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ABSTRACT

This report describes an IBM 650 computer program which makes a least-squares fit to any number of data points (x, y) exact in x and uncertain in y to a function of the form

$$y(x) = \sum_{k=0}^{n} a_k \phi_k(x),$$

where $\phi_k(x)$ is an arbitrary function of x only, and $0 \leq n \leq 10$. It computes the fitted parameters, a_k, and makes tables of y(x) for a range of values of x. It gives the error matrix for the fit, and uses it to propagate errors in the calculated quantities. It also calculates the quantities necessary to make a statistical test of the goodness of fit of the function to the points.

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I. INTRODUCTION

A problem frequently encountered in analyzing experimental results is to fit the data points with an arbitrary function by the method of least squares. This report describes an IBM 650 computer program of sufficient generality to handle the numerical work involved in many specific instances.

The program fits any number of data points, (x, y), to a series of the form

$$y(x) = \sum_{k=0}^{n} a_k \phi_k(x) , \qquad (1)$$

where $\phi_k(x)$ is an arbitrary function of x only. The program was originally written to fit data to a power series, $\phi_k(x) = x^k$, or a Legendre polynomial series, $\phi_k(x) = P_k(x)$. These two functions and a few others are already contained in the program. In addition, provision is made for the inclusion of subroutines which allow any other programmable function, $\phi_k(x)$, to be used. The order of the fit, n, can be chosen to be any number up to 10, this upper limit being set by inherent rounding errors.

The data points are assumed to be of the form $y_i \pm \Delta y_i$ at x_i, where the experimental point y_i has an error Δy_i, and x_i is taken to be exact.

The program determines the parameters of the fit, a_k, and produces tables of calculated values of y(x) by using the fitted parameters. It also gives the error matrix associated with the fit and uses it to assign errors to the calculated values of a_k and y(x). It produces in addition the information necessary to make a chi-square (χ^2) test of the fit when appropriate.

II. THEORY

Most of the theory of least-squares fitting employed here is developed in simple form by Cziffra and Moravcsik.[1] A more general explanation has been given in lucid form by Deming.[2] The following is a summary of the formulae employed in the program.

A least-squares fit is made by defining the weighted sum, S, of the squares of the residuals, $[y(x_i) - y_i]$, as

$$S = \sum_{i=1}^{p} [y(x_i) - y_i]^2/\Delta y_i^2 , \qquad (2)$$

where $y(x)$ is obtained from Eq. (1), and p is the number of data points. The values of the parameters a_k that minimize this sum is found by solving in turn the set of $n+1$ simultaneous equations

$$\partial S/\partial a_k = 0 , \qquad (3)$$

where $k = 0, 1, 2 \cdots, n$, and where the condition $p \geq n + 1$ must hold.

The solution to Eq. (3) may be written in the form

$$a_k = \sum_{j=0}^{n} c_{jk} \sum_{i=1}^{p} y_i \, \phi_j(x_i) / \Delta y_i^2 , \qquad (4)$$

where c_{jk} is the error matrix for the fit, a symmetrical $n + 1$ - square matrix.

The value of χ^2 for the fit is defined as

$$\chi^2 = S/\sigma^2 , \qquad (5)$$

where σ is the standard deviation error for a measurement of unit weight,[2] and S has its minimum value. The degrees of freedom for the fit, d, are the number of points minus the number of parameters,

$$d = p - n - 1 . \qquad (6)$$

The off-diagonal elements of the error matrix, c, are the correlation coefficients between the corresponding parameters. In general, the error $\Delta f(x)$ in any function of x, f(x) is given by[2]

$$\Delta f(x)^2 = \sigma^2 \sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij} \frac{\partial f(x)}{\partial a_i} \frac{\partial f(x)}{\partial a_j} . \qquad (7)$$

In particular, the error Δa_k on a_k is given by

$$\Delta a_k = \sigma \, (c_{kk})^{1/2} , \qquad (8)$$

and the error $\Delta y(x)$ on $y(x)$ is given by

$$\Delta y(x)^2 = \sigma^2 \sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij} \phi_i(x) \phi_j(x) . \tag{9}$$

When the errors Δy_i on the data points y_i are in standard deviations, σ^2 in Eq. (5) becomes unity, and the minimum value of S is then equal to χ^2. In this instance, a χ^2 test may be made to find the probability that a chosen function adequately fits the data. The probability that χ^2 would exceed the value found in a random sample of data can be obtained from tables of χ^2 probabilities for d degrees of freedom.[1] This probability is a test of the goodness of fit of the function. By fitting the data points to increasing orders n, the lowest order necessary to give an adequate fit may be found.

When the errors Δy_i on the data points y_i are only relative errors, but the function chosen, $y(x)$, is assumed to be the proper one to fit the points, an estimate may be made of the magnitude of σ. This external estimate,[2] σ_{ext}, is obtained by setting χ^2 equal to its most probable value, d, the degrees of freedom. From Eq. (5) there then results for σ_{ext} the value

$$\sigma_{ext} = (S/d)^{1/2} . \tag{10}$$

This value may then be used in Eq. (7) to find the error $\Delta f(x)$ on any function of x, f(x).

III. HOW TO USE THE GENERAL LEAST-SQUARES PROGRAM

A. Program Cards

A printout of the program deck is given in Appendix A. It includes a drum-zeroing card, a load routine, and a transfer card that initiates the program. A program deck can be prepared from this printout, or a copy of the program deck may be obtained from the Lawrence Radiation Laboratory Computer Group. How to run this program is discussed in Section C below.

B. Input Cards

Instructions and data are contained in a set of three types of input data cards. These three types of cards are distinguished by a card number from 1 to 3 in column 10 of word 1, as shown in Fig. 1.

Card 1

Word 1 of card 1 must have the number 1 in column 10. Only one of these cards is used in each input-card set. It serves to instruct the program what order of fit, n, is to be made, and what form the function $\phi_k(x)$ is to take.

Word 2 of card 1 must have the order of the fit, n, on its right side (columns 19 and 20). Any order from 0 to 10 may be used. If an order greater than 10 is used it will be replaced with 10, and the fit made to this order.

Word 3 of card 1 must have a function code number on its right-hand side. This code number consists of from one to ten nonzero digits. The form of the function, $\phi_k(x)$, is determined by these digits according to the following scheme. The meanings of the digits from 1 through 6 are already contained in the program. They are the following.

Word 1	Word 2	Word 3	Word 4	Word 5	Word 6	Word 7	Word 8
			INPUT CARDS				
0000000001	000000000n	funct. code					
0000000002	x_i (flt.)	y_i (flt.)	Δy_i (flt.)				
0000000003	x_{min} (flt.)	Δx (flt.)	x_{max} (flt.)	error code			
			OUTPUT CARDS				
0000000004	n (flt.)	p (flt.)	d (flt.)	S (flt.)	$\sqrt{S/d}$ (flt.)		
0000000005	000000000i	a_i (flt.)	Δa_i (flt.)				
0000000006	000000000i	000000000j	c_{ij} (flt.)				
0000000007	x (flt.)	y(x) (flt.)	$\Delta y(x)$ (flt.)				

Fig. 1. Format of input and output cards.

Digit		Functional Meaning
0 (x)	=	x (identity operation)
1 (x)	=	$(x)^k$ (kth power of x)
2 (x)	=	cos (x) (where x must be in radians)
3 (x)	=	$(\pi/180)$ (x) (changes degrees in to radians)
4 (x)	=	$P_k(x)$ where $-1 \leq x \leq +1$ (Legendre polynomial)
5 (x)	=	sin (x) (where x must be in radians)
6 (x)	=	$(\lvert x \rvert)^{1/2}$ (square root of absolute value)

These digits may be combined to give combinations of these functions. In this case, the operation indicated by the right digit are performed first, followed by the remaining digits in order from right to left. A zero terminates this procedure. For example,

$$0000000123\ (x) = [\cos(\pi x/180)]^k,$$

$$0000000423\ (x) = P_k[\cos(\pi x/180)].$$

The remaining digits, 7, 8, and 9, transfer the program to empty drum space which may be used to program other desired functions. The method for doing this is discussed in Appendix B.

Words 4 through 8 of card 1 may be used for any desired numerical information. All columns must be filled with some number; if not otherwise indicated they should be filled with zeros.

Card 2

Word 1 of card 2 must have the number 2 in column 10. One of these cards is used for each data point. Any number of these cards may be used in a given input-card set following a type 1 card. They contain the individual values of x_i, y_i, and Δy_i in machine floating-point number form. (A machine floating-point number is in the following form. The left-hand eight digits constitute the number, expressed with a decimal point to the left of the left digit. The right-hand two digits are the power of ten by which this eight-place number must be multiplied plus fifty. For example,

$$3760000054 = 0.376 \times 10^4 = 3.76 \times 10^3,$$

$$9670000046 = 0.976 \times 10^{-4} = 9.76 \times 10^{-5}.)$$

Word 2 of card 2 contains a value of x_i in floating point.

Word 3 of card 2 contains the corresponding value of y_i in floating point.

Word 4 of card 2 contains the corresponding value of Δy_i in floating point. This value cannot be zero, or an overflow will result.

A type 3 card must follow the end of the series of type 2 cards in any input-card set to indicate the end of the data points.

Card 3

Word 1 of card 3 must have the number 3 in column 10. Any number of these cards may be used. They serve to indicate the end of the data-point cards of type 2, and to instruct the program to make tables of $y(x) \pm \Delta y(x)$ for a range of values of x, from x_{min} to x_{max}, in steps Δx wide. They also indicate the value of σ to use in the error propagation.

Word 2 of card 3 has the value of x_{min} in floating point.

Word 3 of card 3 has the value of Δx in floating point.

Word 4 of card 3 has the value of x_{max} in floating point.

Word 5 of card 3 has an error code number which determines the value of σ to be used. When word 5 is zero, $\sigma = 1$ will be used. When word 5 is any nonzero number (it is convenient to use 0000000001), σ is $(S/d)^{1/2}$, the value of σ_{ext} for a proper function.

C. How to Run the Program

1. Insert standard "80-80 flops" board in the Read-Punch unit.

2. Load the program deck, followed by any number of sets of input cards in the Read Feed of the Read Punch unit.

3. On the console set 70 1951 1951 + on the Storage Entry Switches.

4. On the console set the Programmed, Half-Cycle, and Control switches on the Run position.

5. On the console set the Overflow and Error switches on the Stop position. (All other settings are arbitrary.)

6. On the console press the Computer Reset key.

7. On the console press the Program Start key.

8. On the Read-Punch unit press the Read Start key.

9. On the Read-Punch unit fill the Punch Feed with blank cards.

10. On the Read-Punch unit press the Punch Start key.

11. When the End-of-File light lights on the Read-Punch unit, press the End-of-File key.

D. Output Cards

The output of the program is contained on four types of output data cards. These four types of cards are distinguished by a card number from 4 through 7 in column 10 of word 1, as shown in Fig. 1. These cards have the following format.

Card 4

Word 2 of card 4 has the order of the fit, n, in floating point.

Word 3 of card 4 has the number of data points, p, in floating point.

Word 4 of card 4 has the degrees of freedom, d, in floating point.

Word 5 of card 4 has the minimized sum of the residuals, S, in floating point.

Word 6 of card 4 has the square root of the minimized value of the sum of the residuals divided by the degrees of freedom, $(S/d)^{1/2}$, in floating point.

Card 5

These cards have the values of the fitted parameters, a_k.

Word 2 of card 5 has the index number, k, in column 20.

Word 3 of card 5 has the value of the fitted parameter, a_k, in floating point.

Word 4 of card 5 has the value of the error on the fitted parameter, Δa_k, in floating point.

Card 6

These cards have the elements of the error matrix, c_{ij},

Word 2 of card 6 has the index number, i, in column 20.

Word 3 of card 6 has the index number, j, in column 30.

Word 4 of card 6 has the error matrix element, c_{ij}, in floating point.

Card 7

The cards are the computed table cards.

Word 2 of card 7 has a value of x in floating point.

Word 3 of card 7 has the corresponding value of y(x) in floating point.

Word 4 of card 7 has the corresponding value of $\Delta y(x)$ in floating point.

IV. SAMPLE PROBLEMS

A. A Power-Series Fit

Problem: Fit the following points

x	y
1	9
2	3
3	5
4	7
5	6

to the function

$$y = a_0 + a_1 x + a_2 x^2$$

and calculate the fitted values of y at the given values of x.

Answer: Since no errors are given, assume equal weights on all points, i.e., $\Delta y_i = 1$. The order is $n = 2$. The power series is obtained by the function code number 0000000001. Tables of the output points can be obtained by using the external estimate of the error, σ_{ext}, with the following data cards

word 1	word 2	word 3	word 4	word 5
1	2	1		
2	1000000051	9000000051	1000000051	
2	2000000051	3000000051	1000000051	
2	3000000051	5000000051	1000000051	
2	4000000051	7000000051	1000000051	
2	5000000051	6000000051	1000000051	
3	1000000051	1000000051	5000000051	1

The output of the program will be as shown on the printout given in Fig. 2.

1	2	1			
2	1000000051	9000000051	1000000051		
2	2000000051	3000000051	1000000051		
2	3000000051	5000000051	1000000051		
2	4000000051	7000000051	1000000051		
2	5000000051	6000000051	1000000051		
4	2000000051	5000000051	2000000051	1245714352	2495710651
5		1160000052	5352703051		
5	1	4485714351-	4079115451		
5	2	7142857150	6670067150		
6			4600000051		
6	1		3300000051-		
6	1	1	2671428651		
6	2		5000000050		
6	2	1	4285714350-		
6	2	2	7142857149		
3	1000000051	1000000051	5000000051	1	
7	1000000051	7828571051	2348773051		
7	2000000051	5485714051	1521008251		
7	3000000051	4571428051	1739338451		
7	4000000051	5085714051	1521006151		
7	5000000051	7028571051	2348771351		

Fig. 2. Printout of answer to sample problem A.

B. Legendre Polynomial Fit

Problem: Given the measured $\pi^- - p$ differential cross sections,

c.m. angle, θ^* (deg)	$d\sigma/d\Omega^*(\theta^*)$ (mb/sr)
30.0	3.14±0.17
70.0	1.45±0.06
110.0	1.06±0.05
150.0	2.11±0.07

where the errors are expressed in standard deviations. Are S and P waves sufficient to fit the data? Assuming they are, what are the extrapolated forward differential cross section, $d\sigma/d\Omega^*(0)$, and the total integrated cross section?

Answer: S and P waves imply that the data fit a function of the form

$$d\sigma/d\Omega^*(\theta^*) = \sum_{k=0}^{2} a_k P_k (\cos\theta^*) .$$

Legendre polynomials are chosen, since the total integrated cross section, σ_T, is then simply

$$\sigma_T = 4\pi a_0 .$$

Tables of results starting with $\theta^* = 0$ will give the forward differential cross section, $d\sigma/d\Omega^*(0)$, by extrapolation of the fitted curve. The errors are in standard deviations, so we use $\sigma = 1$ in the program. Then $S = \chi^2$, and a χ^2 test may be made for d degrees of freedom to test the goodness of fit.

The values of θ^* are in degrees. We convert to radians, take the cosine, and then the Legendre polynomial by using the function code number 0000000423. The order, n, is 2, so the data-input cards take the form

word 1	word 2	word 3	word 4	word 5
1	2	423		
2	3000000052	3140000051	1700000050	
2	7000000052	1450000051	6000000049	
2	1100000053	1060000051	5000000049	
2	1500000053	2110000051	7000000049	
3				

The output of the program is as shown on the printout given in Fig. 3. As can be seen, the fit is quite good, so that S and P waves will fit the data adequately. The value of the total cross section is $\sigma T = 21.6 \pm 0.5$ mb, and the forward cross section is $d\sigma/d\Omega^{*}(0) = 3.74 \pm 0.17$ mb/sr .

APPENDIX

A. The Program Deck

Following Fig. 3 is a printout of the program deck. It includes a drum-zeroing card, a load routine, and a transfer card to initiate the program (to 0500). All columns not containing a digit must be filled with zero when this printout is being copied to make a program deck.

1	2	423			
2	3000000052	3140000051	1700000050		
2	7000000052	1450000051	6000000049		
2	1100000053	1060000051	5000000049		
2	1500000053	2110000051	7000000049		
4	2000000051	4000000051	1000000051	2491000049	1578290250
5		1721385351	3884179349		
5	1	5832994050	7774063249		
5	2	1434595151	9129615849		
6			1508684948		
6	1		1515541948		
6	1	1	6043605948		
6	2		1368337748		
6	2	1	3521357948		
6	2	2	8334988548		
3					
7		3739279851	1694041250		

Fig. 3. Printout of answer to sample problem A.

FUNCTION LEAST SQUARES FIT PROGRAM IN 7 WORDS PER CARD LOAD FORM

6019541953	5000011957	6919561955	2420001952	8000008003	2400001958	2029758003	7019511951
6519521953	6919581957	1019548001	6919561955	2419938002	6919521951	2419968001	7019931993
2419981998	6919541953	2419888003	4419918001	2419898002	4219921996	2419901996	5300011998
2419911998	8280031999	2419928003	5100011997	2419948002	8080021995	2419951996	3500061988
2419971998	6959511990	2419998003	5040001992	2419138002	3000041919	2419141996	6980061969
2419151998	6980071970	2419168003	6919271947	2419178002	5280011941	2419181996	4619721935
2419191998	8280021924	2419208003	2119291942	2419218002	6940001923	2419221996	6919811946
2419231998	2439271938	2419248003	2219271980	2419258002	7119401945	2419261996	6919821944
2419351998	6580051964	2419378003	2019591972	2419388002	5100011943	2419391996	7119771920
2419401998	6580061916	2419418003	6580061949	2419428002	2319601913	2419431996	5300011948
2419441998	8280011950	2419458003	6919591917	2419468002	8080011926	2419471996	2319271985
2419481998	4019211925	2419498003	1619601965	2419508002	119501950	2419611996	6780001971
2419621998	8080011967	2419638003	5200011918	2419648002	3500041924	2419651996	4619681922
2419661998	3020001975	2419678003	5280011973	2419688002	8180021984	2419691996	2419821915
2419701998	2419831939	2419718003	6980051974	2419728002	6540001987	2419731996	6519761966
2419741998	2419811914	2419758003	2019361943	2419768002	888888888	2419771996	6975646587
2419781998	6163638474	2419798003		2419808002	3000041937	2419841996	1519591963
2419851998	6919591962	2419868003	8808000888	2419878002	4519401941	2419981996	2420001989
2419931996	6719511994						
70000	2400780099	6700540059	1680020012		6000890093	4000080096	3200090035
70007	3300100037	3900240063	7000000051	2000000051	8080020070	2000030056	2400450002
70014	1180030096	2100890050	3500020073	6900430046	6000280097	2100240027	2100240032
70021	6000780022	4400770034	6000780047	3600000043-	3400780079	58	3200430069
70028	8717829161	3200090085	3900830034	6000240029	3900280082	3400030053	2100830036
70035	4400390040	3200890015	2100030006	4400410092	4600420040	3200430019	6000280091
70042	6900450048	1000000051	3400830094	200	2400280081	3400100061	8080010004
70049	2100890050	6000030007	2000540057	2000830055	3400240025	7	6700540062
70056	2000240052	6980050013	3280020067	8080020017	5080020081	2100240097	4500160067
70063	2100280064	6000680065	3800280066	4600180014	2100090001	1000000010	3900240074
70070	4000230075			1500260058	2100280031	6000430096	
70077	6000280033	7200000043-	3400780030	3200430020	4000840086	2100280087	3432000054
70084	6000240080	2100240021	6000830038	5100010081	2400830090	2834798950-	6700540060
70091	3400830044	6900280088	2102300285	2100830095	6700540011	3900830098	5100010005
70098	2100830049	2100930051	2401040111	2401040107	3500010159	6780030132	

70105	3500010112	3000020184	6901100118	8000030113	3500010116	2001770103	6901650118
70112	2101170120	1901170109	3500020126	4401190129	1021220134		2401210127
70119	3600000140	6001370108	2001770103	1570796318	645963711-	79689679	4673765-
70126	4401300142	6980050187	6580030191	6901920195	5180030135	8080020186	1501360191
70133	1001360141	6080030151	3000010142	1	151484	4001430150	
70140	6780020149	3100010148	4601460147	4101500189	6780020153	4	5000510152
70147	5100510152	6080020157	8080020158	3120100121	4001540155	1680050161	3500020166
70154	5100010113	1901620102	3100020114	2101620115	5100040164	6080030169	1001638003
70161	6080020129		171	4101850168	2001770188	8180030176	1980010105
70168	6001220155	3600000131	6780030178	6701770141	6801770133	6801770141	6701770182
70175	5000500180	6001790183		1501810191	6366197722	3500020138	2
70182	1101360141	1901390175	2101390144	6001620167	6580030156	2401920106	4601700128
70189	6580030150		1401450160				8080010104
70200	1080030208	3162277660	3500010211		1502090213	2002090212	1902090273
70208	1080010214		1580030217	1580010267	4402650266	1180030221	1080010220
70215	3000050224	8080010226	3000020223	1502290236	1502290234	1080010275	3500040230
70222	2402250228	6580020232	1902330206			6480010225	1180030235
70230	6480010202	2102380241	3500020240	10000	3100020243	3500010242	6080020244
70237	6580030252		2402510210	1002450200	1180010248	8080030249	3500020262
70244	1902010269	51	6902510216		3500010205	6022520215	
70252	1002090263	59500413	78300317	93100268	105900236	117200213	127300196
70259	137300182	146000171	154200162	6080020278	1680020271	4402688001	6902180222
70266	6902190222	1002700277	4602680272	3600010276	25	3000030227	2402260279
70273	6080030204	2002290237	3000010231	6580030234	1980020274	1002380246	6980050239
70300	6003030307	6020000308	6103060311	1000000051	6903170320	6903170325	
70307	3420000309	3403150316	2403150323	6503170372	3940000318		5800010319
70314	6903170321		2140000322		3220000345	5200010326	8280010329
70321	8280010327	5200010327	2140000314	2160000360	8280010334	5000010331	5000010332
70328	5800010334	6920000349	2303410336	6580060339	6580060340	4503590380	6940000344
70335	8080010342	6903430350	1603410346	6503430356	1603430348	1603430347	
70342	8880010300		2460000338	2160000313	4503040305	4503010304	4503020353
70349	2403060326	8280010374	8080010357	1980060354	6103060361	1580050362	1580060368
70356	1680060363	3000040364	2403120392	2003700336	5800010366	3403150324	2303170371
70363	4503730367	8280030376	2103700352	6580070337	6503700375	2303430310	2103770351
70371	1603780355	1680060330	5200010328	6903770335	1603780333	3000040365	
70378	1	2403850388	6903850389	2403860390	6903860391	2403870369	6903870393
70388	6980060381	8080010382	6980070383	8280010384	6980050379	8880010312	
70400	2404030406	3000010407	6004030457		2104080461	3965500750	2004110414
70407	2104420545			5100020716		8280010468	1004160471

70414	2104180421	6904032513	3	3230000727			8000000000
70421	6980050677	2104260429	3904760426		2405300783		6780030435
70428	5900010784	6004190423		6904260264	3307330809	6904360439	2104030495
70435	6080020443		6004400445	6906540907	2404420495	1000000051	2404940747
70443	3304460673		4004480426	1000000025	4404010402	3904190469	3905300930
70450	2406530456	2405270730	6006840790	3000090773	4208070458	2105310734	6004090413
70457	6904300483	6504090864	6904620450	2404360940	6904640450	6004080663	3980030417
70464	6004080463	5800010721	2105300883	2108220425	8880010474	2104240427	1580050478
70471	1980060695	2104260479	6904260100	5200010480	6906540957	1745329349	6004240779
70478	3500040690	6004190473	5800020486	6004340489	6980060488	8080010418	6904400493
70485	6080020426	6005030657	6904900264	2405280831	1504110415	2104440497	5300010547
70492	6780030499	2430000903		6004420447	6504090914	2105320735	
70499	6080020758	6904200923					
70511	702	452	652	2404190722	2404190672	2404190422	3754
70518	2404190472	2404190772	1200	1400	1600	6904260101	8840000681
70525	6904280450	1680020485					
70537	6980050543	4204910492	8000040745	6730000655	1680020699		2405270880
70544	6909350738	1180030453	6080020906	6904090962		3905300981	
70650	6904090412	6907040450	6906540908		800000000	6080020763	5300010459
70657	3405040404	6980050714	2407620465	4608150865	7005000951	8280010668	3945500700
70664	6906670441	8080010771	5200010922	6504090863	6906710450	6904090662	8100070895
70671	6045500405	2104260679	4605260477	8800000780		6980050782	2404300433
70678	1580050985	6004190523	6930000953	6904840450	6908350441	2407620715	
70685	3408220872	2405280731	6905400450	1680020797	2105290432	1507930697	7005000802
70693	2405290682	5000010841	1580070703	2405270832	1509010856		2006840887
70700	3230000777	8000060658	6905030460	8080020653	6930000754	6580060913	5300010668
70707	5900010713	2105300834	6907620665	8100070895	2105300884	8280010718	5900010669
70714	2405270980	6907560759	4008000670	6904200823	5200010524		6008730678
70721	5300010877	2104260729	6080020431	5200010795		8100090982	2130000753
70729	8080020437	6907330686	6006840689	4208850736		6508220977	6904380441
70736	6905020855	6980070693	8080010494	2005320785	4008430544	5000010895	
70743	1680070751		6980050451	2405020710	7105260676		
70750	3230000827	4508540705	4008100860	5300010459	2405290482	3946000950	
70757	4807130661	6907110264	2430000454	2404090915	3205020929		6904660264
70764	6004440549	8080010971	2510	6005030808	5200010674		
70771	5100010927	2104260829	8080020481	2126000958			2130000803
70779	5100010445	6908330450	1680020739	2409350688	6904090812	6030000905	2004440735
70786	4009900541	6904090964		8200000795	3204400867	7105260726	
70795	8800000651		8100090954			6904090712	3966000851

70802	8100070895	4206560707	2405300983	2105290732	6904090862	5300010877	4407610500
70809	3304400467	5000010943	5100010720	8280010768	6004440449	6907670441	6908930746
70817	2	6908710450			4507240475		2405360939
70827	2130000853		6704190723	6005310685	8000050537	6904090765	6030000455
70834	4805250538	6580060743	5100010786	3204400917		8200000845	6980050696
70841	2025360891		5000010954		8800010701		1605020660
70850	6906540857	3205300708		4207060757	5800020701	2405280931	2007930546
70857	2405360539	4008110650	4408130664	6906540910	5100010971	8280010818	1680060821
70864	1508170921	6905020760		2106840787	8840010525		
70871	6030000755	6780030879	3405040956				6906800450
70879	6080020487	5840010687	5900010737	1580050890	6005050859	6005050909	5300010818
70887	8101990943		2405280881	6905020400	4006940945		10
70895	6925080911					8800000806	1000
70903	6980050659	8080010410	3946000801	6908500358	2405360789	2405360934	4407640814
70910	2405360691	2425340937	8280010868	1604090963	8000030470	8080010972	
70917	2107330836	5800010525			3500080989	8800010701	2405360691
70927	6980050683		2105020955	2105300664	6904090912	2007330786	6930000804
70934	8000070840			4007410791		8100070895	6508930847
70943	2431990752		6905010960				
70950	3205290805	6905010904		2407560709	2025360740	6005040959	2125500858
70957	2405360839	4008610900	3305020979	8080010766		8880010918	4506660717
70964	8080010720						
70971	6007740882	1680020932					4508300781
70979	4605000934	6909330450	2105300814	1680020841	6980060889		6905020400
70989	2007930496	6007330837					
500							

B. How to Program an Arbitrary Function, $\phi_k(x)$, as a Subroutine

When a function code numbers 7, 8, or 9 is used, the program goes to the following drum locations for its next instruction:

7 to 1200,
8 to 1400,
9 to 1600 .

The drum area from 1200 through 1800 is empty, and may be used to program any desired function, $\phi_k(x)$, or any part thereof.

Upon arrival at the designated drum locations, the following are the contents of the accumulator and distributor.

The upper accumulator has the exit instruction.

The lower accumulator has the value of k in its low-order position (right side).

The distributor has the value of x in floating point.

In programming the subroutines, the contents of all index accumulators used must be stored and replaced before the program goes to the exit instruction. At the end of the subroutine, the value of the function calculated should be placed in the upper accumulator in normalized floating point before the program goes to the exit instruction.

The following standard subroutines are already in the program translated by the amount shown.

Subroutine	Translated by
sine, cosine	100
square root	200
matrix inversion	300

The Legendre polynomial subroutine, $P_k(x)$, is located in positions 0000 to 0099. It is entered by storing the next command in the upper accumulator, storing the order k in the right-hand side of the lower accumulator, storing x in floating point ($-1 \leq x \leq +1$) in the distributor, and transferring control to drum location 0000. The value of $P_k(x)$ obtained appears in floating point in the upper accumulator.

After a function subroutine, has been programmed, it may be put in seven-word-per-card load form and placed in the program deck just before the transfer card (the last card).

REFERENCES

1. P. Cziffra and M. J. Moravscik, A Practical Guide to the Method of Least Squares, UCRL-8523, June 1958.

2. W. E. Deming, Statistical Adjustment of Data (John Wiley and Sons, New York, 1948).