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On the efficiency of intrinsic rotation generation in tokamaks

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A theory of the efficiency of the plasma flow generation process is presented. A measure of the *efficiency* of plasma *self-acceleration* of mesoscale and mean flows from the heat flux is introduced by analogy with engines, using the entropy budget defined by thermal relaxation and flow generation. The efficiency is defined as the ratio of the entropy destruction rate due to flow generation to the entropy production rate due to ∇T relaxation (i.e., related to turbulent heat flux). The efficiencies for two different cases, i.e., for the generation of turbulent driven $E \times B$ shear flow (zonal flow) and for toroidal intrinsic rotation, are considered for a stationary state, achieved by balancing entropy production rate and destruction rate order by order in $O(k_{\parallel}/k_{\perp})$, where k is the wave number. The efficiency of intrinsic toroidal rotation is derived and shown to be $e_{\mathrm{IR}} \sim (\mathrm{Mach})^2_{\mathrm{th}} \sim 0.01$. The scaling of the efficiency of intrinsic rotation generation is also derived and shown to be $\rho_*^2(q^2/\hat{s}^2)(R^2/L_T^2) = \rho_*^2(L_s^2/L_T^2)$, which suggests a machine size scaling and an unfavorable plasma current scaling which enters through the shear length. © *2010 American Institute of Physics.* [doi:10.1063/1.3496055]

I. INTRODUCTION

Turbulence driven mesoscale and mean flows in fusion plasmas, such as $E \times B$ shear flows [zonal flow (ZF)] (Ref. [1]) and intrinsic rotation in toroidal direction,[2,3] play an important role in achieving better confinement and improving stability. The reduction of turbulent transport by radially sheared $E \times B$ flow[4,5] is a widely accepted concept in the fusion community. The reduction of transport by sheared toroidal rotation[4] is also argued based on the idea that the radial force balance relates the toroidal rotation to the radial electric field E_r, which is responsible for the transport reduction. The stabilization of resistive wall modes by toroidal rotation[6] is discussed as a means to achieve and sustain a high β discharge. The need for intrinsic flow in the transport reduction and the stabilization will surely increase for the future larger machines since it becomes harder to drive the plasma rotation by external means (NBI) due to shallow beam penetration and large plasma inertia.

One of the main issues in intrinsic flow physics is to explain its generation processes. The system is characterized by *no* external momentum input, while energy is injected into a system using methods such as radio-frequency heating. To explain the generation of flows, the concept of a wave driven *residual stress* was developed and extensive experimental[2,7,8] and theoretical[9,10] research on this topic is ongoing. The residual stress is a component of momentum flux which is not proportional to either flow or flow shear as

$$\langle \widetilde{V}_r \widetilde{V}_\parallel \rangle = -\chi_\phi \langle V_\parallel \rangle' + U_r \langle V_\parallel \rangle + \Pi_{r\parallel}^{\mathrm{res}}. \tag{1}$$

The first term is diffusive part, the second term is pinch,[11–13] and the last term is the residual stress. Intrinsic torque in

toroidal plasmas, which is related to the residual stress via $\eta = -\nabla \cdot \Pi_{r\parallel}^{\mathrm{res}}$, was observed for a plasma with no flow and unbalanced NBI injection (1 co+2 counter).[7] For a cylindrical plasma, the residual stress was determined by measuring the total flux $\langle \widetilde{V}_r \widetilde{V}_\theta \rangle$ and the diffusive part $-\chi_\theta \langle V_\theta \rangle'$ separately.[8] Note that the direction of intrinsic flow is azimuthal in the case of a cylindrically symmetric plasma. The residual contribution was determined by calculating the difference of the two (i.e., the total flux and the diffusive flux), since there was no radial convection, i.e., no pinch effect, in the experiment. Symmetry breaking mechanisms were identified and shown to induce a nonzero Reynolds stress $\langle \widetilde{V}_r \widetilde{V}_\parallel \rangle \propto \langle k_\parallel k_\theta \rangle$ which includes the residual stress.[10] The momentum conservation theorem was formulated for wave-particle interaction and the resultant momentum flux, which includes the diffusive flux, the pinch, and the wave-driven residual stress, was calculated.[9]

In the framework of residual stress, the generation process of flows can be understood as a conversion of thermal energy, which is injected into a system by heating, into kinetic energy of macroscopic flow by drift wave turbulence excited by ∇T, ∇n, etc. (Fig. 1). From this picture, one may conceptually view the plasma as a type of an *engine*, where energy input drives turbulence, which leads to ∇T relaxation but also to the generation of flow. See Table I for a comparison between a "car" and intrinsic rotation. The idea of attributing flow generation to heat was mentioned by Carnot[14] to explain the general circulation of the Earth's atmosphere. The concept of an engine may be applied to the problem of the solar differential rotation as well. In the case of solar differential rotation, energy is generated by fusion at the core of the sun (an example of fusion which actually works, albeit
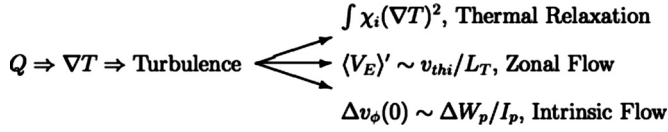
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$$Q \Rightarrow \nabla T \Rightarrow \text{Turbulence} \begin{cases} \int \chi_i (\nabla T)^2, \text{ Thermal Relaxation} \\ \langle V_E \rangle' \sim v_{thi}/L_T, \text{ Zonal Flow} \\ \Delta v_\phi(0) \sim \Delta W_p/I_p, \text{ Intrinsic Flow} \end{cases}$$

FIG. 1. Energy input Q sets temperature profile ∇T which generates turbulence in a system. The turbulence leads to both relaxation and generation of flow.

one using inertial confinement), leading to excitation of turbulence at the convective zone and generation of the solar differential rotation profile. See Table II for a comparison. In the fusion community, some attempts to characterize flow generation in plasmas as the result of the action of a *thermodynamic* engine have been discussed.[15,16] In those, flow generation is treated as analogous to *the work* (power, more precisely) which can be extracted from the exchange of heat between hot and cold parts of plasmas, i.e., the heat flux driven by ∇T. However, these discussions have not given a systematic calculation for the *figure of merit* of the engine.

In this paper, using the physical picture of plasma flow generation as an engine and a simple kinetic model with drift kinetic ions and adiabatic electrons, we formulate an explicit expression for the criterion for engine *efficiency* by comparing *rates of entropy production/destruction* due to thermal relaxation/flow generation. Flow generation reduces entropy since it leads to large scale order in the system. We formulate the entropy budget for the turbulent relaxation process by calculating the time evolution of the mean field entropy.[17] The mean field entropy is the part of entropy defined using only the mean field distribution function as $S_0 \equiv -\int d\Gamma \langle f \rangle \ln \langle f \rangle$, which evolves due to the action of turbulence. Note that S_0 is defined in terms of coarse grained fields. We show that thermal relaxation creates entropy, while intrinsic flow generation decreases the entropy of the system, consistent with the physical picture of flow as an ordered state. We also show that the destruction of entropy due to zonal flow is larger in magnitude than that due to intrinsic toroidal rotation by order of $O(k_\perp/k_\parallel)$, where k is a representative wave number of the drift waves. Given the disparity in their magnitude, we discuss the nature of the stationary state achieved by order-by-order balance in the entropy budget. We show that the lowest order balance, i.e., the balance between the entropy production rate due to the thermal relaxation and the entropy destruction rate due to the zonal flow generation, recovers the conventional stationary state, where turbulence is suppressed by zonal flow shearing. After discussing the class of possible stationary states, we define and calculate the efficiency of plasma flow drive using

TABLE I. Car and intrinsic flow.

	Car	Intrinsic rotation
Fuel	Gas	Heating $\Rightarrow \nabla T$
Conversion	Burn	∇T driven DW turbulence
Work	Cylinder/Cam	Residual stress direction/symmetry breaking
Result	Wheel rotation	Flow

TABLE II. Comparison of differential rotation in the sun and intrinsic rotation in tokamak.

	Sun	Tokamak
Heat source	Fusion reaction at the core	Heat deposition
Turbulence source	∇T	∇T
Threshold	Schwarzschild criteria	ITG
	$\frac{1}{T}\left\|\frac{dT}{dz}\right\| > (\gamma-1)\frac{1}{\rho}\left\|\frac{d\rho}{dz}\right\|$	$R/L_T > R/L_{T,c}$
Turbulence	Convective turbulence	ITG turbulence
Symmetry breaking	Rotation, β	Velocity shear, $\langle V_E \rangle'$
	Stratification	Intensity gradient, $I(x), \ldots$
Resultant flow	Polar differential rotation	Intrinsic rotation
	$v_\phi(\theta)$	$v_\parallel(r)$
B.C.	?	SOL, edge, etc.

the entropy production rate and destruction rate. More precisely, an *upper* bound on the efficiency is calculated, since only the dominant contribution to the entropy production rate is retained. The scaling of intrinsic toroidal rotation generation is derived by using the entropy destruction rate due to wave driven residual stress and shown to be proportional to $\rho_*^2(L_s^2/L_T^2)$. We emphasize that these results are obtained for, and apply only to, a standard, generic model of drift wave turbulence.

The reminder of the paper is organized as follows. In Sec. II, the entropy budget for turbulent relaxation with flow generation is formulated. Using the expression for the entropy budget, we discuss the possible stationary state with coupling to flows. In Sec. III, we define and calculate the efficiency of the plasma flow drive by using the entropy production rate derived in Sec. II. In Sec. IV, we present the discussion and conclusions.

II. ENTROPY BUDGET

In this section, we formulate the entropy budget for the processes of turbulent relaxation and flow generation for a simple model of drift-ITG mode turbulence. In this derivation, we assume simple drift kinetic ions and adiabatic electrons. Given the basic structure of entropy budget, we discuss the possible stationary states with and without flow generation. We derive a coupled set of equations for turbulent fluctuations, δf^2, and shear flow evolution, which are analogous to the conventional predator-prey model for drift wave-zonal flow turbulence system, but are formulated at the level of phase space dynamics. The role of intrinsic toroidal rotation generation in stationary state is discussed as well.

A. Formulation

In kinetic theory, entropy is given as $S \equiv -\int d^3x d^3 v f \ln f$, where f is the distribution function of a system. Here f is normalized to $\int d^3 v f = n$. For a general case, f evolves in time according to the Boltzmann equation $df/dt = C(f)$, where $C(f)$ is a collision operator. For this system—which is open—one can calculate the evolution of entropy as $(d\Gamma \equiv d^3 x d^3 v)$
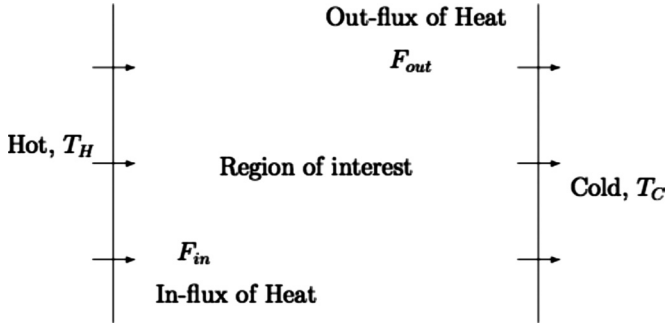
FIG. 2. Heat flux through boundary. The region of interest is surrounded by hot and cold regions with the temperatures T_H and T_C.

$$\partial_t S = - \int d\Gamma C(f)\ln f + \int d^3v \int d\mathbf{A} \cdot (\mathbf{v}f \ln f), \qquad (2)$$

where $\int d\mathbf{A}$ denotes the integral over the surface area. From this relation, one can see that the net entropy of a system changes in two ways, i.e., by collisional entropy production (positive by H-theorem) and by a boundary flux term, which arises as a consequence of outflow of particles, heat, turbulence intensity, etc. In the following analysis the boundary term is dropped by assuming a boundary condition such as $f \propto n \to 0$ or $v_n \to 0$, where v_n is the velocity component normal to the boundary. Before preceding, we offer the observation that the boundary term may play an important role in the entropy budget. For example, the role of the boundary term for thermodynamic systems is described by Ozawa et al.[18] as follows. For a system, as shown in Fig. 2, the region of interest exchanges heat across the boundary between hot and cold regions. The entropy production associated with the heat exchange through the boundaries is $-F_{\text{in}}/T_H$ and F_{out}/T_C, respectively, where $F_{\text{in}} > 0$ is the inflow of heat, $F_{\text{out}} > 0$ is the outflow of heat, T_H is the temperature of hot region, and T_C is the temperature of cold region. Since for a stationary state, the influx and outflux are equal, the total effect of the boundary term on the net entropy balance is

$$-\frac{F_{\text{in}}}{T_H} + \frac{F_{\text{out}}}{T_C} = F\frac{T_H - T_C}{T_C T_H} > 0, \qquad (3)$$

which shows a net contribution to the entropy budget from the boundary terms. Such an effect can be important in tokamak plasmas when one considers an annular region with steep temperature gradient, which suggests a significant difference in temperature across boundary. Here we consider a simplified case with no entropy outflow, so net volume integrated production and dissipation must cancel. As a consequence then, this theory is probably more directly relevant to δf particle simulations—which impose the boundary condition $\nabla\phi = 0$ and so preclude any outflux of entropy—than to actual tokamak plasmas. Indeed, since trends in the evolution of intrinsic rotation appear closely linked to the L-H transition, we note that the drop in the cross-boundary flux $\langle \tilde{v}_r \delta f^2 \rangle / \langle f \rangle$ (which necessarily occurs at the transition) will impact the global entropy budget, and thus should be considered in models of intrinsic rotation evolution.

Since we are interested in turbulent relaxation, we focus on the generation of the "mean field" entropy,[17] $S_0 \equiv -\int d\Gamma \langle f \rangle \ln \langle f \rangle$, where $\langle f \rangle$ is a coarse grained mean distribution function. By decomposing $f = \langle f \rangle + \delta f$, one can approximate the coarse grained entropy as

$$\langle S \rangle = - \int d\Gamma \langle (\langle f \rangle + \delta f)\ln(\langle f \rangle + \delta f) \rangle$$

$$\cong - \int d\Gamma \langle f \rangle \ln \langle f \rangle - \int d\Gamma \frac{\langle \delta f^2 \rangle}{\langle f \rangle} \equiv S_0 + S_2,$$

where $S_2 \equiv -\int d\Gamma \langle \delta f^2 \rangle / \langle f \rangle$ is entropy of fluctuations. Using the decomposition of entropy and a linearized collision operator, i.e., $C(f) = C(\langle f \rangle) + C(\delta f) \cong C(\delta f)$, with $\langle f \rangle$ thus driven to a local Maxwellian, Eq. (2) can be rewritten in terms of S_0 as

$$\partial_t S_0 = -\partial_t S_2 - \int d\Gamma \frac{\langle \delta f C(\delta f) \rangle}{\langle f \rangle}$$

$$= \partial_t \int d\Gamma \frac{\langle \delta f^2 \rangle}{\langle f \rangle} - \int d\Gamma \frac{\langle \delta f C(\delta f) \rangle}{\langle f \rangle}, \qquad (4)$$

which relates the evolution of the mean field entropy to the evolution and collisional dissipation of δf^2. Note that the last term, collisional dissipation, is positive definite, as a consequence of the H-theorem.

To calculate δf^2 generation, we employ a simple drift kinetic equation for ions,

$$\partial_t f + v_\parallel \nabla_\parallel f + \frac{c}{B}\hat{\mathbf{z}} \times \nabla\tilde{\phi} \cdot \nabla f + \frac{|e|}{m_i}\tilde{E}_\parallel \frac{\partial f}{\partial v_\parallel} = C(f), \qquad (5)$$

and assume adiabatic response for electrons,

$$\frac{\delta n_e}{n_0} = \frac{|e|\tilde{\phi}}{T_e}. \qquad (6)$$

Thus, we are interested in ITG turbulence as a specific model of drift wave turbulence. For δf^2 balance, we have

$$\partial_t \left\langle \frac{\delta f^2}{2\langle f \rangle} \right\rangle + \frac{1}{r}\partial_r\left(r\left\langle \tilde{V}_r \frac{\delta f^2}{2\langle f \rangle} \right\rangle \right) - \frac{\langle \delta f C(\delta f) \rangle}{\langle f \rangle}$$

$$= -\langle \tilde{v}_r \delta f \rangle \frac{\langle f \rangle'}{\langle f \rangle} - \frac{|e|}{m_i}\langle \tilde{E}_\parallel \delta f \rangle \frac{1}{\langle f \rangle}\frac{\partial \langle f \rangle}{\partial v_\parallel}, \qquad (7)$$

where a scale separation between mean and fluctuation, i.e., $\partial_t \delta f \gg \partial_t \langle f \rangle$ and $\nabla \delta f \gg \nabla \langle f \rangle$, was assumed. Since we are interested in the evolution of $\int d\Gamma \langle \delta f^2 \rangle / \langle f \rangle$ [see Eq. (4)], we need to integrate Eq. (7) over phase space. Taking the phase space integral, one obtains

$$\partial_t \int d\Gamma \frac{\langle \delta f^2 \rangle}{2\langle f \rangle} = \int d^3x(\mathcal{P} - \mathcal{D}), \qquad (8)$$

where

$$\mathcal{P} \equiv \int d^3v \left(-\langle \tilde{v}_r \delta f \rangle \frac{\langle f \rangle'}{\langle f \rangle} - \frac{|e|}{m_i} \langle \tilde{E}_\parallel \delta f \rangle \frac{1}{\langle f \rangle} \frac{\partial \langle f \rangle}{\partial v_\parallel} \right), \qquad (9a)$$

$$\mathcal{D} = -\int d^3v \frac{\langle \delta f C(\delta f) \rangle}{\langle f \rangle}. \qquad (9b)$$

Here $\mathcal{P}$ is the δf^2 production rate due to the free energy in configuration space (i.e., $\sim \partial \langle f \rangle / \partial r$) and velocity space (i.e., $\sim \partial \langle f \rangle / \partial v_\parallel$). $\mathcal{D}$ is the collisional dissipation.

To calculate $\mathcal{P}$, we assume $\langle f \rangle$ as a local Maxwellian with a mean shear flow, $\langle V_\perp \rangle(r)$ and $\langle V_\parallel \rangle(r)$. With quasineutrality $\tilde{n}_e = \tilde{n}_i$, one obtains

$$\mathcal{P} = -\frac{n}{T_i L_T} Q^i_{\text{turb}} - \frac{n}{v_{\text{thi}}^2} \langle V_\perp \rangle' \langle \tilde{V}_r \tilde{V}_\perp \rangle - \frac{n}{v_{\text{thi}}^2} \langle V_\parallel \rangle' \langle \tilde{V}_r \tilde{V}_\parallel \rangle$$
$$+ \frac{1}{T_i} \langle \tilde{J}_\parallel^i \tilde{E}_\parallel \rangle, \qquad (10)$$

where $v_{\text{thi}} \equiv \sqrt{(T_i/m_i)}$, $L_T^{-1} \equiv (dT_i/dr)/T_i$, $Q^i_{\text{turb}} \equiv n^{-1} \int d^3v E \langle \tilde{V}_r \delta f \rangle = \langle \tilde{V}_r T_i \rangle$, $\langle \tilde{V}_r \tilde{V} \rangle \equiv n^{-1} d^3v (v - \langle V \rangle) \langle \tilde{V}_r \delta f \rangle$, and $\langle \tilde{J}_\parallel^i \tilde{E}_\parallel \rangle = |e| \int d^3v (v_\parallel - \langle V_\parallel \rangle) \langle \delta f \tilde{E}_\parallel \rangle$. Note that the mean *ion* velocity was replaced by the mean *plasma* velocity due to the large ion inertia. The first three terms are related to the spatial inhomogeneity of a local Maxwellian and have the standard form of the entropy production rate $\mathcal{J}_k \mathcal{X}_k$, where $\mathcal{J}_k = \{Q^i_{\text{turb}}, \langle \tilde{V}_r \tilde{V}_\perp \rangle, \langle \tilde{V}_r \tilde{V}_\parallel \rangle\}$ is the flux vector and $\mathcal{X}_k = \{-\langle T \rangle', -\langle V_\perp \rangle', -\langle V_\parallel \rangle'\}$ is the thermodynamic force. In the following, we further simplify the entropy production rate by employing a simple model to calculate the turbulent flux as $\mathcal{J}_k = \mathcal{J}_k[\mathcal{X}_l]$ and discuss their consequences. The last term in Eq. (10) comes from the velocity space dependence in the distribution function and represents the effect of resonant heating. Using Poynting's theorem,[1,9] one can write the heating term as

$$\int d^3x \langle \tilde{J}_\parallel^i \tilde{E}_\parallel \rangle = \int d^3x (-\partial_t W - \nabla \cdot \mathbf{S}_w - \langle \tilde{\mathbf{J}}_\perp \cdot \tilde{\mathbf{E}}_\perp \rangle). \qquad (11)$$

Here W is wave energy density and $\mathbf{S}_w$ is flux of wave energy density. For a stationary state, the first term in the right hand side is zero, $\partial_t W = 0$. The second term also vanishes due to the boundary conditions, $\int d^3x \nabla \cdot \mathbf{S}_w = \int d\mathbf{A} \cdot \mathbf{S}_w|_{\text{boundary}} \to 0$. In other words we assumed there are no outgoing waves, again as enforced in simulations. The third term is also zero, $\langle \tilde{\mathbf{J}}_\perp \cdot \tilde{\mathbf{E}}_\perp \rangle = 0$, since $\mathbf{J}_\perp \propto \mathbf{E} \times \mathbf{B}$ at the lowest order. Thus, the heating term is dropped in the following discussion.

To further simplify the entropy production rate term in Eq. (10), we employ a simple model for flux terms here. The first term in Eq. (10) is related to the thermal relaxation. For simple ITG turbulence, we have a simple flux-gradient relation $Q^i_{\text{turb}} = -\chi_i \nabla T$, where the thermal conductivity χ_i is

$$\chi_i \sim \sum_k \tau_k^{\text{DW}} |\tilde{v}_r|_k^2 \Theta(R/L_T - R/L_{T,c}). \qquad (12)$$

Here τ_k^{DW} is the correlation time for ITG drift wave turbulence and Θ is the step function, which accounts for the threshold condition. Using the flux-gradient relation, it fol-

lows that the production rate due to thermal relaxation is positive definite, i.e.,

$$-\frac{n}{T_i L_T} Q^i_{\text{turb}} = n\chi_i \left(\frac{\nabla T}{T} \right)^2 > 0. \qquad (13)$$

Thus turbulent thermal relaxation produces entropy. The second and the third terms in Eq. (10) are the momentum flux in the perpendicular and parallel directions, which contain information concerning flow generation. For the perpendicular flow, for simplicity we consider only $E \times B$ shear flow or zonal flow, $\langle V_\perp \rangle' = \langle V_E \rangle'$. The momentum flux in the perpendicular direction can be calculated using the wave kinetic equation as $\langle \tilde{V}_r \tilde{V}_\theta \rangle = K \langle V_E \rangle'$ (see Appendix B for the derivation), where

$$K \equiv \sum_k c_s^2 \tau_{\text{ZF}} \frac{\rho_s^2 k_\theta^2}{(1 + k_\perp^2 \rho_s^2)^2} \left(-k_r \frac{\partial \langle \eta_k \rangle}{\partial k_r} \right), \qquad (14)$$

$$\eta_k \equiv (1 + k_\perp^2 \rho_s^2)^2 \left| \frac{e\phi_k}{T_e} \right|^2. \qquad (15)$$

Here τ_{ZF} is the correlation time of the zonal flow, η_k is the fluctuation potential enstrophy, and K is related to the nonlinear growth rate of zonal flow as $\gamma_{\text{ZF}} = q_r^2 K$, with q_r as the radial wave number of the zonal flow. Making *the assumption* that the zonal flow grows ($\gamma_{\text{ZF}} > 0 \Leftrightarrow K > 0 \Leftrightarrow -k_r \partial \langle \eta_k \rangle / \partial k_r$, a standard criterion for the zonal flow growth[19]), one can show that the entropy production rate due to zonal flow growth is negative definite, i.e.,

$$-\frac{n}{v_{\text{thi}}^2} \langle V_\perp \rangle' \langle \tilde{V}_r \tilde{V}_\perp \rangle = -nK \left(\frac{\langle V_E \rangle'}{v_{\text{thi}}} \right)^2 < 0. \qquad (16)$$

Hence, the generation of zonal flow leads to a *destruction* of entropy. This is physically plausible and can be easily understood, since zonal flow shears oppose relaxation of ∇T by reducing transport, and hence act *against* entropy production. Put differently, one can regard zonal flow as a large scale coherent structure, and the generation of a coherent structure may be viewed as a restoring "order" to the system, thus decreasing the entropy of that system. Note that the entropy destruction occurs only in the sense that it opposes entropy production due to other relaxation processes, i.e., thermal relaxation, here. The overall entropy production rate, i.e., the sum of those due to thermal relaxation and zonal flow generation, cannot be negative. The parallel momentum flux can be decomposed as[20]

$$\langle \tilde{V}_r \tilde{V}_\parallel \rangle = -\chi_\phi \langle V_\parallel \rangle' + U \langle V_\parallel \rangle + \Pi_{r\parallel}^{\text{res}}. \qquad (17)$$

The first term is turbulent diffusion of parallel momentum, the second term shows the effect of the pinch, and the third term is residual stress, which leads to generation of intrinsic toroidal rotation. The pinch term is taken to be zero for simplicity hereafter, since it only redistributes momentum by radial convection. For a stationary state, there is no torque input, so we must have

$$\langle \tilde{V}_r \tilde{V}_\parallel \rangle = -\chi_\phi \langle V_\parallel \rangle' + \Pi_{r\parallel}^{\text{res}} = 0 \qquad (18)$$

to get a nontrivial toroidal flow profile, $\langle V_\parallel \rangle' = \Pi_{r\parallel}^{\text{res}}/\chi_\phi$. From

TABLE III. Comparison of δf^2 stationary state.

	$\mathcal{P}=\mathcal{D}$	$\mathcal{P}=0$
Flow generation	Not necessarily	Yes
δf^2 production	∇T relaxation	∇T relaxation
δf^2 destruction	Collisional dissipation (small scale)	Flow generation (mesoscale)

this consideration, we see that the *total* entropy production rate due to parallel momentum flux, $\propto\langle V_{\parallel}\rangle'\langle\widetilde{V}_r\widetilde{V}_{\parallel}\rangle$, is *zero* for a stationary state of intrinsic toroidal rotation. However, it consists of *two* competing parts, i.e., the terms due to the diffusive and residual parts of the momentum flux, respectively. The diffusive part gives rise to viscous *heating* and the resultant entropy production rate is shown to be *positive definite*, i.e.,

$$-\frac{n}{v_{\text{thi}}}\langle V_{\parallel}\rangle'\langle\widetilde{V}_r\widetilde{V}_{\parallel}\rangle|_{\text{diff}} = n\chi_{\phi}\left(\frac{\langle V_{\parallel}\rangle'}{v_{\text{thi}}}\right)^2 > 0. \quad (19)$$

The residual part in the parallel momentum flux leads to the *generation of intrinsic toroidal rotation* and the resultant entropy production rate is shown to be *negative definite*, i.e.,

$$-\frac{n}{v_{\text{thi}}^2}\langle V_{\parallel}\rangle'\langle\widetilde{V}_r\widetilde{V}_{\parallel}\rangle|_{\text{res}} = -\frac{n}{v_{\text{thi}}^2}\langle V_{\parallel}\rangle'\Pi_{r\parallel}^{\text{res}} = -\frac{n}{\chi_{\phi}v_{\text{thi}}^2}\Pi_{r\parallel}^{\text{res}2} < 0, \quad (20)$$

where the stationary condition for the parallel momentum flux $\langle V_{\parallel}\rangle=\Pi_{r\parallel}^{\text{res}}/\chi_{\phi}$ was used.

After the simplification above, we have

$$\mathcal{P} = n\chi_i\left(\frac{\nabla T}{T}\right)^2 - nK\left(\frac{\langle V_E\rangle'}{v_{\text{thi}}}\right)^2 + n\chi_{\phi}\left(\frac{\langle V_{\parallel}\rangle'}{v_{\text{thi}}}\right)^2 - n\frac{\Pi_{r\parallel}^{\text{res}2}}{v_{\text{thi}}^2\chi_{\phi}}. \quad (21)$$

The first term is due to thermal relaxation and is positive definite. The second term is related to the zonal flow generation and is negative definite, given that the zonal flow grows. The third term is due to viscous heating and is positive definite. The fourth term comes from the generation of intrinsic toroidal rotation and is negative definite.

B. Flow generation and stationary state

Since we are interested in the calculation of the efficiency of an engine for a *stationary* state, it would be important to clarify the criteria for stationarity and the physical picture of the system we are concerned with. Here we discuss the class of states which is defined by requiring δf^2 to be stationary. First we discuss the stationary state when the flow generation is *weak*. Then we consider the case with generation of flow.

When the generation of the flow is weak and the instability source is the temperature gradient, the production rate becomes

$$\mathcal{P} \cong n\chi_i\left(\frac{\nabla T}{T}\right)^2 > 0, \quad (22)$$

which is positive definite as long as a supercritical temperature gradient is maintained. To achieve stationarity, we must balance production with dissipation, i.e., $\mathcal{P}=\mathcal{D}$. Note that this is a global balance in phase space. One may understand this balance as a cascade of "phasetrophy" δf^2 in phase space,[21,22] where δf^2 is produced by inhomogeneity in $\langle f\rangle(x)$ at some scale in phase space, transferred to smaller scale by nonlinear interaction and eventually dissipated by collision.

However, by allowing the generation of flow, one can access different types of stationary states, since entropy destruction occurs due to flow generation, as we saw in the last section. With the generation of flow, one can achieve stationary state with $\mathcal{P}\cong0$. See Table III for comparison.

Since $\langle\widetilde{V}_r\widetilde{V}_{\perp}\rangle\propto k_{\theta}k_{\perp}$, $\langle\widetilde{V}_r\widetilde{V}_{\parallel}\rangle\propto k_{\theta}k_{\parallel}$, and $k_{\perp}>k_{\parallel}$ for typical drift wave turbulence, the entropy destruction rate due to zonal flow generation is virtually *always* larger than that due to intrinsic toroidal rotation generation. Alternatively put, since self-generated flows are ultimately driven by wave momentum (i.e., momentum conservation laws relate flow momentum plus turbulence pseudomomentum to sources, sinks, etc.), and since $p_{\theta}=k_{\theta}N$ while $p_{\parallel}=k_{\parallel}N$, poloidal wave momentum naturally exceeds parallel wave momentum. In turn then, absent damping, poloidal and zonal flows, naturally can be expected to exceed intrinsic toroidal flows. Hence, the $\mathcal{P}\cong0$ state can be calculated order by order. To the lowest order in $O(k_{\parallel}/k_{\perp})$,

$$\mathcal{P} \cong n\chi_i\left(\frac{\nabla T}{T}\right)^2 - nK\left(\frac{\langle V_E\rangle'}{v_{\text{thi}}}\right)^2 + O(k_{\parallel}/k_{\perp}).$$

Note that $K>0$ for zonal flow growth. A nontrivial stationary state is evident with $\mathcal{P}\cong0$, i.e., when

$$\langle V_E\rangle'^2 = \frac{\chi_i}{K}\frac{v_{\text{thi}}^2}{L_T^2}. \quad (23)$$

Note that χ_i and K approximately cancel, i.e., $\chi_i/K\sim1$, since $\chi_i\cong\Sigma_k|\widetilde{v}_r|_k^2\tau_k^{\text{DW}}$, $K\cong\Sigma_k|\widetilde{v}_r|_k^2\tau_k^{\text{ZF}}$, and $\tau_{\text{ZF}}\sim\tau_{\text{DW}}$ for a simple model. Here $\tau_k^{\text{ZF}}\sim1/\nu_{\text{eff}}$, $\tau_k^{\text{DW}}\sim1/\Delta\omega_k$, where ν_{eff} is the "Krook" operator for wave-wave scattering process,[9] and $\Delta\omega_k$ is the decorrelation rate. Thus, the stationary flow shear is tied directly to the ∇T force by

$$\langle V_E \rangle'^2 \cong \frac{v_{thi}^2}{L_T^2} \Theta(L_T^{-1} - L_{T,c}^{-1}) = v_{thi}^2 \left(\frac{\nabla T}{T}\right)^2 \Theta(L_T^{-1} - L_{T,c}^{-1}),$$

(24)

where the step function $\Theta(L_T^{-1} - L_{T,c}^{-1})$ accounts for the threshold behavior, originating from the turbulent thermal conductivity χ_i. Note that the profile of zonal flow is relatively smooth. In other words the zonal flow treated here is a large scale flow at the limit of long wavelength.

It is interesting to see how δf^2 evolves in time with the dominant terms in the production rate, i.e., that of ∇T relaxation and zonal flow generation,

$$\partial_t \int d\Gamma \frac{\langle \delta f^2 \rangle}{2\langle f \rangle} = \int d^3x \left[n\chi_i \left(\frac{\nabla T}{T}\right)^2 - nK \left(\frac{\langle V_E \rangle'}{v_{thi}}\right)^2 \right].$$

(25)

Adding the equation for flow shear amplification by Reynolds stress, we have

$$\partial_t \int d\Gamma \frac{\langle \delta f^2 \rangle}{2\langle f \rangle} = \int d^3x \left[n\chi_i \left(\frac{\nabla T}{T}\right)^2 - nK \left(\frac{\langle V_E \rangle'}{v_{thi}}\right)^2 \right],$$

(26a)

$$\partial_t \frac{\langle V_E \rangle'^2}{2} = K\overline{q_r^2}\langle V_E \rangle'^2 - \nu_{col}\langle V_E \rangle'^2,$$

(26b)

where $\overline{q_r^2} \equiv \Sigma_q q_r^2 \langle V_E \rangle_q'^2 / \langle V_E \rangle'^2$ is the spectral average of the radial wave number of the zonal flow, q_r. Note that Eqs. (26a) and (26b) have the same structure as the familiar predator-prey model for the DW-ZF turbulence system.[23] For comparison, recall the standard predator-prey form,

$$\partial_t \varepsilon = \gamma_L \varepsilon - \alpha V'^2 \varepsilon - \Delta\omega(\varepsilon)\varepsilon,$$

(27a)

$$\partial_t V'^2 = \alpha V'^2 \varepsilon - \nu_{col} V'^2,$$

(27b)

where ε is the turbulence intensity, V'^2 is flow shear, γ_L is linear growth rate of a mode, α represents a coupling between flow and fluctuations, $\Delta\omega$ is a decorrelation rate, and ν_{col} is a collisional drag on flow. By comparing the two sets of equations, not surprisingly, we see that the fluctuation entropy or δf^2 plays the same role of the fluctuation intensity ε. In the terminology of the predator-prey system, δf^2 or fluctuation entropy is the "prey" and the zonal flow shear is the "predator." The prey grows with the relaxation process, $n\chi_i(\nabla T/T)^2$, and decreases with the generation of the predator, $n\langle V_E \rangle'^2 K/v_{thi}^2$. The predator increases by consuming the prey (i.e., flow generated by fluctuations), $K\overline{q_r^2}\langle V_E \rangle'^2$, and eventually dissipated by small, but finite, collisional damping, $\nu_{col}\langle V_E \rangle'^2$. The steady state occurs when entropy generation and destruction balance each other. The stationary state is thus

$$\langle V_E \rangle'^2 = \frac{\chi_i}{K} \frac{v_{thi}^2}{L_T^2},$$

(28a)

$$K(|\hat\phi|^2) = \frac{\nu_{col}}{q_r^2},$$

(28b)

which has the same structure as a stationary solution for the familiar predator-prey system, namely,

$$V'^2 = \frac{1}{\alpha}[\gamma_L - \Delta\omega(\varepsilon)],$$

(29a)

$$\varepsilon = \frac{\nu_{col}}{\alpha}.$$

(29b)

The stationary level of the flow has similar structure in both systems through the χ_i/L_T^2 and $\gamma_L - \Delta\omega$ dependence. Both systems show threshold behavior, $\chi_i \propto \Theta(L_{T,c}/L_T - 1)$ and $\gamma_L - \Delta\omega$. The flow level increases as drive of instability is strengthened, as manifested in $1/L_T$ and γ_L. This reflects the fact that the dynamical system naturally couples ∇T free energy to the flow. The stationary level of turbulence $K \sim \chi_i$ in the model and ε in the standard predator-prey is tied to the collisionality in flow, $\chi_i \sim K = \nu_{col}/\overline{q_r^2}$ and $\varepsilon = \nu_{col}/\alpha$, which is consistent with gyrokinetic simulations.[24]

The role of generation of intrinsic *toroidal* rotation in stationary state can be seen by going to the higher order $O(k_\parallel/k_\perp)$ balance in the production rate term. After the cancellation at the lowest order terms, the production rate becomes

$$\mathcal{P} = n\frac{\chi_\phi}{v_{thi}^2}\langle V_\parallel \rangle'^2 - n\frac{\Pi_{r\parallel}^{res2}}{v_{thi}^2 \chi_\phi},$$

(30)

which consists of the production due to turbulent viscous heating and the destruction due to toroidal flow generation. The two terms cancel for a stationary state of intrinsic toroidal rotation, since

$$\langle \tilde V_r \tilde V_\parallel \rangle = -\chi_\phi \langle V_\parallel \rangle' + \Pi_{r\parallel}^{res} = 0.$$

(31)

Hence, the total entropy production rate by the parallel momentum flux vanishes in a stationary state, i.e., the entropy production by intrinsic toroidal flow is balanced by the entropy destruction by intrinsic toroidal flow generation, to order $O(k_\parallel/k_\perp)$.

To summarize, we considered the two classes of stationary state: $\mathcal{P} = \mathcal{D}$ and $\mathcal{P} \cong 0$. The former is the stationary state with the balance between production (positive definite) and total dissipation, without the coupling to the flow generation. One may understand this process as the cascade of the phasetrophy. The latter is achieved by including the effect of flow generation. The $\mathcal{P} = 0$ state is achieved order by order since the entropy destruction rate due to zonal flow and intrinsic toroidal rotation differ by $O(k_\parallel/k_\perp)$. The dominant balance occurs between ∇T relaxation and zonal flow generation. The effect of intrinsic toroidal rotation generation appears in the next order in $O(k_\parallel/k_\perp)$ and vanishes for a stationary state. Given all the terms calculated above, the total production rate becomes

$$\mathcal{P} = n\chi_i \left(\frac{\nabla T}{T}\right)^2 - nK \left(\frac{\langle V_E \rangle'}{v_{thi}}\right)^2 + n\chi_\phi \left(\frac{\langle V_\parallel \rangle'}{v_{thi}}\right)^2 - n\frac{\Pi_{r\parallel}^{res2}}{v_{thi}^2 \chi_\phi}.$$

(32)

The first two terms balance at the lowest order and the next two terms balance at the next order. In the next section, we calculate the efficiency of flow generation for the stationary state with flow, i.e., the $\mathcal{P} = 0$ state.

III. EFFICIENCY OF INTRINSIC FLOW DRIVE

Having established the entropy budget for the flow generation and relaxation process, we are ready to calculate the efficiency of flow generation. In this section, we present a definition and calculation for the plasma flow generation efficiency. First, we define the efficiency using the entropy budget in the last section. After defining the efficiency, we give the actual calculation of its value and *scaling*, both for zonal flow and intrinsic toroidal rotation.

A. Definition of efficiency

With the flow generation terms in the production rate, we have

$$\partial_t S_0 = \int d^3x \left[n\chi_i \left(\frac{\nabla T}{T} \right)^2 - nK \frac{\langle V_E \rangle'^2}{v_{thi}^2} \right.$$
$$\left. + n\chi_\phi \frac{\langle V_\parallel \rangle'^2}{v_{thi}^2} - n\frac{\Pi_{r\parallel}^{res2}}{v_{thi}^2 \chi_\phi} \right]. \tag{33}$$

We calculate the efficiency of flow generation for stationary state with $\mathcal{P} = 0$, where the balance is achieved order by order. Using the expression for the entropy production rate, we define the efficiency of plasma flow generation as follows:

$$e \equiv \frac{\left| \int d^3x \mathcal{P}_{flow} \right|}{\int d^3x \mathcal{P}_{net}}, \tag{34}$$

i.e., the ratio between the magnitude of the entropy destruction rate due to flow generation and the total entropy production rate due to relaxation. Note that the efficiency here is defined using the entropy production *rate* and destruction rate ($\propto \dot{Q}$, where Q is heat, not heat *flux*), while usually the efficiency of a *thermodynamic engine* is defined in terms of *heat and work* ($\propto Q$). In other words, the former is defined using ratios of *power*, while the latter is defined using ratios of *energy*. As for the entropy destruction mechanism, we can consider two cases, i.e., zonal flow generation $\mathcal{P}_{flow} = -(nK\langle V_E \rangle'^2)/v_{thi}^2$ and intrinsic toroidal flow generation $\mathcal{P}_{flow} = -n(\Pi_{r\parallel}^{res})^2/\chi_\phi$. As for the net production rate, we have

$$\mathcal{P}_{net} = n\chi_i \left(\frac{\nabla T}{T} \right)^2 + n\chi_\phi \left(\frac{\langle V_\parallel \rangle'}{v_{thi}} \right)^2. \tag{35}$$

The first term is related to ∇T relaxation due to turbulence. The second term is related to the turbulent viscous heating. The second term is smaller than the first term by order of $(\langle V_\parallel \rangle / v_{thi})^2$, where $(\langle V_\parallel \rangle / v_{thi})^2 \equiv M_{th}^2 \sim 0.01$, typically. This follows, in part, from

$$n\chi_\phi \left(\frac{\langle V_\parallel \rangle'}{v_{thi}} \right)^2 \sim n\chi_\phi \left(\frac{\langle V_\parallel \rangle'}{\langle V_\parallel \rangle} \right)^2 \frac{\langle V_\parallel \rangle^2}{v_{thi}^2} \sim n\chi_i \left(\frac{\nabla T}{T} \right)^2 \frac{\langle V_\parallel \rangle^2}{v_{thi}^2},$$

with $Pr \equiv \chi_\phi / \chi_i \sim 1$ and $\nabla T/T \sim \langle V_\parallel \rangle' / \langle V_\parallel \rangle$. Hereafter, we only keep the dominant contribution to the net entropy balance, i.e., $P_{net} \cong \chi_i (\nabla T/T)^2$. Since we drop the positive definite term (the turbulent viscous heating) in the denominator in Eq. (34), we calculate an *upper bound* for the efficiency.
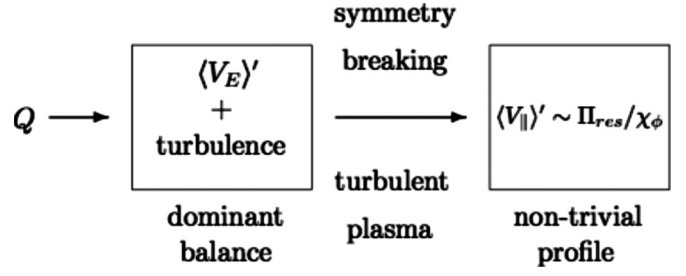

FIG. 3. Turbulent plasma and flow generation.

B. Efficiency of zonal flow generation

As the first case, we consider the efficiency of zonal flow generation, although the outcome is trivial, as shown below. Using the definition given above, we obtain, as an upper bound,

$$e_{ZF} \lesssim \frac{\int d^3x (nK\langle V_E \rangle'^2)/v_{thi}^2}{\int d^3x (n\chi_i)/L_T^2}. \tag{36}$$

Since we are interested in the efficiency at a stationary state, we substitute Eq. (23) for the value of $\langle V_E \rangle'$. With the substitution, one obtains

$$e_{ZF} \lesssim 1. \tag{37}$$

This is the result we should expect given the *assumption* we made, i.e., we considered flow shear dominated state for δf^2 balance,

$$\partial_t \int d\Gamma \frac{\langle \delta f^2 \rangle}{\langle f \rangle} = \int d^3x \left(n\frac{\chi_i}{L_T^2} - \frac{n\langle V_E \rangle'^2}{v_{thi}^2} K \right) = 0, \tag{38}$$

and defined the efficiency to be the ratio of the two terms in the right hand side. Hence,

$$e_{ZF} \lesssim 1 \tag{39}$$

is just the restatement of the fact that we have a stationary state by balancing the entropy production rate due to thermal relaxation and the dominant entropy destruction rate due to zonal flow growth.

C. Efficiency of intrinsic toroidal flow generation

The efficiency of zonal flow production was calculated using dominant terms in the production rate, Eq. (32). By going to next order in $O(k_\parallel / k_\perp)$, we can calculate the efficiency of intrinsic toroidal rotation generation. In this picture, generation of intrinsic toroidal rotation is considered to be a two step process (Fig. 3). First, a stationary state is achieved by a balance between dominant terms in the entropy production rate, i.e., temperature relaxation and zonal flow generation. This is the state given by the stationary solution of Eqs. (26a) and (26b) with $e_{ZF} \sim 1$. As a secondary process, this pre-existing stationary turbulent plasma and shear flow give rise to the wave driven residual stress, which generates an intrinsic toroidal torque. Thus, the efficiency of intrinsic toroidal flow in this process is, from the definition,

$$e_{IR} \cong \frac{\int d^3x n(\Pi_{r\parallel}^{res})^2/(v_{thi}^2 \chi_\phi)}{\int d^3x n\chi_i (\nabla T/T)^2}. \tag{40}$$

We can easily estimate the order of magnitude for e_{IR}. Using the stationary condition for intrinsic flow $\Pi_{r\parallel}^{res} = \chi_\phi \langle V_\parallel \rangle'$ and assuming $\langle V_\parallel \rangle' / \langle V_\parallel \rangle \sim \sigma \nabla T / T$ [where σ is a $O(1)$ constant factor], we can obtain (here $M_i \equiv \langle V_\parallel \rangle / v_{thi}$ is toroidal Mach number)

$$e_{IR} = \frac{\int d^3 x n \chi_\phi \langle V_\parallel \rangle'^2 / v_{thi}^2}{\int d^3 x n \chi_i (\nabla T / T)^2} \sim \sigma^2 M_i^2. \tag{41}$$

For a typical value of $M_i \sim 0.1$ and $\sigma \sim 1$, we have $e_{IR} \sim 0.01$ which states that intrinsic toroidal rotation generation has low efficiency. This is also consistent with the assumption that intrinsic toroidal rotation contribution to entropy generation is smaller than that from zonal flow. Both are a straightforward consequence of the ordering $k_\parallel < k_\perp$. Note that more careful consideration must be given to cases with reversed shear.

In order to explicitly *calculate* the scaling form of e_{IR}, one needs the modeling of residual stress. In doing so, we consider a simple $\langle V_E \rangle'$ driven case, since $\langle V_E \rangle'$ is already given as a consequence of lowest order balance in δf^2 stationarity. In this case, a shift in the spectral envelope, which is required for nonzero Reynolds stress $\langle k_\parallel k_\theta \rangle \propto \langle x \rangle$, originates from the radial electric field shear or $\langle V_E \rangle'$ as[10]

$$\left\langle \frac{k_\parallel}{k_\theta} \right\rangle = -\rho_* \frac{L_s}{2 c_s} \langle V_E \rangle' \tag{42}$$

for simple drift wave turbulence. Here $\rho_* \equiv \rho_s / a$, ρ_s is ion sound Larmor radius, $L_s^{-1} = \hat{s} / (qR)$ is a shear length, and a is the minor radius. This can be further calculated by using the stationary value for the $E \times B$ flow, Eq. (24),

$$\left\langle \frac{k_\parallel}{k_\theta} \right\rangle = \mp \rho_* \frac{L_s}{2 c_s} \sqrt{\frac{\chi_i}{K} \frac{v_{thi}}{L_T}} = \mp \frac{\rho_*}{2\tau} \sqrt{\frac{\chi_i}{K} \frac{L_s}{L_T}}, \tag{43}$$

where $\tau \equiv T_e / T_i$. The sign is ultimately determined by the sign of $E \times B$ shear; however, in the following discussion we only need the squared value of $\langle k_\parallel / k_\theta \rangle$, so the sign is not important. Given the symmetry breaking by $E \times B$ shear, one can calculate the residual stress driven by the wave momentum flux as[9]

$$\Pi_{r\parallel}^{res} = K \langle V_E \rangle' \left\langle \frac{k_\parallel}{k_\theta} \right\rangle, \tag{44}$$

$$= -\rho_* \frac{L_s}{2 c_s} K \langle V_E \rangle'^2, \tag{45}$$

$$= -\rho_* \frac{L_s}{2 c_s} \chi_i \left(\frac{\nabla T}{T} \right)^2 v_{thi}^2. \tag{46}$$

Here we assumed the $E \times B$ flow shear symmetry breaking in the second equality and δf^2 stationarity in the third equality. Note that the residual stress scales directly as the temperature gradient, ∇T. This is due to the fact that to estimate $\langle V_E \rangle'$, we used δf^2 stationarity instead of radial force balance, which would relate $\langle V_E \rangle'$ to the pressure gradient ∇P, rather than ∇T. Use of δf^2 stationarity is more consistent, with both the model under study and with assumptions made in the theory. A recent simulation result by Wang et al.[25] exhibits a similar behavior, albeit the scaling is between intrinsic *torque* ($\propto \nabla \cdot \Pi_{r\parallel}^{res}$) and ion temperature gradient. Wang also noted that intrinsic torque scales with ∇T_e for CTEM turbulence.[26]

One of the consequences of the residual stress modeling here, although somewhat outside of the scope of the paper, is that one can calculate a nontrivial stationary profile of intrinsic toroidal flow as

$$\langle V_\parallel \rangle' = \frac{\Pi_{r\parallel}^{res}}{\chi_\phi} = -\frac{1}{2} \rho_* \frac{\chi_i}{\chi_\phi} \frac{L_s}{c_s} \left(\frac{\nabla T}{T} \right)^2 v_{thi}^2. \tag{47}$$

This simple relation directly relates the intrinsic toroidal flow shear to the temperature gradient—which is consistent with recent experiments on LHD (Ref. 27) and Alcator C-Mod—and the magnetic shear. Note that the intrinsic toroidal flow shear depends strongly on temperature gradient as $\langle V_\parallel \rangle' \propto (\nabla T)^2$, while zonal flow shear is directly proportional to temperature gradient, $\langle V_E \rangle' \propto \nabla T$. This is because in this model, $E \times B$ shear flow plays a dual role in intrinsic toroidal flow shear; i.e., $E \times B$ shear flow breaks symmetry ($\langle k_\parallel k_\theta \rangle \propto \langle V_E \rangle'$) and gives rise to the flux of wave momentum ($\Pi_{r\parallel}^{res} \propto \langle V_E \rangle'$).[9] Hence, $\langle V_\parallel \rangle' \propto \langle V_E \rangle'^2$, which gives the $(\nabla T)^2$ dependence. Note also the explicit ρ_* dependence, which originates from the symmetry breaking. One can also calculate the flow velocity $\langle V_\parallel \rangle$ by integrating once to show

$$\frac{\langle V_\parallel \rangle}{v_{thi}} \cong \frac{1}{2} \rho_* \frac{\chi_i}{\chi_\phi} \frac{L_s}{L_T} \sqrt{\frac{T_i}{T_e}}. \tag{48}$$

Here we used $(T'/T)' = -(T'/T)^2 + T''/T \cong -(T'/T)^2$. The scaling derived here can be compared to Rice scaling[3] $\Delta v_\phi(0) \sim \Delta W_p / I_p$, which shows similar behavior; ∇T is large when confinement is good, such as the H-mode, which tracks the ΔW_p behavior. Current scaling can enter through the geometry of the B field, $L_s \propto q / \hat{s}$, which suggests the scaling, $q \propto B_\theta^{-1} \sim I_p^{-1}$. Note that the scaling calculated here shows the direct dependence on ∇T rather than ∇P, since $\langle \delta f^2 \rangle$ stationarity is used to calculate $\langle V_E \rangle'$ and $\langle \delta f^2 \rangle$ evolves via ITG turbulence. Note also that the expression for the flow contains the information regarding directionality. However, the sign of the flow direction is strongly model dependent.[10] Moreover, this is a consequence of residual stress modeling and is not directly related to the efficiency calculation, which is the main focus of the paper. Indeed, note $e \sim \Pi_{r\parallel}^{res2}$, so e is independent of the sign of $\Pi_{r\parallel}^{res}$. Hence, here we do not pursue a detailed discussion regarding the relation between flow direction and entropy, but rather leave this to a future publication. We also note that a similar scaling $\Delta v_\phi \sim \nabla \langle T \rangle / B_\theta$ was proposed on the basis of the modeling of off-diagonal components in momentum flux.[28,29] That work was concerned with the velocity increment for the change of NBI direction and included the explicit momentum source (NBI torque) in the analysis.

The efficiency can be calculated by using the value for $\Pi_{r\parallel}^{res}$,

$$e_{IR} = \frac{\int d^3 x n \frac{\chi_i}{\chi_\phi} \chi_i (\nabla T/T)^2 \frac{L_s^2}{L_T^2} \phi \frac{\rho_*^2}{4} \frac{v_{thi}^2}{c_s^2}}{\int d^3 x n \chi_i (\nabla T/T)^2} \sim \rho_*^2 \frac{q^2}{\hat{s}^2} \frac{R^2}{L_T^2}, \quad (49)$$

where we assumed that $\chi_i \sim \chi_\phi$, $T_e \sim T_i$, and $\hat{s} \neq 0$. The efficiency depends on (i) machine size, ρ_*, which implies the efficiency will decrease for larger machines. Note that the ρ_* scaling appears, even *after* calculating the ratio of turbulence driven quantities, i.e., it is not a trivial consequence of $|e|\tilde{\phi}/T_e \sim \rho_*$ scaling. In fact, the ρ_* scaling originates from ρ_* dependence in the symmetry breaking correlator, $\langle k_\parallel k_\theta \rangle \propto \langle x \rangle \propto \rho_*$. We speculate the ρ_* dependence is thus inherent to any residual stress modeling based on $k_\parallel$ symmetry breaking of drift wave turbulence. (ii) Geometry of the B field, $q/\hat{s}$. In a simple geometry with $\hat{s} = \text{const} \sim O(1)$, the efficiency varies as $q \propto B_\theta^{-1} \sim I_p^{-1}$, which shows an *unfavorable* current scaling, as in the Rice scaling[3] $\Delta v_\phi(0) \sim \Delta W_p/I_p$. Note that this is a $q(r)$ scaling, not an I_p scaling. (iii) Temperature gradient, R/L_T, which originates from both symmetry breaking and wave momentum flux driven by $V_E' \propto \nabla T$. Plasmas with a steep gradient, i.e., such as H-mode plasmas, are more effective and efficient for driving intrinsic toroidal rotation. The dependence on ∇T can be linked to the ΔW_p dependence in the Rice scaling. Here, the efficiency scaling of intrinsic rotation drive is directly tied to ∇T rather than ∇P. This is a consequence of the fact that the model in this paper is derived for ITG turbulence. The resultant $E \times B$ flow is also driven by ITG turbulence, so it is no surprise that we have $\langle V_E \rangle' \propto \nabla T$. Note that the scaling was evaluated in local form in the last expression. This is a reasonable approximation when a system has a well-defined gradient region, such as for a peaked profile or a transport barrier, for example. Of course the case with reversed shear internal transport barrier is of great interest; however, this is beyond the scope of the paper, which assumes normal shear with $\hat{s} \sim O(1)$.

IV. CONCLUSION

In this paper, by analogy between plasma flow generation and an engine, we introduced the concept of flow generation efficiency by calculating the ratio of the entropy destruction rate due to turbulent flow generation to the entropy production rate due to thermal relaxation. The principal results are the following.

(1) The entropy production rate was calculated and shown to be

$$\mathcal{P} = n\chi_i \left(\frac{\nabla T}{T}\right)^2 - nK\left(\frac{\langle V_E \rangle'}{v_{thi}}\right)^2 + n\chi_\phi \left(\frac{\langle V_\parallel \rangle'}{v_{thi}}\right)^2 - n\frac{\Pi_{r\parallel}^{res2}}{v_{thi}^2 \chi_\phi}.$$

Thermal relaxation and viscous heating produce entropy. Flow generation, driving both zonal flow and intrinsic toroidal rotation, leads to the destruction of entropy. The first two terms are larger than the last two terms by the order of $O(k_\parallel/k_\perp)$. The production rate due thermal relaxation (the first term) and viscous heating (the third term) differs in magnitude by $M_i^2 \equiv (\langle V_\parallel \rangle/v_{thi})^2 \sim 0.01$ for a typical value of $M_i \sim 0.1$, since $\chi_i \sim \chi_\phi$,

$(\langle V_\parallel \rangle'/v_{thi}) \sim M_i(\langle V_\parallel \rangle'/\langle V_\parallel \rangle)$, $(\nabla T/T) \sim \sigma(\langle V_\parallel \rangle'/\langle V_\parallel \rangle)$, and $\sigma \sim 1$.

(2) Coupled equations for phase space density fluctuation intensity δf^2 and zonal flow were formulated based on entropy budget and wave kinetic analysis. They have a similar structure to the familiar predator-prey model,

$$\partial_t \int d\Gamma \frac{\langle \delta f^2 \rangle}{2\langle f \rangle} = \int d^3 x \left[n\chi_i \left(\frac{\nabla T}{T}\right)^2 - nK\left(\frac{\langle V_E \rangle'}{v_{thi}}\right)^2 \right],$$

$$\partial_t \frac{\langle V_E \rangle'^2}{2} = K\bar{q}_r^2 \langle V_E \rangle'^2 - \nu_{col}\langle V_E \rangle'^2,$$

where δf^2 plays the role of the prey population density.

(3) The stationary levels of zonal flow and intrinsic toroidal rotation were calculated for the state achieved by imposing $\mathcal{P} = 0$ order by order. They are

$$\langle V_E \rangle'^2 = \frac{\chi_i}{K} \frac{v_{thi}^2}{L_T^2},$$

$$\langle V_\parallel \rangle' = -\frac{1}{2} \rho_* \frac{\chi_i}{\chi_\phi} \frac{L_s}{c_s} \left(\frac{\nabla T}{T}\right)^2 v_{thi}^2,$$

$$\frac{\langle V_\parallel \rangle}{v_{thi}} \cong \frac{1}{2} \rho_* \frac{\chi_i}{\chi_\phi} \frac{L_s}{L_T} \sqrt{\frac{T_i}{T_e}}.$$

The first relation is obtained from lowest order balance in the entropy production rate. The $E \times B$ shear is tied to the ∇T thermodynamic force directly, since at saturation entropy destruction due to zonal flow balances entropy production due to thermal relaxation. The second relation is calculated from the next order balance in the entropy production rate and the third relation is obtained by integrating the second relation. The intrinsic toroidal flow shows a similar scaling to the Rice scaling $\Delta v_\phi(0) \sim \Delta W_p/I_p$, i.e., $L_T^{-1} \sim \nabla T$ corresponds to ΔW_p and $L_s \propto q \sim B_\theta^{-1}$ for fixed magnetic shear corresponds to I_p^{-1}. Explicit ρ_* scaling originates from the symmetry breaking mechanism invoked in the model.

(4) The efficiency of flow generation is defined as the ratio of entropy destruction rate due to flow generation to entropy production rate due to thermal relaxation. The actual value for the efficiency was calculated for intrinsic toroidal rotation and shown to be $e_{IR} \sim M_i^2 \sim 0.01-0.1$ for a value of $M_i \sim 0.1-0.3$. This indicates that the drive of toroidal rotation is inherently one of the processes of modest efficiency. This finding follows from $k_\parallel < k_\perp$.

(5) The scaling of the intrinsic toroidal flow generation efficiency was derived as

$$e_{IR} = \rho_*^2 \frac{q^2}{\hat{s}^2} \frac{R^2}{L_T^2}.$$

The efficiency of intrinsic toroidal flow generation scales as machine size ρ_*, geometry of the B field $q/\hat{s}$, and temperature profile R/L_T. Related to (3) above, the efficiency exhibits a similar scaling behavior to the Rice scaling, except for the appearance of explicit ρ_* scaling.

Note that the efficiency scaling suggests a possible origin of the unfavorable current scaling through the safety factor[30] q.

As a caveat, the model cannot capture the phenomenology of flow direction dependence on plasma current direction. In particular, the model cannot describe the reversal of flow direction in TCV,[31] since this reversal is likely related to the conversion of drift modes between ion and electron branches. However, the model presented here includes only ITG turbulence. A recent simulation by Wang also showed that the residual stress scaling is strongly dependent upon the kind of driving turbulence, i.e., the residual stress scale with ∇T_i for ITG turbulence and with ∇P_e for CTEM turbulence,[26] which is likely to give a different efficiency scaling for CTEM turbulence. To capture the flow reversal physics and clarify the mode dependence of the efficiency scaling, one would need a further extension of the theory to include the dynamics of nonadiabatic electrons and their role in the entropy budget. The boundary term is dropped throughout the analysis as well. These may also have an impact on the entropy budget. Note that in H-mode, turbulence is unlikely and fluctuation flux, a cause of the boundary terms, is quenched. Note also that the calculation implemented here is a reasonably faithful model of the computer simulation studies by Wang.[25] In that δf PIC simulation using GTS, $\nabla \phi = 0$ is imposed at the boundaries, thus guaranteeing no entropy outflow. Interestingly, that simulation observed symmetry breaking by zonal flow shear $\langle V_E \rangle'$ and a level of intrinsic toroidal rotation $\langle V_\phi \rangle \sim \nabla T$, as calculated here. This suggests that it would be interesting for the simulation to examine the ρ_* and I_p scaling of the intrinsic rotation, as well as to directly calculate the efficiency and compare with theoretical predictions. The role of the boundary term will be pursued in the future publication.

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APPENDIX A: LINEAR MODE

In this section, we review the basic properties of drift waves (DWs) which we need in the calculation of shift in the mode. First we start with DW without symmetry breaking. Susceptibility for DW [$\tilde{n}/n = \chi(|e|\tilde{\phi}/T_e)$] is given by

$$\chi = \frac{\omega_{*e}}{\omega} + \frac{k_\parallel^2 c_s^2}{\omega^2} - 1 - k_\perp^2 \rho_s^2. \tag{A1}$$

In sheared magnetic field, susceptibility takes an operator form,

$$\hat{\chi} = \frac{\omega_{*e}}{\omega} + \frac{k_y^2 c_s^2 x^2}{\omega^2 L_s^2} - 1 - k_y^2 \rho_s^2 + \rho_s^2 \partial_x^2. \tag{A2}$$

Solving the eigenvalue problem $\hat{\chi}\hat{\phi}=0$, one obtains

$$\omega = \frac{\omega_{*e}}{1 + k_y^2 \rho_s^2} - i\frac{|L_n|}{|L_s|}, \tag{A3}$$

$$\phi \propto \exp\left(-i\frac{\mu}{2}x^2\right) \quad \text{with} \quad \mu \equiv \frac{L_n}{\rho_s^2 L_s}. \tag{A4}$$

With $E \times B$ shear flow as a symmetry breaker,

$$\hat{\chi} \cong \frac{\omega_{*e}}{\omega}\left(1 - \frac{k_y \langle V_E \rangle' x}{\omega}\right) + \frac{k_y^2 c_s^2 x^2}{\omega^2 L_s^2} - 1 - k_y^2 \rho_s^2 + \rho_s^2 \partial_x^2, \tag{A5}$$

and the mode will be shifted around a rational surface by

$$x_0 = -\rho_* \frac{L_s^2}{2c_s}\langle V_E \rangle', \tag{A6}$$

$$\phi \propto \exp\left(-i\frac{\mu}{2}(x+x_0)^2\right). \tag{A7}$$

Then the spectral average of $k_\parallel$ is obtained as

$$\left\langle \frac{k_\parallel}{k_\theta} \right\rangle = \left\langle \frac{x_0}{L_s} \right\rangle = -\rho_* \frac{L_s}{2c_s}\langle V_E \rangle'. \tag{A8}$$

APPENDIX B: WAVE KINETIC ANALYSIS OF FLOW GENERATION

In this section, we derive the radial momentum flux of $E \times B$ shear flow, the growth rate of the mean $E \times B$ flow, and the radial momentum flux of toroidal flow based on wave kinetic equations. We start with wave kinetic equation,

$$\partial_t N_k + \frac{\partial \omega_k}{\partial \mathbf{k}} \cdot \frac{\partial N_k}{\partial \mathbf{x}} - \frac{\partial \omega_k}{\partial \mathbf{x}} \cdot \frac{\partial N_k}{\partial \mathbf{k}} = -2\frac{\text{Im }\epsilon}{\partial \epsilon/\partial \omega}N_k + C_w(N_k). \tag{B1}$$

Here $N_k = (\partial \epsilon/\partial \omega)(|E_k|^2/8\pi)$ is wave action density and we allowed wave-wave scattering in the right hand side. At the simplest level one can employ Krook type operator $C_w(N_k) = -\nu_{\text{eff}}\delta N_k$. In the following calculation we assume strong "collisionality" between waves, i.e., ν_{eff}^{-1} is assumed to be the fastest timescale.[9] Note that the dielectric function ϵ is related to the susceptibility in the last section as $\epsilon = 1 - \chi/(k^2\lambda_{\text{De}}^2)$. For example, the wave action density for EDW is

$$N_k = -\frac{\partial \chi}{\partial \omega}\frac{|E_k|^2}{8\pi k^2 \lambda_{\text{De}}^2} = \frac{nT_e}{2\omega_{*e}}(1 + k_\perp^2 \rho_s^2)^2\left|\frac{e\phi_k}{T_e}\right|^2. \tag{B2}$$

Inhomogeneity in medium, such as intensity gradient and $E \times B$ shear flow, builds up inhomogeneity in wave population density. For the general derivation and discussion, see Ref. 9. For the purpose of this paper, it is sufficient to consider a pure $\langle V_E \rangle'(x)$ driven case. For this case, one can solve wave kinetic equation to obtain

$$\delta N_k(x) = \frac{1}{\nu_{\text{eff}}} k_\theta \delta \langle V_E \rangle'(x) \frac{\partial \langle N_k \rangle}{\partial k_r}. \tag{B3}$$

With this, one can calculate the Reynolds stress to drive ZF and the residual stress for toroidal flow.

For the Reynolds stress for ZF, one can calculate as

$$\delta \langle \widetilde{V}_r \widetilde{V}_\theta \rangle(x) = -\sum_k \frac{\rho_s^2 k_r k_\theta c_s^2}{(1 + k_\perp^2 \rho_s^2)^2} \frac{2\omega_{*e}}{nT_e} \delta N_k(x)$$

$$\equiv K \delta \langle V_E \rangle'(x), \tag{B4}$$

where

$$K \equiv \sum_k c_s^2 \tau_{\text{ZF}} \frac{\rho_s^2 k_\theta^2}{(1 + k_\perp^2 \rho_s^2)^2} \left(-k_r \frac{\partial \langle \eta_k \rangle}{\partial k_r} \right), \tag{B5}$$

$\eta_k \equiv (1 + k_\perp^2 \rho_s^2)^2 |(e\phi_k)/T_e|^2$ is potential enstrophy, and $\tau_{\text{ZF}} \equiv \nu_{\text{eff}}^{-1}$. The growth rate is easily obtained with the momentum flux derived above and shown to be

$$\gamma_{\text{flow}} = q_r^2 K. \tag{B6}$$

The instability requires $K > 0$ or $-k_r(\partial \langle \eta_k \rangle / \partial k_r) > 0$, which is the same criterion for zonal flow growth.

For the residual stress in the parallel momentum flux $\Pi_{r\parallel}^{\text{res}}$, one can calculate as[9]

$$\Pi_{r\parallel}^{\text{res}} = \frac{1}{m_i n} \sum_k v_{g,r} k_\parallel \delta N_k$$

$$= \frac{c_s^2}{T_e n} \sum_k -\frac{2\rho_s^2 k_r k_\theta v_*}{(1 + k\perp^2 \rho_s^2)} k_\parallel \delta N_k$$

$$\equiv \left\langle \frac{k_\parallel}{k_\theta} \right\rangle K \delta \langle V_E \rangle'(x), \tag{B7}$$

where

$$\left\langle \frac{k_\parallel}{k_\theta} \right\rangle \equiv \frac{1}{K} \sum_k \frac{k_\parallel}{k_\theta} c_s^2 \tau_{\text{ZF}} \frac{\rho_s^2 k_\theta^2}{(1 + k_\perp^2 \rho_s^2)^2} \left(-k_r \frac{\partial \langle \eta_k \rangle}{\partial k_r} \right). \tag{B8}$$

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