Title
Matrix pencil and system poles

Journal
Signal Processing, 21(2)

ISSN
1872-7557

Authors
Hua, Yingbo
Sarkar, Tapan K

Publication Date
1990-10-01

DOI
10.1016/0165-1684(90)90050-9

Peer reviewed

Signal Processing 21 (1990) 195-198
Elsevier

SHORT COMMUNICATION

MATRIX PENCIL AND SYSTEM POLES*

Yingbo HUA

Department of Electrical Engineering, University of Melbourne, Parkville, Victoria 3052, Australia

Tapan K. SARKAR

Department of Electrical and Computer Engineering, Syracuse University, Syracuse, NY 13244-1240, U.S.A.

Received 2 October 1989
Revised 9 May 1990

Abstract. Poles of a linear time-invariant system can be extracted from a matrix pencil constructed from the transient response of the system. Three subspace filtering techniques applicable to the matrix pencil method are presented briefly in a unified way.

Zusammenfassung. Die Pole eines linearen, zeitinvarianten Systems lassen sich aus einem Matrixpencil gewinnen, der aus der Sprungantwort des Systems konstruiert wird. Es werden in aller Kürze drei Unterraum-Filtertechniken vereinheitlicht vorgestellt, die auf diese Matrixpencil-Methode anzuwenden sind.

Résumé. Les pôles d'un système linéaire invariant dans le temps peuvent être extraits de matrices 'pencil' construites à partir de la réponse transitoire du système. Trois méthodes de filtrage de sous-espace, applicable à cette méthode, sont présentées brièvement et d'une manière unifiée.

Key words. System identification, transient processing, matrix pencil, subspace decomposition.

1. Introduction

It is known that the transient response of linear time-invariant (LTI) system can be generally described by

$$y(t)=\sum_{i=1}^{d}(b_{i,0}+tb_{i,1}+\cdots+t^{m(i)-1}b_{i,m(i)-1})\exp(s_it), \tag{1.1}$$

where $m(i)$ is the multiplicity of the pole s_i, $M=\sum_{i=1}^{d} m(i)$, which is the order of the system or the total number of poles, d is the number of distinct poles. For a defective system, $m(i)>1$ for at least one i. In matrix form, $y(t)$ can be written as

$$y(t)=A(t)\boldsymbol{b}, \tag{1.2}$$

where

$$A(t)=[A_1(t),\ldots,A_d(t)], \tag{1.3}$$

$$A_i(t)=[\exp(s_it),\ldots,t^{m(i)-1}\exp(s_it)], \tag{1.4}$$

$$\boldsymbol{b}=[\boldsymbol{b}_1^{\mathrm{T}},\ldots,\boldsymbol{b}_d^{\mathrm{T}}]^{\mathrm{T}}, \tag{1.5}$$

$$\boldsymbol{b}_i^{\mathrm{T}}=[b_{i,0},\ldots,b_{i,m(i)-1}]. \tag{1.6}$$

The superscript $^{\mathrm{T}}$ denotes the transposition. Note that $A(t)$ is completely determined by s_i, d and $m(i)$. Given $A(t)$, $\boldsymbol{b}$ can be obtained in a least squares sense by

$$\boldsymbol{b}=A^{+}\boldsymbol{y}, \tag{1.7}$$

* This work was supported in part by Aeritalia Corporation and by the Office of Naval Research under contract N00014-79-C-0598.

where the superscript $^+$ denotes the Moore–Penrose inverse [1] and

$$A=[A^{\mathrm{T}}(t_0),\ldots,A^{\mathrm{T}}(t_{N-1})]^{\mathrm{T}}, \tag{1.8}$$

$$y=[y(t_0),\ldots,y(t_{N-1})]^{\mathrm{T}}, \tag{1.9}$$

in which $t_0,\ldots,t_{N-1}$ can be any distinct sample times but $N\geqslant M$.

In this short paper, we will present that the parameters s_i, d and $m(i)$ can be obtained from a matrix pencil constructed from the uniformly sampled data $y_k=y(T_s k)$. T_s is the sampling interval. In particular, three subspace filtering techniques proposed in [3, 7] are shown in a unified way.

2. Matrix pencil method

We define two matrices Y_0 and Y_1 as

$$Y_0=[y_{L-1}\ y_{L-2}\ \cdots\ y_0] =\begin{bmatrix} y_{L-1} & y_{L-2} & \cdots & y_0\\ \vdots & \vdots & & \vdots\\ y_{N-2} & y_{N-3} & \cdots & y_{N-L-1}\end{bmatrix} \quad ((N-L)\times L), \tag{2.1}$$

$$Y_1=[y_L, y_{L-1},\ldots,y_1] =\begin{bmatrix} y_{L} & y_{L-1} & \cdots & y_1\\ \vdots & \vdots & & \vdots\\ y_{N-1} & y_{N-2} & \cdots & y_{N-L}\end{bmatrix} \quad ((N-L)\times L), \tag{2.2}$$

where $M\leqslant L\leqslant N-M$. Then the following analytical decomposition can be shown (see Appendix A):

$$Y_0=Z_{\mathrm{L}}Z_{\mathrm{R}}, \tag{2.3}$$

$$Y_1=Z_{\mathrm{L}}ZZ_{\mathrm{R}}, \tag{2.4}$$

where Z_{L} is a rank-M $(N-L)\times M$ matrix, Z_{R} a rank-M $M\times L$ matrix and Z the companion matrix of the polynomial $\sum_{m=0}^{M} c_m z^{-m}$ with $c_0=1$. c_m's are the M coefficients of the M-order linear prediction equation of y_k.

It is known that Z has d distinct eigenvalues $\{z_i=\exp(s_iT_s),\ i=1,\ldots,d\}$, each of algebraic multiplicity (AM) equal to $m(i)$ and geometric multiplicity (GM) equal to one. It should be noted that Y_0 and Y_1 have the same column space and the same row space, and each of them has the rank M, which implies that M can be estimated by the number of dominant singular values of Y_0 and Y_1 (e.g., see [4, 5]).

In the following, we show three algorithms for extracting the poles z_i (hence s_i, d and $m(i)$) from Y_0 and Y_1, assuming M is known.

Direct algorithm [3]

Y_0 can be numerically (e.g., using SVD [1]) decomposed into

$$Y_0=Y_{\mathrm{L}}Y_{\mathrm{R}}, \tag{2.5}$$

where Y_{L} is a rank-M $(N-L)\times M$ matrix (spanning the column space of Y_0) and Y_{R} a rank-M $M\times L$ matrix (spanning the row space of Y_0). Specifically, $Y_{\mathrm{L}}=Z_{\mathrm{L}}Y$ and $Y_{\mathrm{R}}=Y^{-1}Z_{\mathrm{R}}$, where Y is an $M\times M$ nonsingular matrix (which can be arbitrary otherwise). Now the $M\times M$ matrix P_1, defined by

$$P_1=Y_{\mathrm{L}}^{+}Y_1Y_{\mathrm{R}}^{+}, \tag{2.6}$$

where $Y_{\mathrm{L}}^{+}=(Y_{\mathrm{L}}^{\mathrm{H}}Y_{\mathrm{L}})^{-1}Y_{\mathrm{L}}^{\mathrm{H}}$ and $Y_{\mathrm{R}}^{+}=Y_{\mathrm{R}}^{\mathrm{H}}(Y_{\mathrm{R}}Y_{\mathrm{R}}^{\mathrm{H}})^{-1}$, becomes $P_1=Y^{-1}Z_{\mathrm{L}}^{+}Z_{\mathrm{L}}ZZ_{\mathrm{R}}Z_{\mathrm{R}}^{+}Y=Y^{-1}ZY$. Since P_1 is a similar transform of Z, P_1 has the same eigenvalues as Z, i.e., d distinct eigenvalues $\{z_i\,|\,i=1,\ldots,d\}$ each of AM $m(i)$ and GM one.

Subspace estimation (SE) algorithm [7]

Both Y_0 and Y_1 can be alternatively numerically [1] decomposed into

$$\underset{(2(N-L)\times L)}{\begin{bmatrix} Y_0\\ Y_1\end{bmatrix}} = \underset{(2(N-L)\times M)}{\begin{bmatrix} X_0\\ X_1\end{bmatrix}} \underset{(M\times L)}{X_{\mathrm{R}}}, \tag{2.7}$$

where

$$\operatorname{span}\begin{bmatrix} X_0\\ X_1\end{bmatrix}=\operatorname{span}\begin{bmatrix} Z_{\mathrm{L}}\\ Z_{\mathrm{L}}Z\end{bmatrix}=\operatorname{span}\begin{bmatrix} Y_0\\ Y_1\end{bmatrix}, \tag{2.8}$$

which indicates the subspace estimation process (for both Y_0 and Y_1) inherent in the SE algorithm. Since $X_0 = Z_L X$ and $X_1 = Z_L Z X$, where X is an $M \times M$ nonsingular matrix, the $M \times M$ matrix P_2, defined by

$$P_2 = X_0^+ X_1, \tag{2.9}$$

where $X_0^+ = (X_0^H X_0)^{-1} X_0^H$, becomes $P_2 = X^{-1} Z_L^+ Z_L Z X = X^{-1} Z X$. Therefore, P_2 is similar to Z and has d distinct eigenvalues $\{z_i \mid i = 1, \ldots, d\}$ each of AM $m(i)$ and GM one. X_0 and X_1 are a pair of compressed matrices from Y_0 and Y_1 unless $L = M$.

Subspace estimation and subspace fitting (SESF) algorithm [7]

Further matrix compression can be made on X_0 and X_1 before retrieving the poles. Since X_0 and X_1 should have the same column space, there exist two $M \times M$ nonsingular matrices T_0 and T_1 such that

$$X_0 T_0 - X_1 T_1 = 0, \tag{2.10}$$

which indicates a subspace fitting process inherent in the SESF algorithm. Numerically [1], $\begin{bmatrix} T_0 \\ T_1 \end{bmatrix}$ can be the M non-principal right singular vectors of $[X_0, X_1]$. From (2.10), $T_0 T_1^{-1} = X_0^+ X_1$. Hence, the $M \times M$ matrix P_3, defined by

$$P_3 = T_0 T_1^{-1}, \tag{2.11}$$

is P_2 and has d distinct eigenvalues $\{z_i \mid i = 1, \ldots, d\}$ each of AM $m(i)$ and GM one.

3. Discussions

In the matrix pencil method, L is a free parameter subject to $M \leqslant L \leqslant N - M$. A proper value for L should be such that the column (or row) space of Y_0 and Y_1 has the largest dimension so that the maximum noise components fall into the noise subspace (which is automatically discarded in the numerical computations of the above three algorithms). But for $\frac{1}{3}N \leqslant L \leqslant \frac{2}{3}N$, the robustness of the estimated poles to noise is relatively invariant [2, 3]. On the other hand, a large dimension of Y_0 and Y_1 implies that more computations are needed in performing subspace filtering. It is clear that the three algorithms are in increasing order of computation. Consistent with Roy's observation [7] for wave direction finding, we found [2] that for estimating poles from transient responses, the extra computation used in the SESF algorithm makes it the most robust to noise. But for SNR above a threshold, the three algorithms are equally accurate [2].

If the signal is known to be oversampled, the matrix pencil method can be modified [2] to yield more accurate poles than the well-known Prony's method, without using subspace filtering.

Comparing to the SVD Prony's method [5, 6], the matrix pencil method has been shown in [3] to be more efficient in computation as well as more robust to noise.

Finally, we mention that the above matrix pencil approach is a special case of using a set of matrices as defined in the following:

$$\begin{aligned} Y_{0,G} &= [y_{L-G} \; y_{L-G-1} \; \cdots \; y_0], \\ Y_{1,G} &= [y_{L-G+1} \; y_{L-G} \; \cdots \; y_1], \\ &\vdots \\ Y_{G,G} &= [y_L \; y_{L-1} \; \cdots \; y_G], \end{aligned}$$

where $L - M + 1 \geqslant G \geqslant 1$. These matrices can be analytically decomposed into

$$\begin{aligned} Y_{0,G} &= Z_L Z_{RG}, \\ Y_{1,G} &= Z_L Z Z_{RG}, \\ &\vdots \\ Y_{G,G} &= Z_L Z^G Z_{RG}, \end{aligned}$$

where Z_{RG} is a rank-M $M \times (L - G + 1)$ matrix. It is clear that the previous three algorithms can be applied to any pair in the matrix set $\{Y_{0,G}, Y_{1,G}, \ldots, Y_{G,G}\}$ to extract the poles $\{z_i \mid i = 1, \ldots, d\}$. However, an efficient way of utilizing the whole matrix set $\{Y_{i,G} \mid i = 0, 1, \ldots, G;$

$G=1,2,\ldots,L-M+1\}$ to yield better estimates of $\{z_i\,|\,i=1,\ldots,d\}$ has not yet been found.

Appendix

Since y_k is the transient response of an order-M LTI system, y_k satisfies, for a unique set of coefficients c_m's,

$$y_k=-\sum_{m=1}^{M} c_m y_{k-m} \tag{A.1}$$

or, equivalently,

$$\mathbf{y}_k=-\sum_{m=1}^{M} c_m \mathbf{y}_{k-m}, \tag{A.2}$$

which implies that $\mathbf{y}_j$ for any j can be written as a linear combination of the M independent (assuming $M\leqslant N-L$) vectors $\mathbf{y}_0,\mathbf{y}_1,\ldots,\mathbf{y}_{M-1}$. Hence, there is an $M\times(L-M)$ matrix $Z_{\rm R}'$ such that

$$Y_0=[\mathbf{y}_{M-1}\ \mathbf{y}_{M-2}\ \ldots\ \mathbf{y}_0][Z_{\rm R}',\ I_{M\times M}], \tag{A.3}$$

where $I_{M\times M}$ is the $M\times M$ identity matrix. Defining $Z_{\rm L}=[\mathbf{y}_{M-1}\ \mathbf{y}_{M-2}\ \cdots\ \mathbf{y}_0]$ and $Z_{\rm R}=[Z_{\rm R}',\ I_{M\times M}]$ yields (2.3).

Equation (2.4) follows from

$$Y_1=[\mathbf{y}_M\ \mathbf{y}_{M-1}\ \cdots\ \mathbf{y}_1]Z_{\rm R}=Z_{\rm L}ZZ_{\rm R}, \tag{A.4}$$

where

$$Z=\begin{bmatrix} -c_1 & 1 & & & \\ -c_2 & & 1 & & \\ \vdots & & & \ddots & \\ \vdots & & & & 1 \\ -c_M & & & & 0 \end{bmatrix}. \tag{A.5}$$

References

[1] G.H. Golub and C.F. Van Loan, *Matrix Computations*, Johns Hopkins Univ. Press, Baltimore, MD, 1983.

[2] Y. Hua and T.K. Sarkar, "System identification by matrix pencil method", *Technical Report*, Syracuse University, 1988.

[3] Y. Hua and T.K. Sarkar, "Matrix pencil method for estimating parameters of exponentially damped/undamped sinusoids in noise", *IEEE Trans. Acoust. Speech Signal Process.*, Vol. 38, No. 5, May 1990, pp. 814–824.

[4] K. Konstantinies and K. Yao, "Statistical analysis of effective singular values in matrix rank determination", *IEEE Trans. Acoust. Speech Signal Process.*, Vol. 36, No. 5, May 1988, pp. 757–763.

[5] R. Kumaresan, "Estimating the parameters of exponentially damped or undamped sinusoidal signal in noise", *Ph.D. Dissertation*, University of Rhode Island, 1982.

[6] S.L. Marple, Jr., *Digital Spectral Analysis with Applications*, Prentice-Hall, Englewood Cliffs, NJ, 1987.

[7] R. Roy and T. Kailath, "Total least square ESPRIT", *Proc. of 21st Asilomar Conf. on Systems and Signals*, November 1987.