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# Ricci flows with non-compact initial conditions

by

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Abstract

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First, we show that a Ricci flow can be started from a non-compact complete manifold, if the manifold is non-collapsed and satisfies a lower bound for many known curvature conditions. In this theorem we do not need the manifold to have bounded curvature, which was assumed in an earlier work by Bamler-Cabezas-Rivas-Wilking.

Second, we show that a Ricci flow can be started from a 3d complete manifold with non-negative Ricci curvature. This gave a partial affirmative answer to a conjecture by Topping. we prove it by generalizing the concept of singular Ricci flow by Kleiner and Lott to non-compact initial conditions.

Third, we find a family of 3d steady gradient Ricci solitons that are flying wings. This verifies Hamilton's flying wing conjecture. We also show that the scalar curvature does not vanish at infinity in a 3d flying wing. For dimension $n \geq 4$, we find a family of non-collapsed, $\mathbb{Z}_2 \times O(n-1)$-symmetric, but non-rotationally symmetric n-dimensional steady gradient solitons with positive curvature operator.

This thesis is a composition of the following three papers of the author: "Ricci flow under local almost non-negative curvature conditions", "Producing 3d Ricci flows with non-negative Ricci curvature via singular Ricci flows", "A family of 3d steady gradient solitons that are flying wings" [48, 47, 46].

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# Chapter 1

# Introduction

A Ricci flow is a family of Riemannian metrics $g(t)$, $t \in [0, T]$, on a manifold $M$ evolving by the partial differential equation

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}(g(t)). \tag{1.0.1}$$

The Ricci curvature Ric is a symmetric bilinear form obtained by tracing the full curvature tensor. In a vague sense, the Ricci curvature is the laplacian of the metric, and hence the Ricci flow is the heat equation for a Riemannian manifold.

The Ricci flow was introduced by Hamilton in 1982, and has proven itself to be important in differential geometry. The most remarkable application is the resolution of Poincaré conjecture and the Geometrization conjecture by Perelman. There are many remaining problems in Ricci flow after Perelman's work. In particular, much less is known about Ricci flows with non-copmact initial conditions than those with compact ones.

Ricci flows with non-compact initial conditions are very useful to study the geometry and topology of the initial manifolds. For example, it is widely used to smooth a Riemannian metric and gain more regularity. Moreover, as compact Ricci flows are used by Perelman to solve the Poincaré conjecture, non-compact Ricci flows are also important tools to understand the underlying topology of non-compact Riemannian manifolds.

As the heat equation with non-compact initial data is not always solvable, it is not always possible to start a Ricci flow from a non-compact Riemannian manifold. Therefore, it is crucial to determine under what conditions we can run a Ricci flow from non-compact Riemannian manifolds.

It is well-known that many non-negative curvature conditions guarantee a short-time existence of such flows. Recently, it was discovered that with almost non-negative curvatures we can also produce Ricci flows from non-compact manifolds: Bamler-Cabezas-Rivas-Wilking showed that a Ricci flow can be started from a complete non-compact manifold, assuming the manifold satisfies a volume non-collapsing assumption, and a certain curvature has a negative lower bound [2]. However, for a few curvatures, they had also to assume that the curvature norm is uniformly bounded on the initial manifold. By a different approach, Simon-Topping obtained the same result particularly for Ricci curvature in dimension 3 [64].

In Chapter 3, we show that for several almost non-negative curvature conditions, a Ricci flow can be started from a non-compact, complete, and non-collapsed manifold [48]. We use a combination of methods from the previous two results [2] and [64], and generalize them in the following sense: First, we do not need the bounded curvature assumption as in [2]. Second, our result holds for the 2-nonnegative curvature in all $n \geq 3$, which implies Simon-Topping's result when $n = 3$.

Usually the existence theorems of non-compact Ricci flows need to assume the initial manifold is non-collapsing [64, 2, 48]. In fact, there is a counterexample by Topping when the initial manifold is collapsed and has negative curvature somewhere [65]. However, the non-collapsing assumption seems removable under the non-negative curvature conditions. For example, if the complex sectional curvature is non-negative, then a Ricci flow exists. In dimension 3, the non-negative complex sectional curvature, equivalent to non-negative sectional curvature, is stronger than non-negative Ricci curvature.

Therefore, Topping conjectured that a complete Ricci flow can be started from a complete 3-manifold with non-negative Ricci curvature. In Chapter 4, we give a partial affirmative answer to Topping's conjecture, modulo the completeness of the Ricci flow we construct. Our construction uses the singular Ricci flow developed by Bamler, Kleiner and Lott [44, 45, 4]. We show that the concept of singular Ricci flow can be extended to non-compact initial manifolds. Our generalized singular Ricci flow has similar singularity-forming properties as the ordinary singular Ricci flow. We construct a smooth Ricci flow from a manifold with non-negative curvature by first running our generalized singular Ricci flow and then show that it is actually smooth.

Ricci flows with non-compact initial conditions are also important in singularity analysis of compact Ricci flows. Many singularity models that are non-compact such as the Bryant soliton and the cylindrical flow. Moreover, most of singularity models are self-similar Ricci flows, which are called solitons.

The soliton equation is the elliptic version of the Ricci flow equation. It reads

$$\mathrm{Ric}(g) = \lambda g + \frac{1}{2}\mathcal{L}_V g \tag{1.0.2}$$

for some constant $\lambda$ and a smooth vector field $V$. The soliton equation is a generalization of the Einstein equation $\mathrm{Ric}(g) = \lambda g$, both of which generate self-similar Ricci flows. Depending on $\lambda > 0$, $\lambda < 0$, or $\lambda = 0$, a soliton is called shrinking, expanding or steady. Moreover, a soliton is called a gradient soliton if $V$ is the gradient of some smooth function.

In dimension 2, Hamilton's cigar soliton is the only steady gradient soliton that is non-flat [37]. It is rotationally symmetric and has positive curvature. For all $n \geq 3$, the only non-flat and rotationally symmetric solitons are the Bryant solitons on $\mathbb{R}^n$ [11]. The Bryant solitons have positive curvature. It is a well-known conjecture by Hamilton that there exists a 3d steady gradient soliton that is a so-called flying wing. The term flying wing was used by Hamilton to describe certain steady gradient solitons in Ricci flow and their analogues in mean curvature flow. In Ricci flow and particularly in dimension 3, a flying wing is a steady gradient soliton, which is asymptotic to a 2-dimensional sector with angle $\alpha \in (0, \pi)$.

In mean curvature flow, the study of the analogues of steady gradient solitons had many exciting results in the past two decades. In the collapsed case, the flying wings are first constructed by X.J. Wang in all dimensions [67]. Recently, it is shown independently by Bourni-Langford-Tinaglia [7] and Hoffman-Ilmanen-Martin-White [42] that a flying wing exists within any prescribed width greater than or equal to $\pi$. Moreover, they are unique in $\mathbb{R}^3$ [42]. In the non-collapsed case, many examples are obtained by Hoffman-Ilmanen-Martin-White [42]. Despite the fruitful results in mean curvature flow, their analogues in Ricci flow remain unknown for a longer time.

In Chapter 5, we confirm Hamilton's flying wing conjecture. We also construct some new steady gradient Ricci solitons that are analogous to the above results in mean curvature flow. More specific, we show that there is a family of 3d steady gradient Ricci solitons that are flying wings. The 3d flying wings are collapsed. Moreover, in dimension $n \geq 4$, we find a family of $\mathbb{Z}_2 \times O(n-1)$-symmetric but non-rotationally symmetric n-dimensional steady gradient solitons with positive curvature operator. These solitons are non-collapsed and hence are potential singularity models.

# Chapter 2

# Preliminaries

## 2.1 Ricci flow spacetime

In order to continue the Ricci flow when singularities occur, Hamilton introduced a surgery process [34]. Based on Hamilton's work, Perelman constructed Ricci flow with surgery on any compact 3 dimensional Riemannian manifold, and used it to prove the Geometrization and Poincaré Conjectures.

Perelman's Ricci flow with surgery is a sequence of ordinary compact Ricci flows such that the final time-slice of each flow is isometric, modulo surgery, to the initial time-slice of the next one. The surgery process depends on several parameters, and hence it is not canonical. Perelman conjectured that there exists a Ricci flow with surgery in which the surgeries are done at infinitesimal scale.

Recently, such a canonical flow, named the singular Ricci flow, was constructed by Kleiner and Lott in [44], and shown to be unique by Bamler and Kleiner in [4]. To present the definition of the singular Ricci flow, we need to introduce the concept of spacetime. An n-dimensional Ricci flow $(M, g(t))_{t\in I}$ can be viewed as a partial metric in the horizontal directions on the $(n+1)$- dimensional manifold $M \times I$. We call the manifold $M \times I$ a Ricci flow spacetime. The definitions in Section 2.1 and 2.2 are taken from [44] and [4].

**Definition 2.1.1** (Ricci flow spacetime)**.** A Ricci flow spacetime is a tuple $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ (sometimes abbreviate as $\mathcal{M}$ or $(\mathcal{M}, g(t))$) with the following properties:

1. $\mathcal{M}$ is a smooth 4-manifold with (smooth) boundary $\partial\mathcal{M}$.

2. $\mathfrak{t} : \mathcal{M} \to [0, T)$, where $T$ can be infinity, is a smooth function without critical points. For any $t \geq 0$ we denote by $\mathcal{M}_t := \mathfrak{t}^{-1}(t) \subset \mathcal{M}$ the time-t-slice of $\mathcal{M}$.

3. $\partial\mathcal{M} = \mathcal{M}_0$, i.e. the boundary of $\mathcal{M}$ is equal to the initial time-slice.

4. $\partial_{\mathfrak{t}}$ is a smooth vector field (the time vector field), which satisfies $\partial_{\mathfrak{t}}\mathfrak{t} \equiv 1$.

5. $g$ is a smooth inner product on the spacial subbundle $\ker(d\mathfrak{t}) \subset T\mathcal{M}$. For any $t \geq 0$ we denote by $g(t)$ the restriction of $g$ to the time-t-slice $\mathcal{M}_t$, which is a Riemannian metric.

6. $g$ satisfies the Ricci flow equation: $\mathcal{L}_{\partial_t} g = -2\mathrm{Ric}(g(t))$.

We call the Riemannian metric $G := d\mathfrak{t}^2 + g$ the spacetime metric.

In Definition 2.1.2-2.1.5, we explain what are points, metric balls, parabolic neighborhoods, and curves in a Ricci flow spacetime.

**Definition 2.1.2** (Points in a Ricci flow spacetime)**.** Let $(\mathcal{M}, \mathfrak{t}, \partial_\mathfrak{t}, g)$ be a Ricci flow spacetime and $x \in \mathcal{M}$ be a point. Set $t := \mathfrak{t}(x)$. We sometimes write $x$ as $(x, t)$ to indicate its time, when there is no ambiguity. Consider the maximal trajectory $\gamma_x : I \to \mathcal{M}$, $I \subset [0, \infty)$ of the time-vector field $\partial_\mathfrak{t}$ such that $\gamma_x(t) = x$. Note that $\mathfrak{t}(\gamma_x(t')) = t'$ for all $t' \in I$. For any $t' \in I$ we say that $x$ survives until time $t'$ and we write

$$x(t') := \gamma_x(t'). \tag{2.1.1}$$

Similarly, for a subset $X \subset \mathcal{M}_t$, we say that $X$ survives until time $t'$ if this is true for every $x \in X$, and we write $X(t') = \{x(t') : x \in X\}$.

**Definition 2.1.3** (Distance and metric balls)**.** Let $(\mathcal{M}, \mathfrak{t}, \partial_\mathfrak{t}, g)$ be a Ricci flow spacetime. For any two points $x, y \in \mathcal{M}_t$ we denote by $d_t(x, y)$, or simply $d(x, y)$ the distance between $x, y$ within $(\mathcal{M}_t, g(t))$.

For any $x \in \mathcal{M}_t$ and $r \geq 0$ we denote by $B_t(x, r) \subset \mathcal{M}_t$ the $r$-ball around $x$ with respect to the Riemannian metric $g(t)$.

**Definition 2.1.4** (Parabolic neighborhood)**.** Let $(\mathcal{M}, \mathfrak{t}, \partial_\mathfrak{t}, g)$ be a Ricci flow spacetime. For any $y \in \mathcal{M}$ let $I_y \subset [0, \infty)$ be the set of all times until which $y$ survives. Let $x \in \mathcal{M}$ and $a \geq 0, b \in \mathbb{R}$. Set $t = \mathfrak{t}(x)$. Then we define the *parabolic neighborhood* $P(x, a, b) \subset \mathcal{M}$ to be:

$$P(x, a, b) := \bigcup_{y \in B_t(x,a)} \bigcup_{t' \in [t, t+b] \cap I_y} y(t'). \tag{2.1.2}$$

If $b < 0$, we replace $[t, t + b]$ by $[t + b, t]$. We call $P(x, a, b)$ *unscathed* if $B(x, a)$ is relatively compact in $\mathcal{M}_t$, and $B(x, a)$ survives until $\max\{t + b, 0\}$.

**Definition 2.1.5** (Admissible curve and accessibility)**.** Let $\mathcal{M}$ be a Ricci flow spacetime, we say $\gamma : [c, d] \to \mathcal{M}$ is an admissible curve if $\gamma(t) \in \mathcal{M}_t$ for all $t \in [c, d]$. We say a point $x \in \mathcal{M}$ with $\mathfrak{t}(x) < \mathfrak{t}(x_0)$ is accessible to $x_0$ if there is an admissible curve running from $(x, t)$ to $(x_0, t_0)$.

Let $x_0 \in \mathcal{M}_t$, $t > 0$. We denote by $\mathcal{M}(x_0)$ the subset consisting of all points in $\mathcal{M}$ that are accessible to $x_0$.

In dimension 3, Ricci flows preserve the following Hamilton-Ivey pinching condition. This is important in showing that the curvature is non-negative in finite-time singularities.

**Definition 2.1.6** (Hamilton-Ivey pinching)**.** Let $M$ be a 3 dimensional Riemannian manifold and $\varphi > 0$. We say that the curvature at $x \in M$ is $\varphi$-positive if there is an $X > 0$ with $\mathrm{Rm}(x) \geq -X$ such that

$$R(x) \geq -\frac{3}{\varphi^{-1}} \quad \text{and} \quad R(x) \geq X(\log X + \log(\varphi^{-1}) - 3). \tag{2.1.3}$$

Let $(M, g(t)), t \in [0, T]$ be a 3 dimensional compact Ricci flow and $\varphi \in \mathbb{R}_+ \cup \infty$. We say that the curvature at $(x, t) \in M \times [0, T]$ is $\varphi$-positive if there is an $X > 0$ with $\mathrm{Rm}(x, t) \geq -X$ such that

$$R(x,t) \geq -\frac{3}{\varphi^{-1} + t} \quad \text{and} \quad R(x,t) \geq X(\log X + \log(\varphi^{-1} + t) - 3). \tag{2.1.4}$$

It is useful to see how the pinching-condition changes under rescaling: If a Ricci flow $(M, g(t))_{t\in[0,T]}$ is $\varphi$-positive, then the rescaling $(M, \tilde{g}(t))_{t\in[0,\frac{T}{\lambda^2}]}$, where $\tilde{g}(t) = \lambda^{-2} g(\lambda^2 t)$, is $\lambda^2\varphi$-positive. Moreover, if $(M, g(t))$ is $\varphi_0$-positive, then it is $\varphi$-positive for any $\varphi > \varphi_0$. In particular, if $(M, g(t))$ is $\varphi$-positive for all $\varphi > 0$, then the sectional curvature is non-negative.

For any 3D Ricci flow, Hamilton-Ivey showed that if the curvature is $\varphi$-positive at time 0, then the curvature is $\varphi$-positive at all positive times (see e.g. [43, Appendix B]). The same conclusion also holds for singular Ricci flow [44, Theorem 1.3].

In the next two definitions, we explain what we mean by a normalized 3D manifold. It is a 3D Riemannian manifold which satisfies some normalized curvature and volume conditions. Note that via suitable rescalings, we can normalize any 3D compact manifolds.

**Definition 2.1.7** ($\kappa$-non-collapsed)**.** Let $(M, g)$ be a 3 dimensional Riemannian manifold, $x \in M$ and $\kappa, r_0 > 0$. We say $M$ is $\kappa$-non-collapsed at $x$ at scales less than $r_0$, if $r^{-3}vol(B_g(x, r)) \geq \kappa > 0$, for all $0 < r \leq r_0$ such that $|\mathrm{Rm}| \leq r^{-2}$ holds on $B_g(x, r)$.

**Definition 2.1.8** (Normalized manifold)**.** Let $(M, g)$ be a 3-dimensional compact orientable connected Riemannian manifold that

1. is not a higher spherical space form,
2. has scalar curvature $R < 1$ everywhere,
3. is 1-non-collapsed at scales less than 1 and
4. satisfies the 1-positive curvature condition at time 0.

Then we say $(M, g)$ has normalized geometry. For a Ricci flow spacetime, we say it has normalized initial condition if it starts from a manifold $(M, g)$ with normalized geometry.

By the Hamilton-Ivey pinching theorem we know that the scalar curvature goes to positive infinity when the curvature goes unbounded, and hence the following curvature scale $\rho(x) = R^{-1/2}(x)$ becomes arbitrarily small.

**Definition 2.1.9** (Curvature scale)**.** Let $(M, g)$ be a 3 dimensional Riemannian manifold and $x \in M$ a point. Let the curvature scale at $x$ be

$$\rho(x) = R_+^{-1/2}(x), \tag{2.1.5}$$

where $R_+ = \max\{R, 0\}$, and we use the convention $0^{-1/2} = \infty$.

In a singular Ricci flow we will introduce in the next section, all horizontal curves (curves contained in a fixed time slice) and vertical curves (trajectories of points) are extendable as long as the curvature stays bounded along them. This property is called 0-completeness.

**Definition 2.1.10.** (0-complete) We say a Ricci flow spacetime $\mathcal{M}$ is 0-complete if for any smooth curve $\gamma : [0, s_0) \to \mathcal{M}$ that satisfies $\inf_{[0,s_0)} \rho(\gamma(s)) > 0$ and one of the following

(1) $\gamma([0, s_0))$ is contained in a time-slice $\mathcal{M}_t$, and has finite length with respect to the horizontal metric in $\mathcal{M}_t$, or

(2) $\gamma$ is the integral curve of $-\partial_t$, or $\partial_t$,

then $\lim_{s\to s_0} \gamma(s)$ exists.

Also, we say a spacetime is backward (resp. forward) 0-complete if in case (2), $\gamma$ is only the integral curve of $-\partial_t$ (resp. $\partial_t$).

We say a Riemannian manifold is 0-complete if the conclusion holds under condition (1).

## 2.2 Singular Ricci flow

The surgery process of Perelman's Ricci flow with surgery is regulated by several parameters, one of them being the surgery scale. It is the scale where we cut-off along the thin necks, and replace the high curvature regions with some caps. Perelman showed that the surgery scale can be chosen arbitrarily small. Moreover, he conjectured that the Ricci flow with surgery should converge to a canonical Ricci flow through singularities.

Recently, this conjecture was resolved by Bamler, Kleiner and Lott. The canonical Ricci flow is named singular Ricci flow. It is a 0-complete Ricci flow spacetime, which satisfies the Hamilton-Ivey pinching and the so-called canonical neighborhood assumption. The assumption says that the singular Ricci flow is close to some model solutions where the curvature is high. These model solutions are Perelman's $\kappa$-solutions.

In Definitions 2.2.1-2.2.4, we explain what are $\kappa$-solutions and canonical neighborhood assumption.

**Definition 2.2.1** ($\kappa$-solution)**.** An ancient Ricci flow $(M, g(t)_{t\in(-\infty,0]})$ on a 3 dimensional manifold $M$ is called a $\kappa$-solution if it satisfies the following:

1. $(M, g(t))$ is complete for all $t \in (-\infty, 0]$,
2. $|\mathrm{Rm}|$ is bounded on $M \times (-\infty, 0]$,
3. $\sec \geq 0$ on $M \times (-\infty, 0]$,
4. $(M, g(t))$ is $\kappa$-non-collapsed at all scales for all $t \in (-\infty, 0]$.

**Definition 2.2.2** (Geometric closeness)**.** We say that a pointed Riemannian manifold $(M, g, x)$ is $\epsilon$-close to another pointed Riemannian manifold $(\overline{M}, \overline{g}, \overline{x})$ at scale $\lambda > 0$ if there is a diffeomorphism onto its image

$$\psi : B^{\overline{M}}(\overline{x}, \epsilon^{-1}) \to M \tag{2.2.1}$$

such that $\psi(\overline{x}) = x$ and

$$\|\lambda^{-2}\psi^* g - \overline{g}\|_{C^{[\epsilon^{-1}]}(B^{\overline{M}}(\overline{x},\epsilon^{-1}))} < \epsilon. \tag{2.2.2}$$

Here the $C^{[\epsilon^{-1}]}$-norm of a tensor $h$ is defined to be the sum of the $C^0$-norms of the tensors $h, \nabla^{\overline{g}}h, \nabla^{\overline{g},2}h, ..., \nabla^{\overline{g},[\epsilon^{-1}]}h$ with respect to the metric $\overline{g}$.

Similarly, we say a pointed Ricci flow $(M, g(t), (x, 0))$ is $\epsilon$-close to a pointed Ricci flow $(\overline{M}, \overline{g}(t), (\overline{x}, 0))$ on $[a, b]$ $(a \leq 0 \leq b)$ at scale $\lambda > 0$ if $g(t)$ is defined on $[\lambda^2 a, \lambda^2 b]$, and there is a diffeomorphism onto its image

$$\psi : B^{\overline{M}}_{\overline{g}(0)}(\overline{x}, \epsilon^{-1}) \to M \tag{2.2.3}$$

such that $\psi(\overline{x}) = x$ and

$$\|\lambda^{-2}\psi^* g(\lambda^2 t) - \overline{g}(t)\|_{C^{[\epsilon^{-1}]}(B^{\overline{M}}_{\overline{g}(0)}(\overline{x},\epsilon^{-1}))} < \epsilon \tag{2.2.4}$$

for all $t \in [a, b]$, where the norm is measured with respect to the metric $\overline{g}(t)$. In particular, when $a = -\epsilon^{-1}$ and $b = 0$, we simply say $(M, g(t), (x, 0))$ is $\epsilon$-close to $(\overline{M}, \overline{g}(t), (\overline{x}, 0))$.

**Definition 2.2.3** ($\delta$-neck and strong $\delta$-neck)**.** Let $(M, g)$ be a 3 dimensional Riemannian manifold and $\delta > 0$. Suppose $U \subset M$ is an open subset, $x \in U$. We say $U$ is a $\delta$-neck centered at $x$, if $(U, g)$ is $\delta$-close to the standard cylindrical metric on $(-\delta^{-1}, \delta^{-1}) \times S^2$ at scale $\rho(x)$.

Let $(M, g(t))$ be a Ricci flow. Suppose $U \subset M$ is an open subset and $x$ is a point in $U$. We say that $U$ is a strong $\delta$-neck on $[-c, 0]$ centered at $x$ for some $c > 0$, if $(U, g(t), x)$ is $\delta$-close to the standard cylindrical flow on the time interval $[-c, 0]$ at scale $\rho(x)$. We simply call it a strong $\delta$-neck when $c = -\delta^{-1}$.

**Definition 2.2.4** (Canonical neighborhood assumption)**.** Let $(M, g)$ be a 3 dimensional Riemannian manifold and $\epsilon > 0$. We say that $(M, g)$ satisfies the $\epsilon$-canonical neighborhood assumption at some point $x \in M$ if there is a $\kappa$-solution $(\overline{M}, \overline{g}(t)_{t\in(-\infty,0]})$ and a point $\overline{x} \in \overline{M}$ such that $\rho(\overline{x}, 0) = 1$ and $(M, g, x)$ is $\epsilon$-close to $(\overline{M}, \overline{g}(0), \overline{x})$ at scale $\rho(x) > 0$.

We say that $(M, g)$ satisfies the $\epsilon$-canonical neighborhood assumption at scales $(r_1, r_2)$, for some $r_2 > r_1 > 0$, if $M$ satisfies the $\epsilon$-canonical neighborhood assumption at every point $x \in M$ with $r_1 < \rho(x) < r_2$.

Perelman showed the compactness of $\kappa$-solutions. As a consequence, the two scaling invariant quantities $|\nabla R^{-1/2}|$ and $|\partial_t R^{-1}|$ have uniform upper bounds in all $\kappa$-solutions as well as the flows which are $\epsilon$-close to them. For convenience of later use, we state this in terms of the curvature scale $\rho$ instead of $R$ in the following lemma.

**Lemma 2.2.5.** *(Gradient estimate, see e.g. [4, Lemma 8.1]) There exist $\overline{\epsilon}$, and $\eta > 0$ such that for all $\epsilon \leq \overline{\epsilon}$ the following holds:*

1. *If $M$ is a Riemannian manifold satisfying the $\epsilon$-canonical neighborhood assumption at $x \in M$, then*
$$|\nabla\rho|(x) \leq \eta. \tag{2.2.5}$$

2. *If $\mathcal{M}$ is a Ricci flow spacetime satisfying the $\epsilon$-canonical neighborhood assumption at $x \in \mathcal{M}$, then*
$$|\nabla\rho|(x) \leq \eta, \quad |\partial_t\rho^2(x)| \leq \eta. \tag{2.2.6}$$

Hereafter, we always assume $\epsilon > 0$ to be smaller than the $\overline{\epsilon}$ from the above lemma whenever we talk about the $\epsilon$-canonical neighborhood assumption.

The $\kappa$-solutions can be decomposed as a union of regions which are diffeomorphic to a few manifolds. Roughly speaking, a $\kappa$-solution is a union of some caps and necks. Therefore, for a manifold which satisfies the canonical neighborhood assumption, the high curvature regions are unions of caps and necks.

**Lemma 2.2.6.** *([4, Lemma 8.2]) For every $\delta > 0$ there are constants $C_0(\delta), \epsilon_{can}(\delta) > 0$ such that if $\epsilon \leq \epsilon_{can}(\delta)$, then the following holds.*

*Let $(M, g)$ be a Riemannian manifold that satisfies the $\epsilon$-canonical neighborhood assumption at some point $x \in M$. Then $x$ is contained in a compact, connected domain $V \subset M$ such that $\operatorname{diam}(V) \leq C_0\rho(x)$ and $\rho(y_1) \leq C_0\rho(y_2)$ for all $y_1, y_2 \in V$, and one of the following hold:*

1. $\operatorname{Int}(V)$ *is a $\delta$-neck at scale $\rho(x)$ and $x$ is its center.*

2. *$V$ is a closed manifold without boundary.*

3. *Either $V$ is a 3-disk or is diffeomorphic to a twisted interval bundle over $\mathbb{R}P^2$ and $\partial V$ is a central 2-sphere of a $\delta$-neck. We call* $\mathrm{Int}(V)$ *a $\delta$-cap and $x$ its center. Moreover, for any $y_1, y_2 \in \partial V$, we have $d(y_1, x) + d(y_2, x) \geq d(y_1, y_2) + 100\rho(x)$.*

In the following definition we define $\delta$-tubes and capped $\delta$-tubes, which are the glue-ups of $\delta$-necks and caps. One advantage of considering these objects is that they can be constructed in such a way such that the curvature scales are not too small at the boundaries. This is useful for achieving a Thick-Thin decomposition of a manifold satisfying the canonical neighborhood assumption.

**Definition 2.2.7** ($\delta$-tube and capped $\delta$-tube)**.** (see e.g. [55]) A *$\delta$-tube $T$* in a Riemannian 3-manifold $M$ is a submanifold diffeomorphic to $S^2 \times \mathbb{R}$ which is a union of $\delta$-necks with the central spheres that are isotopic to the 2-spheres of the product structure.

A capped $\delta$-tube is a connected submanifold that is the union of a $\delta$-cap and a $\delta$-tube where the intersection of them is diffeomorphic to $S^2 \times \mathbb{R}$ and contains an end of the $\delta$-tube and an end of the $\delta$-cap.

In a manifold satisfying the canonical neighborhood assumption, the following lemma gives a Tick-Thin decomposition by cutting along central spheres of some $\delta$-necks. These central spheres are of a uniform curvature scale. So by a volume comparison, we obtain an upper bound on their number, which depends only on the curvature scale and the volume of region where we do the decomposition.

**Lemma 2.2.8.** *(High curvature regions are covered by tubes and capped-tubes) For some sufficiently small $\delta > 0$, there exist $\epsilon_{can}(\delta), C(\delta), \lambda(\delta), \Lambda(\delta) > 0$ such that the following holds:*

*Let $(M, g, x_0)$ be a 0-complete 3d Riemannian manifold, $x_0 \in M$. Suppose the $\epsilon_{can}$-canonical neighborhood assumption holds at scales $(0, 1)$ on $B_g(x_0, d)$ for some $d \geq 2$. For any $r_0 \in (0, 1)$ such that $\Lambda r_0 < 1$, there is a collection $S$ of disjoint central spheres of $\delta$-necks with curvature scale $r_0$, such that the following holds:*

1. *Let $\Omega$ be the union of components in $B_g(x_0, d) \setminus \cup_{\Sigma \in S} \Sigma$ that satisfies $\rho \geq \lambda r_0$. Then $\rho \leq \Lambda r_0$ on $B_g(x_0, d) \setminus \Omega$.*

2. *Suppose the number of elements in $S$ is $N$, then $N r_0^2 \leq C\, vol(B_g(x_0, d))$.*

3. *If $\rho(x_0) \geq 2C_0$ where $C_0(\delta)$ is from Lemma 2.2.6. Then $B_g(x_0, d) \setminus \Omega$ is the union of some $\delta$-tubes and capped $\delta$-tubes which are bounded by the central spheres in $S$.*

*Proof of Lemma 2.2.8.* The existence of $S$ and assertion (1)(2) follow from an easy adaptation of the central sphere decomposition Lemma 11.4 in [4]. For assertion (3), if $\rho(x_0) \geq 2C_0$, then for any $x \in B_g(x_0, d) \setminus \Omega$, by Lemma 2.2.6 $x$ must be the center of a $\delta$-neck or a $\delta$-cap. By [55, Proposition 19.21] we see that a non-compact connected subset of points which are centers of $\delta$-necks or $\delta$-caps is contained in a $\delta$-tube or a capped $\delta$-tube. So assertion (3) follows. □

We can now give the definition of the singular Ricci flow. Since the singular Ricci flow satisfies the canonical neighborhood assumption, Lemmas 2.2.5, 2.2.6, 2.2.8 also hold.

**Definition 2.2.9.** If $\epsilon > 0$ and $r(t) : [0, \infty) \to (0, \infty)$ is a non-increasing function. Then we say a Ricci flow spacetime $\mathcal{M}$ is $(r, \epsilon)$-singular if the following holds:

1. $\mathcal{M}_0$ is a compact orientable manifold;
2. $\mathcal{M}$ is 0-complete;
3. $\mathcal{M}_{[0,t]}$ satisfies the $\epsilon$-canonical neighborhood assumption at scales $(0, r(t))$.

We call $t$ a singular time if $\mathcal{M}_t$ is not compact.

The following lemma states the existence and uniqueness of the singular Ricci flow, which are proved by Kleiner-Lott [44] and Bamler-Kleiner [4].

**Lemma 2.2.10** (Singular Ricci flow)**.** *For any $\epsilon > 0$, there is a non-increasing function $\overline{r}_\epsilon(t) : [0, \infty) \to [0, \infty)$ such that the following holds: Let $M$ be a 3D normalized Riemannian manifold. Then there exists a unique Ricci flow spacetime $\mathcal{M}$, called a singular Ricci flow, such that for any $\epsilon > 0$, $\mathcal{M}$ is $(\overline{r}_\epsilon, \epsilon)$-singular.*

## 2.3 Distance distortion estimates

In studying the Ricci flows, we often need to compare the distance between two points at different times. In this section, we review some standard distance distortion estimates under different curvature conditions, which are originally due to Hamilton and Perelman. Moreover, the estimates we present here are local as we only need the curvature conditions to hold in certain balls instead of the entire manifold.

First, we have following elementary observations: Let $(M, g(t))$, $t \in [0, T]$ be a Ricci flow and $x, y \in M$. Let $\gamma : [0, d_0(x, y)] \to M$ be a minimizing geodesic connecting $x, y$ with respect to $g(0)$, which is parametrized by $s$. Then by using the Ricci flow equation, we see that $L_\gamma(t)$, the length of $\gamma$ at time $t$ evolves by

$$\frac{d}{dt} L_\gamma(t) = -\int_0^{d_0(x,y)} Ric\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) ds. \tag{2.3.1}$$

Therefore, we have different distance distortion estimates when the curvature conditions differ.

**Lemma 2.3.1.** *(see e.g. [28, Theorem 18.7]) Let $(M, g(t))_{t\in[0,T]}$ be a Ricci flow of dimension $n$. Let $K, r_0 > 0$.*

1. *Let $x_0 \in M$ and $t_0 \in (0, T)$. Suppose that $\mathrm{Ric} \leq (n-1)K$ on $B_{t_0}(x_0, r_0)$. Then the distance function $d(x, t) = d_t(x, x_0)$ satisfies the following inequality in the outside of $B_{t_0}(x_0, r_0)$:*
$$(\frac{\partial}{\partial t} - \Delta)|_{t=t_0} d \geq -(n-1)(\frac{2}{3}Kr_0 + r_0^{-1}). \tag{2.3.2}$$

2. *Let $t_0 \in [0, T)$ and $x_0, x_1 \in M$. Suppose*
$$\mathrm{Ric}(x, t_0) \leq (n-1)K, \tag{2.3.3}$$
*for all $x \in B_{t_0}(x_0, r_0) \cup B_{t_0}(x_1, r_0)$. Then*
$$\frac{\partial}{\partial t}|_{t=t_0} d_t(x_0, x_1) \geq -2(n-1)(Kr_0 + r_0^{-1}). \tag{2.3.4}$$

The following lemma follows directly by integrating the distance evolution equation under the Ricci flow. It says that the distance can not expand too much if the Ricci curvature is bounded below by $-C$.

**Lemma 2.3.2.** *(Expanding Lemma)(see e.g. [64, Lemma 2.1]). Given $T, K, R > 0$ and $n \in \mathbb{N}$. Let $(M^n, g(t))$ be a Ricci flow for $t \in [-T, 0]$. Suppose for some $x_0 \in M$ we have $B_{g(0)}(x_0, R) \subset\subset M$ and $\mathrm{Ric}_{g(t)} \geq -K$ on $B_{g(0)}(x_0, R) \cap B_{g(t)}(x_0, Re^{Kt})$ for each $t \in [-T, 0]$.*

*Then for all $t \in [-T, 0]$,*
$$B_{g(0)}(x_0, R) \supset B_{g(t)}(x_0, Re^{Kt}), \tag{2.3.5}$$
*or equivalently, for all $y \in B_{g(0)}(x_0, Re^{Kt})$ we have*
$$d_{g(t)}(y, x_0) \geq d_{g(0)}(y, x_0)e^{Kt}. \tag{2.3.6}$$

We can get the following lemma by integrating item 2 in Lemma 2.3.1. It says that the distance can not shrink too much at later times if the Ricci curvature is bounded above by $\frac{C}{t}$.

**Lemma 2.3.3.** *(Shrinking Lemma)(see e.g. [64, Lemma 2.2]). Given $T, c_0, r > 0$ and $n \in \mathbb{N}$, there exists constant $\beta = \beta(n) \geq 1$ such that the following holds: Let $(M^n, g(t))$ be a Ricci flow for $t \in [0, T]$. Suppose for some $x_0 \in M$ we have $B_{g(0)}(x_0, r) \subset\subset M$. Suppose also $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$, or more generally $\mathrm{Ric}_{g(t)} \leq \frac{(n-1)c_0}{t}$, on $B_{g(0)}(x_0, r) \cap B_{g(t)}(x_0, r - \beta\sqrt{c_0 t})$ for each $t \in [0, T]$.*

*Then for all $t \in [0, T]$, we have*
$$B_{g(0)}(x_0, r) \supset B_{g(t)}(x_0, r - \beta\sqrt{c_0 t}), \tag{2.3.7}$$

*or equivalently, for all $y \in B_{g(t)}(x_0, r-\beta\sqrt{c_0 t})$ we have*

$$d_{g(t)}(y, x_0) \geq d_{g(0)}(y, x_0) - \beta\sqrt{c_0 t}. \tag{2.3.8}$$

*More generally, for $0 \leq s \leq t \leq T$, we have*

$$B_{g(s)}(x_0, r-\beta\sqrt{c_0 s}) \supset B_{g(t)}(x_0, r-\beta\sqrt{c_0 t}), \tag{2.3.9}$$

*or equivalently, for all $y \in B_{g(t)}(x_0, r-\beta\sqrt{c_0 t})$ we have*

$$d_{g(t)}(y, x_0) \geq d_{g(s)}(y, x_0) - \beta(\sqrt{c_0 t} - \sqrt{c_0 s}). \tag{2.3.10}$$

The following Hölder distance estimate [64, Lemma 3.1] is also a consequence of item 2 in Lemma 2.3.1. Compared with Lemma 2.3.3, it gives a better lower bound for $d_{g(t)}(x,y)$ when $d_{g(0)}(x,y)$ is much smaller than the time $t$. To do this, we choose an intermediate time $t_0 \in (0,t)$ depending on $d_{g(0)}(x,y)$ and apply different distance estimates on $[0,t_0]$ and $[t_0,t]$ and combine the results.

**Lemma 2.3.4.** *Given $T, c_0, r > 0$ and $n \in \mathbb{N}$, there exist positive constants $\beta = \beta(n)$ and $\gamma = \gamma(c_0, n, T)$ such that the following holds: Let $(M^n, g(t))$ be a Ricci flow for $t \in [0,T]$, not necessarily complete. Suppose for some $x_0 \in M$, we have $B_{g(t)}(x_0, 2r) \subset\subset M$ for all $t \in [0,T]$. Suppose also $|\mathrm{Rm}|_{g(t)}(x) \leq \frac{c_0}{t}$, or more generally $\mathrm{Ric}_{g(t)}(x) \leq \frac{(n-1)c_0}{t}$ for all $x \in B_{g(t)}(x_0, 2r)$ and $t \in [0,T]$.*

*Then for all $x, y \in \bigcap_{s\in[0,T]} B_{g(s)}(x_0, r)$, and $0 \leq t_1 < t_2 \leq T$, we have*

$$d_{g(t_2)}(x,y) \geq d_{g(t_1)}(x,y) - \beta\sqrt{c_0}(\sqrt{t_2} - \sqrt{t_1}), \tag{2.3.11}$$

*Moreover, for all $t \in [0,T]$, we have*

$$d_{g(t)}(x,y) \geq \gamma[d_{g(0)}(x,y)]^{1+2(n-1)c_0}. \tag{2.3.12}$$

**Remark 2.3.5.** *We need the curvature assumption on $B_{g(t)}(x_0, 2r) \subset\subset M$ for all $t$ to estimate the distances change on $\cap_{s\in[0,T]} B_{g(s)}(x_0, r)$. The reason is that there are two ways to make sense of the distance at time $t$ between two points $x, y \in B_{g(t)}(x_0, 2r)$. One is the infimum length of all connecting paths in $M$, and the other is the infimum length of all connecting paths that are contained in $B_{g(t)}(x_0, 2r)$. The former is usually shorter than the latter. These two metrics agree for $x, y \in B_{g(t)}(x_0, r)$ when $B_{g(t)}(x_0, 2r)$ is compactly contained in $M$, and the distance can be realized by a geodesic which lies within $B_{g(t)}(x_0, 2r)$.*

**Remark 2.3.6.** *We can also prove the same conclusion for the Ricci flow defined only for $t \in (0,T]$, where $d_{g(0)}$ in (2.3.12) is replaced by the limit distance of $d_{g(t)}$. The limit exists thanks to bound $|\mathrm{Rm}|_{g(t)} \leq \frac{C}{t}$ in (3.1.2).*

*Proof of Lemma 2.3.4.* We note that there is no ambiguity to talk about $d_{g(t)}(x,y)$ for $x,y \in \bigcap_{s\in(0,T]} B_{g(s)}(x_0,r)$ for all $t \in [0,T]$, because the minimizing geodesic joining $x$ and $y$ with respect to $g(t)$ is contained in $B_{g(t)}(x_0,2r) \subset\subset M$. Inequality (2.3.11) follows by the above Shrinking Lemma. The proof of (2.3.12) follows by splitting $[0,t]$ into two intervals. We choose and fix $t_0 = \frac{1}{c_0}\left[\frac{1}{2\beta}d_{g(0)(x,y)}\right]^2$. Then in the first interval $[0,t_0]$, we integrate the following inequality from Hamilton and Perelman

$$\frac{\partial^+}{\partial t}d_{g(t)}(x,y) \geq -\frac{\beta}{2}\sqrt{\frac{c_0}{t}} \tag{2.3.13}$$

to get

$$d_{g(t_0)}(x,y) \geq \frac{1}{2}d_{g(0)}(x,y). \tag{2.3.14}$$

By $\frac{\partial^+}{\partial t}|_{t_0}F$ we mean $\limsup_{t\to t_0^+}\frac{F(t)-F(t_0)}{t-t_0}$. In the second interval we use the following inequality, which follows from (2.3.1):

$$\frac{\partial^+}{\partial t}d_{g(t)}(x,y) \geq -(n-1)\frac{c_0}{t}d_{g(t)}(x,y), \tag{2.3.15}$$

integrating which we get

$$d_{g(t)}(x,y) \geq d_{g(t_0)}(x,y)\left[\frac{t}{t_0}\right]^{-(n-1)c_0}. \tag{2.3.16}$$

The combination of (2.3.14) and (2.3.16) gives (2.3.12). □

# Chapter 3

# Ricci flow with almost non-negative curvature

## 3.1 Introduction and main results

In general, Ricci flow tends to preserve some kind of positivity of curvatures. For example, positive scalar curvature is preserved in all dimensions. This follows from applying maximum principle to the evolution equation of scalar curvature, which is

$$\frac{\partial}{\partial t}R = \Delta R + 2|\mathrm{Ric}|^2.$$

By developing a maximum principle for tensors, Hamilton [38][34] proved that Ricci flow preserves the positivity of the Ricci tensor in dimension three and positivity of the curvature operator in all dimensions. H. Chen [21] also proved the preservation of 2-non-negative curvature. The invariance of weakly PIC was first shown in dimension four by Hamilton [35], and the general case was obtained independently by Brendle and Schoen [10] and by Nguyen [56]. The curvature conditions weakly $\mathrm{PIC}_1$ and $\mathrm{PIC}_2$ were in turn introduced by Brendle and Schoen in [10] and played a key role in their proof of the differentiable sphere theorem. Finally in the Kähler case, the condition of non-negative holomorphic bisectional curvature, which is a weaker condition than non-negative sectional curvature, is also preserved for compact manifolds. This was shown by Bando [5] in dimension three and by Mok [54] in all dimensions. In [60] Shi generalized this result to the complete Kähler manifolds with bounded curvature. In [49], Lee and Tam proved that any complete non-collapsed Kähler metric with non-negative holomorphic bisectional curvature on a noncompact complex manifold can be deformed by a Ricci flow to a complete Kähler metric with non-negative and bounded holomorphic bisectional curvature.

In this chapter, we study the preservation of almost non-negativity of curvature conditions. We say a quantity is almost non-negative when it has a negative lower bound. The almost non-negative case is less restrictive since it puts no constraints on the topology of the manifold. In [2], Bamler, Cabezas-Rivas, and Wilking studied the complete manifold with bounded curvature, which satisfies global non-collapsedness and almost non-negativity for some curvature conditions. They showed that under the assumption, a Ricci flow exists for a uniform amount of time, during which the curvature can be bounded below by a negative constant depending only on initial conditions. In the same paper, they also established some local results without the complete and curvature bound assumptions.

However, the local cases of almost 2-non-negative curvature and weakly $\mathrm{PIC}_1$ remained unsolved. We verify these two local cases in this chapter. We use $\mathcal{C}$ to denote various non-negative curvature conditions, and write $\mathrm{Rm} \in \mathcal{C}$ to indicate that the curvature operator Rm satisfies the corresponding curvature condition. Then $\mathrm{Rm} + C\mathrm{I} \in \mathcal{C}$ indicates the non-negativity of $\mathrm{Rm} + C\mathrm{I}$, where I is the identity curvature operator whose scalar curvature is $n(n-1)$. Under this notation, our main theorem can be stated as below:

**Theorem 3.1.1.** *Given $n \in \mathbb{N}$, $\alpha_0 \in (0, 1]$ and $v_0 > 0$, there exist positive constants $\tau = \tau(n, v_0, \alpha_0)$ and $C = C(n, v_0)$ such that the following holds: Let $(M^n, g_0)$ be a Riemannian manifold (not necessarily complete) and consider one of the following curvature conditions $\mathcal{C}$:*

1. *non-negative curvature operator;*
2. *2-non-negative curvature operator (i.e. the sum of the lowest two eigenvalues of the curvature operator is non-negative);*
3. *weakly $\mathrm{PIC}_2$ (i.e. non-negative complex sectional curvature, meaning that taking the cartesian product with $\mathbb{R}^2$ produces a non-negative isotropic curvature operator);*
4. *weakly $\mathrm{PIC}_1$ (i.e. taking the cartesian product with $\mathbb{R}$ produces a non-negative isotropic curvature operator).*

*Suppose $B_{g_0}(x_0, s_0) \subset\subset M$ for some $x_0 \in M$ and $s_0 > 4$ such that*

$$\begin{cases} \mathrm{Rm}_g + \alpha_0 \mathrm{I} \in \mathcal{C} \quad \text{on} \quad B_{g_0}(x_0, s_0) \\ Vol_{g_0} B_{g_0}(x, 1) \geq v_0 > 0 \quad \text{for all} \quad x \in B_{g_0}(x_0, s_0 - 1) \,. \end{cases} \tag{3.1.1}$$

*Then there exists a Ricci flow $g(t)$ defined for $t \in [0, \tau]$ on $B_{g_0}(x_0, s_0 - 2)$, with $g(0) = g_0$, such that for all $t \in [0, \tau]$,*

$$\begin{cases} |\mathrm{Rm}|_{g(t)} \leq \dfrac{C}{t} \qquad \text{on} \quad B_{g_0}(x_0, s_0 - 2) \\ \mathrm{Rm}_{g(t)} + C\alpha_0 \mathrm{I} \in \mathcal{C}. \end{cases} \tag{3.1.2}$$

The results of the first and third conditions above were obtained in [2]. In dimensional three, 2-non-negative curvature has the same meaning as non-negative Ricci curvature, where the result was obtained by Simon and Topping in [63] and [64].

For each curvature condition $\mathcal{C}$, we define $\ell(x) \geq 0$ to be the smallest number such that $\mathrm{Rm}_g(x) + \ell(x)\,\mathrm{I} \in \mathcal{C}$. Then in each case the bound $\ell \leq 1$ implies a lower bound on the Ricci curvature. We also observe that each curvature condition implies weakly $\mathrm{PIC}_1$. The method we allows a uniform treatment of all curvature conditions that imply a lower bound for Ricci curvature and weakly $\mathrm{PIC}_1$.

As an application we have the following global existence result on complete manifolds with possibly unbounded curvature. It extends the corresponding results in [2] to the 2-non-negative and weakly $\mathrm{PIC}_1$ cases.

**Corollary 3.1.2.** *Given $n \in \mathbb{N}, \alpha_0 \in (0, 1]$ and $v_0 > 0$, there exist positive constants $C = C(n, v_0)$ and $\tau = \tau(n, v_0, \alpha_0)$ such that the following holds: Let $\mathcal{C}$ be any curvature conditions listed in Theorem 3.1.1, and $(M^n, g)$ be any complete Riemannian manifold (with possibly unbounded curvature) such that*

$$\begin{cases} \mathrm{Rm}_g + \alpha_0 \mathrm{I} \in \mathcal{C} \\ Vol_g B_g(p, 1) \geq v_0 \quad \text{for all } p \in M. \end{cases} \tag{3.1.3}$$

*Then there exists a complete Ricci flow $(M, g(t))_{t \in (0,\tau]}$ with $g(0) = g$ and so that*

$$\begin{cases} \mathrm{Rm}_{g(t)} + C\alpha_0 \mathrm{I} \in \mathcal{C} \quad \text{for all } t \in (0, \tau] \text{ throughout } M \\ |\mathrm{Rm}|_{g(t)} \leq \dfrac{C}{t}. \end{cases} \tag{3.1.4}$$

To prove the corollary we apply the local Ricci flow in Theorem 3.1.1 to a sequence of larger and larger balls on the complete manifold. Thanks to the curvature decay estimate $|\mathrm{Rm}| \leq \frac{C}{t}$ in (3.1.2), we can then take a convergent subsequence and get a globally defined flow.

Another application is the following smoothing result for singular limit spaces of sequences of manifolds with lower curvature bounds, which asserts the limit space is bi-Hölder homeomorphic to a smooth manifold.

**Corollary 3.1.3.** *Given $n \in \mathbb{N}$, $\alpha_0, v_0 > 0$. Let $\mathcal{C}$ be any curvature conditions listed in Theorem 3.1.1, and $(M_i^n, g_i)$ be a sequence of complete Riemannian manifolds such that for all $i$, we have*

$$\begin{cases} \mathrm{Rm}_{g_i} + \alpha_0 \mathrm{I} \in \mathcal{C} \quad \text{throughout } M_i \\ Vol_{g_i} B_{g_i}(x, 1) \geq v_0 \quad \text{for all } x \in M_i \end{cases} \tag{3.1.5}$$

*Then there exists a smooth manifold $M$, a point $x_\infty \in M$, and a continuous distance metric $d_0$ on $M$ such that for some points $x_i \in M_i$, a subsequence of $(M_i, d_{g_i}, x_i)$ converges in* *the pointed Gromov-Hausdorff distance sense to* $(M, d_0, x_\infty)$. *Furthermore, the metric space* $(M, d_0)$ *is bi-Hölder homeomorphic to the smooth manifold* $M$ *equipped with any smooth metric.*

We give the proofs of Corollary 3.1.2 and 3.1.3 in Section 8. We mention here that with some careful local distance distortion arguments, the same conclusion in Corollary 3.1.3 holds provided noncollapsedness of only one ball centered at a point. For detailed proof of this, we refer to [64] where the argument is done for Ricci curvature and carries over to our case.

Finally, we sketch the proof of Theorem 3.1.1 under some additional assumptions. That is, assuming (3.1.1) holds globally and a short time Ricci flow exists up to a uniform time $T < 1$, during which $|\mathrm{Rm}| \leq \frac{C}{t}$ holds, we want to show $\mathrm{Rm}_{g(t)} + C\alpha_0 \mathrm{I} \in \mathcal{C}$ for all $t$. We define $\ell(x,t)$ by

$$\ell(x,t) := \inf\{\varepsilon \in [0,\infty) | \mathrm{Rm}_{g(t)}(x) + \varepsilon \mathrm{I} \in \mathcal{C}\}. \tag{3.1.6}$$

Then it's equivalent to show $\ell(\cdot,t) \leq C\alpha_0$ for all $t$. By [2, Proposition 2.2], $\ell$ satisfies an evolution inequality of the form

$$\frac{\partial}{\partial t}\ell \leq \Delta \ell + R\,\ell + C(n)\ell^2 \tag{3.1.7}$$

in the barrier sense for some dimensional constant $C(n)$. Assuming $\ell(x,t) \leq 1$, then by the maximum principle, $\ell(\cdot,t) \leq e^{C(n)t}h$ on $M \times [0,t)$, where $h$ solves

$$\frac{\partial}{\partial t}h = \Delta h + R\,h, \quad h(\cdot,0) = \ell(\cdot,0). \tag{3.1.8}$$

We can express this solution as

$$h(x,t) = \int_M G(x,t;y,0)\,\ell(y,0)\,d_0y, \tag{3.1.9}$$

where $G(\cdot,\cdot;y,s)$ satisfies

$$(\frac{\partial}{\partial t} - \Delta_{x,t} - R_{g(t)})G(x,t;y,s) = 0 \quad \text{and} \quad \lim_{t \searrow s} G(x,t;y,s) = \delta_y(x). \tag{3.1.10}$$

We say $G(\cdot,\cdot;y,s)$ is the heat kernel of equation (3.1.10). It can be shown with the bound $|\mathrm{Rm}|_{g(t)} \leq \frac{C}{t}$ that $G(x,t;y,s)$ has the following Gaussian upper bound

$$G(x,t;y,s) \leq \frac{C}{(t-s)^{\frac{n}{2}}} exp\left(-\frac{d_s^2(x,y)}{C(t-s)}\right), \tag{3.1.11}$$

substituting which into (3.1.9) we get

$$\ell(x,t) \leq e^{C(n)}h(x,t) \leq \sup_{y\in M}\ell(y,0) \cdot \frac{C}{t^{\frac{n}{2}}}\int_M exp\left(-\frac{d_0^2(x,y)}{Ct}\right)\,d_0y \leq C\sup_{y\in M}\ell(y,0). \tag{3.1.12}$$

To prove Theorem 3.1.1 by adapting the above argument, we need to overcome the difficulties caused by the lack of those additional assumptions. To construct a local Ricci flow, we use an extension method which was introduced in [41] and [64]. Similar methods of constructing local Ricci flows were also used in [50, 51, 53, 39]. The process starts by doing a conformal change to the initial metric, making it a complete metric and leaving it unchanged on $B_{g(0)}(x_0, r_1)$ for some $0 < r_1 < r_0 = s_0$. Then by the following doubling time estimate of Shi in [61], we can then run a complete Ricci flow up to a short time $t_1$.

**Lemma 3.1.4.** *(Doubling time estimate) Let $(M^n, g(0))$ be a complete manifold with bounded curvature $|\mathrm{Rm}|_{g(0)} \leq K$, then there exits a complete Ricci flow $(M^n, g(t))$ such that*

$$|\mathrm{Rm}|_{g(t)} \leq 2K \tag{3.1.13}$$

*for all $0 \leq t \leq \frac{1}{16K}$.*

Of course $t_1$ is uncontrolled and may depend on specific manifold due to the lack of a uniform curvature bound. Next we do another conformal change to complete the metric at $t_1$, leaving it unchanged on $B_{g(0)}(x_0, r_2)$ for some $0 < r_2 < r_1$. Then using the doubling time estimate again, we have another complete Ricci flow from $t_1$ to $t_2$. Repeating the process, we obtain some successive complete Ricci flow pieces $(\{M_i\}_{i=1}^m, \{g_i(t)\}_{i=1}^m)$, with each $M_i$ containing $B_{g(0)}(x_0, r_i)$. Restricting all the $g_i(t)$ on $B_{g(0)}(x_0, r_m)$, we thus obtain a smooth local Ricci flow $g(t)$ defined for all $t \in [0, t_m]$. The inductive construction is carried out in Section 6.

In particular, the curvature decay $|\mathrm{Rm}|_{g(t)} \leq \frac{C}{t}$ in (3.1.2) together with the doubling time estimate enable us to choose $t_{i+1} = t_i(1 + \frac{1}{16C})$ for each $i$. To verify $|\mathrm{Rm}|_{g(t)} \leq \frac{C}{t}$ after each extension step, we use the curvature decay lemma in Section 3, which ensures the existence of $C$ under the assumption of a local upper bound of $\ell(\cdot, t)$.

For the verification of $\ell(\cdot, t) \leq C\alpha_0$ in (3.1.2), we perform a new local integration estimate, in which we use a generalized heat kernel. We know the standard heat kernel $G(x, t; y, s)$ on a complete Ricci flow satisfies the following reproduction formula for all $\mu < s < t$

$$\int G(x, t; y, s)\, G(y, s; z, \mu)\, d_s y = G(x, t; z, \mu). \tag{3.1.14}$$

The standard heat kernel $G(x, t; y, s)$ is well defined by equation (3.1.10) for all $(x, t)$ and $(y, s)$ in a same complete Ricci flow piece $(M_i, g_i(t))$ coming from the above inductive construction. In section 5, we use equation (3.1.14) inductively to make sense of $G(x, t; y, s)$ for $(x, t)$ and $(y, s)$ in different pieces and thus obtain a generalized heat kernel whose definition domain is on the whole $(\{M_i\}_{i=1}^m, \{g_i(t)\}_{i=1}^m)$ and has a Gaussian upper bound.

## 3.2 Preliminaries

### 3.2.1 Extension Lemma

For the metric on a local region, we can modify it by a conformal change that pushes the boundary of the region, on which we have curvature bounds, to infinity in such a way that the modified metric is complete and has bounded curvature. For example, the open Euclidean unit ball can be made into a complete hyperbolic metric under a conformal change. The following conformal change has been used in [41], [64]. In [2], a different conformal change was also used to achieve the local results of the first and third cases listed in Theorem 3.1.1, as a corollary of their corresponding global results.

**Lemma 3.2.1.** *(Conformal Change Lemma) Let $(N^n, g)$ be a smooth (not necessarily complete) Riemannian manifold and let $U \subset N$ be an open set. Assume that for some $\rho \in (0,1]$, we have $\sup_U |\mathrm{Rm}|_g \leq \rho^{-2}$, $B_g(x,\rho) \subset\subset N$ and $\mathrm{inj}_g(x) \geq \rho$ for all $x \in U$. Then there exist a constant $\gamma = \gamma(n) \geq 1$, an open set $\tilde{U} \subset U$ and a smooth metric $\tilde{g}$ defined on $\tilde{U}$ such that each connected component of $(\tilde{U}, \tilde{g})$ is a complete Riemannian manifold satisfying*

1. $|\mathrm{Rm}|_{\tilde{g}} \leq \gamma\rho^{-2}$ *and* $\mathrm{inj}_{\tilde{g}} \geq \frac{1}{\sqrt{\gamma}}\rho$ *for* $x \in \tilde{U}$
2. $U_\rho \subset \tilde{U} \subset U$
3. $\tilde{g} = g$ *on* $\tilde{U}_\rho \supset U_{2\rho}$,

*where* $U_s = \{x \in U | B_g(x,s) \subset\subset U\}$.

### 3.2.2 Some integrations

For later convenience, we include some frequently used inequalities and their proofs in this subsection.

**Lemma 3.2.2.** *Given $K, R, C_1 > 0$, $t \in (0,1]$ and $n \in \mathbb{N}$. There exists positive constant $C = C(K, C_1, n)$ such that the following holds. Let $(M, g)$ be a complete Riemannian manifold with $\mathrm{Ric} \geq -(n-1)K$ on $B_g(x,R)$ for some point $x \in M$. Then*

$$\frac{C_1}{t^{\frac{n}{2}}} \int\limits_{B_g(x,R)} exp\left(-\frac{d_g^2(x,y)}{C_1 t}\right) d_g y \leq C \tag{3.2.1}$$

*Proof.* Let $\hat{g} = \frac{1}{t}g$, then it suffices to show $I := C_1 \int_{B_{\hat{g}}(x,\frac{R}{\sqrt{t}})} exp(-\frac{d_{\hat{g}}^2(x,y)}{C_1}) d_{\hat{g}} y \leq C(C_1, K, n)$. For all $y \in B_{\hat{g}}(x, \frac{R}{\sqrt{t}})$, the minimizing geodesic connecting $x$ and $y$ lies within $B_{\hat{g}}(x, \frac{R}{\sqrt{t}})$ where $\mathrm{Ric} \geq -(n-1)Kt \geq -(n-1)K$. So by Laplacian comparison the volume form $d_{\hat{g}} y \leq sn_{-K}^{n-1}(r(y)) dr \wedge dvol_{n-1} \leq \frac{exp((n-1)\sqrt{K}r)}{(2\sqrt{K})^{n-1}} dr \wedge dvol_{n-1}$, where $r$ is the distance function centered at $x$ and $dvol_{n-1}$ is the standard volume form on $S^{n-1}(1)$. So we can express the integral on the segment domain in $T_xM$ and obtain

$$
\begin{aligned}
I &\leq \frac{C_1}{(2\sqrt{K})^{n-1}} \int\limits_{r\leq \frac{R}{\sqrt{t}}} exp\left(-\frac{r^2}{C_1}\right) exp((n-1)\sqrt{K}r)\, dr \wedge dvol_{n-1} \\
&\leq C(C_1,n,K) \int_{\mathbb{R}} exp\left(-\frac{r^2}{C_1} + (n-1)\sqrt{K}r\right)\, dr \leq C(C_1,n,K)
\end{aligned}
$$

□

**Lemma 3.2.3.** *Given $C_1, C_2 > 0$ and $n \in \mathbb{N}$. Let $(M, g(t)), t \in [0,1]$ be a complete Ricci flow with $|\mathrm{Rm}|_{g(t)} \leq \frac{C_1}{t}$. Then for any $d \geq 2(n-1)\sqrt{C_1}C_2$,*

$$
\frac{C_2}{t^{\frac{n}{2}}} \int\limits_{M - B_{g(t)}(x, \sqrt[4]{t}\, d)} exp\left(-\frac{d_t^2(x,y)}{C_2 t}\right) d_t y \leq C\, exp\left(-\frac{d^2}{C\sqrt{t}}\right) \tag{3.2.2}
$$

*where $C$ is a constant depending on $n$, $C_1$ and $C_2$.*

*Proof.* For convenience, $C$ denotes all the constants depending on $C_1$, $C_2$, and $C_3$. Fix $t$, let $\hat{g} = \frac{1}{t}g(t)$. Then it suffices to show

$$
C_2 \int\limits_{M - B_{\hat{g}}(x, \frac{d}{\sqrt[4]{t}})} exp\left(-\frac{d_{\hat{g}}^2(x,y)}{C_2}\right) d_{\hat{g}} y \leq C\, exp\left(-\frac{d^2}{C\sqrt{t}}\right) \tag{3.2.3}
$$

with $|\mathrm{Rm}|_{\hat{g}} \leq C_1$.

Since $\mathrm{Ric} \geq -(n-1)C_1$, we get by Laplacian comparison that the volume form $d_{\hat{g}}y \leq sn_{-C_1}^{n-1}(r(y))dr \wedge dvol_{n-1} \leq \frac{e^{(n-1)\sqrt{C_1}r}}{(2\sqrt{C_1})^{n-1}} dr \wedge dvol_{n-1}$ Thus by considering the integral over the segment domain in $T_xM$, denoting by $\omega_{n-1}$ the volume of $S^{n-1}(1)$, we get

$$
\begin{aligned}
I &\leq \frac{C_2}{(2\sqrt{C_1})^{n-1}} \int\limits_{r \geq \frac{d}{\sqrt[4]{t}}} exp\left(-\frac{r^2}{C_2}\right) exp((n-1)\sqrt{C_1}r)\, dr \wedge dvol_{n-1} \\
&= \frac{C_2}{(2\sqrt{C_1})^{n-1}} \omega_{n-1} \int\limits_{r \geq \frac{d}{\sqrt[4]{t}}} exp\left(-\frac{r^2}{C_2} + (n-1)\sqrt{C_1}r\right) dr \\
&\leq C \int\limits_{r \geq \frac{d}{\sqrt[4]{t}}} exp\left(-\frac{r^2}{2C_2}\right) dr \leq C\, exp\left(-\frac{d^2}{2C_2\sqrt{t}}\right).
\end{aligned}
$$

□

**Lemma 3.2.4.** *Given $t, T, d, C > 0$ and $n \in \mathbb{N}$ such that $t < T \leq d^2$, there exists positive constant $C_1 = C_1(C, n)$ such that*

$$\frac{C}{t^{\frac{n}{2}}} exp\left(-\frac{d^2}{Ct}\right) \leq \frac{C_1}{T^{\frac{n}{2}}} exp\left(-\frac{d^2}{C_1 T}\right). \tag{3.2.4}$$

*Proof.* It's easy to see there exists $C_1 = C_1(C, n)$ such that for all $x > 0$,

$$\frac{1}{x^{\frac{n}{2}}} exp\left(-\frac{1}{Cx}\right) \leq C_1 exp\left(-\frac{1}{2Cx}\right). \tag{3.2.5}$$

Then (3.2.4) follows immediately from this inequality and the above assumptions. □

### 3.2.3 Weak derivatives

In this section, we assume $(M^n, g(t))$ is a complete Ricci flow with bounded curvature. As we mentioned in introduction, $\ell$ satisfies the evolution inequality (3.1.7) in the barrier sense: for any $(q, \tau) \in M \times (0, T)$ we find a neighborhood $\mathcal{U} \subset M \times (0, T)$ of $(q, \tau)$ and a $C^\infty$ function $\phi : \mathcal{U} \to \mathbb{R}$ such that $\phi \leq \ell$ on $\mathcal{U}$, with equality at $(q, \tau)$ and

$$(\frac{\partial}{\partial t} - \Delta)\phi \leq R\ell + C(n)\ell^2 \quad \text{at} \quad (q, \tau). \tag{3.2.6}$$

Set $\mathcal{L} = e^{-C(n)t}\ell$ and assume $\ell \leq 1$ then by (3.1.7) we have the following inequality which holds in the barrier sense

$$(\frac{\partial}{\partial t} - \Delta)\mathcal{L} \leq R\,\mathcal{L}. \tag{3.2.7}$$

Suppose for a moment that $\mathcal{L}$ is smooth and $\psi(x, t)$ is a non-negative smooth function which is compactly supported in $M$ for each $t$. Then we see from the integration by parts formula that

$$\begin{aligned}
\frac{\partial}{\partial t}\int_U \mathcal{L}\psi \, d_t x &= \int_U (\frac{\partial}{\partial t}\mathcal{L}\psi - \mathcal{L}\psi\, R + \mathcal{L}\frac{\partial}{\partial t}\psi)\, d_t x \\
&\leq \int_U ((\Delta\mathcal{L})\psi + \mathcal{L}\frac{\partial}{\partial t}\psi)\, d_t x \\
&= \int_U \mathcal{L}(\Delta\psi + \frac{\partial}{\partial t}\psi)\, d_t x.
\end{aligned} \tag{3.2.8}$$

We show in Lemma 3.2.6 that some variant of (3.2.8) is still true without the smooth assumptions either on $\ell$ or the test function $\psi$.

First, we give the definitions of inequalities in several weak senses. We say a continuous function $f : M \to \mathbb{R}$ satisfies $\Delta f \leq u$ for some function $u : M \to \mathbb{R}$ in the barrier sense if for any point $x$ and every $\varepsilon > 0$ there exists a neighborhood $\mathcal{U}_\varepsilon \subset M$ of $x$ and a smooth function $h_\varepsilon : \mathcal{U}_\varepsilon \to \mathbb{R}$ such that $h_\varepsilon(x) = f(x)$, $h_\varepsilon \geq f$ in $\mathcal{U}_\varepsilon$ and $\Delta h_\varepsilon(x) \leq u(x) + \varepsilon$.

We say a continuous function $f : M \to \mathbb{R}$ satisfies $\Delta f \leq u$ for some bounded function $u : M \to \mathbb{R}$ in the distributional sense if for any non-negative smooth function $h$ with compact support one has $\int f\Delta h \leq \int uh$. By standard argument, if $f$ satisfies $\Delta f \leq u$ in the barrier sense, then $f$ satisfies it in the distributional sense (see for example [52, Appendix A]).

**Lemma 3.2.5.** *Let $\psi(x,t)$ be a non-negative smooth function which is compactly supported in $M$ for each $t$. $\mathcal{L} = e^{-C(n)t}\ell$ with $\ell \leq 1$. Then we have*

$$\frac{\partial^+}{\partial t}\int \mathcal{L}\psi \, d_tx \leq \int \mathcal{L}(\Delta\psi + \frac{\partial}{\partial t}\psi)\, d_tx \tag{3.2.9}$$

*for all $t \in [a,b) \subset (0,T)$, integrating which we have:*

$$\left(\int \mathcal{L}\psi \, d_tx\right)\Big|_a^b \leq \int_a^b \left(\int \mathcal{L}(\Delta\psi + \frac{\partial}{\partial t}\psi)\right) d_tx\, dt \tag{3.2.10}$$

*Proof.* Let $t_0$ be an arbitrary time in $[a,b)$. Since $\mathcal{L}$ satisfies

$$(\frac{\partial}{\partial t} - \Delta)\mathcal{L} \leq R\,\mathcal{L}$$

in the barrier sense, by the maximum principle for complete manifold with bounded curvature, $\mathcal{L}(\cdot,t) \leq \overline{\mathcal{L}}(\cdot,t)$ for all $t \in [t_0,b]$, where $\overline{\mathcal{L}}$ is the solution to the initial value problem:

$$(\frac{\partial}{\partial s} - \Delta)\overline{\mathcal{L}} = R\overline{\mathcal{L}}, \quad \overline{\mathcal{L}}(\cdot,t_0) = \mathcal{L}(\cdot,t_0). \tag{3.2.11}$$

Then $\overline{\mathcal{L}}$ is smooth for all $t > t_0$ and so we have

$$\frac{\partial^+}{\partial t}\Big|_{t_0}\int \mathcal{L}\psi\, d_tx \leq \frac{\partial^+}{\partial t}\Big|_{t_0}\int \overline{\mathcal{L}}\psi\, d_tx = \lim_{t\to t_0^+}\frac{\partial}{\partial t}\int \overline{\mathcal{L}}\psi\, d_tx. \tag{3.2.12}$$

For each $t > t_0$, we calculate by integration by parts to get

$$\frac{\partial}{\partial t}\int \overline{\mathcal{L}}\psi\, d_tx = \int \overline{\mathcal{L}}(\Delta\psi + \frac{\partial}{\partial t}\psi)\, d_tx, \tag{3.2.13}$$

substituting which into (3.2.12) we have

$$\frac{\partial^+}{\partial t}\Big|_{t_0}\int \mathcal{L}\psi\, d_tx \leq \lim_{t\to t_0^+}\int \overline{\mathcal{L}}(\Delta\psi + \frac{\partial}{\partial t}\psi)\, d_tx = \int \mathcal{L}(\Delta\psi + \frac{\partial}{\partial t}\psi)\, d_tx\Big|_{t_0} \tag{3.2.14}$$

□

**Lemma 3.2.6.** *Let $\psi(x,t)$ be a non-negative continuous function which is compactly supported in $M$ for each $t$, and satisfies $\Delta\psi \le u(x,t)$ and $\frac{\partial}{\partial t}\psi \le v(x,t)$ in the barrier sense, where $v(x,t)$ is continuous with respect to $t$.*

*Then for all $t$ we have*

$$\frac{\partial^+}{\partial t}\int \mathcal{L}(x,t)\psi(x,t)\,d_tx \le \int \mathcal{L}(x,t)(u(x,t)+v(x,t))\,d_tx \tag{3.2.15}$$

*Proof.* Let $t_0$ be an arbitrary time in $(a,b)$. Differentiating at $t_0$ by the product rule we get

$$\frac{\partial^+}{\partial t}\bigg|_{t_0}\int \mathcal{L}(x,t)\psi(x,t)\,d_tx \le \int \mathcal{L}(x,t_0)v(x,t_0)\,d_{t_0}x + \frac{\partial^+}{\partial t}\bigg|_{t_0}\int \mathcal{L}(x,t)\psi(x,t_0)\,d_tx. \tag{3.2.16}$$

Let $\overline{\mathcal{L}}$ be the solution to the initial value problem

$$(\frac{\partial}{\partial s}-\Delta)\overline{\mathcal{L}} = R\overline{\mathcal{L}}, \quad \overline{\mathcal{L}}(\cdot,t_0) = \mathcal{L}(\cdot,t_0). \tag{3.2.17}$$

Then $\overline{\mathcal{L}}$ is smooth for all $t > t_0$. We calculate using the fact that barrier sense implies distributional sense:

$$\begin{aligned}
\frac{\partial^+}{\partial t}\bigg|_{t_0}\int \mathcal{L}(x,t)\psi(x,t_0)\,d_tx &\le \frac{\partial^+}{\partial t}\bigg|_{t_0}\int \overline{\mathcal{L}}(x,t)\psi(x,t)\,d_tx \\
&\le \limsup_{t\to t_0^+}\frac{\partial}{\partial t}\int \overline{\mathcal{L}}(x,t)\psi(x,t_0)d_tx \\
&= \limsup_{t\to t_0^+}\int \Delta\overline{\mathcal{L}}(x,t)\psi(x,t_0)d_tx \\
&\le \limsup_{t\to t_0^+}\int \overline{\mathcal{L}}(x,t)u(x,t_0)d_tx \\
&= \int \mathcal{L}(x,t_0)u(x,t_0)d_{t_0}x
\end{aligned} \tag{3.2.18}$$

where we used the fact that barrier sense implies distributional sense

□

## 3.3 Curvature Decay Lemma

The main result in this section is Lemma 3.3.4, which provides a local estimate on the norm of the Riemann curvature tensor, under the assumption of a local bound for $\ell$. This lemma can be viewed as a weaker version of Theorem 3.1.1 in the sense that we take the two conclusions of the existence of the Ricci flow and the bound of $\ell$, as additional hypotheses, and deduce the remaining conclusion about $|\mathrm{Rm}|$.

We need three ingredients in the proof of Lemma 3.3.4. One is the following Lemma, given in [63, Lemma 5.1] by a point-picking argument.

**Lemma 3.3.1.** *Given $c_0, r_0 > 0$, $n \in \mathbb{N}$, and take $\beta = \beta(n) > 0$ as in Lemma 2.3.3. Let $(M^n, g(t))$, $t \in [0,T]$ be a Ricci flow. Suppose for some $x_0 \in M$ we have $B_{g(t)}(x_0, r_0) \subset\subset M$ for each $t \in [0,T]$.*

*Then at least one of the following assertions is true:*

1. *For each $t \in [0,T]$ with $t < \frac{r_0^2}{\beta^2 c_0}$, we have $B_{g(t)}(x_0, r_0 - \beta\sqrt{c_0 t}) \subset B_{g(0)}(x_0, r_0)$ and*
$$|\mathrm{Rm}|_{g(t)} < \frac{c_0}{t} \quad \text{on } B_{g(t)}(x_0, r_0 - \beta\sqrt{c_0 t}). \tag{3.3.1}$$
2. *There exist $\bar{t} \in (0,T]$ with $\bar{t} < \frac{r_0^2}{\beta^2 c_0}$ and $\bar{x} \in B_{g(\bar{t})}(x_0, r_0 - \frac{1}{2}\beta\sqrt{c_0 \bar{t}})$ such that*
$$Q := |\mathrm{Rm}|_{g(\bar{t})}(\bar{x}) \geq \frac{c_0}{\bar{t}}, \tag{3.3.2}$$
*and*
$$|\mathrm{Rm}|_{g(t)}(x) \leq 4Q = 4|\mathrm{Rm}|_{g(\bar{t})}(\bar{x}), \tag{3.3.3}$$
*whenever $d_{g(\bar{t})}(x, \bar{x}) < \frac{\beta c_0}{8} Q^{-\frac{1}{2}}$ and $\bar{t} - \frac{1}{8} c_0 Q^{-1} \leq t \leq \bar{t}$.*

The second ingredient we need, [63, Lemma 2.3], says that the volume of a ball of fixed radius cannot decrease too rapidly under some curvature hypothesis.

**Lemma 3.3.2.** *Given $K, \gamma, c_0, v_0, T > 0$ and $n \in \mathbb{N}$, there exist positive constants $\varepsilon_0 = \varepsilon_0(v_0, K, \gamma, n)$ and $\hat{T} = \hat{T}(v_0, c_0, K, \gamma, n) \geq 0$ such that the following holds: Let $(M^n, g(t)), t \in [0,T)$ be a Ricci flow such that $B_{g(t)}(x_0, \gamma) \subset\subset M$ for some $x_0 \in M$ and all $t \in [0,T)$. Suppose $\mathrm{Ric}_{g(t)} \geq -K$ and $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$ on $B_{g(t)}(x_0, \gamma)$ for all $t \in [0,T)$, and $Vol_{g(0)} B_{g(0)}(x_0, \gamma) \geq v_0$.*

*Then*
$$Vol_{g(t)} B_{g(t)}(x_0, \gamma) \geq \varepsilon_0 \tag{3.3.4}$$
*for all $t \in [0, \hat{T}] \cap [0,T)$.*

The third ingredient is the following Lemma, which says that the asymptotic volume ratio of a weakly $\mathrm{PIC}_1$ ancient solution is zero. This is proved in [2, Lemma 4.2]. We note that each curvature condition listed in Theorem 3.1.1 implies weakly $\mathrm{PIC}_1$, so the proof of Lemma 3.3.4 is uniform for all $\mathcal{C}$.

**Lemma 3.3.3.** *Let $(M^n, g(t)), t \in (-\infty, 0]$ be a nonflat ancient solution of the Ricci flow with bounded curvature satisfying weakly $\mathrm{PIC}_1$. Then it has non-negative complex sectional curvature. Furthermore, the volume growth is non-Euclidean, i.e. $\lim_{r\to\infty} r^{-n} Vol_{g(0)} B_{g(0)}(x, r) = 0$ for all $x \in M$.*

We now states our main result of this section. In the proof we blow up a contradicting sequence to get a weakly $\mathrm{PIC}_1$ ancient solution with positive asymptotic volume ratio, which is impossible by Lemma 3.3.3.

**Lemma 3.3.4.** *(Curvature Decay Lemma). Given $v_0, K > 0$, there exist positive constants $\tilde{T} = \tilde{T}(v_0, K, n)$, $C_1 = C_1(v_0, K, n)$ and $\eta_0 = \eta_0(v_0, K, n)$ such that the following holds: Let $(M^n, g(t)), t \in [0, T]$ be a Ricci flow (not necessarily complete) such that $B_{g(t)}(x_0, 1) \subset\subset M$ for each $t \in [0, T]$ and some $x_0 \in M$, and*

$$Vol_{g(0)} B_{g(0)}(x_0, 1) \geq v_0 > 0. \tag{3.3.5}$$

*Suppose further that*

$$\ell(x, t) \leq K \quad \text{on} \bigcup_{s \in [0,T]} B_{g(s)}(x_0, 1), \quad \text{for all } \; t \in [0, T]. \tag{3.3.6}$$

*Then for all $t \in (0, T) \cap (0, \tilde{T})$, we have*

$$|\mathrm{Rm}|_{g(t)} < \frac{C_1}{t} \quad \text{on} \quad B_{g(t)}(x_0, \frac{1}{2}), \tag{3.3.7}$$

*and*

$$Vol_{g(t)} B_{g(t)}(x_0, 1) \geq \eta_0 \quad \text{and} \quad \mathrm{inj}_{g(t)}(x_0) \geq \sqrt{\frac{t}{C_1}} \tag{3.3.8}$$

*for all $t \in (0, \min(T, \tilde{T})]$.*

*Proof.* By Bishop-Gromov inequality, $Vol_{g(0)} B_{g(0)}(x_0, \frac{1}{2})$ has a positive lower bound depending only on $v_0$, $n$ and $K$. Applying Lemma 3.3.2 to $g(t)$, we see that there exists $\eta_0 > 0$ depending only on $v_0$, $n$ and $K$ such that for each $C_1 < \infty$, there exist $\tilde{T} = \tilde{T}(v_0, n, C_1)$ such that prior to time $\tilde{T}$ and while $|\mathrm{Rm}|_{g(t)} \leq \frac{C_1}{t}$ still holds on $B_{g(t)}(x_0, \frac{1}{2})$, we have a lower volume bound

$$Vol_{g(t)} B_{g(t)}(x_0, 1) \geq \eta_0. \tag{3.3.9}$$

In particular, $\eta_0$ is independent of $C_1$. From this we deduce that is suffices to prove the lemma with the additional hypothesis that the equation above holds for each $t \in [0, T)$.

Let us assume that the lemma is false, even with the extra hypothesis. Then for any sequence $c_k \to \infty$, we can find Ricci flows that fail the lemma with $C_1 = c_k$ in an arbitrary short time, and in particular within a time $t_k$ that is sufficiently small so that $c_k t_k \to 0$ as $k \to \infty$. By reducing $t_k$ to the first time at which the desired conclusion fails, we have a sequence of Ricci flows $(M_k, \tilde{g}_k(t))$ for $t \in [0, t_k]$ with $t_k \to 0$, and even $c_k t_k \to 0$, and a sequence of points $x_k \in M_k$ with $B_{\tilde{g}_k(t)}(x_k, 1) \subset\subset M_k$ for each $t \in [0, t_k]$, such that

$$Vol_{\tilde{g}_k(t)} B_{\tilde{g}_k(t)}(x_k, 1) \geq \eta_0, \; \text{ for all } \; t \in [0, t_k], \tag{3.3.10}$$

$$\ell(x,t) \leq K, \text{ on } \bigcup_{s\in[0,t_k]} B_{\tilde{g}_k(s)}(x_k,1) \text{ for all } t\in[0,t_k], \tag{3.3.11}$$

and

$$|\mathrm{Rm}|_{\tilde{g}_k(t)} < \frac{c_k}{t} \text{ on } B_{\tilde{g}_k(t)}(x_k,\frac{1}{2}) \text{ for all } t\in[0,t_k], \tag{3.3.12}$$

but so that

$$|\mathrm{Rm}|_{\tilde{g}_k(t_k)} = \frac{c_k}{t_k} \text{ at some point in } \overline{B_{\tilde{g}_k(t)}(x_k,\frac{1}{2})}. \tag{3.3.13}$$

For sufficiently large $k$, we have $\beta\sqrt{c_k t_k} < \frac{1}{4}$. We apply Lemma 3.3.1, to each $\tilde{g}_k(t)$ with $r_0 = \frac{3}{4}$ and $c_0 = c_k$, then it follows by (3.3.13) that Assertion 1 there cannot hold, and thus Assertion 2 must hold for each $k$, giving time $\bar{t}_k \in (0,t_k]$ and points $\bar{x}_k \in B_{\tilde{g}_k(\bar{t}_k)}(x_k, r_0 - \frac{1}{2}\beta\sqrt{c_k\bar{t}_k})$ such that

$$|\mathrm{Rm}|_{\tilde{g}_k(t)}(x) \leq 4|\mathrm{Rm}|_{\tilde{g}_k(\bar{t}_k)}(\bar{x}_k) \tag{3.3.14}$$

on $B_{\tilde{g}(\bar{t}_k)}(\bar{x}_k, \frac{\beta c_k}{8}Q_k^{-\frac{1}{2}})$, for all $t \in [\bar{t}_k - \frac{1}{8}c_kQ_k^{-1}, \bar{t}_k]$, where $Q_k := |\mathrm{Rm}|_{\tilde{g}_k(\bar{t}_k)}(\bar{x}_k) \geq \frac{c_k}{\bar{t}_k} \to \infty$. We also notice that $B_{\tilde{g}(\bar{t}_k)}(\bar{x}_k, \frac{\beta c_k}{8}Q_k^{-\frac{1}{2}}) \subset B_{\tilde{g}(\bar{t}_k)}(x_k,1)$, thus

$$\ell(x,t) \leq K \tag{3.3.15}$$

on $B_{\tilde{g}(\bar{t}_k)}(\bar{x}_k, \frac{\beta c_k}{8}Q_k^{-\frac{1}{2}}) \times [\bar{t}_k - \frac{1}{8}c_kQ_k^{-1}, \bar{t}_k]$. The above conditions at $\bar{t}_k$, together with Bishop-Gromov inequality, imply that we have uniform volume ratio control

$$\frac{Vol_{\tilde{g}_k(\bar{t}_k)}B_{\tilde{g}_k(\bar{t}_k)}(\bar{x}_k,r)}{r^n} \geq \eta > 0 \tag{3.3.16}$$

for all $0 < r < \frac{1}{4}$, where $\eta$ depends on $\eta_0$, $K$ and $n$. A parabolic rescaling on $B_{\tilde{g}(\bar{t}_k)}(\bar{x}_k, \frac{\beta c_k}{8}Q_k^{-\frac{1}{2}}) \times [\bar{t}_k - \frac{1}{8}c_kQ_k^{-1}, \bar{t}_k]$ gives new Ricci flows defined by

$$g_k(t) := Q_k\tilde{g}_k(\frac{t}{Q_k} + \bar{t}_k)$$

for $t \in [-\frac{1}{8}c_k, 0]$. The scaling factor is chosen so that $|\mathrm{Rm}|_{g_k(0)}(\bar{x}_k) = 1$. By (3.3.14), the curvature of $g_k(t)$ is uniformly bounded on $B_{g_k(0)}(\bar{x}_k, \frac{1}{8}\beta c_k) \times [-\frac{1}{8}c_k, 0]$. Condition (3.3.15) transforms to

$$\ell(x,t) \leq \frac{K}{Q_k} \to 0 \tag{3.3.17}$$

on $B_{g_k(0)}(\bar{x}_k, \frac{1}{8}\beta c_k) \times [-\frac{1}{8}c_k, 0]$. The volume ratio (3.3.16) gives

$$\frac{Vol_{g_k(0)}B_{g_k(0)}(\bar{x}_k,r)}{r^n} \geq \eta > 0 \tag{3.3.18}$$

for all $0 < r < \frac{1}{4}Q_k^{\frac{1}{2}} \to \infty$.

With this control we can apply Hamilton's compactness theorem to give convergence $(M_k, g_k(t), \bar{x}_k) \to (N, g(t), x_\infty)$, for some complete bounded-curvature Ricci flow $(N, g(t))$, for $t \in (-\infty, 0]$, and $x_\infty \in N$.

Moreover, the last volume equation passes to limit to force $g(t)$ to have positive asymptotic volume ratio. From (3.3.17) we know that $g(t)$ is a nonflat ancient solution of Ricci flow with bounded curvature satisfying weakly $\text{PIC}_1$. This contradicts Lemma 3.3.3 that the volume ratio of $(N, g(t))$ vanishes, and thus shows the first part of the Lemma. For the second part, the injectivity radius estimate of Cheeger-Gromov-Taylor [19] and the Bishop-Gromov comparison then tell us $\text{inj}_{g(t)}(x) \geq i_0\sqrt{t}$ for some $i_0 = i_0(\eta_0, C) > 0$. $\square$

## 3.4 A cut-off function

In this section we construct a cut-off function on manifolds (not assumed to be complete) evolving by Ricci flow, which helps to localize the integration estimates in section 7.

**Lemma 3.4.1.** *Given $n \in \mathbb{N}$, $c_0, K > 0$, $0 < T < 1$, $0 < R < 1$, $0 < r < \frac{1}{10}$ with $\beta\sqrt{c_0 T} \leq \frac{1}{4}r$, where $\beta = \beta(n)$ is from the Shrinking Lemma, there exists positive constant $C = C(n, K, v_0)$ such that the following holds: Let $(M^n, g(t)), t \in [0, T]$ be a smooth Ricci flow such that $B_{g(0)}(x_0, R + r) \subset\subset M$, and on $B_{g(0)}(x_0, R + r) \times [0, T]$,*

$$\text{Ric}_{g(t)}(x) \geq -K \quad \text{and} \quad |\text{Rm}|_{g(t)} \leq \frac{c_0}{t}, \tag{3.4.1}$$

*and for all $\delta \in [0, r]$ and $x \in B_{g(0)}(x_0, R)$ we have*

$$Vol_{g(0)} B_{g(0)}(x, \delta) \geq v_0 \delta^n. \tag{3.4.2}$$

*Then there exists a continuous function $\phi(y, s) : M \times [0, T] \longrightarrow \mathbb{R}$ with the following properties:*

**(P1)** *$\text{supp}\, \phi(\cdot, s) \subset B_{g(0)}(x_0, R)$ for all $s \in [0, T]$.*

**(P2)** *$\nabla\phi$ exists a.e. and $|\nabla\phi| \leq Cr^{-(n+1)}$.*

**(P3)** *$\Delta\phi \leq Cr^{-(2n+2)}$ in the barrier sense.*

**(P4)** *$\frac{\partial^+}{\partial s}\phi \leq Cr^{-n}$.*

*Moreover, we have the inclusions:*

$$B_{g(s)}(x_0, R - \tfrac{5}{4}r) \subset B_{g(0)}(x_0, R - r) \subset \{y \in M \mid \phi(y, s) = 1\} \tag{3.4.3}$$

*for all $s \in [0, T]$.*

*Proof.* Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a non-increasing smooth function such that $f(z) = 1$ for all $z < \frac{1}{4}$ and $f(z) = 0$ for all $z > \frac{1}{2}$. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a non-decreasing and convex smooth function such that $F(z) = 0$ for all $z \leq 0$ and $F(1) = 1$. Let $C_0$ be a constant such that $|f'|, |f''|, |F'|, |F''| \leq C_0$. Hereafter we use the same letter $C$ to denote the constants depending on $K$, $v_0$, $n$.

Let $\{p_k\}_{k=1}^N$ be a maximal $\frac{r}{4e^K}$-separated set in the annulus $A := B_{g(0)}(x_0, R) - B_{g(0)}(x_0, R - \frac{1}{4}r)$ with respect to $g(0)$. By a $\varepsilon$-separated set we mean a set in which the points are at least $\varepsilon$-distant from each other. It's clear that the $\varepsilon/2$-balls of points in a $\varepsilon$-separated set are disjoint pairwise. By volume comparison we see that $Vol_{g(0)} B_{g(0)}(x_0, R) \leq C$, and furthermore by (3.4.2) $Vol_{g(0)} B_{g(0)}(p_k, \frac{r}{4e^K}) \geq Cr^n$. Hence we have $N \leq Cr^{-n}$.

**Claim 3.4.2.** *$A \subset \bigcup_{k=1}^{N} B_{g(s)}(p_k, \frac{r}{4})$ for all $s \in [0, T]$.*

*Proof of Claim 3.4.2.* By the choice of $\{p_k\}_{k=1}^N$ we see that $A \subset \bigcup_{k=1}^{N} B_{g(0)}(p_k, \frac{r}{4e^K})$. For each $p_k$, the triangle inequality implies that $B_{g(0)}(p_k, \frac{r}{2}) \subset\subset B_{g(0)}(x_0, R + r)$ where $|\mathrm{Rm}|_{g(s)} \leq \frac{c_0}{s}$ and $\mathrm{Ric}_{g(s)} \geq -K$ holds true for all $s \in [0, T]$. Applying the Shrinking Lemma to $g(t)$, we find that $B_{g(s)}(p_k, \frac{r}{2} - \beta\sqrt{c_0 s}) \subset B_{g(0)}(p_k, \frac{r}{2})$ for all $s \in [0, T]$ and in particular $B_{g(s)}(p_k, \frac{r}{4}) \subset B_{g(0)}(p_k, \frac{r}{2})$ due to $\beta\sqrt{c_0 T} \leq \frac{1}{4}r$. So $\mathrm{Ric} \geq -K$ holds on $B_{g(s)}(p_k, \frac{r}{4})$, which gives the condition we need in order to apply the Expanding Lemma to the Ricci flow on $B_{g(s)}(p_k, \frac{r}{4}) \times [0, s]$, giving $B_{g(s)}(p_k, \frac{r}{4}) \supset B_{g(0)}(p_k, \frac{r}{4e^K})$, and thus proves the claim. $\square$

By the Shrinking Lemma and triangle inequality, we have $B_{g(s)}(p_k, \frac{r}{2}) \subset B_{g(0)}(p_k, r) \subset B_{g(0)}(x_0, R + r)$. In view of this together with the definition of $f$, we define the following continuous function on $M$:

$$f_k(y, s) = \begin{cases} f\left(\frac{d_{g(s)}(p_k, y)}{r}\right) & \text{for} \quad y \in B_{g(0)}(p_k, r)\ ; \\ 0 & \text{for} \quad y \notin B_{g(0)}(p_k, r). \end{cases} \tag{3.4.4}$$

By Claim 3.4.2, for each point $y \in A$ and $s \in [0, T]$, there is some $k$ such that $y \in B_{g(s)}(p_k, \frac{r}{4})$, $f_k(y, s) = 1$ and $F(1 - \sum_{k=1}^N f_k(y, s)) = 0$. Based on this we define the following continuous function on $M$:

$$\phi(y, s) = \begin{cases} F(1 - \sum_{k=1}^{N} f_k(y, s)) & \text{for} \quad y \in B_{g(0)}(x_0, R)\ ; \\ 0 & \text{for} \quad y \notin B_{g(0)}(x_0, R)\ . \end{cases} \tag{3.4.5}$$

It's clear that $\phi(y, s)$ satisfies (P1). Below we abbreviate $d_{g(s)}(p_k, y)$ by $d_k$, $f'(\frac{d_{g(s)}(p_k,y)}{r})$ by $f'_k$, and $f''(\frac{d_{g(s)}(p_k,y)}{r})$ by $f''_k$. Using that

$$\nabla\phi = -F' \cdot \sum_{k=1}^{N} f'_k \cdot r^{-1} \cdot \nabla d_k, \tag{3.4.6}$$

and taking into account that $\nabla d_k$ exists a.e. with $|\nabla d_k| = 1$, and $N \leq C \cdot r^{-n}$, we see that $\nabla\phi$ exists a.e. and
$$|\nabla\phi| \leq C \cdot r^{-(n+1)}. \tag{3.4.7}$$

To estimate $\frac{\partial}{\partial s}\phi(y,s)$ and $\Delta\phi(y,s)$, we may assume $y \in B_{g(s)}(p_k, \frac{1}{2}r) - B_{g(s)}(p_k, \frac{1}{4}r)$ without loss of generality. Because otherwise $f'(\frac{d_k(y,s)}{r}) = 0$, and hence $\frac{\partial}{\partial s}\phi(y,s) = \Delta\phi(y,s) = 0$. By the Shrinking Lemma and the choice of $p_k$ we have

$$B_{g(s)}(p_k, \frac{1}{2}r) \subset B_{g(0)}(p_k, r) \subset B_{g(0)}(x_0, R + r). \tag{3.4.8}$$

So the minimizing geodesic connecting $y$ and $p_k$ with respect to $g(s)$ remains within $B_{g(0)}(x_0, R+r)$ where $\mathrm{Ric}_{g(s)} \geq -K$. Hence by the Laplacian comparison and noting that $d_{g(s)}(y, p_k) \geq \frac{1}{4}r$, we have
$$\Delta d_{g(s)}(p_k, y) \leq (n-1)\sqrt{K}\coth(\sqrt{K}d_{g(s)}(p_k, y)) \leq \frac{C}{r} \tag{3.4.9}$$
in the barrier sense. Then using that

$$\Delta\phi = F''|\sum_{k=1}^{N} f_k' \cdot r^{-1} \cdot \nabla d_k|^2 - F' \cdot \sum_{k=1}^{N}(f_k'' \cdot r^{-2} \cdot |\nabla d_k|^2 + f_k' \cdot r^{-1} \cdot \Delta d_k), \tag{3.4.10}$$

and noting $f' \leq 0$, $F' \geq 0$, we can estimate

$$\Delta\phi \leq C \cdot r^{-(2n+2)}. \tag{3.4.11}$$

We see from the Ricci flow equation that

$$\frac{\partial^+}{\partial s} d_{g(s)}(p_k, y) \leq K d_{g(s)}(p_k, y) \leq \frac{1}{2}Kr, \tag{3.4.12}$$

and using that
$$\frac{\partial^+}{\partial s}\phi = -F'\sum_{k=1}^{N} f_k' \cdot r^{-1} \cdot \frac{\partial^+}{\partial s} d_k, \tag{3.4.13}$$

we obtain
$$\frac{\partial^+}{\partial s}\phi \leq C \cdot r^{-n}. \tag{3.4.14}$$

It remains to prove the inclusion (3.4.3). The first inclusion is a consequence of the Shrinking Lemma and $\beta\sqrt{c_0 T} \leq \frac{1}{4}r$. To prove the second inclusion, we note by triangle inequality that
$$B_{g(0)}(x_0, R - r) \cap \bigcup_{k=1}^{N} B_{g(0)}(p_k, \frac{3}{4}r) = \emptyset, \tag{3.4.15}$$

and by the Shrinking Lemma,

$$B_{g(s)}(p_k, \frac{1}{2}r) \subset B_{g(0)}(p_k, \frac{1}{2}r + \beta\sqrt{c_0 T}) \subset B_{g(0)}(p_k, \frac{3}{4}r) \tag{3.4.16}$$

for each $k$ and $s \in [0, T]$. Thus for all $s \in [0, T]$,

$$B_{g(0)}(x_0, R - r) \cap \bigcup_{k=1}^{N} B_{g(s)}(p_k, \frac{1}{2}r) = \emptyset. \tag{3.4.17}$$

Then the second inclusion in (3.4.3) follows immediately from (3.4.17) and the definitions of $f$ and $\phi$.

□

## 3.5 Heat kernel estimates for Ricci flow in expansion

### 3.5.1 An upper bound for the heat kernel of Ricci flow

Let $(M, g(t))$, $t \in [0, T]$, be a complete Ricci flow. Hereafter we denote by $G(x, t; y, s)$, with $x, y \in M$, $0 \le s < t \le T$, the heat kernel corresponding to the backwards heat equation coupled with the Ricci flow. This means that for any fixed $(x, t) \in M \times [0, T]$ we have

$$(\frac{\partial}{\partial s} + \Delta_{y,s})G(x, t; y, s) = 0 \text{ and } \lim_{s \nearrow t} G(x, t; y, s) = \delta_x(y) \tag{3.5.1}$$

Then for any fixed $(y, s) \in M \times [0, T]$ we can compute that $G(\cdot, \cdot; y, s)$ is the heat kernel associated to the conjugate equation

$$(\frac{\partial}{\partial t} - \Delta_{x,t} - R_{g(t)})G(x, t; y, s) = 0 \text{ and } \lim_{t \searrow s} G(x, t; y, s) = \delta_y(x). \tag{3.5.2}$$

Note that in literatures it is more common to consider the fundamental solution of the conjugate heat equation $\frac{\partial}{\partial t}u + \Delta_{x,t}u - Ru = 0$. $G(x, t; y, s)$ has the following property

$$\int_M G(x, t; y, s)\, d_t x = 1 \text{ for all } 0 \le s < t \le T. \tag{3.5.3}$$

In the compact case, this follows from the following simple calculation:

$$\frac{\partial}{\partial t}\int_M G(x, t; y, s)\, d_t x = \int_M ((\Delta_{x,t} + R_{g(t)})G(x, t; y, s) - G(x, t; y, s)\, R_{g(t)})\, d_t x = 0. \tag{3.5.4}$$

The general case follows using an exhaustion and limiting argument.

The heat kernel $G$ has a Gaussian bound by the following proposition from [2].

**Proposition 3.5.1.** *Given $n \in \mathbb{N}$ and $A > 0$, there is a constant $C = C(n, A) < \infty$ such that the following holds: Let $(M^n, g(t))$, $t \in [0, T]$, be a complete Ricci flow satisfying*

$$|\mathrm{Rm}|_{g(t)} \leq \frac{A}{t} \quad \text{and} \quad Vol_{g(t)} B_{g(t)}(x, \sqrt{t}) \geq \frac{(\sqrt{t})^n}{A} \tag{3.5.5}$$

*for all $(x, t) \in M \times (0, T]$. Then*

$$G(x, t; y, s) \leq \frac{C}{(t-s)^{\frac{n}{2}}} exp(-\frac{d_s^2(x, y)}{C(t-s)}) \text{ for all } 0 \leq s < t \leq T. \tag{3.5.6}$$

**Remark 3.5.2.** *We note that* (3.5.5) *is invariant under rescaling and time shifting in the sense that for the Ricci flow $\hat{g}(\tau) = \frac{1}{t-s} g(\tau(t-s) + s), \tau \in [0, 1]$, where $0 \leq s < t \leq T$, the condition* (3.5.5) *still holds true. The right-hand side of the second bound in* (3.5.5) *may change by a controlled factor due to a volume comparison argument.*

### 3.5.2 Generalized heat kernel of Ricci flow in expansion and its upper bound

**Definition 3.5.3.** (Ricci flow in expansion) We say $(\{M_j\}_{j=1}^m, \{g_j(t)\}_{j=1}^m, \nu)$ is a Ricci flow in expansion, if for each $j$, $(M_j, g_j(t))$ is a complete Ricci flow defined on $[t_j, t_{j+1}]$ with $t_1 > 0$, $t_{j+1} = \nu \, t_j$ for $j \geq 1$, and $M_0 \supset M_1 \supset M_2 \supset ... \supset M_m$. Moreover, at each $t_{j+1}$ we have $g_{j+1}(t_{j+1}) \geq g_j(t_{j+1})$ everywhere on $M_{j+1}$.

We call each $t_j$ a expanding time. In the following discussion we will often need to distinguish metrics $g_{j-1}(t_j)$ and $g_j(t_j)$. Without ambiguity, we use $t_j^+$ whenever referring to any geometric quantity with respect to $g_j(t_j)$, and $t_j^-$ for $g_{j-1}(t_j)$ respectively. For example, $B_{t_j^+}(x, r)$ denotes a $r$-ball centered at $x$ with respect to $g_j(t_j)$ and $M_{t_j^+}$ denotes $M_j$.

**Definition 3.5.4.** (Generalized heat kernel) Let $(\{M_j\}_{j=1}^m, \{g_j(t)\}_{j=1}^m, \nu)$ be a Ricci flow in expansion. For any $x \in M_i$ and $t \in (t_i, t_{i+1}]$, we define the generalized heat kernel $G(x, t; \cdot, \cdot)$ as follow: First, $G(x, t; y, s)$ is the standard heat kernel for all $y \in M_i$ and $s \in [t_i, t)$. Next, suppose $G(x, t; z, s')$ has been defined for all $z \in M_j$ and $s' \in [t_j, t_{j+1})$ for some $j \leq i$. Then for $y \in M_{j-1}$ and $s \in [t_{j-1}, t_j)$, we set

$$G(x, t; y, s) = \int_{M_{t_j^+}} G(x, t; z, t_j) G(z, t_j; y, s) d_{t_j^-} z. \tag{3.5.7}$$

Inductively, $G(x, t; \cdot, \cdot)$ is defined on $(\bigcup_{j=0}^{i-1} M_j \times [t_j, t_{j+1})) \cup M_i \times [t_i, t)$ (see Figure 1). It's easy to see that $G(x, t; \cdot, \cdot)$ is continuous on all over its domain, and smooth on each $M_j \times (t_j, t_{j+1})$ for $j \leq i - 1$ and on $M_i \times (t_i, t)$.

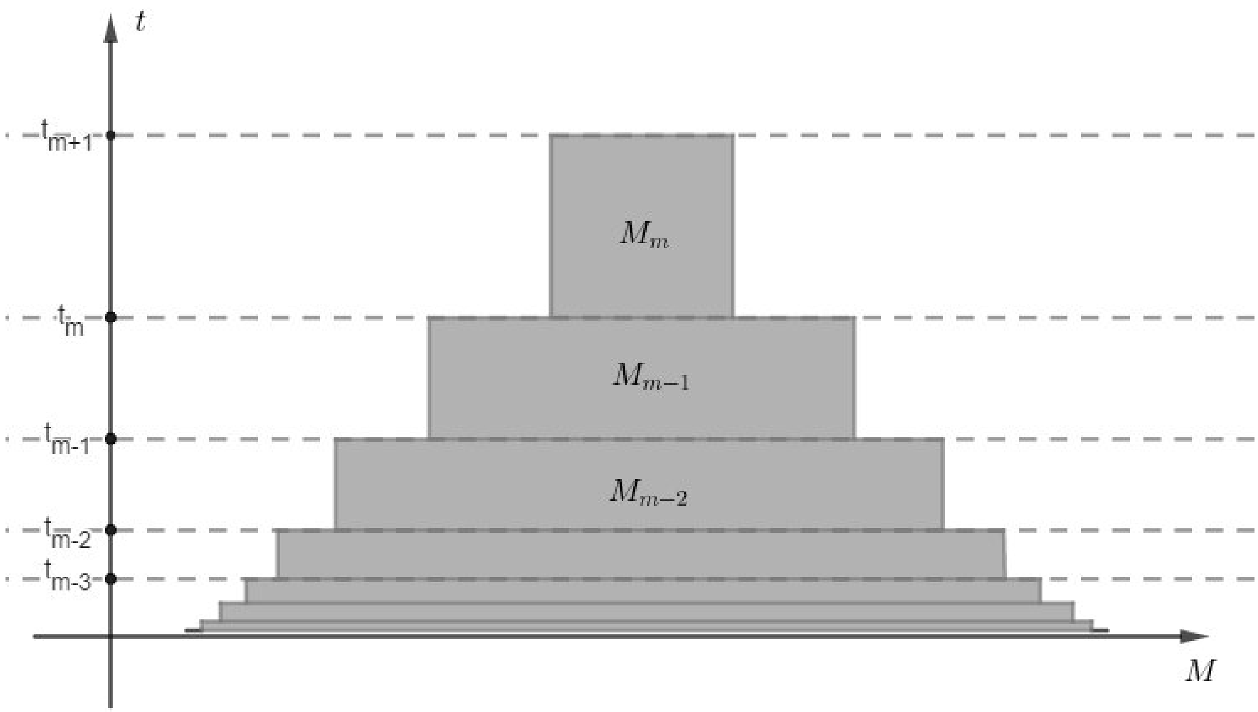


Figure 3.1: Ricci flow in expansion

The goal of this section is to derive a Gaussian bound for the generalized heat kernel. A crucial fact in the proof is that the $L^1$-norm of $G(\cdot, t_j; y, t_{j-1})$ is not bigger than 1 for all $t$, that is,

$$\int_{M_{t_j^+}} G(x, t_j; y, t_{j-1})\, d_{t_j^-} x \le \int_{M_{t_j^-}} G(x, t_j; y, t_{j-1})\, d_{t_j^-} x = 1 \tag{3.5.8}$$

for any $y \in M_{j-1}$.

**Proposition 3.5.5.** *Given $n \in \mathbb{N}$, $A > 0$, and $\nu > 1$, there is a constant $C = C(n, A, \nu) < \infty$ such that the following holds: Let $(\{M_j\}_{j=1}^m, \{g_j(t)\}_{j=1}^m, \nu)$ be a Ricci flow in expansion such that for each $j$ we have*

$$|\mathrm{Rm}|_{g_j(t)} \le \frac{A}{t} \text{ and } Vol_{g_j(t)} B_{g_j(t)}(x, \sqrt{t}) \ge \frac{t^{\frac{n}{2}}}{A} \tag{3.5.9}$$

*for all $x \in M_j$ and $t \in [t_j, t_{j+1}]$. Then for any pairs $(x, t)$ and $(y, s)$ such that $G(x, t; y, s)$ is well defined as above, we have*

$$G(x, t; y, s) \le \frac{C}{(t-s)^{\frac{n}{2}}} exp\left( -\frac{d_{s^+}^2(x, y)}{C(t-s)} \right). \tag{3.5.10}$$

**Remark 3.5.6.** *It may seem surprising that it is not necessary to assume the equality of metrics $g_j(t_{j+1})$ and $g_{j+1}(t_{j+1})$ on $M_{j+1}$. But as we will see in the proof below, the expanding condition $g_j(t_{j+1}) \leq g_{j+1}(t_{j+1})$ is compatible with the application of the Shrinking Lemma and hence sufficient for us to get the conclusion. In later application to the proof of Theorem* 3.1.1*, the metric $g_{j+1}(t_{j+1})$ is the conformally changed metric of $g_j(t_{j+1})$, which is not less than $g_j(t_{j+1})$ everywhere on $M_{j+1}$, and agrees with it on a smaller region.*

*Proof.* For notational convenience, the same letter $C$ will be used to denote constants depending on $n$, $A$ and $\nu$.

**Part 1** Let us first establish the estimate (3.5.10) for $t = t_{k+i}$ and $s = t_i$ for some $i \geq 1$ and $k \geq 1$. Rescaling the flow $g(t), t \in [t_i, t_{k+i}]$ to $\hat{g}(\tau) = \frac{1}{t_{k+i}-t_i} g(\tau(t_{k+i} - t_i) + t_i), \tau \in [0, 1]$, the "expanding time" sequence

$$t_{k+i} > t_{k+i-1} > \cdots > t_{k+i-j} > \cdots > t_{i+1} > t_i$$

becomes

$$1 = \tau_0 > \tau_1 > \cdots > \tau_j > \cdots > \tau_{k-1} > \tau_k = 0$$

where $\tau_j := \frac{t_{k+i-j}-t_i}{t_{k+i}-t_i} = \frac{\nu^{k-j}-1}{\nu^k-1}$, for $j = 0, 1, 2, ..., k$. Then for each $j = 0, 1, ..., k-1$, we have

$$\tau_j - \tau_{j+1} = \frac{\nu^{k-j} - \nu^{k-j-1}}{\nu^k - 1} \leq \nu^{-j}. \tag{3.5.11}$$

To show (3.5.10) for $t = t_{k+i}$ and $s = t_i$, it's equivalent to show the following inequality under the new flow:

$$G(x, 1; y, 0) \leq Cexp\left(-\frac{d^2_{0^+}(x, y)}{C}\right). \tag{3.5.12}$$

We note that by Remark 3.5.2, the new flow $\hat{g}(\tau)$ satisfies the curvature and volume conditions in (3.5.9).

Since $\tau_1 \leq \nu^{-1}$, applying the Gaussian bound (3.5.6) for standard heat kernel we find that

$$G(x, 1; \cdot, \tau_1) \leq \frac{C}{(1-\tau_1)^{\frac{n}{2}}} \leq C_0 := \frac{C}{(1-\nu^{-1})^{\frac{n}{2}}}. \tag{3.5.13}$$

Let $C_0$ be fixed hereafter. Suppose by induction that $G(x, 1; \cdot, \tau_j) \leq C_0$ for some $j \geq 1$. Then for any $z$ such that $G(x, 1; z, \tau_{j+1})$ is well defined, we have

$$\begin{aligned} G(x, 1; z, \tau_{j+1}) &= \int_{M_{\tau_j^+}} G(x, 1; w, \tau_j) G(w, \tau_j; z, \tau_{j+1}) d_{\tau_j^-} w \\ &\leq C_0 \int_{M_{\tau_j^+}} G(w, \tau_j; z, \tau_{j+1}) d_{\tau_j^-} w \leq C_0 \end{aligned} \tag{3.5.14}$$

where we used (3.5.8) in the last inequality. So by induction we obtain

$$G(x,1;\cdot,\tau_j) \leq C_0, \tag{3.5.15}$$

for all $j = 1, 2, ..., k$. In particular, we have $G(x, 1; \cdot, 0) \leq C_0$. This implies (3.5.12) when $d_{0^+}(x, y)$ is controlled. So it remains to show $G(x, 1; y, 0) \leq \ exp\left(-\frac{d^2}{C}\right)$ whenever $d_{0^+}(x, y) \geq 4d(1-(\sqrt[4]{\nu})^{-1})$ for a large number $d$ (which we will specify in the course of proof). For each $j = 1, 2, ..., k$, let

$$r_j = 4d(1 - (\sqrt[4]{\nu})^{-j}). \tag{3.5.16}$$

Then set $B_j = B_{\tau_j^+}(x, r_j)$, $C_j = M_{\tau_j^+} - B_j$ and

$$a_j := \sup_{C_j} G(x, 1; \cdot, \tau_j). \tag{3.5.17}$$

Then it suffices to show the following Claim:

**Claim 3.5.7.** *$a_j \leq Cexp(-\frac{d^2}{C})$, for some constant $C$ independent of $d$, which is uniform for all $j = 1, 2, ..., k$.*

*Proof of Claim 3.5.7.* For each $j$, the expanding condition $g_{j-1}(t_j) \leq g_j(t_j)$ implies $B_j = B_{\tau_j^+}(x, r_j) \subset B_{\tau_j^-}(x, r_j)$. Applying the Shrinking Lemma on $[\tau_{j+1}, \tau_j]$, we find that $B_{\tau_j^-}(x, r_j) \subset B_{\tau_{j+1}^+}(x, r_j + \beta\sqrt{A}\sqrt{\tau_j - \tau_{j+1}})$. Thus for any $z \in C_{j+1}$ and $w \in B_j$, the triangle inequality implies

$$d_{\tau_{j+1}^+}(z, w) \geq r_{j+1} - r_j - \beta\sqrt{A}\sqrt{\tau_j - \tau_{j+1}}. \tag{3.5.18}$$

By (3.5.11), $\sqrt{\tau_j - \tau_{j+1}} \leq (\sqrt{\nu})^{-j} \leq (\sqrt[4]{\nu})^{-j}$, we choose

$$d \geq \frac{\beta\sqrt{A}}{2(1-(\sqrt[4]{\nu})^{-1})},$$

then (3.5.18) gives

$$d_{\tau_{j+1}^+}(z, w) \geq \delta r_j := \frac{2d(1-(\sqrt[4]{\nu})^{-1})}{(\sqrt[4]{\nu})^j}. \tag{3.5.19}$$

To conclude, we have

$$B_j \subset M_{\tau_{j+1}^+} - B_{\tau_{j+1}^+}(z, \delta r_j). \tag{3.5.20}$$

By Definition 3.5.4, we have

$$G(x, 1; z, \tau_{j+1}) = \int_{M_{\tau_j^+}} G(x, 1; w, \tau_j) G(w, \tau_j; z, \tau_{j+1})\, d_{\tau_j^-} w, \tag{3.5.21}$$

for any $z \in C_{j+1}$ fixed. We split the following integral $\mathcal{I}[M_{\tau_j^+}] := G(x, 1; z, \tau_{j+1})$ into the integrals over $C_j$ and $B_j$. We obtain from the definition of $a_j$ and (3.5.8) that

$$\begin{aligned}\mathcal{I}[C_j] &= \int_{C_j} G(x,1;w,\tau_j)G(w,\tau_j;z,\tau_{j+1})d_{\tau_j^-}w \\ &\leq a_j \int_{M_{\tau_j^+}} G(w,\tau_j;z,\tau_{j+1})d_{\tau_j^-}w \leq a_j.\end{aligned} \tag{3.5.22}$$

To estimate $\mathcal{I}[B_j]$, we notice that by (3.5.20), (3.5.15) and (3.5.9) we have

$$\begin{aligned}\mathcal{I}[B_j] &\leq C_0 \int_{B_j} G(w,\tau_j;z,\tau_{j+1})\ d_{\tau_j^-}w, \\ &\leq C_0 \int_{M_{\tau_{j+1}^+}-B_{\tau_{j+1}^+}(z,\delta r_j)} G(w,\tau_j;z,\tau_{j+1})\ d_{\tau_j^-}w \\ &\leq C \int_{M_{\tau_{j+1}^+}-B_{\tau_{j+1}^+}(z,\delta r_j)} G(w,\tau_j;z,\tau_{j+1})\ d_{\tau_{j+1}^+}w.\end{aligned}$$

Then applying the Gaussian bound (3.5.6) to $G(w,\tau_j;z,\tau_{j+1})$ and calculating as in Lemma 3.2.3 we have

$$\mathcal{I}[B_j] \leq Cexp\left(-\frac{(\delta r_j)^2}{C(\tau_j-\tau_{j+1})}\right). \tag{3.5.23}$$

Plugging (3.5.11) and (3.5.19) into (3.5.23) we have

$$\mathcal{I}[B_j] \leq Cexp\left(-\frac{(\sqrt{\nu})^j d^2}{C}\right). \tag{3.5.24}$$

Combining (3.5.22) and (3.5.23), we see that $G(x,1;z,\tau_{j+1}) \leq a_j + Cexp(-\frac{d^2(\sqrt{\nu}^j)}{C})$ for arbitrary $z$ in $C_{j+1}$. Hence by the definition of $a_{j+1}$, there holds

$$\begin{aligned}a_{j+1} &\leq a_j + Cexp\left(-\frac{d^2(\sqrt{\nu})^j}{C}\right) \leq a_1 + C\sum_{l=1}^{j} exp\left(-\frac{d^2(\sqrt{\nu})^l}{C}\right) \\ &\leq a_1 + Cexp\left(-\frac{d^2}{C}\right)\sum_{l=1}^{j} exp\left(-\frac{d^2((\sqrt{\nu})^l-1)}{C}\right) \\ &\leq a_1 + C\,exp\left(-\frac{d^2}{C}\right).\end{aligned}$$

Note $a_1 = \sup_{C_1} G(x,1;\cdot,\tau_1)$. For any $z \in C_1 = M_{\tau_1^+} - B_{\tau_1^+}(x,r_1)$, we have $d_{\tau_1^+}(x,z) \geq r_1 = 4d(1-(\sqrt[4]{\nu})^{-1})$. Substituting this into the ordinary Gaussian bound, we get $G(x,1;z,\tau_1) \leq \frac{C}{(1-\tau_1)^{\frac{n}{2}}}exp(-\frac{d^2}{C(1-\tau_1)})$. This gives $a_1 \leq Cexp(-\frac{d^2}{C})$. Hence $a_{j+1} \leq Cexp(-\frac{d^2}{C})$. This finishes the proof of the claim. □

To summarize, we showed for $t = t_{k+i}$, $s = t_i$, $i \geq 1, k \geq 1$ and $x, y$ such that $G(x, t_{k+i}; y, t_i)$ is defined, we have the Gaussian bound.

$$G(x, t_{k+i}; y, t_i) \leq \frac{C}{(t_{k+i} - t_i)^{\frac{n}{2}}} exp\left(-\frac{d^2_{t_i^+}(x, y)}{C(t_{k+i} - t_i)}\right). \tag{3.5.25}$$

We will use this to derive the Gaussian bound (3.5.10) for arbitrary $t$ and $s$.

**Part 2** To show (3.5.10) for arbitrary $t$ and $s$, there are two cases left. The first is that neither $t$ nor $s$ is an expanding time, and the second is that one of them is an expanding time. Since the second case follows a same but easier route than the first one, we prove the first case below.

Since $t$ and $s$ are not expanding times, we may assume $t \in (t_{k+i}, t_{k+i+1})$ and $s \in (t_i, t_{i+1})$ for some $k$ and $i$. Rescaling the flow on $[s, t]$ to a new flow on $[0, 1]$, for the same reason as in Part 1, it suffices to show for any very large $d$ (which we specify below) and $x, y$ such that $d_0(x, y) \geq 5d$, we have

$$G(x, 1; y, 0) \leq Cexp\left(-\frac{d^2}{C}\right). \tag{3.5.26}$$

Under rescaling, $t_{k+i}$ and $t_{i+1}$ become $\tau_2 := \frac{t_{k+i}-s}{t-s}$ and $\tau_1 := \frac{t_{i+1}-s}{t-s}$, respectively. By Definition 3.5.4 of the generalized heat kernel, we have

$$G(x, 1; y, 0) = \int_{M_{\tau_2^+}} \int_{M_{\tau_1^+}} G(x, 1; z, \tau_2) G(z, \tau_2; w, \tau_1) G(w, \tau_1; y, 0) d_{\tau_1^-} w \, d_{\tau_2^-} z. \tag{3.5.27}$$

We split the integral $\mathcal{I}[M_{\tau_2^+} \times M_{\tau_1^+}] := G(x, 1; y, 0)$ over three regions

$$\begin{aligned} U &= \{(z, w) \,|\, z \in B_{\tau_2^-}(x, d) \; and \; w \in B_{\tau_1^-}(y, d)\}, \\ V &= \{(z, w) \,|\, z \notin B_{\tau_2^-}(x, d)\}, \\ W &= \{(z, w) \,|\, w \notin B_{\tau_1^-}(y, d)\}. \end{aligned} \tag{3.5.28}$$

Then $G(x, 1; y, 0) \leq \mathcal{I}[U] + \mathcal{I}[V] + \mathcal{I}[W]$. Since $\tau_1$ and $\tau_2$ are both expanding times and $\tau_2 - \tau_1$ is bounded below by a positive number depending only on $\nu$, the result from Part 1 implies

$$G(z, \tau_2; w, \tau_1) \leq C\, exp\left(-\frac{d^2_{\tau_1^+}(z, w)}{C}\right). \tag{3.5.29}$$

If we choose $d \geq \beta\sqrt{A}$, then for any $z \in B_{\tau_2^-}(x, d)$ and $w \in B_{\tau_1^-}(y, d)$, the Shrinking Lemma together with the expanding conditions imply $d_{\tau_1^-}(x, z) \leq d_{\tau_1^+}(x, z) \leq d_{\tau_2^-}(x, z) + \beta\sqrt{A} \leq d_{\tau_2^-}(x, z) + d \leq 2d$, and $d_{\tau_1^-}(x, y) \geq d_0(x, y) - \beta\sqrt{A} \geq 4d$. Then by triangle inequality we have

$$d_{\tau_1^+}(z, w) \geq d_{\tau_1^-}(z, w) \geq d_{\tau_1^-}(x, y) - d_{\tau_1^-}(x, z) - d_{\tau_1^-}(y, w) \geq d. \tag{3.5.30}$$

Hence by (3.5.29), (3.5.30), (3.5.6) and (3.5.8) we have

$$\begin{aligned}\mathcal{I}[U] &\le C\,exp\left(-\frac{d^2}{C}\right)\cdot\left(\int_{M_{\tau_2^+}} G(x,1;z,\tau_2)\,d_{\tau_2^-}z\right)\cdot\left(\int_{M_{\tau_1^+}} G(w,\tau_1;y,0)\,d_{\tau_1^-}w\right)\\ &\le C\,exp\left(-\frac{d^2}{C}\right)\cdot C\cdot 1 = C\,exp\left(-\frac{d^2}{C}\right).\end{aligned} \tag{3.5.31}$$

And (3.5.29), (3.5.8), (3.5.6) together with Lemma 3.2.3 imply

$$\mathcal{I}[V] \le C\left(\int_{z\notin B_{\tau_2^-}(x,d)} G(x,1;z,\tau_2)\ d_{\tau_2^-}z\right) \le Cexp\left(-\frac{d^2}{C}\right). \tag{3.5.32}$$

Similarly we have

$$\mathcal{I}[W] \le Cexp\left(-\frac{d^2}{C}\right). \tag{3.5.33}$$

So (3.5.26) follows from (4.7.20), (3.5.32), (3.5.33) immediately. □

### 3.5.3 Gradient of heat kernel

In this subsection, we consider Ricci flow in expansion $(\{M_j\}_{j=1}^m, \{g_j(t)\}_{j=1}^m, \nu)$ and use Proposition 3.5.5 to derive an upper bound for the gradient of the generalized heat kernel. Assume all the conditions are the same as in Proposition 3.5.5. We choose and fix some $x \in M_i$ , $t \in (t_i, t_{i+1}]$ for some $i$. Then $G(x,t;\cdot,\cdot)$ is a solution to the heat equation $\frac{\partial}{\partial s'}G(x,t;z,s') + \Delta_{z,s'}G(x,t;z,s') = 0$ on $M_j \times (t_j, \min(t_{j+1},t)]$, $j = 1,...,i$. For an arbitrary $(y,s) \in M_j \times (t_j, \min(t_{j+1},t)]$, $j = 1,...,i$, applying the standard result of Schauder estimate (see [33] for example), we see that there is a constant $C$ depending on $A$ and $n$ such that

$$|\nabla G|(x,t;y,s) \le \frac{C}{\sqrt{s-t_j}} \sup G(x,t;\cdot,\cdot), \tag{3.5.34}$$

where the supremum is taken over $B_{g(s)}(y, \sqrt{s-t_j}) \times [t_j, s]$.

Since $|\mathrm{Rm}| \le \frac{A}{t}$ on $M_j \times [t_j, t_{j+1}]$,we have a constant $C_1 = C_1(n, A, \nu) > 0$ such that for any $s, s' \in [t_j, t_{j+1}]$, $C_1^{-1} d_{s'} \le d_s \le C_1\, d_{s'}$. Suppose $d_s(x,y) \ge d$ for a large number $d$ satisfying

$$d \ge 2C_1(\sqrt{t_{i+1}-t_i} + \beta\sqrt{A}\sqrt{t_{i+1}-t_i}). \tag{3.5.35}$$

We claim the following Gaussian bound of $|\nabla G|(x,t;y,s)$:

**Claim 3.5.8.** *For each $j = 1,...,i$, we have the following estimate:*

$$|\nabla G|(x,t;y,s) \le \frac{1}{\sqrt{s-t_j}}\frac{C}{t_{i+1}^{\frac{n}{2}}}exp\left(-\frac{d_s^2(x,y)}{Ct_{i+1}}\right) \tag{3.5.36}$$

*for some constant $C$ that only depends on $A$, $\nu$ and $n$.*

*Proof of Claim 3.5.8.* For any $(z, s') \in B_{g(s)}(y, \sqrt{s-t_j}) \times [t_j, s]$, first we have by the Shrinking Lemma that $d_{s'}(y,z) \le d_s(y,z) + \beta\sqrt{A}(\sqrt{s-s'})$. Then the triangle inequality and (3.5.35) we get

$$\begin{aligned} d_{s'}(x,z) &\ge d_{s'}(x,y) - d_{s'}(y,z) \\ &\ge d_{s'}(x,y) - d_s(y,z) - \beta\sqrt{A}(\sqrt{s-s'}) \\ &\ge d_{s'}(x,y) - \sqrt{t_{j+1}-t_j} - \beta\sqrt{A}\sqrt{t_{j+1}-t_j} \\ &\ge C_1^{-1} d_s(x,y) - \sqrt{t_{j+1}-t_j} - \beta\sqrt{A}\sqrt{t_{j+1}-t_j} \\ &\ge \tfrac{1}{2} C_1^{-1} d_s(x,y). \end{aligned} \tag{3.5.37}$$

So by Proposition 3.5.5 we have

$$G(x,t;z,s') \le \frac{C}{(t-s')^{\frac{n}{2}}} exp\left(-\frac{d_{s'}^2(x,z)}{C(t-s')}\right) \le \frac{C}{(t-s')^{\frac{n}{2}}} exp\left(-\frac{d_s^2(x,y)}{C(t-s')}\right). \tag{3.5.38}$$

Since $d_s(x,y) \ge d$ and $t - s' \le t_{i+1}$, Lemma 3.2.4 implies

$$G(x,t;z,s') \le \frac{C}{(t_{i+1})^{\frac{n}{2}}} exp\left(-\frac{d_s^2(x,y)}{Ct_{i+1}}\right). \tag{3.5.39}$$

The claim thus follows by letting $(z, s')$ run over $B_{g(s)}(y, \sqrt{s-t_j}) \times [t_j, s]$. □

## 3.6 Proof of Theorem 3.1.1

First, we consider the conditions given in Theorem 3.1.1. The upper bound on $\ell(x,0)$ implies a lower bound on Ricci curvature, that is, $\ell(x,0) \le \alpha_0 \le 1$ implies $\mathrm{Ric} \ge -K(n)$. So by Bishop-Gromov comparison, reducing $v_0$ to a smaller positive number depending only on the original $v_0$ and $n$, we may assume without loss of generality that

$$Vol_{g(0)} B_{g(0)}(x,r) \ge v_0 r^n \tag{3.6.1}$$

for all $x \in B_{g(0)}(x_0, s_0 - 1)$ and $r \in (0,1]$. We can also assume $\alpha_0$ without loss of generality that

$$\alpha_0 \le \frac{1}{2C_4} < 1 \tag{3.6.2}$$

where $C_4 = C_4(v_0, n) > 2$ is to be determined later. Otherwise, we get the result by applying the above result to a rescaled metric and then scale it back.

By the relative compactness of $B_{g(0)}(x_0, s_0)$, there exists some $\rho \in (0, \frac{1}{2}]$ such that $|\mathrm{Rm}| \le \frac{1}{\rho^2}$, $B_{g(0)}(x,\rho) \subset\subset M$ and $\mathrm{inj}_{g(0)}(x) \ge \rho$ for all $x \in B_{g(0)}(x_0, s_0)$. The constant $\rho$ may depend on $(M, g(0))$, $x_0$ and $s_0$. By applying Lemma 3.2.1, with $U := B_{g(0)}(x_0, s_0)$, we can find a connected subset $\tilde{M} \subset U \subset M$ containing $B_{g(0)}(x_0, s_0 - \frac{1}{2})$, and a smooth, complete metric

$\tilde{g}(0)$ on $\tilde{M}$ with $\sup_{\tilde{M}} |\mathrm{Rm}|_{\tilde{g}(0)} < \infty$ such that on $B_{g(0)}(x_0, r_0)$, where $r_0 := s_0 - 1 > 3$, the metric remains unchanged. Taking Shi's Ricci flow we get a smooth, complete, bounded-curvature Ricci flow $g_0(t)$ on $M_0 := \tilde{M}$, existing for some nontrivial time interval $[0, t_1]$. In view of the boundedness of the curvature, after possibly reducing $t_1$ to a smaller positive value, we may trivially assume that $|\mathrm{Rm}|_{g(t)} \leq \frac{C_3}{t}$ for all $t \in (0, t_1]$ and $\ell(x,t) \leq 2\alpha_0 < 1$ for all $x \in B_{g(0)}(x_0, r_0)$ and $t \in [0, t_1]$. The constant $C_3 = C_3(v_0, n)$ will be given below.

Of course, our flow still lacks a uniform control on its existence time. Below we will carry out an inductive argument to show that $t_1$ could be extended up to a uniform time $t_k$, while the repeating time $k$ may be allowed to depend on $(M, g)$.

Now we begin the proof of Theorem 3.1.1. First, suppose we have constructed a Ricci flow in expansion $(\{M_j\}_{j=1}^i, \{g_j(t)\}_{j=1}^i, \nu)$ with $(M_0, g_0(t))_{t\in[0,t_1]}$ as above. Suppose further the Ricci flow in expansion satisfies the following a priori assumptions:

**(APA 1)** Restricting it on $B_{g(0)}(x_0, r_i)$, we get a smooth Ricci flow $g(t)$ up to $t_{i+1}$;

**(APA 2)** For each complete Ricci flow $(M_j, g_j(t))$, we have $|\mathrm{Rm}|_{g_j(t)} \leq \frac{C_3}{t}$;

**(APA 3)** $\ell(x,t) \leq C_4\alpha_0 < 1$ for all $t \in [0, t_{i+1}]$ and $x \in B_{g(0)}(x_0, r_i)$.

where the constants $C_3, C_4, \nu$ depending on $v_0, n$ will be specified in the course of the proof.

Our goal is to extend it to a new Ricci flow in expansion $(\{M_j\}_{j=1}^{i+1}, \{g_j(t)\}_{j=1}^{i+1}, \nu)$ by adding a complete Ricci flow $(M_{i+1}, g_{i+1}(t))$ piece existing for $[t_{i+1}, t_{i+2}]$, and show that it still satisfies (APA 1)-(APA 3). In the current section, we construct $(M_{i+1}, g_{i+1}(t))$, and then verify (APA 1) and (APA 2), and we leave the verification of (APA 3) to the next section.

Let $C_1 \geq 1$ and $\tilde{T} > 0$ be the constants from the Curvature Decay Lemma (Lemma 3.3.4) when $K = 1$ and $v_0 = v_0$. With this choice of $C_1$, we set $C_2 = \gamma C_1$ and $C_3 = 4C_2 = 4\gamma C_1 > 1$, where $\gamma = \gamma(n) \geq 1$ is the constant from the Conformal Change Lemma (Lemma 3.2.1), and set $\nu = 1 + \frac{1}{4C_3}$. Choose $\tau$ such that

$$\tau \leq \hat{T}, \quad \beta^2 C_3 \tau \leq \frac{1}{16}\sqrt{\tau} \leq 1, \quad \tau \leq \frac{1}{16}, \quad \tau \leq \frac{C_1}{4}, \tag{3.6.3}$$

where $\beta \geq 1$ is the constant from the Shrinking Lemma. We can also assume that $2t_{i+1} \leq \tau$, because otherwise we get the desired uniform existence time $\frac{\tau}{2}$.

In the Claim below, we show that in fact we have a stronger curvature decay bound $|\mathrm{Rm}|_{g(t)} \leq \frac{C_1}{t}$. However, the original curvature decay will nevertheless be used to control the distance distortion.

**Claim 3.6.1.** *For all $x \in U := B_{g(0)}(x_0, r_i - 2\sqrt{\frac{t_{i+1}}{\tau}})$, we have $B_{g(t)}(x, \sqrt{\frac{t}{\tau}}) \subset\subset B_{g(0)}(x_0, r_i)$, $\mathrm{inj}_{g(t)}(x) \geq \sqrt{\frac{t}{C_1}}$ and $|\mathrm{Rm}|_{g(t)}(x) \leq \frac{C_1}{t}$, for all $t \in (0, t_{i+1}]$.*

*Proof of Claim 3.6.1.* For any $x \in B_{g(0)}(x_0, r_i - 2\sqrt{\frac{t_{i+1}}{\tau}})$, the triangle inequality implies that $B_{g(0)}(x, 2\sqrt{\frac{t_{i+1}}{\tau}}) \subset\subset B_{g(0)}(x_0, r_i)$ and hence by assumption (APA 3), $\ell(y,t) \leq 1$ on $B_{g(0)}(x, 2\sqrt{\frac{t_{i+1}}{\tau}})$ for all $t \in [0, t_{i+1}]$. Scaling the solution to $\hat{g}(t) := \frac{\tau}{t_{i+1}} g(t\frac{t_{i+1}}{\tau})$ we see that we have a solution $\hat{g}(t)$ on $B_{g(0)}(x_0, r_i) \supset\supset B_{\hat{g}(0)}(x, 2)$, $t \in [0, \tau]$ with $|\mathrm{Rm}|_{\hat{g}(t)} \leq \frac{C_3}{t}$ and $\ell(\cdot,\cdot) \leq 1$ on $B_{\hat{g}(0)}(x, 2) \times (0, \tau]$.

On the one hand, applying the Shrinking Lemma to $\hat{g}(t)$, we find that $B_{\hat{g}(t)}(x, 2 - \beta\sqrt{C_3 t}) \subset B_{\hat{g}(0)}(x, 2)$ for all $t \in [0, \tau]$, and in particular $B_{\hat{g}(t)}(x, 1) \subset B_{\hat{g}(0)}(x, 2)$ because $\tau \leq \frac{1}{\beta^2 C_3}$. Thus we have $\ell(\cdot,\cdot) \leq 1$ on $\bigcup_{s\in[0,\tau]} B_{\hat{g}(s)}(x, 1) \times [0, \tau]$. On the other hand, the volume inequality (3.6.1) transforms to $Vol_{\hat{g}(0)} B_{\hat{g}(0)}(x, 1) \geq v_0$.

Applying the Curvature Decay Lemma (Lemma 3.3.4) to $\hat{g}(t)$, we have $\mathrm{inj}_{\hat{g}(t)}(x) \geq \sqrt{\frac{t}{C_1}}$ and $|\mathrm{Rm}|_{\hat{g}(t)}(x) \leq \frac{C_1}{t}$ for all $0 < t \leq \tau$. Scaling back, we see that $B_{g(t)}(x, \sqrt{\frac{t_{i+1}}{\tau}}) \subset\subset B_{g(0)}(x_0, r_i)$, $\mathrm{inj}_{g(t)}(x) \geq \sqrt{\frac{t}{C_1}}$ and $|\mathrm{Rm}|_{g(t)}(x) \leq \frac{C_1}{t}$ for $t \in (0, t_{i+1}]$.

□

Specializing the claim 3.6.1 to $t = t_{i+1}$, we have $|\mathrm{Rm}|_{g(t_{i+1})}(x) \leq \frac{C_1}{t_{i+1}}$ and $\mathrm{inj}_{g(t_{i+1})}(x) \geq \sqrt{\frac{t_{i+1}}{C_1}}$ for any $x \in U := B_{g(0)}(x_0, r_i - 2\sqrt{\frac{t_{i+1}}{\tau}})$. Now we apply the Conformal Change Lemma 3.2.1 with $U = B_{g(0)}(x_0, r_i - 2\sqrt{\frac{t_{i+1}}{\tau}})$, $N = B_{g(0)}(x_0, r_i)$, $g(t_{i+1})$ and $\rho^2 := \frac{t_{i+1}}{C_1} \leq 1$, and obtain a new, possibly disconnected, smooth manifold $(\tilde{U}, h)$, each component of which is complete, such that

1. $|\mathrm{Rm}|_h \leq \gamma\frac{C_1}{t_{i+1}} = \frac{C_2}{t_{i+1}}$ and $\mathrm{inj}_h \geq \sqrt{\frac{t_{i+1}}{\gamma C_1}} = \sqrt{\frac{t_{i+1}}{C_2}}$ for all $x \in \tilde{U}$,
2. $U_\rho \subset \tilde{U} \subset U$,
3. $h = g(t_{i+1})$ on $\tilde{U}_\rho \supset U_{2\rho}$

where $U_r = \{x \in U | B_g(x, r) \subset\subset U\}$.

**Claim 3.6.2.** *We have $B_{g(0)}(x_0, r_i - 4\sqrt{\frac{t_{i+1}}{\tau}}) \subset U_{2\rho}$ where the metric $g(t_{i+1})$ and $h$ agree.*

*Proof of Claim 3.6.2.* By definition of $U$, for every $x \in B_{g(0)}(x_0, r_i - 4\sqrt{\frac{t_{i+1}}{\tau}})$, the triangle inequality implies $B_{g(0)}(x, 2\sqrt{\frac{t_{i+1}}{\tau}}) \subset\subset U$. By (APA 2), we have $|\mathrm{Rm}|_{g(t)} \leq \frac{C_3}{t}$ on $B_{g(0)}(x_0, r_i)$, and hence on $B_{g(0)}(x, 2\sqrt{\frac{t_{i+1}}{\tau}})$ for all $t \in (0, t_{i+1}]$. Applying the Shrinking Lemma we have $B_{g(0)}(x, 2\sqrt{\frac{t_{i+1}}{\tau}}) \supset B_{g(t)}(x, 2\sqrt{\frac{t_{i+1}}{\tau}} - \beta\sqrt{C_3 t})$ for all $t \in [0, t_{i+1}]$. Specializing to $t = t_{i+1}$

and use $\beta\sqrt{C_3 t_{i+1}} \leq \sqrt{\frac{t_{i+1}}{\tau}}$ we see that $B_{g(t_{i+1})}(x, \sqrt{\frac{t_{i+1}}{\tau}}) \subset\subset U$. By (3.6.3) this gives $B_{g(t_{i+1})}(x, 2\sqrt{\frac{t_{i+1}}{C_1}}) \subset\subset U$ which means $x \in U_{2\rho}$ by definition of $\rho$. □

In view of Claim 3.6.2 we define the connected component of $(\tilde{U}, h)$ that contains $B_{g(0)}(x_0, r_i - 4\sqrt{\frac{t_{i+1}}{\tau}})$ as $M_{i+1}$. Then we restart the flow from $(M_{i+1}, h)$ using Shi's complete bounded curvature Ricci flow. By the doubling time estimate (Lemma 3.1.4), we have a complete Ricci flow $(M_{i+1}, h(t))$ with $h(0) = h$ existing for $t \in [0, (\nu - 1)t_{i+1}]$ and satisfying

$$|\mathrm{Rm}|_{h(t)}(y) \leq 2\frac{C_2}{t_{i+1}} \quad \text{and} \quad Vol_{h(t)} B_{h(t)}(y, \sqrt{t_{i+1}}) \geq \frac{t_{i+1}^{\frac{n}{2}}}{A_0} \tag{3.6.4}$$

for all $y \in M_{i+1}$, where $A_0$ is a constant depending on $C_2$ and thus on $v_0$ and $n$. Setting $g_{i+1}(t) = h(t - t_{i+1})$ for $t \in [t_{i+1}, t_{i+2}] = [t_{i+1}, \nu t_{i+1}]$, we obtain a new Ricci flow in expansion $(\{M_j\}_{j=1}^{i+1}, \{g_j(t)\}_{j=1}^{i+1}, \nu)$, which clearly satisfies (APA 1). By (3.6.4) and $t_{i+2} = \nu t_{i+1}$ we have

$$|\mathrm{Rm}|_{g(t)}(y) \leq 2\frac{C_2}{t_{i+1}} \leq \frac{C_3}{t} \tag{3.6.5}$$

for all $t \in [t_{i+1}, t_{i+2}]$. Hence we verified (APA 2). For the same reason, we have

$$Vol_{g(t)} B_{g(t)}(y, \sqrt{t}) \geq \frac{t_{i+1}^{\frac{n}{2}}}{A_0} \geq \frac{t^{\frac{n}{2}}}{A} \tag{3.6.6}$$

for all $t \in [t_{i+1}, t_{i+2}]$, where $A = A_0 \nu^{\frac{n}{2}}$ also depends on $v_0$ and $n$. The volume estimate is needed to apply Proposition 3.5.5 in next section.

## 3.7 Induction Step: Verification of (APA 3)

In this section we finish the proof of Theorem 3.1.1 by verifying (APA 3) for $(\{M_j\}_{j=1}^{i+1}, \{g_j(t)\}_{j=1}^{i+1}, \nu)$. More specifically, we determine $r_{i+1}$ such that when restricted on $B_{g(0)}(x_0, r_{i+1})$, the smooth Ricci flow $g(t)$ satisfies $\ell(x, t) \leq C_4\alpha_0 < 1$ for all $t \in [0, t_{i+2}]$. The estimates (3.6.5) and (3.6.6) allow us to apply Proposition 3.5.5 to $(\{M_j\}_{j=1}^{i+1}, \{g_j(t)\}_{j=1}^{i+1}, \nu)$, and get the Gaussian bound for the generalized heat kernel $G(x, t; y, s)$:

$$G(x, t; y, s) \leq \frac{C}{(t - s)^{\frac{n}{2}}} exp(-\frac{d_{s^+}^2(x, y)}{C(t - s)}), \tag{3.7.1}$$

where $C$ depends on $v_0$ and $n$. We will frequently use this inequality implicitly in this section. Also for notational convenience, the same letter $C$ will be used to denote positive constants depending on $n$ and $v_0$. We divide the integration estimates of $\ell$ into two steps.

**Step 1** We derive a rough bound for $\ell$. Specifically, we show that $\ell$ is bounded above by a constant depending only on $v_0$ and $n$. This bound gives a lower bound for Ricci curvature with the same dependence, which will be used in the second step.

**Claim 3.7.1.** *For any* $(x,t) \in B_{g(0)}(x_0, r_i - 4\sqrt{\frac{t_{i+1}}{\tau}} - \sqrt[4]{t_{i+2}}) \times [0, t_{i+2}]$*, we have* $\ell(x,t) \leq C$ *and correspondingly* $\mathrm{Ric} \geq -K$*, where both* $C$ *and* $K$ *are positive constants depending only on* $v_0$ *and* $n$*.*

*Proof.* Since $(\{M_j\}_{j=1}^{i}, \{g_j(t)\}_{j=1}^{i}, \nu)$ satisfies (APA 3), we have $\ell(\cdot,\cdot) \leq 1$ on $B_{g(0)}(x_0, r_i) \times [0, t_{i+1}]$. Thus it only remains to show $\ell(x,t) \leq C$ for $t \in [t_{i+1}, t_{i+2}]$. Recall the evolution inequality of $\ell$.

$$\frac{\partial}{\partial t}\ell(x,t) \leq \Delta \ell(x,t) + R(x,t)\ell(x,t) + C(n)\ell^2(x,t). \tag{3.7.2}$$

Using the curvature decay $|\mathrm{Rm}|_{g(t)} \leq \frac{C_3}{t}$ we verified in Section 6, we have $\ell(x,t) \leq \frac{C}{t_{i+1}}$ for all $(x,t) \in M_{i+1} \times [t_{i+1}, t_{i+2}]$. Substituting this into (3.7.2), we get

$$\frac{\partial}{\partial t}\ell \leq \Delta \ell + R\,\ell + C\ell^2 \leq \Delta \ell + R\ell + \frac{C}{t_{i+1}}\ell \tag{3.7.3}$$

in the barrier sense. For any $t \in [t_{i+1}, t_{i+2}]$, set $\mathcal{L}(x,t) = \ell(x,t)e^{-\frac{C}{t_{i+1}}t}$. Then $\mathcal{L}(x, t_{i+1}) = \ell(x, t_{i+1})e^{-C} \leq e^{-C}$ and

$$\frac{\partial}{\partial t}\mathcal{L} \leq \Delta \mathcal{L} + R\,\mathcal{L} \tag{3.7.4}$$

in the barrier sense. Let $h(x,t) = \int_{M_{i+1}} G(x,t;z,t_{i+1})\mathcal{L}(z,t_{i+1})d_{t_{i+1}}z$, then $h$ solves the following initial value problem:

$$\frac{\partial}{\partial s}h = \Delta h + R\,h \quad \text{and} \quad h(\cdot, t_{i+1}) = \mathcal{L}(\cdot, t_{i+1}). \tag{3.7.5}$$

By the maximum principle, we have

$$\mathcal{L}(x,t) \leq h(x,t) = \mathcal{I}[M_{i+1}] := \int_{M_{i+1}} G(x,t;y,t_{i+1})\mathcal{L}(y,t_{i+1})\, d_{t_{i+1}}y \tag{3.7.6}$$

for all $x \in M_{i+1}$ and $t \in [t_{i+1}, t_{i+2}]$.

Seeing that $B_{g(0)}(x_0, r_i - 4\sqrt{\frac{t_{i+1}}{\tau}}) \subset M_{i+1}$ is where the smooth local flow exists up to $t_{i+2}$, we split the integral $\mathcal{I}[M_{i+1}]$ into two integrals over $\mathcal{B}_{i+1} := B_{g(0)}(x, \sqrt[4]{t_{i+2}})$ and $\mathcal{C}_{i+1} := M_{i+1} - B_{g(0)}(x, \sqrt[4]{t_{i+2}})$. Since $\mathcal{B}_{i+1} \subset B_{g(0)}(x_0, r_i - 4\sqrt{\frac{t_{i+1}}{\tau}})$ where $\mathcal{L}(\cdot, t_{i+1}) \leq \ell(\cdot, t_{i+1}) \leq 1$ by (APA 3), we can estimate

$$\mathcal{I}[\mathcal{B}_{i+1}] \leq \int_{M_{i+1}} G(x,t;y,s)\, d_s y \leq C. \tag{3.7.7}$$

To estimate $\mathcal{I}[\mathcal{C}_{i+1}]$, we first estimate $d_{t_{i+1}}(x,y)$ for any $y \in \mathcal{C}_{i+1}$ by the Shrinking Lemma and (3.6.3):

$$d_{t_{i+1}}(x,y) \geq \sqrt[4]{t_{i+2}} - \beta\sqrt{C_3}\sqrt{t_{i+1}} \geq \frac{1}{2}\sqrt[4]{t_{i+2}} \geq \sqrt{t_{i+2}} \geq \sqrt{t - t_{i+1}}. \tag{3.7.8}$$

Then by Lemma 3.2.4 we have

$$G(x,t;y,t_{i+1}) \leq \frac{C}{(t_{i+2})^{\frac{n}{2}}} exp(-\frac{d^2_{t_{i+1}}(x,y)}{C\, t_{i+2}}) \leq \frac{C}{(t_{i+2})^{\frac{n}{2}}} exp(-\frac{1}{C\sqrt{t_{i+2}}}). \tag{3.7.9}$$

Now we apply Lemma 3.2.3 at $t_{i+1}$, and combining with $\mathcal{L}(y,t_{i+1}) \leq \frac{C}{t_{i+1}}$ to obtain:

$$\mathcal{I}[\mathcal{C}_{i+1}] \leq C\, exp(-\frac{1}{C\sqrt{t_{i+2}}}) \leq C. \tag{3.7.10}$$

Hence Claim 3.7.1 follows by (3.7.7) and (3.7.10).

□

**Step 2** It remains to convert this upper bound in Lemma 3.7.1 to the stronger upper bound as claimed in (APA 3). Using the bound for $\ell$ from Claim 3.7.1, we get the following linearization of the evolution equation for $\ell$ on $B_{g(0)}(x_0, r_i - 4\sqrt{\frac{t_{i+1}}{\tau}} - \sqrt[4]{t_{i+2}}) \times [0, t_{i+2}]$:

$$\frac{\partial \ell}{\partial t} \leq \Delta\ell + R\ell + C(n)\ell^2 \leq \Delta\ell + R\ell + C\ell \tag{3.7.11}$$

in the barrier sense. Setting $\mathcal{L}(\cdot,t) = e^{-Ct}\ell(\cdot,t)$, we get $\frac{\partial}{\partial t}\mathcal{L} \leq \Delta\mathcal{L} + R\mathcal{L}$ on the same region as above, in the barrier sense.

Hereafter, we choose and fix an arbitrary $(x,t) \in B_{g(0)}(x_0, r_i - 4\sqrt{\frac{t_{i+1}}{\tau}} - 6\sqrt[4]{t_{i+2}}) \times (t_{i+1}, t_{i+2}]$. Let $r = \sqrt[4]{t_{i+2}}$ and $R = 3r$, then by triangle inequality, $B_{g(0)}(x, R+2r) \subset\subset B_{g(0)}(x_0, r_i - 4\sqrt{\frac{t_{i+1}}{\tau}} - \sqrt[4]{t_{i+2}})$, where by Claim 3.7.1 we have $\mathrm{Ric}_{g(s)} \geq -K(v_0,n)$ for all $s \in [0, t_{i+2}]$. We now apply Lemma 3.4.1 to the flow on $B_{g(0)}(x, R+2r)$ during $[0, t_{i+2}]$, and obtain a cut-off function $\phi_{i+1}$ such that

$$B_{g(s)}(x,r) \subset B_{g(0)}(x,2r) \subset \{y \,|\, \phi_{i+1}(y,s) = 1\} \tag{3.7.12}$$

and $supp\, \phi_{i+1}(\cdot,s) \subset B_{g(0)}(x,3r)$, for all $s \in [0,t_{i+2}]$. Combining with (3.7.12), we find that the supports of $|\nabla\phi_{i+1}|$, $\frac{\partial}{\partial s}\phi_{i+1}$ and $\Delta\phi_{i+1}$ are all contained in the annulus $A_{2r,3r}(x) := B_{g(0)}(x,3r) - B_{g(0)}(x,2r)$ and we have the following estimates:

**(P1)** $\nabla\phi_{i+1}$ exists a.e. and $|\nabla\phi_{i+1}| \leq= C\, t_{i+2}^{-\frac{n+1}{4}}$;

**(P2)** $\Delta\phi_{i+1} \leq \mu_1 := C\, t_{i+2}^{-\frac{n+1}{2}}$, in the barrier sense;

**(P3)** $\frac{\partial^+}{\partial s}\phi_{i+1} \leq \mu_2 := C\, t_{i+2}^{-\frac{n}{4}}$.

In view of (3.7.12) we have $\phi_{i+1}(x,t)=1$ and hence

$$\mathcal{L}(x,t)=\lim_{s\nearrow t}\int G(x,t;y,s)\mathcal{L}(y,s)\phi_{i+1}(y,s)d_sy. \tag{3.7.13}$$

The integration domain here and below is always $B_{g(0)}(x,3r)$. In particular, for any integral involving $\nabla\phi_{i+1}$, $\frac{\partial}{\partial s}\phi_{i+1}$ or $\Delta\phi_{i+1}$, the actual integration domain is contained in $A_{2r,3r}(x)$ since these derivatives vanish at the outside.

Since $G(x,t;\cdot,\cdot)$ is continuous on $B_{g(0)}(x,3r)\times[0,t)$ and smooth on $B_{g(0)}(x,3r)\times(t_j,\min(t_{j+1},t))$ for each $j\le i+1$, applying Lemma 3.2.6 to $G(x,t;y,s)\phi_{i+1}(y,s)$ and $\mathcal{L}(y,s)$ and using (P1)-(P3) we obtain

$$\int G\,\phi_{i+1}\mathcal{L}\Big|_{t_j}^{\min(t_{j+1},t)}\le\int_{t_j}^{\min(t_{j+1},t)}\int(G\,\mu_1+G\,\mu_2+2\langle\nabla G,\nabla\phi_{i+1}\rangle)\mathcal{L}, \tag{3.7.14}$$

and hence

$$\mathcal{L}(x,t)\le\int G\,\phi_{i+1}\mathcal{L}\Big|_{t_1}+\int_{t_1}^{t}\int(G\,\mu_1+G\,\mu_2+2\langle\nabla G,\nabla\phi_{i+1}\rangle)\mathcal{L}. \tag{3.7.15}$$

To estimate the first term in the RHS of (3.7.15), we first note that on $B_{g(0)}(x,3r)\subset B_{g(0)}(x_0,r_i-4\sqrt{\frac{t_{i+1}}{\tau}}-\sqrt[4]{t_{i+2}})$ we have $\mathcal{L}(\cdot,t_1)\le 2\alpha_0<1$ and hence $\mathrm{Ric}_{g(t_1)}\ge -C(n)$ for some dimensional constant $C(n)$. Then applying Lemma 3.2.2 we get

$$\int G\,\phi_{i+1}\,\mathcal{L}\Big|_{t_1}\le C\cdot 2\alpha_0. \tag{3.7.16}$$

Then we split the second term in the RHS of (3.7.15) into two parts:

$$\mathcal{I}=\int_{t_1}^{t}\int(\mu_1+\mu_2)\,G(x,t;y,s)\,\mathcal{L}(y,s)\,d_sy\,ds, \tag{3.7.17}$$

$$\mathcal{J}=2\int_{t_1}^{t}\int\langle\nabla G(x,t;y,s),\nabla\phi_{i+1}(y,s)\rangle\,\mathcal{L}(y,s)\,d_sy\,ds. \tag{3.7.18}$$

On the one hand, by the Shrinking Lemma, for all $y$ in $A_{2r,3r}(x)$ and $s\in[0,t_{i+2}]$, we have $d_{g(s)}(x,y)\ge d_{g(0)}(x,y)-\sqrt[4]{t_{i+2}}\ge\sqrt[4]{t_{i+2}}$. Thus by Lemma 3.2.4 we have

$$G(x,t;y,s)\le\frac{C}{(t-s)^{\frac{n}{2}}}exp\left(-\frac{\sqrt{t_{i+2}}}{C(t-s)}\right)\le\frac{C}{t_{i+2}^{\frac{n}{2}}}exp\left(-\frac{1}{C\sqrt{t_{i+2}}}\right). \tag{3.7.19}$$

On the other hand, let $V(s)$ be the volume of $A_{2r,3r}(x)$ at time $s\in[0,t_{i+2}]$. By Bishop-Gromov comparison, we have $V(0)\le C(n)$. Then we get $V(s)\le C$ by integrating $V'(s)\le$

$C\,V(s)$, which follows from the evolution equation of volume under Ricci flow and Claim 3.7.1. Combining this with (3.7.19), (P2), (P3) and Claim 3.7.1 in (3.7.17) we can estimate

$$\mathcal{I} \leq C\,exp\left(-\frac{1}{C\sqrt{t_{i+2}}}\right). \tag{3.7.20}$$

Suppose $s \in (t_j, \min(t_{j+1}, t))$ for some $j \leq i+1$. Since $d_{g(s)}(x,y) \geq \sqrt{t_{i+2}}$ for all $y \in A_{2r,3r}(x)$, applying Claim 3.5.8 of the estimate of $|\nabla G|$, we obtain

$$|\nabla G|(x,t;y,s) \leq \frac{C}{\sqrt{s-t_j}} exp(-\frac{1}{C\sqrt{t_{i+2}}}), \tag{3.7.21}$$

where the constant $C$ depending on $n$ and $v_0$ is uniform for all $j$. Then by Claim 3.7.1 and (P1) we have

$$|\langle \nabla G, \nabla \phi_{i+1}\rangle|(y,s)\mathcal{L}(y,s) \leq \frac{C}{\sqrt{s-t_j}} exp(-\frac{1}{C\sqrt{t_{i+2}}}). \tag{3.7.22}$$

Integrating (3.7.22) over $A_{2r,3r}(x) \times [t_j, t_{j+1}]$, and then summing over all $j$, we obtain

$$\begin{aligned}\mathcal{J} &\leq Cexp(-\frac{1}{C\sqrt{t_{i+2}}})\sum_{j=1}^{i+1}\sqrt{t_{j+1}-t_j}\\ &= Cexp(-\frac{1}{C\sqrt{t_{i+2}}})\sqrt{t_{i+2}(1-\frac{1}{\nu})}(1+\frac{1}{\sqrt{\nu}}+\dots)\\ &= Cexp(-\frac{1}{C\sqrt{t_{i+2}}})\frac{\sqrt{(\nu-1)t_{i+2}}}{\sqrt{v}-1} \leq Cexp(-\frac{1}{C\sqrt{t_{i+2}}}).\end{aligned} \tag{3.7.23}$$

Putting the estimates (3.7.16), (3.7.20) and (3.7.23) into (3.7.15), we thus have

$$\mathcal{L}(x,t) \leq Cexp(-\frac{1}{C\sqrt{t_{i+2}}}) + 2\alpha_0 C. \tag{3.7.24}$$

Then there exists positive constant $t(n, v_0, \alpha_0)$, such that the first term can be bounded by $\alpha_0$ when $t_{i+2} \leq t(n, v_0, \alpha_0)$, and hence $\mathcal{L}(x,t) \leq \alpha_0(1+2C)$. Choose $C_4 = 2(1+2C)$, then $\ell(x,t) \leq \mathcal{L}(x,t)e^{Ct} \leq 2\mathcal{L}(x,t) \leq C_4\alpha_0$. Let $r_{i+1} = r_i - 4\sqrt{\frac{t_{i+1}}{\tau}} - 6\sqrt[4]{t_{i+2}}$, then $\ell(y,s) \leq C_4\alpha_0 < 1$ for any $(y,s) \in B_{g(0)}(x_0, r_{i+1}) \times [0, t_{i+2}]$, as claimed in (APA 3). Moreover, by choosing $t(n, v_0, \alpha_0)$ small, we can make sure

$$r_0 - r_{i+1} = \sum_{j=0}^{i} r_j - r_{j+1} \leq \sum_{j=0}^{i} 4\sqrt{\frac{t_{j+1}}{\tau}} + 6\sqrt[4]{t_{j+2}} \leq 1. \tag{3.7.25}$$

So we proved Theorem 3.1.1.

## 3.8 Proof of the global existence and bi-Hölder homeomorphism

In this section we prove Corollary 3.1.2 and 3.1.3. We need two local curvature estimate lemmas stated below, in both of which the Riemannian manifolds $(M^n, g)$ appearing are not necessarily complete.

Lemma 3.8.1 is proved by B.L. Chen in [20, Theorem 3.1] and Simon in [62, Theorem 1.3].

**Lemma 3.8.1.** *Suppose $(M^n, g(t))$ is a Ricci flow for $t \in [0, T]$, not necessarily complete, with the property that for some $y_0 \in M$ and $r > 0$, and all $t \in [0, T]$, we have $B_{g(t)}(y_0, r) \subset\subset M$ and*

$$|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t} \tag{3.8.1}$$

*on $B_{g(t)}(y_0, r)$ for all $t \in (0, T]$ and some $c_0 \geq 1$. Then if $|Rm|_{g(0)} \leq r^{-2}$ on $B_{g(0)}(y_0, r)$, we must have*

$$|\mathrm{Rm}|_{g(t)}(y_0) \leq e^{Cc_0} r^{-2} \tag{3.8.2}$$

*for some $C = C(n)$.*

Lemma 3.8.2 is an non-standard version of Shi's derivative estimates. The proof of it can be found in [66, Lemma A.4] and [27, Theorem 14.16].

**Lemma 3.8.2.** *Suppose $(M^n, g(t))$ is a Ricci flow for $t \in [0, T]$, not necessarily complete, with the property that for some $y_0 \in M$ and $r > 0$, we have $B_{g(0)}(y_0, r) \subset\subset M$ and $|\mathrm{Rm}|_{g(t)} \leq r^{-2}$ on $B_{g(0)}(y_0, r)$ for all $t \in [0, T]$, and so that for some $l_0 \in \mathbb{N}$ we have initially $|\nabla^l \mathrm{Rm}|_{g(0)} \leq r^{-2-l}$ on $B_{g(0)}(y_0, r)$ for all $l \in \{1, 2, ..., l_0\}$. Then there exists $C = C(l_0, n, \frac{T}{r^2})$ such that*

$$|\nabla^l \mathrm{Rm}|_{g(t)}(y_0) \leq C r^{-2-l} \tag{3.8.3}$$

*for each $l \in \{1, 2, ..., l_0\}$ and all $t \in [0, T]$.*

*Proof of Corollary 3.1.2.* Pick any point $x_0 \in M$. We apply Theorem 3.1.1 with $s_0 = k + 2$, for each integer $k \geq 2$, giving a Ricci flow $(B_{g_0}(x_0, k), g_k(t))_{t \in [0,\tau]}$ satisfying

$$\begin{cases} \mathrm{Rm}_{g_k(t)} + C\alpha_0 \mathrm{I} \in \mathcal{C} \\ |\mathrm{Rm}|_{g_k(t)} \leq \dfrac{C}{t} \end{cases} \tag{3.8.4}$$

on $B_{g_0}(x_0, k)$ for all $t \in (0, \tau]$, where $\tau = \tau(n, v_0, \alpha_0) > 0$ and $C = C(n, v_0) > 0$.

Fix some $r_0 > 0$. Since $B_{g_0}(x_0, r_0+2)$ is compactly contained in $M$, we have $\sup |\mathrm{Rm}| \leq \frac{1}{r^2}$ for some $r > 0$, where the supremum is taken over $B_{g_0}(x_0, r_0+2)$. For any $y_0 \in B_{g_0}(x_0, r_0+1)$, by the Shrinking Lemma we have $B_{g_k(t)}(y_0, \frac{1}{2}) \subset B_{g_0}(y_0, 1) \subset B_{g_0}(x_0, r_0 + 2)$ for all $t \in [0, \tau]$, with possibly reducing $\tau$ to a smaller number depending also on $n, \alpha_0$ and $v_0$. Now we can apply Lemma 3.8.1 to $g_k(t)$ centered at $y_0$ and get $|\mathrm{Rm}|_{g_k(t)}(y_0) \le K_0 = K_0(r,n,C)$. In particular, $K_0$ is independent of $k$. Thus, we have $|\mathrm{Rm}|_{g_k(t)} \le K_0$ on $B_{g_0}(x_0, r_0+1)$ for all $t \in [0,\tau]$.

Then we apply Lemma 3.8.2 to $g_k(t)$ centered at each $y_0 \in B_{g_0}(x_0,r_0)$. The outcome is for each $l \in \mathbb{N}$, there exists $K_1 = K_1(n,l,r,\tau)$ such that

$$|\nabla^l \mathrm{Rm}|_{g_k(t)} \le K_1 \tag{3.8.5}$$

on $B_{g_0}(x_0,r_0)$ for all $t \in [0,\tau]$. Again, the $K_1$ is also independent of $k$. Using these derivative estimates in local coordinate charts, by Ascoli-Arzela Lemma we can pass to a subsequence in $k$ and obtain a smooth limit Ricci flow $g(t)$ on $B_{g_0}(x_0,r_0)$ for $t \in [0,\tau]$ with $g(0) = g_0$, which satisfies

$$\begin{cases} \mathrm{Rm}_{g(t)} + C\alpha_0 \in \mathcal{C} \\ |\mathrm{Rm}|_{g(t)} \le \dfrac{C}{t} \end{cases} \tag{3.8.6}$$

on $B_{g_0}(x_0,r_0)$, for all $t \in (0,\tau]$.

We repeat this process for larger and larger radii $r_i \to \infty$, and take a diagonal subsequence to obtain $g_{k_i}(t)$ which converges on each $B_{g_0}(x_0,r_i)$. Its limit is a smooth Ricci flow $g(t)$ on the whole of $M$ for $t \in [0,\tau]$ with $g(0) = g_0$.

By the Shrinking Lemma, $B_{g(t)}(x_0,r) \subset B_{g_0}(x_0, r+\beta\sqrt{Ct}) \subset\subset M$ for all $t \in (0,\tau]$ and $r > 0$. This guarantees that $g(t)$ must be complete for all positive times $t \in (0,\tau]$. ☐

*Proof of Corollary 3.1.3.* We apply Corollary 3.1.2 and Lemma 3.3.2 to each $(M_i, g_i)$ and obtain a sequence of Ricci flows $(M_i, g_i(t))_{[0,T]}$ with $g_i(0) = g_i$ and $T$ uniform for each $i$, and satisfying the following uniform estimates

$$\begin{cases} \mathrm{Rm}_{g_i(t)} + C\alpha_0 \mathrm{I} \in \mathcal{C} \\ Vol_{g_i(t)} B_{g_i(t)}(x,1) \ge v > 0 \\ |\mathrm{Rm}|_{g_i(t)} \le \dfrac{C}{t} \end{cases} \tag{3.8.7}$$

for all $x \in M_i$ and all $t \in [0,T]$, where constant $C > 0$ depends on $n, v_0$, and constants $v, T > 0$ depend on $n, v_0, \alpha_0$. And

$$d_{g_i(t_1)}(x,y) - \beta\sqrt{C}(\sqrt{t_2} - \sqrt{t_1}) \le d_{g_i(t_2)}(x,y) \le e^{K(t_2-t_1)} d_{g_i(t_1)}(x,y) \tag{3.8.8}$$

for any $0 < t_1 \le t_2 \le T$ and any $x, y \in M_i$, where $K$ depends on $n, v_0, \alpha_0$. The curvature decay for all positive times provides a uniform bound on the curvature which allows us to apply Hamilton's compactness theorem. We can pass to a subsequence in $i$ so that $(M_i, g_i(t), x_i) \to (M, g(t), x_\infty)$ in the Cheeger-Gromov sense, where $(M, g(t))$ is a complete

Ricci flow defined over $(0, T]$. $(M, g(t))$ inherits the estimates for curvatures and distance:

$$\begin{cases} \mathrm{Rm}_{g(t)} + C\alpha_0 \mathrm{I} \in \mathcal{C} \\ Vol_{g(t)} B_{g(t)}(x, 1) \geq v > 0 \\ |\mathrm{Rm}|_{g(t)} \leq \dfrac{C}{t} \end{cases} \tag{3.8.9}$$

and

$$d_{g(t_1)}(x, y) - \beta\sqrt{C}(\sqrt{t_2} - \sqrt{t_1}) \leq d_{g(t_2)}(x, y) \leq e^{K(t_2 - t_1)} d_{g(t_1)}(x, y) \tag{3.8.10}$$

for any $0 < t_1 \leq t_2 \leq T$, and any $x, y \in M$. The inequality (3.8.10) tells us that $d_{g(t)}$ converges locally uniformly to some metric $d_0$ as $t \searrow 0$. Also, the inequality (3.8.8) tells us that $d_{g_i(t)}$ converges locally uniformly to $d_{g_i}$. So we have $(M_i, d_{g_i}, x_i) \to (M, d_0, x_\infty)$ in the pointed Gromov-Hausdorff sense.

By Lemma 2.3.4 we have

$$\gamma d_0(x, y)^{1+2(n-1)C} \leq d_{g(t)}(x, y) \leq e^{Kt} d_0(x, y), \tag{3.8.11}$$

where $\gamma$ depends on $n, v_0$ and also on $\alpha_0$ via $T$. Then the claim of bi-Hölder homeomorphism follows immediately from this. □

# Chapter 4

# 3d Ricci flow with non-negative Ricci curvature

## 4.1 Introduction and main results

In this chapter, we first introduce a new weak solution of Ricci flow that we call a generalized singular Ricci flow, which allows the initial manifold to be a complete manifold with possibly unbounded curvature. We have the following existence theorem.

**Theorem 4.1.1.** *For any 3d complete Riemannian manifold $(M, g)$, there is a generalized singular Ricci flow starting from $(M, g)$.*

The generalized singular Ricci flow has many properties similar to those of a singular Ricci flow. In particular, it satisfies the canonical neighborhood assumption in a distance-dependent way. The precise definition of a generalized singular Ricci flow will be given in Definition 4.6.3.

The existence is obtained from a compactness result for singular Ricci flows, which states that a sequence of singular Ricci flows converges to a generalized singular Ricci flow starting from a complete manifold $(M, g)$, if the sequence of their initial manifolds converges to $(M, g)$:

**Theorem 4.1.2.** *Let $\mathcal{M}_i$ be a sequence of singular Ricci flows starting from compact manifolds $M_i$, $x_i \in M_i$. Suppose $(M_i, x_i)$ converges smoothly to a 3d complete manifold $(M, x_0)$ as $i \to \infty$. Then by passing to a subsequence, $(\mathcal{M}_i, x_i)$ converges smoothly to a generalized singular Ricci flow starting from $M$.*

Theorem 4.1.2 can be compared to the convergence of a sequence of singular Ricci flows, when the sequence of their initial time-slices converges to a compact manifold, see [44, Prop 5.39]. In that result, the initial time-slices have uniformly bounded curvature and injectivity radius. So the local geometry in each singular Ricci flow is uniformly controlled by its scalar curvature, which guarantees their convergence to a singular Ricci flow. However, in our case, the initial time-slices may not have uniformly bounded geometry. Instead, we will show that the scalar curvature controls the local geometry in a uniform distance-dependent way, which ensures the convergence in Theorem 4.1.2.

Before stating our next main result, we recall some results of the existence theory of Ricci flow with non-compact initial conditions. Much less is known about it compared to the compact case. In [61], Shi showed that if $(M, g)$ is an n-dimensional complete Riemannian manifold with bounded curvature, then there exists a complete Ricci flow with bounded curvature for a short time. Since then, many efforts have been made to relax the bounded-curvature assumption, in order to obtain a Ricci flow starting from a complete non-compact manifold.

In [13], Cabezas-Rivas and Wilking proved that a smooth complete Ricci flow exists on a complete n-dimensional manifold with non-negative complex sectional curvature, which in dimension 3 is the same as non-negative sectional curvature. Recently, Simon and Topping [64] showed that a complete Ricci flow exists on a complete 3d Riemannian manifold, if its Ricci curvature has a negative lower bound and the volume is globally non-collapsed (i.e. there is a uniform positive lower bound on the volume of every unit ball). In [2], Bamler, Cebazas-Rivas and Wilking proved that the same thing holds in dimension $n$, assuming a certain curvature is bounded below, and the volume is non-collapsed. In [48], by a combination of methods in [64] and [2], the author generalized both works.

The volume non-collapsing assumption should be necessary in the above works [64][2][48], where the curvature is allowed to be negative somewhere. Without the non-collapsing assumption, Topping gave a conjectural counter-example: for any arbitrarily small $\epsilon > 0$ we can construct a complete 3-manifold with $\mathrm{Ric} \geq -\epsilon$, by connecting countably many three-spheres by necks that become longer and thinner. So the necks would want to pinch in a time that converges to zero [65, Example 2.4].

This leads to an open question: whether a smooth complete Ricci flow exists for a 3 dimensional complete manifold with non-negative Ricci curvature, see e.g. [65, Section 7, Conjecture 2]. Our next main result gives a partial affirmative answer to this question:

**Theorem 4.1.3.** *Let $(M, g)$ be a 3d complete Riemannian manifold with $\mathrm{Ric} \geq 0$. There exist $T > 0$ and a smooth Ricci flow $(M, g(t))$ on $[0, T)$, with $g(0) = g$ and $\mathrm{Ric}(g(t)) \geq 0$. Moreover, if $T < \infty$, then $\limsup_{t \nearrow T} |\mathrm{Rm}|(x, t) = \infty$ for all $x \in M$.*

We remark that the completeness of this flow is not guaranteed in this chapter. Instead, we show that it can be embedded in a smooth Ricci flow spacetime with complete time-slices. Also, it is possible for the maximal existence time to be finite, such as the standard solution and the cylindrical solutions.

A common strategy to produce a smooth Ricci flow with a complete non-compact initial condition is by a limiting argument: first construct a sequence of local Ricci flows starting from larger and larger balls in $M$, and then try to get a uniform lower bound on the existence times, as well as an upper bound on the curvature norms. Then by Hamilton's compactness theorem for Ricci flow, we obtain a smooth limit Ricci flow starting from $M$. This argument typically works when there is a non-collapsing assumption [64, 2, 48], or the curvature condition is relatively strong [13].

However, it seems hard to apply the limiting argument to prove Theorem 4.1.3, for $\mathrm{Ric} \geq 0$ is a relatively weak curvature assumption, and there is no uniform lower bound on the volume of all unit balls on certain manifolds, as shown by examples in [29]. In this chapter, we produce a smooth Ricci flow by showing that a generalized singular Ricci flow starting from a complete manifold with $\mathrm{Ric} \geq 0$ is actually smooth. The existence of the generalized singular Ricci flow is guaranteed by Theorem 4.1.1.

This chapter is organized as follows. In Section 4.2 we prove some technical lemmas. In Section 4.3, we generalize Perelman's no local collapsing and canonical neighborhood theorem to singular Ricci flows. It provides a distance-dependent lower bound on the non-collapsing scale and canonical neighborhood scale, assuming the geometry is bounded in a parabolic neighborhood of the base point.

In Section 4.4, we define a heat kernel $H$ for a singular Ricci flow $\mathcal{M}$. For any point $(x_0, t_0) \in \mathcal{M}$, $H(x_0, t_0; \cdot, \cdot)$ is a positive solution to the conjugate heat equation on $\mathcal{M}$, which is a $\delta$-function around $(x_0, t_0)$. We show that the heat kernel decays polynomially fast to zero as the curvature blows up. With this estimate we show that the overall amount of heat is a constant, i.e. the integral of $H(x_0, t_0; \cdot, t)$ at all times $t$ prior to $t_0$ is equal to one. Note that for the ordinary heat kernel of a compact smooth Ricci flow, the constancy of the integral is easily shown by a computation using integration by part. Moreover, with the decay estimate, we can establish some standard properties of the heat kernel as of the ordinary ones for compact Ricci flows. For example, we obtain the symmetry between the heat and adjoint heat kernels, and a semigroup property of them. As an application of the heat kernel, we generalize Perelman's pseudolocality theorem to singular Ricci flow in Section 4.5.

In Section 4.6, we define the generalized singular Ricci flow, and prove Theorem 4.1.1 and 4.1.2. The proofs depend on a compactness theorem, which states that assuming there is a uniform distance-dependent canonical neighborhood assumption in a sequence of pointed singular Ricci flows, then a subsequence converges smoothly to a semi-generalized singular Ricci flow, which satisfies most properties of the generalized singular Ricci flow. The compactness theorem can be proved by first taking a Gromov-Hausdorff limit, and showing that the convergence is smooth on the subset of points which are limits of points with bounded curvature. This induces a semi-generalized singular Ricci flow. To prove Theorem 4.1.2, by applying the compactness theorem, we get a semi-generalized singular Ricci flow in which the base point $x_0$ survives until its curvature goes unbounded. Then a generalized singular Ricci flow is obtained by varying the base points and gluing up all the corresponding semi-generalized singular Ricci flows.

In Section 4.7 we prove Theorem 4.1.3. First, by a maximum principle argument, we show in Lemma 4.7.2 that the generalized singular Ricci flow $\mathcal{M}$ preserves the non-negativity of Ricci curvature. Suppose the curvature blows up in a ball of finite radius. Then by the canonical neighborhood assumption, we can show that the curvature blow-up is due to the asymptotic formation of a cone-like point. Doing a further rescaling at this cone-like point, we obtain a Ricci flow solution whose final time-slice is a part of a non-flat metric cone, which is impossible. So $\mathcal{M}$ is in fact a non-singular Ricci flow spacetime with complete time-slices. Restricting the spacetime on $M$, we obtain a smooth Ricci flow.

## 4.2 Preparatory results

In this section we prove some technical lemmas that will be used later. First, in Lemma 4.2.1-4.2.3, we study manifolds that are 0-complete and satisfy a canonical neighborhood assumption at scales depending on the distance to a base point $x_0$. We show for such manifolds that any metric ball of a fixed radius centered at $x_0$ is uniformly totally bounded (Lemma 4.2.1), and for any point $x\in M$, there exists a minimizing geodesic from $x_0$ to $x$ (Lemma 4.2.2).

Second, we show in Lemma 4.2.4 that for a Ricci flow spacetime with some appropriate canonical neighborhood assumption, the closeness of a time-slice to a $\kappa$-solution implies the closeness in a parabolic region. As a consequence, Lemma 4.2.5 shows that a blow-up sequence in a singular Ricci flow converges to a $\kappa$-solution defined from $-\infty$ to its maximal existence time.

**Lemma 4.2.1.** *(Metric balls are uniformly totally bounded) Let $(M,g,x_0)$ be a 3 dimensional connected Riemannian manifold, $x_0\in M$. Suppose $M$ is 0-complete and 1-positive. Suppose for any $A,\epsilon_{can}>0$, there are $r(A,\epsilon_{can}),\kappa(A)>0$ such that the $\epsilon_{can}$-canonical neighborhood assumption and the $\kappa(A)$-non-collapsing assumption hold at scales less than $r(A,\epsilon_{can})$ on $B_g(x_0,A)$.*

*Then for any given $A,\epsilon>0$, there exist $V(A),N(A,\epsilon)>0$ such that $vol(B_g(x_0,A))\le V(A)$, and the number of elements in any $\epsilon$-separating subset of $B_g(x_0,A)$ is bounded above by $N(A,\epsilon)$.*

*Proof.* Fix some small $\delta>0$ and let $\epsilon_{can}(\delta),C_0(\delta)>0$ be from Lemma 2.2.6. Let $\eta>0$ be from Lemma 2.2.5. Let $r_0=\frac{1}{7\eta}\cdot\min\{r(A+1,\epsilon_{can}),\frac{1}{6}\}$.

We claim that there exists a universal constant $C_1>0$ such that $vol(B_g(x,3r_0))\le C_1$ for all $x\in B_g(x_0,A)$. In fact, suppose first that $\rho(x)<4\eta r_0$, then by Lemma 2.2.5 we get $\rho<7\eta r_0$ on $B_g(x,3r_0)$. By the assumption of $r_0$, we see that $B_g(x,3r_0)$ is contained in some $\delta$-tube or capped $\delta$-tube with diameter less than 1, which has volume less than $C_1$. So the claim holds. Next, suppose $\rho(x)\ge 4\eta r_0$, then by Lemma 2.2.5 we get $\rho\ge\eta r_0$ on $B_g(x,3r_0)$. So the claim follows from the Bishop-Gromov volume comparison.

Now suppose by induction that for some $k \in \mathbb{N}$ with $kr_0 \le A$, there exist $C_k(A) > 0$ and $N_k(A, \epsilon) \in \mathbb{N}_+$ such that the following holds:

$\mathcal{A}$(k): $vol(B_g(x_0, kr_0)) \le C_k$;

$\mathcal{B}$(k): For any $\epsilon \le r_0$, an $\epsilon$-separating subset in $B_g(x_0, (k-1)r_0)$ has at most $N_k$ elements.

To show $\mathcal{A}(k+1)$, consider a maximal $r_0$-separating subset $\{y_j\}$ in $B_g(x_0, (k-1)r_0)$. Then $B_g(x_0, (k-1)r_0)$ is covered by the union of all $B_g(y_j, r_0)$, and by the triangle inequality we get

$$B_g(x_0, (k+1)r_0) \subset \bigcup_j B_g(y_j, 3r_0). \tag{4.2.1}$$

So $\mathcal{A}$(k+1) follows from the above claim and $\mathcal{B}$(k). It remains to establish $\mathcal{B}(k+1)$.

Let $\{x_j\}_{j=1}^m$ be an $\epsilon$-separating set in $B_g(x_0, kr_0)$. First, by the non-collapsing assumption and $\mathcal{A}(k+1)$ we see that the number of $x_j$ with $\rho(x_j) \ge (2C_0)^{-1}\epsilon$ is bounded above in terms of $A, \epsilon$. So we may assume $\rho(x_j) < (2C_0)^{-1}\epsilon$ for all $j$. Then by $\mathcal{A}(k+1)$ and Lemma 2.2.8, we may further assume that all $x_j$ are contained in a single $\delta$-tube or capped $\delta$-tube $V \subset B_g(x_0, (k+1)r_0)$.

Pick a point $y \in \partial V$, we can arrange the order of $\{x_j\}_{j=1}^m$ in a way such that $d_V(y, x_{j+1}) \ge d_V(y, x_j)$ for each $j \le m-1$. Here $d_V$ denotes the length metric in $V$ induced by $g$. We claim that each $x_j$, $j \le m-1$, is the center of a $\delta$-neck. Otherwise, $x_j$ is the center of a $\delta$-cap $\mathcal{C} \subset V$. By Lemma 2.2.6 we have $\mathrm{diam}(\mathcal{C}) \le C_0\rho(x_j) \le \epsilon/2$. Since $d_V(x_j, x_{j+1}) \ge \epsilon$, we get $x_{j+1} \in V - \mathcal{C}$, and $x_{j+1}$ is the center of a $\delta$-neck. Connecting $y$ with $x_j$ by a minimizing geodesic, then it must intersect the central sphere at $x_{j+1}$, which has diameter less than $10(2C_0)^{-1}\epsilon < \epsilon/2$. So it is easy to see $d_V(y, x_{j+1}) < d_V(y, x_j)$, a contradiction.

So by the triangle inequality we get

$$d_V(y, x_{j+1}) \ge d_V(y, x_j) + \epsilon - 2 \cdot 10 \cdot (2C_0)^{-1}\epsilon \ge d_V(y, x_j) + \epsilon/2, \tag{4.2.2}$$

for all $j \le m-1$. In particular, this implies

$$(m-1)\epsilon/2 \le d_V(x_m, y) \le \mathrm{diam}(V) \le 2(k+1)r_0, \tag{4.2.3}$$

and hence $m \le 4(k+1)r_0\epsilon^{-1} + 1$. This established $\mathcal{B}(k+1)$. □

**Lemma 4.2.2.** *Under the same assumptions as Lemma 4.2.1. Then for any $x \in M$, there exists a minimizing geodesic connecting $x$ to $x_0$.*

*Proof.* By Lemma 4.2.1 we have $vol(B_g(x_0, A))$ is bounded above, so by Lemma 2.2.5 it is easy to see that $|\mathrm{Rm}|$ is a proper function restricted on $\overline{B_g(x_0, A)}$ for all $A > 0$. Suppose $d = d_g(x_0, x) > 0$. Fix a sufficiently small number $\delta > 0$, and let $\epsilon_{can} = \epsilon_{can}(\delta), C_0 = C_0(\delta)$ be from Lemma 2.2.6. Let $r_0 = \min\{C_0^{-1}\rho(x_0), C_0^{-1}\rho(x), r(d+2, \epsilon_{can}), 1\}$. Then by Lemma

2.2.8 and Lemma 4.2.1, all points in $B_g(x_0, d+1)$ with $\rho \le r_0$ are contained in the union of a finite collection $S$ of disjoint $\delta$-tubes and capped $\delta$-tubes in $B_g(x_0, d+2)$.

For each $i \in \mathbb{N}$, let $\gamma_i : [0, d] \to M$ be a smooth curve joining $x_0$ and $x$ with constant speed, such that the length of $\gamma_i$ satisfies $L(\gamma_i) < d + \frac{1}{i}$. So $\gamma_i \subset B_g(x_0, d+1)$. We claim that the curvature on $\gamma_i$ is uniformly bounded for all $i$. Suppose not, we may assume there is a sequence of points $x_i \in \gamma_i$ such that $\rho(x_i) \to 0$ as $i \to \infty$. By the finiteness of $S$ we may also assume there is some $\mathcal{T} \in S$ that contains all $x_i$. Then $\mathcal{T}$ must be a $\delta$-tube with curvature blowing up in one end. Taking a point $y \in \mathcal{T}$ such that $\rho(y) < \frac{1}{2}\min\{\rho(x_0), \rho(x)\}$. Then for all large $i$, $\gamma_i$ passes through the $\delta$-neck centered at $y$ at least twice, which contradicts the almost minimality of $\gamma_i$. So the claim holds. Therefore, by the properness of $|\mathrm{Rm}|$, there is a compact set $K \subset M$ such that $\gamma_i \subset K$ for all $i$.

Since $|\gamma_i'| = \frac{L(\gamma_i)}{d} \to 1$, it follows that $\gamma_i$'s are uniformly Lipschiz-continuous on $[0, d]$. Since $\gamma_i$'s are equicontinuous and map into a compact set of $M$, the Arzela-Ascoli Lemma applies. So by passing to a subsequence, we may assume that $\gamma_i$ uniformly converges to some continuous curve $\gamma_\infty : [0, d] \to M$. Since $\int_0^d |\gamma_i'|^2\, dt = \frac{L(\gamma_i)^2}{d} \le 4d$, we can apply weak compactness to the sequence $\{\gamma_i\}$. By passing to a subsequence, we may assume $\gamma_i$ weakly converges to $\gamma_\infty$ in $W^{1,2}$.

Let $E(\gamma) = \int_0^d |\gamma'(t)|^2\, dt$ be the energy function on all $W^{1,2}$-path connecting $x_0$ and $x$. Then by Cauchy-Schwarz inequality we have

$$E(\gamma) = \int_0^d |\gamma'(t)|^2\, dt \ge \frac{(\int_0^d |\gamma'|)^2}{d} = \frac{L(\gamma)^2}{d} \ge d. \tag{4.2.4}$$

By the weak semi-continuity of the $E$-energy, it follows that the $W^{1,2}$-path $\gamma_\infty$ has energy $E(\gamma_\infty) \le d$. So $\gamma_\infty$ minimizes the energy $E$ in $W^{1,2}$. Therefore, $\gamma_\infty$ is a smooth solution to the geodesic equation, and hence it is a minimizing geodesic.

□

The next lemma says that if a ball is scathed, then we can find a minimizing geodesic in the ball along which the curvature blows up, and it is covered by $\delta$-necks.

**Lemma 4.2.3.** *(Minimal geodesic covered by $\delta$-necks) Under the same assumptions as Lemma 4.2.1. Let $\delta > 0$. Suppose $\inf_{B_g(x_0, A)} \rho = 0$ for some $A > 0$. Then there exists a minimizing geodesic $\gamma : [0, 1) \to B_g(x_0, A)$ such that $R(\gamma(s)) \to \infty$ as $s \to 1$, and $\gamma(s)$ is the center of a $\delta$-neck for all $s$ close to* 1.

*Proof.* Let $x_i \in B_g(x_0, A)$ be a sequence of points such that $R(x_i) \to \infty$ as $i \to \infty$. Since by Lemma 4.2.1 the ball $B_g(x_0, A)$ is totally bounded, we may assume by passing to a subsequence that $\{x_i\}$ is Cauchy. So there is a $\delta$-tube $\mathcal{T}$, which blows up at one end, that contains $x_i$ for all large $i$. By Lemma 4.2.2, there exists $\gamma_i : [0, 1] \to M$, which is the minimizing geodesic connecting $x_0$ and $x_i$. Noting that $\gamma_i$ passes through any $\delta$-neck in $\mathcal{T}$ at most once, after passing to a subsequence, $\gamma_i$ converges to a minimizing geodesic $\gamma : [0,1) \to M$. Moreover, by the minimality of $\gamma$, $\gamma(s)$ is the center of a $\delta$-neck for $s$ close enough to 1.

□

The following lemma shows that for a Ricci flow spacetime satisfying some appropriate canonical neighborhood assumption, suppose a time-slice is close enough to that of a $\kappa$-solution, then a parabolic region of a certain size is close to that in the $\kappa$-solution.

**Lemma 4.2.4.** *(Time-slice closeness implies spacetime closeness) Let $(M_\infty, g_\infty(t), x_\infty)$ be a $\kappa$-solution, $t \in (-\infty, T_{\max})$, where $T_{\max}$ is the maximal existence time. For any $\delta > 0$, there exists $\epsilon > 0$ such that the following holds:*

*Let $(\mathcal{M}, g(t))$ be a Ricci flow spacetime, $x_0 \in \mathcal{M}$, $t_0 := \mathfrak{t}(x_0) > 0$, $\rho(x_0) = 1$. Suppose the $\epsilon$-canonical neighborhood assumption holds at scales $(0, 2\sqrt{\eta\delta^{-1}})$, where $\eta$ is from Lemma 2.2.5. Suppose also that $(\mathcal{M}_{t_0}, x_0)$ is $\epsilon$-close to $(M_\infty, g_\infty(0), x_\infty)$. Then $(\mathcal{M}, g(t+t_0), x_0)$ is $\delta$-close to $(M_\infty, g_\infty(t), x_\infty)$ on the time interval $[-\delta^{-1}, \min\{\delta^{-1}, T_{\max} - \delta\}]$.*

*Proof.* Suppose the assertion does not hold, then there is a sequence of spacetimes $(\mathcal{M}_i, g_i(t))$, $x_{0i} \in \mathcal{M}_i$, $\mathfrak{t}(x_{0i}) = t_{0i} > 0$, and a sequence $\epsilon_i > 0$ with $\lim_{i\to\infty} \epsilon_i = 0$, such that the assumptions are satisfied for each $i$, but the conclusion fails.

Let $-a^*, b^*$ be the infimum and supremum of $s_1, s_2 \in [-2\delta^{-1}, \min\{2\delta^{-1}, T_{\max} - \delta/2\}]$, respectively, such that there exists $C$ (may depend on $s_1, s_2$) such that $|\mathrm{Rm}| \le C$ in $\bigcup_{t\in[t_{0i}-s_1, t_{0i}+s_2]} (B_{t_{0i}}(x_{0i}, d))(t)$ for all $d > 0$ and sufficiently large $i$. Then by the gradient estimates we have $a^*, b^* > 0$. So by passing to a subsequence, we may assume $(\mathcal{M}_i, g_i(t+t_{0i}), x_{0i})$ converges to a smooth complete Ricci flow $(\widehat{M}, \widehat{g}(t), \widehat{x})$, $t \in (-a^*, b^*)$, which has bounded curvature in any compact subinterval in $(-a^*, b^*)$.

Note that $(\widehat{M}, \widehat{g}(0), \widehat{x})$ is isometric to $(M_\infty, g_\infty(0), x_\infty)$, and each time-slice of $(\widehat{M}, \widehat{g}(t))$ is a time-slice of a $\kappa$-solution, it is easy to see that the flow $(\widehat{M}, \widehat{g}(t), \widehat{x})$ is isometric to $(M_\infty, g_\infty(t), x_\infty)$ for all $t \in (-a^*, b^*)$.

In particular, $(\widehat{M}, \widehat{g}(t), \widehat{x})$ extends to a complete Ricci flow with bounded curvature on $[-a^*, b^*]$. Applying the gradient estimates in $(\mathcal{M}_i, g_i(t + t_{0i}), x_{0i})$ at times $t$ close to $-a^*$ and $b^*$ we get $a^* = 2\delta^{-1}$ and $b^* = \min\{2\delta^{-1}, T_{\max} - \delta/2\}$. So $(\mathcal{M}_i, g_i(t + t_{0i}), x_{0i})$ is $\delta$-close to $(M_\infty, g_\infty(t), x_\infty)$ on $[-\delta^{-1}, \min\{\delta^{-1}, T_{\max} - \delta\}]$ for sufficiently large $i$, which is a contradiction.

□

**Lemma 4.2.5.** *(Blow-up sequence converges to a $\kappa$-solution) Let $(\mathcal{M}_i, g_i(t))$ be a sequence of singular Ricci flows with normalized initial condition, $x_i \in \mathcal{M}_i$, with $\sup_i \mathfrak{t}(x_i) < \infty$ and $\lim_{i\to\infty} \rho(x_i) = 0$. Let $\widetilde{g}_i(t) := \rho^{-2}(x_i) g(\rho^2(x_i)t + \mathfrak{t}(x_i))$. Then a subsequence $(\mathcal{M}_{i_k}, \widetilde{g}_{i_k}(t), x_{i_k})$*

*converges to a $\kappa$-solution on $(-\infty, T_{\max})$, where $T_{\max} \in (0,\infty]$ is the maximal existence time of the $\kappa$-solution.*

*Proof.* Let $T = \sup_i \mathfrak{t}(x_i)$. Since $\lim_{i\to\infty} \rho(x_i) = 0$, there are sequences $\epsilon_i, \delta_i > 0$ such that $\lim_{i\to\infty} \epsilon_i = \lim_{i\to\infty} \delta_i = 0$, such that $2\sqrt{\eta\delta_i^{-1}}\rho(x_i) < \overline{r}_{\epsilon_i}(T+1)$, where $\overline{r}_{\epsilon_i} : [0,\infty) \to [0,\infty)$ is the function in Lemma 2.2.10.

In particular, the $\epsilon_i$-canonical neighborhood assumption holds at $x_i$. By passing to a subsequence we may assume that $(\mathcal{M}_i, g_i(t), x_i)$ is $\epsilon_i$-close to the time-0-slice of some $\kappa$-solution $(M_\infty, g_\infty(t), x_\infty)$. So we can apply Lemma 4.2.4 and deduce that $(\mathcal{M}, g_i(t), x_i)$ is $\delta_i$-close to $(M_\infty, g_\infty(t), x_\infty)$ on the interval $[-\delta_i^{-1}, \min\{\delta_i^{-1}, T_{\max} - \delta_i\}]$. So the conclusion follows by letting $i \to \infty$. □

## 4.3 Canonical neighborhood theorem

The main results in this section are a canonical neighborhood theorem (Proposition 4.3.1) for singular Ricci flows, and a bounded curvature at bounded distance theorem by reduced volume (Theorem 4.3.2).

Recall that Perelman proved a canonical neighborhood theorem for compact smooth Ricci flows [43, Theorem 26.2], which says that assuming the Ricci flow has normalized initial condition, then for any $T > 0$ there exists $r(T) \geq 0$ such that the canonical neighborhood assumption holds in $\mathcal{M}_{t\leq T}$ at scales less than $r(T)$.

He also proved a local version of the theorem [43, Proposition 85.1], in which he assumed the curvature is bounded in a backward parabolic neighborhood of a point $x_0$, $\mathfrak{t}(x_0) = t_0$, and showed that for all $A > 0$ there exists $r(A) > 0$, such that the canonical neighborhood assumption holds in $B_{t_0}(x_0, A)$ at scales less than $r(A)$.

The following proposition extends the local theorem to singular Ricci flows.

**Proposition 4.3.1.** *(Canonical neighborhood theorem) For any $A > 0$, $\epsilon > 0$, there are constants $\kappa(A), r(A,\epsilon), \overline{r}(A), K(A) > 0$, such that the following holds: Let $\mathcal{M}$ be a singular Ricci flow, $x_0 \in \mathcal{M}$, $\mathfrak{t}(x_0) = t_0 > 0$. Suppose for some $r_0 > 0$ with $2r_0^2 < t_0$ the following holds:*

1. *$\mathcal{M}$ is unscathed on a parabolic neighborhood $P(x_0, r_0, -r_0^2)$.*
2. *$|\mathrm{Rm}| \leq r_0^{-2}$ on $P(x_0, r_0, -r_0^2)$.*
3. *$vol(B_{t_0}(x_0, r_0)) \geq A^{-1} r_0^3$.*

*Then*

*(a) The solution is $\kappa$-non-collapsed at scales less than $r_0$ in $B_{t_0}(x_0, A r_0)$.*

*(b) The $\epsilon$-canonical neighborhood assumption holds in $B_{t_0}(x_0, Ar_0)$ at scales less than $rr_0$.*

*(c) If $r_0 \le \bar{r}\sqrt{t_0}$ then $|\mathrm{Rm}| \le Kr_0^{-2}$ in $B_{t_0}(x_0, Ar_0)$.*

As an application of Proposition 4.3.1, as well as other results by Perelman, we can generalize his bounded curvature at bounded distance theorem by using the reduced volume:

**Theorem 4.3.2.** *(Bounded curvature at bounded distance by reduced volume) For any $A, \kappa > 0$, there exist $\varphi(A,\kappa), K(A,\kappa) > 0$ such that the following holds: Let $\mathcal{M}$ be a singular Ricci flow, $x_0 \in \mathcal{M}$, $t_0 := \mathfrak{t}(x_0)$. Suppose the reduced volume $\tilde{V}_{x_0}(1) \ge \kappa$, and $\mathcal{M}_{t \ge t_0 - 1}$ is $\varphi$-positive.*

*If $|\mathrm{Rm}|(x_0) \le 1$, then $|\mathrm{Rm}| \le K$ in $B_{t_0}(x_0, A)$.*

In the theorem, $\tilde{V}_{x_0}(1)$ is the reduced volume at $\tau = 1$, for the base point $x_0$. It is defined by

$$\tilde{V}_{x_0}(\tau) = \int_{\mathcal{M}_{t_0-\tau}} \tau^{-\frac{3}{2}} e^{-\ell_{x_0}(x)}\, d_{t_0-\tau}x. \tag{4.3.1}$$

To prove Proposition 4.3.1 and Theorem 4.3.2, we first prove some results in $\mathcal{L}$-geometry for singular Ricci flows.

**Definition 4.3.3.** The $\mathcal{L}_+$-length of an admissible curve $\gamma : [t_0 - \tau, t_0] \to \mathcal{M}$ is

$$\mathcal{L}_+(\gamma) = \int_{t_0-\tau}^{t_0} \sqrt{t_0 - t}(R_+(\gamma(t)) + |\gamma'(t)|^2)\, dt. \tag{4.3.2}$$

**Lemma 4.3.4.** *Given constants $T, E$ and $\Lambda > 0$, there exists $r = r(\Lambda, T, E)$ such that the following holds. Let $\mathcal{M}$ be a singular Ricci flow with normalized initial condition. Suppose $\gamma : [a,b] \to \mathcal{M}$ is a smooth admissible curve with $b \le T$. Suppose $b - a \ge E$, and $\rho(\gamma(a)) \le r$.*

*Then $\int_a^b R_+(\gamma(t)) + |\gamma'(t)|^2\, dt > \Lambda$.*

*Proof.* Suppose for some $\Lambda, T, E > 0$, the conclusion does not hold, then we can find a sequence of singular Ricci flows $(\mathcal{M}_k, g_k(t))$ and a sequence of smooth curves $\gamma_k : [a_k, b_k] \to \mathcal{M}_k$ satisfying the assumptions in the theorem. In particular, we have $\rho(\gamma_k(a_k)) \le r_k$, and $r_k \to 0$ as $k \to \infty$, but

$$\int_{a_k}^{b_k} (R_+(\gamma_k(t)) + |\gamma_k'(t)|^2)\, dt \le \Lambda. \tag{4.3.3}$$

We rescale each $(\mathcal{M}_k, g_k(t), \gamma_k(a_k))$ by $\rho_k^{-2} := \rho^{-2}(\gamma_k(a_k))$ and then shift time $a_k$ to 0, to obtain a sequence of Ricci flow spacetimes $(\tilde{\mathcal{M}}_k, \tilde{g}_k(t), \gamma_k(a_k))$ defined on $[0, \infty)$. Then $\gamma_k : [0, (b_k - a_k)\rho_k^{-2}] \to \tilde{\mathcal{M}}_k$ is an admissible curve and $(b_k - a_k)\rho_k^{-2} \to \infty$ as $k \to \infty$. Note that (4.3.3) is invariant under rescaling. Applying Lemma 4.2.5 and passing to a subsequence we may assume that $(\tilde{\mathcal{M}}_k, \tilde{g}_k(t), \gamma_k(a_k))$ converges to a $\kappa$-solution $(M_\infty, g_\infty(t), x_\infty)$ on $[0, T_{\max})$, where $T_{\max} \in (0, \infty]$ is the maximal existence time. So by [8], $(M_\infty, g_\infty(t))$ is either a Bryant soliton [8], or has finite extinction time.

Suppose first that $(M_\infty, g_\infty(t))$ is a Bryant soliton. Let $\mathcal{M}_\infty$ denote the Ricci flow spacetime associated to it. Fix a large $A > 0$ and let $\mathcal{P}_A := \bigcup_{t\in[0,A^{3/2}]} B_t(x_\infty, A)$. Then for large $k$, there exists a time-preserving diffeomorphism $\phi_k : U_k \to V_k$, with $\phi_k(x_\infty) = \gamma_k(a_k)$, where $U_k$ and $V_k$ are open subsets of $\mathcal{M}_\infty$ and $\tilde{\mathcal{M}}_k$ respectively, such that given any compact subset $K \subset \mathcal{M}_\infty$ and $\delta > 0$, we have $K \subset U_k$ for all large $k$, and $\|\phi_k^* G_k - G_\infty\|_{C^{[\delta^{-1}]}(K,G_\infty)}$, where $G_k$ and $G_\infty$ are spacetime metrics of $\tilde{\mathcal{M}}_k$ and $\mathcal{M}_\infty$ respectively. So we have $\mathcal{P}_A \subset U_k$ for all large $k$. Let $\widehat{\gamma}_k \subset U_k$ be the image of $\gamma_k|_{V_k}$ under $\phi_k^{-1}$.

Since $(b_k - a_k)\rho_k^{-2} \to \infty$ as $k \to \infty$, we see that $\gamma_k$ must exit $\phi_k(\mathcal{P}_A)$ at some time $T_k \in [0, A^{3/2}]$, and accordingly $\widehat{\gamma}_k$ exits $\mathcal{P}_A$ at the time $\widehat{T}_k \in [0, A^{3/2}]$. Then for sufficiently large $k$, we have

$$\begin{aligned} \int_0^{T_k} |\gamma_k'(t)|^2 dt &\geq \frac{1}{2}\int_0^{\widehat{T}_k} |\widehat{\gamma}_k'(t)|^2 dt, \\ \int_0^{T_k} R(\gamma_k(t)) dt &\geq \frac{1}{2}\int_0^{\widehat{T}_k} R(\widehat{\gamma}_k(t)) dt. \end{aligned} \tag{4.3.4}$$

Suppose $T_k < A^{3/2}$. Since $\mathrm{Ric} \geq 0$, we have $\frac{\partial}{\partial t} g_\infty(t) \leq 0$ for all $t \geq 0$. So $|\widehat{\gamma}_k'(t)|_{g_\infty(t)} \geq |\widehat{\gamma}_k'(t)|_{g_\infty(T_k)}$ for all $t \in [0, T_k]$, and hence

$$\int_0^{T_k} |\widehat{\gamma}_k'(t)|^2\, dt \geq \int_0^{T_k} |\widehat{\gamma}_k'(t)|^2_{T_k}\, dt \geq \frac{d^2_{T_k}(\widehat{\gamma}_k(0), \widehat{\gamma}_k(T_k))}{T_k} \geq A^{1/2}. \tag{4.3.5}$$

Otherwise, we have $T_k = A^{3/2}$, and $\widehat{\gamma}_k(t) \in B_t((\widehat{\gamma}_k(0))(t), A)$ for all $t \in [0, A^{3/2}]$. Since $(M_\infty, g_\infty(t))$ is a Bryant soliton, we have $R(\widehat{\gamma}_k(t)) \geq \frac{C}{A}$, where $C$ is a constant depending only on the curvature of the tip. So we have

$$\int_0^{A^{3/2}} R(\widehat{\gamma}_k(t))\, dt \geq \int_0^{A^{3/2}} \frac{C}{A}\, dt = CA^{1/2}. \tag{4.3.6}$$

In both cases, taking $A$ sufficiently large, it follows by (4.3.4) that

$$\int_0^{T_k} (R(\gamma_k(t)) + |\gamma_k'|^2)\, dt > \Lambda, \tag{4.3.7}$$

a contradiction to (4.3.3).

Now suppose $(M_\infty, g_\infty(t))$ has a finite extinction time $T_\infty < \infty$. Let $\theta \in (0, T_\infty)$ and consider $\mathcal{P}_A := \bigcup_{t=0}^{\theta} B_t(x_\infty, A)$. Let $T_k$ and $\widehat{T}_k$ be defined as above. Then if $\widehat{T}_k < \theta$, then (4.3.5) holds. Otherwise, since the scalar curvature blows up at the rate of $(T_\infty - t)^{-1}$ when the time goes up to $T_\infty$, we have

$$\int_0^{\theta} R(\widehat{\gamma}_k(t)) dt \geq \int_0^{\theta} \frac{C}{T_\infty - t} dt = -C\log(T - \theta). \tag{4.3.8}$$

By taking $\theta$ sufficiently close to $T_\infty$ and $A$ sufficiently large, we get a contradiction.

□

The following lemma says that an admissible curve that contains a point of large curvature must have large $\mathcal{L}_+$-length.

**Lemma 4.3.5.** *For all $\Lambda < \infty, \overline{r}, T > 0$, there is a constant $\delta = \delta(\Lambda, \overline{r}, T)$ with the following property:*

*Let $\mathcal{M}$ be a singular Ricci flow with normalized initial condition, and $x_0 \in \mathcal{M}_{t_0}$ with $0 < t_0 \leq T$. Suppose $r_0 \geq \overline{r}$, and $P(x_0, r_0, -r_0^2)$ is unscathed, and $|\mathrm{Rm}| \leq r_0^{-2}$ on $P(x_0, r_0, -r_0^2)$. Suppose also $\gamma : [t_1, t_0] \to \mathcal{M}$ is an admissible curve ending at $(x_0, t_0)$, and there exists $t \in [t_1, t_0]$ such that $\rho(\gamma(t)) < \delta$.*

*Then $\mathcal{L}_+(\gamma) > \Lambda$.*

*Proof.* First, let $\Delta t = \frac{1}{4}10^{-4}\overline{r}^4\Lambda^{-2}$. By taking $\delta < \overline{r}$, we see that $\gamma$ must exit $P(x_0, r_0, -r_0^2)$ at some time $\tilde{t}$. First, suppose $\tilde{t} > t_0 - \Delta t$. Then by the Schwarz inequality we get

$$\begin{aligned} \int_{\tilde{t}}^{t_0} \sqrt{t_0 - s}|\gamma'(s)|^2\, ds &\geq \left(\int_{\tilde{t}}^{t_0} |\gamma'(s)|\, ds\right)^2 \left(\int_{\tilde{t}}^{t_0} (t_0 - s)^{-1/2}\, ds\right)^{-1} \\ &\geq \frac{1}{2}10^{-2} r_0^2 (\Delta t)^{-1/2} > \Lambda, \end{aligned} \tag{4.3.9}$$

where the factor $10^{-2}$ comes from the distance distortion on $P(x_0, r_0, -r_0^2)$.

So now we may assume that $\gamma$ exits $P(x_0, r_0, -r_0^2)$ at a time $\tilde{t} \leq t_0 - \Delta t$. Then the conclusion follows immediately from Lemma 4.3.4.

□

*Proof of Proposition 4.3.1.* Under a suitable rescaling, we may assume without loss of generality that $\mathcal{M}$ has normalized initial condition. Note by Lemma 4.3.5 and the properness of scalar curvature from [44, Theorem 1.3], a minimizing sequence of admissible curves between any two points converges to a smooth minimizing $\mathcal{L}$-geodesic. Now the rest of proof for part (a) is the same as [43, Proposition 85.1(a)].

For part (b), suppose that for some $A > 0$ the claim is not true. Then there is a sequence of singular Ricci flows $\mathcal{M}_k$ which provide a counterexample. In particular, there exists $x_k \in B_{t_{0k}}(x_{0k}, Ar_{0k})$ with $\rho(x_k) \leq r_k r_{0k}$ but at which the $\epsilon$-canonical neighborhood assumption does not hold, where $r_k \to 0$ as $k \to \infty$.

Omitting the subscripts for a moment, by Lemma 4.2.5 we can apply a point-picking and find points $\overline{x} \in B_{\overline{t}}(x_0, 2Ar_0)$, $\overline{t} \in [t_0 - r_0^2/2, t_0]$ with $\overline{\rho} := \rho(\overline{x}, \overline{t}) \leq rr_0$ satisfying the following: the $\epsilon$-canonical neighborhood assumption does not hold at $(\overline{x}, \overline{t})$, but it holds at all points in $\overline{P}$ with $\rho \leq \overline{\rho}/2$, where

$$\overline{P} = \{x \in \mathcal{M}_t : d_t(x_0(t), x) \leq d_{\overline{t}}(x_0(\overline{t}), \overline{x}) + r^{-1}\overline{\rho}, t \in [\overline{t} - \frac{1}{4}r^{-2}\overline{\rho}^2, \overline{t}]\}. \tag{4.3.10}$$

The rest of proof is the same as [43, Prop 85.1(b)], in which we can extract a convergent subsequence of $(\overline{P}_k, g_k(t), \overline{x}_k)$ that converges to a $\kappa$-solution. Then this contradicts the assumption of $\overline{x}_k$ for large $k$, and thus proves part (b).

Part (c) follows from (b) in the same way as [43, Prop 85.1(c)].

□

The following lemma is a corollary of Proposition 4.3.1(c). Given the curvature and reduced volume bound at a single point, it provides curvature bound in a backward parabolic neighborhood of this point.

**Lemma 4.3.6.** *For any $\kappa > 0$, there exists $r(\kappa) > 0$ such that the following holds: Let $\mathcal{M}$ be a singular Ricci flow, $x_0 \in \mathcal{M}$, $\mathfrak{t}(x_0) = t_0$. Suppose the reduced volume $\tilde{V}_{x_0}(1) \geq \kappa$, and $\mathcal{M}_{t \geq t_0 - 1}$ is 1-positive.*

*If $|\mathrm{Rm}|(x_0) \leq 1$, then $|\mathrm{Rm}|(x) \leq r^{-2}$ for all $x \in P(x_0, r, -r^2)$.*

*Proof.* We will first show that $|\mathrm{Rm}|(x) \leq r^{-2}$ for all $x \in B_{t_0}(x_0, r)$. Then the assertion in the theorem follows immediately from this by [43, Theorem 54.2], which gives upper bound on curvature at earlier smaller balls, assuming lower bounds for the volume and curvature in a ball.

For any $x \in \mathcal{M}$, put $\overline{\rho}(x) = \sup\{r > 0 : |\mathrm{Rm}|(y) \leq r^{-2}$ for all $y \in B_{\mathfrak{t}(x)}(x, r)\}$. Suppose the assertion is not true. Then there is a sequence of singular Ricci flows $\mathcal{M}_i$ and points $x_{0i} \in \mathcal{M}_i$, $\mathfrak{t}(x_{0i}) = t_{0i}$, that satisfy the assumptions, but $\lim_{i\to\infty} \overline{\rho}(x_{0i}) = 0$.

By the reduced volume comparison theorem (see e.g. [43, Lem 78.11]), $\tilde{V}_{x_0}(1) \geq \kappa$ implies that there exists $\kappa'(\kappa) > 0$ such that $\mathcal{M}_i$ is $\kappa'$-non-collapsed at $x_{0i}$ at scales less than 1. Rescale and do a time-shifting to the flows $(\mathcal{M}_i, x_{0i})$ to get a sequence of new flows, which are still denoted by $(\mathcal{M}_i, x_{0i})$, such that in the new flows we have $\overline{\rho}(x_{0i}) = 1$ and $\mathfrak{t}(x_{0i}) = 0$. So $vol(B_0(x_{0i}, 1)) \geq \kappa' > 0$ for all $i$.

By [43, Theorem 54.2], we can find $C(\kappa) > 0$ such that $|\mathrm{Rm}| \leq C$ in $P(x_{0i}, C^{-1/2}, -C^{-1})$. So by applying Proposition 4.3.1(c) and some distance distortion estimates, we can find a smooth Ricci flow $(U, g_\infty(t), x_\infty)$ (possibly incomplete), $t \in [-c, 0]$, for some $c > 0$, such that there are diffeomorphisms $\phi_i : U \to \mathcal{M}_i$ such that $\phi_i(x_\infty) = x_{0i}$ and $\lim_{i\to\infty} \|\phi_i^* g_i(t) - g_\infty(t)\|_{C^k(U)} = 0$ for all $k \in \mathbb{N}$ and $t \in [-c, 0]$, and $B_0(x_\infty, 2)$ is relatively compact in $U$.

By the 1-positive pinching assumption, we see that $\mathrm{Rm}(x,t) \geq 0$ for all $x \in U$ and $t \in [-c, 0]$. Also, we have $|\mathrm{Rm}|(x_\infty, 0) = 0$, $\overline{\rho}(x_\infty, 0) = 1$, and hence $|\mathrm{Rm}|(y, 0) = 1$ for some $y \in U$. However, this contradicts with the strong maximum principle, see e.g. [55, Theorem 4.18]. □

Theorem 4.3.2 is a immediate consequence of combining Lemma 4.3.6 and Proposition 4.3.1(c).

## 4.4 heat kernel for singular Ricci flow

Let $\mathcal{M}$ be a singular Ricci flow. We find a heat kernel $H$ of $\mathcal{M}$ (Theorem 4.4.1), such that for any $x_0 \in \mathcal{M}$, $\mathfrak{t}(x_0) = t_0 > 0$, $H(x_0, \cdot)$ is a smooth solution to the conjugate heat equation $(-\frac{\partial}{\partial t} - \Delta + R)H(x_0, \cdot) = 0$ on $\mathcal{M}_{t<t_0}$ and it is a $\delta$-function at $x_0$. We also find an adjoint heat kernel $H^*$ with similar properties.

A key estimate is in Theorem 4.4.6, where we show that for any integer $m \geq 1$,

$$H(x_0, x)R^m(x) \leq C_m \tag{4.4.1}$$

holds for all $x$ outside of a parabolic neighborhood around $x_0$, where $C_m > 0$ depends on $m$. This indicates that $H(x_0, \cdot)$ is sufficiently small at points where the curvature is sufficiently large. With this estimate we are able to show that the heat kernel of singular Ricci flow shares many standard properties as the ordinary heat kernel for compact smooth Ricci flows.

For a compact Ricci flow, it is easy to see by an integration by parts computation that the integral of its heat kernel in each time-slice is equal to 1, we show in Corollary 4.4.7 that this is also true for $H$. Moreover, by applying (4.4.1) we show in Corollary 4.4.9 the symmetry between the heat and adjoint heat kernel, that is, $H(x, y) = H^*(y, x)$ for all $x, y \in \mathcal{M}$, $\mathfrak{t}(x) > \mathfrak{t}(y)$.

### 4.4.1 Construction of the heat kernel

**Theorem 4.4.1.** *Let $\mathcal{M}$ be a singular Ricci flow. For any $x_0 \in \mathcal{M}$, $t_0 := \mathfrak{t}(x_0) > 0$, there exists a function $H(x_0, \cdot) : \mathcal{M}_{t<t_0} \to \mathbb{R}$ which is a smooth solution to the conjugate heat equation $(-\frac{\partial}{\partial t} - \Delta + R)H(x_0, \cdot) = 0$ and*

1. *$H(x_0, x) > 0$ for all $x \in \mathcal{M}(x_0)$, and $H(x_0, x) = 0$ otherwise, where $\mathcal{M}(x_0)$ is the subset of all points that are accessible to $x_0$.*

2. *$\lim_{t \nearrow t_0} H(x_0, \cdot) = \delta_{x_0}$, in the sense that*

$$\lim_{t \nearrow t_0} \int_{\mathcal{M}_t} H(x_0, x)h(x)d_t x = h(x_0) \tag{4.4.2}$$

*for all $h \in C^0_c(\mathcal{M})$.*

3. *Suppose $\mathcal{M}$ has normalized initial condition, and for some $r_0, T > 0$ we have $|\mathrm{Rm}| \leq r_0^{-2}$ on $\mathcal{P}_0 := P(x_0, r_0, -r_0^2)$, and $t_0 < T$. Then there exists $C_0(r_0, T) > 0$ such that $H(x_0, \cdot) \leq C_0$ on $\mathcal{M}_{t<t_0} - \mathcal{P}_0$.*

We say $H$ is the heat kernel of $\mathcal{M}$. The next theorem gives the existence of the adjoint heat kernel $H^*$. With some decay estimates of $H$ at high curvature regions in next subsection, we will show in Corollary 4.4.9 that $H$ is symmetric to $H^*$ in the sense that $H(x,y) = H^*(y,x)$ for all $x, y \in \mathcal{M}$ with $\mathfrak{t}(x) > \mathfrak{t}(y)$.

**Theorem 4.4.2.** *Let $\mathcal{M}$ be a singular Ricci flow. For any $x_0 \in \mathcal{M}$, $\mathfrak{t}(x_0) = t_0 \geq 0$, there exists a function $H^*(x_0, \cdot) : \mathcal{M}_{t>t_0} \to \mathbb{R}$ which is a smooth solution to the heat equation $(\frac{\partial}{\partial t} - \Delta) H^*(x_0, \cdot) = 0$ and*

1. *$H^*(x_0, x) > 0$ for all $x \in \mathcal{M}_{t>t_0}$ that is accessible to $x_0$, and $H^*(x_0, x) = 0$ otherwise.*
2. *$\lim_{t \searrow t_0} H^*(x_0, \cdot) = \delta_{x_0}$, in the sense that*

$$\lim_{t \searrow t_0} \int_{\mathcal{M}_t} H^*(x_0, x) h(x) d_t x = h(x_0) \tag{4.4.3}$$

   *for all $h \in C_c^0(\mathcal{M})$.*

3. *Suppose $\mathcal{M}$ has normalized initial condition, and for some $r_0, T > 0$ we have $|\mathrm{Rm}| \leq r_0^{-2}$ on $\mathcal{P}_0 := P(x_0, r_0, r_0^2)$, and $t_0 < T$. Then there exists $C_0(r_0, T) > 0$ such that $H^*(x_0, \cdot) \leq C_0$ on $\mathcal{M}_{t>t_0} - \mathcal{P}_0$.*

For simplicity, we use the following variant of Perelman's Ricci flow with surgery, in which the surgeries are done slightly before singular times (the times where curvature blows up), instead of exactly at them. This Ricci flow with surgery can be obtained with little modification to that of Perelman's. It is also constructed for complete manifold with bounded geometry in [6].

**Definition 4.4.3** (Ricci flow with surgery)**.** A Ricci flow with surgery is given by

1. A collection of Ricci flows $\{(M_k \times [t_k^-, t_k^+], g_k(\cdot))\}_{1 \leq k \leq N}$, where $N \leq \infty$, $M_k$ is a compact (possibly empty) manifold, $t_k^+ = t_{k+1}^-$ for all $1 \leq k \leq N$.
2. A collection of isometric embeddings $\{\psi_k : Y_k^+ \to Y_{k+1}^-\}_{1 \leq k \leq N}$ where $Y_k^+ \subset M_k$ and $Y_{k+1}^- \subset M_{k+1}$, $1 \leq k < N$, are compact 3 dimensional submanifold with boundary. The $Y_k^{\pm}$'s are the subsets which survive the transition from one flow to the next, and the $\psi_k$'s give the identification between them.

We call each final time $t_k^+$ a surgery time. Let $X_k^+$ and $X_{k+1}^-$ denote the interior of $Y_k^+$ and $Y_{k+1}^-$ respectively for $1 \leq k < N$. We can associate a Ricci flow spacetime $\mathcal{M}$ to the Ricci flow with surgery by taking $\mathcal{M}$ to be the disjoint union of

$$(M_k \times [t_k^-, t_k^+)) \cup (X_k^+ \times \{t_k^+\}) \tag{4.4.4}$$

for $1 \leq k \leq N$, and removing the following subset

$$(M_{k+1} - X_{k+1}^-) \times \{t_{k+1}^-\} \tag{4.4.5}$$

for all $1 \leq k < N$ (making identifications using the $\psi_k$'s as gluing maps).

*Proof of Theorem 4.4.1.* Let $\mathcal{M}_i$ be a sequence Ricci flow with surgery spacetimes starting from $(M, g)$, with the surgery scale $\delta_i \to 0$ as $i \to \infty$. Then $\mathcal{M}_i$ have the local control required for the application of the spacetime compactness theorem in [44, Theorem 2.20], and hence $\mathcal{M}_i$ converges to the singular Ricci flow $\mathcal{M}$ as $i \to \infty$. Assume $x_{0i} \in \mathcal{M}_i$ converges to $x_0$ when $i \to \infty$. We shall construct smooth non-negative solutions to the conjugate heat equation on each $\mathcal{M}_i$ starting from $x_{0i}$, and then take a limit of them to obtain the desired heat kernel on $\mathcal{M}$.

Let $\mathcal{M}_i$ be fixed below, and assume the compact Ricci flows that form $\mathcal{M}_i$ are $\{(M_k \times [t_k^-, t_k^+], g_k(\cdot))\}_{1 \leq k \leq N}$, where $t_N^+ = t_{0i} = \mathfrak{t}(x_{0i})$. We shall define $u_i$ on each of these Ricci flows and then restrict it to $\mathcal{M}_i$ to get a smooth function.

First, on $M_N \times [t_N^-, t_N^+)$, let $u_i$ be the ordinary heat kernel of $(M_N, g_N(t))$ which starts from $(x_{0i}, t_{0i})$. Note that $u_i$ vanishes at $(x, t)$ if $x$ are not in the same component with $x_{0i}$. Then suppose by induction that $u_i$ has been defined on $M_j \times [t_j^-, t_j^+)$ for all $j = k, ..., N$ such that the following holds:

1. $u_i \geq 0$ is a smooth solution to the conjugate heat equation on $\mathcal{M}_{i, t \geq t_k^-}$.
2. $u_i$ vanishes at points that are not accessible to $x_{0i} \in \mathcal{M}_i$.
3. $\int_{M_j} u_i(x, t) d_t x \leq 1$ for all $t \in [t_j^-, t_j^+)$, and $j = k, ..., N$.

Then we define $u_i$ on $M_{k-1} \times \{t_{k-1}^+\}$ by letting $u_i(x, t_{k-1}^+) = u_i(x, t_k^-)$ if $x \in X_{k-1}^+$, and $u_i(x, t_{k-1}^+) = 0$ if $x \in M_{k-1} - X_{k-1}^+$. Then for any $(x, t) \in M_{k-1} \times [t_{k-1}^-, t_{k-1}^+)$, set

$$u_i(x, t) = \int_{X_{k-1}^+} u_i(y, t_{k-1}^+) H(y, t_{k-1}^+; x, t) \, d_{t_{k-1}^+} y, \tag{4.4.6}$$

where $H(\cdot, \cdot; \cdot, \cdot)$ is the ordinary heat kernel for the smooth compact Ricci flow $(M_{k-1}, g_{k-1}(t))$, $t \in [t_{k-1}^-, t_{k-1}^+]$. So $u_i$ is a smooth solution to the conjugate heat equation on $M_{k-1} \times$

$[t_{k-1}^-, t_{k-1}^+) \cup X_{k-1}^+ \times \{t_{k-1}^+\}$, and hence it is smooth on $\mathcal{M}_{i, t \geq t_{k-1}^-}$. Moreover, by (4.4.6) we get for all $t \in [t_{k-1}^-, t_{k-1}^+)$ that

$$\begin{aligned} \int_{M_{k-1}} u_i(x,t) d_t x &\leq \int_{M_{k-1}} \int_{X_{k-1}^+} u_i(y, t_{k-1}^+) H(y, t_{k-1}^+; x, t) \, d_{t_{k-1}^+} y \, d_t x \\ &= \int_{X_{k-1}^+} u_i(y, t_{k-1}^+) \, d_{t_{k-1}^+} y \leq 1. \end{aligned} \tag{4.4.7}$$

It is clear that assumptions (1) and (2) also hold for $j = k-1$. So by induction, we obtain a smooth non-negative solution $u_i$ on $\mathcal{M}_i$ which satisfies $\int_{\mathcal{M}_i(t)} u_i(x,t) d_t x \leq 1$, and $u_i \equiv 0$ on $\mathcal{M}_i - \mathcal{M}_i(x_{0i})$.

Next, suppose $r_0, T$ are the constants in assertion (3). Then since $(\mathcal{M}_i, x_{0i})$ converges to $(\mathcal{M}, x_0)$, we have $|\mathrm{Rm}| \leq 2r_0^{-2}$ on $\mathcal{P}_{0i} := P(x_{0i}, r_0, -r_0^2)$ for all large $i$. Since $\mathcal{M}_i$ has the normalized initial condition, the scalar curvature satisfies $R \geq -6$ anywhere on $\mathcal{M}_i$, see [43, Lemma 79.11]. So $\tilde{u}_i := u_i e^{-6(t_0 - t)}$ satisfies

$$(-\frac{\partial}{\partial t} - \Delta)\tilde{u}_i \leq 0. \tag{4.4.8}$$

Let $\Gamma$ be the parabolic boundary of $\mathcal{P}_{0i}$, i.e.

$$\Gamma = \left( \bigcup_{t \in [t_{0i} - r_0^2, t_{0i})} \partial B_{t_{0i}}(x_{0i}, r_0)(t) \right) \cup B_{t_{0i}}(x_{0i}, r_0)(t_{0i} - r_0^2). \tag{4.4.9}$$

Then since $\int u_i \, d_t x \leq 1$, we can apply the parabolic mean value inequality (see e.g. [28, Theorem 25.2]) at any points in $\Gamma$, and hence find a constant $C_1(r_0) > 0$ such that $\tilde{u}_i \leq C_1$ on $\Gamma$.

Suppose by induction that $\tilde{u}_i \leq C_1$ on $\mathcal{M}_{i,[t_k^-, t_0)} - \mathcal{P}_{0i}$ for some $k \leq N$. Without loss of generality, we may assume that $t_{0i} - r_0^2$ is a surgery time. Then if $M_{k-1} \times [t_{k-1}^-, t_{k-1}^+)$ does not intersect $\mathcal{P}_{0i}$, by the inductive assumption we can apply the maximum principle and get $\tilde{u}_i \leq C_1$ on $\mathcal{M}_{i,[t_{k-1}^-, t_{k-1}^+)}$. Otherwise, apply the maximum principle on $\bigcup_{t \in [t_{k-1}^-, t_{k-1}^+)} (M_k(t) \setminus B_{t_{0i}}(x_{0i}, r_0)(t))$, whose boundary is contained in $\Gamma$. Then by inductive assumption and $\tilde{u}_i \leq C_1$ on $\Gamma$, we get $\tilde{u}_i \leq C_1$ on $\mathcal{M}_{i,[t_{k-1}^-, t_{k-1}^+)}$. So by induction, $\tilde{u}_i \leq C_1$ holds on $\mathcal{M}_{i,t<t_{0i}} - \mathcal{P}_{0i}$. Let $C_0 = C_1 e^{6T}$, then $u_i \leq C_0$ on $\mathcal{M}_{i,t<t_{0i}} - \mathcal{P}_{0i}$.

Then applying the interior Hölder estimate, we can bound the derivatives of $u_i$ in terms of its $C^0$-norm and the curvature norm nearby. So by passing to a subsequence, we may assume that $u_i$ converges smoothly to a non-negative smooth solution $u$ to the conjugate heat equation on $\mathcal{M}$. It follows immediately that

$$u(x) \leq C_0 \tag{4.4.10}$$

for all $x \in \mathcal{M} - \mathcal{P}_0$, which proves assertion (3) in the theorem.

Now we establish the properties claimed in Theorem 4.4.1. First, we show that $u$ is a $\delta$-function at $x_0$. For large $i$, let $\phi_i$ be a cut-off function whose support is contained in $B_{t_{0i}}(x_{0i}, r_0)$, $\phi_i \equiv 1$ on $B_{t_{0i}}(x_{0i}, \frac{r_0}{2})$, and $|\nabla \phi_i|$ and $|\Delta \phi_i|$ are bounded above in terms of $r_0$. Here the derivatives and norms are considered with respect to $g(t_{0i})$. By the choice of $r_0$, there exists a universal constant $c > 0$ such that $\frac{1}{2}g(t) \leq g(t_{0i}) \leq 2g(t)$ for all $t \in [t_{0i} - cr_0^2, t_{0i}]$ on $B_{t_{0i}}(x_{0i}, r_0)$. So for all $t \in [t_{0i} - cr_0^2, t_{0i})$, a direct computation using integration by parts shows that there is a constant $C = C(r_0) > 0$ such that

$$\frac{\partial}{\partial t} \int_{B_{t_{0i}}(x_{0i}, r_0)} u_i \phi_i \, d_t x = \int_{B_{t_{0i}}(x_{0i}, r_0)} -u_i \Delta_{g(t)} \phi_i \, d_t x \leq |\Delta_{g(t)} \phi_i| \leq C, \tag{4.4.11}$$

where we also used $\int_{\mathcal{M}_{i,t}} u_i(x,t) \, d_t x \leq 1$. Integrating this we get for all $t \in [t_{0i} - cr_0^2, t_{0i})$ that $\int_{B_{t_{0i}}(x_{0i}, r_0)} u_i(x,t) \, d_t x \geq 1 - C(t_{0i} - t)$. Passing to the limit this implies

$$\int_{(B_{t_0}(x_0, r_0))(t)} u(x) \, d_t x \geq 1 - C(t_0 - t). \tag{4.4.12}$$

Letting $t \nearrow t_0$, we get

$$\lim_{t \nearrow t_0} \int_{\mathcal{M}_t} u(x) \, d_t x = 1. \tag{4.4.13}$$

By a same argument we can show that $\lim_{t \nearrow t_0} \int_{\mathcal{M}_t} u(x) h(x) \, d_t x = h(x_0)$, for all smooth function $h$ that has compact support. So $u$ is a $\delta$-function at $x_0$, which verifies assertion (2) in Theorem 4.4.1.

Now we verify assertion (1) in Theorem 4.4.1. First, the positivity of $u$ on $\mathcal{M}(x_0)$ is an easy consequence of the Harnack inequality for parabolic equations. So it remains to show that $u$ vanishes on $\mathcal{M} - \mathcal{M}(x_0)$.

To show this, let $y \in \mathcal{M} - \mathcal{M}(x_0)$, $\mathfrak{t}(y) = t \in [0, t_0)$. Since by [45, Theorem 6.3] the non-singular times, at which the time-slices have bounded curvature, are dense, we may assume without loss of generality that $t$ is a non-singular time. Otherwise, we can find a sequence of non-singular times $s_k > t$ that converges to $t$ as $k \to \infty$, and $y(s_k) \in \mathcal{M} - \mathcal{M}(x_0)$. So $\mathcal{M}_t$ has finitely many connected components which are closed manifolds. Since $y \in \mathcal{M} - \mathcal{M}(x_0)$, it is easy to see that $x_0(t)$ and $y$ are in different connected components in $\mathcal{M}_t$.

Then by [44, Theorem 1.3], there is a sequence of time-preserving diffeomorphisms $\{\phi_i : U_i \to \mathcal{M}_i\}$, where $U_i$ are open subsets of $\mathcal{M}$ such that given any $\bar{t}, \overline{R} > 0$, if $i$ is sufficiently large then

$$U_i \supset \{x \in \mathcal{M} : \mathfrak{t}(x) \leq \bar{t} \text{ and } R(x) \leq \overline{R}\}, \tag{4.4.14}$$

and $\{\phi_i^* g_i\}$ converges smoothly on compact subsets of $\mathcal{M}$ to $g$. So $\mathcal{M}_t$ is contained in $U_i$ for all large $i$ and $\phi_i(\mathcal{M}_t)$ is a finite union of closed manifolds. In particular, $\phi_i(x_0(t))$ and $\phi_i(y)$ are in different connected components in $\phi_i(\mathcal{M}_t)$, which implies $\phi_i(y) \in \mathcal{M}_i - \mathcal{M}(x_{0i})$. So $u_i$ vanishes at $\phi_i(y)$. Letting $i \to \infty$ we get $u(y) = 0$. $\square$

The proof of Theorem 4.4.2 follows along the same line as Theorem 4.4.1.

### 4.4.2 Further properties of the heat kernel

In this subsection we investigate more properties of the heat kernel in Theorem 4.4.1. The first main result is Theorem 4.4.5, which is a semi-local maximum principle for the heat kernel. Then in Theorem 4.4.6 we derive from Theorem 4.4.5 a polynomial decay estimate of the heat kernel. Corollary 4.4.7 and 4.4.8 are applications of Theorem 4.4.6.

The main ingredient in the proof of the semi-local maximum principle is the following vanishing theorem of the Bryant soliton:

**Proposition 4.4.4.** *(Vanishing theorem on Bryant soliton) Let $(M, g(t)), t \in \mathbb{R}$ be a Bryant soliton with tip $x_0 \in M$, $R(x_0, 0) = 1$, and $u(x,t) : M \times [0,\infty) \to \mathbb{R}$ be a smooth non-negative solution to the conjugate heat equation. Suppose there are constants $C > 0$ and $m \in \mathbb{N}_+$ such that $u(x,t)R^m(x,t) \le C$ for all $x \in M$ and $t \in [0,\infty)$.*

*Then $u \equiv 0$.*

*Proof.* Let $C$ denote all the constants that depend only on the constants $C$ and $m$ in the assumption. Without loss of generality, it suffices to show $u(x, 0) = 0$ for all $x \in M$.

Let $H(y, t; z, s)$, $t > s$ be the heat kernel of $(M, g(t))$. Then the following holds for all $t > 0$

$$\begin{aligned} u(x,0) &= \int H(y,t;x,0)u(y,t)\,d_ty \\ &= \int_{d_t(y,x_0)\le 1} H(y,t;x,0)u(y,t)\,d_ty + \int_{d_t(y,x_0)>1} H(y,t;x,0)u(y,t)\,d_ty \\ &= \mathcal{I}(t) + \mathcal{J}(t). \end{aligned} \tag{4.4.15}$$

We shall show that $\mathcal{I}(t)$ and $\mathcal{J}(t)$ converge to 0 as $t \to \infty$. First, to estimate $\mathcal{I}(t)$ we note that $d_t(y, x_0) \le 1$ implies $R(y,t) > C^{-1}$. So it follows from the assumption $uR^m \le C$ and the Bishop-Gromov volume comparison that

$$\mathcal{I}(t) \le C \int_{d_t(y,x_0)\le 1} H(y,t;x,0)\,d_ty \le C \sup_{d_t(y,x_0)\le 1} H(y,t;x,0). \tag{4.4.16}$$

By the Gaussian bound for heat kernel of a Ricci flow with bounded curvature in [16, Theorem 3.1], we have $\lim_{t\to\infty} H(y,t;x,0) = 0$ uniformly for all $y \in M$. So $\mathcal{I}(t) \to 0$, as $t \to \infty$.

Next we estimate $\mathcal{J}(t)$. Since $d_t(y, x_0) \ge 1$, it follows that $C^{-1}d_t(y,x_0)^{-1} \le R(y,t) \le Cd_t(y,x_0)^{-1}$, which together with $uR^m \le C$ implies

$$\mathcal{J}(t) \le C \int_{d_t(y,x_0)>1} H(y,t;x,0)d_t^m(y,x_0)\,d_ty. \tag{4.4.17}$$

We claim that the following estimate holds for all $(y,t)$ such that $t$ is sufficiently large and $d_t(y,x_0)\geq 1$:

$$H(y,t;x,0)\leq C\exp\left(-\frac{d_0^2(x,y)}{Ct}\right). \tag{4.4.18}$$

Assume for a moment that the claim is true, and we use it to prove the proposition. For any $(y,t)$ such that $t$ is sufficiently large and $d_t(y,x_0)\geq 1$, we have $d_s(y,x_0)\geq 1$ for all $s\in[0,t]$ because of $\mathrm{Ric}\geq 0$. Since $R\geq C^{-1}$ on $B_s(x_0,1)$ for any $s\in\mathbb{R}$, by the distance distortion estimate we get

$$\frac{\partial^-}{\partial s}d_s(x_0,y)\leq-\int_0^{d_s(x_0,y)}\mathrm{Ric}(\sigma'(r),\sigma'(r))\,dr\leq-\int_0^1\mathrm{Ric}(\sigma'(r),\sigma'(r))\,dr\leq -C^{-1}, \tag{4.4.19}$$

where $\sigma(r)$ is a minimizing geodesic from $x_0$ to $y$ with respect to $g(s)$. So this implies

$$d_s(x_0,y)\geq C^{-1}(t-s) \tag{4.4.20}$$

for all $s\in[0,t]$. In particular, we have $d_0(x_0,y)\geq C^{-1}t$, and for all $t\geq 2Cd_0(x_0,x)$ and $y\in M$ such that $d_t(x_0,y)\geq 1$ we have

$$d_0(x,y)\geq d_0(x_0,y)-d_0(x,x_0)\geq d_0(x_0,y)(1-\frac{d_0(x,x_0)}{C^{-1}t})\geq\frac{1}{2}d_0(x_0,y), \tag{4.4.21}$$

substituting which into (4.4.18) we get

$$H(y,t;x,0)\leq C\exp\left(-\frac{d_0^2(x_0,y)}{Ct}\right). \tag{4.4.22}$$

Putting this into (4.4.17) and using the Bishop-Gromov volume comparison theorem we get

$$\mathcal{J}(t)\leq\int_{d_0(y,x_0)>C^{-1}t}C\exp\left(-\frac{d_0^2(x_0,y)}{Ct}\right)d_0^m(x_0,y)\ \ d_0y\leq Ce^{-\frac{t}{C}}. \tag{4.4.23}$$

Hence $\mathcal{J}(t)\to 0$ as $t\to\infty$. Therefore, by letting $t\to\infty$, we obtain $u(x,0)=0$.

Now we establish (4.4.18) to finish the proof. Fix a pair $(y,t)$ where $t$ is sufficiently large and $d_t(y,x_0)\geq 1$. The value of $C$ will be determined later. For any $s\in[0,1]$ and $z\in B_s(y,1)$, let $\ell(z,s)$ be the reduced length from $(z,s)$ to $(y,t)$, and let $\gamma:[s,t]\to M$ be a curve such that $\gamma|_{[2,t]}\equiv y$ and $\gamma|_{[s,2]}$ is a minimal geodesic connecting $y$ and $z$ with respect to $g(0)$.

For $\tau\in[0,t-s]$, we have $R(y,t-\tau)\leq Cd_{t-\tau}^{-1}(y,x_0)$ and $d_{t-\tau}(y,x_0)\geq C^{-1}\tau$, and hence $R(y,t-\tau)\leq\frac{C}{\tau}$. For $\tau\in[t-2,t-s]$, we have $R(\gamma(t-\tau),t-\tau)\leq C$ and $|\gamma'|(t-\tau)\leq C$. Putting these together we can estimate the $\mathcal{L}$-length of $\gamma$:

$$\begin{aligned}\mathcal{L}(\gamma)&=\int_0^{t-2}\sqrt{\tau}R(y,\tau)\,d\tau+\int_{t-2}^{t-s}\sqrt{\tau}(R(\gamma(t-\tau),t-\tau)+|\gamma'|^2)\,d\tau\\&\leq\int_0^{t-2}\sqrt{\tau}\frac{C}{\tau}\,d\tau+\int_{t-2}^{t-s}C\sqrt{\tau}\,d\tau\leq C\sqrt{t},\end{aligned} \tag{4.4.24}$$

and hence

$$\ell(z,s) = \frac{\mathcal{L}(z,s)}{2\sqrt{t-s}} \le \frac{\mathcal{L}(\gamma)}{2\sqrt{t-s}} \le C. \tag{4.4.25}$$

Recall the heat kernel lower bound by Perelman in [58, Corollary 9.5] we get:

$$H(y,t;z,s) \ge \frac{1}{4\pi(t-s)^{3/2}} e^{-\ell(z,s)} \ge \frac{C}{t^{\frac{3}{2}}}, \tag{4.4.26}$$

for all $s \in [0,1]$ and $z \in B_s(y,1)$. So by the Bishop-Gromov volume comparison we get

$$\int_{B_s(y,1)} H(y,t;z,s)\, d_s z \ge \frac{C}{t^{3/2}} \ge \frac{C}{d_0(x,y)^{3/2}}, \tag{4.4.27}$$

for all $s \in [0,1]$.

Note by the multiplication inequality for the heat kernel in [40, Theorem 1.30] we have

$$\left(\int_{B_s(y,1)} H(y,t;z,s)\, d_s z\right)\left(\int_{B_s(x,1)} H(y,t;z,s)\, d_s z\right) \le C \exp\left(-\frac{(d_s(x,y)-2)^2}{4C(t-s)}\right). \tag{4.4.28}$$

So by substituting (4.4.27) into (4.4.28) we get

$$\left(\int_{B_s(x,1)} H(y,t;z,s)\, d_s z\right) \le C\, d_0(x,y)^{\frac{3}{2}} \exp\left(-\frac{d_0(x,y)^2}{4C(t-s)}\right) \le C \exp\left(-\frac{d_0(x,y)^2}{4C(t-s)}\right), \tag{4.4.29}$$

where we also used $d_0(x,y) \ge C^{-1}t \ge 4C_0$ for very large $t$, and hence

$$d_s(x,y) - 2 \ge C^{-1} d_0(x,y) - 2 \ge (2C)^{-1} d_0(x,y). \tag{4.4.30}$$

Integrating this for all $s \in [0,1]$, and then applying the parabolic mean value inequality (see e.g. [28]) to $H(y,t;\cdot,\cdot)$ at $(x,0)$, we obtain

$$H(y,t;x,0) \le C \exp\left(-\frac{d_0^2(x,y)}{4Ct}\right), \tag{4.4.31}$$

which confirms claim (4.4.18) and hence completes the proof. □

**Proposition 4.4.5.** *(A semi-local Maximum Principle) Given $r_0, T > 0$, $r_0^2 < T$ and $m \in \mathbb{N}$, there exist $\epsilon = \epsilon(r_0,T,m) > 0$ and $r = r(r_0,T,m) > 0$ such that for any $t_0 \in (r_0^2, T)$ the following holds:*

*Let $\mathcal{M}$ be a singular Ricci flow with normalized initial condition, $H$ be the heat kernel. Let $x_0 \in \mathcal{M}$, $\mathfrak{t}(x_0) = t_0 > 0$, and $|\mathrm{Rm}| \le r_0^{-2}$ on $\mathcal{P}_0 := P(x_0, r_0, -r_0^2)$. Then for any $x \in \mathcal{M}_{t<t_0} - \mathcal{P}_0$ with $R(x) > r^{-2}$, there exists $y \in \mathcal{M}_{t<t_0} - \mathcal{P}_0$ with $\mathfrak{t}(y) \ge \mathfrak{t}(x)$ such that*

$$\begin{cases} H(x_0,y) \ge (1+\epsilon) H(x_0,x) \quad \text{and} \\ H(x_0,y) R^m(y) \ge (1+\epsilon) H(x_0,x) R^m(x). \end{cases} \tag{4.4.32}$$

*Proof.* Suppose the conclusion is not true, then there are sequences $\{\epsilon_k\}$ and $\{r_k\}$ both converging to zero, and a sequence of Ricci flow spacetimes $(\mathcal{M}_k, g_k(t))$, $x_{0k} \in \mathcal{M}_k$, $t_{0k} = \mathfrak{t}(x_{0k})$, along with the heat kernels $H_k$, which together contradict the lemma at points $x_k \in \mathcal{M}_{k,t<t_{0k}} - \mathcal{P}_{0k}$, $\mathfrak{t}(x_k) = t_k$. This means that $\rho(x_k) < r_k$, and for any $y \in \mathcal{M}_{k,t<t_{0k}} - \mathcal{P}_{0k}$ with $\mathfrak{t}(y) \geq t_k$, we have either

$$u_k(y) < (1+\epsilon_k)u_k(x_k), \tag{4.4.33}$$

where $u_k = H_k(x_{0k}, \cdot)$, or

$$(u_k R^m)(y) < (1+\epsilon_k)(u_k R^m)(x_k). \tag{4.4.34}$$

This implies $u(x_k) > 0$, and by item (1) in Theorem 4.4.1 we get $x_k \in \mathcal{M}_k(x_{0k})$.

Let $\rho_0 = \min\{\frac{1}{2}r_0, \overline{r}_{\epsilon_{can}}(T+1)\}$, where $\overline{r}_{\epsilon_{can}}(t)$ is the canonical neighborhood scale function for singular Ricci flow with normalized initial condition in Lemma 2.2.10, and $\epsilon_{can} > 0$ is sufficiently small. Let $P_k = P(x_k, \frac{\rho_0}{4\eta}, \frac{\rho_0^2}{4\eta}) \subset \mathcal{M}_k$, where $\eta > 0$ is from Lemma 2.2.5. Then by Lemma 2.2.5 we have $\rho < r_0$ on $P_k$ for all large $k$. Seeing that $\rho \geq r_0$ on $\mathcal{P}_{0k}$, it implies that $P_k \subset \mathcal{M}_{k,t<t_{0k}} - \mathcal{P}_{0k}$. Now rescaling the spacetimes in $P_k$ by $R(x_k)$ and shifting the times $t_k$ to 0, we get a sequence of Ricci flow spacetimes $(\tilde{P}_k, \tilde{g}_k(s)_{s\geq 0}, (x_k, 0))$, where $\tilde{g}_k(s)$ denotes the horizontal Riemannian metric. Since $r_k \to 0$ as $k \to \infty$, by Lemma 4.2.5 we may assume by passing to a subsequence that $(\tilde{P}_k, \tilde{g}_k(s)_{s\geq 0}, (x_k, 0))$ converges to a $\kappa$-solution $(M_\infty, g_\infty(s)_{s\geq 0}, (x_\infty, 0))$.

Let $\tilde{u}_k(y) = \frac{u_k(y)}{u_k(x_k)}$ for all $y \in \tilde{P}_k$. Then for all $y \in \tilde{P}_k$ we have either

$$\tilde{u}_k(y) < 1 + \epsilon_k, \tag{4.4.35}$$

or

$$(\tilde{u}_k R^m)(y) < 1 + \epsilon_k. \tag{4.4.36}$$

Then since $R > 0$ on $(M_\infty, g_\infty(s), x_\infty)$, we deduce that $\tilde{u}_k$ has locally bounded $C^0$-norm. By Hölder estimate this implies that the $C^k$-norm of $\tilde{u}_k$ is locally bounded bound for any $k \in \mathbb{N}$. So by passing to a subsequence we may assume that $\tilde{u}_k$ converges smoothly to a smooth non-negative solution $\tilde{u}$ to the conjugate heat equation of the flow $(M_\infty, g_\infty(s)_{s\geq 0}, x_\infty)$, and $\tilde{u}(x_\infty, 0) = 1$. Since $\epsilon_k \to 0$ as $k \to \infty$, we have that for all $y \in M_\infty$ and $s \geq 0$ one of the following holds

$$\tilde{u}(y,s) \leq 1, \tag{4.4.37}$$

or

$$(\tilde{u}R^m)(y,s) \leq 1. \tag{4.4.38}$$

We claim that $(M_\infty, g_\infty(s))$ is either a cylindrical solution (the standard solution on $S^2 \times \mathbb{R}$, or its quotient by the map that is a reflection on $\mathbb{R}$ and an antipodal map on $S^2$), or the Bryant soliton. Using the classification result of non-compact $\kappa$-solutions [8], it suffices to show that $M_\infty$ is not compact.

Suppose this is not true. On the one hand, by the compactness of $M_\infty$, for large $k$ there exists a diffeomorphism $\phi_k : M_\infty \to U_k$ such that $\phi_k(x_\infty) = x_k$, where $U_k = \phi_k(M_\infty)$ is a connected component in $\mathcal{M}_{k,t_k}$ and $x_k \in U_k$. Also, for any given $\delta > 0$, the following holds:

$$\|r_k^{-2}\phi_k^*(g_k(t_k)) - g_\infty(0)\|_{C^{[\delta^{-1}]}(M_\infty, g_\infty(0))} \le \delta. \tag{4.4.39}$$

On the other hand, since $u_k(x_k) > 0$ and by item (1) in Theorem 4.4.1 we see that $x_k \in \mathcal{M}_k(x_{0k})$. By the component stability theorem, [44, Proposition 5.32], for any $t < t_{0k}$, the time-t-slice of $\mathcal{M}_k(x_{0k})$ is the connected component of $\mathcal{M}_{k,t}$ that contains $x_{0k}(t)$. So we deduce that $U_k$ is equal to $\mathcal{M}_k(x_{0k})(t_k)$, which is the time-$t_k$-slice of $\mathcal{M}_k(x_{0k})$.

Since $\inf_{M_\infty} R(\cdot, 0) \ge c$ for some $c > 0$, by (4.4.39) we get

$$\inf_{\mathcal{M}_k(x_{0k})(t_k)} R \ge \frac{1}{2} c r_k^{-2}, \tag{4.4.40}$$

for all large $k$. Then by the maximum principle for scalar curvature we get

$$R(x_{0k}) \ge \inf_{\mathcal{M}_k(x_{0k})(t \ge t_k)} R \ge \frac{1}{2} c r_k^{-2}. \tag{4.4.41}$$

For sufficiently large $k$, this contradicts the assumption $R(x_{0k}) \le r_0^{-2}$. So $M_\infty$ must be non-compact.

So first we suppose $(M_\infty, g_\infty(s)), s \in [0,\infty)$ is the Bryant soliton. Since the curvature is uniformly bounded everywhere, if $\tilde{u}(y,s) \le 1$ for some $(y,s) \in M_\infty \times [0,\infty)$, then

$$\tilde{u}(y,s) R^m(y,s) \le C, \tag{4.4.42}$$

where $C$ depends only on the curvature at the tip of $(\mathcal{M}_\infty, g_\infty(0))$. Combining with (4.4.38), we see that (4.4.42) holds at all $(y,s) \in M_\infty \times [0,\infty)$. By the vanishing theorem, Proposition 4.4.4, we get $\tilde{u}(x_\infty, 0) = 0$, contradiction.

Next, suppose $(M_\infty, g_\infty(s))$, is a cylindrical solution with $R(x_\infty, 0) = 1$. Then the flow exists on $[0, \frac{3}{2})$, and $R(y,s) \ge 1$ for all $y \in M_\infty$ and $s \in [0, \frac{3}{2})$. So (4.4.38) implies (4.4.37), and hence $\tilde{u}(y,s) \le 1$ for all $(y,s) \in M_\infty \times [0, \frac{3}{2})$. Noting $\tilde{u}(x_\infty, 0) = 1$, we can apply the maximum principle at $(x_\infty, 0)$ and get

$$(-\frac{\partial}{\partial t} - \Delta)\tilde{u} \ge 0, \quad \text{at} \quad (x_\infty, 0). \tag{4.4.43}$$

This is impossible seeing that $(-\frac{\partial}{\partial t} - \Delta + R)\tilde{u} = 0$ and $\tilde{u}(x_\infty, 0) R(x_\infty, 0) > 0$. □

**Theorem 4.4.6.** *Given $r_0, T > 0$, $r_0^2 < T$, $m \in \mathbb{N}$, and $K > 0$, there exist $C_m = C(r_0, T, m, K) > 0$ such that for any $t_0 \in (r_0^2, T)$ the following holds:*

*Let $\mathcal{M}$ be a singular Ricci flow, and $H$ be the heat kernel. Let $x_0 \in \mathcal{M}$, $\mathfrak{t}(x_0) = t_0 > 0$, and $|\mathrm{Rm}| \le r_0^{-2}$ on $\mathcal{P}_0 := P(x_0, r_0, -r_0^2)$. Suppose $(\mathcal{M}, Kg(K^{-1}t))$ has normalized initial condition. Then the following holds for all $x \in \mathcal{M}_{t<t_0} - \mathcal{P}_0$:*

$$H(x_0, x) R^m(x) \le C_m. \tag{4.4.44}$$

*Proof.* Let $C_m = C_0 r^{-2m}$, where $C_0 = C_0(r_0, T)$ is from item (3) in Theorem 4.4.1, and $r = r(r_0, T, m) > 0$ is from Theorem 4.4.5. Let $u = H(x_0, \cdot)$. It is clear that $uR^m \le C_m$ for all the points in $\mathcal{M}_{t<t_0} - \mathcal{P}_0$ that satisfy $R \le r^{-2}$. We shall show that $uR^m \le C_m$ holds everywhere on $\mathcal{M}_{t<t_0} - \mathcal{P}_0$. Suppose by contradiction that this is not true. Then there exists $x_1 \in \mathcal{M}_{t<t_0} - \mathcal{P}_0$ such that $uR^m(x_1) > C_m$ and $R(x_1) > r^{-2}$.

Suppose by induction that there are $\{x_k\} \subset \mathcal{M}_{t<t_0} - \mathcal{P}_0$, $t_k = \mathfrak{t}(x_k)$, $k = 1, 2, ..., N$, such that $t_k \ge t_{k-1}$, and the following holds for all $k$:

$$\begin{cases} uR^m(x_k) \ge (1+\epsilon) uR^m(x_{k-1}), \quad \text{and} \\ u(x_k) \ge (1+\epsilon) u(x_{k-1}), \end{cases} \tag{4.4.45}$$

where $\epsilon = \epsilon(r_0, T, m) > 0$ is from Theorem 4.4.5. Since $uR^m(x_N) \ge uR^m(x_1) > C_m$, it follows from the definition of $C_m$ that $R(x_N) > r^{-2}$. This allows us to apply Proposition 4.4.5 and get a point $x_{N+1} \in \mathcal{M}_{t<t_0} - \mathcal{P}_0$, $\mathfrak{t}(x_{N+1}) = t_{N+1} \ge t_N$ which together with $x_N$ satisfies (3.2.16). So by induction we get an infinite sequence $\{x_k\}_{k=1}^{\infty}$ which satisfies (3.2.16). Then we can deduce from the second inequality in (3.2.16) that $u(x_k) \to \infty$ as $k \to \infty$, which contradicts item (3) in Theorem 4.4.1. □

**Corollary 4.4.7.** *Let $\mathcal{M}$ be a singular Ricci flow, and $H$ be the heat kernel. Let $x_0 \in \mathcal{M}$, $\mathfrak{t}(x_0) = t_0 > 0$. Then*

$$\int_{\mathcal{M}_t} H(x_0, x)\, d_t x = 1, \tag{4.4.46}$$

*for all $t \in [0, t_0)$.*

*Proof.* Without loss of generality we may assume $\mathcal{M}$ has normalized initial condition. After proper rescaling we may assume the assumptions in Proposition 4.4.5 holds. It suffices to show the claim for $t = 0$. First, we fix some small $\delta_\# > 0$ and $\epsilon < \epsilon_{can}(\delta_\#)$ from Lemma 2.2.6. Let $\eta$ be from Lemma 2.2.5. Let $m \in \mathbb{N}$ be greater than 1. We use $C$ to denote all the constants depending on $\delta_\#, r_0, T, m$ and $vol(\mathcal{M}_0)$.

Let $\delta > 0$, whose value will be determined in the course of the proof. Choose a division of $[0, t_0]$ by $0 = t_1 < t_2 < ... < t_N = t_0$, such that $t_{i+1} - t_i \le \delta^2$ for all $i = 1, ..., N-1$, and $N \le (t_0 + 1)\delta^{-2} \le (T+1)\delta^{-2}$.

Let $\lambda = \lambda(\delta_\#), \Lambda = \Lambda(\delta_\#) > 0$ be from Lemma 2.2.8, and assume $\delta$ sufficiently small such that $r_0 := \frac{2\sqrt{\eta}\delta}{\lambda} < \frac{1}{\Lambda}\overline{r}_\epsilon(T)$, where $\overline{r}_\epsilon(t)$ is the function from Lemma 2.2.10. Then we can apply Lemma 2.2.8 to $\mathcal{M}_{t_{2i+1}}$ and find a collection of central spheres $\{\Sigma_k\}_{k=1}^{N_0}$ of $\delta_\#$-necks of curvature scale $r_0$, an open domain $\Omega \subset \mathcal{M}_{t_{2i+1}}$ whose boundary is a union of $\{\Sigma_k\}_{k=1}^{N_0}$ such that $N_0 \le C$, $\rho \ge \lambda r_0 = 2\sqrt{\eta}\delta$ on $\Omega$, and $\rho \le \Lambda r_0 \le C\delta$ on $\mathcal{M}_{t_{2i+1}} - \Omega$. So we have $\text{Area}_{t_{2i+1}}(\partial\Omega) \le C\delta^2$. Moreover, by Lemma 2.2.5 we see that $\sqrt{\eta}\delta \le \rho(x(t)) \le C\delta$ for all $x \in \Omega$ and $t \in [t_{2i}, t_{2i+1}]$. So $\Omega$ survives until time $t_{2i}$, and $\text{Area}_t(\partial\Omega(t)) \le C\delta^2$.

Let $u(x) = H(x_0, x)$. Applying Theorem 4.4.6 on $\partial\Omega(t)$ and using the interior Hölder estimate, we get $|\nabla u| \leq C\delta^{2m-1}$ on $\partial\Omega(t)$. Let $t \in [t_{2i}, t_{2i+1}]$, then

$$\frac{\partial}{\partial t}\int_{\Omega(t)} u(x)\, d_t x = \int_{\Omega(t)} -\Delta u(x)\, d_t x = \int_{\partial\Omega(t)} \frac{\partial u}{\partial \vec{n}}\, d_t S \leq C \cdot \delta^{2m-2}, \tag{4.4.47}$$

where $\vec{n}$ is the inwards unit normal vector field on $\partial\Omega(t)$. Integrating this on $[t_{2i}, t_{2i+1}]$, we get

$$\int_{\Omega(t_{2i})} u(x)\, d_{t_{2i}} x \geq \int_{\Omega} u(x)\, d_{t_{2i+1}} x - C\delta^{2m}. \tag{4.4.48}$$

Also, applying Theorem 4.4.6 on $\mathcal{M}_{t_{2i+1}} - \Omega$ and using $vol(\mathcal{M}_{t_{2i+1}}) \leq C$, we get

$$\int_{\mathcal{M}_{t_{2i+1}}-\Omega} u(x)\, d_{t_{2i+1}} x \leq C\delta^{2m}, \tag{4.4.49}$$

which combining with (4.4.48) gives

$$\int_{\mathcal{M}_{t_{2i}}} u(x)\, d_{t_{2i}} x \geq \int_{\mathcal{M}_{t_{2i+1}}} u(x)\, d_{t_{2i+1}} x - C\delta^{2m}. \tag{4.4.50}$$

Note $\lim_{t \nearrow t_0} u(x)\, d_t x = 1$, by induction we have

$$\int_{\mathcal{M}_0} u(x)\, d_0 x \geq 1 - CN\delta^{2m} \geq 1 - C(T+1)\delta^{2m-2}. \tag{4.4.51}$$

Letting $\delta \to 0$, the conclusion follows immediately. □

**Corollary 4.4.8.** *Let $\mathcal{M}$ be a singular Ricci flow, and $H$ be the heat kernel. For any $x_0 \in \mathcal{M}$, $\mathfrak{t}(x_0) > 0$, let $f$ be a smooth function on $\mathcal{M}(x_0)$ such that $H(x_0, x) = (4\pi(t_0 - t))^{-3/2} e^{-f(x)}$ for all $x \in \mathcal{M}(x_0)$, where $t = \mathfrak{t}(x)$. Then*

$$v = [(t_0 - t)(2\Delta f - |\nabla f|^2 + \mathbb{R}) + f - n]H(x_0, \cdot) \leq 0. \tag{4.4.52}$$

*Proof.* Since $H(x_0, x) > 0$ for all $x \in \mathcal{M}(x_0)$, we see that $f$ is well defined. Suppose the assertion does not hold. Then without loss of generality we may assume that there exists $x_1 \in \mathcal{M}_0$ such that $v(x_1) > 0$. Let $h_0 \geq 0$ be a smooth function on $\mathcal{M}_0$ which is supported in a neighborhood of $x_1$ in which $v > 0$, and $h_0(x_1) > 0$. Then $\int_{\mathcal{M}_0} h_0 v\, d_0 x > 0$. In the same way we constructed $u$, we can find a smooth and bounded function $h \geq 0$ on $\mathcal{M}$ with $h(x) = h_0(x)$ for all $x \in \mathcal{M}_0$, which solves the heat equation $(\frac{\partial}{\partial t} - \Delta)h = 0$.

Since $(-\frac{\partial}{\partial t} - \Delta + \mathbb{R})v \leq 0$, see e.g. [43, Prop 29.5], for any open domain $\Omega \subset \mathcal{M}_t$ with smooth boundary, we have

$$\frac{\partial}{\partial t}\int_{\Omega} -hv\, d_t V \leq \int_{\partial\Omega} \left(\frac{\partial h}{\partial \vec{n}} v - \frac{\partial v}{\partial \vec{n}} h\right) d_t S. \tag{4.4.53}$$

Applying Theorem 4.4.6 in the same way as Corollary 4.4.7, we get

$$\int_{\mathcal{M}_t} hv\, d_t x \le \lim_{s \nearrow t_0} \int_{\mathcal{M}_s} hv\, d_s x \tag{4.4.54}$$

for all $t \in [0, t_0)$. It was shown in [57] that $\int_{\mathcal{M}_t} hv\, d_t x$ approaches to zero as $t$ goes up to $t_0$. So (4.4.54) implies $\int_{\mathcal{M}_0} hv\, d_0 x \le 0$, a contradiction.

□

As another corollary of Theorem 4.4.6, we establish the symmetry between heat kernel and adjoint heat kernels on singular Ricci flow, as well as a semigroup property of the heat kernel.

**Corollary 4.4.9.** *(Symmetry and semigroup property of the heat kernel) Let $\mathcal{M}$ be a singular Ricci flow and $H, H^*$ be the heat kernel and adjoint heat kernel in Theorem 4.4.1 and 4.4.2. Then for any $x, y \in \mathcal{M}$ with $\mathfrak{t}(x) > \mathfrak{t}(y)$ we have*

$$H(x,y) = H^*(y,x), \tag{4.4.55}$$

*and for all $t \in (\mathfrak{t}(y), \mathfrak{t}(x))$ we have*

$$H(x,y) = \int_{\mathcal{M}_t} H(x,z)\, H(z,y)\, d_t z. \tag{4.4.56}$$

*Proof.* Note that for any open domain $\Omega \subset \mathcal{M}_t$ with smooth boundary we get by computation

$$\frac{\partial}{\partial t}\int_{\Omega} H(x,z)\, H^*(y,z)\, d_t z = \int_{\Omega} (-H^*\Delta_z H + H\Delta_z H^*)\, d_t z = \int_{\partial\Omega} \left( H^* \frac{\partial H}{\partial \vec{n}} - H\frac{\partial H^*}{\partial \vec{n}} \right) d_t S. \tag{4.4.57}$$

Hence, applying Theorem 4.4.6 as in Corollary 4.4.7, we get that $\int_{\mathcal{M}_t} H(x,z)\, H^*(y,z)\, d_t z$ is constant in $t$. Using (4.4.2) and (4.4.3), we see that $H^*(y,x)$ and $H(x,y)$ are the limits of $\int_{\mathcal{M}_t} H(x,z)\, H^*(y,z)\, d_t z$ as $t \nearrow \mathfrak{t}(x)$ and $t \searrow \mathfrak{t}(y)$. So we have

$$H^*(y,x) = H(x,y), \tag{4.4.58}$$

from which (4.4.56) follows immediately. □

## 4.5 Pseudolocality theorem on singular Ricci flow

In this section, we generalize Perelman's pseudolocality theorem for compact Ricci flows to singular Ricci flows. The main ingredient is the heat kernel in Section 4.4, especially Corollary 4.4.7 and 4.4.8.

**Theorem 4.5.1.** *(Pseudolocality theorem) For every $\alpha > 0$ there exists $\delta, \epsilon > 0$ with the following property. Let $(\mathcal{M}, g(t))$ be a singular Ricci flow and $x_0 \in \mathcal{M}_{t_0}$ for some $t_0 \geq 0$. Suppose $R \geq -1$ on $B_{t_0}(x_0, 2)$, and for any $\Omega \subset B_{t_0}(x_0, 2)$ we have $vol(\partial\Omega)^3 \geq (1-\delta)c_3 vol(\Omega)^2$, where $c_3$ is the Euclidean isoperimetric constant at dimension 3. Then $\bigcup_{t\in[t_0,t_0+\epsilon^2]} B_t(x_0(t), \epsilon)$ is unscathed, and $|\mathrm{Rm}|(x) < \frac{\alpha}{\mathfrak{t}(x)-t_0} + \epsilon^{-2}$ holds for all $x \in \bigcup_{t\in[t_0,t_0+\epsilon^2]} B_t(x_0(t), \epsilon)$.*

**Corollary 4.5.2.** *Under the same assumption as in Theorem 4.5.1. Assume furthermore that $|\mathrm{Rm}| \leq 1$ on $B_{t_0}(x_0, 2)$. Then there is $r_0 > 0$ such that $P(x_0, r_0, r_0^2)$ is unscathed and $|\mathrm{Rm}| \leq r_0^{-2}$ on $P(x_0, r_0, r_0^2)$.*

*Proof of Corollary 4.5.2.* The corollary follows immediately from Theorem 4.5.1 and a local curvature estimate, see e.g. [20, Theorem 3.1]. □

As the proof has a lot in common with that of Perelman's pseudolocality theorem, we will focus on the differences, especially the places where the generalized heat kernel comes into play, see [43, Section 30-34] for details of the parts which we are brief about.

*Proof.* Without loss of generality, we assume $t_0 = 0$, and $\alpha < \frac{1}{300}$. Suppose the assertion is not true. Then there are sequences $\epsilon_k \to 0$ and $\delta_k \to 0$, and pointed singular Ricci flows $(\mathcal{M}_k, (x_{0k}, 0), g_k(\cdot))$ which satisfy the hypotheses of the theorem but for which there is a point $x_k$ in the unscathed set $\bigcup_{t\in[0,\epsilon_k^2]} B_t(x_{0k}(t), \epsilon_k)$ with $|\mathrm{Rm}|(x_k) \geq \alpha t_k^{-1} + \epsilon_k^{-2}$. By reducing $\epsilon_k$ if needed, we may also assume that

$$|\mathrm{Rm}|(x) < \alpha t_k^{-1} + 2\epsilon_k^{-2}, \tag{4.5.1}$$

for all $x \in \bigcup_{t\in[0,\epsilon_k^2]} B_t(x_{0k}(t), \epsilon_k)$. We abbreviate $d_t(x_{0k}(t), x)$ as $d(x,t)$.

Let $A_k = \frac{1}{300\epsilon_k}$. We say a point $y$ is an $\alpha$-large point if $|\mathrm{Rm}|(y) \geq \frac{\alpha}{\mathfrak{t}(y)}$. First, suppose $\mathcal{P}_k := \bigcup_{t\in[0,\epsilon_k^2]} B_t(x_{0k}(t), (2A_k+1)\epsilon_k)$ is unscathed. Then by a point-picking we can find an $\alpha$-large point $\overline{x}_k \in \mathcal{P}_k$, $\mathfrak{t}(\overline{x}_k) = \overline{t}_k$, such that

$$|\mathrm{Rm}|(y) \leq 4|\mathrm{Rm}|(\overline{x}_k) := 4Q_k, \tag{4.5.2}$$

holds for all $\alpha$-large points $y$, $\mathfrak{t}(y) = s$, with $s \in (0, \overline{t}_k]$ and $d(y,s) \leq d(\overline{x}_k, \overline{t}_k) + A_k Q_k^{-1/2}$. By a distance distortion estimate we can show that (4.5.2) also holds on $P(\overline{x}_k, \frac{1}{10}A_k Q_k^{-1/2}, -\frac{1}{2}\alpha Q_k^{-1/2})$.

Next, suppose $\mathcal{P}_k$ is scathed. They by Lemma 4.2.5, we can also find an $\alpha$-large point $\overline{x}_k \in \mathcal{P}_k$ so that for large $k$, (4.5.2) holds on $P(\overline{x}_k, \frac{1}{10}A_k Q_k^{-1/2}, -\frac{1}{2}\alpha Q_k^{-1/2})$.

Let $H_k$ be the heat kernel of $\mathcal{M}_k$, whose existence is given by Theorem 4.4.1. Let $u_k = H_k(\overline{x}_k, \cdot)$ and $f_k$ be such that $u_k = (4\pi(\overline{t}_k - t))^{-n/2} e^{-f_k}$, and $v_k$ be defined by (4.4.52). The following lemma says that a local integral of $v_k$ has a negative upper bound at some time earlier than $\overline{t}_k$.

**Lemma 4.5.3.** *([43, Lemma 33.4]) There is some $\beta > 0$ so that for all sufficiently large $k$, there is some $\tilde{t}_k \in [\bar{t}_k - \frac{1}{2}\alpha Q_k^{-1}, \bar{t}_k)$ with $\int_{B_k} v_k \, dV_k \leq -\beta$, where $B_k$ is the time-$\tilde{t}_k$ ball of radius $\sqrt{\bar{t}_k - \tilde{t}_k}$ centered at $\bar{x}_k(\tilde{t}_k)$.*

We drop the subscript $k$ for a moment and consider a fixed $\mathcal{M}_k$ for $k$ large. Let $\phi$ be a smooth non-increasing function on $\mathbb{R}$ such that: $\phi$ is 1 on $(-\infty, 1]$ and 0 on $[2, \infty)$, and $\phi'' \geq -10\phi$ and $(\phi')^2 \leq 10\phi$. Put $h(y) = \phi\left(\frac{d(y,\mathfrak{t}(y))+600\sqrt{\mathfrak{t}(y)}}{10A\epsilon}\right)$ on $\mathcal{M}_{t\leq\epsilon^2}$. Then

$$(\frac{\partial}{\partial t} - \Delta)h = \frac{1}{10A\epsilon}\left(d_t - \Delta d + \frac{300}{\sqrt{t}}\right)\phi' - \frac{1}{(10A\epsilon)^2}\phi''. \tag{4.5.3}$$

By (4.5.1) and Lemma 2.3.1, we get

$$d_t - \Delta d + \frac{300}{\sqrt{t}} \geq 0, \tag{4.5.4}$$

for all points $y$, $t = \mathfrak{t}(y)$, such that $d(y,t) > \epsilon \geq \sqrt{t}$. In particular, if $\phi'\left(\frac{d(y,t)+600\sqrt{t}}{10A\epsilon}\right) \neq 0$, then $9A\epsilon < d(y,t) < 20A\epsilon$, and hence (4.5.4) holds at the point. So we have

$$(\frac{\partial}{\partial t} - \Delta)h \leq \frac{-\phi''}{(10A\epsilon)^2} \leq \frac{10\phi}{(10A\epsilon)^2}. \tag{4.5.5}$$

First, for any open domain $\Omega \subset \mathcal{M}_t$ with smooth boundary, we can compute that

$$\frac{\partial}{\partial t}\int_\Omega hu \, d_tV = \int_\Omega (\frac{\partial}{\partial t}h - \Delta h)u \, d_tV + \int_{\partial\Omega}\left(-\frac{\partial h}{\partial \vec{n}}u + \frac{\partial u}{\partial \vec{n}}h\right) d_tS, \tag{4.5.6}$$

where $\vec{n}$ is the inwards unit normal vector field on $\partial\Omega$. By (4.5.5), $h \leq 1$, $|\nabla h| \leq \frac{\phi'}{10A\epsilon}$, and a same argument as in Corollary 4.4.7 using Theorem 4.4.6, we get

$$\int hu \, d_tV\bigg|_{t=0} \geq \int hu \, d_tV\bigg|_{t=\bar{t}} - \frac{\bar{t}}{(A\epsilon)^2} \geq 1 - A^{-2}. \tag{4.5.7}$$

Similarly, by using $(-\partial_t - \Delta + R)v \leq 0$ we get

$$\frac{\partial}{\partial t}\int_\Omega -hv \, d_tV \leq \int_{\partial\Omega}\left(\frac{\partial h}{\partial \vec{n}}v - \frac{\partial v}{\partial \vec{n}}h\right) d_tS. \tag{4.5.8}$$

Suppose $t$ is not a singular time, this implies

$$\frac{\partial}{\partial t}\int -hv \, d_tV \leq \frac{10}{(10A\epsilon)^2}\int -hv \, d_tV, \tag{4.5.9}$$

and hence

$$\int -hv\, d_tV\Big|_{t=0} \geq \exp\left(\frac{-\tilde{t}}{10(A\epsilon)^2}\right) \int -hv\, d_tV\Big|_{t=\tilde{t}} \geq (1-A^{-2}) \int -hv\, d_tV\Big|_{t=\tilde{t}}. \tag{4.5.10}$$

Also, replacing the function $h$ by $\overline{h}(y) = \phi\left(\frac{d(y,\mathrm{t}(y))+600\sqrt{\mathrm{t}(y)}}{5A\epsilon}\right)$, we can show for some constant $C > 0$,

$$\int_{B_0(x_0,10A\epsilon)} u\, dV \geq 1 - CA^{-2}. \tag{4.5.11}$$

By some distance distortion estimates and Lemma (4.5.1) we can establish the following inclusion

$$B_{\tilde{t}}(\overline{x}(\tilde{t}), \sqrt{\overline{t}-\tilde{t}}) \subset B_{\tilde{t}}(x_0(\tilde{t}), 9A\epsilon). \tag{4.5.12}$$

Since $h(\cdot,\tilde{t}) = 1$ on $B_{\tilde{t}}(x_0(\tilde{t}), 9A\epsilon)$ and $v \leq 0$, Lemma 4.5.3 implies $\int -hv\, d_tV\big|_{t=\tilde{t}} \geq \beta$. Hence by (4.5.10) we get

$$\int -hv\, d_tV\Big|_{t=0} \geq \beta(1-A^{-2}). \tag{4.5.13}$$

Let $\tilde{u}(x) = h(x)u(x)$ for all $x \in \mathcal{M}_0$, and define $\tilde{f}(x)$ by $\tilde{u} = (2\pi)^{-\frac{n}{2}}e^{-\tilde{f}}$. Then a direct computation shows

$$\int_{\mathcal{M}_0} -hv\, d_0V = \int_{\mathcal{M}_0} (-\overline{t}|\nabla\tilde{f}|^2 - \tilde{f} + 3)\tilde{u}\, d_0V + \int_{\mathcal{M}_0} \left(\overline{t}\left(\frac{|\nabla h|^2}{h} - Rh\right) - h\log h\right) u\, d_0V. \tag{4.5.14}$$

By (4.5.11) and $-h\log h \leq 1$ when $h \leq 1$, we have

$$\begin{aligned}\int_{\mathcal{M}_0} -uh\log h\, d_0V &= \int_{B_0(x_0,20A\epsilon)-B_0(x_0,10A\epsilon)} -uh\log h\, d_0V \leq \int_{\mathcal{M}_0-B_0(x_0,10A\epsilon)} u\, d_0V \\ &\leq 1 - \int_{B_0(x_0,10A\epsilon)} u\, d_0V \leq CA^{-2}.\end{aligned} \tag{4.5.15}$$

Seeing also that $\frac{|\nabla h|^2}{h} \leq \frac{10}{(10A\epsilon)^2}$, and $R \geq -1$ on $B_0(x_0, 20A\epsilon)$, we can bound the second integral in (4.5.14) above by $(1+C)A^{-2} + \epsilon^2$. This combining with (4.5.13) implies

$$\beta(1-A^{-2}) \leq \int_{\mathcal{M}_0} (-\overline{t}|\nabla\tilde{f}|^2 - \tilde{f} + 3)\tilde{u}\, dV + (1+C)A^{-2} + \epsilon^2. \tag{4.5.16}$$

Put $\widehat{g} = \frac{1}{2\overline{t}}g(0)$, $\widehat{u} = (2\overline{t})^{\frac{n}{2}}\tilde{u}$, and define $\widehat{f}$ by $\widehat{u} = (2\pi)^{-\frac{n}{2}}e^{-\widehat{f}}$. Restoring the subscript $k$, then $\widehat{u}_k$ are supported in $B_0(x_{0k}, 2)$, and by (4.5.7) we get

$$\lim_{k\to\infty} \int_{B_0(x_{0k},2)} \widehat{u}_k\, d\widehat{V}_k = 1. \tag{4.5.17}$$

Moreover, (4.5.16) implies the following for large $k$,

$$\frac{1}{2}\beta \le \int_{B_0(x_{0k},2)} \left(-\frac{1}{2}|\nabla \widehat{f}_k|^2 - \widehat{f}_k + 3\right) \widehat{u}_k \, d\widehat{V}_k. \tag{4.5.18}$$

This contradicts with the isoperimetric inequality in the assumption.

□

## 4.6 Generalized singular Ricci flow

### 4.6.1 Generalized singular Ricci flow: the definition and properties

In this subsection, we give the definition and some properties of the generalized singular Ricci flow.

**Definition 4.6.1.** Let $(M, g)$ be a Riemannian manifold. For any $x \in M$, let

$$\overline{\rho}(x) = \sup\{r > 0 : B_g(x, r) \text{ is relatively compact and } |\mathrm{Rm}| \le r^{-2} \text{ on } B_g(x, r)\}.$$

Recall $\rho = R_+^{-1/2}$, it's clear that $c_0\overline{\rho} \le \rho$ for some universal constant $c_0 > 0$.

**Definition 4.6.2.** We say a Ricci flow spacetime $\mathcal{M}$ is backward (resp. forward) 0-complete if each time-slice of $\mathcal{M}$ is 0-complete (see Definition 2.1.10), and for any smooth curve $\gamma : [0, s_0) \to \mathcal{M}$, which is the integral curve of $-\partial_t$ (resp. $\partial_t$), and satisfies $\inf_{[0,s_0)} \overline{\rho}(\gamma(s)) > 0$. Then $\lim_{s\to s_0} \gamma(s)$ exists.

**Definition 4.6.3.** A generalized singular Ricci flow is a Ricci flow spacetime $(\mathcal{M}, g(t))$, which satisfies:

1. $\mathcal{M}_0 = M$ is a complete orientable manifold.
2. $g(t)$ satisfies the Hamilton-Ivey pinching condition (2.1.4) with $\varphi = \infty$.
3. $\mathcal{M}$ is forward 0-complete, and weakly backward 0-complete.
4. For any $x_0 \in \mathcal{M}$, there exist $N \in \mathbb{N}$ and a sequence of points $\{x_j\}_{j=0}^N$ with $\mathfrak{t}(x_j) = t_j$, such that $t_0 \ge t_1 \ge \cdots \ge t_N = 0$, $x_j$ survives until $t_{j+1}$, and $x_j(t_{j+1}), x_{j+1}$ are in the same connected component in $\mathcal{M}_{t_{j+1}}$.
5. For any $x_0 \in \mathcal{M}$ surviving on $[t_1, t_0]$ for some $t_1 < t_0$, and any $A, \epsilon_{can} > 0$, there is $r > 0$ such that the $\epsilon_{can}$-canonical neighborhood assumption holds at scales less than $r$ on $B_t(x_0(t), A)$ for all $t \in [t_1, t_0]$.

**Definition 4.6.4.** A semi-generalized singular Ricci flow is a Ricci flow spacetime $(\mathcal{M}, g(t), x_0)$ with $\mathfrak{t}(\mathcal{M}) = [0, t_0)$ for some $t_0 \in [0, \infty]$, and $x_0 \in \mathcal{M}_0$, which satisfies the following properties:

1. $x_0$ survives until $t$ for all $t \in [0, t_0)$, .
2. $g(t)$ satisfies the Hamilton-Ivey pinching condition (2.1.4) with $\varphi = \infty$.
3. $\mathcal{M}$ is weakly backward 0-complete.
4. $\mathcal{M}_t$ is connected for each $t \in [0, t_0)$.
5. For any $t_1 \in [0, t_0)$, and any $A, \epsilon_{can} > 0$ there is $r > 0$ such that the $\epsilon_{can}$-canonical neighborhood assumption holds at scales less than $r$ on $B_t(x_0(t), A)$ for all $t \in [0, t_1]$.

For a singular Ricci flow, it's obvious that it satisfies property (1)(2)(3)(5) in Definition 4.6.3. It also satisfies property (4) there, because by [44, Theorem 7.1] there are at most countably many points in each time-slice that can not flow back to initial manifold. So a singular Ricci flow is a generalized singular Ricci flow.

Moreover, let $\mathcal{M}$ be a singular Ricci flow, $x_0 \in \mathcal{M}_0$. Suppose $x_0$ survives on $[0, t_0)$ for some $t_0 \in [0, \infty]$, and let $\mathcal{M}_{x_0} = \bigcup_{[0,t_0)} \bigcup_{A>0} B_t(x_0(t), A)$. By the component stability [44, Prop 5.17], the connected components are preserved when going backwards in time, it is clear that $\mathcal{M}_{x_0}$ is a semi-generalized singular Ricci flow.

The following properties can be derived directly from the definition of the semi-generalized singular Ricci flow.

**Lemma 4.6.5.** *Let $(\mathcal{M}, g(t), x_0)$ be a semi-generalized singular Ricci flow on $[0, t_0)$. Let $t \in (0, t_0)$, then*

(i) *For any $A > 0$, the scalar curvature is proper on $\overline{B_t(x_0(t), A)}$.*

(ii) *For any $A > 0$, there exist $\overline{Q}, C > 0$ such that for any $x \in B_t(x_0(t), A)$, letting $Q = \max\{\overline{Q}, R(x)\}$, then $R \leq CQ$ in $P(x, (CQ)^{-1/2}, -(CQ)^{-1})$, which is contained in $\bigcup_{s\in[0,t]} B_s(x_0(s), 2A)$.*

*Proof.* For any $C > 0$, consider the subset $K := \overline{B_t(x_0(t), A)} \cap \{y \in \mathcal{M} : R(y) \leq C\}$, equipped with the metric induced by the length metric on $\overline{B_t(x_0(t), A)}$. On the one hand, Lemma 4.2.1 implies that $B_t(x_0(t), A)$ is totally bounded. So $K$ is totally bounded. On the other hand, by the gradient estimate there exists $c > 0$ such that for any $x \in K$, the ball $B_t(x, c)$ is unscathed and $R \leq 2C$ in $B_t(x, c)$. From this it is easy see that $K$ is complete as a metric space. So $K$ is compact, which established (i).

For any $A > 0$, by the gradient estimate, and the distance distortion estimate, and seeing that $\mathcal{M}$ is weakly backward 0-complete, we can find $\overline{Q}, C > 0$ such that $(C\overline{Q})^{-1} <$

$t/2$, and the following holds: For any $x \in B_t(x_0(t), A)$, $Q = \max\{\overline{Q}, R(x)\}$, $x$ survives on $[t-(CQ)^{-1}, t]$, and $R \leq CQ$ in $B_s(x(s), (CQ)^{-1/2})$, which is contained in $B_s(x_0(s), 2A)$. By another distance distortion estimate this implies assertion (ii).

□

The next proposition says that the connected components of a Ricci flow spacetime are preserved when going backwards in time, assuming the spacetime is weakly backward 0-complete and satisfies a distance-dependent canonical neighborhood assumption. In particular, this component stability holds for generalized singular Ricci flows.

**Proposition 4.6.6.** *(Component stability when going backwards in time) Let $\mathcal{M}$ be a Ricci flow spacetime, $x_0, x_1 \in \mathcal{M}_{t_1}$ for some $t_1 > 0$. Suppose that $\mathcal{M}$ is weakly backward 0-complete. Suppose both $x_0, x_1$ survive until some $t_2 < t_1$. Suppose also for any $A, \epsilon_{can} > 0$ there exists $r > 0$ such that the $\epsilon_{can}$-canonical neighborhood assumption holds in $B_t(x_0(t), A)$ at scales less than $r$ for all $t \in [t_2, t_1]$.*

*Suppose $x_0, x_1$ are in the same connected component of $\mathcal{M}_{t_1}$. Then $x_0(t_2), x_1(t_2)$ are in the same connected component of $\mathcal{M}_{t_2}$.*

*Proof.* Without loss of generality we may assume that $x_0(t), x_1(t)$ are in the same connected component of $\mathcal{M}_t$ for all $t \in (t_2, t_1]$. Put

$$\rho_0 = \min\{\inf_{[t_2,t_1]} \overline{\rho}(x_0(t)), \inf_{[t_2,t_1]} \overline{\rho}(x_1(t)), 1\} > 0. \tag{4.6.1}$$

So by the distance distortion estimate we can find $A > 0$ such that $d_t(x_0(t), x_1(t)) \leq A$ for all $t \in (t_2, t_1]$. Fix a small $\delta > 0$ and let $C_0(\delta) > 0$ and $\epsilon_{can}(\delta) > 0$ be from Lemma 2.2.6. Choose some $\overline{r} \in (0, C_0^{-1}\rho_0)$ such that the $\epsilon_{can}$-canonical neighborhood assumption holds in $B_t(x_0(t), 6A)$ at scales less than $4\overline{r}$ for all $t \in [t_2, t_1]$. We may also assume $t_1 - t_2 < c$ for some $c(\overline{r}, A) > 0$ whose value will be determined in the course of the proof.

By Lemma 4.2.2, there exists a minimizing geodesic $\sigma : [0,1] \to \mathcal{M}_{t_1}$ between $x_0$ and $x_1$. Choose a division of $[0,1]$ by $0 = \alpha_0 \leq \alpha_1 \leq ... \leq \alpha_m = 1$ such that one of the following two cases holds for each $i = 0, 1, ..., m-1$:

1. $\rho(x) \geq \overline{r}$ for all $x \in \sigma_i := \sigma([\alpha_i, \alpha_{i+1}])$;
2. $\rho(x) \leq 2\overline{r}$ for all $x \in \sigma([\alpha_i, \alpha_{i+1}])$, and $\rho(\sigma(\alpha_i)) = \rho(\sigma(\alpha_{i+1})) = 2\overline{r}$.

Next, suppose by induction that for $\sigma_{i-1}$, $i \geq 1$, the following assumptions hold:

(a) $\sigma_{i-1}$ survives backwards until $t_2$.

(b) $\sigma_{i-1}(t) \subset B_t(x_0(t), 6A)$ for all $t \in [t_2, t_1]$.

Suppose $\sigma_i$ satisfies case (1) and assume $c$ sufficiently small. Then by the gradient estimate, the distance distortion estimate, and the weakly backward 0-completeness of $\mathcal{M}$, we get that (a)(b) hold for $\sigma_i$. In particular, (a)(b) hold for $\sigma_0$. So we can assume $\sigma_i$ satisfies case (2), $i \geq 1$. Let $t_3$ be the infimum of all times $t \in [t_2, t_1]$ such that $\sigma_i$ survives until $t$, and $\sigma_i(s) \subset B_s(x_0(s), 6A)$ for all $s \in [t, t_1]$.

First, since $\overline{r} < C_0^{-1}\rho_0$ and $\sigma$ is a minimizing geodesic, it follows from Lemma 2.2.6 that $y_1 := \sigma(\alpha_i)$ and $y_2 := \sigma(\alpha_{i+1})$ are both centers of $\delta$-necks. Taking $c$ small, then by the gradient estimate we have for all $t \in (t_3, t_1]$ that $\rho(y_1(t)), \rho(y_2(t)) \in [\overline{r}, 4\overline{r}]$, and $\rho(x(t)) \leq 4\overline{r}$ for all $x \in \sigma_i$. Since the $\epsilon_{can}$-canonical neighborhood assumption holds in $B_t(x_0(t), 6A)$ at scales less than $4\overline{r}$, by Lemma 4.2.4 that $y_1(t), y_2(t)$ are centers of $2\delta$-necks when $c$ is taken sufficiently small. Moreover, by [55, Proposition 19.21] we know that $\sigma_i(t)$ is contained in a $2\delta$-tube or a capped $2\delta$-tube. Since $y_1(t), y_2(t)$ are the centers of $2\delta$-necks, it is easy to see that $\sigma_i(t)$ is contained in a $2\delta$-tube. Then the evolution equation of scalar curvature implies $\partial_t R(x(t)) > 0$ for all $x \in \sigma_i$ and $t \in (t_3, t_1]$. Therefore, $\sigma_i$ survives until $t_3$.

Next, for any $t \in [t_3, t_1]$, let $\mathcal{T}_t \subset \mathcal{M}_t$ be a $2\delta$-tube that contains the $100\overline{r}$-neighborhood of $\sigma_i(t)$, and let $\widehat{d}_t$ denote the length metric induced by $g(t)$ in $\mathcal{T}_t$. Then for any $z_1, z_2 \in \sigma_i(t)$, $\widehat{d}_t(z_1, z_2)$ is realized by a smooth geodesic in $\mathcal{T}_t$. Let $\gamma_t$ be such a minimizing geodesic connecting $y_1(t)$ and $y_2(t)$. Then all the second variations along $\gamma_t$ are non-negative since it has the minimal length among all smooth curves in a neighborhood around it. So a distance distortion estimate as Lemma 2.3.1 shows

$$\partial_t \widehat{d}_t(y_1(t), y_2(t)) \geq -C\overline{r}^{-1}, \tag{4.6.2}$$

for some universal constant $C > 0$. Integrating this and taking $c$ sufficiently small, we get

$$\widehat{d}_t(y_1(t), y_2(t)) \leq \widehat{d}_{t_1}(y_1, y_2) + C\overline{r}^{-1}(t_1 - t) \leq 2A. \tag{4.6.3}$$

Moreover, since $\rho(x) \leq 4\overline{r}$ for all $x \in \sigma_i(t)$, by the triangle inequality we get

$$\widehat{d}_t(y_1(t), x) \leq \widehat{d}_t(y_1(t), y_2(t)) + 10 \cdot 2 \cdot 4\overline{r} \leq 3A. \tag{4.6.4}$$

By the distance distortion estimate we get $d_t(x_0(t), y_1(t)) \leq 2A$ and

$$d_t(x, x_0(t)) \leq d_t(x_0(t), y_1(t)) + d_t(y_1(t), x) \leq 2A + \widehat{d}_t(y_1(t), x) \leq 5A. \tag{4.6.5}$$

So $\sigma_i(t) \subset B_t(x_0(t), 6A)$ for all $t \in [t_3, t_1]$. By the infimum assumption of $t_3$, we get $t_3 = t_2$. Hence (a)(b) hold for $\sigma_i$. So by induction the entire $\sigma$ survives backwards until $t_2$. It follows that $x_0(t_2), x_1(t_2)$ are in the same connected component of $\mathcal{M}_{t_2}$. □

The following corollary of Proposition 4.6.6 gives the relation between a semi-generalized singular Ricci flow and a generalized singular Ricci flow.

**Corollary 4.6.7.** *A Ricci flow spacetime* $(\mathcal{M}, g(t))$ *is a generalized singular Ricci flow if and only if it satisfies property* (1)(2)(3)(4) *in Definition 4.6.3, and for any* $x_0 \in \mathcal{M}$*, which survives on* $[t_1, t_0)$*, let* $\mathcal{M}_{x_0} = \bigcup_{t\in[t_1,t_0)} \bigcup_{A>0} B_t(x_0(t), A)$*, then* $(\mathcal{M}_{x_0}, x_0)$ *is a semi-generalized singular Ricci flow.*

*Proof.* The 'if' part is obvious by the definitions. For the 'only if' part, we need to show $\mathcal{M}_{x_0}$ is a semi-generalized singular Ricci flow. It suffices to show $\mathcal{M}_{x_0}$ is weakly backward 0-complete. Let $x \in \mathcal{M}_{x_0}$ be an arbitrary point. Suppose $\mathfrak{t}(x) = t_1$, $x$ survives on $(t_2, t_1]$ in $\mathcal{M}_{x_0}$ for some $t_2 < t_1$, and $\inf_{t\in(t_2,t_1]} \overline{\rho}(x(t)) > 0$. Since $\mathcal{M}$ is weakly backward 0-complete, $x(t_2) = \lim_{t\searrow t_2} x(t)$ exists. By Proposition 4.6.6, $x(t_2) \in \mathcal{M}_{x_0}$. So $\mathcal{M}_{x_0}$ is weakly backward 0-complete. □

### 4.6.2 Compactness and existence theorems

First, we show a compactness theorem which gives a criterion for a sequence of singular Ricci flows to have a subsequence that converges to a semi-generalized singular Ricci flow. Then we apply the compactness theorem to show the existence of generalized singular Ricci flows.

**Definition 4.6.8.** (Partial convergence) We say a sequence of Ricci flow spacetimes $(\mathcal{M}_i, G_i)$ partially converges to a Ricci flow spacetime $(\mathcal{M}, G)$ if the following holds: There is a sequence of diffeomorphisms $\phi_i : U_i \to V_i \subset \mathcal{M}_i$, where $U_i$ and $V_i$ are open domains in $\mathcal{M}$ and $\mathcal{M}_i$ respectively, such that given any compact subset $K \subset \mathcal{M}$, $k \in \mathbb{N}$ and $\epsilon > 0$, we have $K \subset U_i$ for all large $i$, and $\|\phi_i^* G_i - G\|_{C^k(K,G)} \leq \epsilon$.

Let $x_0 \in \mathcal{M}$ and $x_{0i} \in \mathcal{M}_i$. We say the sequence of pointed spacetimes $(\mathcal{M}_i, x_{0i})$ partially converges to $(\mathcal{M}, x_0)$, if $x_0 \in U_i$, $x_{0i} \in V_i$, and $\phi_i(x_0) = x_{0i}$.

**Theorem 4.6.9** (Compactness theorem)**.** *Let* $\{(\mathcal{M}_i, x_{0i})\}_{i=1}^{\infty}$ *be a sequence of singular Ricci flows,* $x_{0i} \in \mathcal{M}_i$*,* $\mathfrak{t}(x_{0i}) = t_{0i}$*. Suppose*

*(a) For some* $r_0 > 0$*,* $x_{0i}$ *survives until* $t_{0i} + r_0^2$*, and* $|\mathrm{Rm}| \leq r_0^{-2}$ *on* $P(x_{0i}, r_0, r_0^2)$*.*

*(b) For any* $\epsilon_{can} > 0$ *and* $A > 0$*, there exist* $r(A, \epsilon_{can})$ *and* $\kappa(A)$ *such that the* $\epsilon_{can}$*-canonical neighborhood assumption and the* $\kappa(A)$*-non-collapsedness hold at scales less than* $r(A, \epsilon_{can})$ *in* $B_t(x_{0i}(t), A)$ *for all* $t \in [t_{0i}, t_{0i} + r_0^2]$*.*

*(c) For any* $A > 0$*, there is* $C(A) > 0$ *such that for all* $m \in \mathbb{N}$ *and* $m \leq A$*, we have* $|\nabla^m \mathrm{Rm}|(x) \leq C(A)$ *for all* $x \in B_{t_{0i}}(x_{0i}, Ar_0)$*.*

*Then there exists a semi-generalized singular Ricci flow* $(\mathcal{M}, x_0)$ *on* $[0, r_0^2)$*,* $x_0 \in \mathcal{M}_0$*, such that a subsequence of* $(\mathcal{M}_{i,t\geq t_{0i}}, x_{0i})$ *partially converges to* $(\mathcal{M}, x_0)$*.*

*Proof to Theorem 4.6.9.* We may assume $t_{0i} = 0$ and $r_0 \leq 1$ without loss of generality. Let $d_{G_i}$ be the length metric on $\mathcal{M}_i$ induced by the spacetime metric $G_i$. For any $A > 0$, restrict the metric $d_{G_i}$ on the subset

$$\mathcal{P}_i(A) = \bigcup_{t\in[0,r_0^2)} B_t(x_{0i}(t), A) \cap \{x \in \mathcal{M}_i : |\mathrm{Rm}|(x) \leq A^2\}. \tag{4.6.6}$$

Then the diameter of every $(\mathcal{P}_i(A), d_{G_i})$ is bounded above by $2(A + r_0^2)$. Moreover, the following lemma shows that they are uniformly totally bounded.

**Lemma 4.6.10.** *For any $A, \epsilon > 0$, there exists $N = N(A, \epsilon) \in \mathbb{N}$ such that for all $i$, any $\epsilon$-separating subset in $(\mathcal{P}_i(A), d_{G_i})$ has at most $N$ elements.*

*Proof.* On the one hand, by a combination of assumption (b), the gradient estimate and the distance distortion estimate, we may assume that $\epsilon$ is sufficiently small (depending on $A$) so that the following holds: First, for any $x \in \mathcal{P}_i(A)$, $\mathfrak{t}(x) \geq \epsilon/8$, the backward parabolic neighborhood $P(x, \epsilon/8, -(\epsilon/8)^2)$ is unscathed and contained in $\mathcal{P}_i(2A)$. Second, for any $x \in \mathcal{P}_i(A)$ with $\mathfrak{t}(x) \leq \epsilon/8$, $x$ survives until 0, and $x(0) \in \mathcal{P}_i(2A)$. Furthermore, there exists $c(A, \epsilon) > 0$ such that

$$vol_{G_i}\left(P\left(x, \epsilon/8, -(\epsilon/8)^2\right)\right) \geq c(A, \epsilon). \tag{4.6.7}$$

On the other hand, assumption (b) allows us to apply Lemma 4.2.1 on each time-slice $\mathcal{M}_{i,t}$, and deduce that $B_t(x_{0i}(t), 2A)$ is uniformly totally bounded, and there is a constant $v(A) > 0$ such that

$$vol_{g_i(t)} B_t(x_{0i}(t), 2A) \leq v(A) \tag{4.6.8}$$

for all $i$ and $t \in [0, r_0^2)$. Integrating this we get

$$vol_{G_i}(\mathcal{P}_i(2A)) \leq \int_0^{r_0^2} vol_{g_i(t)} B_t(x_{0i}(t), 2A)\, dt \leq v(A). \tag{4.6.9}$$

Now suppose $\{x_k\}_{k=1}^{N_i}$ is an $\epsilon$-separating subset of $(\mathcal{P}_i(A), d_{G_i})$, and $t_k = \mathfrak{t}(x_k)$. Let $\{x_{k_j}\}_{j=1}^{J_i}$ be all $x_k$ with $t_k < \epsilon/8$, then each $x_{k_j}$ survives backwards until 0 and $x_{k_j}(0) \in B_0(x_{0i}, 2A)$. Since $d_{G_i}(x_{k_j}, x_{k_l}) > \epsilon$ for any $j \neq l$, by the triangle inequality, $\{x_{k_j}(0)\}_{j=1}^{J_i}$ is an $3\epsilon/4$-separating subset of $B_0(x_{0i}, 2A)$. Since $B_0(x_{0i}, 2A)$ is uniformly totally bounded, there is $C(A, \epsilon) > 0$ such that $J_i \leq C$. Therefore, in order to bound $N_i$ we may assume that $t_k \geq \epsilon/8$ for all $k$.

Then each $P(x_k, \epsilon/8, -(\epsilon/8)^2)$ is unscathed, and $d_{G_i}(x_k, y) \leq \epsilon/8 + (\epsilon/8)^2 < \epsilon/4$ for all $y \in P(x_k, \epsilon/8, -(\epsilon/8)^2)$. Since $d_{G_i}(x_k, x_j) > \epsilon$ for any $k \neq j$, by the triangle inequality, we see that $P(x_k, \epsilon/8, -(\epsilon/8)^2)$, $k = 1, 2, ..., N_i$, are pairwise disjoint. Therefore, combining (4.6.7) and (4.6.9), we conclude that there is $N(A, \epsilon) > 0$ such that $N_i \leq N(A, \epsilon)$ for all $i$.

□

Now since $(\mathcal{P}_i(A), d_{G_i})$ have uniformly bounded diameter and are uniformly totally bounded for all $i$, by Gromov's compactness theorem [59, Proposition 44], we may assume $(\mathcal{P}_i(A), d_{G_i}, x_{0i})$ converges to a metric space $(X(A), d_A, x_0)$ in the pointed Gromov-Hausdorff sense. Since for any $A_1 \geq A_2$, $(\mathcal{P}_i(A_2), d_{G_i}, x_{0i})$ isometrically embeds into $(\mathcal{P}_i(A_1), d_{G_i}, x_{0i})$, we get $(X(A_2), d_{A_2}, x_0)$ isometrically embeds into $(X(A_1), d_{A_1}, x_0)$. Let $(X, d) = \bigcup_{A>0}(X(A), d_A)$, and $\mathcal{N}_i = \bigcup_{A>0} \mathcal{P}_i(A) \subset \mathcal{M}_i$, then $(\mathcal{N}_i, d_{G_i}, x_{0i})$ converges to $(X, d, x_0)$ in the pointed Gromov-Hausdorff sense as $i \to \infty$.

Let $x \in X$, and suppose $x \in X(A)$ for some $A > 0$. We say $x$ is a smooth point if there are a $\delta > 0$, and a sequence of points $x_{i_k} \in \mathcal{P}_{i_k}(A)$ with $|\mathrm{Rm}|(x_{i_k}) \leq \delta^{-2}$ converging to $x$ (modulo the Gromov-Hausdorff approximations). By Lemma 2.3.1, the canonical neigborhood assumption in $\mathcal{P}_{i_k}(A)$, and the gradient estimate, we can find $\overline{\delta} = \overline{\delta}(\delta, A)$ such that $|\mathrm{Rm}| \leq \overline{\delta}^{-2}$ in $P(x_{i_k}, \overline{\delta}, -\overline{\delta}^2) \cap \mathfrak{t}^{-1}([t_{0i_k}, t_{0i_k} + r_0^2])$. Moreover, by the non-collapsing assumption in $\mathcal{P}_{i_k}(A)$ and Corollary 4.5.2 of the Pseudolocality theorem 4.5.1, we get $|\mathrm{Rm}| \leq \overline{\delta}^{-2}$ in $P(x_{i_k}, \overline{\delta}, \overline{\delta}^2)$ with a possibly smaller $\overline{\delta}$. Let $U(x_{i_k}, \overline{\delta}) = P(x_{i_k}, \frac{1}{2}\overline{\delta}, -\frac{1}{4}\overline{\delta}^2) \cup P(x_{i_k}, \frac{1}{2}\overline{\delta}, \frac{1}{4}\overline{\delta}^2) \cap \mathfrak{t}^{-1}([t_{0i_k}, t_{0i_k} + r_0^2])$, then by Shi's derivative estimate and assumption (c), we get uniform bounds on the derivatives of Rm in $U(x_{i_k}, \overline{\delta})$. So we obtain a smooth limit of $U(x_{i_k}, \overline{\delta})$ in the Cheeger-Gromov sense by passing to a subsequence. This defines a Ricci flow spacetime metric in a neighborhood of $x$ in $X$, which is isometric to that on $X$ by the uniqueness of the Gromov-Hausdorff limit.

Let $X_0 \subset X$ be the set of all smooth points. Then we obtain a global Ricci flow spacetime metric on $X_0$, denoted by $G_\infty = d\mathfrak{t}^2 + g(t)$. In particular, $P(x_{0i}, r_0, r_0^2)$ converges smoothly to $P(x_0, r_0, r_0^2) \subset X_0$. Moreover, by a standard gluing argument (see e.g. [59, Theorem 72]) we get a sequence of diffeomorphisms under which a subsequence of $\mathcal{M}_i$ partially converges to $X_0$, in the sense of Definition 4.6.8.

It implies that for any $\epsilon_{can}, A > 0$, there exists $r(A, \epsilon_{can}) > 0$ such that the $\epsilon_{can}$-canonical neighborhood assumption holds at scales less than $r$ in $X(A) \cap X_0$. Furthermore, for any $A, C > 0$, there exists $c = c(A, C) > 0$ such that for all $x \in B_t(x_{0i}(t), A) \subset \mathcal{N}_i$ with $|\mathrm{Rm}|(x) \leq C$, the backward parabolic neighborhood $P(x, c, -c^2)$ in $\mathcal{N}_i$ is unscathed. So for all $x \in X(A) \cap X_0$ with $|\mathrm{Rm}|(x) \leq C$, the region $P(x, c, -c^2)$ in $X_0$ is unscathed. From this it is easy to see that $X_0$ is weakly backward 0-complete.

Let $\mathcal{M} = \bigcup_{t \in [0, r_0^2)} \bigcup_{A>0} B_t(x_0(t), A)$ be a subset in $X_0$. Then $\mathcal{M}$ is a smooth Ricci flow spacetime with connected time-slices, and a subsequence of $\mathcal{M}_i$ partially converge to $\mathcal{M}$. It is clear that $\mathcal{M}$ satisfies property (1)(2)(4)(5) in Definition 4.6.4. Moreover, applying Proposition 4.6.6 to $X_0$, we see that $\mathcal{M}$ is also weakly backward 0-complete and hence satisfies property (3). This proved Theorem 4.6.9. □

The next lemma shows that the convergence of the initial manifolds implies the convergence of the singular Ricci flows to some semi-generalized singular Ricci flow.

**Lemma 4.6.11.** *Let $(\mathcal{M}_i, g_i(t))$ be a sequence of singular Ricci flows. Suppose for some $t_1 \geq 0$ and $x_{0i} \in \mathcal{M}_{i,t_1}$, the sequence of time-slices $(\mathcal{M}_{i,t_1}, g_i(t_1), x_{0i})$ smoothly converges to a 0-complete manifold $(M, g, x_0)$. Then there exists a semi-generalized singular Ricci flow $(\mathcal{M}, g(t), x_0)$ on $[0, t_0)$ for some $t_0 > 0$, such that $\mathcal{M}_0 = M$ and a subsequence of $(\mathcal{M}_{i,t\geq t_1}, x_{0i})$ partially converges to $(\mathcal{M}, x_0)$. Moreover, $t_0$ can be chosen to be equal to $\infty$, or such that $\inf_{[0,t_0)} \rho(x_0(t)) = 0$, i.e. $\limsup_{t\nearrow t_0} |\mathrm{Rm}|(x_0(t)) = \infty$.*

*Proof.* Without loss of generality we may assume $t_1 = 0$. On the one hand, by Corollary 4.5.2 of the pseudolocality theorem 4.5.1, there exist $r_0, t_0 > 0$ such that for all large $i$ the domain $P(x_{0i}, r_0, t_0) \subset \mathcal{M}_i$ is unscathed and $|\mathrm{Rm}| \leq r_0^{-2}$ holds there. Moreover, for any fixed $A > 0$, there exists $t_A \in (0, t_0)$ such that the geometry $P(x_{0i}, 2A, t_A) \subset \mathcal{M}_i$ is uniformly bounded for all large $i$. By a distance distortion estimate this implies the uniformly bounded geometry on $\bigcup_{t\in[0,t_A]} B_t(x_{0i}(t), A) \subset \mathcal{M}_i$ for a possibly smaller $t_A$.

On the other hand, for any $t \in (t_A, t_0)$, by Proposition 4.3.1 there are constants $r, \kappa > 0$, such that the $\epsilon_{can}$-canonical neighborhood assumption and the $\kappa$-non-collapsing assumption hold at scales less than $r$ in $B_t(x_{0i}(t), A)$. So by Theorem 4.6.9 there is a subsequence of $(\mathcal{M}_i, x_{0i})$ which partially converges to $(\mathcal{M}, x_0)$.

If $\inf_{[0,t_0)} \rho(x_0(t)) > 0$, then by Lemma 4.3.6 we have $\inf_{[0,t_0)} \overline{\rho}(x_0(t)) > 0$. So there exist $\kappa', r' > 0$ such that $\mathcal{M}$ is $\kappa'$-non-collapsed at $x_0(t)$ at scales less than $r'$ for all $t \in [0, t_0)$. Repeating the above argument at $t$ sufficiently close to $t_0$, we can extend $\mathfrak{t}(\mathcal{M})$ to $[0, t_1)$ with $t_1 > t_0$. So we may assume $\inf_{[0,t_0)} \rho(x_0(t)) = 0$ or $t_0 = \infty$. □

**Theorem 4.6.12.** *(Existence of a semi-generalized singular Ricci flow) Let $(M, g)$ be a 3d orientable complete Riemannian manifold, $x_0 \in M$. Then there exists a semi-generalized singular Ricci flow $(\mathcal{M}, g(t), x_0)$ on $[0, t_0)$ for some $t_0 > 0$ with $\mathcal{M}_0 = M$.*

*Moreover, if $M$ is the double cover of a non-orientable manifold, and $\sigma : M \to M$ is the corresponding deck transformation which acts as an isometry. Then there is a semi-generalized singular Ricci flow $\mathcal{M}$ with $\mathcal{M}_0 = M$ such that $\sigma$ extends to an isometry on $\mathcal{M}$, which acts free on the open domain $\{x \in \mathcal{M} : x$ survives until $t = 0\}$.*

*Proof.* Let $(M_i, g_i, x_i)$ be a sequence of compact manifolds which smoothly converges to $(M, g, x_0)$, and $(\mathcal{M}_i, g_i(t), x_i)$ be a sequence of singular Ricci flows starting from $(M_i, g_i, x_i)$. Then the first assertion follows from applying Lemma 4.6.11 to $(\mathcal{M}_i, g_i(t), x_i)$. It only remains to establish the assertion about the $\mathbb{Z}_2$-symmetry. For this we assume $(M, g, x_0)$ is the double cover of a non-orientable manifold $(N, \overline{g})$, and $\sigma : M \to M$ is the non-trivial deck transformation in $\mathbb{Z}_2$, which acts as an isometry. Let $N_i \subset N$ be a compact 3 dimensional submanifold with smooth boundary that contains $B_{\overline{g}}(\pi(x_0), i)$. Take $i > d_g(x_0, \sigma(x_0))$, then $\pi^{-1}(N_i)$ is a compact connected orientable manifold which has smooth orientable boundary $\pi^{-1}(\partial N_i)$, and $B_g(x_0, i) \cup B_g(\sigma(x_0), i) \subset \pi^{-1}(N_i)$. First, we extend $\pi^{-1}(N_i)$ and the metric past a collar of its boundary, and assume the new metric $g_i$ is isometric to the product of a metric on $\pi^{-1}(\partial N_i)$ with an interval. Next, since $\pi^{-1}(\partial N_i)$ is $\sigma$-invariant, we can extend the action of $\sigma$ to the collar neighborhood $\pi^{-1}(\partial N_i) \times [0,1]$ such that $\sigma(x,s) = \sigma(x,0)$ for all $x \in \pi^{-1}(\partial N_i)$ and $s \in [0,1]$. Then by replacing $g_i$ with $\frac{1}{2}(g_i + \sigma^* g_i)$, we may assume $g_i$ is $\sigma$-invariant, and it is still a product metric near the new boundary. Therefore, by doubling the extended manifold, we get a closed, connected and orientable manifold $(M_i, g_i, x_0)$ with a deck transformation $\sigma_i$ which is an isometry, and $g_i = g$, $\sigma_i = \sigma$ on $B_{g_i}(x_0, i)$.

Let $(\mathcal{M}_i, g_i(t), x_0)$ be a sequence of singular Ricci flows starting from $(M_i, g_i, x_0)$. Then by Lemma 4.6.11 there is $t_0 > 0$ such that $\{(\mathcal{M}_i, g_i(t), x_0)\}$ converges to a semi-generalized singular Ricci flow $(\mathcal{M}, x_0)$ on $[0, t_0)$. Moreover, by the uniqueness of singular Ricci flow in [4], each $\sigma_i : M_i \to M_i$ can be uniquely extended to an isometry $\sigma_i : \mathcal{M}_i \to \mathcal{M}_i$. So for any $x_1, x_2 \in M_i$, if $\sigma_i(x_1) = x_2$ and $x_1$ survives until $t > 0$, then $x_2$ also survives until $t$ and $\sigma_i(x_1(t)) = \sigma_i(x_2(t))$. Therefore, $\sigma_i$ converges to an isometry $\sigma : \mathcal{M} \to \mathcal{M}$, which acts free on $\{x \in \mathcal{M} : x \text{ survives back to } \mathcal{M}_0 = M\}$ □

The next lemma shows that for two Ricci flow spacetimes $(\mathcal{N}_j, x_j)$, $j = 1, 2$, which have connected time-slices, suppose they are limits of a same sequence of Ricci flow spacetimes $\mathcal{M}_i$ under the partial convergence. Then they are isometric if the preimages of $x_1, x_2$ under the diffeomorphisms are contained in a parabolic region in $\mathcal{M}_i$.

**Lemma 4.6.13.** *Let $\mathcal{M}_i$ be a sequence of Ricci flow spacetimes, $x_{1,i}, x_{2,i} \in \mathcal{M}_{i,0}$. Suppose $(\mathcal{M}_i, x_{j,i})$ partially converges to a Ricci flow spacetime $(\mathcal{N}_j, x_j)$ on $[0,T)$, for some $T > 0$, $x_j \in \mathcal{N}_{j,0}$, and each time-slice $\mathcal{N}_{j,t}$ is connected, $j = 1, 2$. Suppose also there is $D > 0$ such that $x_{2,i} \in B_0(x_{1,i}, D)$ for all $i$, and $P(x_1, D, T - \delta) \subset \mathcal{N}_1$ is unscathed for any $\delta > 0$. Then $\mathcal{N}_1$ is isometric to $\mathcal{N}_2$.*

*Proof.* Let $H_i$ be the spacetime metric of $\mathcal{M}_i$, and $G_j$ the spacetime metric of $\mathcal{N}_j$, $j = 1, 2$. Let $\phi_{j,i} : \mathcal{N}_j \supset U_{j,i} \to V_{j,i} \subset \mathcal{M}_i$ be the two corresponding diffeomorphism sequences such that $\cup_{i=1}^{\infty} U_{j,i} = \mathcal{N}_j$ and $\|\phi_{j,i}^* H_i - G_j\| \leq \epsilon_i \to 0$. Let $P_{j,k} = \bigcup_{t \in [0, T - k^{-1}]} \overline{B_t(x_j, k)} \cap \{x : \rho(x) \geq k^{-1}\}$, then $\cup_{k=1}^{\infty} P_{j,k} = \mathcal{N}_j$. By the assumption of $x_{1,i}$ and $x_{2,i}$, for a given $k$ there exists $\ell(k, D) \in \mathbb{N}$ such that for all large $i$, we have $\phi_{1,i}(P_{1,k}) \subset \phi_{2,i}(P_{2,\ell})$. So the maps $\phi_{2,i}^{-1} \circ \phi_{1,i} : \mathcal{N}_1 \supset P_{1,k} \to \mathcal{N}_2$ are well-defined, and $\|(\phi_{2,i}^{-1} \circ \phi_{1,i})^* G_2 - G_1\| \leq \delta_i \to 0$. Moreover, $\phi_{2,i}^{-1} \circ \phi_{1,i}(P_{1,k})$ form an exhaustion of $\mathcal{N}_2$ as $i, k \to \infty$. So $\mathcal{N}_1$ is isometric to $\mathcal{N}_2$. □

**Theorem 4.6.14.** *(Theorem 4.1.1 and 4.1.2, Existence of generalized singular Ricci flow) Let $(M, g)$ be a 3d orientable complete Riemannian manifold, $x_0 \in M$. Let $(\mathcal{M}_i, g_i(t), x_{0i})$ be a sequence of singular Ricci flows with $(\mathcal{M}_{i,0}, g_i(0), x_{0i})$ smoothly converging to $(M, g, x_0)$. Then there exists a generalized singular Ricci flow $\mathcal{M}$ with $\mathcal{M}_0 = M$, such that $(\mathcal{M}_i, x_{0i})$ partially converges to $(\mathcal{M}, x_0)$.*

*Moreover, if $M$ is the double cover of a non-orientable manifold, then the same conclusion as Theorem 4.6.12 holds.*

*Proof.* Let $x_0 \in M$, by Lemma 4.6.11 there exist $t_0 \in (0,\infty]$ and a semi-generalized singular Ricci flow $(\mathcal{M}^1, x_0)$ on $[0,t_0)$ such that $(\mathcal{M}_i, G_i, x_0)$ partially converges to $(\mathcal{M}^1, x_0)$, and $\inf_{[0,t_0)} \rho(x_0(t)) = 0$ if $t_0 < \infty$.

Suppose by induction that there is a sequence of Ricci flow spacetimes $\{\mathcal{M}^j\}_{j=1}^{k-1}$ such that $\mathcal{M}^{j-1} \subset \mathcal{M}^j$ and the followings hold for all $j = 2, ..., k-1$:

1. A subsequence of $\mathcal{M}_i$ partially converges to $\mathcal{M}^j$.
2. $\mathcal{M}^j$ is weakly backward 0-complete.
3. For any $x \in \mathcal{M}^{j-1}$, let $a$ be the supremum of all times $t$ until which $x$ survives until in $\mathcal{M}^j$. Then $\inf_{[\mathfrak{t}(x),a)} \rho(x(t)) = 0$.
4. For any $x \in \mathcal{M}^{j-1}$, suppose $x$ survives until some $t > \mathfrak{t}(x)$. Let $\mathcal{M}_x \subset \mathcal{M}^j$ be the subset $\bigcup_{s\in[\mathfrak{t}(x),t)} \bigcup_{A>0} B_s(x(s), A)$, then $(\mathcal{M}_x, x)$ is a semi-generalized singular Ricci flow on $[\mathfrak{t}(x), t)$.

Let $\{x_j\}_{j=1}^{\infty}$ be a dense subset in $\mathcal{M}^{k-1}$. For each $x_j$, by Lemma 4.6.11 there is a subsequence of $\{(\mathcal{M}_i, x_{j,i})\}_{i=1}^{\infty}$ that partially converges to a semi-generalized singular Ricci flow $(\mathcal{N}_j, x_j)$, such that $x_j$ survives in $\mathcal{N}_j$ until $R(x_j(t))$ goes unbounded. So by a diagonal argument we may assume that $\{(\mathcal{M}_i, x_{j,i})\}_{i=1}^{\infty}$ converges to $(\mathcal{N}_j, x_j)$ for all $x_j$.

For any $y_1, y_2 \in \mathcal{M}^{k-1} \sqcup \coprod_{j=1}^{\infty} \mathcal{N}_j$, we say $y_1 \sim y_2$ if there is a sequence of points $w_i \in \mathcal{M}_i$ such that modulo the diffeomorphism maps we have $w_i \to y_1$ and $w_i \to y_2$ as $i \to \infty$. This defines an equivalent relation in $\mathcal{M}^{k-1} \sqcup \coprod_{j=1}^{\infty} \mathcal{N}_j$. If $y_1 \sim y_2$, then by the uniqueness of the smooth limit, there is $\delta > 0$ such that the neighborhoods of $P(y_i, \delta, \delta^2) \cup P(y_i, \delta, -\delta^2)$, $i = 1, 2$, are unscathed and the spacetime metrics on them are isometric. So there is a well-defined smooth Ricci flow spacetime metric on the quotient space $\mathcal{M}^k := \left(\mathcal{M}^{k-1} \sqcup \coprod_{j=1}^{\infty} \mathcal{N}_j\right) / \sim$. So (1) holds for $j = k$.

Since each connected component of $\mathcal{M}^k_t$ is isometric to either $\mathcal{M}^{k-1}_t$ or some $\mathcal{N}_{j,t}$, we get that $\mathcal{M}^k_t$ is 0-complete. For any $x \in \mathcal{M}^k$, $\mathfrak{t}(x) = t_0$, suppose $x$ survives on $(t_1, t_0]$ and $\lim_{t\to t_1} \overline{\rho}(x(t)) > 0$. Assume $x \in \mathcal{M}^{k-1}$, then since $\mathcal{M}^{k-1}$ is weakly backward 0-complete, it follows that $x(t) \in \mathcal{M}^{k-1}$ and $\lim_{t\to t_1} x(t)$ exists. Otherwise, assume $x \in \mathcal{N}_j$ for some $j \in \mathbb{N}$, and let $t_2 \in (t_1, t_0]$ be the infimum of time $t$ such that $x(t) \in \mathcal{N}_j$. Then $x(t_2) = \lim_{t\to t_2} x(t)$ exists because $\mathcal{N}_j$ is weakly backward 0-complete. If $t_2 > t_1$, then we have $x(t_2) \in \mathcal{N}_j \cap \mathcal{M}^{k-1}$, and the existence of $\lim_{t\to t_1} x(t)$ exists by the weakly backward 0-completeness of $\mathcal{M}^{k-1}$. So $\mathcal{M}^k$ is weakly backward 0-complete, and hence (2) holds.

It is clear that (3)(4) hold for each $x_j$. We claim that (3)(4) hold for every point in $\mathcal{M}^{k-1}$. To verify (3), let $x \in \mathcal{M}^{k-1}$ be an arbitrary point, $\mathfrak{t}(x) = t_1$. Let $t_2 > t_1$ be the supremum of all times until which $x$ survives in $\mathcal{M}^k$. Suppose by contradiction that $\inf_{[t_1,t_2)} \rho(x(t)) > 0$. Let $x_i \in \mathcal{M}_i$ be a sequence of points that converge to $x$ modulo the diffeomorphisms. Then by Lemma 4.6.11 there is $\delta > 0$ such that by passing to a subsequence, $(\mathcal{M}_i, x_i)$ partially converges to a semi-generalized singular Ricci flow $\mathcal{N}$ on $[t_1, t_2 + \delta^2)$, and $P(x, \delta, t_2 - t_1 + \delta^2)$ is unscathed. By the density of $\{x_j\}_{j=1}^{\infty}$, there exists $x_j \in P(x, \delta, \delta^2) \subset \mathcal{M}^{k-1}$. Then $x_j$ survives on $[\mathfrak{t}(x_j), t_2 + \delta^2)$ in $\mathcal{N}_j$. So it follows from Lemma 4.6.13 that $\mathcal{N}_j$ is isometric to $\mathcal{N}$ on $[t_1 + \delta^2, t_2 + \delta^2)$. In particular, $x \in \mathcal{N}_j \subset \mathcal{M}^k$ survives until $t_2 + \delta^2/2$, contradicting with the supremum assumption of $t_2$. This verifies (3).

To verify (4), let $x \in \mathcal{M}^{k-1}$ be an arbitrary point, $\mathfrak{t}(x) = t_1$, and assume $x$ survives until some $t_2 > t_1$, and $\mathcal{M}_x$ is defined as in (4). Choose $\delta > 0$ such that $P(x, \delta, t_2 - t_1)$ is unscathed, and pick some $x_j \in P(x, \delta, \delta^2)$ by the density of $\{x_j\}_{j=1}^{\infty}$. Then by Lemma 4.6.13, $\mathcal{M}_x$ is isometric to $\mathcal{N}_j$ on $[t_1 + \delta^2, t_2)$, and hence $(\mathcal{M}_x, x)$ is a semi-generalized singular Ricci flow on $[t_1 + \delta^2, t_2)$. Letting $\delta \to 0$, it implies that $\mathcal{M}_x$ is a semi-generalized singular Ricci flow on $[t_1, t_2)$. This verifies (4).

So by induction we obtain an infinite sequence of spacetimes $\{\mathcal{M}^k\}_{k=1}^{\infty}$ with $\mathcal{M}^{k-1} \subset \mathcal{M}^k$, which satisfies all inductive assumptions. Let $\mathcal{M} = \bigcup_{k=1}^{\infty} \mathcal{M}^k$, then by passing to a subsequence $\mathcal{M}_i$ partially converges to $\mathcal{M}$. By the 'if' part of Corollary 4.6.7, it is clear that $\mathcal{M}$ is a generalized singular Ricci flow. The assertion about the $\mathbb{Z}_2$-symmetry follows in the same way as Theorem 4.6.12.

□

## 4.7 Ricci flows with non-negative Ricci curvature

In this section, we prove Theorem 4.1.3. First, by adapting the maximum principle argument in [21] and [22] to a generalized singular Ricci flow, we show in Lemma 4.7.1 and 4.7.2 that it preserves the non-negativity of scalar curvature and Ricci curvature.

Then we prove Lemma 4.7.3, which is the last ingredient needed to prove Theorem 4.1.3. It says that in a 3-dimensional manifold with $\mathrm{Ric} \geq 0$, no singularity can form within finite distance along a minimizing geodesic covered by final time-slices of strong $\delta$-necks.

**Lemma 4.7.1.** *Let $(M, g)$ be a 3 dimensional complete Riemannian manifold with $R \geq 0$. Let $(\mathcal{M}, g(t))$ be a generalized singular Ricci flow starting from $(M, g)$. Then $R \geq 0$ on $\mathcal{M}$.*

*Proof.* By property (4) in Definition 4.6.3 and Corollary 4.6.7, it suffices to prove the lemma for a semi-generalized singular Ricci flow $(\mathcal{M}, g(t), x_0)$ on $[0, t_0)$, $x_0 \in \mathcal{M}_0 = M$. We may assume that there is $r_0 > 0$ such that $\bigcup_{t \in [0, t_0)} B_t(x_0(t), r_0)$ is unscathed and $\mathrm{Ric}(x) \leq r_0^{-2}$ there. Then by Lemma 2.3.1, we have

$$(\partial_t - \Delta) d_t(x_0(t), x) \geq -\frac{10}{3} r_0^{-1}, \tag{4.7.1}$$

for all $x \in \mathcal{M}_t$ with $d_t(x, x_0(t)) > r_0$.

Let $A \geq \frac{80}{3} r_0^{-2} t_0 + 2$ and define the following function on $\mathcal{M}$

$$u(x) = \varphi\left(\frac{d_t(x_0(t), x) + \frac{10}{3} r_0^{-1} t}{A r_0}\right) R(x), \tag{4.7.2}$$

for all $x \in \mathcal{M}_t$, $t \in [0, t_0)$, where we choose $\varphi$ to be a smooth non-negative non-increasing function such that $\varphi = 1$ on $(-\infty, \frac{7}{8}]$, $\varphi = 0$ on $[1, \infty)$ and $\left|\frac{2\varphi'^2}{\varphi} - \varphi''\right| \leq C\sqrt{\varphi}$. Then with the choice of $A$, we have $u(x) = R(x)$ for all $x \in B_t(x_0(t), \frac{3}{4} A r_0)$, and $u(x) = 0$ for all $x \in \mathcal{M}_t \setminus B_t(x_0(t), A r_0)$.

Let $u_{\min}(t) := \min\{\inf_{\mathcal{M}_t} u(\cdot), 0\}$, $t \in [0, t_0)$. If $u_{\min}(t) < 0$, we claim that $\inf_{\mathcal{M}_t} u(\cdot)$ can be achieved. Suppose not, then there exists a sequence of points $x_i \in B_t(x_0(t), A r_0)$ such that $u(x_i) \to u_{\min}(t)$ as $i \to \infty$. By Lemma 4.6.5, the properness of scalar curvature, we may assume that $R(x_i) \to \infty$. So $u(x_i) \geq 0$ for all large $i$, a contradiction.

Then we claim the following holds for all $t \in (0, t_0)$:

$$u_{\min}(t) \leq \liminf_{s \searrow t} u_{\min}(s). \tag{4.7.3}$$

Suppose this is not true at some $t \in (0, t_0)$. Then there exist some $\epsilon > 0$ and a sequence of times $s_i > t$ which converges to $t$ as $i \to \infty$ such that

$$u_{\min}(t) > u_{\min}(s_i) + \epsilon, \tag{4.7.4}$$

for all $i$. Let $x_i \in B_{s_i}(x_0(s_i), A r_0)$ be a point such that

$$u(x_i) \leq u_{\min}(s_i) + \frac{\epsilon}{2} < u_{\min}(t) - \frac{\epsilon}{2}. \tag{4.7.5}$$

If $\mathbb{R}(x_i)$ is not uniformly bounded, then $u(x_i) \geq 0$ for large $i$, which implies $u_{\min}(t) \geq \epsilon > 0$, a contradiction. So we may assume $R(x_i)$ is uniformly bounded, and hence by Lemma 4.6.5 there is a $\delta > 0$ such that $R \leq \delta^{-2}$ in $P(x_i, \delta, -\delta^2) \subset\subset \bigcup_{t \in [0, t_0)} B_t(x_0(t), 2 A r_0)$. So $u$ is uniformly continuous on $\bigcup_i P(x_i, \delta, -\delta^2)$. Since $s_i - t \to 0$, this implies $u(x_i(t)) \leq u(x_i) + \frac{\epsilon}{2}$ for all large $i$. So

$$u_{\min}(t) \leq u(x_i(t)) \leq u(x_i) + \frac{\epsilon}{2}, \tag{4.7.6}$$

which contradicts with (4.7.5). So claim (4.7.3) is true.

Now we argue by maximum principle that the following holds for all times:

$$u_{\min}(t) \geq -\frac{2C_0}{(A r_0)^2}, \tag{4.7.7}$$

where $C_0 > 0$ will be specified below. Suppose not and let $T$ be the supremum of all times $t$ such that (4.7.7) is true on $[0, t]$. Then $T > 0$ and there exists a sequence $t_i > T$ converging to $T$ as $i \to \infty$ such that $u_{\min}(t_i) < -\frac{2C_0}{(Ar_0)^2}$. Using inequality (4.7.3) at $T$, we have that $u_{\min}(T) \leq -\frac{2C_0}{(Ar_0)^2} < 0$.

Since $u_{\min}(T) < 0$, there exists $x_T \in \mathcal{M}_T$ such that $u_{\min}(T) = u(x_T)$. Then by the choice of $T$ it is easy to see the followings hold at $x_T$: $\nabla u = 0$, $\Delta u \geq 0$, and $\frac{\partial}{\partial t}u \leq 0$. By a direct computation we get the following at $x_T$,

$$\begin{aligned} &\mathbb{R}\nabla\varphi + \varphi\nabla R = 0, \\ &2\nabla\varphi \cdot \nabla R = -2\frac{|\nabla\varphi|^2}{\varphi}R = -2\frac{\varphi'^2}{\varphi}\frac{1}{(Ar_0)^2}R. \end{aligned} \tag{4.7.8}$$

By the evolution equation $(\frac{\partial}{\partial t} - \Delta)R = 2|\mathrm{Ric}|^2$, we get

$$\begin{aligned} (\frac{\partial}{\partial t} - \Delta)u =&\varphi'\mathbb{R}\frac{1}{Ar_0}[(\frac{\partial}{\partial t} - \Delta)d_t(x_0(t), x) + \frac{10}{3}r_0^{-1}] \\ &- \varphi''\frac{1}{(Ar_0)^2}\mathbb{R} + 2\varphi|\mathrm{Ric}|^2 - 2\nabla\varphi\nabla\mathbb{R}, \end{aligned} \tag{4.7.9}$$

restricting which at $x_T$ and using (4.7.1), (4.7.8) and $3|\mathrm{Ric}|^2 \geq \mathbb{R}^2$, we obtain the following

$$\begin{aligned} 0 \geq (\frac{\partial}{\partial t} - \Delta)u &\geq \frac{2}{3}\varphi R^2 - \varphi''\frac{1}{(Ar_0)^2}\mathbb{R} + 2\frac{\varphi'^2}{\varphi}\frac{1}{(Ar_0)^2}\mathbb{R} \\ &\geq \frac{2}{3}\varphi R^2 - \frac{C}{(Ar_0)^2}\sqrt{\varphi}\mathbb{R} \\ &\geq \frac{1}{3}(u_{\min}^2(T) - \frac{C_0^2}{(Ar_0)^4}), \end{aligned} \tag{4.7.10}$$

where $C_0 = \frac{3C}{2}$, and we have used $|\frac{2\varphi'^2}{\varphi} - \varphi''| \leq C\sqrt{\varphi}$, and Cauchy inequality $\frac{C}{(Ar_0)^2}\sqrt{\varphi}\mathbb{R} \leq \frac{1}{3}\varphi\mathbb{R}^2 + \frac{C_0^2}{3(Ar_0)^4}$. Since $u_{\min}(T) < 0$, (4.7.10) implies $u_{\min}(T) \geq \frac{-C_0}{(Ar_0)^2}$, a contradiction. So $u_{\min}(t) \geq \frac{-2C_0}{(Ar_0)^2}$ for all $t \in [0, t_0)$, and in particular it implies

$$\mathbb{R}(x) \geq -\frac{2C}{(Ar_0)^2}, \tag{4.7.11}$$

for all $x \in B_t(x_0(t), \frac{3}{4}Ar_0)$, $t \in [0, t_0)$. Letting $A$ go to infinity, we get $\mathbb{R}(x) \geq 0$, for all $x \in \mathcal{M}$. □

**Lemma 4.7.2.** *Let $(M, g)$ be a 3 dimensional complete Riemannian manifold with* Ric $\geq 0$. *Let $(\mathcal{M}, g(t))$ be a generalized singular Ricci flow starting from $(M, g)$. Then* Ric $\geq 0$ *on $\mathcal{M}$.*

*Proof.* For the same reason as in Lemma 4.7.1, it suffices to prove the lemma for a semi-generalized singular Ricci flow $(\mathcal{M}, g(t), x_0)$ on $[0, t_0)$, $x_0 \in M$. We may assume that there is $r_0 > 0$ such that $\bigcup_{t\in[0,t_0)} B_t(x_0(t), r_0)$ is unscathed and $\mathrm{Ric}(x) \leq r_0^{-2}$ there.

Let $\lambda \geq \mu \geq \nu$ be the three eigenvalues of the curvature operator. Then it suffices to show that the following inequality holds on $\mathcal{M}$ for any $a \geq 0$,

$$\mathbb{R} + a(\mu + \nu) \geq 0 \tag{4.7.12}$$

In fact, if this is true, then we have $\mathbb{R} + \epsilon^{-1}(\mu+\nu) \geq 0$ for any $\epsilon > 0$. Multiplying both sides by $\epsilon$ and letting $\epsilon$ go to zero we get $\mu + \nu \geq 0$, i.e. $\mathrm{Ric} \geq 0$.

Now suppose by contradiction that (4.7.12) does not hold for all $a \geq 0$, then we can find $a, a' \geq 0$ with $a < a' < a + \frac{1}{100}$ such that (4.7.12) holds for $a$ but not for $a'$.

By Lemma 2.3.1 we have

$$(\frac{\partial}{\partial t} - \Delta) d_t(x_0(t), x) \geq -\frac{10}{3} r_0^{-1}, \tag{4.7.13}$$

whenever $d_t(x_0(t), x) > r_0$. Choose $\varphi : \mathbb{R} \to \mathbb{R}$ to be a smooth non-negative non-increasing function such that $\varphi = 1$ on $(-\infty, \frac{7}{8}]$, $\varphi = 0$ on $[1, \infty)$ and $\frac{2|\varphi'|^2}{\varphi} + |\varphi''| \leq C_0$.

Let $u : \mathcal{M} \to \mathbb{R}$ be defined by

$$u(x) = \varphi\left(\frac{d_t(x_0(t), x) + \frac{10}{3} r_0^{-1} t}{A r_0}\right) (\mathbb{R} + a'(\mu + \nu)), \tag{4.7.14}$$

and $u_{\min}(t) = \min\{\inf_{\mathcal{M}_t} u(\cdot), 0\}$.

By the same reasoning as Lemma 4.7.1 we can show the following inequality for all $t \in (0, t_0)$:

$$u_{\min}(t) \leq \liminf_{s \searrow t} u_{\min}(s). \tag{4.7.15}$$

Let $T$ be the supremum of all $t$ such that $u(s) \geq -\frac{2C_1}{(Ar_0)^2}$ for all $s \in [0, t]$, where $C_1$ will be specified later. Then $T > 0$ and by (4.7.15) we get

$$u_{\min}(T) \leq -\frac{2C_1}{(Ar_0)^2}. \tag{4.7.16}$$

Since $u_{\min}(T) < 0$, the minimum of $u$ is obtained at some point $x_T \in B_T(x_0(T), Ar_0)$. Let $\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3$ be the orthonormal eigenvectors of Rm corresponding to eigenvalues $\lambda \geq \mu \geq \nu$ at the tangent space of $x_T$. We extend them smoothly to a neighborhood $\mathcal{P}$ around $x_T$ in the following way: first extend them to a neighborhood of $x_T$ in $\mathcal{M}_T$ by parallel translation along radial geodesic emanating from $x_T$ using $\nabla^{g(T)}$, and then extend them in time to make them constant in time in the sense that $\nabla_t \mathbb{V}_i = 0$, $i = 1, 2, 3$, where $\nabla_t$ is the natural space-time extension of $\nabla^{g(t)}$ such that it is compatible with the metric, i.e. $\frac{\partial}{\partial t}\langle X, X\rangle_{g(t)} = 2\langle \nabla_t X, X\rangle_{g(t)}$. Then $\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3$ is an orthonormal basis on $\mathcal{P}$, and $\Delta \mathbb{V}_i = 0$ at $x_T$, $i = 1, 2, 3$.

Let $\tilde{u}(x) = [\mathrm{Rm}(\mathbb{V}_1, \mathbb{V}_1) + \mathrm{Rm}(\mathbb{V}_2, \mathbb{V}_2) + \mathrm{Rm}(\mathbb{V}_3, \mathbb{V}_3) + a'(\mathrm{Rm}(\mathbb{V}_2, \mathbb{V}_2) + \mathrm{Rm}(\mathbb{V}_3, \mathbb{V}_3))] \cdot \varphi\left(\frac{d_t(x_0(t),x) + \frac{10}{3} r_0^{-1} t}{A r_0}\right)$ for all $x \in \mathcal{P}$. Then it is easy to see that $\tilde{u}(x) \geq u(x)$ in $\mathcal{P}$, and the equality is achieved at $x_T$.

We can compute that

$$
\begin{aligned}
(\frac{\partial}{\partial t}-\Delta)\tilde{u} &= -2\nabla\varphi\nabla(\frac{\tilde{u}}{\varphi})+\varphi\cdot(\frac{\partial}{\partial t}-\Delta)[\mathrm{Rm}(\mathbb{V}_1,\mathbb{V}_1)+(a'+1)(\mathrm{Rm}(\mathbb{V}_2,\mathbb{V}_2)+\mathrm{Rm}(\mathbb{V}_3,\mathbb{V}_3))] \\
&\quad +[\mathrm{Rm}(\mathbb{V}_1,\mathbb{V}_1)+(a'+1)(\mathrm{Rm}(\mathbb{V}_2,\mathbb{V}_2)+\mathrm{Rm}(\mathbb{V}_3,\mathbb{V}_3))]\cdot(\frac{\partial}{\partial t}-\Delta)\varphi \\
&= -2\nabla\varphi\nabla(\frac{\tilde{u}}{\varphi})+\varphi\cdot\mathcal{I}+\mathcal{J}\cdot(\frac{\partial}{\partial t}-\Delta)\varphi.
\end{aligned} \tag{4.7.17}
$$

We estimate each term in (4.7.17) at $x_T$. First, recall that Rm evolves by $(\nabla_t-\Delta)\mathrm{Rm}=\mathrm{Rm}^2+\mathrm{Rm}^{\#}$ under Ricci flow, see [55, Proposition 3.19], where

$$
M^2+M^{\#}=\begin{bmatrix}\lambda^2+\mu\nu & 0 & 0\\ 0 & \mu^2+\lambda\nu & 0\\ 0 & 0 & \nu^2+\lambda\mu\end{bmatrix},\quad \text{for any matrix}\quad M=\begin{bmatrix}\lambda & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \nu\end{bmatrix}. \tag{4.7.18}
$$

So by $\nabla\mathbb{V}_i=\Delta\mathbb{V}_i=\frac{\partial}{\partial t}\mathbb{V}_i=0$ at $x_T$ we get

$$
(\frac{\partial}{\partial t}-\Delta)(\mathrm{Rm}(\mathbb{V}_i,\mathbb{V}_i))=((\nabla_t-\Delta)\mathrm{Rm})(\mathbb{V}_i,\mathbb{V}_i)=(\mathrm{Rm}^2+\mathrm{Rm}^{\#})(\mathbb{V}_i,\mathbb{V}_i) \tag{4.7.19}
$$

at $x_T$, $i=1,2,3$, and hence

$$
\begin{aligned}
\mathcal{I}(x_T) &= (\lambda^2+\mu\nu)+(a'+1)(\mu^2+\lambda\nu+\nu^2+\lambda\mu)\\
&\geq \lambda[\lambda+(a+1)(\mu+\nu)]+(a'+1)(\mu^2+\nu^2)+(a'-a)\lambda(\mu+\nu)\\
&\geq (a'+1)(\mu^2+\nu^2)+(a'-a)\lambda(\mu+\nu),
\end{aligned} \tag{4.7.20}
$$

where we used $\lambda+(a+1)(\mu+\nu)\geq 0$. Since $u(x_T)<0$, we have $\lambda<(a'+1)|\mu+\nu|$ at $x_T$ and hence

$$
|(a'-a)\lambda(\mu+\nu)|\leq (a'-a)(a'+1)(\mu+\nu)^2\leq \frac{a'+1}{100}(\mu+\nu)^2\leq \frac{a'+1}{50}(\mu^2+\nu^2). \tag{4.7.21}
$$

Substituting this into (4.7.20) and using $\lambda<(a'+1)|\mu+\nu|$ at $x_T$ again we get

$$
\begin{aligned}
\mathcal{I}(x_T) &\geq \frac{49}{50}(a'+1)(\mu^2+\nu^2)\geq \frac{49}{100}(a'+1)(\mu+\nu)^2\\
&\geq \frac{49}{200(a'+1)}\{[(a'+1)(\mu+\nu)]^2+\lambda^2\}\\
&\geq \frac{49}{400(a'+1)}[(a'+1)(\mu+\nu)+\lambda]^2\\
&= \frac{49}{400(a'+1)\varphi^2}u_{\min}^2(T).
\end{aligned} \tag{4.7.22}
$$

Then we estimate $\mathcal{J}\cdot(\frac{\partial}{\partial t}-\Delta)\varphi$ at $x_T$ by using (4.7.13), $\varphi'\leq 0$, $0<\varphi\leq 1$ at $x_T$, and $u_{\min}(T)<0$ as below

$$\begin{aligned}(\mathcal{J}\cdot(\frac{\partial}{\partial t}-\Delta)\varphi)(x_T)&=[\lambda+(a'+1)(\mu+\nu)](\frac{\partial}{\partial t}-\Delta)\varphi\\&=\frac{1}{Ar_0}\left[\varphi'\left((\frac{\partial}{\partial t}-\Delta)d_t(x_0(t),x)+\frac{10}{3}r_0^{-1}\right)-\varphi''\frac{1}{Ar_0}\right]u_{\min}(T)\\&\geq\frac{1}{(Ar_0)^2\varphi}|\varphi''|u_{\min}(T).\end{aligned}\tag{4.7.23}$$

Next, since $\tilde{u}$ obtains its minimum on $\mathcal{P}$ at $x_T$ and $\tilde{u}(x_T)=u(x_T)=u_{\min}(T)$, we get

$$\left(-2\nabla\varphi\nabla(\frac{\tilde{u}}{\varphi})\right)(x_T)=2\frac{|\nabla\varphi|^2}{\varphi^2}u_{\min}(T)=2\frac{|\varphi'|^2}{\varphi^2}\frac{1}{(Ar_0)^2}u_{\min}(T).\tag{4.7.24}$$

Now applying the maximum principle at $x_T$ and using (4.7.22), (4.7.23) and (4.7.24), we get

$$\begin{aligned}0\geq(\frac{\partial}{\partial t}-\Delta)\tilde{u}(x_T)&\geq\frac{49}{400(a'+1)\varphi}u_{\min}^2(T)+\frac{1}{(Ar_0)^2\varphi}|\varphi''|u_{\min}(T)+2\frac{|\varphi'|^2}{(Ar_0)^2\varphi^2}u_{\min}(T)\\&\geq\frac{49}{400(a'+1)\varphi}\left[u_{\min}^2(T)+\frac{400(a'+1)}{49(Ar_0)^2}\left(\frac{2|\varphi'|^2}{\varphi}+|\varphi''|\right)u_{\min}(T)\right]\\&\geq\frac{49}{400(a'+1)\varphi}\left[u_{\min}^2(T)+\frac{400(a'+1)C_0}{49(Ar_0)^2}u_{\min}(T)\right],\end{aligned}\tag{4.7.25}$$

where we have used $\frac{2|\varphi'|^2}{\varphi}+|\varphi''|\leq C_0$. Since $u_{\min}(T)<0$, (4.7.25) implies immediately $u_{\min}(T)\geq-\frac{C_1}{(Ar_0)^2}$, where $C_1=\frac{400(a'+1)C_0}{49}$. This contradicts with (4.7.16). So $u_{\min}(t)\geq-\frac{2C_0}{(Ar_0)^2}$ for all $t\in[0,t_0)$. Letting $A\to\infty$ we get $R+a'(\mu+\nu)\geq 0$ on $\mathcal{M}$, which contradicts the assumption of $a'$. So (4.7.12) holds for all $a\geq 0$, and hence by the argument at beginning the conclusion of the Lemma follows.

□

The next lemma says that in a 3-dimensional manifold with Ric $\geq 0$, no singularity can form within finite distance along a minimizing geodesic covered by final time-slices of strong $\delta$-necks. We prove it by a contradiction argument, suppose the assertion does not hold, then by the condition of Ric $\geq 0$, we can show that the blow-up limit of the 'singularity' is a smooth cone, and there is a Ricci flow whose final time-slice is in the smooth part of the cone, which is impossible.

**Lemma 4.7.3.** *For any sufficiently small $\delta > 0$ the following holds: Let $(M, g)$ be a 3 dimensional Riemannian manifold with* $\mathrm{Ric} \geq 0$. *Let $\gamma : [0, s_0) \to M$ (where $s_0 \in \mathbb{R}_+ \cup \{\infty\}$) be a unit speed minimizing geodesic such that $R(\gamma(s))$ does not stay bounded for $s \to s_0$, and assume there are constants $c, \varphi > 0$ such that all points on $\gamma$ are centers of strong $\delta$-necks on the time interval $[-c, 0]$, and the strong $\delta$-necks have $\varphi$-positive curvature.*

*Then $s_0 = \infty$.*

*Proof.* Suppose by contradiction that $s_0 < \infty$. Let $\eta$ be from Lemma 2.2.5.

Since every point on $\gamma$ is the center of some strong $\delta$-neck, we get that $\gamma$ lies inside some open subset $N \subset M$ that is diffeomorphic to $S^2 \times (0, 1)$ and which is covered by final time-slices of strong $\delta$-necks. Consider the length metric induced by the Riemannian metric on $N$, and then $N'$ be the completion of $N$. Then $N'$ is a disjoint union of $\overline{N}$ and a single point $p$.

Consider the rescalings $iN'$ for all $i \in \mathbb{N}$. Then by the Bishop-Gromov volume comparison we can deduce that for any $d > 0$, the $d$-balls $B^{iN'}(p, d)$ in $iN'$ are uniformly totally bounded. Therefore, by Gromov's compactness theorem, we have the following Gromov-Hausdorff convergence by passing to a subsequence $\{i_k N'\}$:

$$(i_k N', p) \xrightarrow{k \to \infty} (X, p_\infty) \tag{4.7.26}$$

We shall show that $X$ is a smooth metric cone with cone point $p_\infty$, and the convergence is actually smooth on $X_0 = X - \{p_\infty\}$.

Let $x \in B(p, s_0) - \{p\} \subset N'$, then by Lemma 2.2.5 we get

$$R^{-1/2}(x) \leq \eta\, d(p, x) \qquad \text{on} \qquad B(p, s_0) - \{p\}. \tag{4.7.27}$$

We claim that there exists $C > 0$ such that

$$R^{-1/2}(x) \geq C^{-1} d(p, x) \qquad \text{on} \qquad B(p, s_0) - \{p\}. \tag{4.7.28}$$

Suppose not, then there exists a sequence $\{x_k\} \subset B(p, s_0) - \{p\}$ such that

$$R^{-1/2}(x_k) \leq C_k^{-1} d(p, x_k), \tag{4.7.29}$$

where $C_k \to \infty$ as $k \to \infty$. We abbreviate $d(p, x_k)$ as $d_k$ and $R(x_k)$ as $R_k$.

Since $x_k$ is the center of a $\delta$-neck, there is a diffeomorphism onto its image $\phi_k : (-\delta^{-1}, \delta^{-1}) \times S^2 \to N'$ under which $(N', x_k)$ is $\delta$-close to $(-\delta^{-1}, \delta^{-1}) \times S^2$ at scale $R_k^{-1/2}$. Let $U_k = \phi_k((-100, 100) \times S^2)$, then $U_k$ separates $N'$ into two components.

Suppose $x \in B(p, s_0) - \{p\}$ is not in $U_k$, then it is easy to see either

$$d(x, p) > d_k + 10R_k^{-1/2}, \qquad \text{or} \qquad d(x, p) < d_k - 10R_k^{-1/2}. \tag{4.7.30}$$

In other words, we have

$$B(p, d_k + 10R_k^{-1/2}) - B(p, d_k - 10R_k^{-1/2}) \subset U_k. \tag{4.7.31}$$

Applying the Bishop-Gromov volume comparison on $N'$, we have $r^{-2}vol(\partial B(p,r))$ is non-increasing for all $r \in (0, s_0)$. In particular, let $v_0 = s_0^{-2}vol(\partial B(p, s_0))$, then $r^{-2}vol(\partial B(p,r)) \geq v_0$ for all $0 < r < s_0$. So by (4.7.31) we can estimate the volume of $U_k$ from below:

$$\begin{aligned} vol(U_k) &\geq \int_{d_k-10R_k^{-1/2}}^{d_k+10R_k^{-1/2}} vol(\partial B(p,r))\,dr \geq \int_{d_k-10R_k^{-1/2}}^{d_k+10R_k^{-1/2}} v_0 r^2\,dr \\ &\geq \frac{9}{16}\int_{d_k-10R_k^{-1/2}}^{d_k+10R_k^{-1/2}} v_0 d_k^2\,dr = \frac{45}{4} v_0 d_k^2 R_k^{-1/2}, \end{aligned} \tag{4.7.32}$$

where in the third inequality we used (4.7.29), which implies $d_k - R_k^{-1/2} \geq \frac{3}{4} d_k$ for large $k$.

By the closeness of the metric on $U_k$ with the standard cylindrical metric at scale $R_k^{-1/2}$, we get an upper bound on the volume of $U_k$:

$$vol(U_k) \leq 2 \cdot R_k^{-3/2} \cdot 200 \cdot 8\pi = 3200\pi R_k^{-3/2} < 3200\pi C_k^{-2} d_k^2 R_k^{-1/2}, \tag{4.7.33}$$

where we used (4.7.29) in the last inequality. Combining (4.7.32) with (4.7.33) we get $C_k^2 \leq \frac{12800\pi}{45 v_0}$, which is impossible for large $k$. Thus there exists $C > 0$ such that (4.7.28) holds.

Therefore, by (4.7.28) we see that the convergence on $X_0$ is smooth. So there is $v_1 > 0$ such that $d^{-2}vol(\partial B^X(p_\infty, d)) = v_1$ for all $d \in (0, \infty)$. So by the rigidity of volume comparison, we see that any Jacobi field along any geodesic emanating from $p_\infty \in X$ has linear growth, which implies that $X$ is a smooth metric cone. By (4.7.27), $X_0$ is nowhere flat.

All points in a neighborhood of $p$ are centers of strong $\delta$-necks on $[-c, 0]$, which has $\varphi$-positive curvature. So under the blow-up rescalings this implies that any point $x \in X_0$ is the center of a strong $2\delta$-neck on $[-\frac{1}{2}c, 0]$, which has non-negative sectional curvature. This contradicts the fact that open pieces in non-flat cones cannot arise as the result of Ricci flow with non-negative curvature [55, Prop 4.22].

□

**Theorem 4.7.4.** *(Theorem 4.1.3) Let $(M, g)$ be a 3d complete Riemannian manifold with* $\mathrm{Ric} \geq 0$*. There exist $T \in (0, \infty]$ and a smooth Ricci flow $(M, g(t))$ with $g(0) = g$ defined on $[0, T)$. Moreover, if $T < \infty$, then* $\limsup_{t \nearrow T} |\mathrm{Rm}|(x,t) = \infty$ *for all $x \in M$.*

*Proof.* First we assume $M$ is orientable. By Theorem 4.6.14 there is a generalized singular Ricci flow $(\mathcal{M}, g(t))$ starting from $(M, g)$, and by Lemma 4.7.2, $\mathcal{M}$ has non-negative Ricci curvature.

Let $x_0 \in M$. Suppose $x_0$ survives until $t_0 > 0$ in $\mathcal{M}$. We claim that the component of $\mathcal{M}_t$ that contains $x_0(t)$ is complete for all $t \in (0, t_0]$. Suppose not, then $\sup_{B_t(x_0(t),A)} R = \infty$ for some $A > 0$ and $t \in (0, t_0]$. By Lemma 4.2.3 and 4.2.4, we can find a minimizing geodesic $\gamma : [0, 1) \to \mathcal{M}_t$ such that $\lim_{s \to 1} R(\gamma(s)) = \infty$, and there exist $c, \varphi > 0$ such that for all $s$ close to 1, $\gamma(s)$ are centers of strong $\delta$-necks on $[-c, 0]$, which have $\varphi$-positive curvature. This contradicts Lemma 4.7.3.

Since $\mathrm{Ric} \geq 0$, for any $A > 0$, the parabolic neighborhood $P(x_0, A, t_0)$ is contained in $\bigcup_{t \in [0,t_0]} B_t(x_0(t), A)$, which is relatively compact. So every point in $M$ survives until $t_0$. Let $T \in (0, \infty]$ be the supremum of all times until which $x_0$ survives. Then $T$ is also the supremum of the survival times of points in $M$. Suppose $T < \infty$, since $\mathcal{M}$ is forward 0-complete, we have $\limsup_{t \nearrow T} |\mathrm{Rm}|(x(t)) = \infty$ for all $x \in M$. So the spacetime restricted on the subset $\bigcup_{t \in [0,T)} M(t)$ is the desired smooth Ricci flow.

Now suppose $M$ is not orientable. Let $\widehat{M} \to M$ be the 2-fold orientation covering. By Theorem 4.6.14, there are a generalized singular Ricci flow $(\widehat{\mathcal{M}}, g(t))$ starting from $\widehat{M}$, and an isometry $\sigma : \widehat{\mathcal{M}} \to \widehat{\mathcal{M}}$ that acts free on the subset of points that can survive back to $\widehat{M}$. As before, there exists $T \in (0, \infty]$ such that $\widehat{M}$ survives on $[0, T)$, and $\limsup_{t \nearrow T} |\mathrm{Rm}|(x(t)) = \infty$ for all $x \in \widehat{M}$ if $T < \infty$. The smooth Ricci flow claimed in the theorem is the quotient of $\bigcup_{t \in [0,T)} \widehat{M}(t)$ by the free action of $\sigma$.

□

# Chapter 5

# Steady gradient Ricci solitons with positive curvature operator

## 5.1 Introduction and main results

Ricci solitons are self-similar solutions of the Ricci flow equation, and they often arise as singularity models of Ricci flows. In particular, a steady gradient soliton is a smooth complete Riemannian manifold $(M, g)$ satisfying

$$\mathrm{Ric} = \nabla^2 f \tag{5.1.1}$$

for some smooth function $f$ on $M$, which is called a potential function. The soliton generates a Ricci flow for all time by $g(t) = \phi_t^*(g)$, where $\{\phi_t\}_{t\in(-\infty,\infty)}$ is the one-parameter group of diffeomorphisms generated by $-\nabla f$ with $\phi_0$ the identity.

In dimension 2, the only non-flat rotationally symmetric steady gradient soliton is Hamilton's cigar soliton [37]. In any dimension $n \geq 3$, the only non-flat rotationally symmetric steady gradient soliton is the Bryant soliton, which is constructed by Bryant [11]. It is an open problem whether there are any 3d steady gradient solitons other than the 3d Bryant soliton and quotients of $\mathbb{R} \times$ Cigar, see e.g. [14, 17, 24, 31].

Hamilton conjectured that there exists a 3d flying wing, which is a $\mathbb{Z}_2 \times O(2)$-symmetric 3d steady gradient soliton asymptotic to a sector with angle $\alpha \in (0, \pi)$. The term flying wing is also used by Hamilton to describe certain translating solutions in mean curvature flow. A lot of important progress has been made for the mean curvature flow flying wings in the past two decades. For example, the flying wings in $\mathbb{R}^3$ are completely classified by the works of X.J. Wang [67] and Hoffman-Ilmanen-Martin-White [42]. Moreover, higher dimensional examples were constructed independently by Bourni-Langford-Tinaglia [7] and Hoffman-Ilmanen-Martin-White [42].

Despite many analogies between the Ricci flow and mean curvature flow, Hamilton's flying wing conjecture remains open. A proposed approach is to obtain the flying wings as limits of solutions of elliptic boundary value problems. This is how the flying wings in mean curvature flow are constructed, where the solutions can be parametrized as graphs [67]. However, it seems hard to choose such a parametrization in Ricci flow to get a strictly elliptic equation. In this chapter, we confirm Hamilton's conjecture by using a different approach.

Our first theorem finds a family of non-rotationally symmetric $n$-dimensional steady gradient solitons with prescribed Ricci curvature at a point in all dimensions $n \geq 3$. This gives an affirmative answer to the open problem by Cao whether there exists a non-rotationally symmetric steady Ricci soliton in dimensions $n \geq 4$ [15]. Throughout this section, the quadruple $(M, g, f, p)$ denotes a steady gradient soliton, where $f$ is the potential function and $p$ is a critical point of $f$.

**Theorem 5.1.1.** *Given any $\alpha \in (0,1)$, there exists an $n$-dimensional $\mathbb{Z}_2 \times O(n-1)$-symmetric steady gradient soliton $(M, g, f, p)$ with positive curvature operator, such that $\lambda_1 = \alpha\lambda_2 = \cdots = \alpha\lambda_n$, where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of the Ricci curvature at $p$.*

The 3d steady gradient solitons from Theorem 5.1.1 are collapsed, which is an easy consequence of its asymptotic geometry. This also follows from the uniqueness of the Bryant soliton among 3d non-collapsed steady gradient solitons by Brendle [9]. Moreover, we show that the n-dimensional steady gradient solitons from Theorem 5.1.1 are non-collapsed for all $n \geq 4$. They are analogous to the non-collapsed translators in mean curvature flow constructed by Hoffman-Ilmanen-Martin-White [42].

Our second theorem says that a $\mathbb{Z}_2 \times O(2)$-symmetric 3d steady gradient soliton must be a Bryant soliton if the asymptotic cone is a ray. So the family of 3d steady gradient solitons from Theorem 5.1.1 are all flying wings, which confirms Hamilton's conjecture. Figure 1 is the picture of a 3d flying wing.

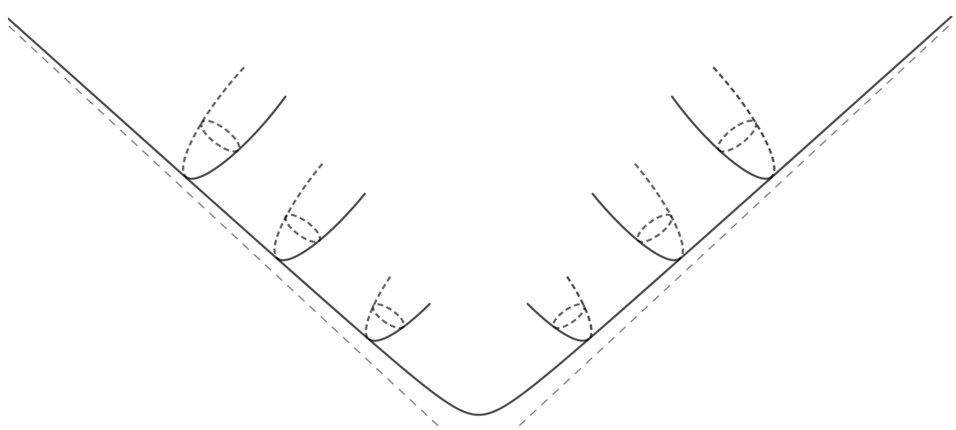

Figure 5.1: A 3d flying wing

**Theorem 5.1.2.** *Let $(M, g, f, p)$ be a $\mathbb{Z}_2 \times O(2)$-symmetric 3d steady gradient soliton. Suppose its asymptotic cone is a ray. Then it is isometric to the Bryant soliton.*

**Corollary 5.1.3.** *A $\mathbb{Z}_2 \times O(2)$-symmetric but non-rotationally symmetric 3d steady gradient soliton with positive curvature operator is a flying wing. In particular, the 3d steady gradient solitons from Theorem 5.1.1 are all flying wings.*

It has been wondered whether the scalar curvature vanishes at infinity in all 3d steady gradient solitons. By Theorem 5.1.4 we see that this fails in 3d flying wings. More precisely, Theorem 5.1.4 shows that the scalar curvature has a positive limit along the edges of the wing, and there is a quantitative relation between this limit and the angle of the asymptotic cone.

**Theorem 5.1.4.** *Let $(M, g, f, p)$ be a $\mathbb{Z}_2 \times O(2)$-symmetric 3d steady gradient soliton, whose asymptotic cone is a metric cone over the interval $[-\frac{\alpha}{2}, \frac{\alpha}{2}]$ for some $\alpha \in [0, \pi]$. Let $\Gamma : (-\infty, \infty) \to M$ be the complete geodesic fixed by the $O(2)$-action, then*

$$\lim_{s \to \infty} R(\Gamma(s)) = R(p) \sin^2 \frac{\alpha}{2}. \tag{5.1.2}$$

We prove in the following corollary that the asymptotic geometry of a 3d flying wing is uniquely determined by the angle of the asymptotic cone. In particular, it converges to $\mathbb{R} \times$ Cigar along the edges. This is analogous to mean curvature flow flying wings, where the asymptotic geometry is uniquely determined by the width of the slab that contains the wing [7].

**Corollary 5.1.5.** *Let $(M, g, f, p)$ be a 3d flying wing, whose asymptotic cone is a sector with angle $\alpha \in (0, \pi)$. Then for any sequence of points $q_i \in \Gamma$ going to infinity, the sequence of pointed Riemannian manifolds $(M, g, q_i)$ smoothly converges to $\mathbb{R} \times \mathrm{Cigar}$, where the scalar curvature at the tip of the cigar is $R(p) \sin^2 \frac{\alpha}{2}$.*

As an application of Theorem 5.1.2 and 5.1.4, we construct a sequence of 3d flying wings whose asymptotic cones have arbitrarily small angles.

**Corollary 5.1.6.** *There exists a sequence of 3d flying wings $\{(M_i, g_i)\}_{i=1}^{\infty}$, whose asymptotic cone is a sector with angle $\alpha_i \in (0, \pi)$ such that $\lim_{i \to \infty} \alpha_i = 0$.*

The concept of flying wing can be naturally generalized to all dimensions $n \geq 3$. We say an $n$-dimensional $O(n-2) \times O(2)$-symmetric steady gradient soliton with positive curvature operator is a flying wing if its asymptotic cone is a metric cone over a geodesic ball of radius $r \in (0, \frac{\pi}{2})$ in $S^{n-2}$. Then we have

**Theorem 5.1.7.** *In dimension $n \geq 3$, there exists an $n$-dimensional flying wing.*

Moreover, similar conclusions in Corollary 5.1.5 and 5.1.6 also hold for all higher dimensional flying wings.

The structure of this chapter is as follows. In Section 5.2, we prove Theorem 5.1.1 by obtaining the steady gradient solitons as limits of appropriate expanding gradient solitons, whose construction is based on Deruelle's results [32]. More specifically, we choose a sequence of expanding gradient solitons whose asymptotic volume ratio goes to zero, and prove that by passing to a subsequence they converge to a steady gradient soliton. In dimension 3, the sequence of expanding gradient solitons is between two sequences converging respectively to the 3d Bryant soliton and $\mathbb{R}\times$ Cigar.

In Section 5.3, we study the asymptotic geometry of $\mathbb{Z}_2 \times O(2)$-symmetric 3d steady gradient solitons that are not Bryant solitons. We prove a dimension reduction theorem which shows that the soliton smoothly converges to $\mathbb{R}\times$ Cigar at infinity. We also show that the higher dimensional solitons from Theorem 5.1.1 are non-collapsed.

In Section 5.4, we first prove Theorem 5.1.4 and then use it to prove Theorem 5.1.2 and all the corollaries. To prove Theorem 5.1.4, we study the variations of $\nabla f$ along certain minimizing geodesics. By the soliton equation this amounts to computing the integral of the Ricci curvature along the geodesics. Then Theorem 5.1.4 follows by estimating this integral. Our main tools are the dimension reduction theorem, curvature comparison arguments, and Perelman's curvature estimates for Ricci flows with non-negative curvature operator.

Theorem 5.1.2 is proved by a bootstrap argument. Suppose the soliton is not a Bryant soliton. So the dimension reduction theorem applies. By the $\mathbb{Z}_2 \times O(2)$-symmetry, the soliton away from the edges is a warped-product metric with $S^1$-fibers. First, by using the dimension reduction theorem and some computations we obtain an estimate on the length of the $S^1$-fibers, which shows that it increases slower than the square root of the distance to the critical point.

Second, by using the estimate from the first step and similar computations we obtain a better estimate, which shows that the length function stays bounded at infinity. Since the length function is concave by the non-negativity of the curvature, this implies that the scalar curvature does not vanish along the edges. This by Theorem 5.1.4 contradicts the assumption that the asymptotic cone is a ray, hence proves Theorem 5.1.2.

## 5.2 A family of non-rotaionally symmetric steady gradient solitons

The main result in this section is Theorem 5.1.1. The outline of the proof is as follows. We first construct a sequence of smooth families of expanding gradient solitons $\{(M_{i,\mu}, g_{i,\mu}, p_{i,\mu}), \mu \in [0,1]\}_{i=0}^{\infty}$ with positive curvature operator, such that $(M_{i,0}, g_{i,0}, p_{i,0})$ converges to a Bryant soliton, and $(M_{i,1}, g_{i,1}, p_{i,1})$ converges to the product of $\mathbb{R}$ and an (n-1)-dimensional Bryant soliton if $n \geq 4$, or a cigar soliton if $n = 3$. Moreover, we require that the asymptotic volume ratio of each expanding gradient solitons tends to zero uniformly as $i \to \infty$.

Let $\alpha_i(\mu)$ be the quotients of the smallest and largest eigenvalues of the Ricci curvature at $p_{i,\mu}$ in $(M_{i,\mu}, g_{i,\mu}, p_{i,\mu})$, then $\alpha_i(\mu)$ is a smooth function in $\mu$ for each fixed $i$. Then for any $\alpha \in (0,1)$, there is some $\mu_i \in (0,1)$ such that $\alpha_i(\mu_i) = \alpha$. Since the asymptotic volume ratio of $(M_{i,\mu_i}, g_{i,\mu_i}, p_{i,\mu_i})$ goes to zero, we can show that it subconverges to an n-dimensional steady gradient soliton $(M, g, p)$ with positive curvature operator. In particular, the quotients of the smallest and largest eigenvalues of the Ricci curvature at $p$ in $(M, g, p)$ is equal to $\alpha$.

To construct the expanding gradient solitons we use Deruelle's work [32]. He showed that for any $(n-1)$-dimensional smooth simply connected Riemannian manifold $(X_1, g_{X_1})$ with $\mathrm{Rm} > 1$, there exists a unique expanding gradient soliton $(M_1, g_1, p_1)$ with positive curvature operator that is asymptotic to the cone $(C(X_1), dr^2 + r^2 g_{X_1})$. Moreover, there is a one-parameter smooth family of expanding gradient solitons connecting $(M_1, g_1, p_1)$ to an expanding gradient soliton $(M_0, g_0, p_0)$, whose asymptotic cone is rotationally symmetric. By Chodosh's work the soliton $(M_0, g_0, p_0)$ is rotationally symmetric, and hence is a Bryant expanding soliton [23].

### 5.2.1 Preliminaries

In this subsection we fix some notions that will be frequently used. First, we recall some standard notions and facts from Alexandrov geometry: Let $(M, g)$ be a non-negatively curved Riemannian manifold, then for any triple of points $o, p, q \in M$, the comparison angle $\widetilde{\measuredangle} poq$ is the corresponding angle formed by minimizing geodesics with lengths equal to $d(o,p), d(o,q), d(p,q)$ in Euclidean space. Let $op, oq$ be two minimizing geodesics in $M$ between $o, p$ and $o, q$, and $\measuredangle poq$ be the angle between them at $o$, then $\measuredangle poq \geq \widetilde{\measuredangle} poq$. Moreover, for any $p' \in op$ and $q' \in oq$, the monotonicity of angle comparison implies $\widetilde{\measuredangle} p'oq' \geq \widetilde{\measuredangle} poq$.

For a non-negatively curved Riemannian manifold $(M, g, p)$ and two rays $\gamma_1, \gamma_2$ with unit speed starting from $p$, the limit $\lim_{r\to\infty} \widetilde{\measuredangle}\gamma_1(r)p\gamma_2(r)$ exists and we say it is the angle at infinity between $\gamma_1$ and $\gamma_2$. Moreover, the space $(X, d_X)$ of equivalent classes of rays is a compact length space, where two rays are equivalent if and only if the angle at infinity between them is zero, and the distance between two rays is the limit of the angle at infinity between them. The asymptotic cone is a metric cone over the space of equivalent classes of rays, and it is isometric to the Gromov-Hausdorff limit of any blow-down sequence of the manifold, see e.g. [43].

Next, we define what we mean by a Riemannian manifold to be $\mathbb{Z}_2 \times O(n-1)$-symmetric. First, we define an $O(n-1)$-action on the Euclidean space $\mathbb{R}^n = \{(x_1, ..., x_n) : x_i \in \mathbb{R}\}$, by extending the standard $O(n-1)$-action on $\mathbb{R}^{n-1} = \{x_n = 0\} \subset \mathbb{R}^n$ in the way such that it fixes the $x_n$-axis. Then we define a $\mathbb{Z}_2 \times O(n-1)$-action on $\mathbb{R}^n$ by futhermore defining a $\mathbb{Z}_2$-action to be generated by a reflection that fixes the hypersurface $\{x_n = 0\}$.

Let $\Gamma_0 = \{x_1 = \cdots = x_{n-1} = 0\}$, $N_0 = \{x_1 = \cdots = x_{n-2} = 0, x_{n-1} > 0\}$ and

$\Sigma_0 = \{x_n = 0\}$. Then $\Gamma_0$ is the fixed point set of the $O(n-1)$-action, $\Sigma_0$ is the fixed point set of the $\mathbb{Z}_2$-action, and $N_0$ is one of the two connected components of the fixed point set of a subgroup isomorphic to $O(n-2)$.

**Definition 5.2.1.** We say that an $n$-dimensional Riemannian manifold $(M^n, g)$ is $\mathbb{Z}_2 \times O(n-1)$-symmetric if there exist an isometric $\mathbb{Z}_2 \times O(n-1)$-action, and a diffeomorphism $\Phi : M^n \to \mathbb{R}^n$ such that $\Phi$ is equivariant with the two actions, where the action on $\mathbb{R}^n$ is defined as above.

Let $\Gamma = \Phi^{-1}(\Gamma_0)$, $\Sigma = \Phi^{-1}(\Sigma_0)$, and $N = \Phi^{-1}(N_0)$. Then it is easy to see that

1. $\Gamma$ is a geodesic that goes to infinity at both ends.
2. $\Sigma$ is a rotationally symmetric $(n-1)$-dimensional totally geodesic submanifold.
3. $N$ is a totally geodesic surface diffeomorphic to $\mathbb{R}^2$.
4. $\Phi^{-1}(0)$ is the unique fixed point of the $\mathbb{Z}_2 \times O(n-1)$-action, at which $\Gamma$ intersects orthogonally with $\Sigma$.

Moreover, consider the projection $\pi : M \to N$, which maps a point $x \in M$ to a point $y \in N$, which is the image of $x$ under some action in $O(n-1)$. Equip $N$ with the induced metric $g_N$, then $\pi$ is a Riemannian submersion, and $N$ is an integral surface of the horizontal distribution. So there is a smooth positive function $\varphi : N \to \mathbb{R}$ such that $g = g_N + \varphi^2 g_{S^{n-2}}$ on $M \setminus \Gamma$, where $g_{S^{n-2}}$ is the standard round metric on $S^{n-2}$.

In this chapter, we study n-dimensional expanding or steady gradient soliton $(M^n, g)$ with non-negative curvature operator, whose potential function $f$ has a critical point $p$. We denote it by a quadruple $(M^n, g, f, p)$ (and sometimes a triple $(M^n, g, p)$). In the case of a steady gradient soliton, $R$ attains its maximum at $p$ by the identity $R + |\nabla f|^2 = \text{const.}$, and $p$ is the unique critical point of $f$ if $\text{Rm} > 0$. In the case of an expanding gradient soliton, by the soliton equation $\nabla^2 f = \text{Ric} + cg$, $c > 0$, and $\text{Rm} \geq 0$, it follows that $\nabla^2 f \geq cg$ and hence $p$ is the unique critical point of $f$, and $f$ attains its minimum at $p$. Then by using the identity $\nabla f(R) = -2\text{Ric}(\nabla f, \nabla f)$ we see that $R$ is non-increasing along any integral curve of $\nabla f$. So $R$ attains its maximum at $p$.

We assume $(M^n, g, f, p)$ is $\mathbb{Z}_2 \times O(n-1)$-symmetric, and fix the notions $\Gamma, N, \varphi, \Sigma$ from above, and assume $\Gamma : (-\infty, \infty) \to M$ has unit speed and $\Gamma(0) = p$. Assume $\text{Rm} > 0$. Then it is easy to see that $p$ is the unique point fixed by the $\mathbb{Z}_2 \times O(n-1)$-action. Moreover, by the soliton equation $\nabla^2 f = \text{Ric} + cg$, $c \geq 0$, it follows that the potential function $f$ is invariant under the actions. So the geodesic $\Gamma$, and all the unit speed geodesics in $\Sigma$ starting from $p$ are integral curves of $\frac{\nabla f}{|\nabla f|}$.

Moreover, use $i, j, k, l$ for indices on $N$, and $\alpha, \beta$ and $g_{\alpha\beta}$ for indices and metric components on $S^{n-2}$ with the standard round metric with radius one. Then by a computation the nonzero components of the curvature tensor of $(M \setminus \Gamma, g)$ are

$$R^M_{ijkl} = R^N_{ijkl}, \quad R^M_{i\alpha\beta j} = -g_{\alpha\beta}(\varphi\nabla^2_{i,j}\varphi), \quad R^M_{\alpha\beta\beta\alpha} = (1-|\nabla\varphi|^2)\varphi^2(g_{\alpha\alpha}g_{\beta\beta} - g^2_{\alpha\beta}). \tag{5.2.1}$$

So by $\mathrm{Rm} \geq 0$ and the second equation we have $\nabla^2\varphi \leq 0$ and $\varphi$ is concave.

### 5.2.2 Proof of Theorem 5.1.1

To prove Theorem 5.1.1, we will take a limit of a sequence of expanding gradient solitons with $R(p) = 1$, where $p$ is the critical point of the potential function. To do this, we need an injectivity radius lower bound and a uniform curvature bound. The curvature bounds follows directly from $R_{\max} = R(p) = 1$. Since the curvature is positive, by a well-known fact of Gromoll and Meyer (see [18]), we always have an injectivity radius estimate

$$\inf_{x\in M} \mathrm{inj}_g(x) \geq \frac{\pi}{\sqrt{R_{\max}}}. \tag{5.2.2}$$

Recall that if $(M^n, g, f, p)$ is an expanding gradient soliton satisfying

$$\mathrm{Ric} + \lambda g = \nabla^2 f \tag{5.2.3}$$

for some $\lambda > 0$. Then it generates a Ricci flow $g(t) := (2\lambda t)\phi^*_{t-\frac{1}{2\lambda}} g$, $t \in (0,\infty)$, where $\{\phi_s\}_{s\in\left(-\frac{1}{2\lambda},\infty\right)}$ is the one-parameter diffeomorphisms generated by the time-dependent vector field $\frac{-1}{1+2\lambda s}\nabla f$ with $\phi_0$ the identity. Moreover, $g(t)$ is an expanding gradient soliton satisfying

$$\mathrm{Ric}(g(t)) + \frac{1}{2t}g(t) = \nabla^2 f_t, \tag{5.2.4}$$

where $f_t = \phi^*_{t-\frac{1}{2\lambda}} f$.

Let $(M^n_i, g_i, f_i, p_i)$ be a sequence of $\mathbb{Z}_2 \times O(n-1)$-symmetric expanding gradient solitons with positive curvature operator, which satisfies $R(p_i) = 1$ and the asymptotic volume ratio $\mathrm{AVR}(g_i) \to 0$ as $i \to \infty$. Let $C_i > 0$ be the constant such that $(M^n_i, g_i, f_i, p_i)$ satisfies the soliton equation

$$\mathrm{Ric}(g_i) + \frac{1}{2C_i} g_i = \nabla^2 f_i. \tag{5.2.5}$$

Then the following lemma shows $C_i \to \infty$ as $i \to \infty$, and hence there is a subsequence of $(M^n_i, g_i, f_i, p_i)$ smoothly converging to a steady gradient soliton.

**Lemma 5.2.2.** *Let $(M^n_i, g_i, f_i, p_i)$ be a sequence of $\mathbb{Z}_2 \times O(n-1)$-symmetric expanding gradient solitons with positive curvature operator. Suppose $R_{g_i}(p_i) = 1$ and $\mathrm{AVR}(g_i) \to 0$ as $i \to \infty$. Then a subsequence of $(M_i, g_i, f_i, p_i)$ smoothly converges to an $n$-dimensional $\mathbb{Z}_2 \times O(n-1)$-symmetric steady gradient soliton $(M, g, f, p)$.*

*Proof.* Suppose $(M_i^n, g_i, f_i, p_i)$ satisfies

$$\mathrm{Ric}(g_i) + \frac{1}{2C_i} g_i = \nabla^2 f_i \tag{5.2.6}$$

for some constant $C_i > 0$. Let $(M_i, \widetilde{g}_i(t), f_{i,t}, p_i)$, $t \in (0, \infty)$, be the Ricci flow generated by $(M_i, g_i, f_i, p_i)$, where $\widetilde{g}_i(t) = \frac{t}{C_i}\phi^*_{i,t-C_i} g_i$, $f_{i,t} = \phi^*_{i,t-C_i} f_i$, and $\{\phi_{i,s}\}_{s\in(-C_i,\infty)}$ is the family of diffeomorphisms generated by $\frac{-s}{s+C_i}\nabla f_i$ with $\phi_0$ the identity. By a direct computation we can show

$$\mathrm{Ric}(\widetilde{g}_i(t)) + \frac{1}{2t}\widetilde{g}_i(t) = \nabla^2 f_{i,t}, \tag{5.2.7}$$

for all positive time $t$. In particular, we have $\widetilde{g}_i(C_i) = g_i$ and $R_{\widetilde{g}_i(1)}(p_i) = C_i$.

We claim that $C_i \to \infty$ as $i \to \infty$: Suppose this is not true. Then by passing to a subsequence we may assume $C_i \le C$ for some constant $C > 0$ and all $i$. We shall use $C$ to denote all positive constant that is independent of $i$.

First, by (5.2.2) we have $\mathrm{inj}_{\widetilde{g}_i(1)}(p_i) \ge C^{-1}$ and

$$R_{\widetilde{g}_i(t)}(x) \le R_{\widetilde{g}_i(t)}(p_i) \le \frac{C}{t}, \tag{5.2.8}$$

for all $x \in M_i$ and $t \in (0, \infty)$. So by Hamilton's compactness for Ricci flow we may assume after passing to a subsequence that $(M_i, \widetilde{g}_i(t), p_i)$, $t \in (0, \infty)$, converges to a smooth Ricci flow $(M_\infty, g_\infty(t), p_\infty)$ on $(0, \infty)$. Assume $f_{i,1}(p_i) = 0$, then by $|\nabla f_{i,1}|(p_i) = 0$ and $\mathrm{Ric}_{\widetilde{g}_i(1)} + \frac{1}{2}\widetilde{g}_i(1) = \nabla^2 f_{i.1}$, we can apply Shi's derivative estimates to get bounds for higher derivatives of curvatures, and thus bounds for higher derivatives of $f_{i,1}$. So we may assume $f_{i,1}$ converges to a smooth function $f_\infty$ satisfying $\mathrm{Ric}_{g_\infty(1)} + \frac{1}{2}g_\infty(1) = \nabla^2 f_\infty$, which makes $(M_\infty, g_\infty(t), p_\infty)$ an expanding gradient soliton. Since $R_{\widetilde{g}_i(t)} \le \frac{C}{t}$, it follows that $R_{g_\infty(t)} \le \frac{C}{t}$.

This curvature condition combined with Hamilton's distance distortion estimate gives us a uniform double side control on $d_{\widetilde{g}_i(t)}$ and $d_{g_\infty(t)}$, which implies the following pointed Gromov-Hausdorff convergences

$$(M_i, \widetilde{g}_i(t), p_i) \xrightarrow[t\searrow 0]{pGH} (C(X_i), o_i), \quad (M_\infty, g_\infty(t), p_\infty) \xrightarrow[t\searrow 0]{pGH} (C(X), o), \tag{5.2.9}$$

where $X_i, X$ are some compact length spaces, and $o_i, o$ are the cone points of the metric cones $C(X_i), C(X)$. In particular, the first convergence is uniform for all $i$, which implies $(C(X_i), o_i) \xrightarrow[i\to\infty]{pGH} (C(X), o)$.

Let $\mathcal{H}_n(\cdot)$ denote the n-dimensional Hausdorff measure. Then since it is weakly continuous under the Gromov-Hausdorff convergence [12], we have

$$\mathcal{H}_n(B(o,1)) = \lim_{i\to\infty} vol(B(o_i,1)) = \lim_{i\to\infty} \mathrm{AVR}(C(X_i)) = \lim_{i\to\infty} \mathrm{AVR}(g_i) = 0. \tag{5.2.10}$$

However, since $(M_\infty, g_\infty)$ is an expanding gradient soliton with Ric $\geq 0$, it must have positive asymptotic volume ratio by a result of Hamilton [26, Prop 9.46]. So by volume comparison we have

$$\mathcal{H}_n(B(o,1)) = \lim_{t\searrow 0} \mathcal{H}_n(B_t(p_\infty,1)) \geq \mathrm{AVR}(g_\infty(t)) > 0, \tag{5.2.11}$$

a contradiction. This proves the claim at beginning that $C_i \to \infty$ when $i \to \infty$.

Let $\widehat{g}_i(t) = \widetilde{g}_i(t + C_i)$, $t \in (-C_i, \infty)$, then $\widehat{g}_i(0) = g_i$, $R_{\widehat{g}_i(0)}(p_i) = 1$, and

$$R_{\widehat{g}_i(t)}(x) = R_{\widetilde{g}_i(t+C_i)}(x) \leq \frac{C_i}{t + C_i} \leq 2, \tag{5.2.12}$$

for all $x \in M_i$ and $t \in (-\frac{C_i}{2}, \infty)$. This together with the injectivity radius estimate (5.2.2) there is a subsequence of $(M_i, \widehat{g}_i(t), p_i)$ which smoothly converges to a Ricci flow $(M, g(t), p)$, $t \in (-\infty, \infty)$. Moreover, by the equation (5.2.6) and Shi's derivative estimates we obtain uniform bounds for all higher derivatives of $f_i$. Since $C_i \to \infty$ as $i \to \infty$, we may assume by passing to a subsequence that $f_i$ smoothly converges to a function $f$ on $M$ which satisfies $\mathrm{Ric}(g) = \nabla^2 f$. So $(M, g(0), f, p)$ is a steady gradient soliton. The $\mathbb{Z}_2 \times O(n-1)$-symmetry is an easy consequence of the smooth convergence.

□

Now we prove Theorem 5.1.1.

*Proof of Theorem 5.1.1.* We claim that there is a sequence of smooth families of $\mathbb{Z}_2 \times O(n-1)$-symmetric Riemannian manifolds $\{X_{i,\mu}, \mu \in [0,1]\}_{i=0}^\infty$ diffeomorphic to $S^{n-1}$, satisfying the following:

1. $X_{i,0}$ is a rescaled round (n-1)-sphere;
2. $\mathrm{diam}(X_{i,1}) \to \pi$ as $i \to \infty$;
3. $K(X_{i,\mu}) > 1$, where $K$ denotes the sectional curvature;
4. $\lim_{i\to\infty} \sup_{\mu\in[0,1]} vol(X_{i,\mu}) = 0$.

We say $X_{i,\mu}$ is $\mathbb{Z}_2 \times O(n-1)$-symmetric if it is rotationally symmetric, and there is a $\mathbb{Z}_2$-isometry that maps the two centers of rotations to each other. We prove the claim in dimension $n = 3$ below, and the case for $n > 3$ follows in the same way.

First, we construct a sequence of smooth $\mathbb{Z}_2 \times O(2)$-symmetric surfaces $\{X_{i,1}\}_{i=1}^\infty$ with $K(X_{i,1}) > 1$, $\mathrm{diam}(X_{i,1}) \to \pi$ and $vol(X_{i,1}) \to 0$ as $i \to \infty$. For each large $i \in \mathbb{N}$, let $g_i$ be the metric of the surface of revolution $(i^{-1}\sin r \cos\theta, i^{-1}\sin r \sin\theta, r)$, $r \in [0,\pi]$ and $\theta \in [0, 2\pi]$. Then by a direct computation we see that $K_{\min}(g_i) = (i^{-2}+1)^{-2}$. Then by some standard smoothing arguments and suitable rescalings, we obtain the desired sequence $\{X_{i,1}\}_{i=1}^\infty$.

Second, for each large $i$, let $h_i(t)$ be the Ricci flow with $h_i(0) = X_{i,1}$, and assume its curvature blows up at $T_i > 0$. Let $K_i(t)$ be the minimum of $K(h_i(t))$, and $V_i(t)$ be the volume with respect to $h_i(t)$. Then we can find a smooth function $r_i : [0, T_i] \to \mathbb{R}_+$ such that $r_i(0) = 1$, $r_i(t) \le \min\{\sqrt{\frac{K_i(t)}{K_i(0)}}, \sqrt{\frac{V_i(0)}{V_i(t)}}\}$ for all $t \in [0, T_i]$, and $r_i(t) = \sqrt{\frac{V_i(0)}{V_i(t)}}$ when $t$ is close to $T_i$ (note $\sqrt{\frac{V_i(0)}{V_i(t)}} < \sqrt{\frac{K_i(t)}{K_i(0)}}$ when $i$ is sufficiently large since $\lim_{i\to\infty} V_i(0) = 0$ and $\limsup_{i\to\infty} K_i(0) \le 1$). Then the rescaled Ricci flow $r_i^2(t)h_i(t)$ converges to a smooth round 2-sphere when $t \to T_i$. Moreover, by letting $X_{i,\mu} = r_i^2(T_i(1-\mu))h_i(T_i(1-\mu))$, $u \in [0, 1]$, we obtain a smooth family of $\mathbb{Z}_2 \times O(2)$-symmetric surfaces $\{X_{i,u}\}$ with $K(X_{i,\mu}) > 1$, $vol(X_{i,\mu}) \le vol(X_{i,1})$, and $X_{i,0}$ is a round 2-sphere. So the claim holds.

Therefore, for each fixed $i$, by applying Deruelle's result [32, Theorem 1.4] to $X_{i,\mu}$, $\mu \in [0, 1]$, we obtain a smooth family of n-dimensional expanding gradient solitons $(M_{i,\mu}, g_{i,\mu}, p_{i,\mu})$, $\mu \in [0, 1]$, with positive curvature operator, and asymptotic to $C(X_{i,\mu})$. Moreover, by [32, Theorem 1.3], the Ricci flow generated by an expanding gradient soliton coming out of $C(X_{i,\mu})$ is unique. So any isometry of $C(X_{i,\mu})$ is an isometry at any positive time of the Ricci flow. In particular, it implies that $(M_{i,\mu}, g_{i,\mu}, p_{i,\mu})$ is $\mathbb{Z}_2 \times O(n-1)$-symmetric and $(M_{i,0}, g_{i,0}, p_{i,0})$ is rotationally symmetric.

By some suitable rescalings we may assume $R(p_{i,\mu}) = 1$, and by item (4) we have $\lim_{i\to\infty} \sup_{\mu\in[0,1]} \mathrm{AVR}(g_{i,\mu}) = \lim_{i\to\infty} \sup_{\mu\in[0,1]} \mathrm{AVR}(C(X_{i,\mu})) = 0$. So we can apply Lemma 5.2.2 and by passing to a subsequence, we may assume $(M_{i,0}, g_{i,0}, p_{i,0})$ and $(M_{i,1}, g_{i,1}, p_{i,1})$ smoothly converge to two steady gradient solitons $(M_{\infty,0}, g_{\infty,0}, p_{\infty,0})$ and $(M_{\infty,1}, g_{\infty,1}, p_{\infty,1})$ respectively. On the one hand, since $(M_{i,0}, g_{i,0}, p_{i,0})$ is rotationally symmetric, it follows that $(M_{\infty,0}, g_{\infty,0}, p_{\infty,0})$ is rotationally symmetric, and hence is a Bryant soliton, see e.g. [26].

On the other hand, since $\mathrm{diam}(X_{i,1}) \to \pi$ when $i \to \infty$, the asymptotic cone for each $(M_{i,1}, g_{i,1}, p_{i,1})$ converges to a half-plane, or equivalently a cone over the interval $[0, \pi]$. So for each $j \in \mathbb{N}$ and all sufficiently large $i$, we can find points $q_{i,j}, r_{i,j} \in M_{i,1}$ such that $d(q_{i,j}, p_{i,1}) = d(r_{i,j}, p_{i,1}) = j$ and $\widetilde{\measuredangle} q_{i,j} p_{i,1} r_{i,j} \ge \pi - j^{-1}$. Passing to the limit we obtain points $q_j, r_j \in M_{\infty,1}$ with $d(q_j, p_{\infty,1}) = d(r_j, p_{\infty,1}) = j$ and $\widetilde{\measuredangle} q_j p_{\infty,1} r_j \ge \pi - j^{-1}$. Then letting $j \to \infty$ and passing to a subsequence, the geodesics $p_{\infty,1}q_j, p_{\infty,1}r_j$ converge to two rays which together form a line passing through $p_{\infty,1}$. Then by the strong maximum principle of Ricci flow, $(M_{\infty,1}, g_{\infty,1})$ is the product of $\mathbb{R}$ and an (n-1)-dimensional rotationally symmetric steady gradient soliton with positive curvature operator, which is an (n-1)-dimensional Bryant soliton if $n > 3$, and a cigar soliton if $n = 3$, see e.g. [26].

For a $\mathbb{Z}_2 \times O(n-1)$-symmetric expanding or steady gradient soliton $(M, g, p)$ with non-negative curvature operator, we write $\lambda_1(g), \lambda_2(g) = \cdots = \lambda_n(g)$ to be the $n$ eigenvalues of the Ricci curvature at $p$ in the directions of $\Gamma'(0)$ and its orthogonal complement subspace $T_p\Sigma = (\Gamma'(0))^\perp$. For any $\alpha \in (0, 1)$, since $\frac{\lambda_1}{\lambda_2}(g_{\infty,0}) = 1$ and $\frac{\lambda_1}{\lambda_2}(g_{\infty,1}) = 0$, we have $\frac{\lambda_1}{\lambda_2}(g_{i,0}) > \alpha$ and $\frac{\lambda_1}{\lambda_2}(g_{i,1}) < \alpha$ when $i$ is sufficiently large. Since $\frac{\lambda_1}{\lambda_2}(g_{i,\mu})$ is a continuous function of $\mu$ for each fixed $i$, there is some $\mu_i \in (0, 1)$ such that $\frac{\lambda_1}{\lambda_2}(g_{i,\mu_i}) = \alpha$. Applying Lemma 5.2.2 to the sequence $(M_{i,\mu_i}, g_{i,\mu_i}, p_{i,\mu_i})$ and taking a limit, we obtain an n-dimensional $\mathbb{Z}_2 \times O(n-1)$-symmetric steady gradient soliton $(M, g, p)$ with $\frac{\lambda_1}{\lambda_2}(g) = \alpha$. This proves Theorem 5.1.1. □

By a similar argument we obtain a family of n-dimensional $O(n-2) \times O(2)$-symmetric steady gradient solitons with positive curvature operator.

**Proposition 5.2.3.** *Given any $\alpha \in (0, 1)$, there exists an n-dimensional $O(n-2) \times O(2)$-symmetric steady gradient soliton $(M, g, f, p)$ with positive curvature operator, such that $\lambda_1 = \cdots = \lambda_{n-2} = \alpha\lambda_{n-1} = \alpha\lambda_n$, where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of the Ricci curvature at $p$.*

*Proof.* The proof follows the same line as that of Theorem 5.1.1. First, we can construct a sequence of smooth families of $O(n-2) \times O(2)$-symmetric Riemannian manifolds $\{X_{i,\mu}, \mu \in [0, 1]\}_{i=0}^{\infty}$ diffeomorphic to $S^{n-1}$, satisfying the four conditions as in the proof of Theorem 5.1.1. We say $X_{i,\mu}$ is $O(n-2) \times O(2)$-symmetric if away from a closed geodesic fixed by the $O(2)$-action, the metric has the form $g = dr^2 + \varphi_1^2(r)d\theta_1^2 + \varphi_2^2(r)d\theta_2^2$, $\theta_j \in (0, 2\pi)$, $j = 1, 2$.

Then this gives a sequence of smooth families of $O(n-2) \times O(2)$-symmetric expanding gradient solitons with positive curvature operator $(M_{i,\mu}, g_{i,\mu}, p_{i,\mu})$ and $R(p_{i,\mu}) = 1$, which is asymptotic to $C(X_{i,\mu})$. In particular, $(M_{i,0}, g_{i,0}, p_{i,0})$ is rotationally symmetric.

By passing to a subsequence, we may assume $(M_{i,0}, g_{i,0}, p_{i,0})$ and $(M_{i,1}, g_{i,1}, p_{i,1})$ smoothly converge to two steady gradient solitons $(M_{\infty,0}, g_{\infty,0}, p_{\infty,0})$ and $(M_{\infty,1}, g_{\infty,1}, p_{\infty,1})$ respectively. Then $(M_{\infty,0}, g_{\infty,0}, p_{\infty,0})$ is a Bryant soliton, and $(M_{\infty,1}, g_{\infty,1}, p_{\infty,1})$ is the product of $\mathbb{R}^{n-2}$ and a cigar soliton. Now the conclusion follows by a continuity argument. □

## 5.3 Asymptotic geometry of steady gradient solitons

In this section, we study the asymptotic geometry of n-dimensional $\mathbb{Z}_2 \times O(n-1)$-symmetric steady gradient solitons. We show that such a soliton strongly dimension reduces along an edge to an $(n-1)$-dimensional ancient Ricci flow (see below for definitions). In particular, when $n = 3$, the 2d ancient Ricci flow is the cigar soliton, assuming in additional that the scalar curvature does not vanish at infinity. See also [25] for discussions of dimension reductions of 4d non-collapsed steady gradient solitons.

**Definition 5.3.1.** Let $(M^n, g, p)$ be an $n$-dimensional $\mathbb{Z}_2 \times O(n-1)$-symmetric steady gradient soliton. We say that it **strongly dimension reduces** along $\Gamma$ to an $(n-1)$-dimensional ancient Ricci flow $(N, g(t))$, if for any sequence $s_i \to \infty$, a subsequence of $(M, K_i g(K_i^{-1}t), \Gamma(s_i))$, $t \in (-\infty, 0]$, where $K_i = R(\Gamma(s_i))$, smoothly converges to the product of $\mathbb{R}$ and $(N, g(t))$.

We also say an $(n-1)$-dimensional ancient Ricci flow $(N, h(t))$ is a ***dimension reduction*** of $(M^n, g, p)$ along $\Gamma$, if there exists $s_i \to \infty$ such that $(M, K_i g(K_i^{-1}t), \Gamma(s_i))$, $t \in (-\infty, 0]$, where $K_i = R(\Gamma(s_i))$, smoothly converges to the product of $\mathbb{R}$ and $(N, h(t), p_\infty)$.

First we prove a lemma about the relations between the potential function and distance function.

**Lemma 5.3.2.** *Let $(M^n, g, f, p)$ be an n-dimensional steady gradient soliton with positive curvature operator. Suppose $\gamma : (0, \infty) \to M$ is an integral curve of $\frac{\nabla f}{|\nabla f|}$, and $\lim_{s\to 0} \gamma(s) = p$. Then for any $\epsilon > 0$, there exists $s_0 > 0$ such that for any $s_1 > s_2 > s_0$ we have*

$$(1-\epsilon)(s_2 - s_1) \le d(\gamma(s_1), \gamma(s_2)) \le (s_2 - s_1). \tag{5.3.1}$$

*In particular, we have $(1-\epsilon)s \le d(p, \gamma(s)) \le s$ for all $s \ge s_0$. Moreover, let $\sigma$ be a unit speed minimizing geodesic between $p$ and $\sigma(0) := \gamma(s)$. Then*

$$\measuredangle(\sigma'(0), \nabla f) \le \epsilon. \tag{5.3.2}$$

*Proof.* Without loss of generality, we may assume $f(p) = 0$ and $\lim_{s\to\infty} |\nabla f|(\gamma(s)) = 1$ after a suitable rescaling. We use $\epsilon = \epsilon(s)$ to denote all functions such that $\lim_{s\to\infty} \epsilon(s) = 0$.

On the one hand, for any $s_2 > s_1 \ge 0$, let $\sigma : [0, D] \to M$ be a minimizing geodesic from $\gamma(s_1)$ to $\gamma(s_2)$, where $D = d(\gamma(s_1), \gamma(s_2))$. Since $\frac{d}{dr}\langle \nabla f, \sigma'(r)\rangle = \nabla^2 f(\sigma'(r), \sigma'(r)) \ge 0$, we obtain

$$f(\gamma(s_2)) - f(\gamma(s_1)) = \int_0^D \langle \nabla f, \sigma'(r)\rangle\, dr \le D\, \langle \nabla f, \sigma'(D)\rangle, \tag{5.3.3}$$

which by $|\nabla f| \le 1$ implies

$$f(\gamma(s_2)) - f(\gamma(s_1)) \le d(\gamma(s_1), \gamma(s_2)). \tag{5.3.4}$$

On the other hand, since $\lim_{s\to\infty} |\nabla f|(\gamma(s)) = 1$, there is $s_0 > 0$ such that $|\nabla f|(\gamma(s)) > 1 - \epsilon$ for all $s \ge s_0$. Therefore, for all $s_2 > s_1 \ge s_0$ we have

$$f(\gamma(s_2)) - f(\gamma(s_1)) = \int_{s_1}^{s_2} \langle \nabla f, \gamma'(r)\rangle\, dr = \int_{s_1}^{s_2} |\nabla f|(\gamma(r))\, dr \ge (1-\epsilon)(s_2 - s_1), \tag{5.3.5}$$

which together with (5.3.3) proves the first inequality in (5.3.1), where the second inequality is an easy consequence of $|\gamma'(s)| = 1$. The inequality of $d(p, \gamma(s))$ follows (5.3.1) and a triangle inequality.

Now let $\sigma : [0, d(p, \gamma(s))] \to M$ be a minimizing geodesic from $p$ to $\gamma(s)$. Then (5.3.3) implies

$$f(\gamma(s)) \le d(p, \gamma(s))\, \langle \nabla f, \sigma'(d(p, \gamma(s)))\rangle. \tag{5.3.6}$$

Moreover, by (5.3.5) and $\lim_{s\to\infty} f(\gamma(s)) = \infty$ we have

$$d(\gamma(s_0), \gamma(s)) \le s - s_0 \le (1+\epsilon)(f(\gamma(s)) - f(\gamma(s_0))) \le (1+\epsilon) f(\gamma(s)) \tag{5.3.7}$$

for all $s$ sufficiently large, which by triangle inequality and $\lim_{s\to\infty} d(p, \gamma(s)) = \infty$ implies

$$d(p, \gamma(s)) \le d(p, \gamma(s_0)) + d(\gamma(s_0), \gamma(s)) \le (1+\epsilon) d(\gamma(s_0), \gamma(s)) \le (1+\epsilon) f(\gamma(s)). \tag{5.3.8}$$

This combining with (5.3.6) and $|\nabla f| \le 1$ yields

$$\left\langle \frac{\nabla f}{|\nabla f|}, \sigma'(d(p,\gamma(s))) \right\rangle \ge \frac{f(\gamma(s))}{d(p,\gamma(s))\,|\nabla f|} \ge 1-\epsilon, \tag{5.3.9}$$

which proves (5.3.2). □

The following lemma shows that all dilation sequence along $\Gamma$ smoothly converges to a limit after passing to a subsequence. The limits are all products of a line and some rotationally symmetric ancient solution.

Our main tool is Perelman's curvature estimate for Ricci flows with non-negative curvature operator, see for example [43, Corollary 45.1(b)], or a more general result in [1, Proposition 3.2]. It implies that for a Ricci flow with non-negative curvature operator $(M, g(t))$, $t \in [-1, 0]$, assume $B_{g(0)}(x_0, 1) \ge \kappa > 0$ for some $x_0 \in M$, then there is $C(\kappa) > 0$ such that $R(x_0, 0) \le C$.

**Lemma 5.3.3.** *Let $(M^n, g, p)$ be a non-flat $\mathbb{Z}_2 \times O(n-1)$-symmetric $n$-dimensional steady gradient soliton. Then there is $C > 0$ such that the following holds:*

*For any $s_i \to +\infty$, a subsequence of $(M, K_i g(K_i^{-1}t), \Gamma(s_i))$, $t \in (-\infty, 0]$, $K_i = R(\Gamma(s_i))$, smoothly converges to an ancient Ricci flow $(\mathbb{R} \times g_\infty(t), p_\infty)$, where $g_\infty(t)$ is an $(n-1)$-dimensional ancient Ricci flow with positive curvature operator and $R \le C$. Moreover, $R^{-1/2}(\Gamma(s_i))\Gamma'(s_i)$ smoothly converges to a unit vector in the $\mathbb{R}$-direction of $\mathbb{R} \times g_\infty(t)$, and $g_\infty(t)$ is rotationally symmetric around $p_\infty$.*

*Proof.* If $\mathrm{Rm} > 0$ does not hold, then by the strong maximum principle the soliton is $\mathbb{R} \times$ Bryant for $n \ge 4$, or $\mathbb{R} \times$ Cigar for $n = 3$. The conclusion clearly holds in these cases, so we may assume $\mathrm{Rm} > 0$.

Let $r(s) = \sup\{\rho > 0 : vol(B(\Gamma(s), \rho)) \ge \frac{\omega}{2}\rho^n\}$ where $\omega$ is the volume of the unit ball in the Euclidean space $\mathbb{R}^n$. Since the asymptotic volume ratio of any non-flat ancient Ricci flow with non-negative curvature operator is zero by [43, Corollary 45.1(b)], we have $r(s) < \infty$ for each $s$, and $\lim_{s\to\infty} \frac{r(s)}{s} = 0$. Moreover, by the choice of $r(s)$ we have $vol(B(\Gamma(s), r(s))) = \frac{\omega}{2} r^n(s)$.

For any $D > 0$ and any $x \in B(\Gamma(s), Dr(s))$, by the volume comparison we have $vol(B(x, r(s))) \ge C_1^{-1} r^n(s)$ for some $C_1(D) > 0$. Therefore, by [43, Corollary 45.1(b)] we can find constants $C_2(D) > 0$ such that $R \le C_2 r^{-2}(s)$ in $B(\Gamma(s), Dr(s))$. By Hamilton's Harnack inequality $\frac{d}{dt} R(\cdot, t) \ge 0$ for ancient complete Ricci flow with non-negative curvature operator [36], this implies $R(x, t) \le C_2 r^{-2}(s)$ for all $x \in B(\Gamma(s), Dr(s))$ and $t \in (-\infty, 0]$. In particular, there is $C_0 > 0$ such that $C_0^{-1} r(s) \le R^{-1/2}(\Gamma(s))$, and $\mathrm{inj}_g(\Gamma(s)) \ge C_0^{-1} r(s)$ by the volume bound.

Therefore, for any $s_i \to \infty$, by Shi's derivative estimates and Hamilton's compactness theorem for Ricci flow, a subsequence of $(M, r^{-2}(s_i) g(r^2(s_i)t), \Gamma(s_i))$, $t \in (-\infty, 0]$, converges to an ancient solution $h_\infty(t)$. Let $\Gamma_i(s) = \Gamma(r(s_i)s + s_i)$, $s \in (-\infty, \infty)$. Suppose $\Gamma_i$ converges to the geodesic $\Gamma_\infty$ in $h_\infty(0)$ as $i \to \infty$, modulo the diffeomorphisms. We claim that $\Gamma_\infty$ is a line: Since $\lim_{s\to\infty} \frac{r(s)}{s} \to 0$, we have $s_i - Dr(s_i) \to \infty$, by which we can apply Lemma 5.3.2 and deduce that for any $D > 0$ that $\widetilde{\measuredangle}\Gamma_i(-D)\Gamma_i(0)\Gamma_i(D) \to \pi$ as $i \to \infty$. So $d(\Gamma_\infty(-D), \Gamma_\infty(D)) = 2D$. Letting $D \to \infty$, this implies $\Gamma_\infty$ is a line.

Next we claim that there is some $C_3 > 0$ such that $R^{-1/2}(\Gamma(s)) \le C_3 r(s)$ for all large $s$. Suppose by contradiction this does not hold, then there is a sequence $s_i \to \infty$ such that $\lim_{i\to\infty} \frac{R^{-1/2}(\Gamma(s_i))}{r(s_i)} = 0$. Then by taking a subsequence we may assume $(r^{-2}(s_i)g, \Gamma(s_i))$ converges to $(\mathbb{R} \times g_\infty(t), p_\infty)$, where $g_\infty(t)$ is some $(n-1)$-dimensional ancient solution.

On the one hand, as a consequence of taking the limit, we have $vol(B(p_\infty, 1)) = \frac{\omega}{2}$ and $R(p_\infty) = 0$, which by the strong maximum principle implies that $g_\infty(t)$ is flat. On the other hand, since $\Gamma_i$ converges to a line, we can find a sequence $D_i \to \infty$ such that $\Sigma_i := \exp_{\Gamma(s_i)}(\Gamma'(s_i)^\perp) \cap B(\Gamma(s_i), D_i r(s_i))$ with the metric $g_{\Sigma_i}$ induced by $g$ is a smooth surface which is rotationally symmetric around $\Gamma(s_i)$, and $(r^{-2}(s_i)g_{\Sigma_i}, \Gamma(s_i))$ smoothly converges to $(g_\infty(0), p_\infty)$. So $g_\infty(0)$ is rotationally symmetric around $p_\infty$. Since $g_\infty(0)$ is flat, it must be isometric to $\mathbb{R}^{n-1}$, which implies $vol(B(p_\infty, 1)) = \omega > \frac{\omega}{2}$, a contradiction.

Then it follows from $C_0^{-1} r(s) \le R^{-1/2}(\Gamma(s)) \le C_3 r(s)$ that $(M, K_i g(K_i^{-1}t), \Gamma(s_i))$, $t \in (-\infty, 0]$, $K_i = R(\Gamma(s_i))$, smoothly converges to an ancient Ricci flow $(\mathbb{R} \times g_\infty(t), p_\infty)$ as claimed. Since $g_\infty(t)$ is rotationally symmetric and has positive curvature, the uniform curvature bound $R \le C$ follows easily by applying [43, Corollary 45.1(b)]. □

As a corollary of Lemma 5.3.3, we show that the n-dimensional steady gradient solitons from Theorem 5.1.1 are all non-collapsed if $n \ge 4$.

**Definition 5.3.4.** A Riemannian manifold $(M^n, g)$ is non-collapsed if there exists a constant $\kappa > 0$ such that for any $x \in M$ and $r > 0$, if $|\mathrm{Rm}| \le r^{-2}$ in the ball $B_g(x, r)$, then $vol_g(B_g(x, r)) \ge \kappa r^n$. Otherwise we say $(M, g)$ is collapsed.

**Corollary 5.3.5.** *For any $n \ge 4$, let $(M^n, g, p)$ be an n-dimensional non-flat $\mathbb{Z}_2 \times O(n-1)$-symmetric steady gradient soliton. Then it is non-collapsed.*

*Proof.* Let $\omega$ be the volume of the unit ball in $\mathbb{R}^n$. Suppose the conclusion is not true, then there is a sequence of points $x_i \in M$ such that $\frac{r_i}{\overline{r}_i} \to \infty$ as $i \to \infty$, where $\overline{r}_i = \sup\{\rho > 0 : vol_g(B_g(x_i, \rho)) \ge \frac{\omega}{2}\rho^n\}$, and $r_i = \sup\{\rho > 0 : |\mathrm{Rm}| \le \rho^{-2} \text{ in } B_g(x_i, \rho)\}$. Then by the same limiting argument as Lemma 5.3.3, we may assume by passing to a subsequence that $(M, \overline{r}_i^{-2} g, x_i)$ smoothly converges to a manifold $(M_\infty, g_\infty, x_\infty)$, which is flat and satisfies $vol(B(x_\infty, 1)) = \frac{\omega}{2}$.

Let $g_i = \overline{r}_i^{-2} g$. We first assume that there are a constant $C > 0$ and $y_i \in \Gamma$ such that $d_{g_i}(x_i, y_i) \le C$ for all $i$. Then a subsequence of $(M, g_i, y_i)$ converges to $(M_\infty, g_\infty, y_\infty)$ for some $y_\infty \in M_\infty$. By Lemma 5.3.3, $(M_\infty, g_\infty)$ is a product of $\mathbb{R}$ and an $(n-1)$-dimensional rotationally symmetric manifold. Since $(M_\infty, g_\infty)$ is flat, it must be isometric to $\mathbb{R}^n$, which contradicts the choice of $\omega$.

Next, assume $\lim_{i\to\infty} d_{g_i}(x_i,\Gamma)=\infty$. Let $h_i$ be the metric induced by $g_i$ on the totally geodesic surface $N$, and assume $x_i \in N$. Then $g_i = h_i + \varphi_i^2 g_{S^{n-2}}$ on $B_{g_i}(x_i, \frac{1}{2}d_{g_i}(x_i,\Gamma))$, where $\varphi_i = \overline{r}_i^{-1}\varphi$. So it follows easily that $vol_{h_i}(B_{h_i}(x_i,1)) \geq c(\omega)$ for some $c(\omega)>0$. Since $B_{h_i}(x_i, \frac{1}{2}d_{g_i}(x_i,\Gamma))$ is relatively compact in $N$, it follows by the same curvature estimates as Lemma 5.3.3 that a subsequence of $(N, h_i, x_i)$ smoothly converges to a complete manifold $(N_\infty, h_\infty, x_\infty)$, which is diffeomorphic to $\mathbb{R}^2$. Since $(N_\infty, h_\infty)$ is totally geodesic in $(M_\infty, g_\infty)$, it is isometric to $\mathbb{R}^2$.

If $\varphi_i(x_i)\to\infty$ as $i\to\infty$, it is easy to see that $(M_\infty, g_\infty)$ is isometric to $\mathbb{R}^n$, a contradiction. Otherwise, there is $C>0$ such that $\varphi_i(x_i)\leq C$ for all $i$. Then by the curvature estimates and (5.2.1), a subsequence of $\varphi_i$ smoothly converges to a positive function $\varphi_\infty$, such that $g_\infty = g_{\mathbb{R}^2} + \varphi_\infty^2 g_{S^{n-2}}$. Since $n\geq 4$, this contradicts the fact that $(M_\infty, g_\infty)$ is flat, hence proves the corollary.

□

To rephrase the statement of Lemma 5.3.3 and use it to prove a more accurate dimension reduction theorem in dimension 3, we introduce the definition of $\epsilon$-closeness between two Ricci flows.

**Definition 5.3.6.** For any $\epsilon>0$, we say a pointed Ricci flow $(M_1, g_1(t), p_1)$, $t\in[-T,0]$, is ***$\epsilon$-close*** to a pointed Ricci flow $(M_2, g_2(t), p_2)$, $t\in[-T,0]$, if there is a diffeomorphism onto its image $\phi: B_{g_2(0)}(p_2,\epsilon^{-1})\to M_1$, such that $\phi(p_2)=p_1$ and $\|\phi^* g_1(t)-g_2(t)\|_{C^{[\epsilon^{-1}]}(U)}<\epsilon$ for all $t\in[-\min\{T,\epsilon^{-1}\},0]$, where the norms and derivatives are taken with respect to $g_2(0)$.

By this definition, Lemma 5.3.3 shows that $(R(\Gamma(s))g(R^{-1}(\Gamma(s))t),\Gamma(s))$ is $\epsilon$-close to the product of $\mathbb{R}$ and a dimension reduction for all sufficiently large $s$. Moreover, a dimension reduction $(M_\infty, g_\infty(t), p_\infty)$ is an $(n-1)$-dimensional ancient solution with positive curvature operator and it is rotationally symmetric around $p_\infty$.

In dimension 3, the next theorem shows that $M_\infty$ is non-compact, if the original soliton is not a Bryant soliton. Moreover, if $\lim_{s\to\infty}R(\Gamma(s))>0$, then the soliton strongly dimension reduces along $\Gamma$ to a cigar soliton.

**Theorem 5.3.7.** *(Dimension Reduction) Let $(M,g,f,p)$ be a non-flat 3d $\mathbb{Z}_2\times O(2)$-symmetric steady gradient soliton, which is not a Bryant soliton. Then any dimension reduction of $(M,g,p)$ along $\Gamma$ is non-compact. In particular, if $\lim_{s\to\infty}R(\Gamma(s))>0$, then $(M,g,p)$ strongly dimension reduces along $\Gamma$ to a cigar soliton $(M_\infty, g_\infty(t), p_\infty)$, $t\in(-\infty,0]$, with $R(p_\infty,0)=1$.*

*Proof.* Let $\epsilon>0$ be sufficiently small. We denote by $\epsilon_\#$ all positive constants that depend on $\epsilon$ such that $\epsilon_\#\to 0$ as $\epsilon\to 0$.

For each sufficiently large $s$, by Lemma 5.3.3 there is a dimension reduction $(h_s(t), p_s)$ of $(M, g, p)$ along $\Gamma$, such that $(R(\Gamma(s))g(R^{-1}(\Gamma(s))t), \Gamma(s))$ is $\epsilon$-close to $(\mathbb{R} \times h_s(t), p_s)$. By Lemma 5.3.3, $(h_s(t), p_s)$ is a 2d ancient Ricci flow rotationally symmetric around $p_s$ and $R(p_s, 0) = 1$. Note the choice of $h_s(t)$ may not be unique for a fixed $s$, but any two such solutions are $\epsilon_{\#}$-close to each other. Let

$$F(s) = \mathrm{diam}(h_s(0)) \in (0, \infty]. \tag{5.3.10}$$

First, if $\limsup_{s\to\infty} F(s) < \frac{1}{100\epsilon}$, then there is $\kappa = \kappa(\epsilon) > 0$ such that all $h_s(0)$ is $\kappa$-non-collapsed. This implies easily that $(M, g, p)$ is $\kappa$-non-collapsed, and hence is a Bryant soliton, as a consequence of the uniqueness of the Bryant soliton among 3d non-collapsed steady gradient solitons [9], or among 3d $\kappa$-solutions [3, 8]. This is a contradiction. So $\limsup_{s\to\infty} F(s) \geq \frac{1}{100\epsilon} > 100\pi$.

Next, we claim that $F(s) \geq D := \frac{1}{1000\epsilon}$ for all large $s$: First, choose $s_0$ such that $F(s_0) \geq 3D$, and let

$$s_1 = \sup\{s \geq s_0 \mid F(\mu) \geq 2D \text{ for all } \mu \in [s, s_0]\}. \tag{5.3.11}$$

Then $F(s_1) \in [2(1-\epsilon_{\#})D, 2(1+\epsilon_{\#})D]$ and $(h_{s_1}(t), p_{s_1})$ is a Rosenau solution by the classification of compact ancient 2d Ricci flows [30]. Moreover, assume $\epsilon$ is sufficiently small, then $1 - \epsilon_{\#} \leq R(p_{s_1}, t) \leq 1$ for all $t \leq 0$, see e.g. [26, Chap 4.4], and

$$\mathrm{diam}(h_{s_1}(t))R^{1/2}(p_{s_1}, t) \geq (1 - \epsilon_{\#})F(s_1) \geq 2(1 - \epsilon_{\#})D \tag{5.3.12}$$

for all $t \leq 0$. Moreover, by a distance distortion estimate, see e.g. [43, Lem 27.8], we can find a $t_1 \in [-\epsilon^{-1}, 0)$ such that

$$\mathrm{diam}(h_{s_1}(t_1))R^{1/2}(p_{s_1}, t_1) = 4D. \tag{5.3.13}$$

Since $g(t) = \phi_t^*(g)$, where $\{\phi_t\}_{t\in(-\infty,\infty)}$ is the flow of $-\nabla f$ with $\phi_0$ the identity. We see that $(g(t), \Gamma(s))$ is isometric to $(g, \phi_t(\Gamma(s)))$, and since $\Gamma$ is the integral curve of $\frac{\nabla f}{|\nabla f|}$, by a direct computation we obtain

$$\phi_t(\Gamma(s)) = \Gamma\left(s - \int_0^t |\nabla f|(\phi_\mu(\Gamma(s)))\, d\mu\right). \tag{5.3.14}$$

Let $s_2 = s_1 - \int_0^{T_1} |\nabla f|(\phi_\mu(\Gamma(s_1)))\, d\mu$, where $T_1 = t_1 R^{-1}(\Gamma(s_1)) < 0$. Then $s_2 > s_1$, $\phi_{T_1}(\Gamma(s_1)) = \Gamma(s_2)$, and $(g(T_1), \Gamma(s_1))$ is isometric to $(g, \Gamma(s_2))$. The conditions (5.3.12)(5.3.13) imply $F(s) \geq 2(1 - \epsilon_{\#})D \geq D$ for all $s \in [s_1, s_2]$, and $F(s_2) \geq 4(1 - \epsilon_{\#})D \geq 3D$. In particular, this implies $s_2 - s_1 \geq R^{-1/2}(\Gamma(s_1)) \geq R^{-1/2}(p)$.

Therefore, by induction we find a sequence $\{s_{2k}\}_{k=0}^{\infty}$, such that $s_{2k} - s_{2(k-1)} \geq R^{-1/2}(p)$ for all $k \geq 1$ and

$$F(s) \geq D \text{ for all } s \in [s_{2(k-1)}, s_{2k}], \quad F(s_{2k}) \geq 3D. \tag{5.3.15}$$

This implies $F(s) \geq D = \frac{1}{1000\epsilon}$ for all large $s$. Letting $\epsilon \to 0$, it follows that any dimension reduction along $\Gamma$ is non-compact.

Now assume $\lim_{s\to\infty} R(\Gamma(s)) > 0$. Suppose $(g_\infty(t), p_\infty)$ is a dimension reduction, and $(M, R(\Gamma(s_i))g(R^{-1}(\Gamma(s_i))t), \Gamma(s_i))$ smoothly converges to $(M_\infty, \mathbb{R}\times g_\infty(t), p_\infty)$ for a sequence $s_i \to \infty$. Let $f_i = f - f(\Gamma(s_i))$. Then $f_i$ smoothly converges to a function $f_\infty$ on $M_\infty$ satisfying $\mathrm{Ric} = \nabla^2 f_\infty$ with respect to the metric $\mathbb{R} \times g_\infty(0)$. So $g_\infty(0)$ is a 2d non-flat steady gradient soliton, which must be a cigar soliton [37].

□

## 5.4 Existence of 3d flying wings

In this section, we prove Theorem 5.1.2, 5.1.4 and all the corollaries. The asymptotic cone of a 3d $\mathbb{Z}_2 \times O(2)$-symmetric steady gradient soliton is a metric cone over $[-\frac{\alpha}{2}, \frac{\alpha}{2}]$ for some $\alpha \in [0, \pi]$ (see Lemma 5.4.2). Theorem 5.1.2 shows that the soliton must be a Bryant soliton, if the asymptotic cone is a ray. So the family of 3d steady gradient solitons from Theorem 5.1.1 are all flying wings, which confirms Hamilton's conjecture.

Throughout this section we assume $(M, g, p)$ is a non-flat $\mathbb{Z}_2 \times O(2)$-symmetric 3d steady gradient soliton, and $\Gamma$ and $\Sigma$ are the fixed point sets of the $O(2)$ and $\mathbb{Z}_2$-action respectively.

The next lemma shows that the integral of scalar curvature in metric balls increases at least linearly in radius. We remark that this is also a consequence of [17], which shows that the only 3d steady gradient solitons satisfying $\liminf_{s\to\infty} \frac{1}{s}\int_{B(p,s)} R\, dvol_M = 0$ are quotients of $\mathbb{R}^3$ and $\mathbb{R}\times$Cigar. The proof below is self-contained and more direct under the symmetric assumption.

**Lemma 5.4.1.** *There exists $C > 0$ such that $\int_{B(p,s)} R\, dvol_M \geq C^{-1}s$ for sufficiently large $s$.*

*Proof.* Fix some small $\epsilon > 0$ and let $s_0 > 0$ be large enough such that Lemma 5.3.2 holds for $\epsilon$. Consider the covering of $\Gamma([s_0, s])$ by $\{\Gamma([\mu - R^{-1/2}(\Gamma(\mu)), \mu + R^{-1/2}(\Gamma(\mu))])\}_{\mu\in[s_0,s]}$. Let $\{\Gamma([\mu_i - R^{-1/2}(\Gamma(\mu_i)), \mu_i + R^{-1/2}(\Gamma(\mu_i))])\}_{i=1}^m$ be a Vitali covering of it, which is disjoint from each other and $\Gamma([s_0, s])$ is covered by $\{\Gamma([\mu_i - 5R^{-1/2}(\Gamma(\mu_i)), \mu_i + 5R^{-1/2}(\Gamma(\mu_i))])\}_{i=1}^m$. So for any $\mu_i < \mu_j$,

$$\mu_j - \mu_i \geq R^{-1/2}(\Gamma(\mu_i)) + R^{-1/2}(\Gamma(\mu_j)) \geq R^{-1/2}(\Gamma(\mu_j)), \tag{5.4.1}$$

and

$$s - s_0 \leq \sum_{i=1}^{m} 10R^{-1/2}(\Gamma(\mu_i)). \tag{5.4.2}$$

Let $c = \frac{1-\epsilon}{4}$, we claim that $B(\Gamma(\mu_i), cR^{-1/2}(\Gamma(\mu_i)))$ and $B(\Gamma(\mu_j), cR^{-1/2}(\Gamma(\mu_j)))$ are disjoint: Suppose not, then $d(\Gamma(\mu_i), \Gamma(\mu_j)) < 2cR^{-1/2}(\Gamma(\mu_j))$, and by Lemma 5.3.2 we get

$$\mu_j - \mu_i \leq (1-\epsilon)^{-1} d(\Gamma(\mu_i), \Gamma(\mu_j)) \leq 2(1-\epsilon)^{-1} c\, R^{-1/2}(\Gamma(\mu_j)) < R^{-1/2}(\Gamma(\mu_j)), \tag{5.4.3}$$

which contradicts (5.4.1).

By Theorem 5.3.7 and Shi's derivative estimates, there is some $C_1 > 0$ such that

$$\int_{B(\Gamma(s),cR^{-1/2}(\Gamma(s)))} R\, dvol_M \geq C_1^{-1} R^{-1/2}(\Gamma(s)). \tag{5.4.4}$$

Since $\lim_{s\to\infty} \frac{R^{-1/2}(\Gamma(s))}{s} = 0$, which can be seen from the proof of Lemma 5.3.3, we have $B(\Gamma(\mu_i), cR^{-1/2}(\Gamma(\mu_i))) \subset B(p, 2s)$ for all $i$. Therefore, by (5.4.2) and (5.4.4) we obtain

$$\int_{B(p,2s)} R\, dvol_M \geq \sum_{i=1}^{m} \int_{B(\Gamma(\mu_i),cR^{-1/2}(\Gamma(\mu_i)))} R\, dvol_M \geq C_2^{-1} s \tag{5.4.5}$$

for some $C_2 > 0$. □

The next lemma shows that for any non-flat $\mathbb{Z}_2 \times O(2)$-symmetric 3d steady gradient soliton $(M, g, p)$, the space of equivalent classes of rays is an interval $[-\frac{\alpha}{2}, \frac{\alpha}{2}]$, where $\alpha \in [0, \pi]$. So the asymptotic cone is a sector with angle $\alpha \in [0, \pi]$. Moreover, the minimizing geodesics between $p$ and points going to infinity along $\Gamma$ and $\Sigma$ converge to a ray in the class $\pm\frac{\alpha}{2}$ and 0 respectively.

**Lemma 5.4.2.** *The asymptotic cone of $(M, g, p)$ is a metric cone $C(X)$ over the interval $X = [-\frac{\alpha}{2}, \frac{\alpha}{2}]$ for some $\alpha \in [0, \pi]$, and*

1. *For any sequence $s_i \to +\infty$, the geodesics between $p$ and $\Gamma(s_i)$ converge to the equivalent class $\frac{\alpha}{2} \in X$.*
2. *For any sequence $q_i \in \Sigma$ and $q_i \to \infty$, the geodesics between $p$ and $q_i$ converge to the equivalent class $0 \in X$.*
3. *For any $q_i \in \Sigma$, $q_i \to \infty$, and $o_i = \Gamma(s_i)$, $s_i \to \infty$, with $C^{-1} d(p, o_i) \leq d(p, q_i) \leq C\, d(p, o_i)$, we have $\lim_{i\to\infty} \widetilde{\measuredangle} q_i p o_i = \frac{\alpha}{2}$.*

*Proof.* The conclusion clearly holds for $\mathbb{R} \times$ Cigar with $\alpha = \pi$, so we may assume $(M, g, p)$ has positive sectional curvature. For any $s_i \to \infty$, let $p_i = \Gamma(s_i)$ and $\overline{p}_i = \Gamma(-s_i)$. Assume after passing to a subsequence that the minimizing geodesics $pp_i, p\overline{p}_i$ converge to rays $\gamma_1, \overline{\gamma}_1$ respectively. Let $(X, d_X)$ be the space of the equivalent classes of rays, and $\gamma_2, \overline{\gamma}_2 \in X$. We claim that $d_X(\gamma_1, \overline{\gamma}_1) > d_X(\gamma_2, \overline{\gamma}_2)$ unless $\{\gamma_1, \overline{\gamma}_1\} = \{\gamma_2, \overline{\gamma}_2\}$. If the claim holds, it follows that $X = [-\frac{\alpha}{2}, \frac{\alpha}{2}]$ for some $\alpha \in [0, \pi]$.

Let $\gamma_i$ be a minimizing geodesic connecting $p_i$ and $\overline{p}_i$, then $d(p, \gamma_i) \to \infty$ as $i \to \infty$, because otherwise $\gamma_i$ would converge to a line, which contradicts with $\mathrm{Rm} > 0$. So for large $i$, the two rays $\gamma_2, \overline{\gamma}_2$ intersect with $\sigma$ at $q_i, \overline{q}_i \neq p$ respectively. Assume $d(p_i, q_i) \leq d(p_i, \overline{q}_i)$ by passing to a subsequence if necessary. Then it is easy to see

$$\widetilde{\measuredangle} p_i p \overline{p}_i \geq \widetilde{\measuredangle} p_i p q_i + \widetilde{\measuredangle} q_i p \overline{q}_i + \widetilde{\measuredangle} \overline{p}_i p \overline{q}_i, \tag{5.4.6}$$

which implies the following when $i \to \infty$

$$d_X(\gamma_1, \overline{\gamma}_1) \geq d_X(\gamma_1, \gamma_2) + d_X(\gamma_2, \overline{\gamma}_2) + d_X(\overline{\gamma}_1, \overline{\gamma}_2) \geq d_X(\gamma_2, \overline{\gamma}_2). \tag{5.4.7}$$

In particular, the equalities hold if and only if $d_X(\gamma_1, \gamma_2) = d_X(\overline{\gamma}_1, \overline{\gamma}_2) = 0$, which proves the claim.

Assertion (2) follows immediately from the fact that $\Sigma$ is the fixed point set of the $\mathbb{Z}_2$-action. Assertion (3) is a consequence of (1) and (2) and the fact that $C(X)$ is isometric to the Gromov-Hausdorff limit of $(M, \lambda_i g, p)$ for any sequence $\lambda_i \to 0$. □

From now on we fix a minimizing geodesic $\gamma : [0, \infty) \to \Sigma$ starting from $p$ such that $\gamma((0, \infty)) \subset N$, and two functions $h_1(s) = d(\gamma(s), \Gamma)$ and $h_2(s) = \varphi(\gamma(s))$ that can be thought of as "dimensions" of the soliton. For example, we have $h_1(s) \approx s^{1/2}$, $h_2(s) \approx s^{1/2}$ in a Bryant soliton, and $h_1(s) \approx s$, $\lim_{s\to\infty} h_2(s) < \infty$ in $\mathbb{R} \times$ Cigar. We establish inequalities between these two functions and $R(\gamma(s))$ in the following three lemmas, when $s$ is sufficiently large.

For convenience, in the rest proofs we shall often use $\epsilon(s)$ to denote all functions such that $\lim_{s\to\infty} \epsilon(s) = 0$, and use $C$ to denote all positive constants.

**Lemma 5.4.3.** *There exists $C > 0$ such that $h_1^2(s)R(\gamma(s)) \leq C$ for all large $s$.*

*Proof.* Without loss of generality we may assume $\alpha < \pi$, because otherwise $(M, g, p)$ is $\mathbb{R} \times$ Cigar, where the assertion follows from the exponential decay of the scalar curvature.

Let $p_1 = \gamma(s)$ and $p_2 = \Gamma(s \cos \frac{\alpha}{2})$. On the one hand, since $\alpha < \pi$, we have $\Gamma(s \cos \frac{\alpha}{2}) \to \infty$ as $s \to \infty$, which allows us to apply Lemma 5.3.2 and deduce $\left|\frac{d(p,p_1)}{s} - 1\right| + \left|\frac{d(p,p_2)}{s} - \cos\frac{\alpha}{2}\right| < \epsilon(s)$. Moreover, since $|\widetilde{\measuredangle} p_1 p p_2 - \frac{\alpha}{2}| < \epsilon(s)$ by Lemma 5.4.2, it follows that $\left|\widetilde{\measuredangle} p p_1 p_2 - (\frac{\pi}{2} - \frac{\alpha}{2})\right| \leq \epsilon(s)$. Choose $p', p_2'$ in the minimizing geodesics between $p, p_1$ and $p_1, p_2$ such that $d(p_1, p_2') = d(p_1, p') = h_1(s)$. Then by angle comparison $\widetilde{\measuredangle} p' p_1 p_2' \geq \widetilde{\measuredangle} p p_1 p_2 \geq \frac{\pi}{2} - \frac{\alpha}{2} - \epsilon(s)$, and hence $\partial B_N(p_1, h_1(s)) \geq d(p', p_2') \geq C^{-1} h_1(s)$. So by volume comparison we get

$$vol(B_N(p_1, h_1(s))) \geq C^{-1}\, h_1^2(s). \tag{5.4.8}$$

On the other hand, let $\widetilde{M_0} \longrightarrow M_0 := M \backslash \Gamma$ be the universal covering, and $(\widetilde{M_0}, \widetilde{g}(t), \widetilde{p}_1)$ be the pull-back Ricci flow of $(M_0, g(t), p_1)$, $t \in (-\infty, 0]$, where $g(t)$ is the Ricci flow associated to $(M, g, p)$ with $g(0) = g$. Then $\widetilde{g}(0) = g_N + \varphi^2 d\theta^2$, $\theta \in (-\infty, \infty)$, and by using (5.4.8) we get

$$vol(B_{\widetilde{g}(0)}(\widetilde{p}_1, h_1(s))) \geq \frac{1}{2} h_1(s)\, vol(B_N(p_1, \frac{1}{2} h_1(s))) \geq C^{-1}\, h_1^3(s). \tag{5.4.9}$$

So by applying Corollary 45.1(b) in [43], we obtain $R(p_1) = R(\widetilde{p}_1) \leq C\, h_1^{-2}(s)$. □

**Lemma 5.4.4.** *Suppose $(M, g, p)$ is not a Bryant soliton. Then $\frac{h_2(s)}{h_1(s)} \to 0$ as $s \to \infty$.*

*Proof.* Suppose by contradiction that there is a sequence $s_i \to \infty$ such that $\frac{h_2(s_i)}{h_1(s_i)} \geq C^{-1} > 0$ for some $C > 0$ and all $i$. Let $\sigma_i$ be a minimizing geodesic from $\gamma(s_i)$ to some $q_i \in \Gamma$ such that $h_1(s_i) = d(\gamma(s_i), q_i)$. Then $\sigma_i$ intersects with $\Gamma$ orthogonally at $q_i$. Let $\Sigma_i = \phi^{-1}(\sigma_i)$, where $\phi : (M \setminus \Gamma, g) \to (N, g_N)$ is the Riemannian submersion. Then $(\Sigma_i, g_i)$ is a smooth rotationally symmetric surface with non-negative curvature, where $g_i$ is the metric induced by $g$. Then by Theorem 5.3.7, $(\Sigma_i, R(\Gamma(s_i))g_i)$ smoothly converges to the time-0-slice of a non-compact ancient Ricci flow $g_\infty(t)$.

Moreover, by Theorem 5.3.7 we know that any blow-down limit along $\Gamma$ is a product of $\mathbb{R}$ and a non-compact ancient Ricci flow, from which it follows that $\lim_{s\to\infty} h_1(s)R^{1/2}(\Gamma(s)) = \infty$. This combining with $\frac{h_2(s_i)}{h_1(s_i)} \geq C^{-1}$ and a volume comparison implies that the asymptotic volume ratio of $g_\infty(0)$ is positive, and hence $g_\infty(t)$ is flat, a contradiction. □

**Lemma 5.4.5.** *Suppose the asymptotic cone of $(M, g, p)$ is a ray. Then there is some $C > 0$ such that $h_1(s)h_2(s) \geq C^{-1}s$ for all large $s$.*

*Proof.* The assertion clearly holds when $(M, g, p)$ is a Bryant soliton, so we may assume below that $(M, g, p)$ is not a Bryant soliton.

On the one hand, since $h_1(s) = d(\gamma(s), \Gamma)$, we have $d(q, \gamma(s)) = h_1(s)$ for some $q \in \Gamma$. Let $\overline{q}$ be the image of $q$ under the $\mathbb{Z}_2$-action, and $\sigma : [-\frac{1}{2}d(q, \overline{q}), \frac{1}{2}d(q, \overline{q})]$ be a minimizing geodesic from $q$ to $\overline{q}$. Then by the $\mathbb{Z}_2$-symmetry it follows that $\sigma$ intersects orthogonally with $\Sigma$ at $\sigma(0)$ and

$$d(q, \sigma(0)) = d(q, \Sigma) = \frac{1}{2}d(q, \overline{q}). \tag{5.4.10}$$

Moreover, by replacing $\sigma$ with its image under some $O(2)$-action, we may assume $\sigma(0) \in \gamma$. So we have

$$\frac{1}{2}d(q, \overline{q}) = d(q, \gamma) \leq d(q, \gamma(s)) = h_1(s). \tag{5.4.11}$$

Since the asymptotic cone is a ray, by Lemma 5.4.2 and $h_1(s) = d(\gamma(s), \Gamma) \leq d(\gamma(s), \Gamma(s))$, we see $h_1(s) \leq \epsilon(s)s$. So by Lemma 5.3.2 and using triangle inequality we obtain

$$d(p, \sigma(0)) \leq d(p, \gamma(s)) + d(\gamma(s), q) + d(q, \sigma(0)) \leq d(p, \gamma(s)) + 2h_1(s) \leq (1 + \epsilon(s))s. \tag{5.4.12}$$

Suppose $\sigma(0) = \gamma(s')$ for some $s' > 0$, then by Lemma 5.3.2 this implies $s' \leq (1 + \epsilon(s))s$, which by the concavity of $h_2$ yields

$$h_2(s) \geq (1 - \epsilon(s))h_2(s') \geq \frac{1}{2}h_2(s'). \tag{5.4.13}$$

On the other hand, let $\Omega(s) \subset M$ be the domain bounded by $\phi^{-1}(\sigma)$, where $\phi : (M \setminus \Gamma, g) \to (N, g_N)$ is the Riemannian submersion, then

$$d(\partial\Omega(s), p) \geq d(p, \sigma(0)) - d(q, \sigma(0)) \geq (1 - \epsilon(s))s - h_1(s) \geq (1 - \epsilon(s))s, \tag{5.4.14}$$

which implies $\Omega(s) \supset B(p, \frac{1}{2}s)$, So by Stokes' theorem, $R = \Delta f$, and Lemma 5.4.1 we obtain

$$\text{Area}(\partial\Omega(s)) \geq \int_{\partial\Omega(s)} \langle \nabla f, \vec{n} \rangle = \int_{\Omega(s)} \Delta f \, dvol_M \geq \int_{B(p,\frac{1}{2}s)} R \, dvol_M \geq C^{-1} s. \tag{5.4.15}$$

By the $\mathbb{Z}_2$-symmetry we have $\frac{d}{dr} \mid_{r=0} \varphi(\sigma(r)) = 0$, which combining with the concavity of the warping function $\varphi$ implies $\varphi(\sigma(r)) \leq \varphi(\sigma(0)) = h_2(s')$ for all $r \in [-\frac{1}{2}d(p,\overline{p}), \frac{1}{2}d(p,\overline{p})]$. So

$$\text{Area}(\partial\Omega(s)) = \int_0^{2\pi} \int_{-\frac{1}{2}d(q,\overline{q})}^{\frac{1}{2}d(q,\overline{q})} \varphi(\sigma(r)) \, dr \, d\theta \leq 2\pi \, d(q,\overline{q}) h_2(s') \leq C h_1(s) h_2(s), \tag{5.4.16}$$

where we used (5.4.11) and (5.4.13) in the last inequality. This together with (5.4.15) proves the lemma.

□

**Lemma 5.4.6.** *Suppose the asymptotic cone is a ray, and* $\lim_{s\to\infty} h_2(s) < \infty$. *Then* $\lim_{s\to\infty} R(\Gamma(s)) > 0$.

*Proof.* Suppose $s$ is sufficiently large, and assume $\lim_{r\to\infty} \varphi(\gamma(r)) = \lim_{r\to\infty} h_2(r) = C$ for some $C > 0$. Let $p_1 = \Gamma(s)$, $p_2 = \Gamma(-s)$, and $\sigma : [0, d(p_1, p_2)] \to M$ be a minimizing geodesic from $p_1$ to $p_2$. Let $pp_1, pp_2, p_1p_2 = \sigma$ be minimizing geodesics between these points. Then since $\widetilde{\measuredangle} p_1pp_2 \leq \epsilon(s)$, we have $\measuredangle pp_1p_2 \geq \widetilde{\measuredangle} pp_1p_2 \geq \frac{\pi}{2} - \epsilon(s)$.

For some $s' >> s$, take $q = \gamma(s')$, and let $qp_1, qp_2$ be minimizing geodesics between these point. By replacing $\sigma = p_1p_2$ and $pp_1$ with their image under suitable $O(2)$-actions, we may assume that $\measuredangle pp_1p_2 + \measuredangle qp_1p_2 \leq \pi$. Since by angle comparison $\measuredangle p_2p_1q \geq \widetilde{\measuredangle} p_2p_1q \geq \frac{\pi}{2} - \epsilon(s)$, it follows that $\left|\measuredangle pp_1p_2 - \frac{\pi}{2}\right| \leq \epsilon(s)$. Note by Lemma 5.3.2 we have $\measuredangle(\nabla f(p_1), pp_1) \leq \epsilon(s)$, so by triangle inequality we obtain

$$|\langle \nabla f, \sigma'(r) \rangle(0)| + |\langle \nabla f, \sigma'(r) \rangle(d(p_2, p_1))| \leq \epsilon(s). \tag{5.4.17}$$

By the dimension reduction Theorem 5.3.7 we have $R^{-1/2}(\Gamma(s)) < \frac{1}{2}d(p_1, p_2)$ and

$$\varphi(\sigma(R^{-1/2}(\Gamma(s)))) \geq C^{-1} R^{-1/2}(\Gamma(s)). \tag{5.4.18}$$

By the $\mathbb{Z}_2$-symmetry it follows that $\sigma$ intersects with $\Sigma$ orthogonally at $\sigma\left(\frac{1}{2}d(p_1, p_2)\right)$, and $\frac{d}{dr}\Big|_{r=\frac{1}{2}d(p_1,p_2)} \varphi(\sigma(r)) = 0$. So by the concavity of $\varphi$ we get

$$\varphi(\sigma(R^{-1/2}(\Gamma(s)))) \leq \varphi\left(\sigma\left(\frac{1}{2}d(p_1, p_2)\right)\right) \leq \lim_{r\to\infty} \varphi(\gamma(r)) = C, \tag{5.4.19}$$

which together with (5.4.18) implies the lemma. □

Now we prove Theorem 5.1.4 of the equation $\lim_{s\to\infty} R(\Gamma(s)) = \sin^2\frac{\alpha}{2}$.

*Proof of Theorem 5.1.4.* Without loss of generality we may assume $\mathrm{Rm} > 0$, and $(M, g, f, p)$ is not a Bryant soliton, since the theorem clearly holds for $\mathbb{R}\times$ Cigar and the Bryant soliton. We may also assume $R(p) = 1$.

For each fixed $s$ sufficiently large, let $\sigma : [0, d(p_1, p_2)] \to M$ be a minimizing geodesic from $p_1 = \Gamma(s)$ to $p_2 = \Gamma(-s)$. By the soliton equation $\nabla^2 f = \mathrm{Ric}$ and by integration by parts we obtain

$$\langle \nabla f, \sigma'(r)\rangle \,|_0^{d(p_2,p_1)} = \int_0^{d(p_2,p_1)} \mathrm{Ric}(\sigma'(r), \sigma'(r))\, dr. \tag{5.4.20}$$

First, we claim

$$\left|\langle \nabla f, \sigma'(r)\rangle \,|_0^{d(p_2,p_1)} - 2|\nabla f|(\Gamma(s)) \sin\frac{\alpha}{2}\right| \le \epsilon(s). \tag{5.4.21}$$

If $\alpha = 0$, the claim holds by Lemma 5.4.6. So we may assume $\alpha > 0$.

Let $p_3 = \Gamma(2s)$, and $pp_2, pp_1, p_1p_2, p_1p_3, p_2p_3$ be minimizing geodesics between these points, where $p_1p_2 = \sigma$ in particular. On the one hand, by replacing geodesics $pp_1, p_1p_3$ with their images under suitable $O(2)$-actions (note $p, p_1, p_3 \in \Gamma$ are fixed under $O(2)$-actions), we may assume $\measuredangle pp_1p_2 + \measuredangle p_2p_1p_3 \le \pi$. On the other hand, by Lemma 5.3.2 and Lemma 5.4.2 we obtain

$$\left|\frac{d(p,p_1)}{s} - 1\right| + \left|\frac{d(p,p_3)}{s} - 2\right| + \left|\frac{d(p_1,p_2)}{s} - \sqrt{2-2\cos\alpha}\right| + \left|\frac{d(p_2,p_3)}{s} - \sqrt{5-4\cos\alpha}\right| \le \epsilon(s).$$

Since $\alpha > 0$, we have $\sqrt{2-2\cos\alpha} > 0$. So by the cosine formula we obtain

$$\left|\widetilde{\measuredangle} pp_1p_2 - \frac{\pi-\alpha}{2}\right| + \left|\widetilde{\measuredangle} p_2p_1p_3 - \frac{\pi+\alpha}{2}\right| \le \epsilon(s). \tag{5.4.22}$$

Then by the angle comparison it follows that $\measuredangle pp_1p_2 \ge \frac{\pi-\alpha}{2} + \epsilon(s)$ and $\measuredangle p_2p_1p_3 \ge \frac{\pi+\alpha}{2} + \epsilon(s)$, which combining with $\measuredangle pp_1p_2 + \measuredangle p_2p_1p_3 \le \pi$ implies

$$\left|\measuredangle pp_1p_2 - \left(\frac{\pi-\alpha}{2}\right)\right| \le \epsilon(s). \tag{5.4.23}$$

Note by Lemma 5.3.2 the angle between $\nabla f$ and the tangent vector of $pp_1$ at $p_1$ is smaller than $\epsilon(s)$, this implies claim (5.4.21).

Next, by the Dimension Reduction Theorem 5.3.7, $(M, R(\Gamma(s))g, \Gamma(s))$ is $\epsilon(s)$-close to $\mathbb{R}\times$Cigar, so we can find $D(s) < \min\{\frac{1}{2}d(p_2,p_1), \frac{1}{2}\epsilon(s)^{-1}\}$ such that $\lim_{s\to\infty} D(s)R^{1/2}(\Gamma(s)) = \infty$. So it follows that $d(\sigma(D(s)), \Gamma) \ge \frac{1}{2}D(s)\cos\frac{\alpha}{2}$. Then by the same argument as in Lemma 5.4.3 we get $R \le C(D(s))^{-2}$ in the two metric balls of radius $\frac{1}{2}\cos\frac{\alpha}{2}D(s)$ which are centered at $\sigma(D(s))$ and $\sigma(d(p_2,p_1) - D(s))$. This implies by the second variation formula that

$$\int_{D(s)}^{d(p_2,p_1)-D(s)} \mathrm{Ric}(\sigma'(r), \sigma'(r))\, dr \le \frac{C}{D(s)} \le \epsilon(s)R^{1/2}(\Gamma(s)). \tag{5.4.24}$$

If $\lim_{s\to\infty} R(\Gamma(s)) = 0$, by the uniform curvature bound for all dimension reductions we have

$$R^{-1/2}(\Gamma(s)) \int_I \mathrm{Ric}(\sigma'(r), \sigma'(r))\, dr \le C, \tag{5.4.25}$$

where $C > 0$ is a constant independent of $s$, and $I = [0, D(s)] \cup [d(p_1, p_2) - D(s), d(p_1, p_2)]$ This combining with (5.4.21)(5.4.24) and (5.4.20) implies $\alpha = 0$. So the theorem holds in this case.

If $\lim_{s\to\infty} R(\Gamma(s)) > 0$, the dimension reduction is a cigar soliton with scalar curvature equal to 1 at the tip, and it follows that

$$\left| \left( R^{-1/2}(\Gamma(s)) \int_I \mathrm{Ric}(\sigma'(r), \sigma'(r))\, dr \right) - 2\cos\frac{\alpha}{2} \right| \le \epsilon(s), \tag{5.4.26}$$

where we used the fact that for a cigar soliton with the sectional curvature $K$ equal to $\frac{1}{2}$ at the tip, the integral of $K$ along a geodesic emanating from $p$ is $\int_0^\infty K\, dr = \int_0^\infty \frac{1}{2}\mathrm{sech}^2(\frac{1}{2}r)\, dr = 1$. This combining with (5.4.24) implies

$$\left| \int_0^{d(p_2,p_1)} \mathrm{Ric}(\sigma'(r), \sigma'(r))\, dr - 2R^{1/2}(\Gamma(s)) \cos\frac{\alpha}{2} \right| \le \epsilon(s). \tag{5.4.27}$$

Combining (5.4.21)(5.4.27) in (5.4.20) and letting $s \to \infty$ we obtain

$$\lim_{s\to\infty} |\nabla f|(\Gamma(s)) \sin\frac{\alpha}{2} = \lim_{s\to\infty} R^{1/2}(\Gamma(s)) \cos\frac{\alpha}{2}. \tag{5.4.28}$$

By the identity $R + |\nabla f|^2 = R(p) = 1$, this implies $\lim_{s\to\infty} R^{1/2}(\Gamma(s)) = \sin\frac{\alpha}{2}$ and $\lim_{s\to\infty} |\nabla f|(\Gamma(s)) = \cos\frac{\alpha}{2}$, which proves the theorem.

□

Corollary 5.1.5 follows immediately from Theorem 5.1.4 and Theorem 5.3.7.

Now we prove Theorem 5.1.2 by a bootstrap argument: First, since $g = g_N + \varphi^2 d\theta^2$ on $M \setminus \Gamma$, the vector field $\frac{\partial}{\partial\theta}$ is a killing field. Then by the killing equation we can establish the following relation between the Ricci curvature and the warping function $\varphi$, when they are restricted on $\gamma \subset \Sigma$:

$$\mathrm{Ric}\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right) = |\nabla f|(\gamma(s)) h_2(s) h_2'(s). \tag{5.4.29}$$

Recall we define $h_2(s) = \varphi(\gamma(s))$.

Suppose that the soliton is not a Bryant soliton, then by combining the estimates from Lemma 5.4.3-5.4.5 in the equation (5.4.29), we obtain that $h_2(s) << s^{1/2}$. Replacing Lemma 5.4.4 with this new upper bound, then the same argument shows that $h_2(s) \le C$. This implies $\lim_{s\to\infty} R(\Gamma(s)) > 0$, and by Theorem 5.1.4 we obtain a contradiction.

*Proof of Theorem 5.1.2.* Let $\epsilon(s)$ be constants that converge to 0 as $s\to\infty$, and let $C$ denote all constants that are uniform for all large $s$. Suppose by contradiction that $M$ is not a Bryant soliton. We shall use the notations in Lemma 5.4.3-5.4.5. Since $g=g_N+\varphi^2d\theta^2$ on $M\setminus\Gamma$, it follows that $X:=\frac{\partial}{\partial\theta}$ is a killing field. So by the identity of killing field we have

$$\langle\nabla_X X,\nabla f\rangle+\langle\nabla_{\nabla f}X,X\rangle=0. \tag{5.4.30}$$

Note that $\langle X,\nabla f\rangle=0$ and $\nabla^2 f=\mathrm{Ric}$, this gives the identity

$$\mathrm{Ric}\left(\frac{X}{|X|},\frac{X}{|X|}\right)=\frac{\nabla f(|X|)}{|X|}. \tag{5.4.31}$$

Restrict the LHS of (5.4.31) on $\gamma(s)$ and abbreviate it by $\widetilde{R}(s)$. Then by the relations among $h_1(s),h_2(s)$ and $R(\gamma(s))$ from Lemma 5.4.5, 5.4.4, and 5.4.3 we obtain

$$s\,\widetilde{R}(s)\le s\,R(\gamma(s))\le Ch_1(s)h_2(s)R(\gamma(s))\le\epsilon(s)h_1(s)^2R(\gamma(s))\le\epsilon(s), \tag{5.4.32}$$

which by (5.4.31), $\lim_{s\to\infty}|\nabla f|(\gamma(s))=C>0$, and $h_2'(s)\ge0$ implies

$$\frac{h_2'(s)}{h_2(s)}\le\frac{\nabla f(h_2(s))}{|\nabla f|\cdot h_2(s)}=\frac{2\widetilde{R}(s)}{|\nabla f|}<\frac{\epsilon(s)}{Cs}<\frac{\epsilon_0}{s}, \tag{5.4.33}$$

for all large $s$ and some $\epsilon_0\in(0,\frac{1}{2})$. So $h_2(s)<Cs^{\epsilon_0}$ for all large $s$.

Next, by using $h_2(s)<Cs^{\epsilon_0}$ and applying Lemma 5.4.5 again we obtain $h_1(s)\ge C^{-1}s^{1-\epsilon_0}$, which combining with Lemma 5.4.3 again gives

$$\widetilde{R}(s)\le R(\gamma(s))\le Cs^{-2+2\epsilon_0}. \tag{5.4.34}$$

Now substituting this into equation (5.4.31) we obtain

$$\frac{h_2'(s)}{h_2(s)}<Cs^{-2+2\epsilon_0}, \tag{5.4.35}$$

which implies $h_2(s)<Ce^{-Cs^{-1+2\epsilon_0}}$, and hence $\lim_{s\to\infty}h_2(s)<\infty$. This by Lemma 5.4.6 implies $\lim_{s\to\infty}R(\Gamma(s))>0$, which by Theorem 5.1.4 yields a contradiction. □

Corollary 5.1.3 follows directly from Theorem 5.1.2. It is easy to see that the conclusions in Theorem 5.1.3 and Theorem 5.1.4 also hold for $n$-dimensional $O(n-2)\times O(2)$-symmetric steady gradient solitons with positive curvature operator. So Theorem 5.1.7 follows from Proposition 5.2.3. It remains to prove Corollary 5.1.6.

*Proof of Corollary 5.1.6.* First, by the proof of Theorem 5.1.1 and Theorem 5.1.2 there exists a sequence of $\mathbb{Z}_2\times O(2)$-symmetric 3d expanding gradient solitons with positive curvature operator $\{(M_{1k}, g_{1k}, p_{1k})\}_{k=1}^{\infty}$, which smoothly converges to a 3d flying wing $(M_1, g_1, p_1)$. We may assume $R_{g_{1k}}(p_{1k}) = R_{g_1}(p_1) = 1$, and the asymptotic cone of $(M_1, g_1, p_1)$ is a sector with angle $\alpha_1 \in (0, \pi)$. This by Theorem 5.1.4 implies $\lim_{s\to\infty} R_{g_1}(\Gamma(s)) = \sin^2 \frac{\alpha_1}{2}$.

Let $(M_0, g_0, p_0)$ be a Bryant soliton with $R_{g_0}(p_0) = 1$, since $\lim_{s\to\infty} R_{g_0}(\Gamma(s)) = 0$, we can find $s_1 > 0$ such that $R_{g_0}(\Gamma(s_1)) < \frac{1}{2}\sin^2 \frac{\alpha_1}{2}$. Choose a constant $\widehat{R} \in (R_{g_0}(\Gamma(s_1)), \frac{1}{2}\sin^2 \frac{\alpha_1}{2})$. Then by the convergence to $(M_1, g_1, p_1)$ and the continuity argument in Theorem 5.1.1, we can find a sequence of $\mathbb{Z}_2 \times O(2)$-symmetric expanding gradient solitons $(M_{2k}, g_{2k}, p_{2k})$ with positive curvature operator, which smoothly converges to a 3d flying wing $(M_2, g_2, p_2)$, with $R_{g_2}(p_2) = R_{g_{2k}}(p_{2k}) = 1$ and $R_{g_2}(\Gamma(s_1)) = \widehat{R}$. Assume the asymptotic cone of $(M_2, g_2, p_2)$ is a sector with angle $\alpha_2 \in [0, \pi]$. Then $\alpha \in (0, \pi)$ by Theorem 5.1.2. Moreover, by Theorem 5.1.4 we have

$$\sin^2 \frac{\alpha_2}{2} = \lim_{s\to\infty} R_{g_2}(\Gamma(s)) \leq \widehat{R} < \frac{1}{2}\sin^2 \frac{\alpha_1}{2}. \tag{5.4.36}$$

Therefore, by induction we obtain a sequence of 3d flying wings $(M_i, g_i, p_i)$ whose asymptotic cone is a sector with angle $\alpha_i$ satisfying $\sin^2 \frac{\alpha_{i+1}}{2} < \frac{1}{2}\sin^2 \frac{\alpha_i}{2}$ for all $i$. So $\alpha_i \to 0$ as $i \to \infty$. □

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