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Publication Date

1993-08-01

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UCB-ITS-PWP-93-5

This work was performed as part of the California PATH Program of the University of California, in cooperation with the State of California Business, Transportation, and Housing Agency, Department of Transportation; and the United States Department of Transportation, Federal Highway Administration.

AUGUST 1993

ISSN 1055-1417

PROPERTIES OF LINK TRAVEL TIME FUNCTIONS UNDER DYNAMIC LOADS*

by

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(July 2, 1993)
(Revised July 29, 1993)

Abstract

This note examines a general form of link travel time functions considered in the dynamic traffic assignment literature and shows that it only makes some physical sense in the special case where each function denotes either a link with no spatial dimension containing a point queue, or a link with constant travel time and no queueing. Roadway segments exhibiting both phenomena must be represented by two links in series.

* Research supported by PATH MOU 90, Institute of Transportation Studies, Berkeley, CA.

1. INTRODUCTION

Many works in the dynamic traffic assignment literature (see Ran, 1993, for a recent overview) use optimization algorithms of one form or another to estimate travel patterns over transportation networks. The approaches essentially apply the methods that have proved successful in tackling the static assignment problem, with suitable modifications, to the dynamic case. In these works, feasible (time-dependent) network flows are defined by means of mathematical relations, and an equilibrium condition which extends the user optimum principle (Wardrop, 1952) to the dynamic case is formulated.

To reduce the equilibrium condition to a tractable form, researchers have assumed that the amount of time that a vehicle entering a link at time t spends on that link can be expressed as a function of the "state" of the link at time t. Most works express the relationship as a special case of the following:

$$\tau(t) = f(x(t),u(t),v(t)), \tag{1}$$

where $\tau(t)$ represents the link travel time for a vehicle entering the link at time t, $x(t)$ is the number of vehicles on the link at time t, $u(t)$ and $v(t)$ are the link arrival and departure rates at time t, and "f" is a non-negative differentiable function. For obvious reasons, f is defined to be non-decreasing in x and u, and non-increasing in v.

Although Eq.(1) is rather general, we shall argue that only two special cases of it depict transportation-like phenomena. To make our point, we consider the simplest possible network; one with only one origin and one destination, joined by a single link.

The input to the link is defined by an increasing function, $A(z)$, that gives the arrival time (to the link) of vehicle z, where z is a real valued variable. Note that the link arrival rate when vehicle z arrives is $1/A'(z)$ wherever $A'(z)$ exists. This representation of the origin-destination table is not common in the literature but it is convenient for our purposes. It allows us to express the schedule of arrivals at the destination (the link departures) by a simple expression that relates z to its link departure time, $D(t)$:

$$D(z) = A(z) + \tau^z, \tag{2}$$

where τ^z is the travel time of vehicle z. See Fig. 1.

2. INDEPENDENCE OF u

We first show that f cannot depend on u because, if it does, an $A(z)$ can be found that causes $D(z)$ to decrease as z increases. This, of course, is absurd since it would mean that all the drivers who depart the origin in a certain time interval (say between 8:00 and 8:05 AM) pass and arrive earlier at the destination than all the drivers who had departed in an earlier time interval (say between 7:55 AM and 8:00 AM). This effect is similar to Smeed's paradox (Smeed, 1972)[1].

The derivations below show that if the partial derivative f_u is greater than zero, a curve $A(z)$ can be found which induces a negative derivative for $D(z)$.

If we let x^z denote the number of cars in the link at the time when vehicle z arrives, and u^z and v^z the arrival and departure rates at that same instant, we can rewrite (1) as:

$$\tau^z = f(x^z, u^z, v^z). \tag{3}$$

Let z^o denote the vehicle arriving at $t = 0$ and use (τ^o, x^o, u^o, v^o) for the corresponding variables, as shown in Fig. 1. We assume that the link is in a steady state with $\tau > 0$ for $t < 0$. This means that A' is constant until $t = 0$, which in turn implies —by virtue of Eq.(2)—that the departure curve remains straight until $z=z^o$. As shown in the figure, this occurs at time τ^o.

Suppose that after time t=O the arrival rate u^z begins a rapid decline — shown in Fig. 1 by an upward curving $A(z)$—assumed to be of the form:

1 Smeed (1972) apparently believed that vehicles could arrive at a destination earlier by leaving later if the travel time on a road was given by a speed-density curve. It should be stressed, however, that such a model feature does not arise from the hydrodynamic theory of traffic flow of Lighthill and Witham (1955) and Richards (1956), which also uses a speed-density curve. This has been pointed out by Newell (1985). Ben-Akiva and DePalma (1986) are also on record against such a model feature. Their argument can be paraphrased as follows: if a driver can postpone a trip in order to arrive earlier there should be an instant when the driver would be at the same location under both scenarios; because from then on the system would have to evolve identically in both cases, the two departure times could not differ.

$$A(z) = (z-z^o)/u^o + M(z-z^o)^2, \quad (4)$$

where $M > 0$. Substitution of (4) and (3) into (2) yields:

$$D(z) = (z-z^o)/u^o + M(z-z^o)^2 + f(x^z,u^z,v^z). \quad (5)$$

We are particularly interested in the derivative of $D(z)$ at $t=0^+$, i.e. for vehicle $[z^o]^+$, which can be expressed as:

$$D'([z^o]^+) = 1/u^o + f_x[dx^z/dz]^+ + f_u[du^z/dz]^+ + f_v[dv^z/dz]^+,$$

where the the partial derivatives f_x, f_u and f_v are taken at $(x^z,u^z,v^z) = (x^o,u^o,v^o)$, and the terms in brackets are evaluated at $Z = [z^o]^+$.

Recalling that v^z is defined as the deperture rate when z arrives, we note from Fig. 1 that v^z is constant in the neigborhood of z^o (i.e. in the neighborhood of $t = 0$). Thus, $[dv^z/dz]^+ = 0$. Figure 1 also reveals that $x^z = Z - D^{-1}(A(z))$, and thus: $dx^z/dz = 1 - u^o(1/u^o) = 0$. Thus, the expression for $D'([z^o]^+)$ only has two non-zero terms.

Since $du^z/dz = d[1/A']/dz = -A''/A'^2 = 2M(u^o)^2$, we can write:

$$D'([z^o]^+) = 1/u^o - 2f_uM(u^o)^2. \quad (7)$$

This equation indicates that if f_u is positive, one can always choose a sufficiently large M to induce a decreasing departure schedule. This will happen whenever flow drops from a high steady state value to zero in a short time, as when a traffic light turns red.

3. INDEPENDENCE OF v

Having shown that f should be independent of u (i.e. travel time is a function g of x and v only: $f(x,u,v) = g(x,v)$), we now argue that g should also be independent of v. To see this, consider an initially empty link into which vehicles start to flow at time t=O at rate u, and let us observe its behavior at the instant when the first vehicle is about to exit. Immediately before this time, the formula would predict a travel time $g(x,0)$, since $v=0$; and immediately afterward a travel time $g(x,u)$, since $v=u$. This results in a negative jump that does not correspond to anything real. The only way in which it can be eliminated for all values of u is by setting $g(x,v) = h(x)$.

4. DISCUSSION

We have shown that **(1)** should be of the form:

$$\tau(t) = h(x(t)), \tag{8}$$

which is a common model, often termed "the point queue model". Unfortunately even this simple link cost function, adopted in a number of works, needs to be restricted. We argue below that the curve $h(x)$ must either pass through the origin or be constant.

If $h(0) > 0$ and the system is empty at $t = 0$, then the travel time of a vehicle arriving to the system at time $t = h(0)$ is determined by the number of vehicles to have arrived in $(0, h(0))$ and nothing else — since all those vehicles must still be in the system at time $h(0)$, independent of their arrival times. That is, conditional of the number of arrivals, the form of the input has no effect. This property is reasonable if the system is uncongested — when a vehicle's travel time is constant — but bears little relation to real-life congestion phenomena. In actuality, if the link is congested, entries close to $t = 0$ would have a lesser impact on delay than those close to $t = h(0)$, since the early ones would have a better chance to clear the bottleneck before the arrival of our vehicle.

The above suggests that to represent properly a point queue network, one should separate the effects of point queues from those of distance travel. Thus, a road segment including a bottleneck should be represented by two links in series: one with a constant travel time, representing the time needed to overcome distance, and another with $h(x) = 0$ for $x = 0$, representing the bottleneck-caused delay.

Of course, the point queue representation still falls short of realism when delays are caused by "too many cars on the road". It does not recognize that traffic often crawls because cars cannot enter a link when too many are in it already. Approximations based on the hydrodynamic representation of traffic flow hold promise in this respect.

REFERENCES

BEN-AKIVA, M. and A. DePALMA (1986) "Some circumstances in which vehicles will reach their destinations earlier by starting later: revisited" Trans. Sci. 20(1), 52-55.

LIGHTHILL, M.J. and J.B. WHITHAM (1955) "On kinematic waves. I Flow movement in long rivers. II A theory of traffic flow on long crowded roads" Proc. Royal Soc. A 229, 281-345.

NEWELL, G.F. (1985) "Unpublished private communication".

RAN, B. (1993) "Dynamic transportation network models for advanced traveler information systems" PhD thesis, Univ. of Illinois, Chicago, 11.

RICHARDS, P.I. (1956) "Shockwaves on the highway" Oms. Res. 4, 42-51.

SMEED, (1972) "Some circumstances in which vehicles will reach their destinations earlier by starting later" Trans. Sci. 1, 308.

WARDROP, J.G. (1952) "Some theoretical aspects of road traffic research", Proc. Inst. Civil Eng., Part II, 1, 325-378.

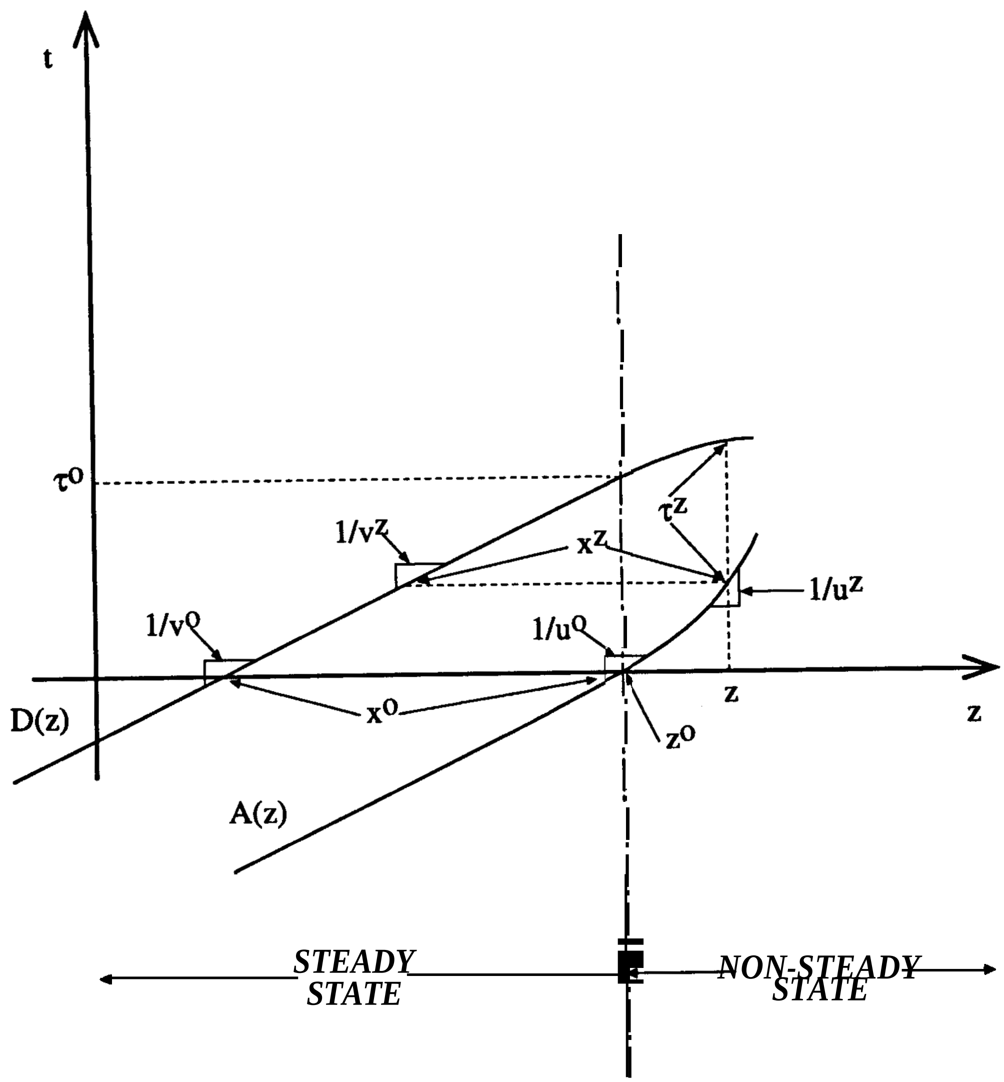
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Figure 1