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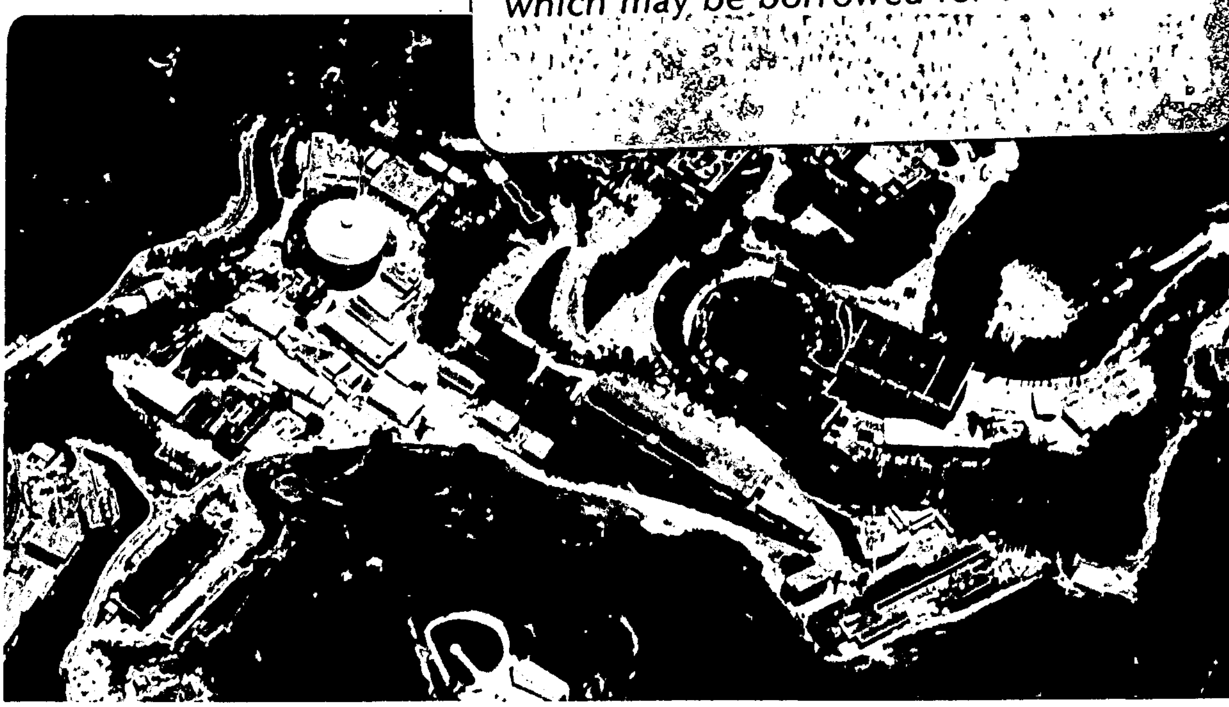

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Continuous and Discontinuous Disappearance of Capillary Surfaces[1]

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Continuous and Discontinuous Disappearance of Capillary Surfaces

Paul Concus and Robert Finn

1. We consider the problem of finding a capillary surface $u(x)$ in a cylinder Z with section Ω, in the absence of gravity. The surface is to meet Z in (constant) contact angle γ and project simply onto Ω. It is known that to each Ω there exists a critical angle $\gamma_0 \in [0,\pi/2]$ such that a surface exists if $\gamma_0 < \gamma \leq \pi/2$, while no surface exists if $0 \leq \gamma < \gamma_0$. We show here that if $0 < \gamma_0 < \pi/2$ and if Ω is smooth then there is no surface at $\gamma = \gamma_0$, while if Ω has corners a surface can in some cases be found. In the former case, the surface disappears "continuously" and always becomes unbounded in a subdomain Ω^* of positive measure, as $\gamma \downarrow \gamma_0$. In the latter case the surface can be bounded, and even analytic with the exception of the corner points. Some applications of the result are given and the exceptional case $\gamma_0 = 0$ is discussed.

2. In the presence of a gravity field the free surface $u(x)$ in a capillary tube of section Ω is determined as a solution of the governing equations under very general conditions on the form of Ω, see, e.g. [1, 2, 3]. If the gravity field is removed, there may be no surface satisfying the physical requirements, even for convex analytic Ω, see [4, 5, 6].

In the absence of gravity, the governing equations take the form

$$\operatorname{div} Tu = \frac{\Sigma}{\Omega} \cos\gamma , \quad Tu = \frac{Du}{\sqrt{1+|Du|^2}} \tag{1}$$

in Ω, and

$$\nu \cdot Tu = \cos\gamma , \tag{2}$$

on $\Sigma = \partial\Omega$. Here γ is the prescribed boundary angle, ν is unit exterior normal on Σ. We use the symbols, $\Sigma, \Omega, \cdots$ to denote alternatively a set or its measure. We may assume $0 \leq \gamma \leq \pi/2$, see, e.g. [4].

The material of this paper derives from the following condition for existence, see [6, Theorem 7.1].

Suppose that an inequality

$$\int_\Sigma |f| \, ds \leq \mu \int_\Omega |Df| + \Upsilon \int_\Omega |f| \tag{3}$$

holds for every f of bounded variation, $f \in BV(\Omega)$, with $\mu < 1/\cos\gamma$, $\Upsilon = \text{const}$ depending only on Ω. Suppose further than an isoperimetric inequality

$$\min(\Omega^*, \Omega \setminus \Omega^*) < C(\Omega \cap \partial\Omega^*)^2 \tag{4}$$

holds for every Cacciopoli set $\Omega^* \subset \Omega$. Then a solution of the capillary problem (1, 2) *exists if and only if the functional*

$$\Phi\,[\Omega^*; \gamma] \equiv \Gamma - \Sigma^* \cos\gamma + \left(\frac{\Sigma}{\Omega}\cos\gamma\right)\Omega^* \tag{5}$$

satisfies $\Phi > 0$ for every Cacciopoli set $\Omega^* \subset \Omega$ with $\Omega^* \neq \emptyset, \Omega$.

Here

$$\Gamma = \int_\Omega |D\phi| \quad , \quad \Sigma^* = \int_\Sigma \phi \, ds \, ,$$

where ϕ in the characteristic function of Ω^*; in the second integral, ϕ is taken to be the trace on Σ^* from within Ω, see [7, ch. 1]. When it exists, the solution is real analytic and bounded in Ω, satisfies (1) strictly and (2) in a generalized sense, and is unique up to an additive constant among all solutions in that class, in particular among all functions that satisfy (1, 2) strictly.

An inequality of the form (3) appears first in Emmer [1], who proved it for a Lipschitz domain with $\mu = \sqrt{1+L^2}$, L = Lipschitz constant. Thus (3) holds for any smooth domain for any $\mu > 1$. An inequality (4) holds in considerable generality, and certainly for smooth domains, cf. [7, p. 25]. Thus, we conclude from the above theorem and from the form of (5) that to every smooth Ω there corresponds $\gamma_0 \in [0, \pi/2]$ such that existence holds if $\gamma_0 < \gamma \le \pi/2$, whereas existence fails if $0 \le \gamma < \gamma_0$. Simple examples show that the case $0 < \gamma_0 < \pi/2$ occurs commonly; that is, it must be expected that solutions of (1, 2) will exist for some values of γ in the admissible physical range, but not for others.

3. We ask what happens at $\gamma = \gamma_0$ and find a result that depends in an unusual way on the smoothness of Ω.

THEOREM. *Let Ω be such that* $0 < \gamma_0 < \pi/2$. *Suppose* (3) *holds with* $\mu < 1/\cos\gamma_0$ *and* (4) *also holds. Then the zero-gravity capillary problem* (1, 2) *for Ω admits no solution when $\gamma = \gamma_0$.*

Proof. From the condition for existence given in Sec. 2, we find that if $\gamma < \gamma_0$ then there exists $\Omega^* \subset \Omega$ with $\Phi[\Omega^*; \gamma] \le 0$, $\Omega^* \ne \emptyset, \Omega$. Taking now a sequence $\gamma_j \uparrow \gamma_0$, we find that the Ω_j^* form a family of sets of bounded perimeter in Ω, hence there is a subsequence that converges to Ω_0^*, in the sense that the characteristic functions converge in $L^1(\Omega)$.

Suppose $\Omega_0^* = \emptyset$. We choose f in (3) to be the characteristic function ϕ_j of Ω_j^*, obtaining

$$\Phi[\Omega_j^*;\gamma_j] \ \ge \ (1 - \mu \cos\gamma_j)\,\Gamma - C\ \Omega_j^*$$

for some fixed constant C. By (4) we have, since $\Omega_j^* \to \emptyset$, that $\Omega_j^* < C\ \sqrt{\Omega_j^*}\,\Gamma$ for all sufficiently large j. Since $\mu < 1/\cos\gamma_0$ there follows $\Phi[\Omega_j^*;\gamma_j] > 0$ for large enough j, which is a contradiction.

Suppose $\Omega_0^* = \Omega$. We introduce the "adjoint" functional

$$\Psi[\Omega^*;\gamma] \equiv \Gamma + \Sigma^* \cos\gamma - \left(\frac{\Sigma}{\Omega}\cos\gamma\right)\Omega^*$$

and observe that $\Phi[\Omega^*;\gamma] = \Psi[\Omega\setminus\Omega^*;\gamma]$. Using (4), we thus find

$$\Phi[\Omega_j^*;\gamma] \geq (1 - C\sqrt{\Omega\setminus\Omega_j^*})\,\Gamma > 0$$

for large enough j, again a contradiction.

Thus $\Omega_0^* \neq \emptyset, \Omega$. We now use an observation that first appears in Gerhardt [8]. We set $\beta_0 = \cos\gamma_0$, $\beta_j = \cos\gamma_j$ and consider

$$\begin{aligned}\Phi_0 - \Phi_j = &\int_\Omega |D\phi_0| - \int_\Omega |D\phi_j| - \beta_0\int_\Sigma(\phi_0 - \phi_j)\,ds \\ &+ (\beta_j - \beta_0)\int_\Sigma \phi_j\,ds \\ &+ \frac{\Sigma}{\Omega}\beta_j\int_\Omega(\phi_0 - \phi_j)dx - \frac{\Sigma}{\Omega}(\beta_j - \beta_0)\int_\Omega \phi_0\,dx.\end{aligned}$$

Let $A_\delta \subset \Omega$ denote a strip of width δ adjacent to Σ. Let $\eta(x) \in C^\infty(\overline{\Omega})$, with $\eta(x) \equiv 1$ on Σ, $\eta(x) \equiv 0$ in $\Omega_\delta = \Omega\setminus A_\delta$, $0 \leq \eta \leq 1$ in Ω. Since $\beta_j \to \beta_0$, $0 < \beta_j < 1$, and $\phi_j \to \phi_0$ in $L^1(\Omega)$, we may write

$$\begin{aligned}\Phi_0 - \Phi_j \leq &\int_{\Omega_\delta} |D\phi_0| - \int_{\Omega_\delta} |D\phi_j| + \int_{A_\delta} |D\phi_0| - \int_{A_\delta} |D\phi_j| \\ &+ \beta_0\int_\Sigma |(\phi_0 - \phi_1)|\,ds + \epsilon_j,\end{aligned}$$

where $\epsilon_j \to 0$ with j. To the integral over Σ we apply (3), obtaining

$$\beta_0 \int_{\Sigma} |(\phi_0 - \phi_j)\eta| \; ds \;\le\; (1-\epsilon) \int_{A_\delta} |D\,\phi_0| + (1-\epsilon) \int_{A_\delta} |D\,\phi_j|$$

$$+ \quad C(\Omega;\delta) \int_{\Omega} |\phi_0 - \phi_j| \; dx$$

for some $\epsilon > 0$. We have further

$$\limsup_j \left\{ \int_{\Omega_\delta} |D\,\phi_0| - \int_{\Omega_\delta} |D\,\phi_j| \right\} \le 0$$

by the semicontinuity of the length integral (cf [7], Theorem 1.9). Thus

$$\limsup_j \{\Phi_0 - \Phi_j\} \le (2-\epsilon) \int_{A_\delta} |D\,\phi_0| \; .$$

Since δ is arbitrary, we find

$$\Phi[\Omega_0;\, \gamma_0] \le \limsup_j \Phi[\Omega_j\,;\, \gamma_j] \le 0 \; ,$$

and since $\Omega_0 \neq \emptyset, \Omega$, the existence condition in Sec. 1 yields that no solution exists at γ_0. The theorem is proved.

Since $0 < \cos\gamma_0 < 1$, we conclude immediately from the Theorem and from the remarks preceding it that for a smooth domain there is no surface of the type sought for the critical angle γ_0. It is remarkable that if Ω is allowed to contain corners, then a solution at γ_0 can in some cases be found. A simple example is obtained by choosing for Ω a regular polygon with interior corner angle 2α. A lower hemisphere whose equatorial circle circumscribes Ω then provides an explicit (bounded) solution of (1, 2), analytic except at the corner, and with $\gamma = \pi/2 - \alpha$. It is shown in [4] that in the case of an interior boundary angle 2α no solution can exist if $\gamma < \pi/2 - \alpha$. Thus in the example considered, $\gamma_0 = \pi/2 - \alpha$ and a solution exists at γ_0.

In such a situation the solution is seen to disappear discontinuously as γ decreases through γ_0. In the former (nonexistence) case, e.g., for smooth boundaries, we may regard the disappearance as continuous, in the sense that each γ for which existence holds lies in a γ-interval of existence.

The Theorem is sharp, in the sense that the result can fail when $\mu = 1/\cos\gamma_0$; that can be seen from the above example of the regular polygon. Nevertheless, the result can hold in much more generality than the case of Lipschitz continuity that seems to be indicated. For example, an inward cusp with opening angle 2π can be admissible; see, e.g., the discussion in Chapter 6 of [9]. Precise geometric conditions for admissibility have not yet been established.

4. As an illustration of continuous disappearance, consider the configuration of Figure 1, bounded by line segments L_1, L_2 and by tangent circular arcs C_1, C_2 of radius 1, ρ respectively. It can be shown that if h is sufficiently large then there is a unique γ_0 and circular arc C_0 meeting the boundary Σ in angle γ_0 (as in Figure 1), such that $\Phi[\Omega^*; \gamma_0] = 0$. If $\gamma > \gamma_0$ a solution of (1, 2) exists, while if $\gamma \leq \gamma_0$ there is no solution. As $\gamma \downarrow \gamma_0$ the solution $u(x;\gamma)$ can be normalized to converge to a solution $u^0(x)$ throughout $\Omega \setminus \Omega^*$ and to infinity throughout Ω^*.

In the above example, for given ρ there holds $\lim_{\alpha \to 0} \gamma_0 = \pi/2$. Thus, the more "parallel" the segments L_1, L_2 are to each other, the smaller is the range of γ for existence. Nevertheless, if $\alpha = 0$ a solution exists for any γ, regardless of h. This singular behavior can be seen in the curves of Figure 2 relating γ_0 to ρ for differing values of α. (The computed tabular points for the curves are spaced with increment 0.025 in ρ and connected with straight line segments). As $\alpha \to 0$ the curves tend to the

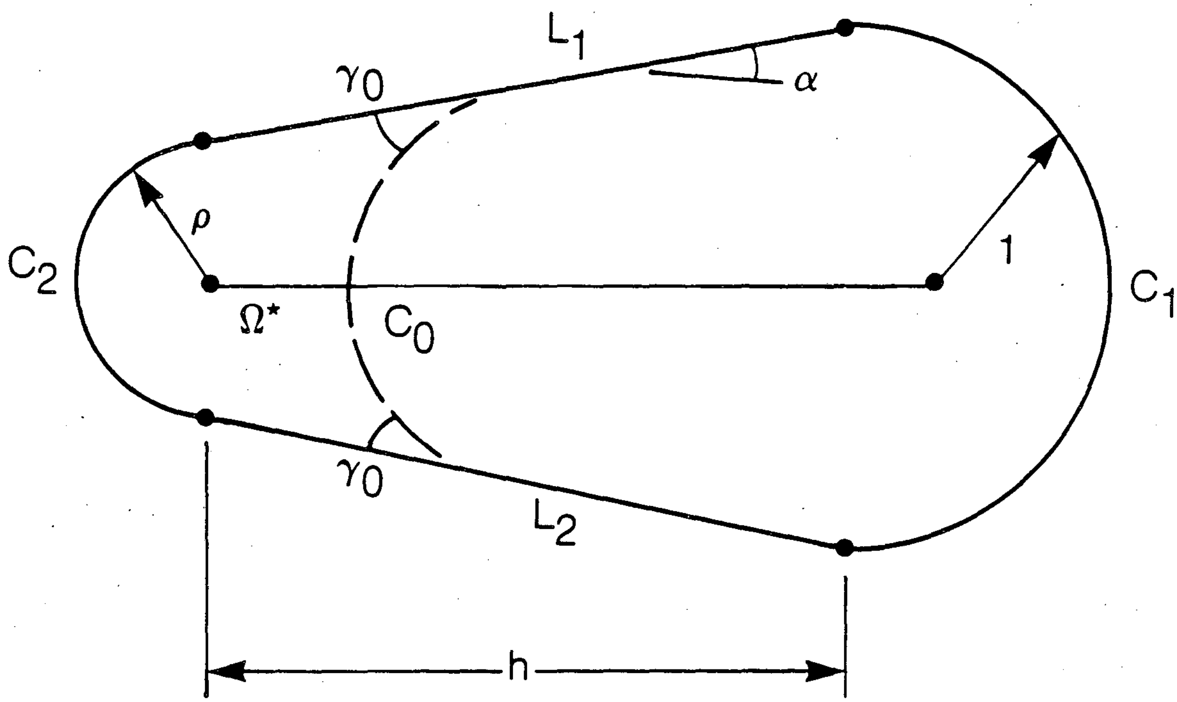
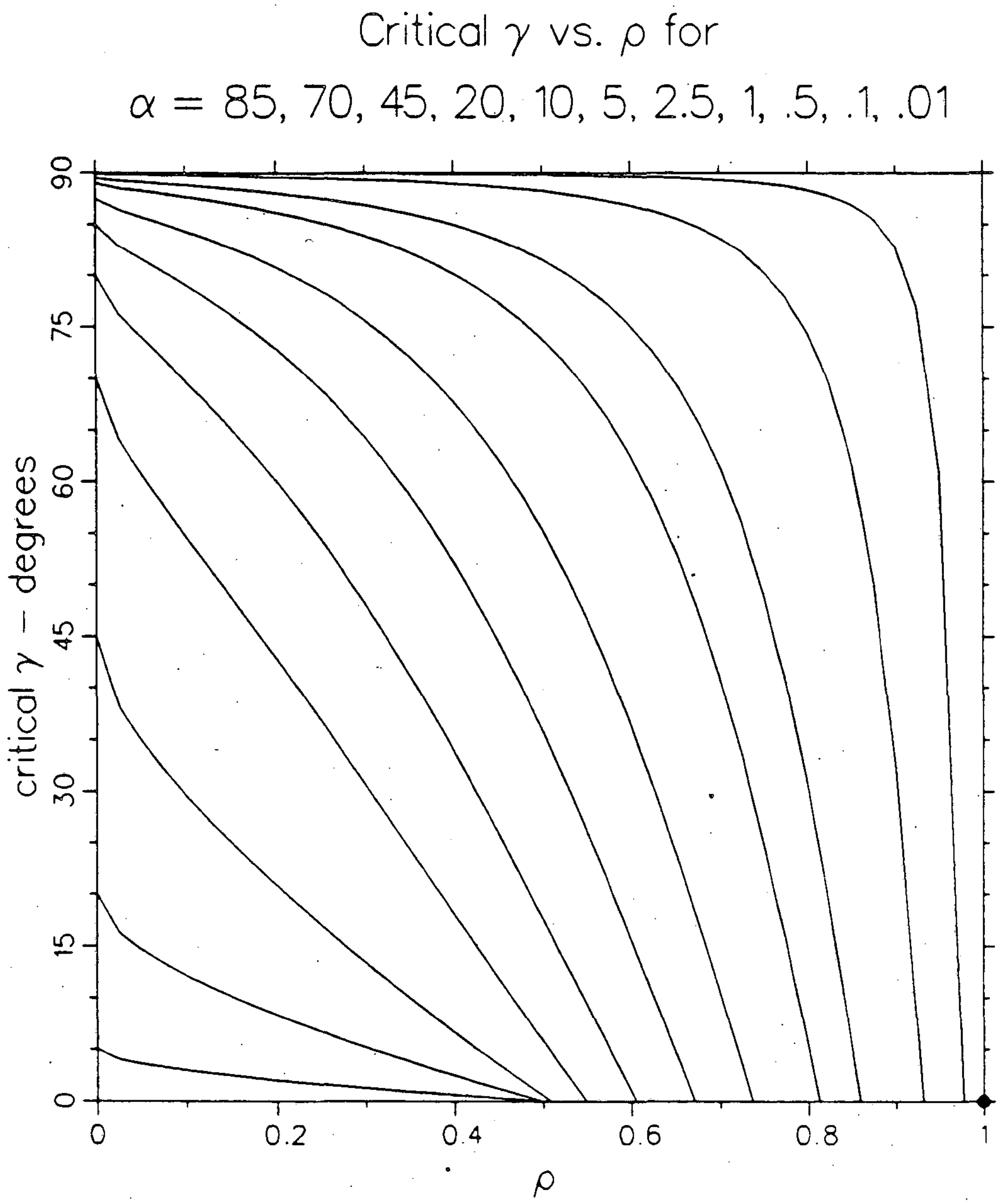


Figure 1



Critical γ vs. ρ for
α = 85, 70, 45, 20, 10, 5, 2.5, 1, .5, .1, .01
critical γ – degrees
0
15
30
45
60
75
90
0
0.2
0.4
0.6
0.8
1
ρ

Figure 2

upper and right hand boundary segments; however $\alpha = 0$ yields only the indicated single point (1,0).

5. The varied behavior that can occur in the case $\gamma = 0$ is illustrated further by the example of Figure 3, in which C_1 is a semicircle of radius 1 tangent to the two parallel line segments, and C_2 is a circular arc of radius ρ.

One can show that there exists a unique $\rho_0 = 1.974...$ such that if $\rho < \rho_0$ a solution surface exists for $\gamma = 0$ (and, *a fortiori*, for all $\gamma > 0$) for any value of h. If $\rho = \rho_0$ then an "extremal" Ω_0^*, for which $\Phi[\Omega_0^*; 0] = 0$, is obtained by inscribing a semicircular arc of radius 1 at any point in the strip as indicated. Thus, no solution can exist at $\gamma_0 = 0$. If $\gamma > 0$, one can show that $\Phi[\Omega^*, \gamma] > 0$ for any possible "extremal" Ω^*, regardless of h, so that a solution exists for this case. One can show, as for the example of Sec. 4, that as $\gamma \downarrow 0$ the solution $u(x;\gamma)$ can be normalized to converge to a solution $u^0(x)$ throughout $\Omega \setminus \Omega^*$, and to infinity throughout the Ω^* that is determined in the strip by the arc C_0.

6. An elaboration of the above configuration yields the example indicated in Figure 4. Here $\delta = 60^\circ$, ABC is a semicircle; AF, CG are circular arcs of unit radius.

If $\rho = 1.974$, calculations show that for $h = 10$ a solution exists if $\gamma > \gamma_0 \approx 13.76^\circ$. At $\gamma = \gamma_0$ an "extremal" arc AC with $\Phi = 0$ appears, of radius $\rho_1 \approx 1.028$. A sequence u^j of solutions corresponding to data $\gamma_j \downarrow \gamma_0$ can be normalized so that u^j tends to a solution to the left of AC, and to infinity to the right of AC.

If $\rho = 1.984$, a solution exists for $h = 10$ if $\gamma > \gamma_0 \approx 15.25^\circ$. At this angle an "extremal" arc DE with $\Phi = 0$ appears, of radius $\rho_2 \approx 1.039$. A sequence u^j of solutions corresponding to data $\gamma_j \downarrow \gamma_0$ can be normalized so that u^j tends to a solution to

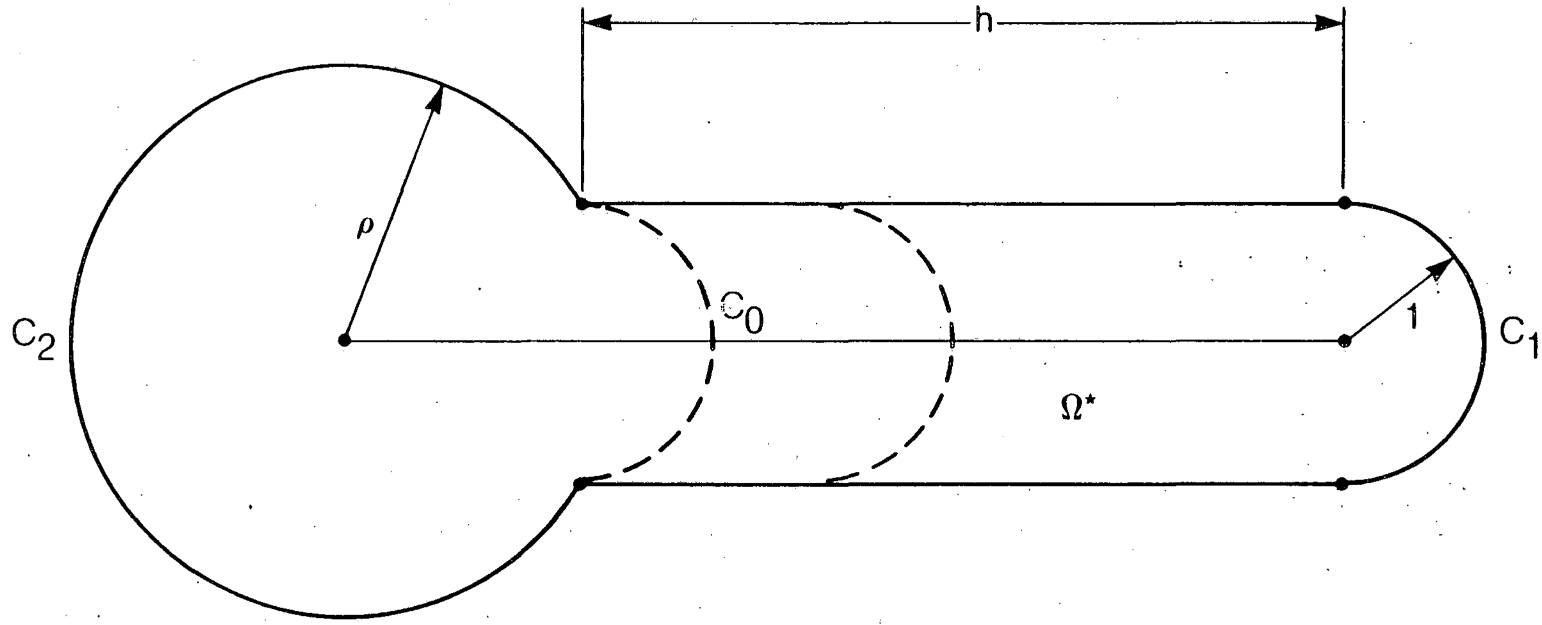


Figure 3

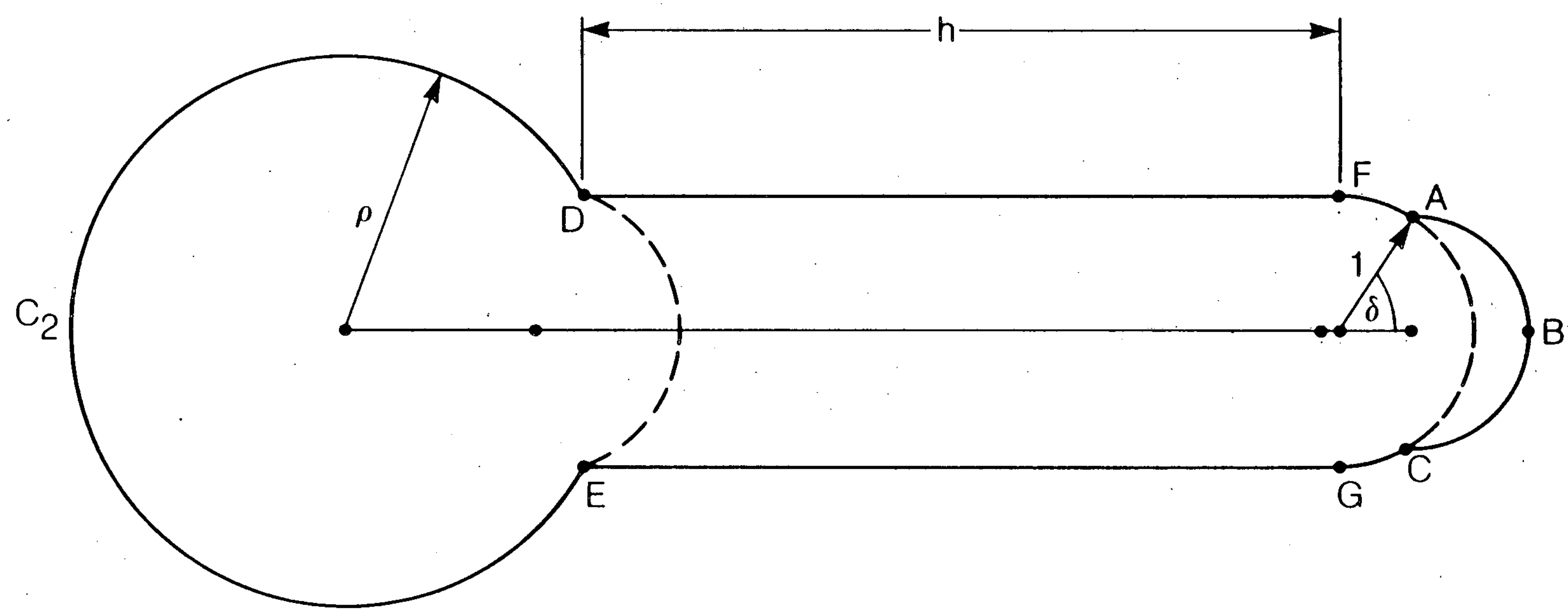


Figure 4

the left of DE, and to infinity throughout the region to the right of DE.

Thus, the region in which the solution becomes infinite can be made to shift, essentially discontinuously, with small changes in the domain and data.

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