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ENACTIVE PERCEPTION AS MATHEMATICS LEARNING

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ABSTRACT: Recent developments in theory of cognitive sciences, interactive technological media, and empirical research instruments are enabling the Learning Sciences to investigate whether manipulation-based mathematics learning might be appropriate beyond early elementary school. Drawing on the embodiment turn in epistemology, the chapter supports and extends Piaget's implication of sensorimotor activity as grounding conceptual development. If, as per enactivism, perception consists in perceptually guided action, and cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided, then mathematics pedagogy should stage conditions that foster this learning process. After overviewing the rationales of embodied learning, the chapter discusses cumulative findings from a design-based research project evaluating the action-based genre of the embodied-design framework. Implementing embodied design would require systemic change in how we conceptualize cognition, what activities we create, how we facilitate these activities, how we prepare teachers, what classroom epistemic norms we sanction, and how we assess learning.

> I've used the metaphor of an egg yolk frying in a pool of oil, or a jetski riding ocean waves, to understand the behaviour of a fine-scaled or high-frequency component of a wave when under the influence of a lower frequency field, and how it exchanges mass, energy, or momentum with its environment. In one extreme case, I ended up rolling around on the floor with my eyes closed in order to understand the effect of a gauge transformation that was based on this type of interaction between different frequencies.
>
> Terence Tao (2016), Recipient, 2006, Fields Medal

Recent developments in the philosophy of cognitive science (embodied cognition), technological media (interaction devices), and research methods (multimodal learning analytics) have created new opportunities for design-based research of mathematical cognition, learning, and teaching. This confluence of scholarship, intervention, and measurement can be understood as challenging implicit assumptions underlying mainstream curriculum administration. Whereas manipulation-based instruction has been the privileged realm of elementary school years, a burgeoning generation of educational devices now offers advanced students opportunities to develop sensorimotor capacity believed to constitute early grips on mathematical concepts. This chapter draws on embodiment theory, in particular enactivism, to discuss the possibility of expanding constructivist pedagogy through embodied-design-based research, later into advanced mathematics subject matter studies. The thesis is contextualized in emerging findings from evaluation studies of three activities, in which the Mathematics Imagery Trainer activity architecture (Abrahamson & Trninic, 2011) was employed to implement the action-based genre of the embodied-design framework (Abrahamson, 2009, 2014, 2015) in the form of concept-oriented learning environments. Corroborating arguments from dynamic system research on motor action, the studies foreground the formative role of sensory perception in coordinating the enactment of movement. Students develop these perceptual routines through solving motor-control problems. In turn, these perceptual structures come to constitute ontological entities that students can mathematize using cultural forms. Students thus ground new mathematical notions.

Constructivist Pedagogy: On Beyond Arithmetic?

Constructivism, Jean Piaget's theory of genetic epistemology (Piaget, 1968), is often cited by educational designers and practitioners as both their motivating rationale and heuristic framework for student-centered instructional methodology (Kamii & DeClark, 1985). Constructivist learning environments typically include materials, activities, and facilitation practices that purportedly optimize for conceptual development. These environments are organized so as to engage students in tasks that solicit their knowledge, skills, and beliefs; as they attempt to carry out a task, however, students encounter unanticipated problems of enactment; these problems stem from embedded activity features, such as emergent properties of the environment as well as functional relations among these properties, which hamper students' straightforward realization of their available personal resources. Struggling to overcome these problems of enactment, students undergo the core learning process intended by the designers of the learning environment to be the activity's pedagogical objective. Students adapt to the constraints they have encountered by modifying their coping capacity, including their sensorimotor dexterity, perceptual patterns, phenomenal categories, and forms of reasoning: they thus assimilate the constraints by accommodating their skills, ergo, they learn.

Still, situated learning of this ilk may be tethered to the narrow local circumstances of a particular activity context, and it may as yet be constituted only as inscrutable action schemes and fleeting, ineffable insight (Morgan & Abrahamson, 2018). Teachers therefore encourage students to reflect on these experiences to draw out portable principles. By engaging students in discourse and supplementing into the learning environment various epistemic forms, such as a table or a diagram, teachers usher students to articulate and document their fledgling inferences in normative semiotic registers, such as verbal utterance and symbolic notation; these, in turn, mediate, form, and possibly transform students' situated know-how into disciplinary know-that (Bamberger & diSessa, 2003; Bartolini Bussi & Mariotti, 2008). Students thus generate formal expressions to signify empirically validated quantitative properties of situations they have learned to control. For example, students discover, capture, and mathematize phenomenal invariants of objects they have transformed in accord with stipulations of an assigned task. In this sense, the new mathematical subject matter content is said to be grounded: the notion is a personally meaningful and empirically validated generalization expressed in normative semiotic register of the disciplinary practice in question, mathematics (Noss & Hoyles, 1996). A new mathematical object comes to be a quasi-real onto–semiotic entity that abstracts and coordinates polysemous meanings from the subjective experiential contexts to enable productive disciplinary discourse across domains of prospective application (Font et al., 2013).

The pedagogical essence of educational activities dubbed as constructivist is often characterized through juxtaposition with instructional practices associated with traditional, “business-as-usual” methods. Unlike mainstream instruction, constructivist methodology is said to evoke children’s curiosity, interest, and exploration; respect, sanction, and celebrate students’ diverse intellectual capacity; create material, epistemic, and affective ecologies auspicious to occasioning engaged “bottom up” meaningful learning rather than “top down” direct teaching of meaningless information and procedures; transpire through students’ self-directed actions of trial, error, discovery, and insight; and, more generally, foster agency, ownership, critical reasoning, expressivity, and collaboration competencies, which are to carry over as the child’s prospective can-do orientation toward problematic situations in this and other disciplines.

Constructivist education proliferates. In more ways than not, its methodology bears family resemblance to educational vision propagated by John Dewey (1944), progressive regimens of Maria Montessori (1967), didactical phenomenology of mathematical structures (Freudenthal, 1983; Gravemeijer, 1999), theory of formalization (Diénès, 1971), theory of didactical situations (Brousseau, 1997), pedagogical philosophy of constructionism (Papert, 1980), and, retroactively to 1837, formative praxis of kindergarten (Froebel, 2005). In turn, empirical studies evaluating constructivist pedagogy create a context for educational researchers to build and refine theory of cognition, teaching, and learning. In particular, various constructivist and neo-constructivist scholars of mathematics learning have been qualifying and expanding Piaget’s thesis, to explain how students generate situated abstractions (Noss & Hoyles, 1996), to hypothesize the epigenesis of central conceptual structures (Case & Okamoto, 1996), to monitor the growth of schemas (APOS theory, Arnon et al., 2013), and to model the attainment of situated intermediary learning objectives (Chase & Abrahamson, 2018). Although the rationale and efficacy of constructivist pedagogy, or aspects thereof, continue to be debated (Abrahamson & Kapur, 2018; Klahr, 2010; Nathan, 2012), its principles are widely adopted, more so recently, through the proliferation of game design for mobile STEM learning (Bano et al., 2018).

Putting aside the polemics of mathematics pedagogy, which query the rigor and practicability of reform-oriented inquiry activities (Schoenfeld, 2004), my point here is to consider an apparent inconsistency within the constructivist camp, an inconsistency that, I believe, may be delimiting both the pedagogical application of constructivist principles and, reflexively, delimiting its empirical argumentation. This inconsistency, as I will explain, appears to stem from a historical misreading of Piaget. A renewed reading of Piaget, invigorated and informed by the recent 4E turn in the cognitive sciences (embodied, embedded, extended, and enactive cognition), may substantiate, refine, and expand the implementation of constructivist pedagogy.

Here’s the rub. Notwithstanding its continuing theorization and evaluation, constructivist learning is generally considered child play, a privilege of elementary school, when – popular wisdom offers – students still need hands-on activities in order to understand new ideas. Childhood’s end, as per the educational system at large and even among progressive institutions, arrives thereafter, whence interactive learning with concrete and virtual materials gives way to toiling with noninteractive media – paradigmatically paper and pencil. What might be the reasons for this view, and what may we be missing by it?

Ardent champions of constructivist pedagogy allude to Piaget’s stage theory of development to argue that once children arrive at the formal operational stage during early adolescence, they no longer require sensorimotor learning. Sensorimotor learning is henceforth shunned as being childish. To engage students in sensorimotor learning is inappropriate, inefficient, and infantilizing. With the impending advent of adulthood, we clean up our toys, sharpen our pencils, and sit down to write. We get serious.

My issue is not regarding the contested validity of stage theory (e.g., Smith et al., 1999) nor its pedagogical implications (Bruner, 1960). Rather, I believe that the problem with this reading of Piaget is that it throws out the sensorimotor baby with the formal operational bathwater. That is, without a sensorimotor grip on a notion, there is nothing there to formalize. Even if an instructional process begins with a meaningless formal proposition expressed in symbolic notation, still the meaning-making process, that is, learning, necessitates animating this expression "down" into sensorimotor modalities, what Pirie and Kieren (1994) call folding back. I thus wish to characterize the Childhood's-End position as manifesting a certain ontological or developmental fallacy, an implicit category error confusing semiosis with phenomenology. When individuals are able to represent their reasoning symbolically, still this reasoning continues to be grounded in simulated multimodal interaction. The crux of my argument is that, on beyond arithmetic and through to the most arcane concepts, mathematical learning continues to transpire as perceptual struggle to gain motor control over situations. Note the opening motto from Terence Tao, or read Hadamard (1945). Sense making is, not metaphorically alone, getting a grip on things (Abrahamson, 2021; Abrahamson & Sánchez-García, 2016; Hutto, 2019).

In the remainder of this chapter, I will first draw on 4E literature to substantiate constructivist pedagogy and elaborate my thesis, and then I will contextualize this discussion in the form of findings from research studies on the action-based genre of the embodied-design framework (Abrahamson, 2009, 2014, 2015) as implemented in the content domains of proportion, parabolas, and trigonometry.

Regrounding Mathematics in Sensorimotor Experiences: A Collusion of 4E Theory

Artificial intelligence, a later twentieth-century computer-science paradigm for engineering information machinery, bore a lasting mark on the cognitive sciences, a discipline concerned with biological intelligence. The CS algorithmic rationale of symbolic information processing (input–procedure–output), along with the robotics design principle of structural modularity (sensation–computation–action), appealed to numerous cognitive scientists, who mapped the AI rationale onto our organic species to discern ready analogs in human functioning and anatomy. Consequently, computational models of biological learning and reasoning proliferated. Unsurprisingly, these AI models of human cognition were estranged from holistic or systemic views of intelligence as embodied, situated, and emergent – such as those found in phenomenological philosophy (Merleau-Ponty, 1964) or ecological psychology (Heft, 1989) – and in particular were inimical to Piaget's structuralist theory of genetic epistemology, which thus fell out of favor (Gopnik, 1996). Despite valiant defiance from some philosophical quarters (Dreyfus & Dreyfus, 1986), by and large disembodied epistemology prevailed.

In recent decades, however, AI models of human cognition have been increasingly queried to be incompatible with empirical evidence for the mind's intrinsic corporeality (Newen et al., 2018; Núñez & Freeman, 1999; Shapiro, 2014). Rather than consider the mind as a central processing unit encapsulated in the brain organ, philosophers of cognitive science are seeking to model the mind as a task-oriented, systemic architecture extending through the body into natural and cultural ecology, including peers, artifacts, and norms (Anderson et al., 2012). The very ontological characterization of cognitive content as the fodder of intelligent behavior is being interrogated as largely unsubstantiated, giving way to enactivist conceptions of the mind as a form of activity (Chemero, 2009; Hutto et al., 2015; Hutto & Myin, 2013, 2017). These scholars model the mind not as transcending matter but as evolutionarily forged by and for our species' adaptive embodied coping in our terrestrial habitat (Wilson & Golonka, 2013). *No matter* how imaginative, counter-factual, and abstract our concepts may be, the phenomenology of inventing and reasoning with these concepts is ineluctably perceptuomotor, though this activity may be covert. Even the manipulation of mathematical symbols is just that – a simulation of manual grasping and moving (Landy & Goldstone, 2007). Disembodied mind is but a conceit – a chimera.

If conceptual reasoning is seeded in embodied coping with environmental contingencies, then perhaps motor action should play a greater role in both modeling cognitive development theoretically and fostering it pedagogically – hence, a renewed interest, post-AI, in Piaget's theory of genetic epistemology (Allen & Bickhard, 2013, 2015; Arsalidou & Pascual-Leone, 2016; Di Paolo et al., 2014). And hence a renewed opportunity for learning scientists to endorse and kindle the sensorimotor grounds of mathematical learning, teaching, reasoning, and problem solving (Abrahamson et al., 2020; Leung et al., 2013; Nathan et al., 2019; Nathan & Walkington, 2017; Nemirovsky et al., 2013; Schansker & Bikner-Ahsbahs, 2016; Sinclair, 2014).

In considering cognitive activity as situated engagement, I wish, here, to foreground the formative role of sensory perception in conceptual development. Numerous scholars have argued that expertise in the disciplines requires perceiving the environment in specialized ways that are conducive to enacting professional practices; apprenticeship into the disciplines therefore demands enculturating novices into appropriate perceptual routines (Goldstone et al., 2009; Goodwin, 1994; Goodwin & Goodwin, 1996; Radford, 2010; Sfard, 2002; Stevens & Hall, 1998). Yet, once we model conceptual reasoning as being inherently embodied, embedded, extended, and enactive (viz. 4E), perception becomes more than a disciplinarily meaningful sensory impression per se – a thing to think *about*. Rather, perception emerges in goal-oriented action to enable the action; for thinking *through*. Perception inheres, implicates, and guides motor action – neurally, evolutionarily, and phenomenologically. As Varela et al. (1991, p. 173) explain enactivism: "(1) perception consists in perceptually guided action and (2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided" (p. 173).

Foregrounding the phenomenology of perception bears educational implications. In particular, the enactive philosophy of cognition implies a tentative pedagogical hypothesis: to foster new cognitive structures, such as early grips on mathematical concepts, we need to create opportunities for students to develop new perception-for-action. The design corollary would then be to create educational activities, where particular forms of perceiving a learning environment would prove conducive to achieving designated task objectives. This hypothetical enactivist methodology for mathematics education is supported by empirical findings in dynamical systems research on movement learning (Mechsner et al., 2001; Turvey, 1992). Namely, what we often call motor learning is, in fact, epiphenomenal to perceptual learning – once the mind has established in the environment a perceptual structure that facilitates enacting goal-effective movement, appropriate motor actions will come forth spontaneously as a means of realizing the movement (Abrahamson & Mechsner, in press). It is upon this functional principle of the human cognitive architecture that the action-based genre of embodied design is based, as I now explain and demonstrate.

Learning Is Moving in New Ways: Designing for Perceptuomotor Grounding

The overarching pedagogical rationale motivating the action-based genre of embodied design is that learning a new mathematics concept begins with goal-oriented change in sensorimotor capacity. The design objective is to bring about this change in sensorimotor capacity. The desired change is achieved through engaging students in a motor-control task that challenges their entering sensorimotor capacity. Performing the motor-control task requires students to move in a way they have never moved before, such as a particular bimanual coordination. Activities are designed so that the physical solution movement for the motor-control task-problem constitutes a dynamical instantiation of the mathematical notion in question. Students need not initially know that their movements are mathematically meaningful – these meanings will arise for students only once when they discuss and represent these movements, using mathematical tools as frames of reference to describe and monitor their enactment. The epistemic shift from enactment to symbolization is thus mediated through the introduction of symbolic artifacts into the activity space (Newman et al., 1989). Students adopt these symbolic artifacts as their means of enhancing the enactment, explanation, or evaluation of their movement; yet in the course of doing so, they may reformulate their actions in accord with quantitative features of these artifacts – for example, switching from continuous movement to discretized movement (Abrahamson & Bakker, 2016; Abrahamson et al., 2011).

Empirical evaluation of designs for movement-based mathematics learning has highlighted the formative role of sensory perception as governing the enactment of new coordinated motor actions. That is, to move in a new way, students need to perceive the world in a new way. Using eye-tracking methods, combined with micro-ethnographic analysis of multimodal behaviors captured in audio–video recordings, we have been able to document the emergence of new perceptual routines that students develop to enable the enactment of new movements that satisfy task demands, routines we call *attentional anchors* (Abdu et al., in review; Abrahamson et al., 2016; Duijzer et al., 2017). Moreover, through appropriate intervention, students become conscious of these attentional anchors, which they gesture and describe as imaginary objects that they are grasping and manipulating (Abrahamson et al., 2012). Students are able to depict these shapes by using various concrete media, such as paper and pencil (Morgan & Abrahamson, 2016). In turn, inscribing the shapes then enables further mathematical modeling (Bongers et al., 2018).

Later in the chapter, we will briefly survey three activities, each for a different mathematical concept, which were designed to foster action-based embodied learning (Abrahamson, 2019). All activities incorporate a learning environment called the Mathematics Imagery Trainer (Abrahamson & Trninic, 2011; Howison et al., 2011). In Trainer activities, students learn to move in new ways by interacting with a technologically enabled environmental regimen, where only very specific movement forms consistently yield a prescribed goal state, often a green monitor color. For each of these activities, we will state students' spontaneous attentional anchors as measured and triangulated through micro-ethnographic analysis of clinical and eye-tracking instruments.

Example 1: Proportion

Figure 14.1The Mathematics Imagery Trainer for Proportion: Parallels problem – raising both hands while increasing the interval makes the screen green

Figure 14.1 features four schematic images illustrating a paradigmatic learning sequence of a student working with a Mathematics Imagery Trainer for proportion. Here, the Trainer's computational interaction function is set as the mathematical concept of ratio, and the quantitative parameter is set at 1:2. As such, the activity task's desirable sensory feedback (a green background, here light grey) favors bimanual positionings where the right hand is twice as high along the monitor as the left hand. Typically, students (a) position their hands at non-favorable locations (dark-grey feedback); (b) stumble upon a favorable position (light grey); (c) raise their hands maintaining a fixed interval between them (dark grey); and (d) correct the position (light grey). The designers of the Trainer perceive students' behavioral change – from a non-favorable routine of maintaining a fixed interval to a favorable routine that adjusts the interval correlative to the height – as a formative perceptuomotor foundation for expanding their enactive potential from additive to multiplicative reasoning and, specifically, for learning the concept of proportionality.

Once students demonstrate facility in enacting the new bimanual movement form, a series of symbolic artifacts are layered onto the monitor – first a grid and, later, numbers running up the grid's *y* axis. Students latch onto the grid as their pragmatic means of better performing and justifying their in-green strategy or, when working in pairs, as a shared means of coordinating their joint action through joint attention. In so doing, students appropriate the mathematical artifacts as frames of reference that, in turn, modify their perceptual construction of the working space. The frame of reference furthermore enables students to coordinate among their various movement strategies and, in so doing, arrive at important conceptual connections (Abrahamson et al., 2014).

Eye-tracking instruments enabled the researchers to document a range of perceptual routines governing the study participants' coordinated solutions to the motor-control problem (Shayan et al., 2017). In particular, students who engaged with this activity frequently attended to the spatial interval between the cursors, using it as their means of organizing the bimanual coordination. They reported the experience of increasing the interval as they raised *it* and of decreasing the interval as they lowered *it* back down (Flood, 2018). Note that although the interval is not an actual sensory entity but only two veritable visual stimuli connected by an imaginary line, for these students the interval was a phenomenologically present Gestalt (Mechsner, 2003). The interval came forth as an affordance that the students generated-cum-detected as their means of tightening their grip on the environment (Dreyfus & Dreyfus, 1999; Heft, 1989) and as a spontaneously evoked "steering wheel" for manipulating the world (Abrahamson & Bakker, 2016; Abrahamson & Sánchez-García, 2016; Abrahamson & Trninic, 2015; Turvey, 1992). Perceptual attunement emerges through goal-oriented interaction as adaptive kinesthetic enmeshing with the world. Perception for action is the epistemic seed of conceptual learning.

Example 2: Parabola

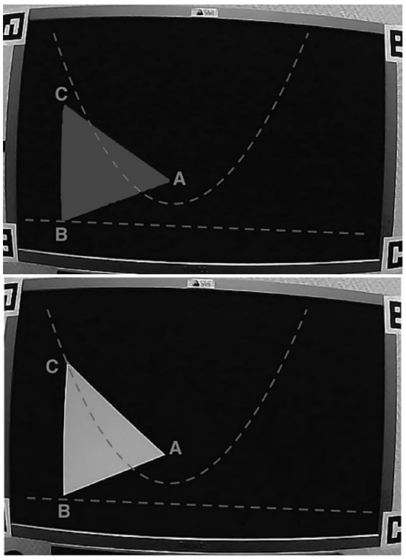

Figure 14.2 A Mathematics Imagery Trainer for Parabola – moving through isosceles triangles keeps the screen green.

Figure 14.2 features two configurations of a Trainer for parabolas. Here, the triangle is green (= light grey) only when BC = AC. A is fixed at the parabola's focus, B runs along the horizontal dashed line immediately below C, and the student manipulates Vertex C. By keeping the triangle green while moving Vertex C, the student effectively inscribes a parabola. (Note that the labels A, B, and C as well as the dashed lines in this figure are used only here to illustrate the design for readers of this text: these lines are never shown to the students as they engage in this stage of the activity.) Participant college students learned to move in green, and then they were guided to derive a definition of the parabola from geometrical properties of the isosceles triangle and auxiliary constructions (Shvarts & Abrahamson, 2019). The key cognitive event, along this solution process, was perceiving the isosceles triangle. Once they saw it, participants immediately became more fluent in operating the device according to task specifications.

Example 3: Trigonometry

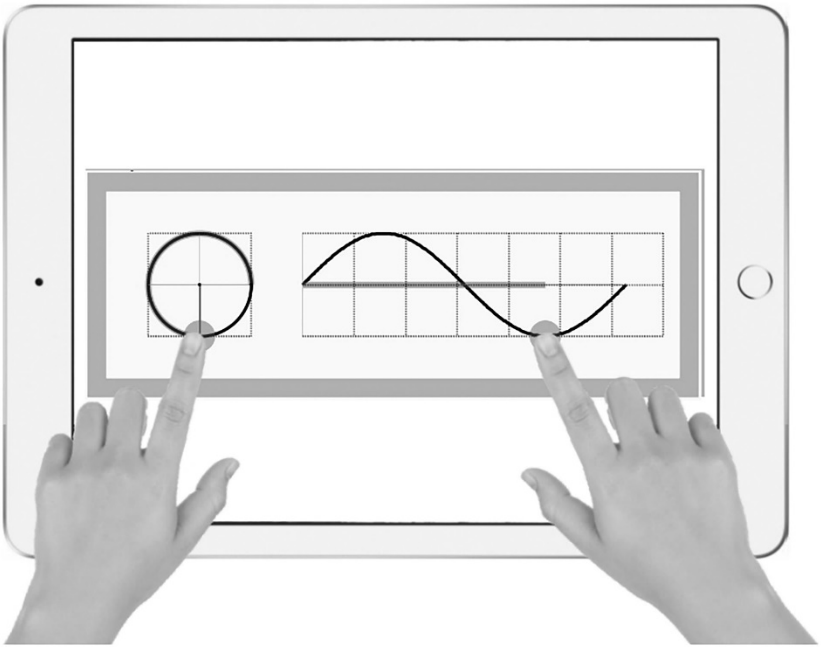

Figure 14.3 A Mathematics Imagery Trainer for Trigonometry – moving both hands equal distances away from their starting points keeps the screen green

Figure 14.3 features a Trainer activity for trigonometry. Here, the student slides their left-hand fingertip on the perimeter of a unit circle, while sliding the right-hand fingertip on a sine graph. Whenever the radian value on the circle corresponds to the *x*-value in the sine graph, the rectangular frame around the interactive zone becomes green. The student needs to keep the frame green while moving both hands. Analysis of data from a pilot study with participant college students suggests that they imagined a horizontal line segment connecting the two fingertips (see dashed illustrative line in Figure 14.3). This attentional anchor seemed to help the participants keep the two fingers at the same height. Mathematized, this imaginary line then came to mean that the left- and right fingertip positions are equally high or low on the grid, thus sharing the same *y*-value, which is *sin(x).* This awareness appeared further to support the enactment of green-keeping movement (Alberto et al., 2019).

Moving Forwards: Proactive Embodied-Design-Based Research

My objective, with this chapter, was not to ask, let alone answer the question "Was Piaget right?" Rather, my objective has been to ask, and attempt to answer, "What if Piaget was right?" and, moreover, "What if, per leading mathematicians, Piaget's implication of sensorimotor activity as formative of cognition extended on beyond arithmetic?" Because if he was right – and the research literature is increasingly supporting this conjecture – then mathematics education is disserving students by denying them visceral understandings of the curricular content they are studying. I state this boldly, and perhaps blithely, given that embodied design is still being evaluated (Alberto et al., 2021; Shvarts & van Helden, 2021), because the matter is urgent and the stakes too high.

Embodied design (Abrahamson, 2009, 2014, 2015, 2019) is a formidable project, requiring systemic change in how we conceptualize cognition, what activities we create, how we facilitate these activities, how we prepare teachers, what classroom epistemic norms we sanction, and how we assess learning. Because we need to redesign these multiple components, even as we research them, I view design-based research as the way forward (Bakker, 2018). It is not enough to analyze the shortcomings of education – one is mandated to respond proactively. Reacting to a *Mathematical Thinking and Learning* special issue on early mathematics, Clements and Sarama (2015) admonish the contributing authors, writing, "Mathematics education should not be an 'implication' tagged on to the end of studies from developmental and cognitive psychology. Mathematics education research and cognitive research should be interwoven enterprises" (p. 251). In like vein, Stetsenko (2017) assumes a *transformative activist stance* on sociocultural scholarship to re-read Vygotsky's project not as disinterested science but as "an explicitly dialectical and, more implicitly, ideologically non-neutral perspective on the core questions about human development, mind, and learning" (p. 4). Embodied design enlists in this transformative activist stance. Moreover, embodied design reframes how we serve students of sensorial and cognitive diversity (Abrahamson et al., 2019; Lambert et al., 2022; Tancredi et al., 2021).

Technology offers powerful resources for pro-activist educational research. Yet, technology is only as useful as its design rationale. Specifically, the success of technologies that are to interact intelligently with human intelligence is contingent on the *epistemological* assumptions of their human engineers. I am calling on educational engineers to consider 4E cognition in formulating their epistemological rationales. Dreyfus and Dreyfus (1999) wryly muse that "Until cognitive scientists recognize [the] essential role of the body, their work will remain a mixed bag of ad hoc successes and, to them, incomprehensible failures" (p. 118). With similar sentiment, Glenberg (2006) comments thus on educational technology: "One can view most of my reasons for skepticism as challenges for the future development of technology that is sensitive to the principles of *biological* cognitive systems" (p. 271, my italics).

Which brings us back to movement, the marker of biological organisms; the source and goal of biological intelligence; and, thus, the incipience of all learning. Moving forward, we need to better understand how thinking springs forth for and through movement. Movement, in space, in time, is the "elusive obvious" (Feldenkrais, 1981) – we enact it without knowing how, yet we learn from reflecting on it. "What is essential is a phenomenology of kinesthetic learning, a fleshing out of the developing awarenesses – felt, perceptual, cognitive – that constitute the knowledge, skills, and abilities of everyday life" (Sheets-Johnstone, 2015, p. 31). Teachers could enable students to tap their phenomenology of kinesthetic learning (Morgan & Abrahamson, 2016). Enactivist epistemologist Petitmengin (2007) offers this on the educational implications of the neurophenomenological first-person introspective methodology.

> [A]re our teaching methods well adapted? For at present, teaching consists in most cases of transmitting conceptual and discursive contents of knowledge. The intention is to fix a meaning, not to initiate a movement. Which teaching methods, instead of *transmitting* contents, could elicit the gestures which allow access to the source experience that gives these contents coherence and meaning? Such a teaching approach, based more on initiation than transmission, by enabling children and students to come into contact with the depth of their experience, could re-enchant the classroom.
>
> (p. 79, original italics)

I have attempted to explain and demonstrate how and why students should learn mathematics through movement and to posit the central role that perception comes to play in developing new ways of moving and, hence, new ways of thinking. If Piaget *was* essentially right, then time is of the essence.

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