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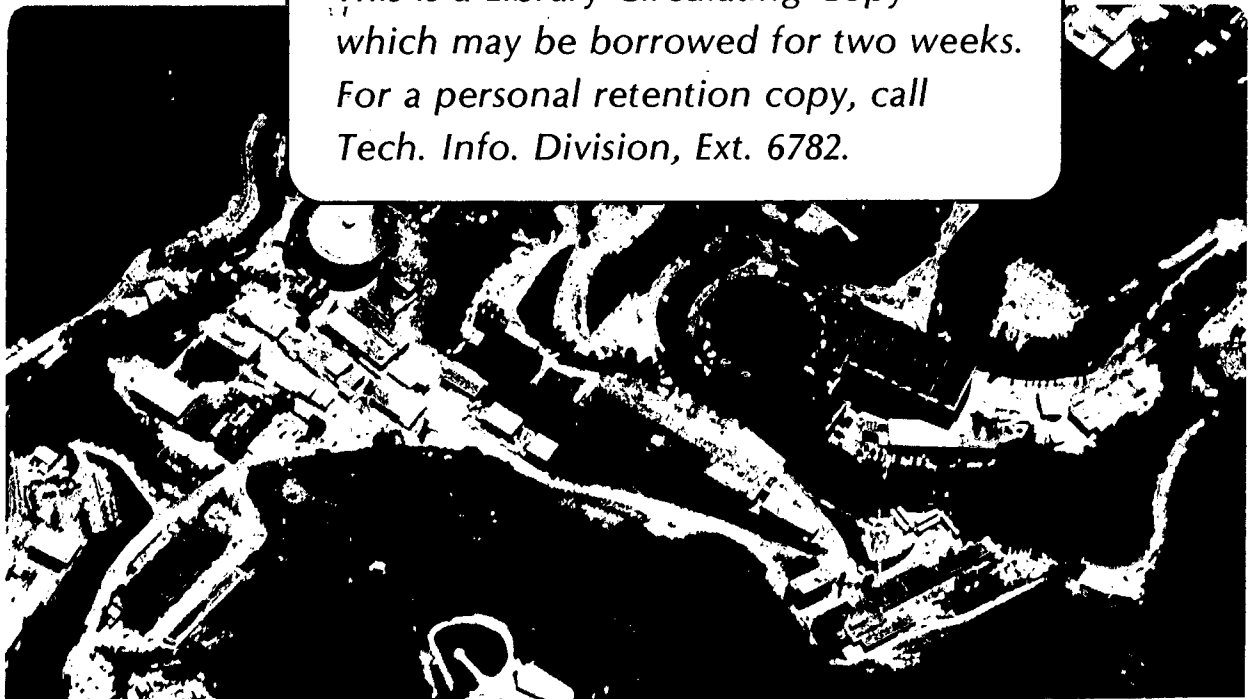
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RECURSION RELATIONS AND A CLASS OF ISOSPECTRAL MANIFOLDS
FOR SCHRODINGER'S EQUATION¹

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RECURSION RELATIONS AND A CLASS OF ISOSPECTRAL MANIFOLDS
FOR SCHRÖDINGER'S EQUATION

The purpose of this note is to show that (properly chosen) eigenfunctions $\phi(x,k)$ for the Schrödinger problem

$$\frac{d^2\phi}{dx^2} + V\phi = k^2\phi \quad (1)$$

satisfy a recursion relation of the form

$$\sum_{-M}^M b_i(k) \phi(x,k+i) = \Theta(x)\phi(x,k) \quad (2)$$

if V is an "N soliton potential" with eigenvalues $\{j^2/4, j=1, \dots, N-1\} \cup \{(N^2-N+4)/4\}^2$ and *arbitrary* normalization constants. The order M of the recursion relation is given by $2M = N^2 - N + 2$, the coefficients $b_i(k)$ are given explicitly, and so is $\Theta(x)$. Relation (2) can be seen as an extension of the three-term recursion relation satisfied by the classical orthogonal polynomials.

N Soliton Potentials

Pick c_1, \dots, c_N ; k_1, k_2, \dots, k_N with $c_j > 0$, $k_j > 0$ and define the matrices M and $M(k)$ by the rules

$$M_{ij} = \delta_{ij} + c_j \frac{e^{(k_i+k_j)x}}{k_i+k_j}$$

$$M_{ij}(k) = \delta_{ij} + c_j \left(\frac{k-k_j}{k+k_j} \right) \frac{e^{(k_i+k_j)x}}{k_i+k_j}$$

Now define

$$\tau(x) = \det M \quad (3a)$$

$$\tau(x,k) = \det M(k) \quad (3b)$$

$$V(x) = 2 \frac{d^2}{dx^2} \log \tau(x) = 2 \left(\frac{\tau'}{\tau} \right)' \quad (4)$$

It is then known, see [1] and its references, that

$$\phi(x,k) \equiv e^{kx} \frac{\tau(x,k)}{\tau(x)}$$

satisfies (1) and of course,

$$\phi(x,k) \sim e^{kx}, \quad x \rightarrow \infty.$$

The expressions above give the most general reflectionless potential $V(x)$ with eigenvalues k_j^2 .

A Special Isospectral Manifold

Now make the choice

$$k_j = j/2, \quad j = 1, \dots, N-1, \quad k_N = \frac{N^2 - N + 4}{4} \quad (5)$$

Set

$$M = \frac{N^2 - N + 2}{2}$$

Given our choice of k_j 's, one gets from (3a)

$$\tau(x) = \sum_0^{2M} a_i e^{ix} \quad (3')$$

for a set of constants a_i expressible in terms of the c_i 's, namely

$$a_i = \sum \alpha(i; i_1 \dots i_\nu) c_{i_1} \dots c_{i_\nu} \quad (6)$$

with $\{i_1, \dots, i_\nu\}$ an arbitrary subset of $\{1, \dots, n\}$ and $\alpha(i_1, \dots, i_\nu)$ fixed once the choice (5) has been made. The sum extends over all the subsets of $\{1, \dots, n\}$. One has

$$a_M = 0 .$$

Now define

$$\Theta(x) = \sum_0^{2M} \frac{a_i}{i-M} e^{(i-M)x} . \quad (7)$$

Notice that

$$\Theta'(x) = e^{-Mx} \tau(x) .$$

In view of (6) and (7) one can write

$$\Theta(x) = \sum_{i=-M}^M b_i e^{ix} \quad (8)$$

with

$$b_i = \sum \beta(i; i_1 \dots i_\nu) c_{i_1} \dots c_{i_\nu} . \quad (9)$$

Now for any choice of subset $\{i_1, \dots, i_\nu\}$ from the set $\{1, \dots, n\}$, define

$$\gamma(i_1, \dots, i_\nu; k) \equiv \frac{(k - k_{i_1})}{(k + k_{i_1})} \dots \frac{(k - k_{i_\nu})}{(k + k_{i_\nu})}$$

The coefficients in the recursion relation (2) are given by an appropriate modification of the terms in (9), namely

$$b_i(k) = \sum \beta(i; i_1, \dots, i_\nu) c_{i_1} \dots c_{i_\nu} \frac{\gamma(i_1 \dots i_\nu; k)}{\gamma(i_1 \dots i_\nu; k+i)}$$

The choice of potentials $V(x)$ such that a family of eigenfunctions of (1) enjoys property (2) is not limited to pure soliton potentials, much less to those specified by the choice (5).

For instance, if $M = 1$ the most general choice of $V(x)$ is given by

$$V(x) = \frac{A}{\sinh^2(x+c)} + \frac{B}{\cosh^2(x+c)}$$

and in general this is not a pure soliton potential.

Among the N soliton potentials, the choice

$$k_j = j, \quad j = 1, \dots, N$$

followed by the *very special* election of normalization constants

$$c_j = 2j(-1)^{j-1} \prod_{i \neq j} \frac{i+j}{i-j}$$

leads to the celebrated

$$V(x) = N(N+1)\operatorname{sech}^2(x)$$

In this case $\phi(x, k)$ are given in terms of Legendre functions and (2) holds with $M = 1$.

The special feature of the choice given by (5) is that it singles out an N-dimensional isospectral manifold of potentials $V(x)$, all of which enjoy property (2). Property (2) is thus preserved — on this manifold — by the whole Korteweg-de Vries hierarchy [2]. These nonlinear evolution equations also play a role in discussions of a continuous version of (2), see [3]. Some of the potentials discussed here as well as in [3] were first considered by Bargmann, see [4]. Proofs and several examples will appear elsewhere.

References

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