1784236149-	5005690350
58.0	8989700050-	2206835649-	7205377848-	1070917349	4996643550
58.5	8290660050-	1438628749-	5042349348-	4751506849	4985235150
59.0	7483513050-	6100581048	1069780848	1048950050	4967681050
59.5	6741596050-	4516811049	1298053549	1835611550	4943918550
60.0	6390706050-	1079273650	3237820749	2774907750	4915927750

x	h_6	h_7	h_8	h_9	h_{10}
44.0	5000032750	5000115250	4999883850	5000070050	4999995450
46.0	5000271250	4999843650	5000167650	5000228250	5000316050
48.0	4999813250	4997934750	5002093950	4999157050	5000469450
50.0	4995344150	4999943250	4999924450	4995237250	4995451050
51.0	4992921550	5010905450	4988726450	4996224050	4989619050
52.0	4997161450	5033543750	4965867150	5007665650	4986657250
53.0	5019900250	5058460950	4941202750	5038640750	5001159750
54.0	5072283850	5043178850	4958102850	5087128350	5057439350
55.0	5140567350	4895790050	5109364650	5110299150	5170835550
56.0	5137660550	4506946550	5503534950	4985365250	5289955950
57.0	4852738050	3913843350	6097537350	4509858950	5195617150
58.0	3993056450	3672542150	6320744950	3560733750	4425379150
58.5	3305425150	4122057950	5848412350	3002884150	3607966150
59.0	2515832450	5333715950	4601646150	2580019250	2451645650
59.5	1797677250	7654005150	2233831950	2576509350	1018845150
60.0	1474778450	1139156951	1559713950-	3417470850	4679140049-

$X_e = 80.0$

x	h_1	h_2	h_3	h_4	h_5
62.0	9999986050-	2994639644	8863875043	9179722344	4999999750
64.0	1000003151-	1842421845	5807079444	1174747645	4999999750
66.0	1000026651-	2581763745-	7217220044-	1357146546-	5000003750
68.0	9999832050-	3351967846-	1042343446-	3645470746-	5000009750
70.0	9995421050-	7550000042-	1272232045-	1768747547	4999954150
71.0	9992947050-	1799981947	5371381546	5031345047	4999871250
72.0	9996974050-	5113097447	1695383747	8234378247	4999792150
73.0	1001934351-	9583678947	3011603347	4808641647	4999880350
74.0	1007151951-	6985531647	2344458247	1868324248-	5000459050
75.0	1014057051-	1749646448-	5036139247-	7674940148-	5001860250
76.0	1014033651-	8196216648-	2492862048-	1592945349-	5003810250
77.0	9859303050-	1800983049-	5589909448-	1764708349-	5004166250
78.0	8999163050-	2190452849-	7002382748-	1060885649	4997527550
78.5	8305962050-	1431367849-	4831240748-	4713724749	4989084350
79.0	7504700050-	6018140048	1241394748	1041782750	4976027950
79.5	6767429050-	4487951649	1303976649	1825023250	4958269250
80.0	6418417050-	1074213850	3222641349	2761768650	4937244350

x	h_6	h_7	h_8	h_9	h_{10}
62.0	4999986450	5000017750	4999981750	4999991750	4999981150
64.0	5000031050	5000110250	4999889250	5000065850	4999996250
66.0	5000262150	4999848850	5000158650	5000218850	5000305450
68.0	4999822150	4997998550	5002020950	4999196750	5000447550
70.0	4995466750	4999953650	4999954650	4995390450	4995543050
71.0	4993075950	5010671150	4989061350	4996298750	4989853150
72.0	4997182250	5032870750	4966713550	5007354550	4987009950
73.0	5019462950	5057382450	4942378250	5037532550	5001393350
74.0	5071059850	5042372250	4958545850	5085126550	5056993150
75.0	5138709450	4896881450	5106839050	5108492650	5168926250
76.0	5136525650	4512037250	5495583250	4986953950	5286097350
77.0	4855137350	3923576450	6084756050	4519742750	5190531950
78.0	4001634950	3683255850	6311799250	3581491950	4421777950
78.5	3316877750	4130263650	5847905050	3027003350	3606752150
79.0	2528672350	5337116350	4614939550	2603156050	2454188650
79.5	1809159550	7651040250	2265498250	2591545550	1026773550
80.0	1481173250	1138252751	1508038550-	3414758050	4524116049-

$X_e = 100.0$

x	h_1	h_2	h_3	h_4	h_5
82.0	9999987050-	2902126044	8618144043	8901761244	4999999850
84.0	1000003051-	1795187445	5593069844	1144708245	4999999850
86.0	1000025951-	2517662245-	7156860044-	1325639346-	5000002850
88.0	9999834050-	3287619046-	1014119846-	3574539946-	5000007450
90.0	9995502050-	8910000043-	9982510044-	1739046547	4999964950
91.0	9993067050-	1772412547	5295243246	4955364647	4999901050
92.0	9997036050-	5439967247	1663985347	8123206747	4999839550
93.0	1001911851-	9474165747	2948089747	4752604247	4999907150
94.0	1007070151-	6923708247	2270883847	1847907748-	5000357450
95.0	1013906651-	1732085348-	5029792547-	7602984548-	5001454850
96.0	1013883351-	8132158448-	2466341748-	1580261349-	5002992450
97.0	9859828050-	1789849649-	5516968748-	1753101849-	5003285450
98.0	9004781050-	2180669949-	6883234248-	1054991949	4998043150
98.5	8315056050-	1427007549-	4707083048-	4691293249	4991342150
99.0	7517304050-	5969117048	1342466848	1037513750	4980949250
99.5	6782809050-	4470699149	1307441749	1818705250	4966772750
100.0	6434919050-	1071186550	3213559749	2753926050	4949938050

x	h_6	h_7	h_8	h_9	h_{10}
82.0	4999986850	5000017250	4999982450	4999992050	4999981650
84.0	5000030250	5000107550	4999892150	5000063850	4999996650
86.0	5000256550	4999851750	5000153950	5000213650	5000299450
88.0	4999826050	4998034850	5001980050	4999218450	5000435450
90.0	4995537050	4999959650	4999970250	4995477150	4995596950
91.0	4993165650	5010535550	4989266550	4996342750	4989988550
92.0	4997196250	5032479350	4967199750	5007180150	4987212350
93.0	5019210850	5056752150	4943062150	5036899350	5001522350
94.0	5070344050	5041899650	4958815250	5083969350	5056718750
95.0	5137611150	4897529750	5105379950	5107432350	5167789950
96.0	5135840450	4515062950	5490921950	4987859950	5283820950
97.0	4857542650	3929375650	6977195250	4525524550	5187560750
98.0	4006737550	3689641250	6306445050	3593743550	4419731650
98.5	3323713750	4135137650	5847546650	3041288750	3606138750
99.0	2536354950	5339096250	4622802250	2616901950	2455806950
99.5	1816036450	7649192250	2284353250	2600501450	1031571450
100.0	1484981550	1137705751	1477181050-	3413117350	4431543049-

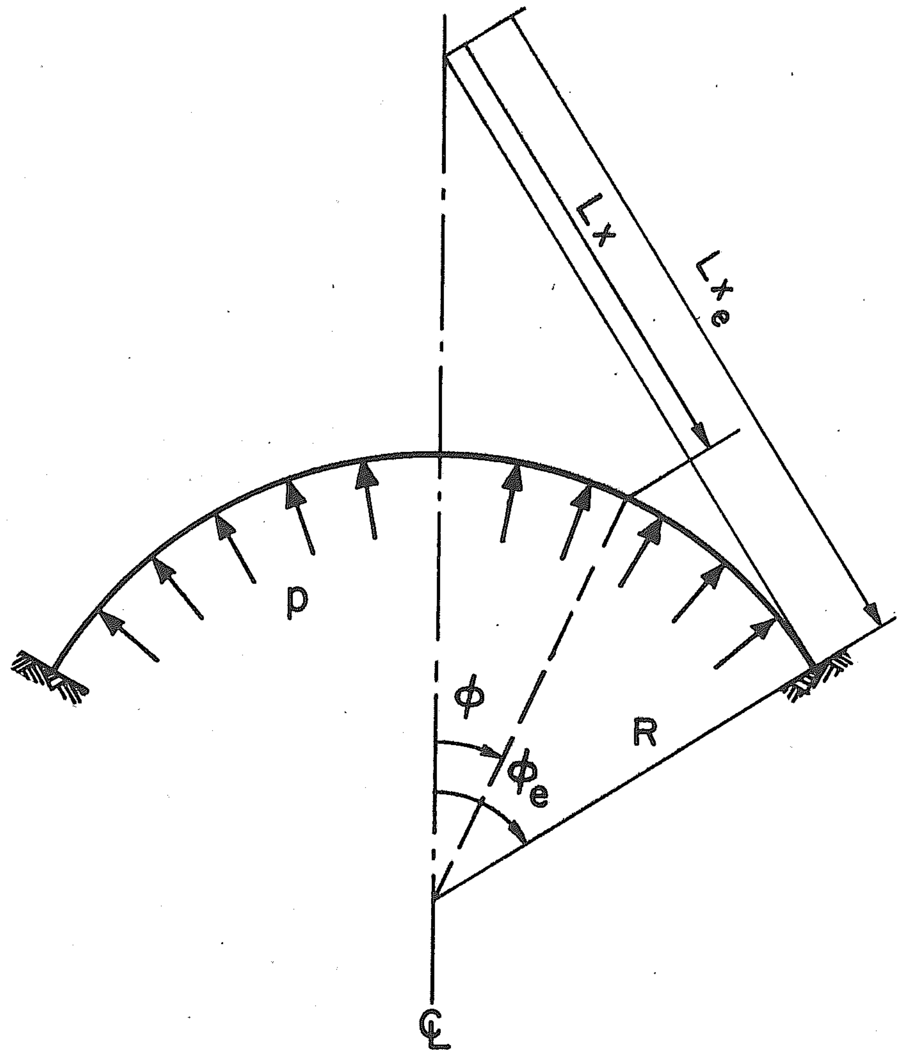
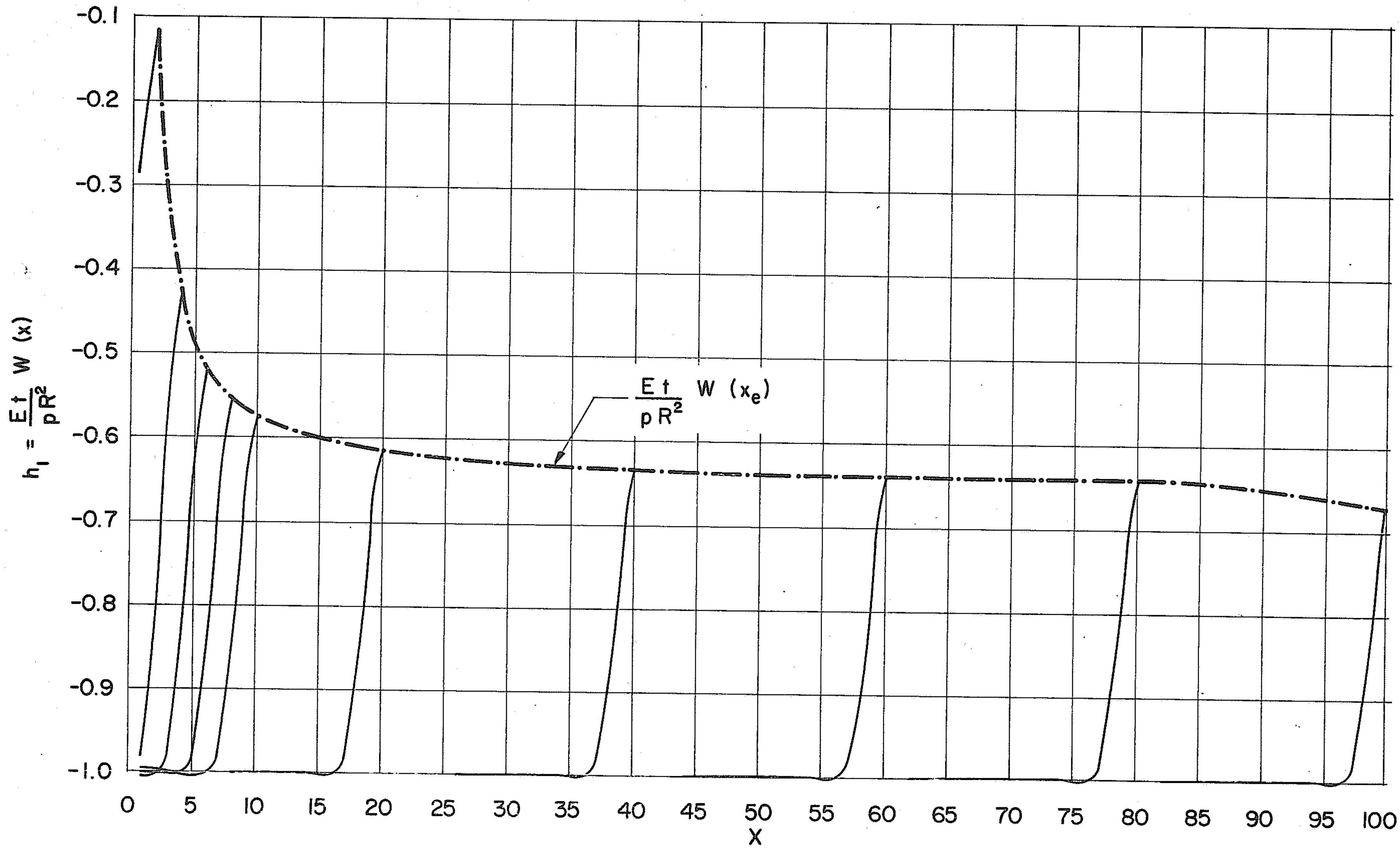
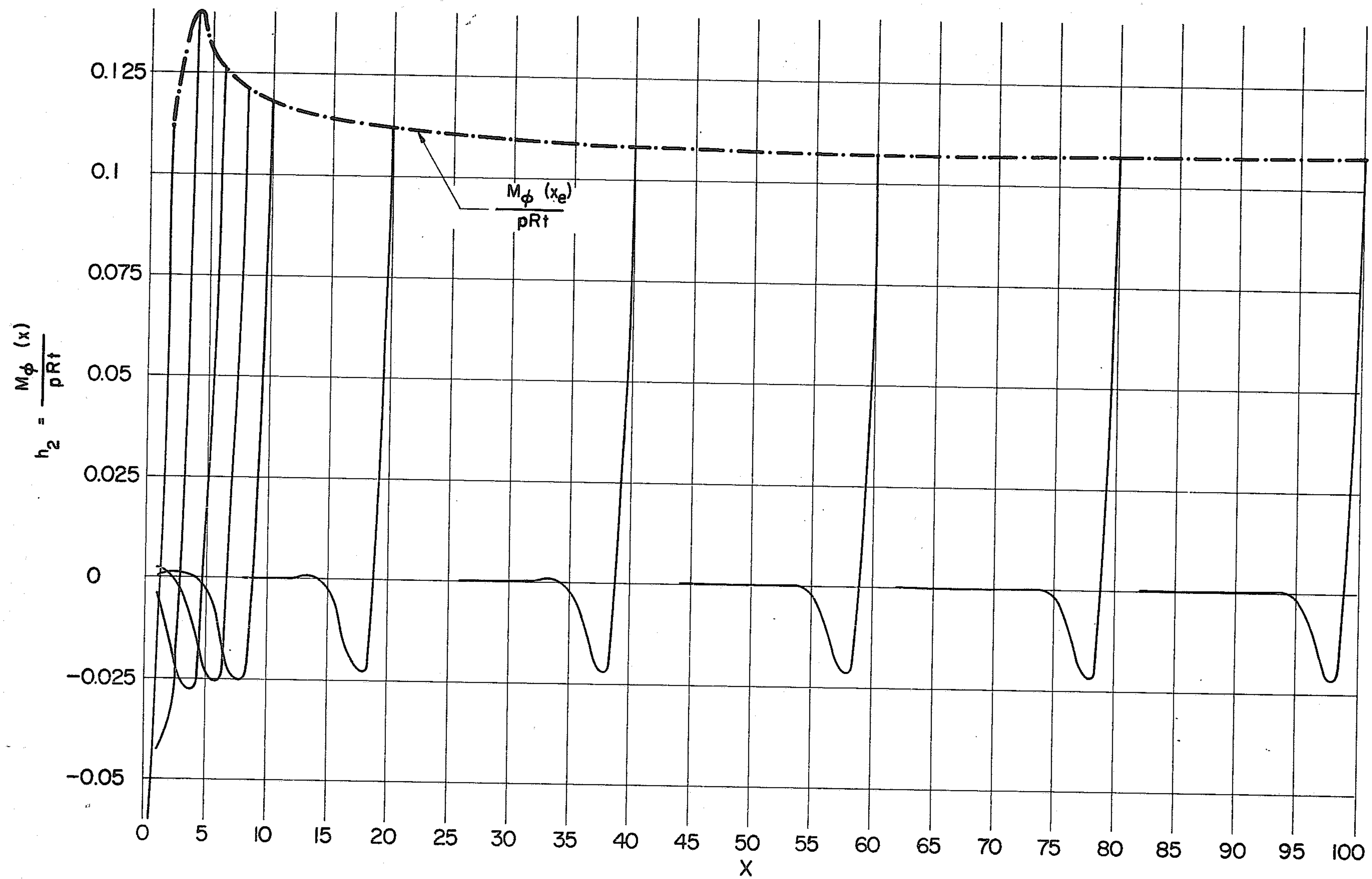


FIG. 28 – UNIFORMLY LOADED SPHERICAL CAP



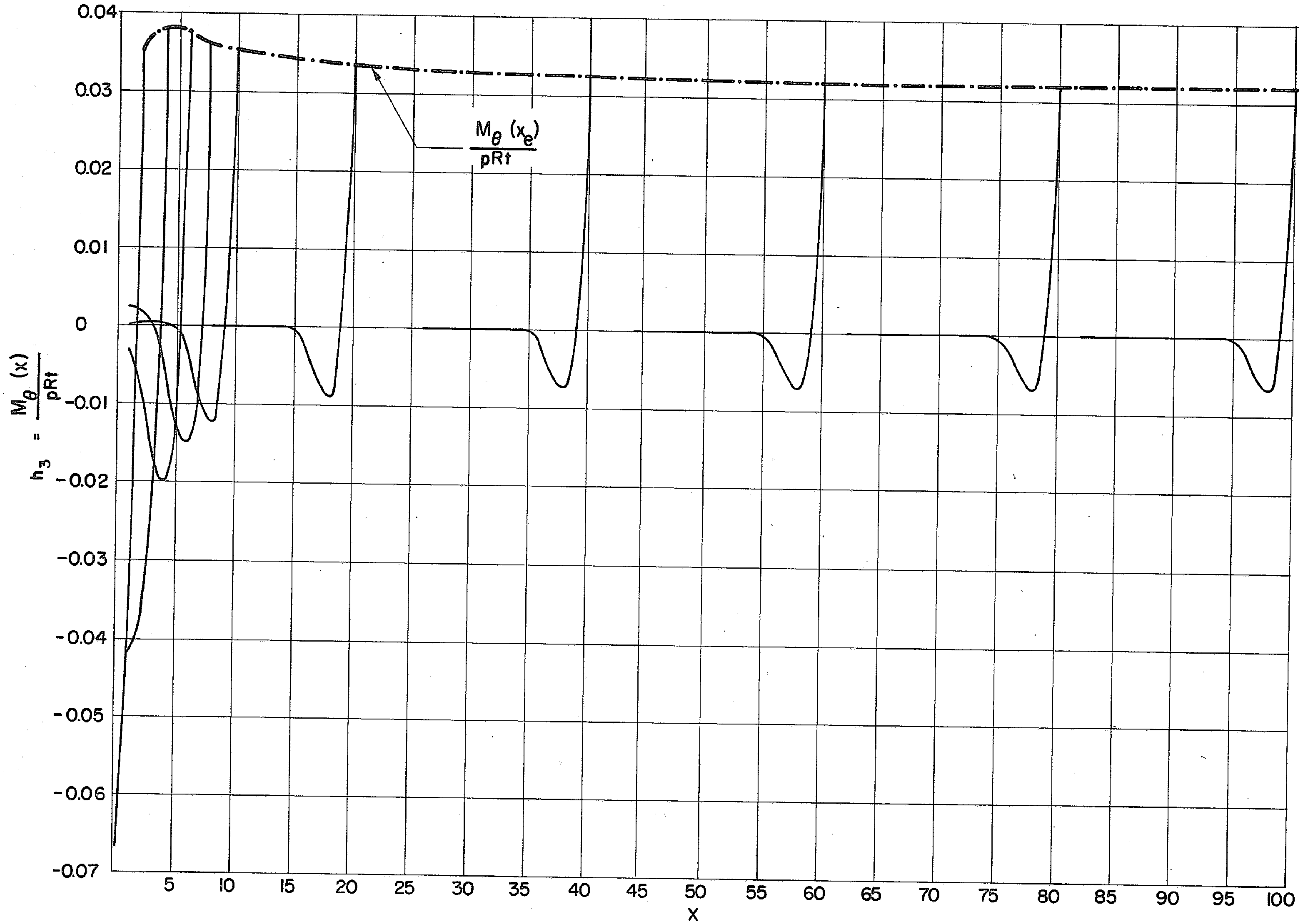
RADIAL DEFLECTION W FOR A UNIFORMLY LOADED SPHERICAL CAP

FIG. 29



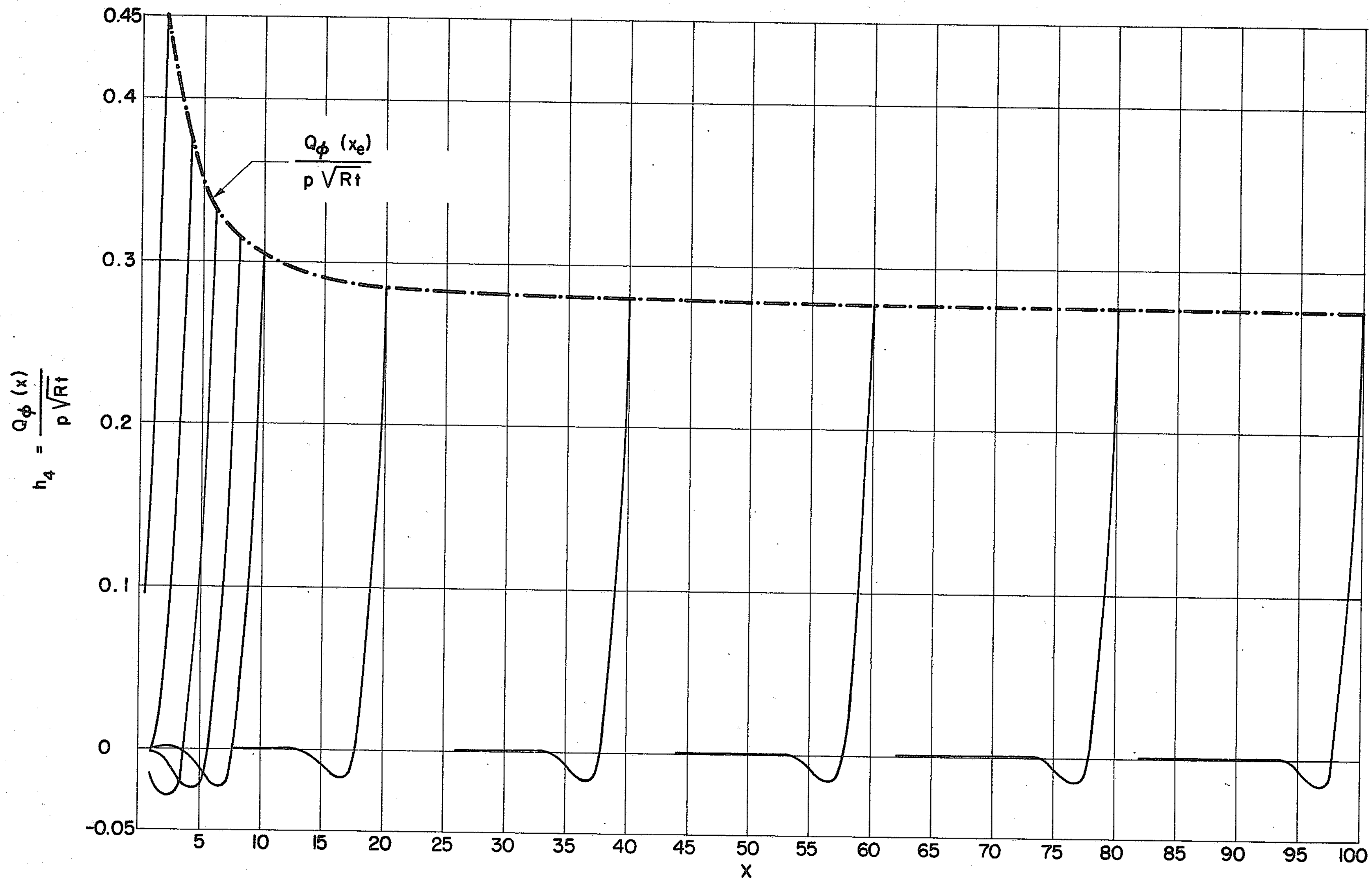
MERIDIONAL MOMENT M_ϕ FOR A UNIFORMLY LOADED SPHERICAL CAP

FIG. 30



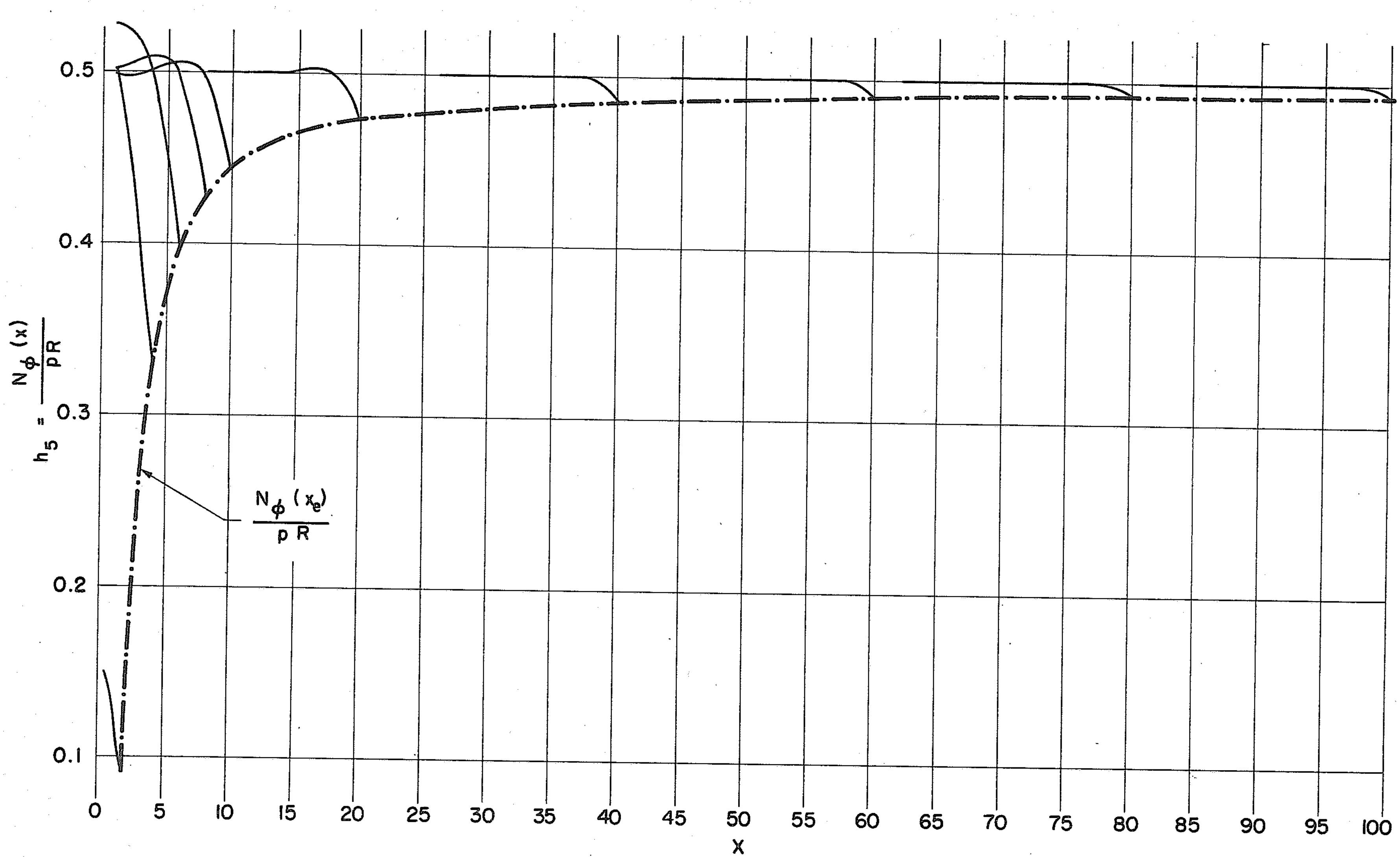
CIRCUMFERENTIAL MOMENT M_θ FOR A UNIFORMLY LOADED SPHERICAL CAP

FIG. 31



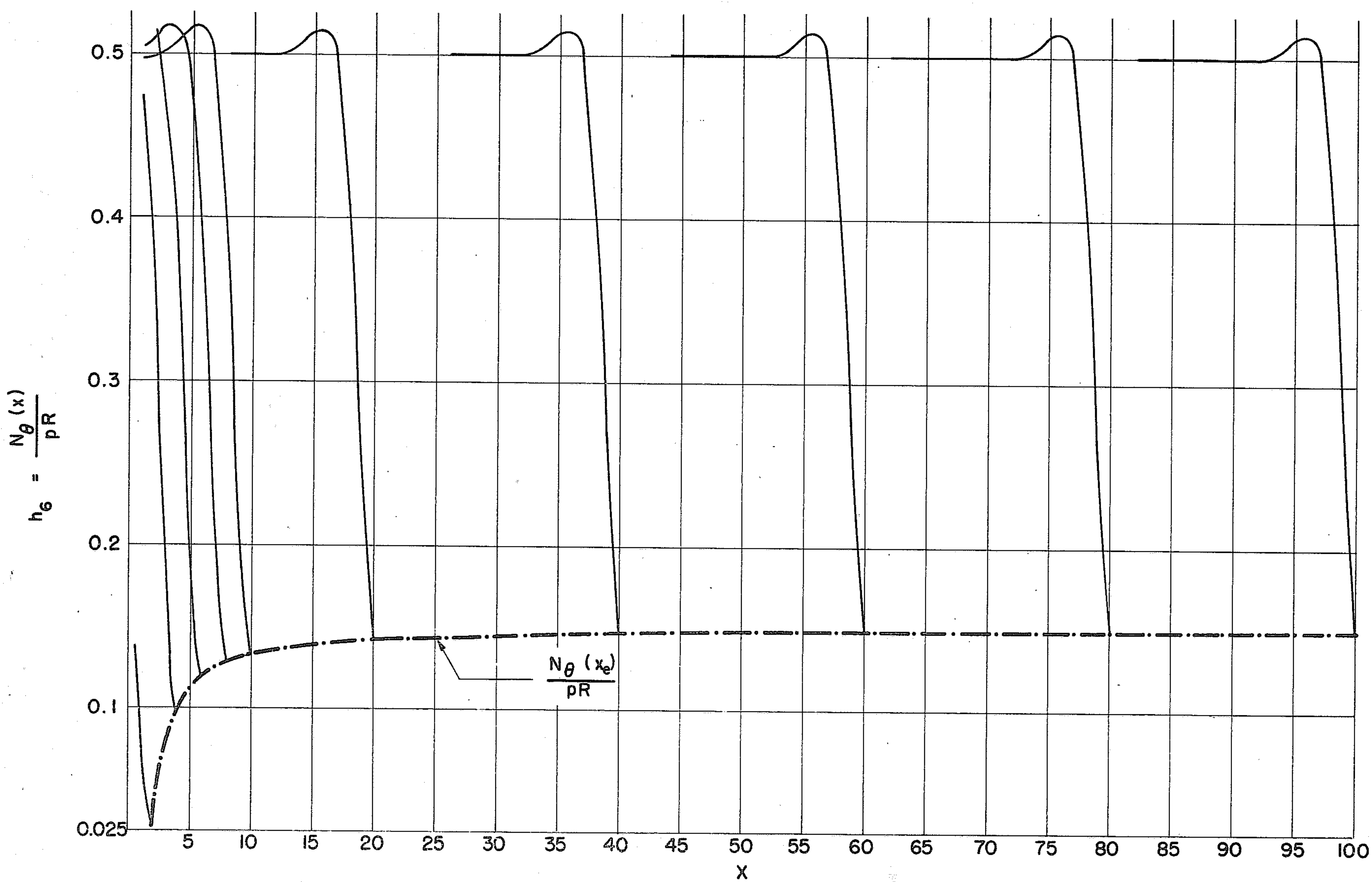
SHEAR RESULTANT Q_ϕ FOR A UNIFORMLY LOADED SPHERICAL CAP

FIG. 32



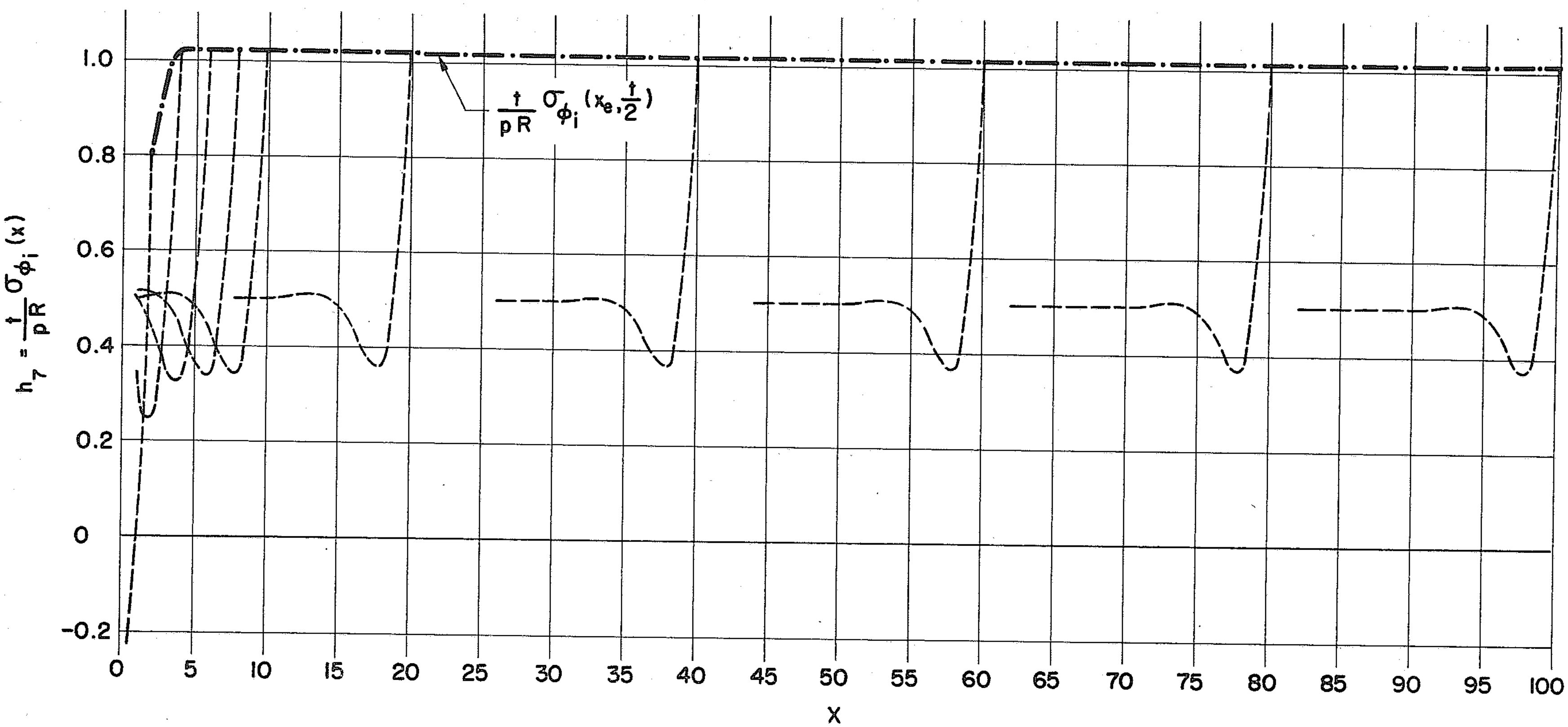
MERIDIONAL DIRECT FORCE N_ϕ FOR A UNIFORMLY LOADED SPHERICAL CAP

FIG. [illegible]



CIRCUMFERENTIAL DIRECT FORCE N_θ FOR A UNIFORMLY LOADED SPHERICAL CAP

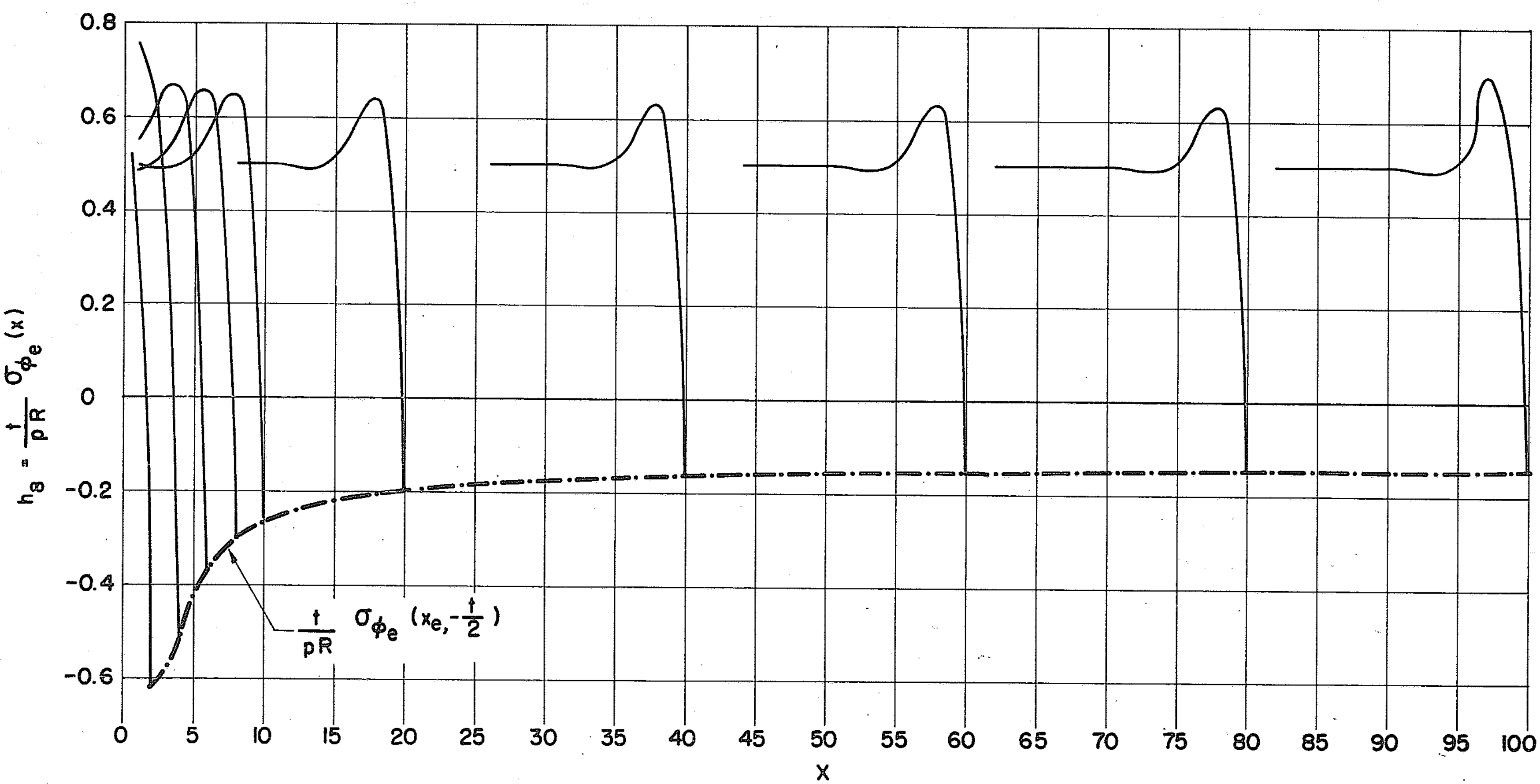
FIG. 33a



MERIDIONAL STRESS σ_{ϕ_i} AT THE INTERIOR FACE FOR A

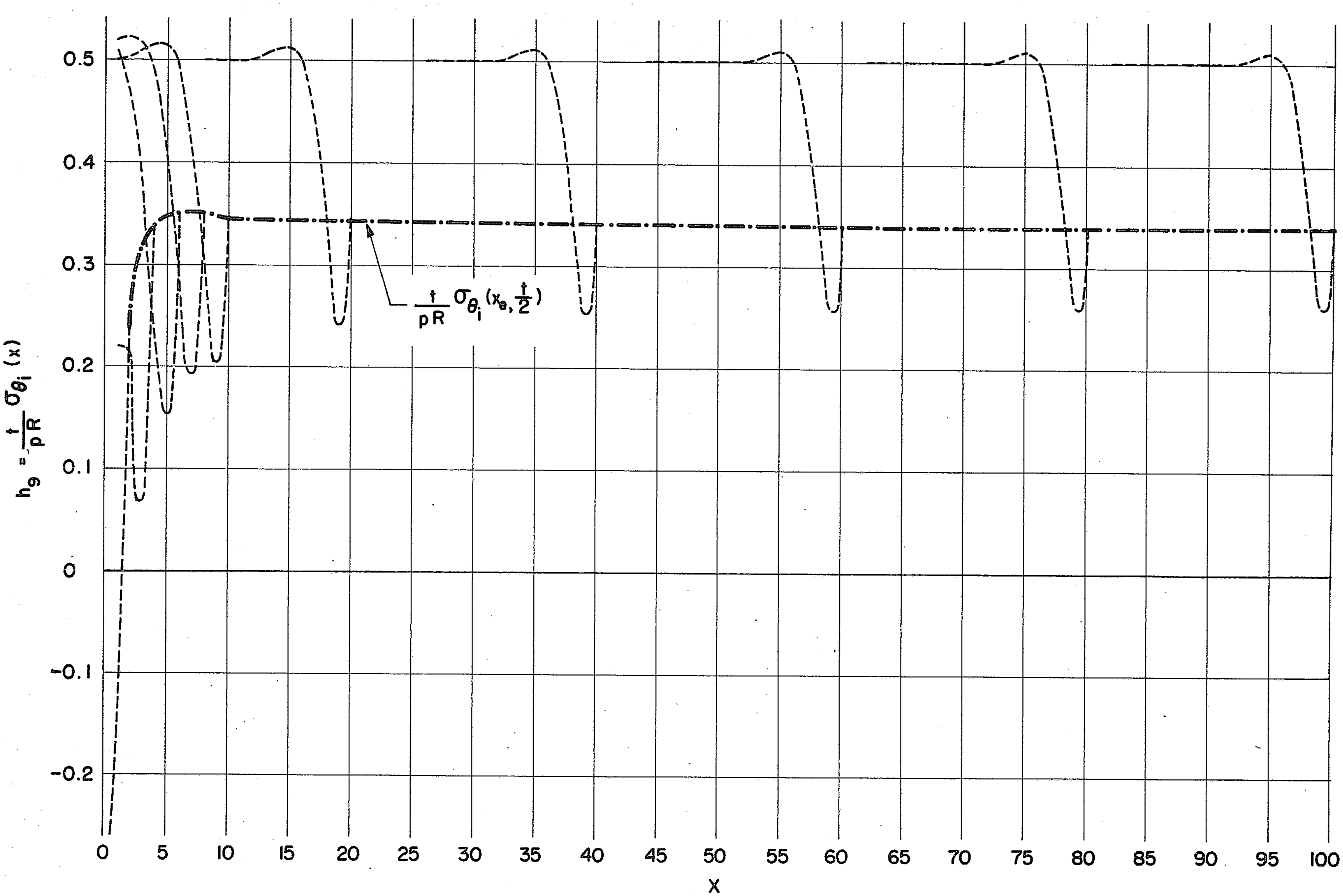
UNIFORMLY LOADED SPHERICAL CAP

FIG. 34



MERIDIONAL STRESS σ_{ϕ_e} AT THE EXTERIOR FACE FOR A UNIFORMLY LOADED SPHERICAL CAP

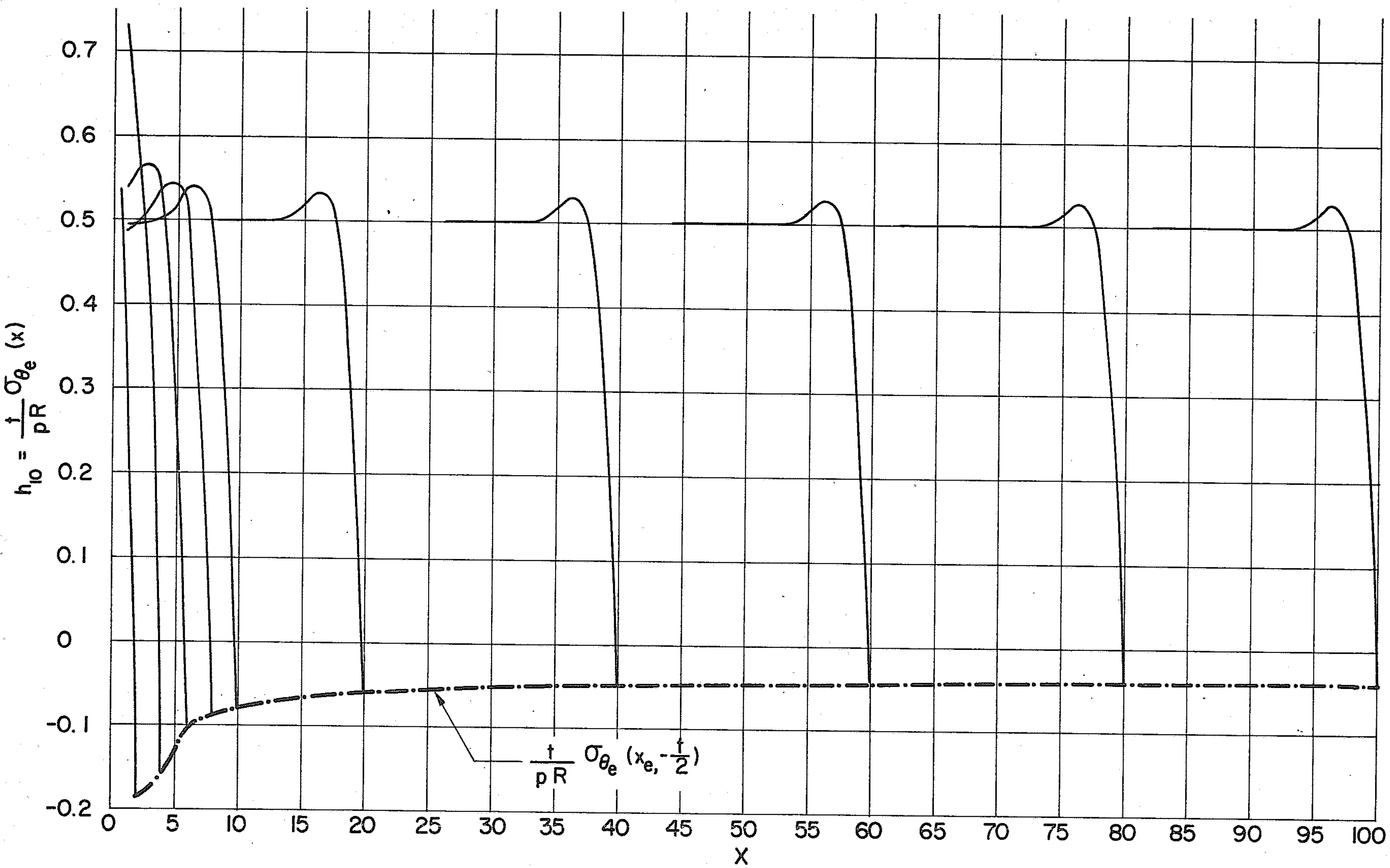
FIG. 34a



CIRCUMFERENTIAL STRESS σ_{θ_i} AT THE INTERIOR FACE FOR

A UNIFORMLY LOADED SPHERICAL CAP

FIG. 35



CIRCUMFERENTIAL STRESS σ_{θ_e} AT THE EXTERIOR FACE FOR

A UNIFORMLY LOADED SPHERICAL CAP

FIG. 35a

Table V

$X_o = 0.2$

External moment on an insert

x	d_1	d_2	d_3	d_4
0.2	1663547050	1424388651	4273166150	
0.4	2734910750	4982947350	3871643350	1828276050-
0.6	3189234050	2829196150	2659921450	1393202050-
0.8	3337700650	1816562150	1908904050	1049181150-
1.0	3301697050	1210526150	1414114050	8084278149-
1.2	3149632650	8056475849	1066535650	6344933549-
2.0	2090582050	5304230848	3559459149	2615640249-
3.0	8816884849	1458624849-	6855330048	8639388048-
4.0	2314984449	1148028749-	2654106047-	2445134948-
5.0	6205673447-	5434873448-	1032362248-	4600767947-

x	d_5	d_6	d_7	d_8
0.2	1333608052-	1294655452-	1279800050-	3838398749-
0.4	3459893751-	3095428651-	8569200049-	1877986350-
0.6	1610784451-	1279274051-	9634080049-	2225826150-
0.8	9465768250-	6511828350-	1040369050-	2297330850-
1.0	6257516150-	3670135350-	1074405050-	2227292650-
1.2	4413280950-	2181621550-	1075161050-	2074472250-
2.0	1352206450-	2859519849-	9012060049-	1189376150-
3.0	2362264549-	3603479748	5868903149-	2947981449-
4.0	5073106448	4106115148	3404859949-	1089875549
5.0	7630274948	1808676948	1872873049-	1934929749

x	d_9	d_{10}	d_{11}	d_{12}
0.2	8418351651	8674311651-	2525515751	2602283751-
0.4	2904076451	3075460451-	2135187451	2510784651-
0.6	1601176951	1793858551-	1373370251	1818535451-
0.8	9859004050	1193974251-	9156093050	1375075551-
1.0	6188751650	8337561650-	6257391450	1071197751-
1.2	3758724550	5909046550-	4324741450	8473685850-
2.0	5829521549-	1219459950-	9462994049	3325051650-
3.0	1462065250-	2882845749	1165216649	7061179449-
4.0	1029303250-	3483312349	9306291048	1249121949
5.0	5133797049-	1388051049	1315512449	2554347049

$X_o = 0.4$

x	d_1	d_2	d_3	d_4
0.4	2075913850	6812505350	2043751750	2118296443-
0.6	2775750250	3383242550	2123876850	8530031249-
0.8	3053173650	2061762650	1689247850	8236300749-
1.0	3098089050	1345741850	1307688850	6953586649-
1.2	3001758950	8916008449	1010143950	5712797249-
1.4	2818492350	5800287149	7801978849	4662994349-
2.0	2053621450	8024088548	3511369849	2515882449-
3.0	8844354149	1369320449-	6989495648	8536505348-
4.0	2393509949	1126540249-	1783289047-	2462532348-
5.0	1438196247-	5443240748-	1009571048-	4795394247-

x	d_5	d_6	d_7	d_8
0.4	3443029051-	3096011351-	1596870050-	4790465749-
0.6	1603261351-	1283469451-	1064228050-	1711522050-
0.8	9431007150-	6556478050-	1017901050-	2035273950-
1.0	6244914850-	3710394150-	1024127050-	2073962450-
1.2	4414029950-	2216069050-	1021843050-	1979915550-
1.4	3235789150-	1358612350-	1002765250-	1815727250-
2.0	1370340150-	3016061349-	8704955049-	1183125950-
3.0	2502087249-	3124684648	5755165649-	3089188149-
4.0	4414790648	4000552448	3370144749-	9766348048
5.0	7430473048	1804643048	1865462249-	1879844149

x	d_9	d_{10}	d_{11}	d_{12}
0.4	3927816251	4247190251-	1178346351	1274155751-
0.6	1923522751	2136368351-	1103173951	1445478351-
0.8	1135267551	1338847751-	8100213050	1217076151-
1.0	7050323850	9098577850-	5772170450	9920095250-
1.2	4327762050	6371448050-	4080947950	8040778950-
1.4	2477407150	4482937550-	2865460150	6496914550-
2.0	3890501949-	1351940850-	9236960049	3289947850-
3.0	1397108850-	2460756849	1104509349	7282885549-
4.0	1012938650-	3389096549	8696374648	1083632149
5.0	5131406649-	1400482249	1274101549	2485586749

$X_o = 0.6$

x	d_1	d_2	d_3	d_4
0.6	2113061950	4238504450	1271551350	
0.8	2591900150	2437670650	1336217550	4653202749-
1.0	2763795550	1552074750	1134389950	5144365149-
1.2	2755511150	1022595150	9168568449	4692909849-
1.4	2636643250	6717104849	7275449949	4046398849-
1.6	2450773250	4262598249	5711479949	3409769649-
2.0	1986070950	1226216549	3419614949	2347500149-
3.0	8846803649	1222648049-	7168189448	8335487548-
4.0	2505913449	1087577649-	4063421446-	2478537548-
5.0	6061865047	5431327448-	9690992947-	5079222647-

x	d_5	d_6	d_7	d_8
0.6	1584053951-	1284153851-	1625427050-	4876341049-
0.8	9332942950-	6596466150-	1149867050-	1442033750-
1.0	6196352850-	3756586850-	1028435050-	1735360750-
1.2	4394881550-	2260008550-	9842880049-	1771223850-
1.4	3235060250-	1397594850-	9530946049-	1683548650-
1.6	2428839050-	8716961349-	9183798049-	1532393450-
2.0	1392532450-	3248222549-	8292229049-	1156848150-
3.0	2710336649-	2357734248	5582504449-	3264298749-
4.0	3359731948	3816209148	3313818849-	8079053048
5.0	7082211348	1789992648	1852267849-	1791649249

x	d_9	d_{10}	d_{11}	d_{12}
0.6	2380559951	2705645351-	7141673750	8116941950-
0.8	1347615751	1577589151-	6575271350	9459338750-
1.0	8284013250	1034088351-	5070978750	8541700150-
1.2	5151282650	7119858650-	3729917250	7272364850-
1.4	3077168350	4983357550-	2681721350	6048818550-
1.6	1639179150	3475938750-	1894494550	4959281350-
2.0	9349300048-	1564952850-	8949208049	3208617050-
3.0	1291839250-	1753383649	1036614949	7565212349-
4.0	9839284449-	3211646849	7835247748	8322858348
5.0	5111064249-	1406528649	1210189649	2373108849

$X_o = 0.8$

x	d_1	d_2	d_3	d_4
0.8	1981631050	2909351050	8728052649	
1.0	2316530950	1809769650	9042084349	2779058349-
1.2	2421989850	1185987650	7912236149	3349440949-
1.4	2386952050	7863155249	6554511649	3227357749-
1.6	2264804650	5122140549	5289460549	2886783449-
1.8	2090422050	3173468249	4196378149	2495644449-
2.0	1887644750	1766580749	3281851549	2116828849-
3.0	8800094749	1026352849-	7358533748	8028916748-
4.0	2636937349	1031132949-	1374045147	2485233248-
5.0	1571276748	5385325548-	9113737847-	5417106647-

x	d_5	d_6	d_7	d_8
0.8	9153696150-	6610918550-	1524329050-	4573017049-
1.0	6098711150-	3795059950-	1135993050-	1180538350-
1.2	4345379350-	2304008850-	9926230049-	1429366750-
1.4	3215881850-	1440055650-	9228633049-	1464088750-
1.6	2429249550-	9099339349-	8744695049-	1390335150-
1.8	1851908050-	5725299849-	8302174049-	1260204550-
2.0	1413339950-	3529708649-	7847256049-	1102919150-
3.0	2963977049-	1354055048	5367994249-	3432100149-
4.0	1978360548	3556647548	3239245049-	6023077048
5.0	6592565448	1760983348	1833411249-	1676283649

x	d_9	d_{10}	d_{11}	d_{12}
0.8	1593177751	1898043551-	4779529950	5694133350-
1.0	9722625050	1199461151-	4244712350	6605788950-
1.2	6123302650	8108548650-	3317975050	6176708450-
1.4	3795029850	5640756450-	2468618350	5396795750-
1.6	2198814850	3947753850-	1783341250	4564011450-
1.8	1073863550	2734298350-	1257622450	3778031450-
2.0	2752228049	1844674050-	8661918049	3072030050-
3.0	1152611150-	7901226048	9830201048	7847220349-
4.0	9426042449-	2947552449	6847504148	5198649948
5.0	5064606549-	1397784149	1129459349	2223107949

$X_o = 1.0$

x	d_1	d_2	d_3	d_4
1.0	1780289050	2099874350	6299623149	
1.2	2017953950	1369698950	6397547149	1760346749-
1.4	2081029750	9154454849	5673240149	2251464449-
1.6	2034035050	6095268649	4764875949	2258688649-
1.8	1918201050	3940182349	3884451149	2078766449-
2.0	1761375850	2387068049	3101281849	1834739549-
3.0	8692776449	7909491748-	7536708848	7622735548-
4.0	2774145449	9590293048-	3445084147	2478151148-
5.0	2688964248	5299308548-	8384924047-	5779233147-

x	d_5	d_6	d_7	d_8
1.0	5948108750-	3817235550-	1369453050-	4108354549-
1.2	4261510550-	2341580250-	1062757050-	9551968049-
1.4	3174200450-	1481009850-	9238076049-	1157222150-
1.6	2415162450-	9491349149-	8459364049-	1188098650-
1.8	1855987950-	6078603249-	7904166049-	1127784450-
2.0	1429165850-	3837285249-	7423850049-	1018990950-
3.0	3242425549-	1737589147	5127749649-	3565026349-
4.0	3532219047	3232184148	3150643449-	3764979048
5.0	5981187648	1716544648	1809567849-	1540671449

x	d_9	d_{10}	d_{11}	d_{12}
1.0	1122979351	1396869951-	3368938550	4190609450-
1.2	7155436450	9280950450-	2883331550	4793725150-
1.4	4568865350	6416480550-	2246722050	4561166250-
1.6	2811224850	4503097650-	1670826950	4047024150-
1.8	1573692850	3154526050-	1202886350	3458455150-
2.0	6898558049	2174625850-	8417782049	2879760050-
3.0	9873444649-	3820546048-	9569990048	8087051649-
4.0	8904819249-	2603532449	5832029548	1697928548
5.0	4989152949-	1370017349	1037576049	2043766849

$X_o = 3.0$

x	d_1	d_2	d_3	d_4
3.0	4077348849	2214164549	6642493648	
3.2	4217697349	1612664349	5873820148	7967900147-
3.4	4147874249	1134733949	5006198348	1232305148-
3.6	3933815649	7581743848	4135613648	1431607948-
3.8	3626660849	4650652548	3317919748	1479018648-
4.0	3265607849	2406128448	2583124948	1431758748-
5.0	1444702549	2249703948-	3231108347	7677092547-

x	d_5	d_6	d_7	d_8
3.0	5169691049-	1341665649-	3136422749-	9409259048-
3.2	4330810449-	9613779548-	2828993549-	1388703849-
3.4	3580228349-	6725818048-	2591169649-	1556704649-
3.6	2917138349-	4551342348-	2395507649-	1538308149-
3.8	2338674849-	2932305648-	2225659449-	1401001549-
4.0	1840368849-	1744166248-	2072061649-	1193546149-
5.0	3353114648-	5388058447	1427383649-	1731890047-

x	d_9	d_{10}	d_{11}	d_{12}
3.0	1014856450	1642141050-	3044570349	4926422149-
3.2	6846992349	1250497950-	2135588349	4912995949-
3.4	4217233849	9399573049-	1447014449	4560423649-
3.6	2153538749	6944553949-	9430601048	4019676349-
3.8	5647321048	5016050949-	5897503048	3391753349-
4.0	6283846048-	3515738649-	3563288048	2743421049-
5.0	2777205949-	7756130047-	1765476048	2111854048-

$X_o = 5.0$

x	d_1	d_2	d_3	d_4
5.0	1269279149	5590488948	1677146848	2118296440-

x	d_5	d_6	d_7	d_8
5.0	1318703149-	2032523048-	9763686048-	2929105048-

x	d_9	d_{10}	d_{11}	d_{12}
5.0	2377924749	4330661949-	7133776048	1299198649-

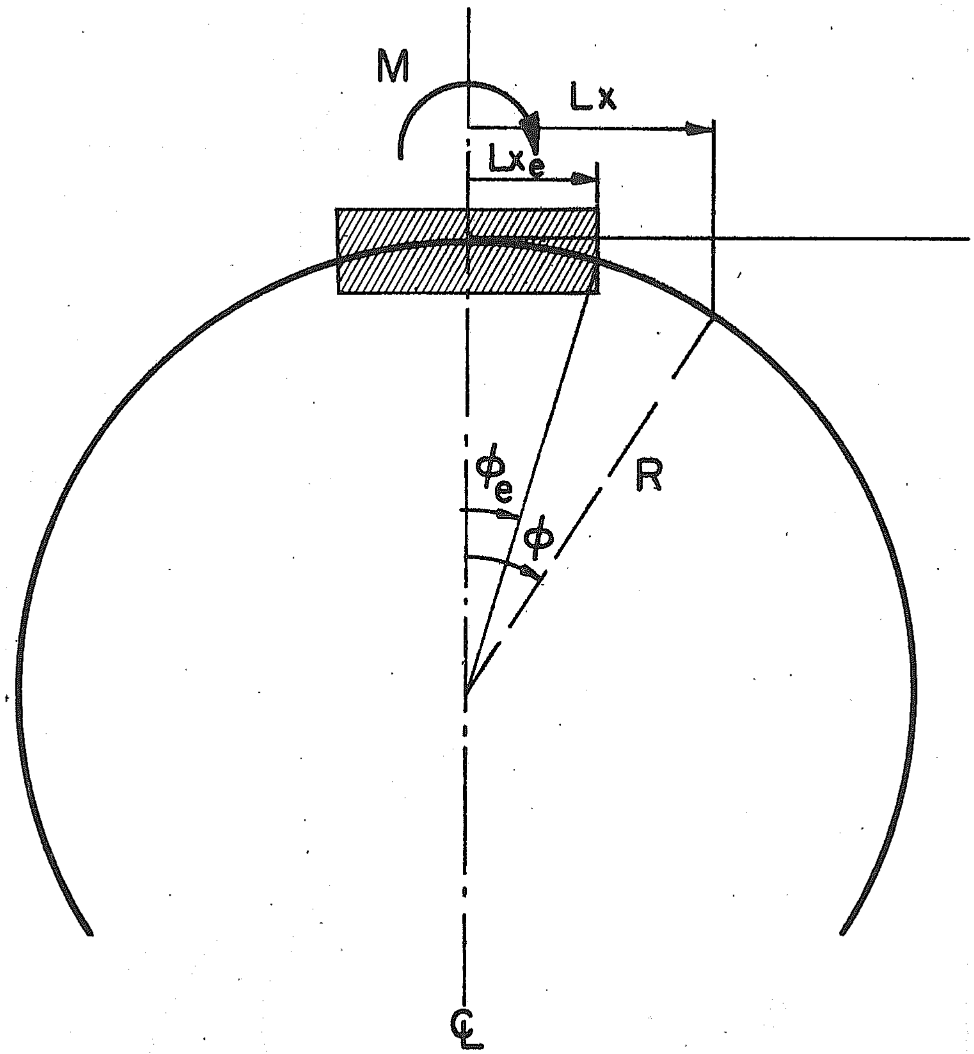
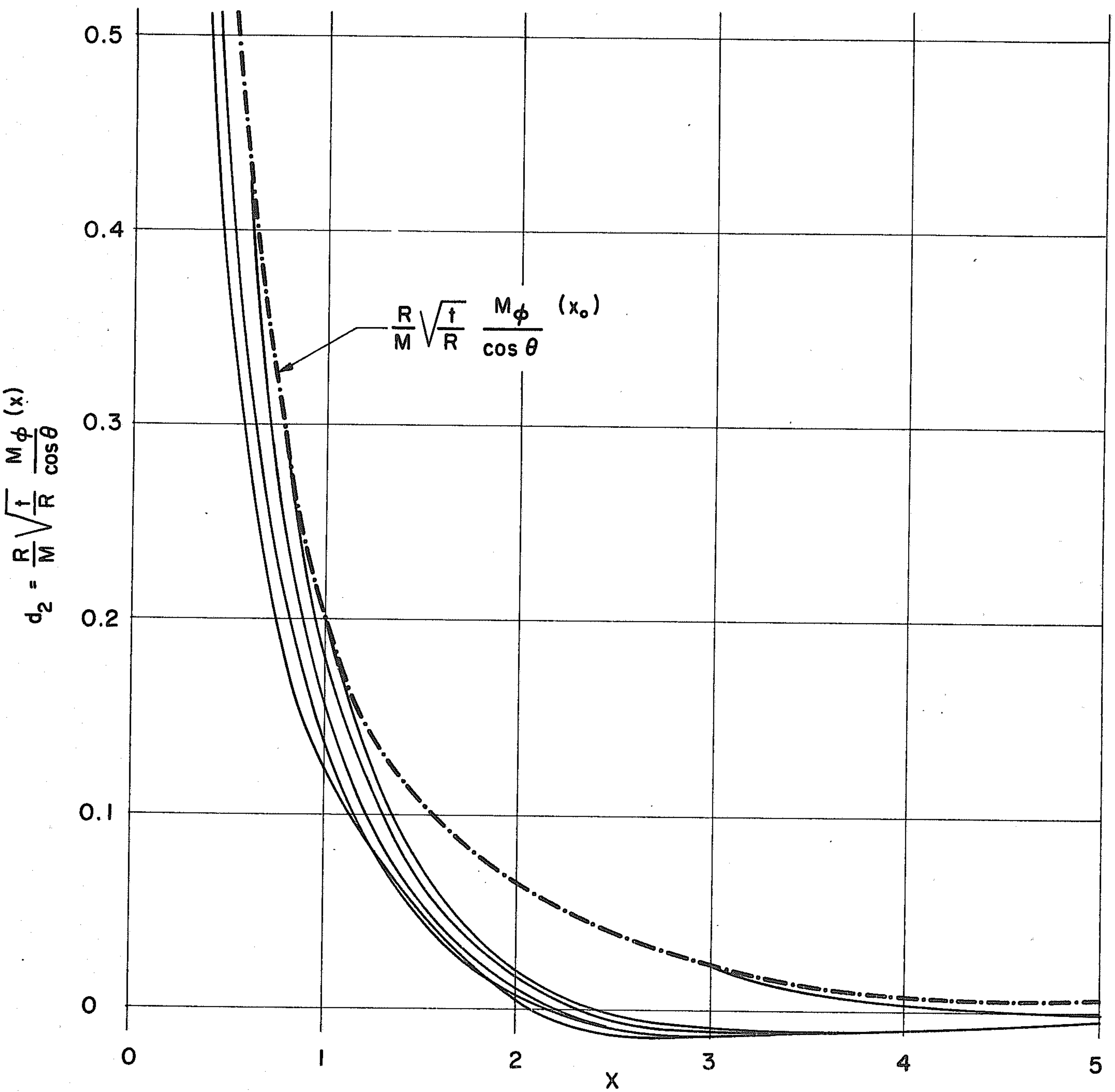
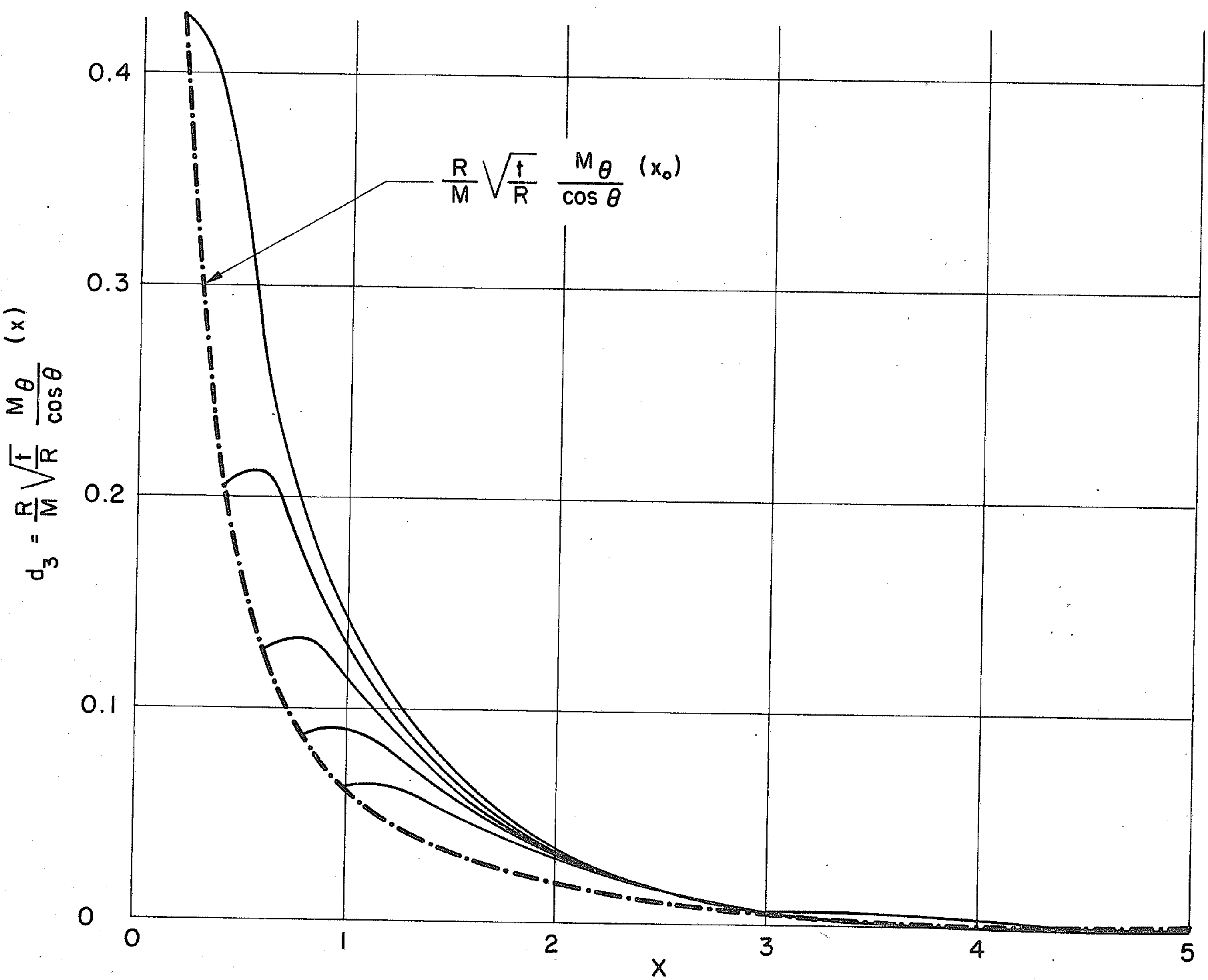


FIG. 36 – EXTERNAL MOMENT ON AN INSERT IN A SPHERICAL SHELL



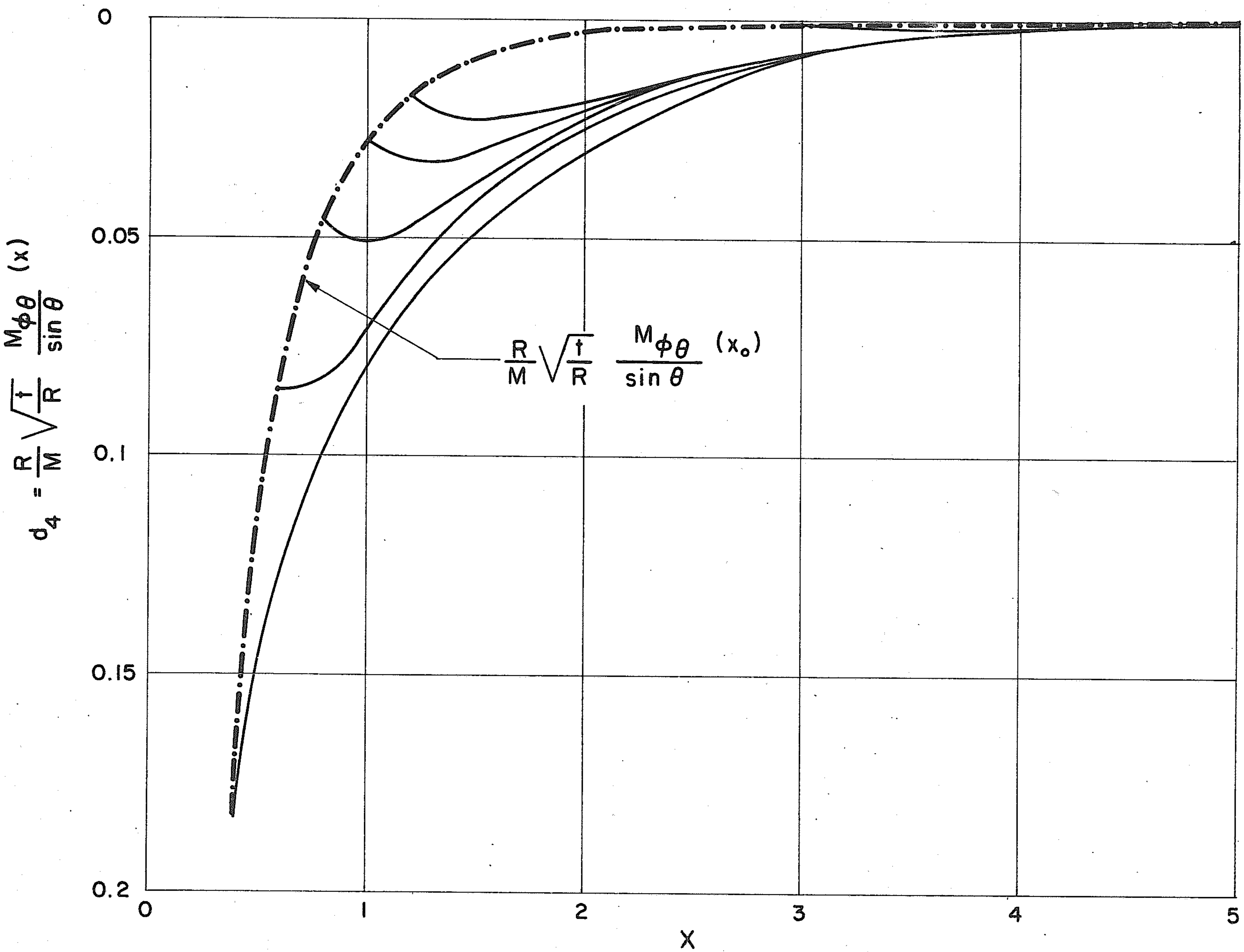
MERIDIONAL MOMENT M_ϕ FOR AN EXTERNAL MOMENT ON AN INSERT IN A SPHERICAL SHELL

FIG. 38



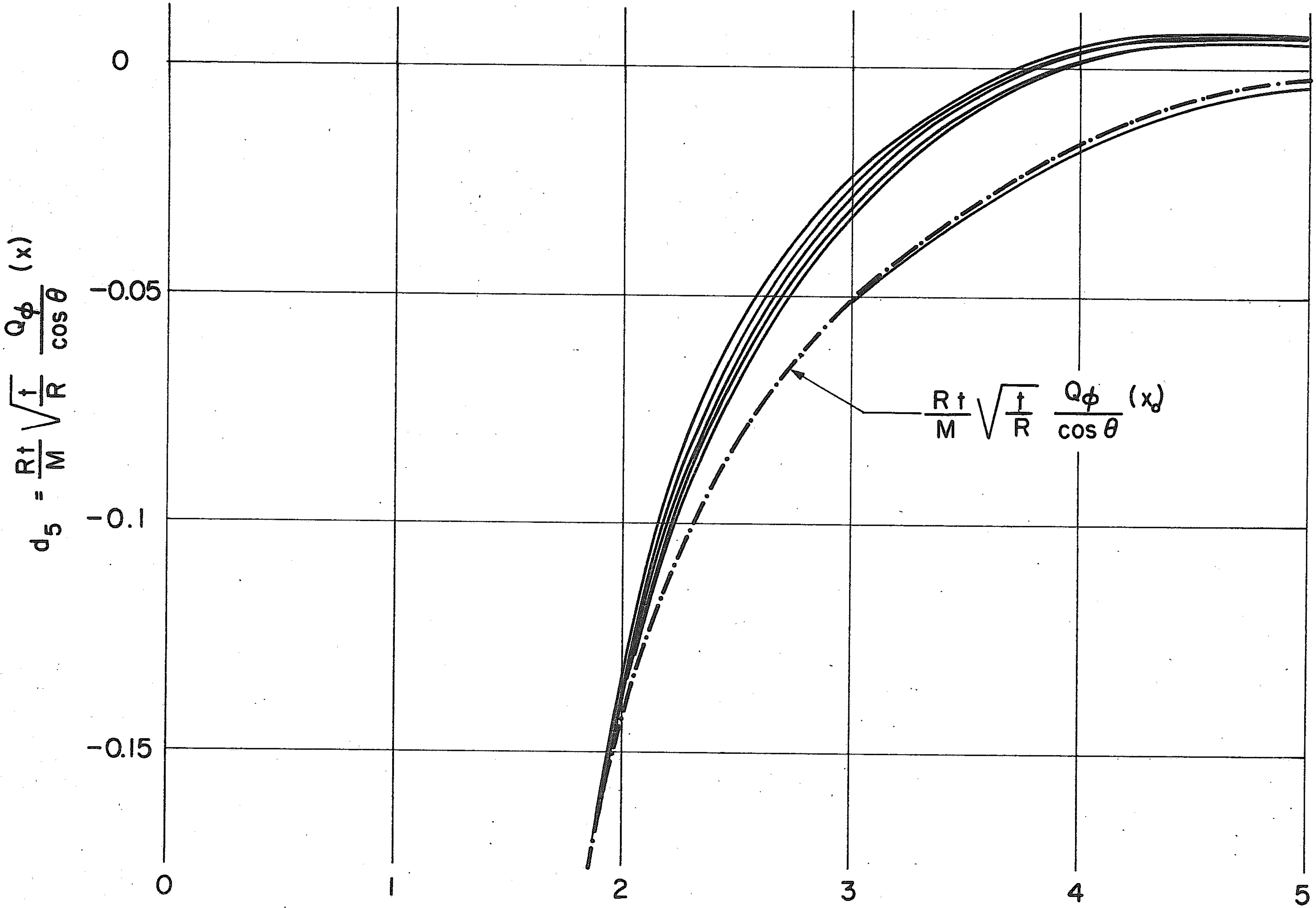
CIRCUMFERENTIAL MOMENT M_Θ FOR AN EXTERNAL MOMENT ON AN INSERT IN A SPHERICAL SHELL

FIG. 39



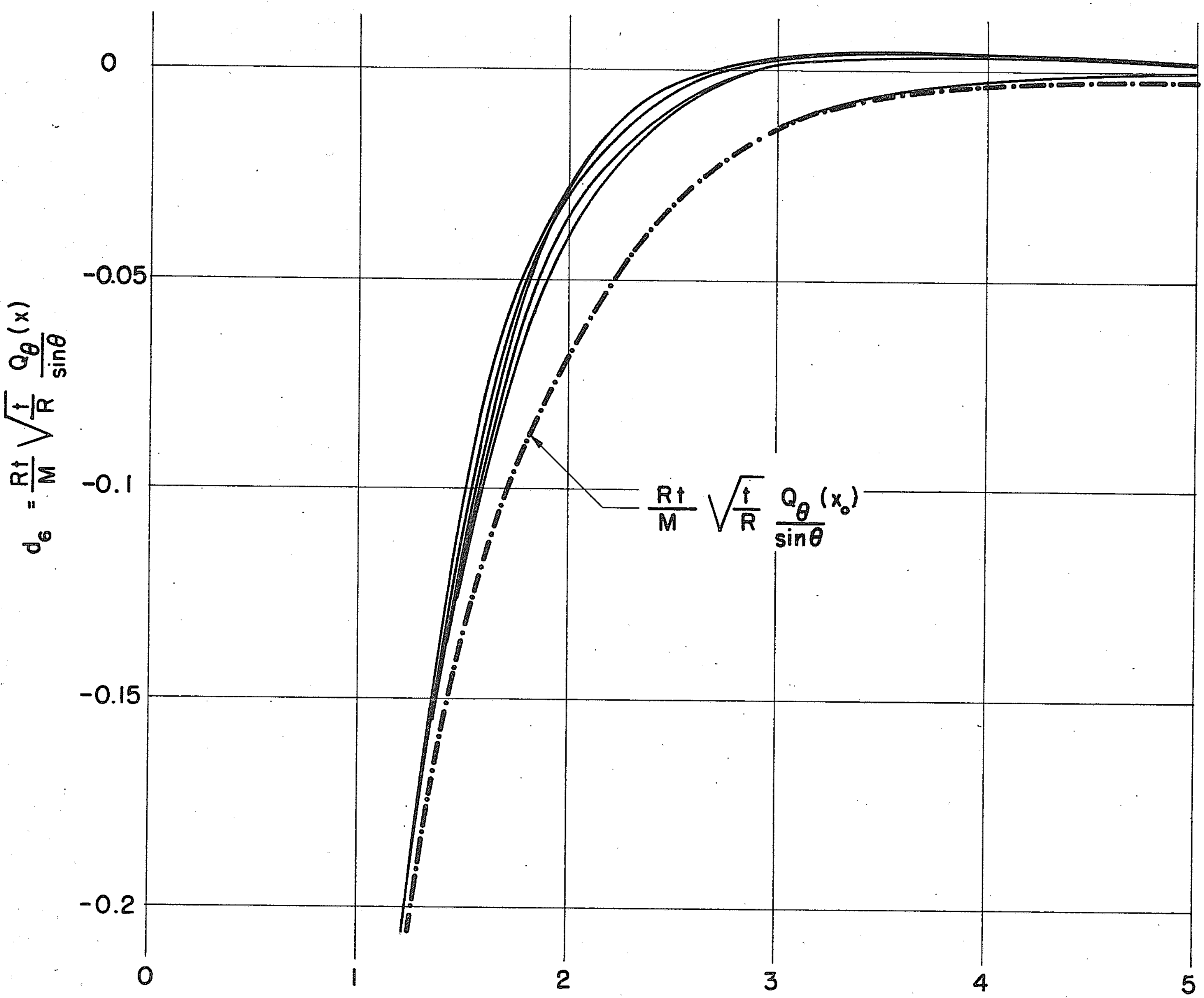
TWISTING MOMENT $M_{\phi\Theta}$ FOR AN EXTERNAL MOMENT ON AN INSERT IN A SPHERICAL SHELL

FIG. 40



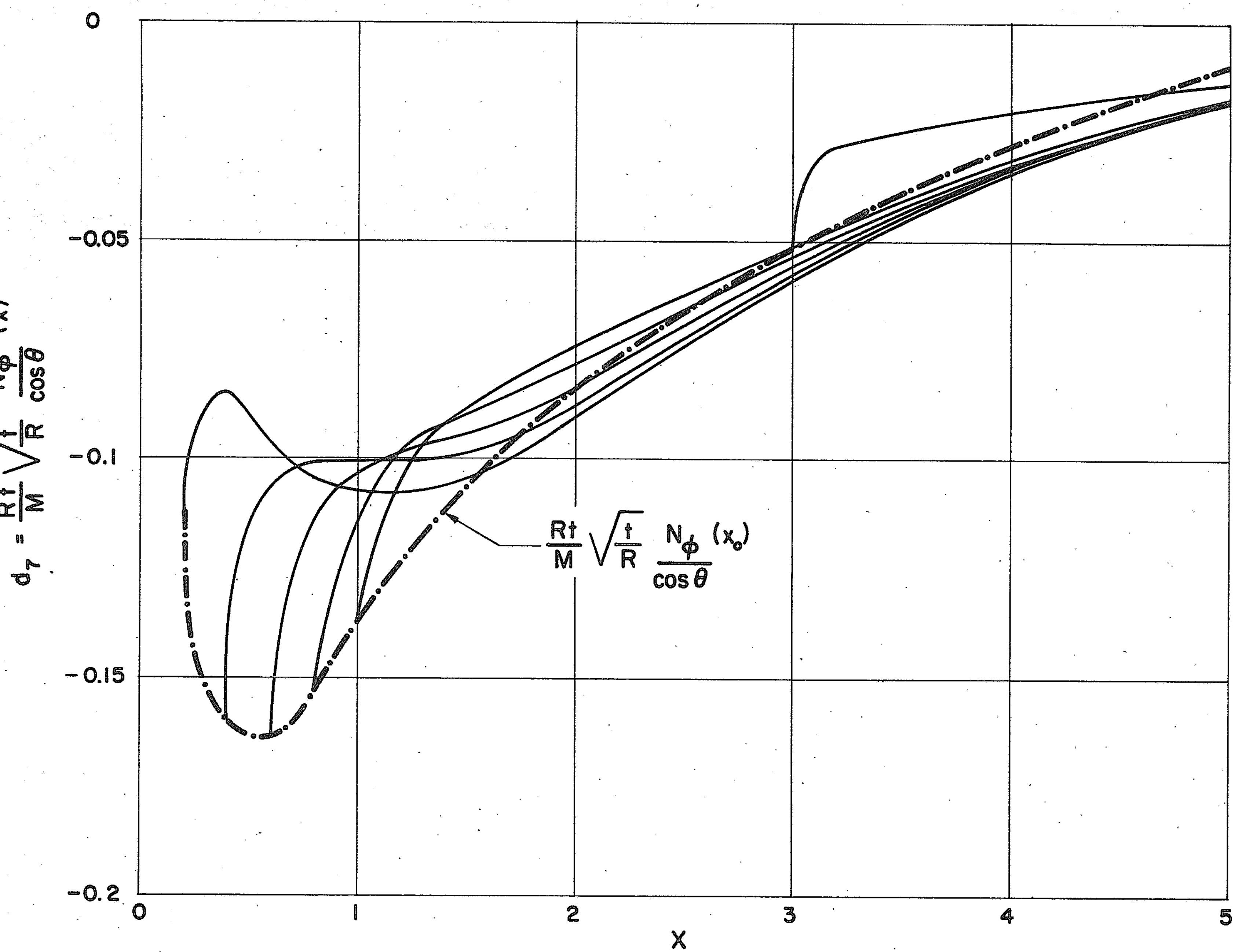
MERIDIONAL CROSS SHEAR Q_ϕ FOR AN EXTERNAL MOMENT ON AN INSERT IN A SPHERICAL SHELL

FIG. 41



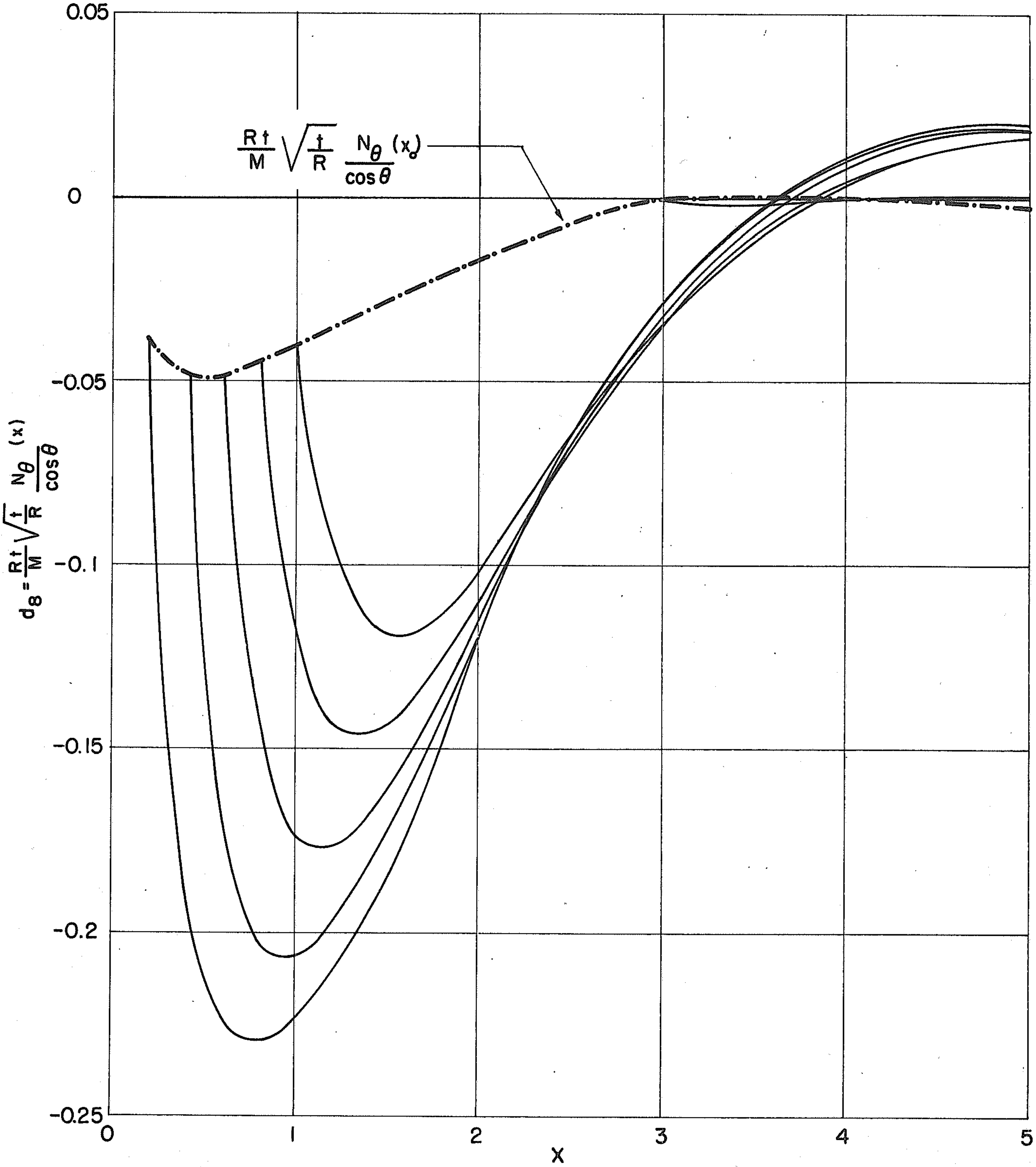
CIRCUMFERENTIAL CROSS SHEAR Q_Θ FOR AN EXTERNAL MOMENT ON AN INSERT IN A SPHERICAL SHELL

FIG. 42



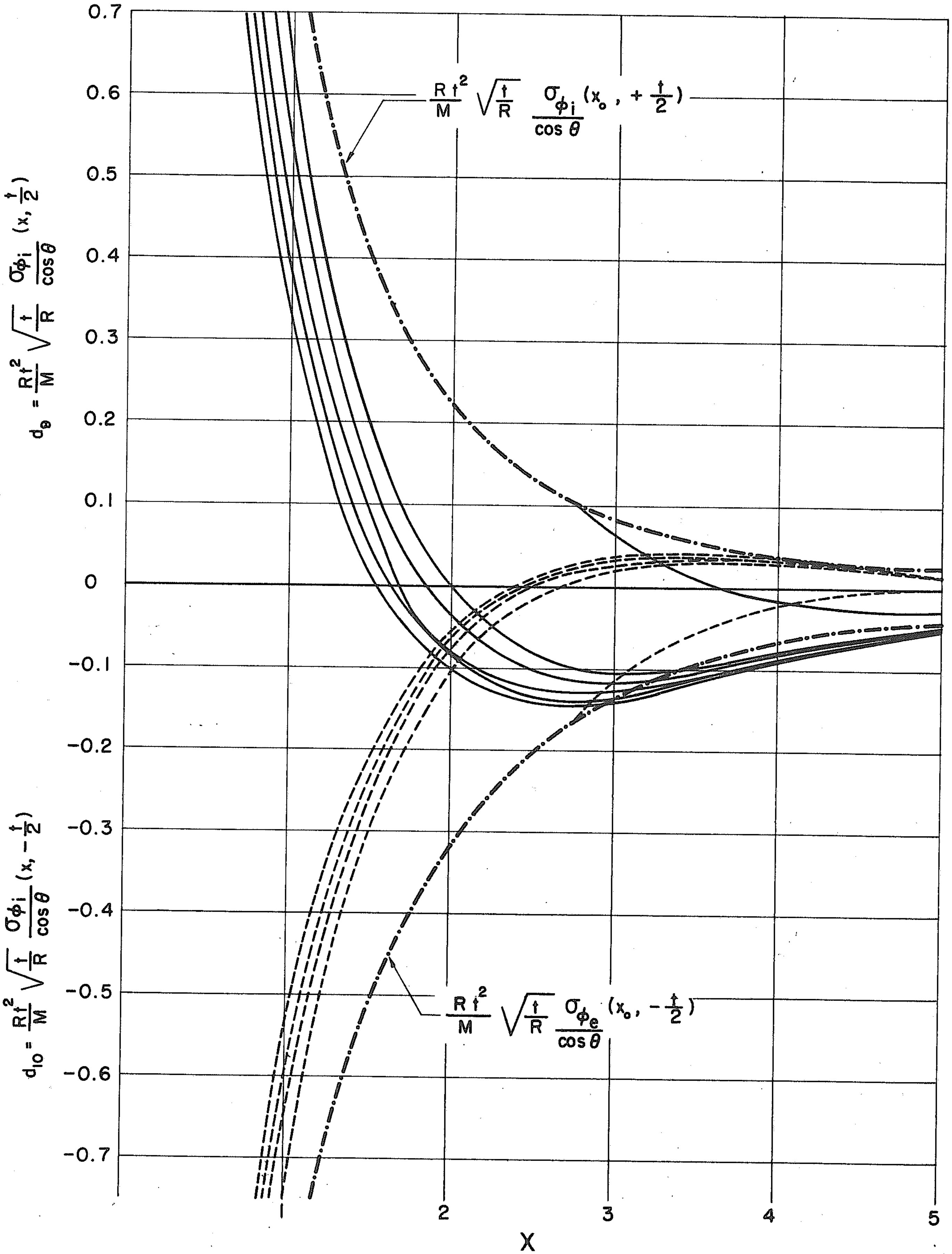
MERIDIONAL DIRECT FORCE N_ϕ FOR AN EXTERNAL MOMENT ON AN INSERT IN A SPHERICAL SHELL

FIG. 43



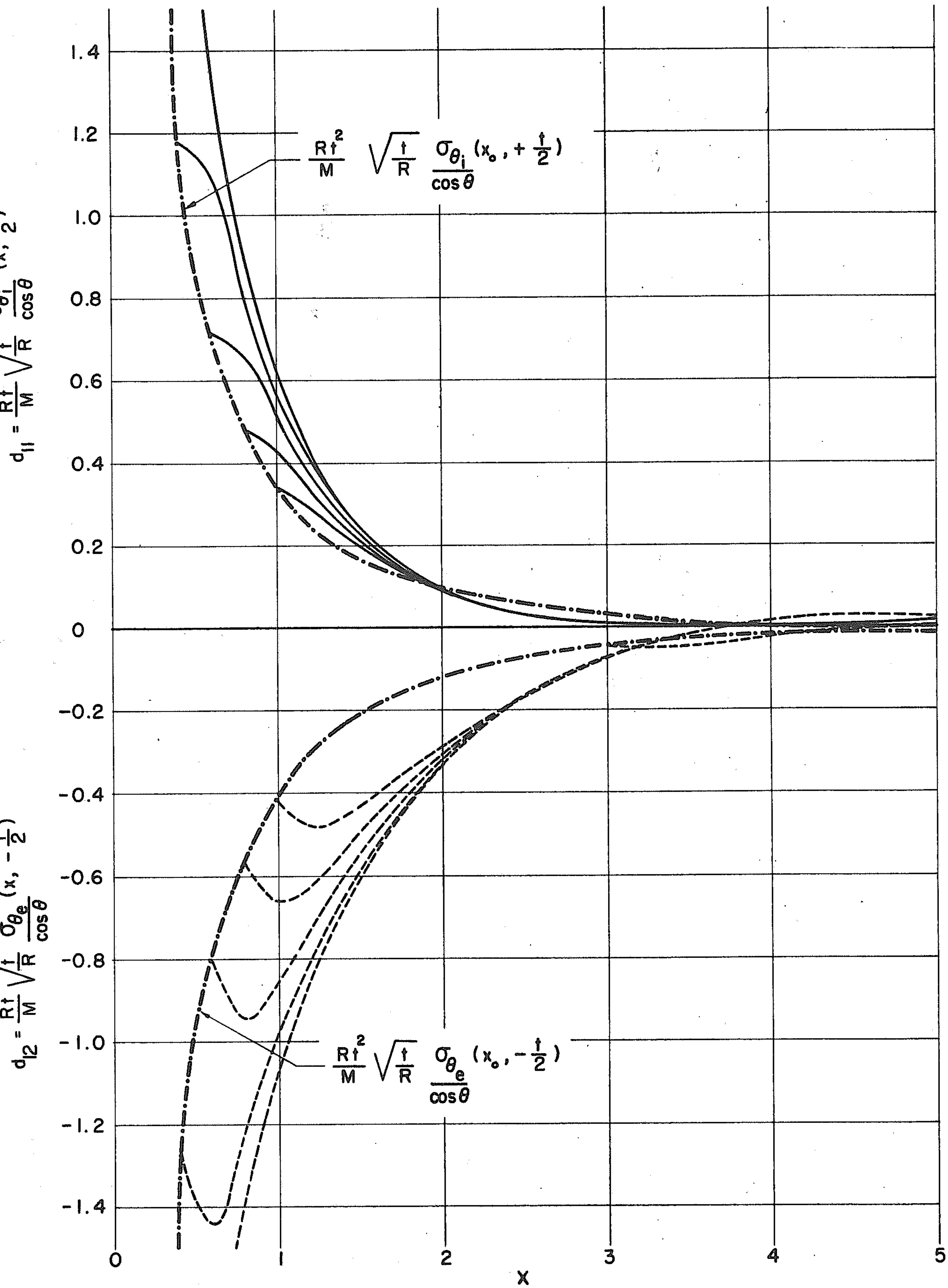
CIRCUMFERENTIAL DIRECT FORCE N_Θ FOR AN EXTERNAL MOMENT ON AN ON AN INSERT IN A SPHERICAL SHELL

FIG. 44



MERIDIONAL STRESS σ_ϕ FOR AN EXTERNAL MOMENT ON AN INSERT IN A SPHERICAL SHELL

FIG. 45



CIRCUMFERENTIAL STRESS σ_θ FOR AN EXTERNAL MOMENT ON AN INSERT IN A SPHERICAL SHELL

FIG. 46

Table VI

Tangentially Loaded Insert

$X_o = 0.2$

x	e_1	e_2	e_3	e_4
0.2	1006824448	8620791748	2586237548	7835610149-
0.4	1655243248	3015816748	2343224948	1873438549-
0.6	1930212148	1712307348	1609857648	7742518048-
0.8	2020068248	1099433348	1155321148	3941137548-
1.0	1998277948	7326437147	8558606947	2221266948-
1.2	1906244348	4876000647	6454967147	1320377348-
2.0	1265277748	3210266446	2154282447	1730660047-
3.0	5336221247	8827998146-	4149034346	2180925046
4.0	1401092247	6948185346-	1606337945-	2485133846
5.0	3755844345-	3289334946-	6248140345-	1094661046

x	e_5	e_6	e_7	e_8
0.2	8071362349-		7746000047-	2323000047-
0.4	2094022849-	1106522948-	5429903050-	5413350650
0.6	9748910348-	8432041547-	4292026650-	4272724550
0.8	5728943348-	6349932247-	3396745650-	3376543950
1.0	3787220848-	4892827447-	2783957550-	2763974750
1.2	2671039048-	3840128447-	2350520650-	2331458150
2.0	8183924647-	1583057347-	1437579550-	1424926850
3.0	1429707447-	5228795146-	9636599049-	9583236849
4.0	3070383646	1479862946-	7235480349-	7221469349
5.0	4618052046	2784511245-	5788441449-	5788816949

x	e_9	e_{10}	e_{11}	e_{12}
0.2	1735754451	5095015049	5249935049-	1528512549
0.4	7954176550	5248954050-	5610852050-	5553944150
0.6	5139936550	4189288250-	4394769050-	4369316050
0.8	3792264450	3330778650-	3462710650-	3445863250
1.0	3002406950	2739998950-	2827916150-	2815326350
1.2	2483542850	2321264650-	2379776650-	2370187950
2.0	1464614250	1435653350-	1439505750-	1437852550
3.0	9665965949	9689567049-	9583631049-	9608131049
4.0	7222856549	7277169449	7193791249-	7220505549
5.0	5775979649	5808177449	5768705449-	5785068049

X_o = 0.2 (Cont'd.)

	e_{13}
0.2	1574972549-
0.4	5272757150
0.6	4176133050
0.8	3307224650
1.0	2712623150
1.2	2292728350
2.0	1412001150
3.0	9558342649
4.0	7222433149
5.0	5792565849

$X_o = 0.4$

x	e_1	e_2	e_3	e_4
0.4	5025601148	1649246449	4947740048	7495165449-
0.6	6719842348	8190526948	5141715148	3107164349-
0.8	7391459848	4991342248	4089517548	1587264549-
1.0	7500195448	3257920248	3165797248	8982531148-
1.2	7266989248	2158485648	2445467748	5364904148-
1.4	6823317148	1404197548	1888789048	3289078348-
2.0	4971633048	1942559847	8500711147	7301613847-
3.0	2141138848	3315001947-	1692094147	7564580946
4.0	5794473047	2727252847-	4317182645-	9684978346
5.0	3481743845-	1317759747-	2444080846-	4368878746

x	e_5	e_6	e_7	e_8
0.4	8335264349-	1000000041-	3865790048-	1159790048-
0.6	3881351649-	2065044048-	2704636250-	2637437850
0.8	2283162149-	1993934548-	2737000750-	2663086150
1.0	1511837949-	1683400948-	2455066250-	2380064350
1.2	1068597049-	1383017048-	2167835950-	2095166050
1.4	7833555048-	1128869148-	1922136150-	1853902950
2.0	3317470448-	6090725547-	1409801450-	1360085050
3.0	6057328547-	2066611347-	9611820349-	9397706449
4.0	1068781247	5961569346-	7242214149-	7184269449
5.0	1798850747	1160921946-	5794492849-	5794841049

x	e_9	e_{10}	e_{11}	e_{12}
0.4	1011067551	9508899449	1028205750-	2852665049
0.6	6092370750	2213204650-	3196067850-	2945940750
0.8	4320059550	2437520250-	3036481250-	2908457250
1.0	3331298250	2259591050-	2650541450-	2570012150
1.2	2702764450	2038326850-	2297345050-	2241894150
1.4	2268750450	1837884350-	2006388050-	1967230250
2.0	1520170650	1398146050-	1421456850-	1411089350
3.0	9740302049	9810720449-	9412920249-	9499232149
4.0	7195921049	7405849349-	7078578949-	7181679149
5.0	5745722249	5873558449-	5715427249-	5780176549

$X_o = 0.4$

	e_{13}
0.4	3084623049-
0.6	2328934950
0.8	2417715150
1.0	2190116550
1.2	1948437950
1.4	1740575650
2.0	1309080750
3.0	9296180849
4.0	7186859749
5.0	5809505549

$X_o = 0.6$

x	e_1	e_2	e_3	e_4
0.6	1150995049	2308733849	6926291048	6994847349-
0.8	1411820449	1327810949	7278441048	3593126949-
1.0	1505452649	8454225448	6179076248	2046230949-
1.2	1500940249	5570124548	4994163348	1231037549-
1.4	1436192249	3658839248	3962972448	7612766048-
1.6	1334948049	2321857748	3111070648	4748170648-
2.0	1081822449	6679260247	1862680748	1769322448-
3.0	4818896448	6659822647-	3904547347	1284269147
4.0	1364983148	5924087547-	2213365045-	2078707547
5.0	3301927146	2958470147-	5278730546-	9750175546

x	e_5	e_6	e_7	e_8
0.6	8628417449-		8853760048-	2656190048
0.8	5083698449-	2534621948-	1644842750-	1503660750
1.0	3375183049-	2802160448-	1907655950-	1757110650
1.2	2393913149-	2556250748-	1861853550-	1711759550
1.4	1762152849-	2204092848-	1738902250-	1595283050
1.6	1323000349-	1857317948-	1603979950-	1470485150
2.0	7585190848-	1278694548-	1361566150-	1253383850
3.0	1476333448-	4540380047-	9562264849-	9080375149
4.0	1830062147	1350071247-	7250719349-	7114221049
5.0	3857714647	2766677046-	5803934749-	5800632749

x	e_9	e_{10}	e_{11}	e_{12}
0.6	7662029950	1296702750	1473777950	3890101649
0.8	5193785950	8481562049-	2441529250-	1940367250
1.0	3878708550	1400402450-	2414909450-	2127855250
1.2	3069943250	1527646050-	2196061050-	2011409350
1.4	2525277850	1519371950-	1958432650-	1833061350
1.6	2135053350	1464668450-	1743291450-	1657149350
2.0	1616641250	1321490550-	1401641750-	1365144650
3.0	9888968549	9961854249-	9162675449-	9314647949
4.0	7161900349	7606164649-	6895274149-	7112893049
5.0	5698513149	5981442949-	5626426549-	5768960349

$X_o = 0.6$ (Cont'd)

	e_{13}
0.6	4421339649 –
0.8	1066954250
1.0	1386366050
1.2	1412109750
1.4	1357504750
1.6	1283820950
2.0	1141623050
3.0	8846102349
4.0	7115549049
5.0	5832305149

$X_o = 0.8$

x	e_1	e_2	e_3	e_4
0.8	1918940349	2817310949	8451932048	6401775949-
1.0	2243245249	1752515849	8756029048	3674999549-
1.2	2345367849	1148467749	7661925048	2231119249-
1.4	2311438449	7614396948	6347153648	1394498149-
1.6	2193155349	4960096748	5122123648	8811472948-
1.8	2024289549	3073072548	4063621748	5544174448-
2.0	1827927349	1710693148	3178027248	3418042748-
3.0	8521695548	9938831047-	7125739347	1311218247
4.0	2553515248	9985120047-	1330575846	3444129547
5.0	1521567947	5214955547-	8825415746-	1705272947

x	e_5	e_6	e_7	e_8
0.8	8864109849-	1000000042	1476105049-	4428360048-
1.0	5905772649-	2691140048-	1151551150-	9272266049
1.2	4207909149-	3243478348-	1435558250-	1201021550
1.4	3114144249-	3125257348-	1481130850-	1249987050
1.6	2352397849-	2795457448-	1440859750-	1221544150
1.8	1793321249-	2416692448-	1370222750-	1167793750
2.0	1368627549-	2049860948-	1291126550-	1108333850
3.0	2870208748-	7774914347-	9477966749-	8625797249
4.0	1915773247	2406610547-	7257314149-	7001962649
5.0	6384003447	5245731346-	5815774449-	5800558749

x	e_9	e_{10}	e_{11}	e_{12}
0.8	6391571650	1542776050	1837997050-	4628323249
1.0	4634813350	1000416049-	2203060650-	1452588350
1.2	3581497650	7464776049-	2124638850-	1660737050
1.4	2886157850	1024267050-	1937994650-	1630816250
1.6	2396045750	1143253950-	1738465550-	1528871550
1.8	2034104750	1185838450-	1554607150-	1411611050
2.0	1757520250	1188484950-	1393768150-	1299015450
3.0	1014186350	1007429750-	8881636849-	9053341649
4.0	7135521049	7856421349-	6658206949-	7009946149
5.0	5639314349	6128671749-	5502877149-	5747606249

$X_o = 0.8$

	e_{13}
0.8	5513995249-
1.0	4018648649
1.2	7413060049
1.4	8691578049
1.6	9142167049
1.8	9239764049
2.0	9176522049
3.0	8198252849
4.0	6993979149
5.0	5853511249

$X_o = 1.0$

x	e_1	e_2	e_3	e_4
1.0	2693699649	31772544492	9531763048	5775740149-
1.2	3053303249	2072448749	9679929048	3542972049-
1.4	3148741449	1385132049	8584003048	2240869549-
1.6	3077634949	9222561248	7209585048	1436106449-
1.8	2902369949	5961767348	5877441748	9197345148-
2.0	2665082949	3611798348	4692452648	5806076648-
3.0	1315276949	1196760648-	1140355948	2629089346
4.0	4197472948	1451077448-	5212649446	4890517047
5.0	4068587847	8018218547-	1268696847-	2597250247

x	e_5	e_6	e_7	e_8
1.0	8999897549-	1000000042	2072077049-	6216230048-
1.2	6447958649-	2663525648-	8974924049-	5921621049
1.4	4802783649-	3406620748-	1151970550-	8370964049
1.6	3654307049-	3417551248-	1229891650-	9221281049
1.8	2808237449-	3145316548-	1230831450-	9405944049
2.0	2162426149-	2776087148-	1197271350-	9307630049
3.0	4906012848-	1153372448-	9348255649-	8032978749
4.0	5344487846	3749613047-	7257609849-	6837862549
5.0	9049948147	8744376046-	5828709949-	5788024049

x	e_9	e_{10}	e_{11}	e_{12}
1.0	5579156750	1699144950	2113560350-	5097434849
1.2	4227176650	3459768049	2140961650-	1172957850
1.4	3346982250	3208913049-	1983049750-	1352136650
1.6	2733594750	6765379049-	1783245350-	1354703250
1.8	2285009150	8731254049-	1588537450-	1293240950
2.0	1945230850	9805634049-	1413979250-	1212310250
3.0	1053099650	1006631250-	8630199249-	8717192249
4.0	7134338349	8128256249-	6386963449-	6869138449
5.0	5574637049	6309803049-	5347616849-	5711902249

$X_o = 1.0$

	e_{13}
1.0	6340680849-
1.2	1136636048
1.4	3220562249
1.6	4895530049
1.8	5879479049
2.0	6492158449
3.0	7348765249
4.0	6806586649
5.0	5864145849

$X_o = 3.0$

x	e_1	e_2	e_3	e_4
3.0	5552379249	3015165249	9045496148	1827029449-
3.2	5743500449	2196065249	7998745648	1309168049-
3.4	5648418149	1545237649	6817251048	9158963848-
3.6	5356920849	1032453149	5631721648	6197845148-
3.8	4938649449	6333083048	4518217548	3993102648-
4.0	4446980749	3276574348	3517601548	2375139448-
5.0	1967341149	3063561548-	4400000747	7337253847

x	e_5	e_6	e_7	e_8
3.0	7039889249-	2000000041-	4271060949-	1281318049-
3.2	5897533249-	1085038548-	4947247549-	7962529048-
3.4	4875419149-	1678106548-	5412981449-	2354367048-
3.6	3972448349-	1949509348-	5717743749-	3608227048
3.8	3184718849-	2014071648-	5899113549-	9604643048
4.0	2506144449-	1949714748-	5986072549-	1539091649
5.0	4566144148-	1045437348-	5647029949-	3679688749

x	e_9	e_{10}	e_{11}	e_{12}
3.0	2576258050	1381993050	2236205250-	4145979749
3.2	2214182450	8229144049	1812363950-	4002994549
3.4	1903704450	3858444249	1468440750-	3854913949
3.6	1638456250	4769749048	1191246250-	3739855749
3.8	1412882850	2099263749-	9698963349-	3671394849
4.0	1222048750	4020127949-	7952917149-	3649652549
5.0	6483036849	7485166849-	3808893049-	3943688749

$X_o = 3.0$

	e_{13}
3.0	6708615749-
3.2	5595500349-
3.4	4325787349-
3.6	3018210349-
3.8	1750466249-
4.0	5714693048-
5.0	3415688749

$X_o = 5.0$

x	e_1	e_2	e_3	e_4
5.0	4801266349	2114698649	6344096248	7688367648-

x	e_5	e_6	e_7	e_8
5.0	4988221149-	1000000041-	3693282349-	1107983849-

x	e_9	e_{10}	e_{11}	e_{12}
5.0	1625177450	8994910049	1638147450-	2698473949

	e_{13}
5.0	4914441549 -

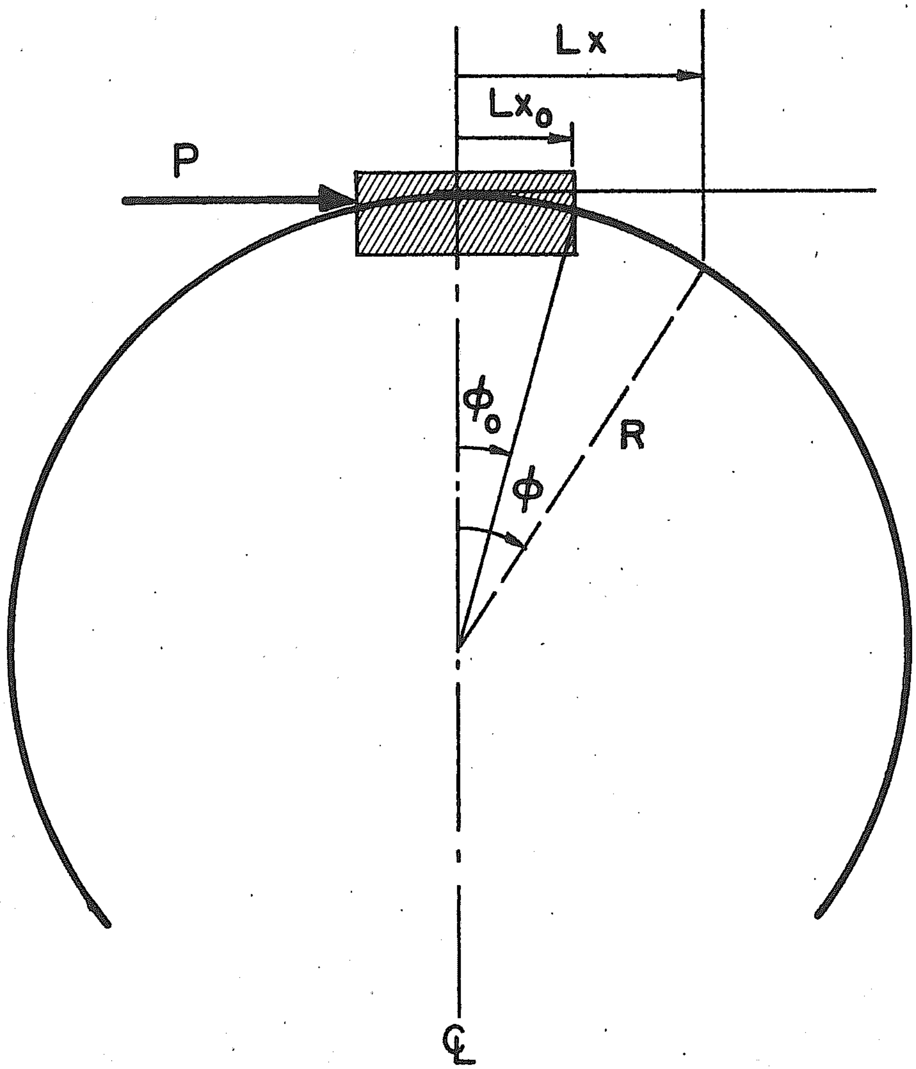


FIG. 47 – TANGENTIAL LOADING OF AN INSERT IN A SPHERICAL SHELL

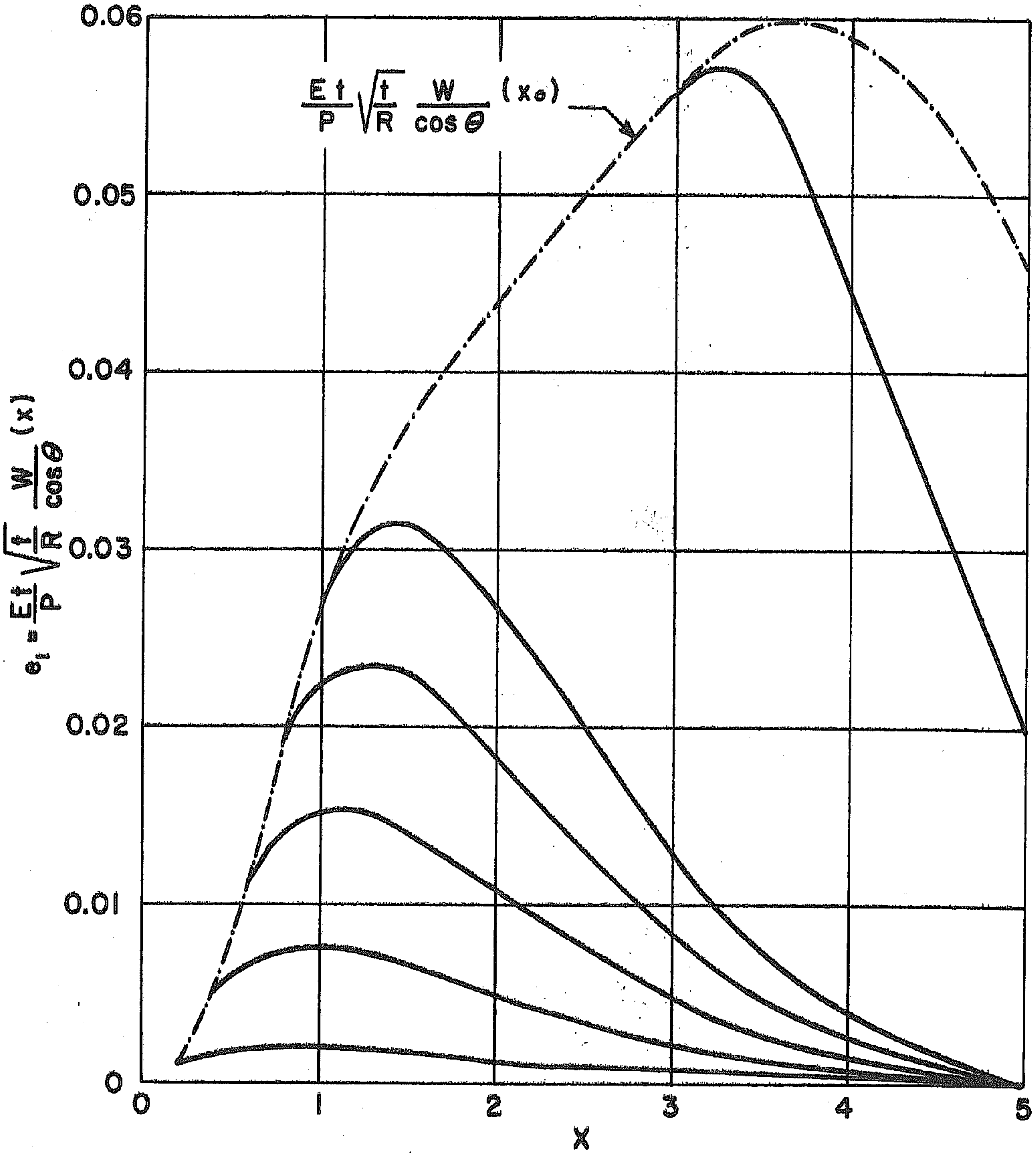


FIG. 48 RADIAL DEFLECTION W FOR A TANGENTIALLY LOADED INSERT IN A SPHERICAL SHELL

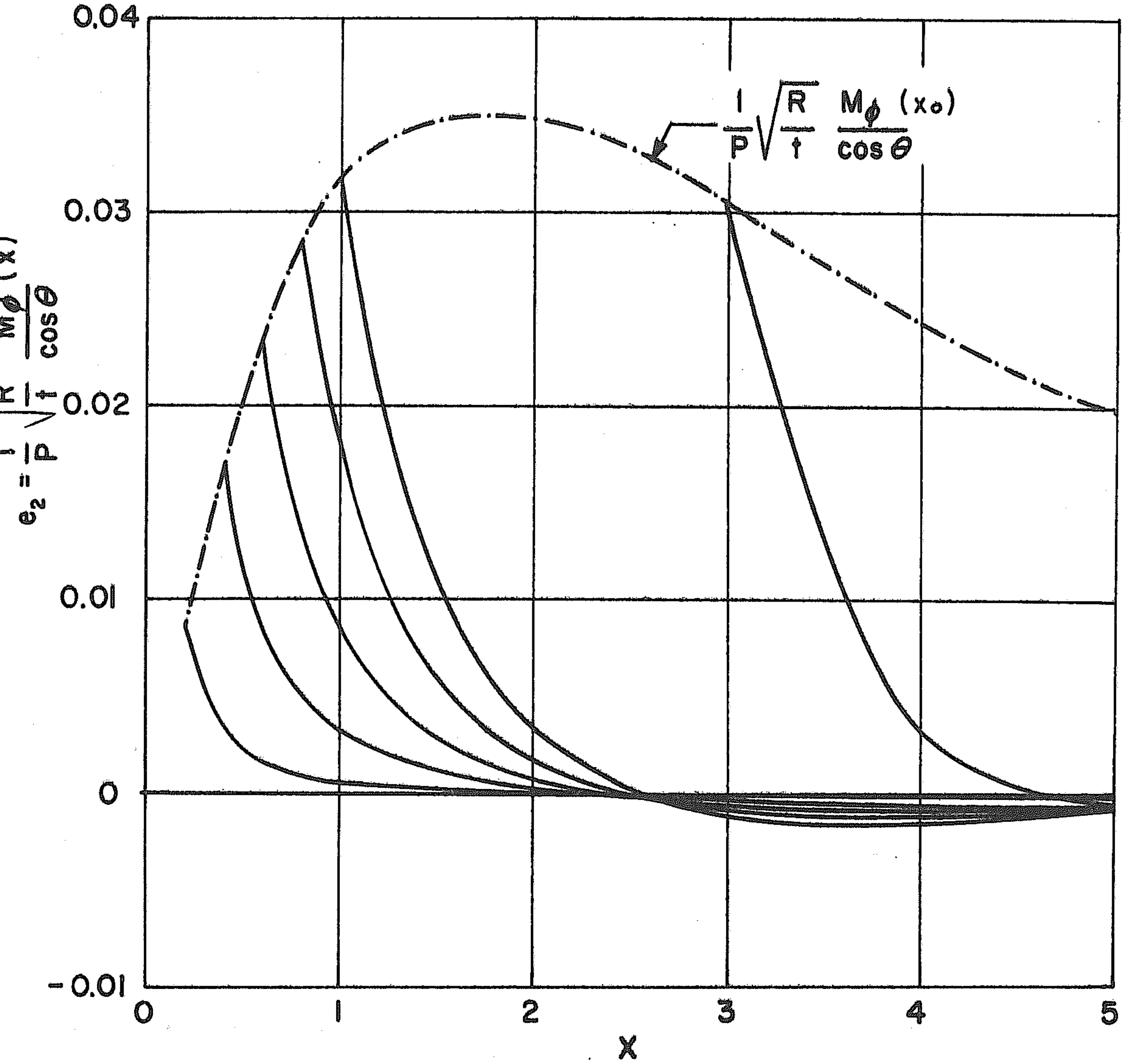


FIG. 49 MERIDIONAL MOMENT M_ϕ FOR A TANGENTIALLY LOADED INSERT IN A SPHERICAL SHELL

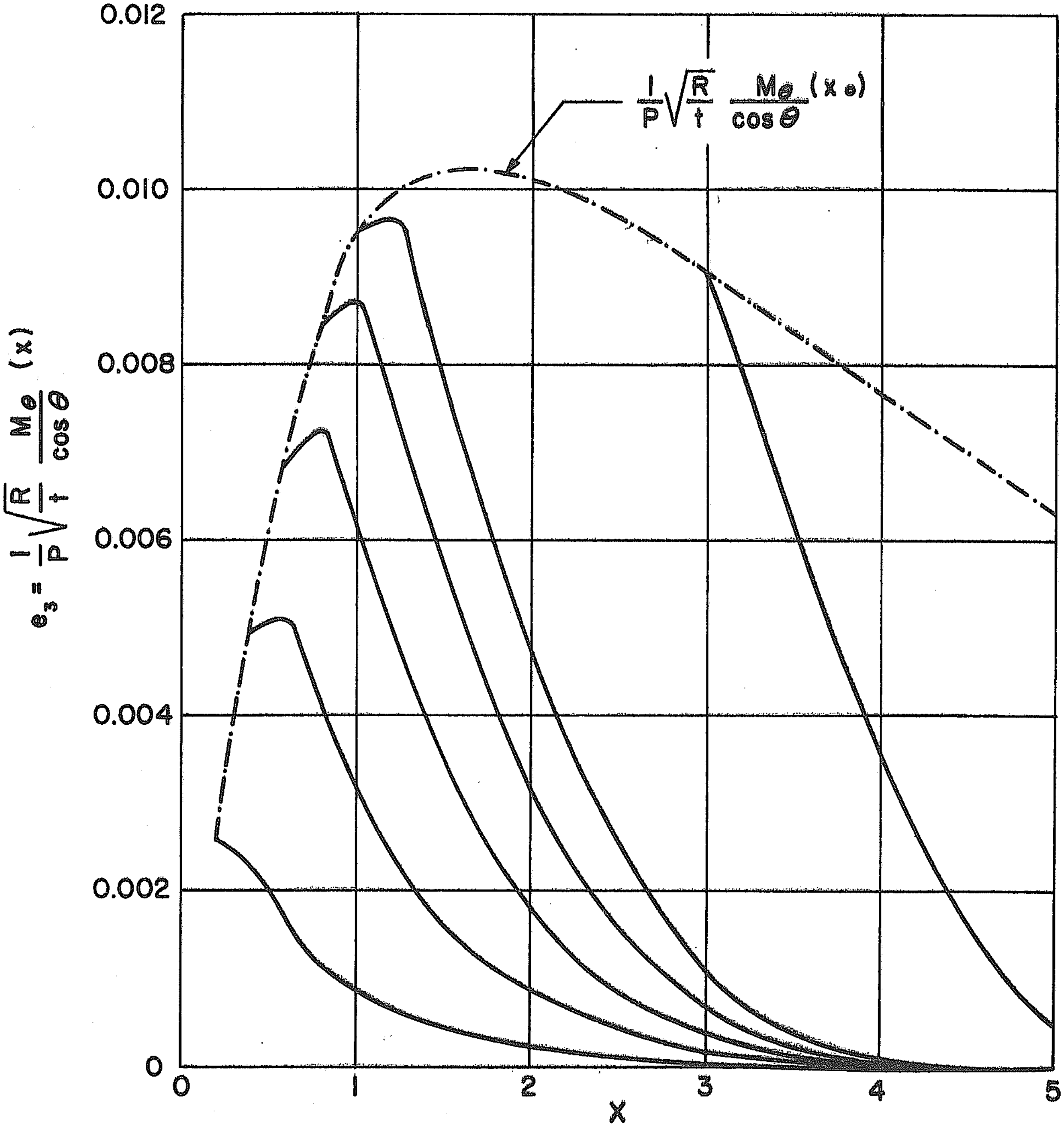


FIG. 50 CIRCUMFERENTIAL MOMENT M_θ FOR A TANGENTIALLY LOADED INSERT IN A SPHERICAL SHELL

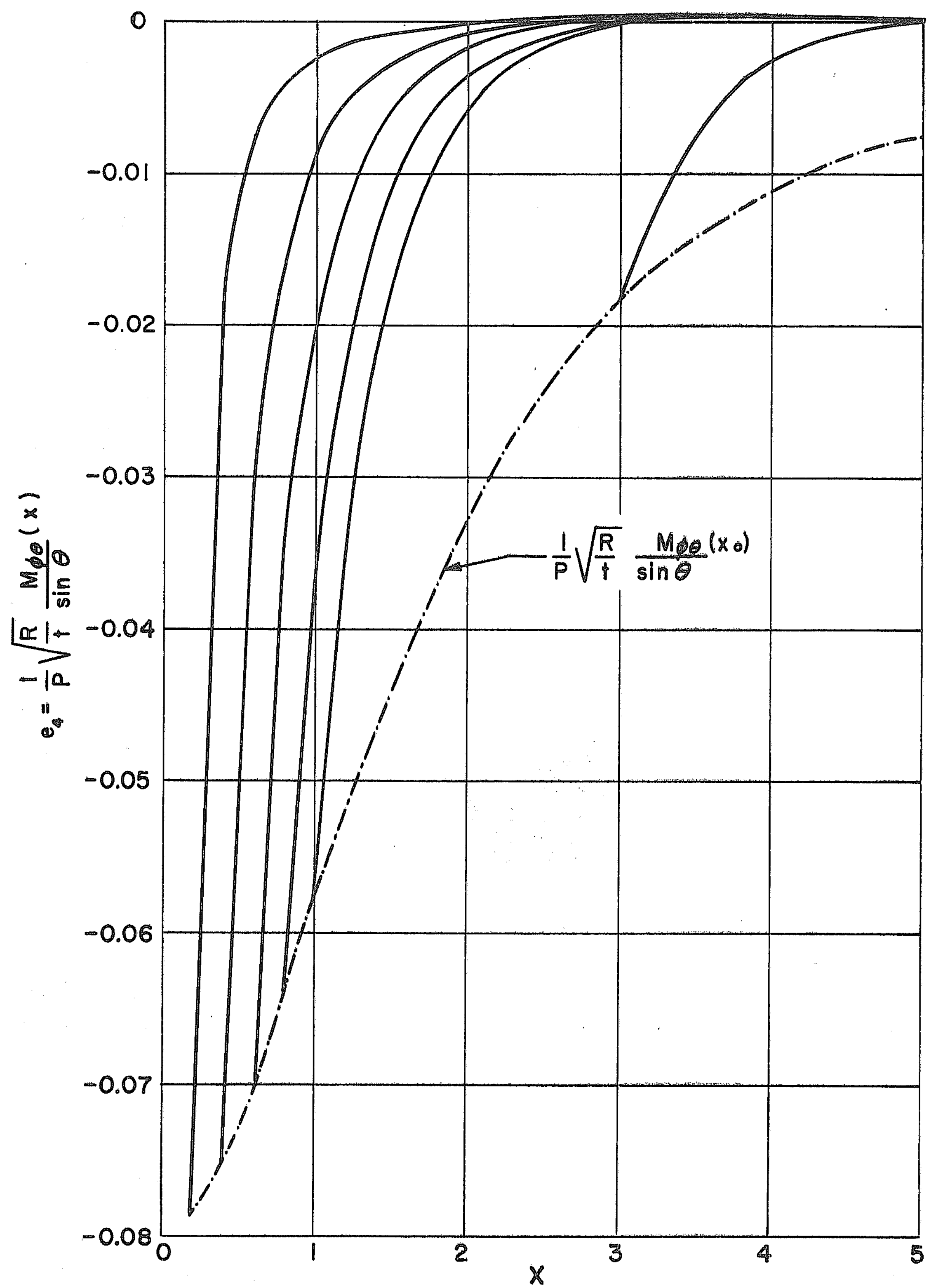


FIG. 51. TWISTING MOMENT $M_{\phi\theta}$ FOR A TANGENTIALLY LOADED INSERT IN A SPHERICAL SHELL

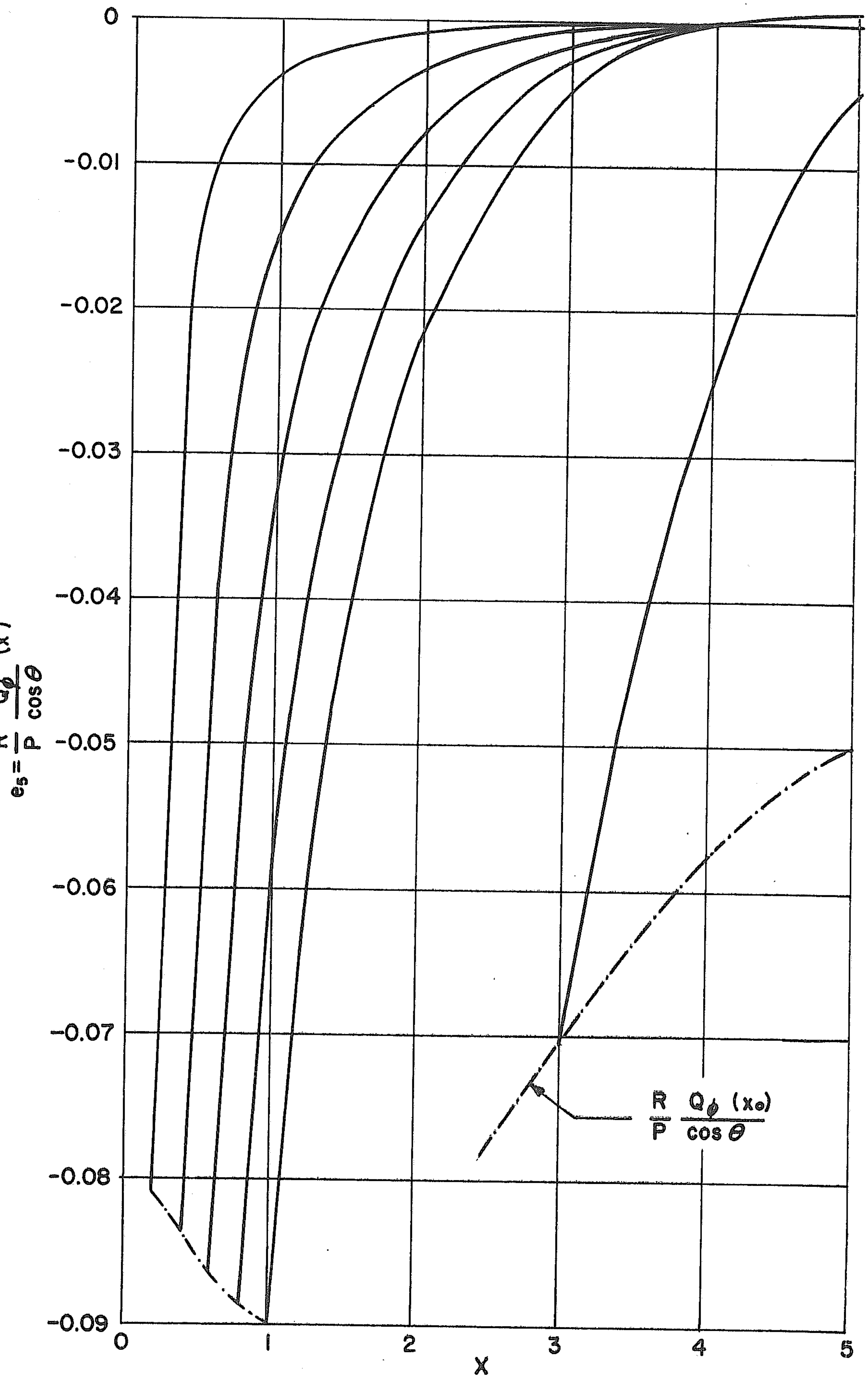


FIG. 52. MERIDIONAL CROSS SHEAR Q_ϕ FOR A TANGENTIALLY LOADED INSERT IN A SPHERICAL SHELL

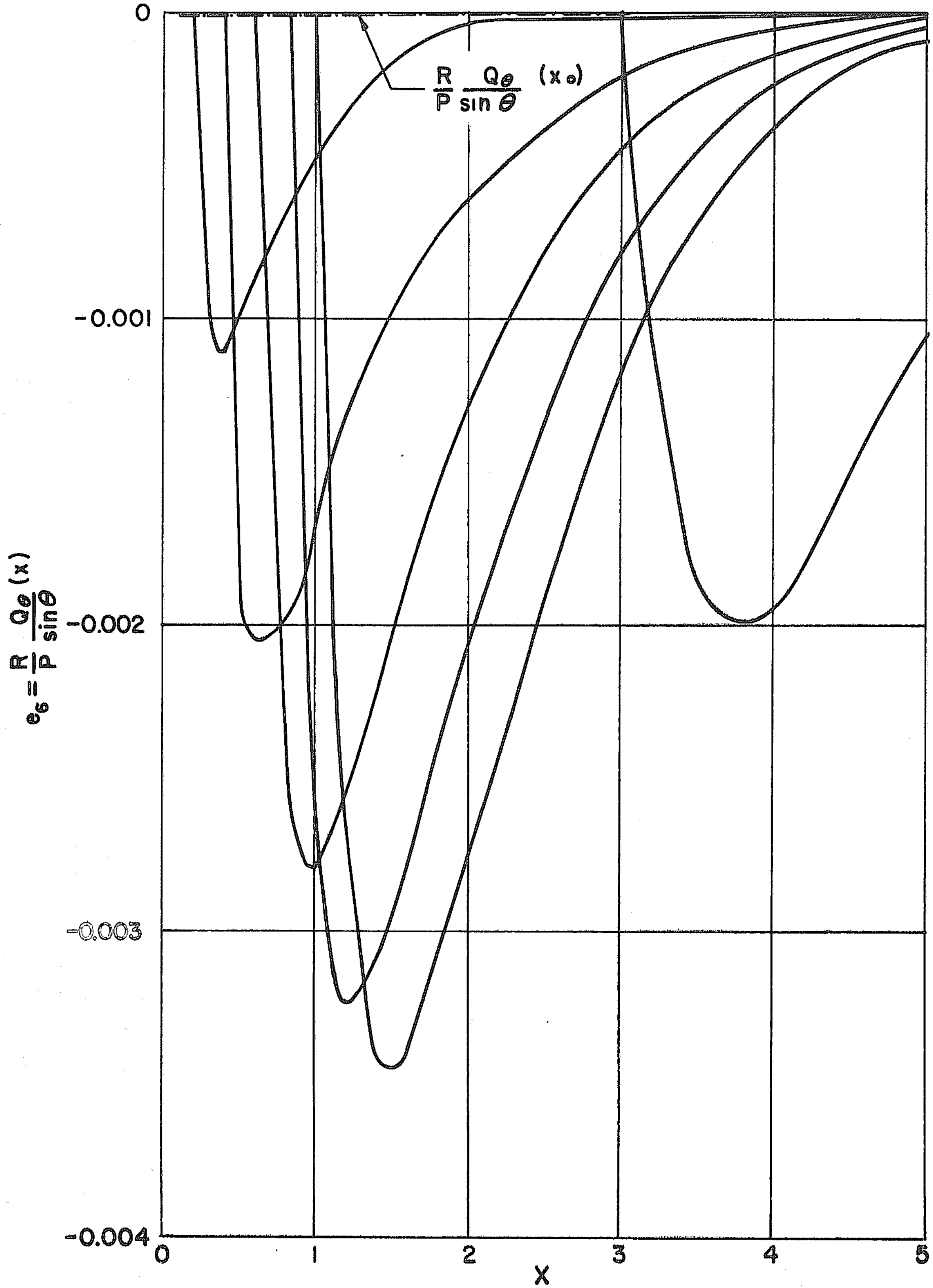


FIG. 53. CIRCUMFERENTIAL CROSS SHEAR Q_Θ FOR A TANGENTIALLY LOADED INSERT IN A SPHERICAL SHELL

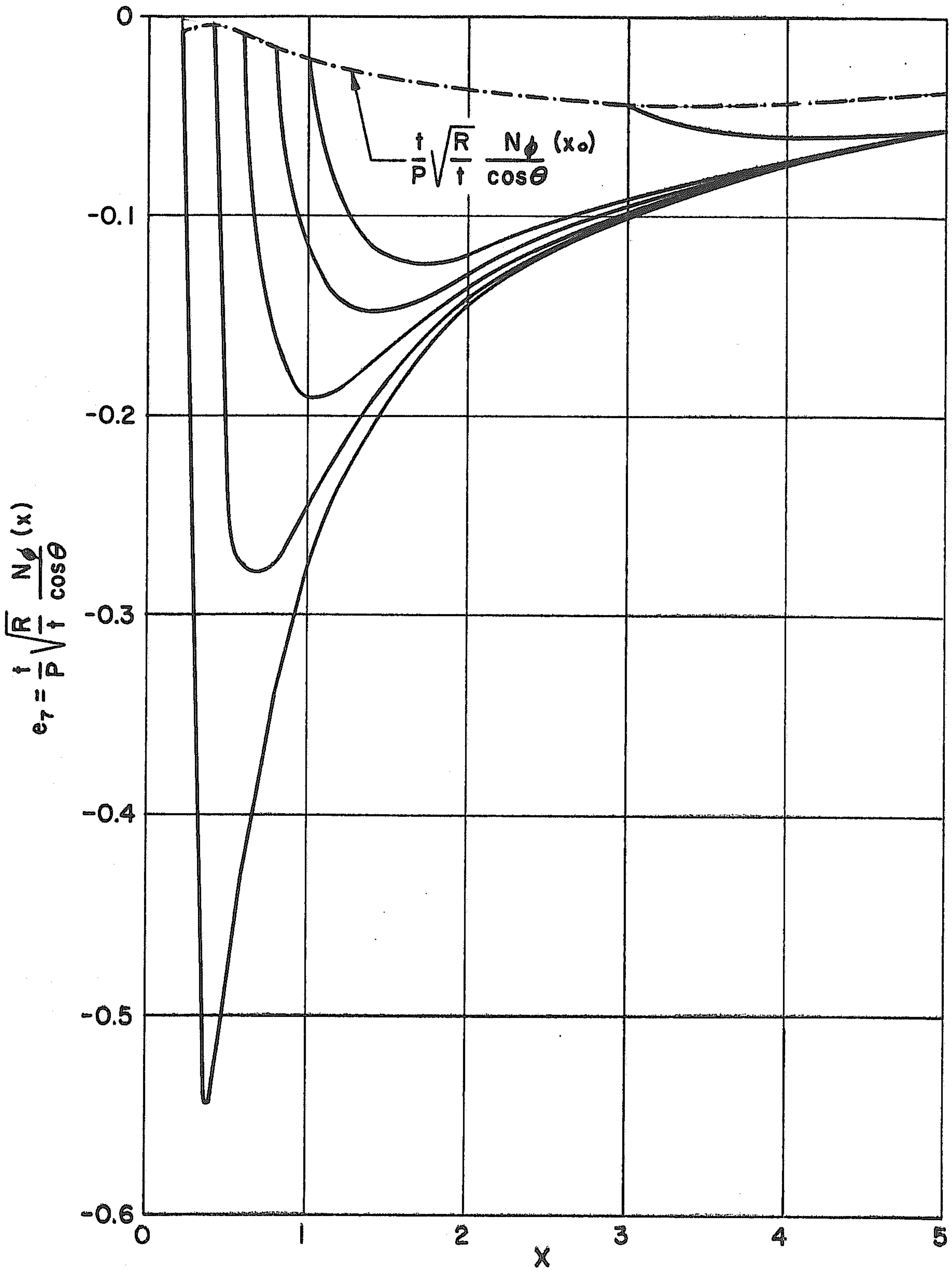


FIG. 54. MERIDIONAL DIRECT FORCE N_ϕ FOR A TANGENTIALLY LOADED INSERT IN A SPHERICAL SHELL

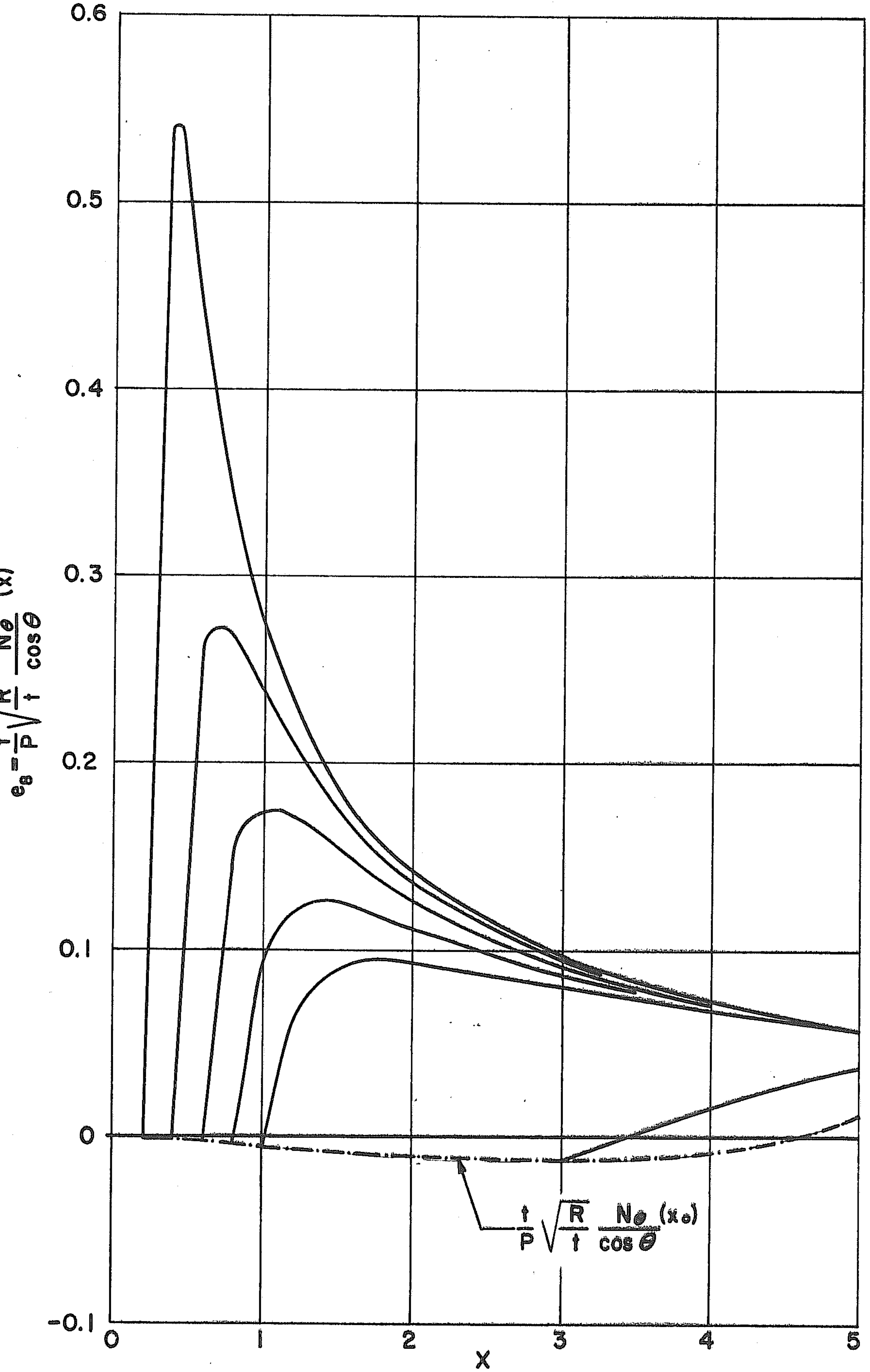


FIG. 55. CIRCUMFERENTIAL DIRECT FORCE N_Θ FOR A TANGENTIALLY LOADED INSERT IN A SPHERICAL SHELL

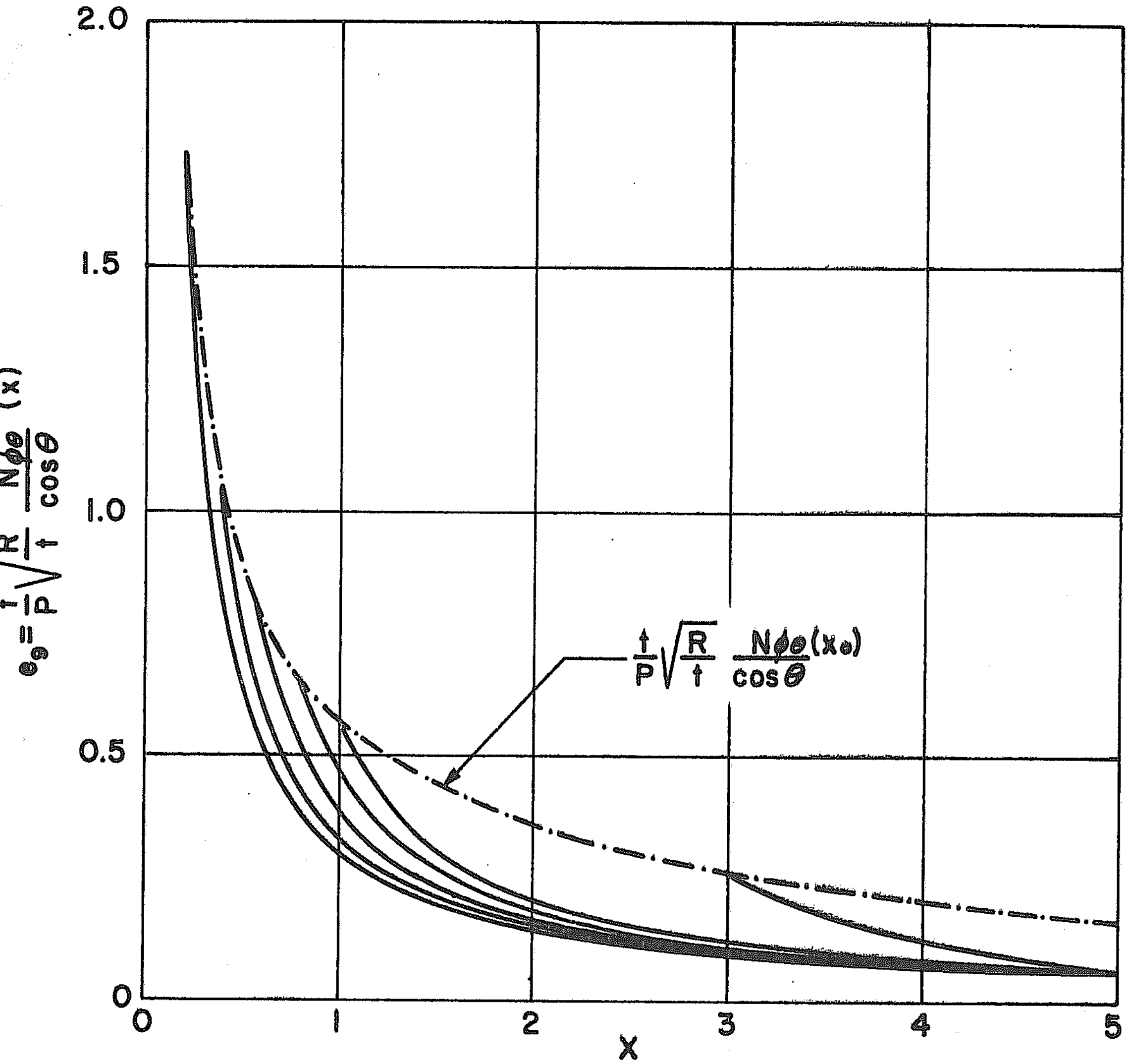


FIG. 55 a. IN-PLANE SHEAR FORCE $N_{\phi\theta}$ FOR A TANGENTIALLY LOADED INSERT IN A SPHERICAL SHELL

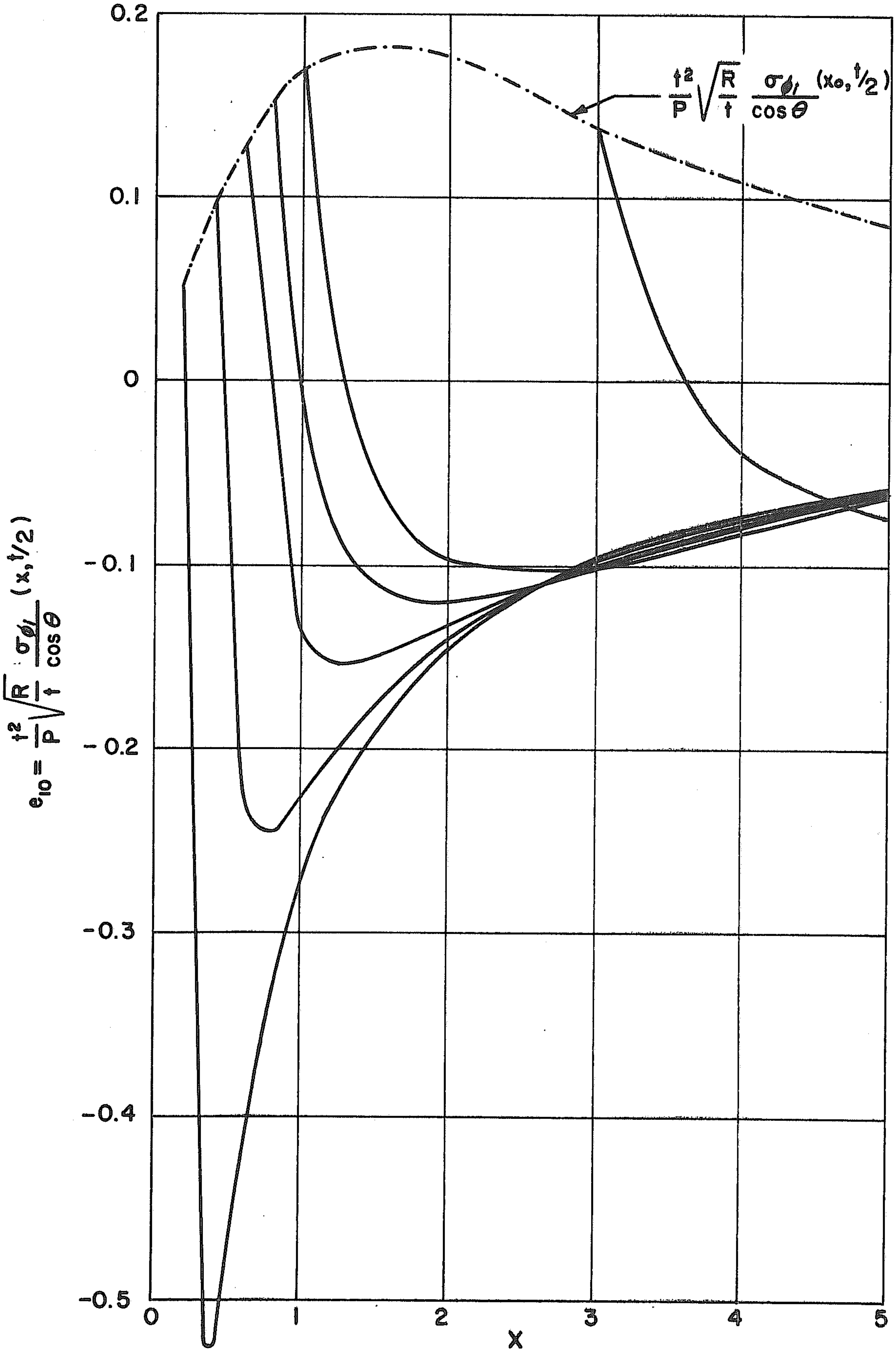


FIG. 56. MERIDIONAL STRESS σ_{ϕ_i} AT THE INTERIOR FACE FOR A TANGENTIALLY LOADED INSERT IN A SPHERICAL SHELL

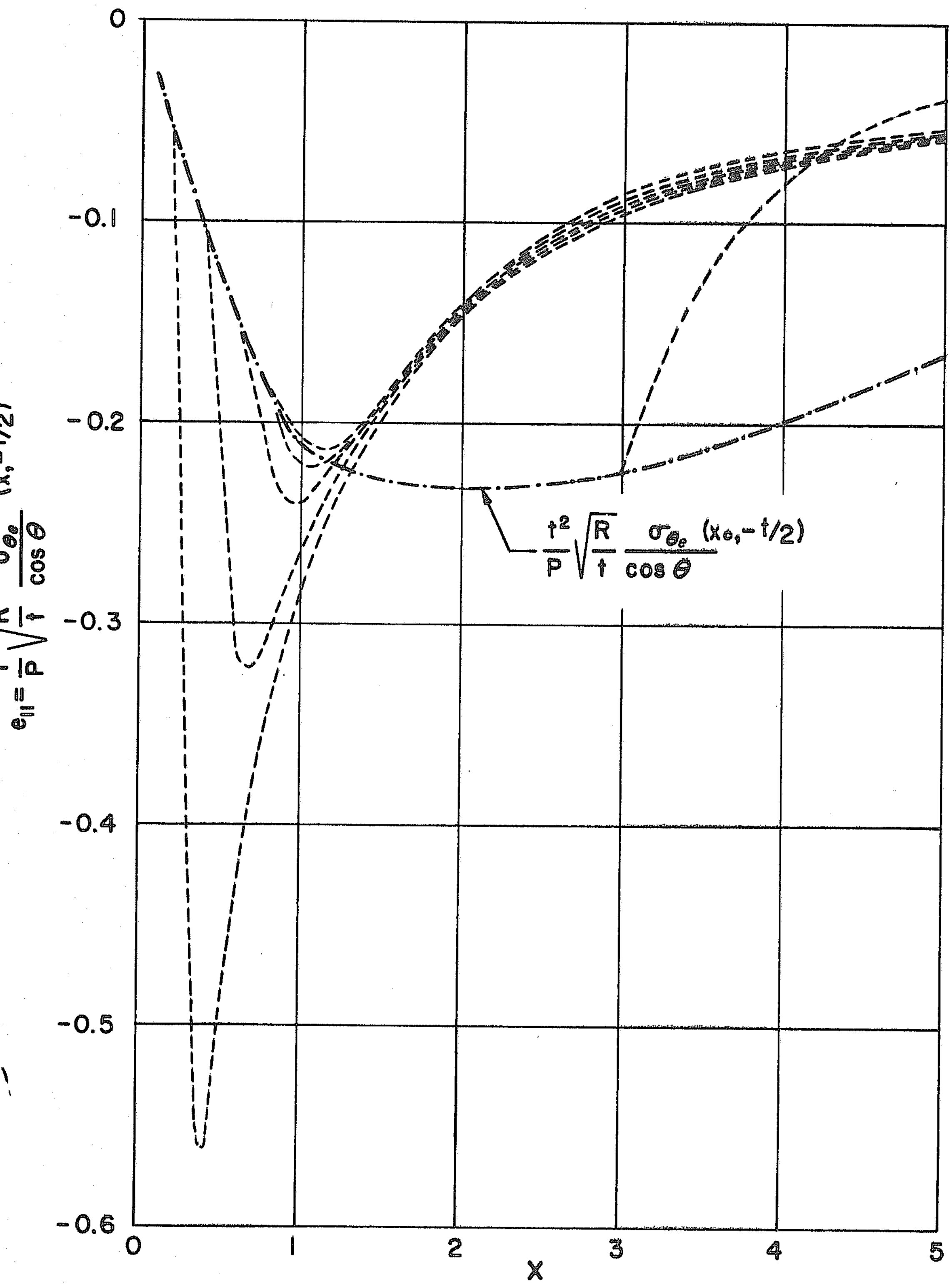


FIG. 56a. MERIDIONAL STRESS σ_{ϕ_e} AT THE EXTERIOR FACE FOR A TANGENTIALLY LOADED INSERT IN A SPHERICAL SHELL

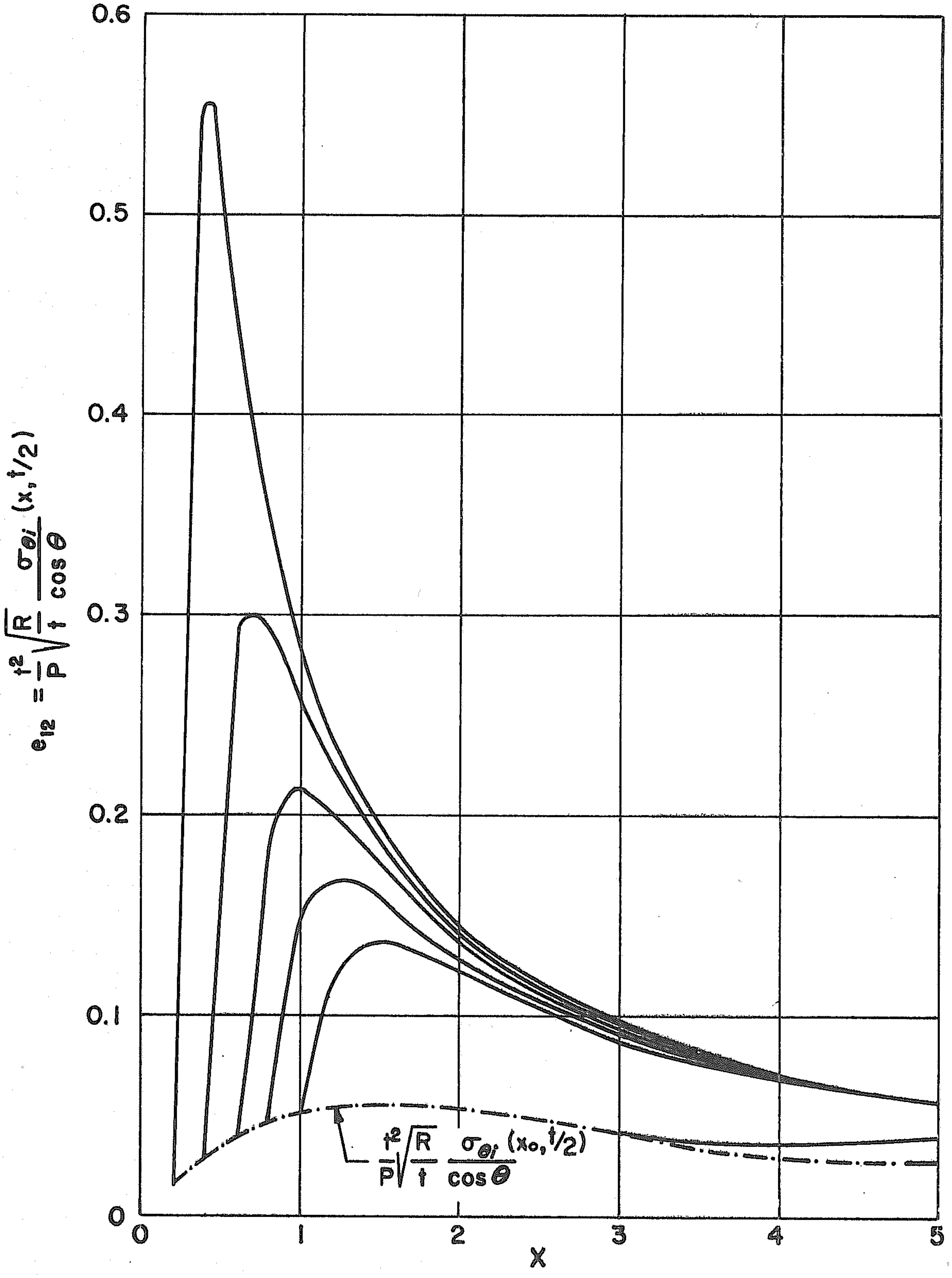


FIG. 57. CIRCUMFERENTIAL STRESS $\sigma_{\theta_{i}}$ AT THE INTERIOR FACE FOR A TANGENTIALLY LOADED INSERT IN A SPHERICAL SHELL

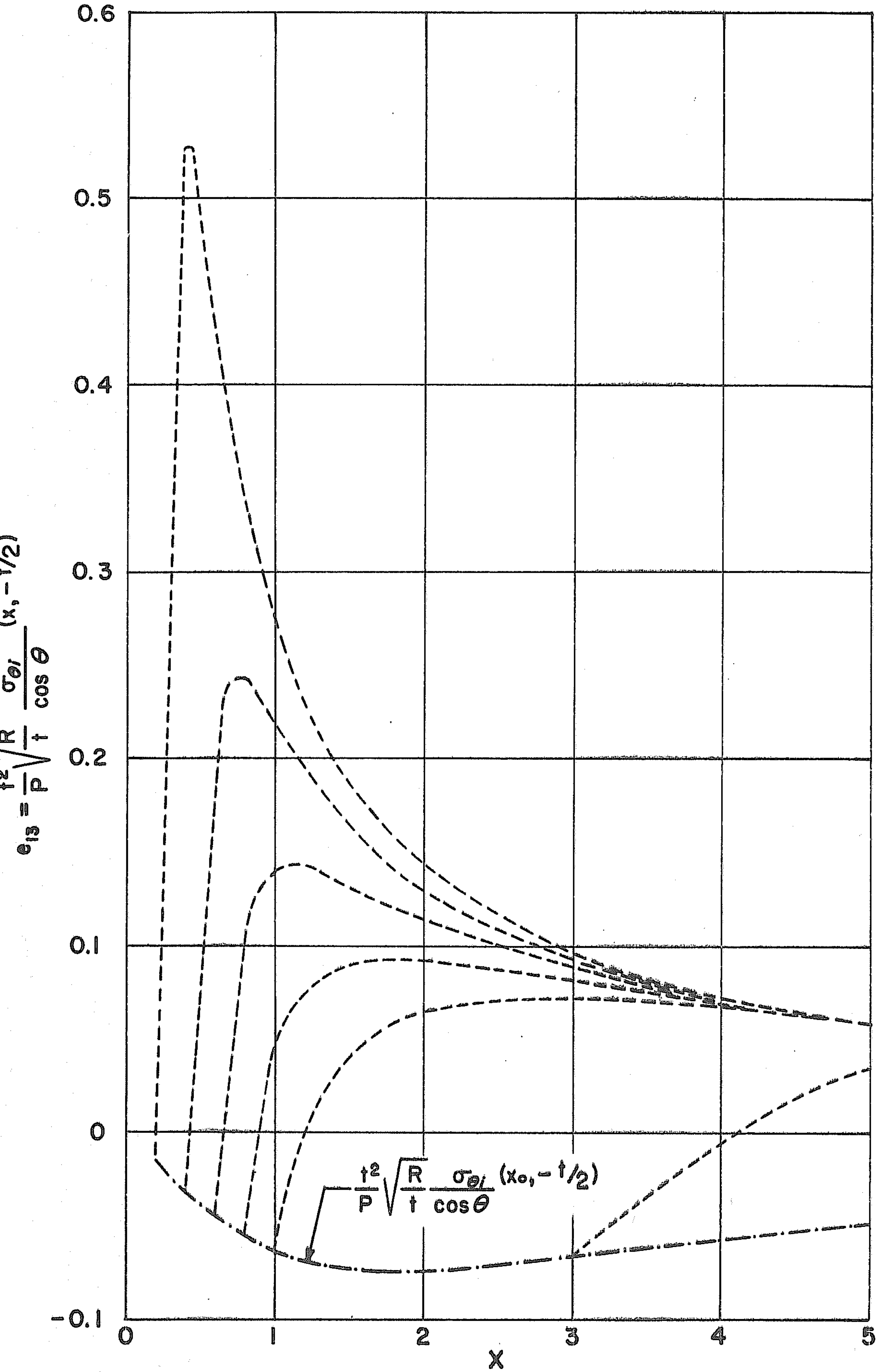


FIG. 57a. CIRCUMFERENTIAL STRESS σ_{θ_e} AT THE EXTERIOR FACE FOR A TANGENTIALLY LOADED INSERT IN A SPHERICAL SHELL

Example (1): A spherical pressure vessel has a rigid insert 2 in. in diameter. Determine the maximum stresses when $p = 200$ psi, $R = 25$ ins., $t = 0.132$ ins. and $\nu = 0.3$.

The ordinate to the edge of the insert in the conical co-ordinate system is given by

$$x_e = \frac{R}{L} \tan \phi_e$$

where $L = \dfrac{\sqrt{Rt}}{\left[12(1-\nu^2)\right]^{1/4}} = 0.9993$ ins.

$$\sin \phi_e = \frac{1}{25} \qquad \therefore \tan \phi_e = 0.04003$$

$$\therefore x_e = \frac{25 \times 0.04003}{0.9993} = 1.001 \simeq 1.0.$$

For $x_e = 1.0$ from Fig. 10-11. we note that the maximum stress occurs in the meridional direction at the insert edge, on the inner surface of the shell. The corresponding coefficient from Table I is

$a_7 = 0.93601850.$

From Eq. (19)

$$\sigma_{\phi_i}(x_e) = 0.9360185 \cdot \frac{pR}{t} = \underline{35.46 \times 10^3 \text{ psi tension}}$$

<u>Example (3)</u>:

Determine the maximum stresses corresponding to a deflection of 7 mils at the apex of a spherical shell loaded by means of a ring load 1/2 in. in diameter, given R = 4 ins., t = 8.5 mils, E = 10^7 psi, ν = 0.3.

Now $x_e = \frac{R}{L} \tan \phi_e = 2.4657$

From Eqs. 35(a) and 40(a), when x = 0, we have

$$w_1 = - \frac{PL^2}{2\pi D} \text{ kei } x_e = 0.007.$$

$$P = 0.1149 \frac{Et^2}{R}$$

Now corresponding to x_e = 2.4657, from Figs. 26 and 27, it can be seen that the maximum stress occurs at the edge of the ring load, in the meridional direction, on the exterior surface of the shell.

The coefficients in Eqs. (41) and (42) for $x = x_e = 2.4657$ are given by

$f_7 = g_7 = 0.081$; $f_8 = g_8 = -0.157$; $f_9 = g_9 = 0.008$

and $f_{10} = g_{10} = -.088$.

Hence the stresses at the edge of the ring load are:

$\sigma_{\phi_i} = 23.27 \times 10^3$ psi tension

$\sigma_{\phi_e} = 45.10 \times 10^3$ psi compression

$\sigma_{\theta_i} = 2.30 \times 10^3$ psi tension

$\sigma_{\theta_e} = 25.28 \times 10^3$ psi compression

No correction as suggest in page 42 is necessary, as ϕ_e is very small and the correction is of the order of 0.4%.

Example (4): Determine the variation of the meridional moment M_ϕ and circumferential inplane force N_θ for a uniformly loaded spherical shell for which $R = 90$ in., $t = 3$ in., $\phi_e = 35^\circ$, $p = -1$ psi, and $\nu = 1/6$.

As $\phi_e > \frac{\pi}{6}$, it is preferable to use Eq. 4(a) to calculate the conical co-ordinates corresponding to respective meridional angle ϕ. With

$$x = \frac{R}{L} (\tan \phi_e + \phi - \phi_e)$$

we have

ϕ°	0	5	10	15	20	25	30	35
x	0.904463	1.78777	2.67118	3.55448	4.43789	5.32130	6.20461	7.08802

Now, as $\nu = 1/6$, we cannot use the tables and charts presented. Hence we resort to direct calculations from the fundamental equations presented. With $x_e = 7.08802$, from Eqs. 46, we get

$$C_1 = -0.0230783 \frac{pR^2}{Et}$$

$$C_2 = -0.0208111 \frac{pR^2}{Et}$$

Using Eqs. 10(b) and 10(c) we can write

$$M_\phi = 1.82429\left\{\text{bei } x + (1 - \nu) \frac{\text{ber}' x}{x}\right\} - 1.64508\left\{\text{ber } x - (1 - \nu) \frac{\text{bei}' x}{x}\right\}$$

$$N_\theta = -2.07705\left\{\text{ber } x - \frac{\text{bei}' x}{x}\right\} + 1.87300\left\{\text{bei } x + \frac{\text{ber}' x}{x}\right\} - 45.0$$

The variation of M_ϕ and N_θ, with the meridional angle ϕ, is tabulated as shown

ϕ°	M_ϕ in lb/in	N_θ lbs/in			Exact *			
		Membrane	Bending	Total	M_ϕ in lb/in	N_θ lbs/in Membrane	Bending	Total
0	-0.580	-45.000	-1.394	-46.394	0.294	-45.000	-2.456	-47.456
5	0.396		-1.863	-46.863	0.377		-2.497	-47.497
10	2.461		-1.944	-46.944	2.364		-2.166	-47.166
15	5.484		-0.080	-45.080	5.451		-0.021	-45.021
20	8.158		5.774	-39.226	8.135		5.950	-39.050
25	6.738		17.104	-27.896	6.687		17.258	-27.742
30	-5.663		31.805	-13.195	-5.756		31.900	-13.100
35	-37.328		38.926	- 6.074	-37.675		38.920	- 6.080

* Exact solution as given by Hetenyi[9] obtained by using 10 terms of a hypergeometric series solution.

Discussion

In this report equations and solutions are presented for the elastic analysis of small displacements in thin spherical shells under bending. Emphasis is given to the cases of stress concentrations in the shell near the loaded zone and the solutions are developed with this in mind.

In the solutions presented it was found that the role of the parameter $\frac{R}{L}$ is negligible for thin shells except for very small meridional angle ϕ. On this basis, terms containing $\frac{L^2}{R^2} \sin \phi$ can be neglected for all values of ϕ. This fact facilitates the satisfaction of the equilibrium equations in terms of the stress function F. This is achieved by neglecting the shear terms in the equilibrium equations which are usually of smaller order than the direct forces.

The use of the conical co-ordinate system transforms the spherical shell into a conical one. The solutions will be quite accurate in a region which closely approximates the corresponding actual spherical shell region. The solutions are valid in a shallow region around the point of tangency of the conical co-ordinate system to the spherical shell. The solution can be termed exact at the point of tangency corresponding to the meridional angle ϕ_e. The solutions presented reduce to well-known approximate theories for particular values of ϕ_e as noted below.

When $\phi_e = 0$, the conical co-ordinate surface becomes a plane. In this case Eq. 4(a) reduces to $x = \frac{R}{L} \phi$ and the governing differential equations become those derived by Geckler[6] for a shallow shell. Similarly Eq. 4(b) reduces to $x = \frac{R}{L} \sin \phi$ and the governing equations become those derived by Reissner[7] for a shallow shell.

For the case $\phi_e = \frac{\pi}{2}$, the cone becomes a cylinder. Here $\tan \phi_e = \infty$ and it is convenient to introduce a new variable y such that

$$y = x - \frac{R}{L} \tan \phi_e .$$

In this case the equations reduce to the approximate solution proposed by Geckler (8) for a deep shell. The governing differential equations also can be reduced to the form of the classical Donnell equations for a cylinder.

Example (4) serves as a check on the validity of the solutions presented.

It should be noted that in our solutions the expressions for the inplane forces N_ϕ, N_θ and $N_{\phi\theta}$ contain terms with coefficients C_5 and C_6. These coefficients C_5 and C_6 are selected to achieve equilibrium with the external singular loadings. Deformations corresponding to these terms are not directly accounted for in the expressions for the deformations, moments and shears, except through the circumferential strain conditions at the boundary of the zone under consideration. This is due to the fact that the homogeneous solution for no external surface loading is itself made to take care of the external singular loadings also. This is an approximation, the error due to which may be significant as the insert diameter reduces. However, it should not be too difficult to determine a particular solution for the given singular external loading and superimpose the same properly on the solutions presented in this report. This aspect of importance for very small insert diameters will be treated separately.

The effect of compliance or of non-rigid inserts, would be to reduce the bending stresses presented in this report. Hence, these results represent an upper bound for stresses due to the stress concentrations considered. The magnitude of the effect of elastic inclusions is considered in detail separately.

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