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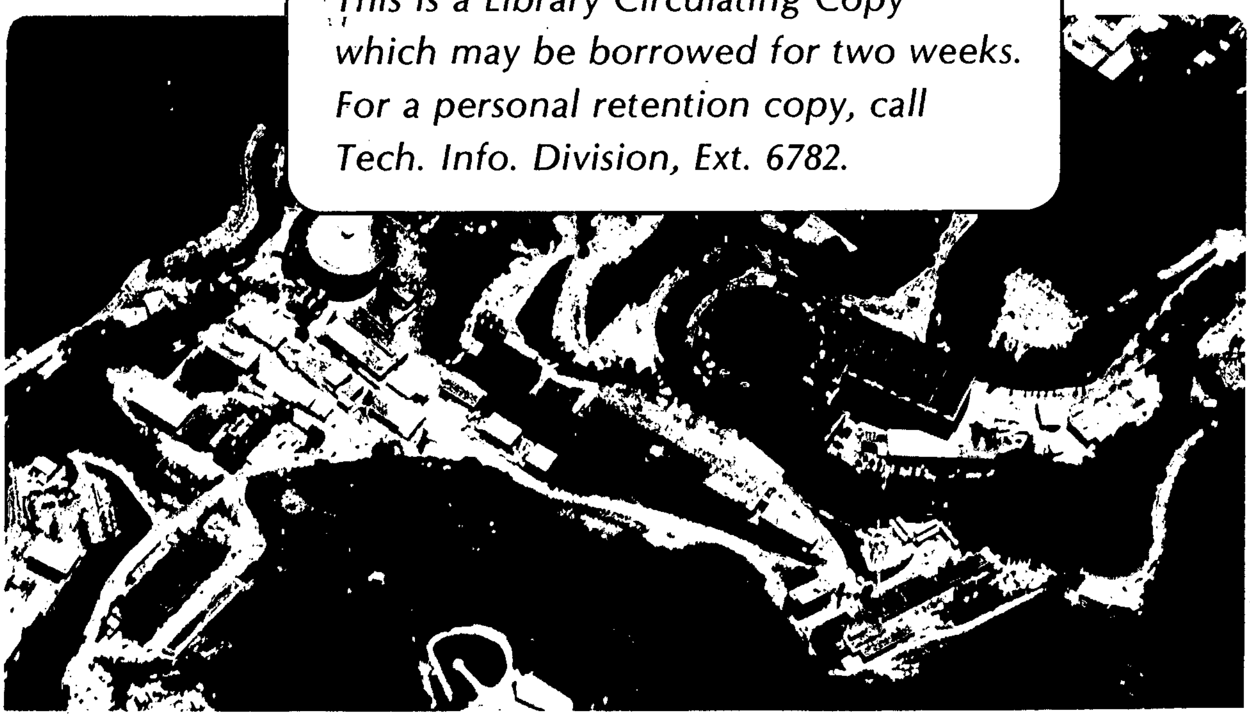

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RECURSION RELATIONS AND A CLASS OF ISOSPECTRAL MANIFOLDS FOR SCHRODINGER'S EQUATION[1]

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RECURSION RELATIONS AND A CLASS OF ISOSPECTRAL MANIFOLDS FOR SCHRÖDINGER'S EQUATION

The purpose of this note is to show that (properly chosen) eigenfunctions $\phi(x,k)$ for the Schrödinger problem

$$\frac{d^2\phi}{dx^2} + V\phi = k^2\phi \tag{1}$$

satisfy a recursion relation of the form

$$\sum_{-M}^{M} b_i(k)\ \phi(x,k+i) = \Theta(x)\phi(x,k) \tag{2}$$

if V is an "N soliton potential" with eigenvalues $\{j^2/4,\ \ j=1,\ldots,N-1\} \cup \{(N^2-N+4)/4\}^2$ and *arbitrary* normalization constants. The order M of the recursion relation is given by $2M = N^2-N+2$, the coefficients $b_i(k)$ are given explicitly, and so is $\Theta(x)$. Relation (2) can be seen as an extension of the three-term recursion relation satisfied by the classical orthogonal polynomials.

N Soliton Potentials

Pick $c_1,\ldots,c_N$; $k_1,k_2,\ldots,k_N$ with $c_j > 0$, $k_j > 0$ and define the matrices M and $M(k)$ by the rules

$$M_{ij} = \delta_{ij} + c_j \frac{e^{(k_i+k_j)x}}{k_i + k_j} ,$$

$$M_{ij}(k) = \delta_{ij} + c_j\left(\frac{k - k_j}{k + k_j}\right) \frac{e^{(k_i+k_j)x}}{k_i + k_j} .$$

Now define

$$\tau(x) = \det M \tag{3a}$$

$$\tau(x,k) = \det M(k) \tag{3b}$$

$$V(x) = 2 \frac{d^2}{dx^2} \log \tau(x) = 2\left(\frac{\tau'}{\tau}\right)' \tag{4}$$

It is then known, see [1] and its references, that

$$\phi(x,k) \equiv e^{kx} \frac{\tau(x,k)}{\tau(x)}$$

satisfies (1) and of course,

$$\phi(x,k) \sim e^{kx} , \qquad x \to \infty .$$

The expressions above give the most general reflectionless potential $V(x)$ with eigenvalues k_j^2.

<u>A Special Isospectral Manifold</u>

Now make the choice

$$k_j = j/2 , \quad j = 1,\ldots,N-1 , \qquad k_N = \frac{N^2-N+4}{4} \tag{5}$$

Set

$$M = \frac{N^2-N+2}{2}$$

Given our choice of k_j's, one gets from (3a)

$$\tau(x) = \sum_0^{2M} a_i e^{ix} \tag{3'}$$

for a set of constants a_i expressible in terms of the c_i's, namely

$$a_i = \sum \alpha(i;\, i_1 \ldots i_\nu)\ c_{i_1} \ldots c_{i_\nu} \tag{6}$$

with $\{i_1,\ldots,i_\nu\}$ an arbitrary subset of $\{1,\ldots,n\}$ and $\alpha(i_1,\ldots,i_\nu)$ fixed once the choice (5) has been made. The sum extends over all the subsets of $\{1,\ldots,n\}$. One has

$$a_M = 0 \ .$$

Now define

$$\Theta(x) = \sum_0^{2M} \frac{a_i}{i-M}\, e^{(i-M)x} \ . \tag{7}$$

Notice that

$$\Theta'(x) = e^{-Mx}\, \tau(x) \ .$$

In view of (6) and (7) one can write

$$\Theta(x) = \sum_{i=-M}^{M} b_i\, e^{ix} \tag{8}$$

with

$$b_i = \sum \beta(i;\, i_1 \ldots i_\nu)\ c_{i_1} \ldots c_{i_\nu} \ . \tag{9}$$

Now for any choice of subset $\{i_1,\ldots,i_\nu\}$ from the set $\{1,\ldots,n\}$, define

$$\gamma(i_1,\ldots,i_\nu;k) \equiv \frac{(k-k_{i_1})}{(k+k_{i_1})} \cdots \frac{(k-k_{i_\nu})}{(k+k_{i_\nu})}$$

The coefficients in the recursion relation (2) are given by an appropriate modification of the terms in (9), namely

$$b_i(k) = \sum \beta(i;\, i_1,\ldots,i_\nu) c_{i_1} \ldots c_{i_\nu} \frac{\gamma(i_1 \ldots i_\nu;\, k)}{\gamma(i_1 \ldots i_\nu;\, k+i)} \quad .$$

The choice of potentials $V(x)$ such that a family of eigenfunctions of (1) enjoys property (2) is not limited to pure soliton potentials, much less to those specified by the choice (5).

For instance, if $M = 1$ the most general choice of $V(x)$ is given by

$$V(x) = \frac{A}{\sinh^2(x+c)} + \frac{B}{\cosh^2(x+c)}$$

and in general this is not a pure soliton potential.

Among the N soliton potentials, the choice

$$k_j = j \quad , \qquad j = 1,\ldots,N$$

followed by the *very special* election of normalization constants

$$c_j = 2j(-1)^{j-1} \prod_{i \neq j} \frac{i+j}{i-j}$$

leads to the celebrated

$$V(x) = N(N+1)\,\mathrm{sech}^2(x) \quad .$$

In this case $\phi(x,k)$ are given in terms of Legendre functions and (2) holds with $M = 1$.

The special feature of the choice given by (5) is that it singles out an N-dimensional isospectral manifold of potentials $V(x)$, all of which enjoy property (2). Property (2) is thus preserved — on this manifold — by the whole Korteweg-de Vries hierarchy [2]. These nonlinear evolution equations also play a role in discussions of a continuous version of (2), see [3]. Some of the potentials discussed here as well as in [3] were first considered by Bargmann, see [4]. Proofs and several examples will appear elsewhere.

References

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