LBNL-46310
Preprint

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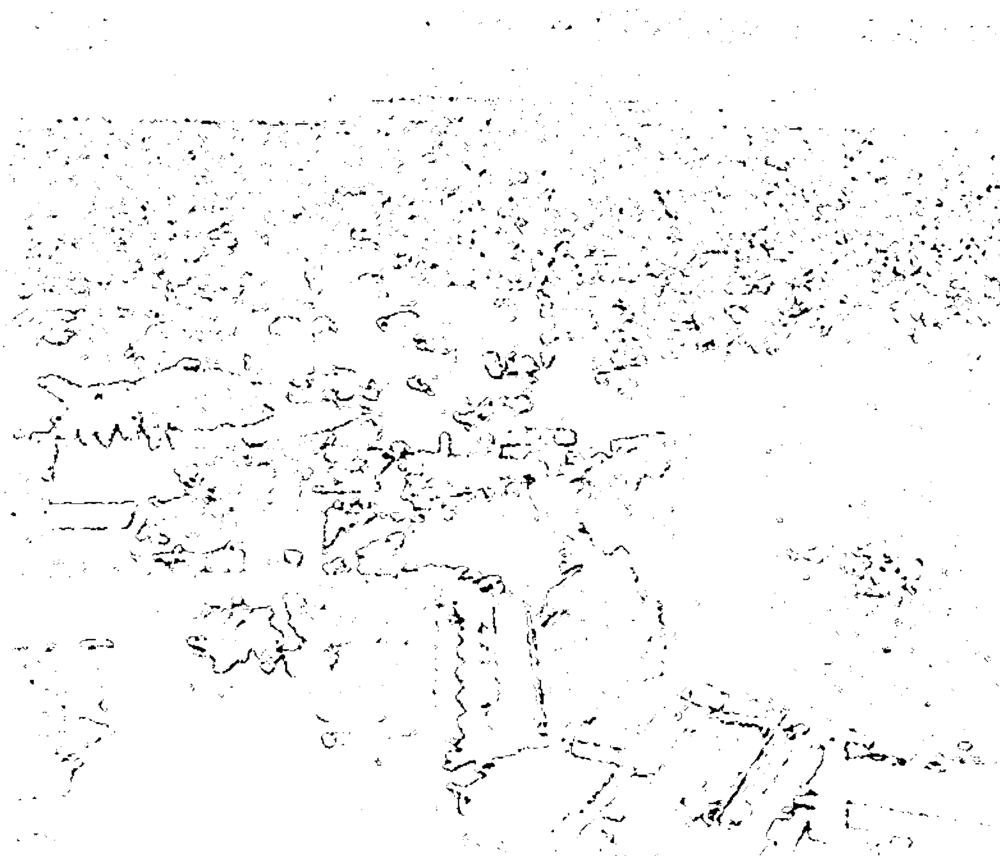
August 2000

Submitted for publication



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Exact Averaging of Stochastic Equations for Transport in Random Fields

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August 2000

This work was supported by JNC (Japan Nuclear Fuel Cycle Corporation) and the U.S. Department of Energy under Contract No. DE-AC03-76SF00098.

ABSTRACT.

The present paper considers the process of transport of a conservative solute in random fields. Here we present new examples of exactly averaged multi-dimensional equation for the mean concentration when the components of random flow-velocity are the functions of time. We assume that local dispersion exists. The functional approach and technique for decoupling the correlations are used.

The exactly averaged differential equation is linear and is first order in time, and in general infinite order with respect to the spatial variables. The coefficients of the derivative are dependent on the cummulants function of flow-velocity random process. In general the averaged equation is non-local. We study the special cases where the averaged equation can be localized and reduced to differential equation of finite order, and the problem of evolution of the initial plume (Cauchy problem) can be solved exactly.

We present in detail the results of numerical analyses of two cases (for Gaussian and telegraph random flow-velocity with the identical exponential correlation function) of exactly averaged problems. We studied the behavior of different initial plumes for all times (evolutions and convergence) and showed that they approach the same asymptotic limit for the both stochastic distributions of flow-velocity.

A comparison between exact solutions and solutions derived by the method of perturbation is discussed.

1. INTRODUCTION

Effective description of flow and transport in irregular porous media involves interpretation of the porosity and permeability fields as random functions of the spatial coordinates and the flow velocity as a random function of the spatial coordinates and time. It also involves averaging of the stochastic system of flow and transport equations containing these functions (conservation laws, Darcy`s law and closing relations). The averaging problem consists of finding the relations between the non-random functionals of the unknown and the given fields – means, variations, distributions, densities, etc. The greatest interest attaches to the averaged description in which the equations for the functionals are invariant with respect to a certain set of conditions that uniquely determines the process in the framework of such a description.

It is apparent that in general this splitting is impossible and thus an averaged description is used for computing the nonrandom characteristics (functionals) of random flow and transport processes for estimating the uncertainty of the processes.

Different variations of this approach and many results have been widely developed. For example see books by *Shvidler*[1964 and 1985], *Matheron* [1967], *Dagan* [1989] and *Gelhar* [1993] . Basically the methods of averaging are approximate. It should be noted that the approximate methods of averaging and derivation of the averaged equations of transport, using either Lagrangian or Eulerian approach, are connected with one or another modification to the method of perturbation. Obviously the similar approach is not always successful because the solution is only approximate due to the truncation of the perturbation series. It is well to bear in mind that the convergence of the perturbation series for transport problem is not clearly understood nor the accuracy of the approximate averaged equations. In this instance it is common to use: 1) comparison of the solutions of approximate averaged equations with the results by Monte-Carlo procedures that, strictly speaking, are also approximate 2) comparison of the solutions of the approximate averaged equations with the exact results using the theory of averaging. Of course besides this function the exact equations have independent values in themselves .

The number of the exact results in stochastic transport theory is very small. Exhaustive examples of averaging one-dimensional transport were described by *Indelman and Shvidler* [1985]. For a special distribution of random porosity *Indelman* [1986] derived one-point and two-point probability density, first four moments, and the correlation function for the position of moving particles. The exact moments for the travel time of moving particles in one-dimensional field with random porosity were derived by *Cvetkovic and al.,*[1991]. The quasi one-dimensional transport in stratified media (longitudinal advection along homogeneous layers with longitudinal and transversal local dispersion) was described by *Matheron and de`Marsily* [1980] . In the multi-dimensional case it is well known that the classical Einstein-Fokker-Plank diffusion-advection equation is only valid if the flow velocity is a Gaussian random field and delta-correlated in time (for example see *Klyatskin,*[1980], *Rytov, Kravtsov and Tatarskii* ,[1989].

Later we will study some problem of stochastic transport for which the exact averaging is accomplished. As is often the case we can obtain the desired results by analyzing special processes. For example in this paper we consider that the flow-velocity is a random function of time, which essentially simplifies the problem. But the problem of the averaging remains sufficiently complicated and only relatively simple solutions can be found for some special examples.

. We studied the problem of exact averaging of the evolution equations for stochastic transport in random fields. The paper considers the process of transport of neutral admixture in porous media. The functional approach and the technique of decoupling the correlations is used. A number of exact functional equations corresponding to the distributions of the random parameters of special forms are obtained. In some cases the functional equations can be localized and reduced to differential equations of fairly high order. We present and analyze the solutions of exact equations.

2. Mathematical Statement of the Problem

The concentration distribution of a non-reactive solute $c(x,t)$ in the region $t \geq t_0, |\mathbf{x}| < \infty$ is described by the equations:

$$\phi \frac{\partial c(\mathbf{x},t)}{\partial t} + v_i(t) \frac{\partial c(\mathbf{x},t)}{\partial x_i} = \frac{\partial}{\partial x_i}\left[d_{ij} \frac{\partial c(\mathbf{x},t)}{\partial x_j} \right] \tag{1}$$

$$c(\mathbf{x},t_0) = f(\mathbf{x}) \tag{2}$$

where ϕ is the porosity , $\mathbf{d}$ is the local dispersion tensor , $\mathbf{v}(t)$ is the vector of the random Darcy- velocity , $f(\mathbf{x})$ is the non-random initial concentration . Here we assume ϕ and $\mathbf{d}$ are constant.

After averaging (1) and (2) we have the following non-closed system

$$\phi \frac{\partial u(\mathbf{x},t)}{\partial t} + V_i(t) \frac{\partial u(\mathbf{x},t)}{\partial x_i} + \frac{\partial}{\partial x_i}\left\langle v_i'(t) c(\mathbf{x},t) \right\rangle = \frac{\partial}{\partial x_i}\left[d_{ij} \frac{\partial u(\mathbf{x},t)}{\partial x_j} \right] \tag{3}$$

$$u(\mathbf{x},t_0) = f(\mathbf{x}) \tag{4}$$

here $u(\mathbf{x},t) = \langle c(\mathbf{x},t) \rangle$ is the mean concentration, where symbol $\langle \ \rangle$ represent the ensamble mean, and $V_i(t) = \langle v_i(t) \rangle$, $v_i'(t) = v_i(t) - V_i(t)$.

According *Klyatskin* [1980], *Rytov et. al*[1989] the correlation moment between random function $v_i'(t)$ and the functional $c(\mathbf{x},t)$ from function $\mathbf{v}'(t)$ can be written in the form:

$$\langle v_i'(t)c(\mathbf{x},t)\rangle = \sum_{n=1}^{\infty}\frac{1}{n!}\int ...\int K_{n+1}^{i,j_1,...,j_n}(t,\tau_1,...,\tau_n)\left\langle \frac{\delta^n c(\mathbf{x},t)}{\delta v_{j_1}'(\tau_1)...\delta v_{j_n}'(\tau_n)}\right\rangle d\tau_1...d\tau_n \tag{5}$$

where $\dfrac{\delta^n c(\mathbf{x},t)}{\delta v_{j_1}'(\tau_1)...\delta v_{j_n}(\tau_n)}$ is the variational derivatives of the functional $c(\mathbf{x},t)$ and $K_{n+1}^{i,j_1,...,j_n}(t,\tau_1,...,\tau_n)$ are the cumulants of the random process $v'(t)$ that is described by following equations:

$$K_{n+1}^{i,j_1,...,j_n}(t,\tau_1,...,\tau_n) = \frac{1}{i^{n+1}}\frac{\delta F[\boldsymbol{\lambda}(\tau)]}{\delta\lambda_i(t)\delta\lambda_{j_1}(\tau_1)...\delta\lambda_{j_n}(\tau_n)}\Big|_{\lambda=0} \tag{6}$$

$$F[\boldsymbol{\lambda}(\tau)] = \ln\Phi[\boldsymbol{\lambda}(\tau)], \qquad \Phi[\boldsymbol{\lambda}(\tau)] = \left\langle \exp\left\{i\int \mathbf{v}'(\tau)\boldsymbol{\lambda}(\tau)d\tau\right\}\right\rangle \tag{7}$$

Taking into account the structure of the functional $c(\mathbf{x},t) = S[\mathbf{z}(t,\tau),t]$, where $S[\mathbf{z}(t,\tau),t]$ is some non-random function of the random variable $\mathbf{z}(t,\tau) = \mathbf{x} - \phi^{-1}\int_{t_0}^{t}\mathbf{V}(\tau)d\tau - \phi^{-1}\int_{t}^{t}\mathbf{v}'(\tau)d\tau$ and t we can calculate the variational derivatives in the following manner.

$$\left\langle \frac{\delta^n c(\mathbf{x},t)}{\delta v_{j_1}'(\tau_1)...\delta v_{j_n}'(\tau_n)}\right\rangle = (-1)^n\phi^n\frac{\partial^n u(\mathbf{x},t)}{\partial x_{j_1}...\partial x_{j_n}} \tag{8}$$

Now we can write the closed equation for the mean concentration $u(x,t)$

$$\phi\frac{\partial u(x,t)}{\partial t} + V_i(t)\frac{\partial u(x,t)}{\partial x_i} + \sum_{1}^{\infty}\frac{(-1)^n}{\phi^n n!}\frac{\partial^{n+1}u(x,t)}{\partial x_i\partial x_{j_1}...\partial x_{j_n}}\int_{t_0}^{t}...\int_{t_0}^{t}K_{n+1}^{i,j_1,...,j_n}(t,\tau_1,...,\tau_n)d\tau_1...d\tau_n = \frac{\partial}{\partial x_i}\left[d_{ij}\frac{\partial u(x,t)}{\partial x_j}\right] \tag{9}$$

Thus in the general case the averaged equation is first order in the time and infinite order with respect to the spatial variables , which means non-local. Moreover, the coefficients of the derivatives with respect to the spatial variable depend on time t and time t_0. This circumstance renders the problem of not being

invariant with respect to the initial condition. This noninvariance is due to the fact that the system at time t_0 is deterministic, but for $t \rangle t_0$, it is stochastic.

In exceptional cases the exact averaged equation is a differential equation of a finite order and it can take various forms depending on the property of the random velocity. Therefore, the approximate averaged equations that only utilize the first moments or cummulants for the random fields can not provide a universal model of the process in the general case.

3. FLOW-VELOCITY - GAUSSIAN PROCESS

If $v_i(t)$ are the Gaussian processes, the cummulants

$$K_1^i(t)=0\ ,\ K_2^{i,j}(t,\tau)=B^{ij}(t,\tau)\ ,\ K_n^{i,j_1,\dots,j_n}(t,\tau_1,\dots,\tau_n)=0\ \ if\ \ n\geq 3 \qquad (5)$$

where $B^{ij}(t,\tau)=\langle v_i'(t)v_j'(\tau)\rangle$ are the correlation tensor-function of the velocity $\mathbf{v}'(t)$.

Now we have from (4) the second order equation

$$\phi\frac{\partial u(\mathbf{x},t)}{\partial t}+V_i(t)\frac{\partial u(\mathbf{x},t)}{\partial x_i}=\left[d_{ij}+\phi^{-1}\int_{t_o}^{t}B^{ij}(t,\tau)\,d\tau\right]\frac{\partial^2 u(\mathbf{x},t)}{\partial x_i\,\partial x_j} \qquad (10)$$

and the initial condition $u(\mathbf{x},t_0)=f(\mathbf{x})$.

It should be noted that the question of the invertibility of the averaged equations is discussed by *Indelnan and Shvidler*[1985] . They considered a simple example where the process of dispersion is invertible. Here we consider different example. Let $v(t)=V+v\cos\omega t$ where v is a Gaussian random number with $\langle v\rangle=0$ and ω is non-random and constant. It can be seen that

$$\phi\frac{\partial u(x,t)}{\partial t}+V\frac{\partial u(x,t)}{\partial x}=\left(d+\frac{\langle v^2\rangle}{2\phi\omega}\sin 2\omega t\right)\frac{\partial^2 u(x,t)}{\partial x^2} \qquad (11)$$

Thus for $2\phi\omega d/\langle v^2\rangle < 1$ and any number k the coefficient of dispersion is negative if time t satisfies the inequality $(2k-1)\pi/2\omega < t < k\pi/\omega$. In this case the Cauchy problem for equation (11) is improperly posed.

4. FLOW-VELOCITY-THE TELEGRAPH PROCESS.

Now we consider the one-dimensional transport along the x-axis by neglecting the local dispersion d. Let the random velocity $v(t)$ have the form

$$v(t) = V + \alpha(-1)^{n(t;t_0)} \tag{12}$$

where V is constant and the random variable α has the probability distribution

$$\varphi(\alpha) = \frac{1}{2}\left[\delta(\alpha+\alpha_0) + \delta(\alpha-\alpha_0)\right] \tag{13}$$

Then $n(t;t_0)$ is the number of the jumps, the random process of Poisson point flux, in the function $v(t)$ in the interval (t, t_0) . In this case for any $t_1 < t_2 < t_3$ we have the following conditions: $n(t_1;t_2)$ and $n(t_2;t_3)$ are statistically independent and $n(t_1;t_3) = n(t_1;t_2) + n(t_2;t_3)$. For any integer number $m \geq 0$ the probability is:

$$P\{n(t_1;t_2) = m\} = \frac{\upsilon^m |t_2-t_1|^m}{m!}\exp\left[-\upsilon|t_2-t_1|\right] \tag{14}$$

In this case the velocity $v(t)$ (so called telegraph process) is a Markovian stationary process with the correlation function:

$$B(t,\tau) = \alpha_0^2 \exp(-2\upsilon\ |t-\tau|) \tag{15}$$

Fig.1 shows the typical realization of the fluctuation in the telegraph process $v(t) - V$. Following *Klyatskin* [1980] and taking into account that in one- dimensional case when d =0 the concentration can be written as $c(x,t)=f\left(x-\phi^{-1}\int_{t_0}^{t}v(u)du\right)$, we can find the correlation moment for the telegraph process $v'(t)$ and the functional $c(x,t)$:

$$\langle v'(t)c(x,t)\rangle = -\phi^{-1}\int_{t_0}^{t}B(t,\tau)\frac{\partial^2 u\left[x-V\phi^{-1}(t-\tau),\tau\right]}{\partial x^2}d\tau \tag{16}$$

And after using (16) in equation (3) we can write the closed equation and the initial condition as:

$$\phi\frac{\partial u\ (x,t)}{\partial\ t} + V\frac{\partial u(x,t)}{\partial x} = \phi^{-1}\ \frac{\partial}{\partial x}\int_{t_0}^{t}B(t,\tau)\ \frac{\partial u\left[x-V\phi^{-1}(t-\tau)\ ,\tau\right]}{\partial x}d\tau \tag{17}$$

$$u(x,t_0)=f(x) \tag{18}$$

The equation (17) is non-local. The integral summarize the derivatives of $u\ (x,t\ )$ with the weighting function $B(t,\tau)$. It is obvious that the non-local measure of the integral is the "memory" of the function $B(t,\tau)$ and that the expression (10) depends on the scale of correlation, which is $(2\upsilon)^{-1}$.

Note the equation (17) is similar to the averaged equation of transport obtained by *Shvidler*[1975], who used the second order perturbation method. For the one-dimensional transport if the velocity is a telegraph process, the both equations are identical, i.e., the method of perturbation leads to the exact result in this case.

It is interesting to add that *Dagan and Neuman* [1991] examined equation (17) and concluded that this equation was unsuitable for describing the transport process. On the contrary we show in this paper that the equation (17) is exact for the random velocity wich is one-dimensional telegraph process.

The exponential correlation $B(t,\tau)$ in (15) and the linear shift $-V\phi^{-1}(t-\tau)$ from t provide a way of localizing the equation (17). To eliminate the integral from this equation we differentiate the equation with respect to t and x and obtain the differential equation of second order in time t and second order in x, where the coefficients of the derivatives are constant.

$$\phi\frac{\partial u(x,t)}{\partial t} + V\frac{\partial u(x,t)}{\partial x} + \frac{1}{2\upsilon}\left[\phi\frac{\partial^2 u(x,t)}{\partial t^2} + 2V\frac{\partial^2 u(x,t)}{\partial x\partial t} + \frac{\left(V^2-\alpha_0^2\right)}{\phi}\frac{\partial^2 u(x,t)}{\partial x^2}\right] = 0 \qquad (19)$$

The set of initial conditions for (19) is:

$$u(x,t_0) = f(x) \ , \ \frac{\partial u(x,t_0)}{\partial t} = -\frac{V}{\phi}f'(x) \qquad (20)$$

(The second initial condition is obtained from the equation (17) by setting $t=t_0$)

Since the discriminant for (19) is $\alpha_0^2 > 0$ the equation (19) is hyperbolic an has two real families of characteristic lines:

$$x - V_1 t = const \ , \ x - V_2 t = const \qquad (21)$$

where $V_1 = \phi^{-1}\left(V - |\alpha_0|\right)$ and $V_2 = \phi^{-1}\left(V + |\alpha_0|\right)$

The change of variables $y = x - V\phi^{-1}(t-t_0)$ *and* $\tau = t - t_0$ transform the equation of transport (19) to the classical telegraph equation in the canonical form:

$$\frac{\phi}{\chi^2}\frac{\partial u(y,\tau)}{\partial \tau} + \frac{1}{b^2}\frac{\partial^2 u(y,\tau)}{\partial \tau^2} = \frac{\partial^2 u(y,\tau)}{\partial y^2} \qquad (22)$$

where $\chi^2 = \alpha_0^2 / 2\upsilon\phi$, $b^2 = \alpha_0^2/\phi^2$

The initial conditions in terms of the new variables are

$$u(y,0) = f(y) \ , \ \frac{\partial u(y,0)}{\partial \tau} = 0 \qquad (23)$$

The parameter $b=|\alpha_0|/\phi$ determines the finite velocity of the propagation of perturbation in the moving coordinate system. The parameter χ^2 specifies the dispersion of perturbation. It is pertinent to note that the parameter χ^2 in the hyperbolic equation (22) is, strictly speaking , not the coefficient of solute dispersion. On the contrary, it easy to show after computing the first and second space moments of the mean concentration $u(y,\tau)$, the coefficient of solute dispersion is $D(\tau)=\chi^2(1-e^{-2\upsilon\tau})$ and only when $\tau>>\upsilon^{-1}$ we have $D\to\chi^2$.

If $\chi^2=const$ for $\upsilon\to\infty$ *and* $|\alpha_0|\to\infty$, the equation (22) is parabolic as the telegraph process in some respect becomes equal to delta-correlated Gaussian process [Klyatskin, 1980].

Now we study in detail the transport problem in the case where the mean concentration satisfies the equation (22) with the initial conditions (23).

Using the Rieman`s method [see for example *Rubinstein and Rubinstein* ,1993], we find the mean concentration

$$u(y,t)=\frac{1}{2}\exp(-\upsilon\tau)\left\{f(y+b\tau)+f(y-b\tau)+\upsilon\int_{-\tau}^{\tau}f(y+b\lambda)\left[I_0(z)+\upsilon\tau z^{-1}I_1(z)\right]d\lambda\right\} \tag{24}$$

where $z=\upsilon\sqrt{\tau^2-\lambda^2}$ and $I_0(z)$, $I_1(z)$ are the modified Bessel`s functions of the first kind of order zero and one respectively.

Let the initial plume be extremely small, that is $f(x)=q\delta(x-x_0)$.

Then for $|x-x_0-V\tau|\le b\tau$ we obtain from equation (24)

$$u(x,t)=\frac{q}{2}\exp(-\upsilon\tau)\left\{\delta(x-x_0-V_1\tau)+\delta(x-x_0-V_2\tau)+\upsilon b^{-1}\left[I_0(\bar{z})+\frac{\upsilon\tau}{\bar{z}}I_1(\bar{z})\right]\right\} \tag{25}$$

where $\bar{z}=\upsilon b\left[b^2\tau^2-\left(x-x_0-\phi^{-1}V\tau\right)^2\right]^{1/2}$

For $\left|x-x_0-\phi^{-1}V\tau\right|>b\tau$ we have:

$$u(x,t)=0 \tag{26}$$

Then if $\left|x-x_0-\phi^{-1}V\tau\right|<<b\tau$ and $b\tau>>1$, we obtain

$$u(x,t)\approx\frac{q\sqrt{\phi}}{2\chi\sqrt{\pi\tau}}\exp\left[-\frac{\phi\left(x-x_0-\phi^{-1}V\tau\right)^2}{4\chi^2\tau}\right] \tag{27}$$

This solution shows that the plume remains in a finite spatial interval . At the front and rear boundaries of the moving plume there are spikes of infinite concentration. For $\tau>>\upsilon^{-1}$ these portions of plume are very small. Between and far from the spikes the distribution of concentration is like Gaussian for large values of $\upsilon\,\tau$.

For small $\upsilon\,\tau$ the portion of the plume in the spikes is dominant and the movement approximately that of a wave mechanism.

5. FIELD WITH SOURCES OF SOLUTE

Let us now assume that the field have non-random solute sources and the local concentration is described by the stochastic transport equations

$$\phi\frac{\partial\ c(\mathbf{x},t)}{\partial\ t}+v_i(t)\frac{\partial\ c(\mathbf{x},t)}{\partial x_i}=\psi(\mathbf{x},t) \tag{28}$$

$$c(\mathbf{x},t_0)=f(\mathbf{x}) \tag{29}$$

We can use the same functional approach and after averaging the equations we find that :1) If the flow-velocity $v(t)$ is Gaussian process

$$\phi\frac{\partial u(\mathbf{x},t)}{\partial t}+V_i(t)\frac{\partial u(x,t)}{\partial x_i}-\phi^{-1}\frac{\partial^2 u(\mathbf{x},t)}{\partial x_i\partial x_j}\int_{t_0}^{t}B^{ij}(t,\lambda)\,d\lambda=\psi(\mathbf{x},t)-\phi^{-2}\frac{\partial^2}{\partial x_i\partial x_j}\int_{t_0}^{t}\int_{\lambda}^{t}B^{ij}(t,\lambda)\langle\psi(\gamma,\mu)\rangle\,d\lambda\,d\mu \tag{30}$$

where $\gamma = \mathbf{x} - \phi^{-1} \int_{\mu}^{t} \mathbf{v}(\tau)\, d\tau$

For any x , t and μ the parameter γ is Gaussian and in one-dimensional case we can calculate

$$\langle \psi(\gamma,\mu) \rangle = \frac{1}{\sqrt{2\pi}\sigma_\gamma} \int_{-\infty}^{\infty} \psi(\gamma,\mu) \exp\left[-\frac{(\gamma - \langle \gamma \rangle)^2}{2\sigma_\gamma^2} \right] d\gamma \tag{31}$$

where $\langle \gamma \rangle = x - V\phi^{-1}(t-\mu)$ *and* $\sigma_\gamma^2 = (\gamma - \langle \gamma \rangle)^2 = \phi^{-2} \int_{\mu}^{t}\int_{\mu}^{t} B(\theta,\theta') d\theta d\theta'$

2) If the flow velocity $v(t)$ is the telegraph process , we obtain the averaged equation

$$\phi \frac{\partial u(x,t)}{\partial t} + V \frac{\partial u(x,t)}{\partial x} + \frac{1}{2\upsilon} \left[\phi \frac{\partial^2 u(x,t)}{\partial t^2} + 2V \frac{\partial^2 u(x,t)}{\partial x \partial t} + \frac{(V^2 - \alpha_0^2)}{\phi} \frac{\partial^2 u(x,t)}{\partial x^2} \right] = \psi^*(x,t) \tag{32}$$

$$\psi^*(x,t) = \psi(x,t) + \frac{1}{2\phi\upsilon} \left[\phi \frac{\partial \psi(x,t)}{\partial t} + V \frac{\partial \psi(x,t)}{\partial x} \right] \tag{33}$$

$$u(x,t_0) = f(x) \quad , \quad \frac{\partial u(x,t_0)}{\partial t} = -V\phi^{-1} f'(x) + \phi^{-1} \psi(x,t_0) \tag{34}$$

In general we have the new additional fictitious sources to compensate for the solute particles introduced into the system at different times.

6. TRANSPORT IN THE SPACE WITH BOUNDARY

Let the concentration distributions in the region $x \geq x_0$, $t \geq t_0$ is described by the one-dimensional equation of transport

$$\phi \frac{\partial c(x,t)}{\partial t} + v(t) \frac{\partial c(x,t)}{\partial x} = \psi(x,t) \quad ; \; x > x_0 \quad , \quad t > t_0 \tag{35}$$

and the initial condition

$$c(x,t_0) = f(x) \quad , \quad x \geq x_0 \tag{36}$$

We consider two types of boundary conditions :

a) $c \ (x_0,t) = g(t) \ , \ t \geq t_0$ (37)

b) $v(t)c(x_o,t) = q(t) \quad , \quad t \geq t_0$ (38)

We assume that the given non-random functions $f(x)$, $g(t)$ and $q(t)$ satisfy the conditions

$f(x_0) = 0 \quad , \quad g(t_0) = 0 \quad , \quad v(t_0)\, f(x_0) = q(t_0) = 0$.

We use the analysis method that leads from the boundary problem to a Cauchy problem in an unbounded space:

1. We consider the continuation of the concentration $c(x,t)$ in the region $x < x_0$. For $v(t) \geq 0$ the natural conclusion is

$$\bar{c}(x,t) = \{ \ c(x,t) \ , if \ \ x \geq x_0 \ \ and \ \ 0 \ , \ if \ \ x < x_0 \ \} \tag{39}$$

2. We introduce a new function

$$\hat{c}(x,t) = \bar{c}(x,t) \ \ h_+(x - x_0)\, h_+(t - t_0) \tag{40}$$

where $h_+(y) = \{ \ 1, if \ \ y \geq 0 \ \ and \ \ 0, if \ \ y < 0 \ \}$ so-called asymmetric Heavisade function.

3. For random function $\hat{c}(x,t)$ we obtain the stochastic equation and the initial condition in all space ,i.e., that is the Cauchy problem .

4. After averaging the modified transport equation when $v(t)$ is the telegraph process we obtain the non-local equation for the mean modified concentration . $\hat{u}(x,t) = \langle \hat{c}(x,t) \rangle$

5. Localization of this non-local equation leads to a differential equation for $\hat{u}(x,t)$.

6. The final step is the slope in space $\bar{u}(x,t)$ and $u(x,t)$. After this we obtain the differential equation for mean concentration $u(x,t)$ in the region $x > x_0$, $t > t_0$.

The averaged equation for $u(x,t)$ in case a) is

$$\phi\frac{\partial u(x,t)}{\partial t}+V\frac{\partial u(x,t)}{\partial t}+\frac{1}{2\nu}\left[\phi\frac{\partial^2 u(x,t)}{\partial t^2}+2V\frac{\partial^2 u(x,t)}{\partial x\partial t}+\frac{V^2-\alpha_0^2}{\phi}\frac{\partial^2 u(x,t)}{\partial x^2}\right]=\psi^*(x,t) \tag{41}$$

where

$$\psi^*(x,t)=\psi+\frac{1}{2\nu\phi}\left[\phi\frac{\partial\psi}{\partial t}+V\frac{\partial\psi}{\partial x}\right]-\frac{4\phi\nu\alpha_0^2 g(t)}{V^2}\exp\left[-\frac{2\phi\nu(x-x_0)}{V}\right]h_+\left[x_0-x+V\phi^{-1}(t-t_0)\right]$$

The initial conditions are for $x>x_0$

$$u(x,t_0)=f(x)\ ,\ \frac{\partial u(x,t_0)}{\partial\ t}=-\frac{V}{\phi}f'(x)+\frac{1}{\phi}\psi(x,t_0) \tag{42}$$

and the boundary condition is

$$u(x_0,t)=g(t) \tag{43}$$

In case b) we have transport equation for region $x>x_0\ ,t>t_0$

$$\phi\frac{\partial u(x,t)}{\partial t}+V\frac{\partial u(x,t)}{\partial x}+\frac{1}{2\upsilon}\left[\phi\frac{\partial^2 u(x,t)}{\partial t^2}+2V\frac{\partial^2 u(x,t)}{\partial x\partial t}+\frac{V^2-\alpha_0^2}{\phi}\frac{\partial^2 u(x,t)}{\partial x^2}\right]=\tilde{\psi}(x,t) \tag{44}$$

where

$$\tilde{\psi}(x,t)=\psi(x,t)+\frac{1}{2\phi\upsilon}\left[\phi\frac{\partial\psi(x,t)}{\partial t}+V\frac{\partial\psi(x,t)}{\partial x}\right]-\frac{\alpha_0^2}{V^2-\alpha_0^2}q(t)\delta_+(x-x_0)$$

The initial conditions are for $x>x_0$

$$u(x,t_0)=f(x)\ ,\ \frac{\partial u(x,t_0)}{\partial t}=-\frac{V}{\phi}f'(x)+\frac{1}{\phi}\psi(x,t_0) \tag{45}$$

and boundary conditions are for $t>t_0$

$$u(x_0,t)=q(t)\frac{V}{V^2-\alpha_0^2}\quad ,\quad \frac{\partial u(x_0,t)}{\partial x}=-q'(t)\ \frac{\phi(V^2+\alpha_0^2)}{(V^2-\alpha_0^2)^2}+\frac{V}{V^2-\alpha_0^2}\psi(x_0,t) \tag{46}$$

We show that in both cases a) and b) the sources $\psi^*(x,t)$ and $\tilde{\psi}(x,t)$ are dependent on the boundary and the initial conditions for the stochastic equations and the initial conditions for the averaged equations are dependent on the initial sources .

7. ANALYSIS OF THE EXACT SOLUTIONS

The two cases discussed in sections 3 and 4 (Gaussian and telegraph processes as the models for flow-velocity) were briefly studied by *Shvidler and Karasaki* [1995 and 1996]. In this paper we present the results of more detailed investigation. We compare u_T - the exact solution of telegraph problem (22) and (23) with u_G - the exact solution for the one-dimensional transport equation (11) when $d = 0$ and the exponential equation $B(t,\tau)$ is defined by (15). Obviously the u_G is the mean concentration of solute with the assumption that the velocity *v(t)* is a Gaussian random process and that by (15) it is so called Ornstein-Ulenbeck process , which describes the Brownian motion for particles with inertia. In this case the particle velocity exists and has a finite variance contrary to the Winer`s process [Yaglom,1987] .

To compare both distributions $u_T(x,t)$ and $u_G(x,t)$ we introduce the new dimensionless variables

$$\bar{y} = (y - x_0)/l \ , \ \bar{\tau} = \upsilon\tau \ , \ \tilde{u} = ul/q \tag{47}$$

where $l = b/\upsilon = \alpha_0/\phi\upsilon$ is the scale of length

In these variables with the initial condition $f(x) = q\delta(x - x_o)$, we obtain from (24) the function $\bar{u}_T(\bar{y},\bar{\tau})$ in the form

$$\tilde{u}_T(\bar{y},\bar{\tau}) = \tilde{u}_{TS}(\bar{y},\bar{\tau}) + \tilde{u}_{TR}(\bar{y},\bar{\tau}) \tag{48}$$

where the singular part $\tilde{u}_{TS}(\bar{y},\bar{\tau})$ is

$$\tilde{u}_{TS}(\bar{y},\bar{\tau}) = \frac{e^{-\bar{\tau}}}{2}\left[\ \delta(\bar{y}+\bar{\tau}) + \delta(\bar{y}-\bar{\tau})\right] \tag{49}$$

For the regular part $\tilde{u}_{TR}(\bar{y},\bar{\tau})$ when $|\bar{y}| \le \bar{\tau}$., we have

$$\tilde{u}_{TR}(\bar{y},\bar{\tau}) = \frac{e^{-\bar{\tau}}}{2}\left[I_0\left(\sqrt{\bar{\tau}^2-\bar{y}^2}\right) + \frac{\bar{\tau}}{\sqrt{\bar{\tau}^2-\bar{y}^2}} I_1\left(\sqrt{\bar{\tau}^2-\bar{y}^2}\right)\right] \tag{50}$$

and when $|\bar{y}| > \bar{\tau}$

$$\tilde{u}_{TR}(\bar{y},\bar{\tau}) = 0 \tag{51}$$

It is obvious that $\eta(\bar{\tau})$, the amount of solute in the boundaries at $\bar{y} = \pm\bar{\tau}$, is:

$$\eta(\bar{\tau}) = e^{-\bar{\tau}} \tag{52}$$

From (14) , we have that $\eta(\bar{\tau})$ is equal the probability $P\{n(0;\bar{\tau}) = 0\}$ - the fraction of realizations for with the velocity $v(t)$ has no jumps in the interval $(0,\bar{\tau})$. For small $\bar{\tau}$ this term is dominant and describes the wave nature of the dispersion. However, with large $\bar{\tau}$ this term is small and the solute is dissipated from the boundaries to a regular distribution between boundaries.

To solve for $\tilde{u}_G(\bar{y},\bar{\tau})$ we set $d = 0$ and substitute the exponential correlation function $B(t,\tau)$ from (15) in to equation (11) and for the initial δ-distribution we obtain the solution in the new dimensionless variables

$$\tilde{u}_G(\bar{y},\bar{\tau}) = \pi^{-1/2}\left(e^{-2\bar{\tau}} + 2\bar{\tau} - 1\right)^{-\frac{1}{2}} \exp\left[-\bar{y}^2/\left(e^{-2\bar{\tau}} + 2\bar{\tau} - 1\right)\right] \tag{53}$$

For $\bar{\tau} >> 1$ and any $\bar{y}$ the distribution (53) tends to Gaussian limit

$$\tilde{u}_G(\bar{y},\bar{\tau}) = (2\pi\bar{\tau})^{-1/2} \exp\left(-\bar{y}^2/2\bar{\tau}\right) \tag{54}$$

wich is exactly the same as the main term of the telegraph equation solution when we use the dimensionless variables. It should be pointed out that the asymptotic solution (54) is valid for any $\bar{y}$, whereas the expression for velocity - telegraph process tends to the Gaussian limit (54) with the additional condition $|\bar{y}| >> 1$ that is inside the plume and far from the fronts $\bar{y} = \pm\bar{\tau}$.

Figures 2, 3, 4 and 5 show the dimensionless concentrations $\tilde{u}_{TR}(\bar{y},\bar{\tau})$ and $\tilde{u}_G(\bar{y},\bar{\tau})$ for times $\bar{\tau}$ = 0.1, 1, 2 *and* 5 respectively . According the equation (49) for $\tilde{u}_{TS}(\bar{y},\bar{\tau})$ the fronts are at $\bar{y} = \pm\bar{\tau}$, where there is a spike.

It is clear that for $\bar{\tau} = 0.1$ (Fig. 2) the mean concentration is strongly dependent on the distribution of the random velocity $v(t)$. For $\bar{\tau} = 0.1$ and for the telegraph velocity process the amount of solute remaining at the points $\bar{y} = \pm 0.1$ is $\eta/2 = 0.4524$ and only the amount $1-\eta = 0.095$ is located between the fronts .

For $\bar{\tau} = 2$ (Fig.3) the frontier portions of the solute is noticeably decreased: $\eta/2 = 0.1839$ at points $\bar{y} = \pm 1$. The concentration distribution in $-\bar{\tau} < \bar{y} < \bar{\tau}$ is slightly curved and we can see the maximum at $\bar{y} = 0$ .The amount of the solute between boundaries $\bar{y} = \pm 1$ is $1-\eta = 0.6321$. It is possible to decide that the distributions of the mean concentration by setting $\bar{\tau} = 1$, however, there still exists significant difference between the Gaussian and telegraph distributions of the velocity.

By $\bar{\tau} = 2$ (Fig.4) the distributions of concentrations draw together in the central part of figure but the amount of solute at the boundary points $\bar{y} = \pm 2$ is still significant and is $\eta/2 = 0.06767$. The amount of the solute between boundaries is $1-\eta = 0.8647$.

For $\bar{\tau} = 5$ (Fig.5) the difference between $\tilde{u}_{TR}(\bar{y},\bar{\tau})$ and $\tilde{u}_G(\bar{y},\bar{\tau})$ is small. The amounts of solute at boundaries $\bar{y} = \pm 5$ are small: $\eta/2 = 0.003369$. The amount of the solute in the interval -5 < $\bar{y}$ <5 is therefore $\eta = 0.9933$. However the distributions are different in the immediate neighborhood of the points $\bar{y}=0$ and $\bar{y}=\pm 5$.

Remember that both distributions of the concentration have the same Gaussian limit when $\bar{\tau} >> 1$ and $|\bar{y}| << \bar{\tau}$.

Now we examine the behavior of the mean concentration when the initial concentration $f(x)$ is a continuous function. For example consider an exponential function with $C_0 = const$.

$$f(x) = C_0 \exp\left[-|x - x_0| / a\right] \tag{55}$$

By using the initial concentration in the exponential form, with a constant q – (the initial amount of solute in space), we can find that $C_0 = q / 2a$.

It is clear that for the Gaussian velocity and for one-dimensional case when d=0 , the function $\tilde{u}_G(\bar{y},\bar{\tau})$ in (53) is the Green`s function for the equation (10) for $\bar{\tau} > 0$. Therefore, the mean concentration for any initial function $f(x)$ is convolution of the $\tilde{u}(\bar{y},\bar{\tau})$ and $f(x)$. After calculating the convolution for the exponential $f(x)$ in (55) we have:

$$\tilde{u}_G(\bar{y},\bar{\tau}) = \frac{1}{2}\exp\left(\frac{\bar{g}}{2}\right)\left\{\exp\left(-\frac{\bar{y}}{\bar{a}}\right)\left[1 - erf\left(\sqrt{\frac{\bar{g}}{2}} - \frac{\bar{y}}{\bar{a}\sqrt{2\bar{g}}}\right)\right] + \exp\left(\frac{\bar{y}}{\bar{a}}\right)\left[1 - erf\left(\sqrt{\frac{\bar{g}}{2}} + \frac{\bar{y}}{\bar{a}\sqrt{2\bar{g}}}\right)\right]\right\} \tag{56}$$

where $\bar{g} = g / 2\bar{a}^2$, $g = e^{-2\bar{\tau}} + 2\bar{\tau} - 1$, $erf\zeta = \frac{2}{\sqrt{\pi}}\int_0^{\zeta} e^{-u^2} du$

In the case of the telegraph random velocity and for the same exponential initial condition we can write the following expression for $\tilde{u}_T(\bar{y},\bar{\tau})$

$$\tilde{u}_T(\bar{y},\bar{\tau})=\frac{1}{2}e^{-\bar{\tau}}\left\{\exp\left(-\frac{|\bar{y}+\bar{\tau}|}{\bar{a}}\right)+\exp\left(-\frac{|\bar{y}-\bar{\tau}|}{\bar{a}}\right)+\int_{-\bar{\tau}}^{\bar{\tau}}\left[I_0(z)+\frac{\bar{\tau}}{z}I_1(z)\right]\exp\left(-\frac{|\bar{y}+\bar{\lambda}|}{\bar{a}}\right)d\bar{\lambda}\right\} \quad (57)$$

where $z=\sqrt{\bar{\tau}^2-\bar{\lambda}^2}$

We calculate $\tilde{u}_T(\bar{y},\bar{\tau})$ and $\tilde{u}_G(\bar{y},\bar{\tau})$ for $\bar{a}=a/l=0.1$. In this case the scale of the initial distribution is essentially a little less than l - the scale length of the stochastic process $v(t)$. In other words such initial distribution is similar to the delta – distributions in the sense that a significant part of solute in the initial time is located near the point $\bar{y}=0$. However in contrast to the case where the initial distributions is Dirak's δ – function, the mean concentration curves $\tilde{u}_T(\bar{y},\bar{\tau})$ for $v(t)$ – telegraph process are continuous, but they are similar to the initial exponential distribution (55) and are not smooth.

As noted above the definite part of realization of the telegraph process $v(t)$, namely with fraction $e^{-\bar{\tau}}$, for time $\bar{\tau}$ is free from jumps. Therefore one half of these realizations have velocity $(V+\alpha_0)$ and the second half has $(V-\alpha_0)$. The both parts are transported with the initial distribution of concentration without change of the velocity. For this reason on all curves $\tilde{u}_T(\bar{y},\bar{\tau})$ at $\bar{y}=\pm\bar{\tau}$ (see Fig.6, 7, 8, 9 and 10) we can find local maximums - the continuous analog of δ - functions in the same border points discussed in the previous analysis. But if the initial concentration is δ - distribution at point $\bar{y}=0$, for any $\bar{\tau}$ the mean concentration has local maximum. For the initial distribution is exponential for $\bar{\tau}\le 1$ there exist a local minimum at point $\bar{y}=0$ (Fig.6,7). If time $\bar{\tau}>1$ we can see that at $\bar{y}=0$ the local maximum is the global maximum for large $\bar{\tau}$ (Fig.10, 11).

The distributions $\tilde{u}_G(\bar{y},\bar{\tau})$ and $\tilde{u}_t(\bar{y},\bar{\tau})$ are relatively close even for $\bar{\tau}=5$ and only slightly different near $\bar{y}=0$ and $\bar{y}=\pm 5$ (Fig.10). For $\bar{\tau}=10$ both distributions are practically congruent (Fig.11).

Now let the initial non-random exponential distribution of concentration be slightly sloping comparatively with the previous analysis. For this purpose we use the parameter $\bar{a}=1$, which means that the scale of initial distribution of concentration a and the length scale of random velocity l are the same.

The functions $\tilde{u}_T(\bar{y},\bar{\tau})$ and $\tilde{u}_G(\bar{y},\bar{\tau})$ for $\bar{a}=1$ are plotted in Fig.12-16. The corresponding distributions are similar in shape and close except in the interval $-\bar{\tau}<\bar{y}<\bar{\tau}$,where the differences are also small.

Thus in all the above cases examined we can see the convergence of the mean concentrations derived for the two different (Gaussian and telegraph) random velocities $v(t)$. Of course this phenomenon is linked to a fact that in the above examined cases the random concentration $c(x,t)$ is a non-random function with the random argument $z(t)=x-\phi^{-1}\int_{t_0}^{t}v(\theta)d\theta$. If $v(t)$ is Gaussian random process, $z(t)$ is also Gaussian for any time t . If $v(t)$ is telegraph process the argument $z(t)$ for $\upsilon(t-t_0)>>1$ is the sum of a large number of uncorrelated summonds and tend to a Gaussian by the central limit theorem. The asymptotical convergence of $c(x,t)$ for different velocity fields leads to the convergence of the functionals like moments, densities of probability etc.

To note the comparison $u_T(x,t)$ and $u_G(x,t)$ as well as the corresponding probability density functions presented earlier by *Shvidler and Karasaki* [1997] is interesting also because both solutions are exact for different distributions of velocity $v(t)$; namely telegraph and

Gaussian (or so called Ornstein-Ulenbeck) processes that have the equal means and exponential autocorrelation function. The solution depends on the full description of the flow-velocity $v(t)$.

If we study transport problem for $v(t)$ -telegraph process, the function $u_T(x,t)$ is the exact mean concentration and $u_G(x,t)$ is the approximation derived with the method of perturbation with an approximate localization in the same order. It is reminded that with exponential correlation function the method of perturbation leads to the exact mean concentration, if the localization is performed exactly.

If the flow-velocity $v(t)$ is a Gaussian process, the function $u_G(x,t)$ is the exact mean concentration and the function $u_T(x,t)$ will be the approximation derived with the method of perturbation with exact localization.

It should be noted that for the non-local transport equation derived with the method of perturbation was discussed in *Shvidler*[1975]. The method of approximation with the same order of localization was presented by *Shvidler*[1993] .

The results reported in this paper and analysis thereof can not of course lead to a final conclusion about accuracy and practicality of the method of perturbation in general. It is obvious that the efficiency of this method essentially depends on the type of the problem (that we study) and the properties of the flow-velocity field. This is also true for estimation with any stable method of approximation.

SUMMARY

We presented new examples of an exactly averaged multi-dimensional equation for the mean concentration when the components of random flow-velocity are functions of time. The exactly averaged equation is a linear differential equation of first order in time and in general of infinite order with respect to the spatial variables. The coefficients of the derivatives are dependent on the cumulants function of flow-velocity random process. In general the exactly averaged equation is non-local. In some exceptional cases the averaged equation is a differential or integro-differential equation of finite order.

We presented and analyzed in detail two one-dimensional cases (for Gaussian and telegraph random flow-velocity with the identical exponential correlation function) where the exactly averaged equations are second order parabolic and hyperbolic types , respectively . We studied the behavior of different initial plumes, the evolutions and convergence of them for large time . We illustrated the process how the mean concentration distribution for both flow-velocity cases approaches a unique asymptotic limit.

Acknowledgements

Authors would like to thank Dmitry Silin of Lawrence Berkeley National Laboratory for his critical review. This work was partially supported by JNC (Japan Nuclear Fuel Cicle Corporation). The work was conducted under the U.S. Department of Energy Contract No.DE-AC03-76SF00098.

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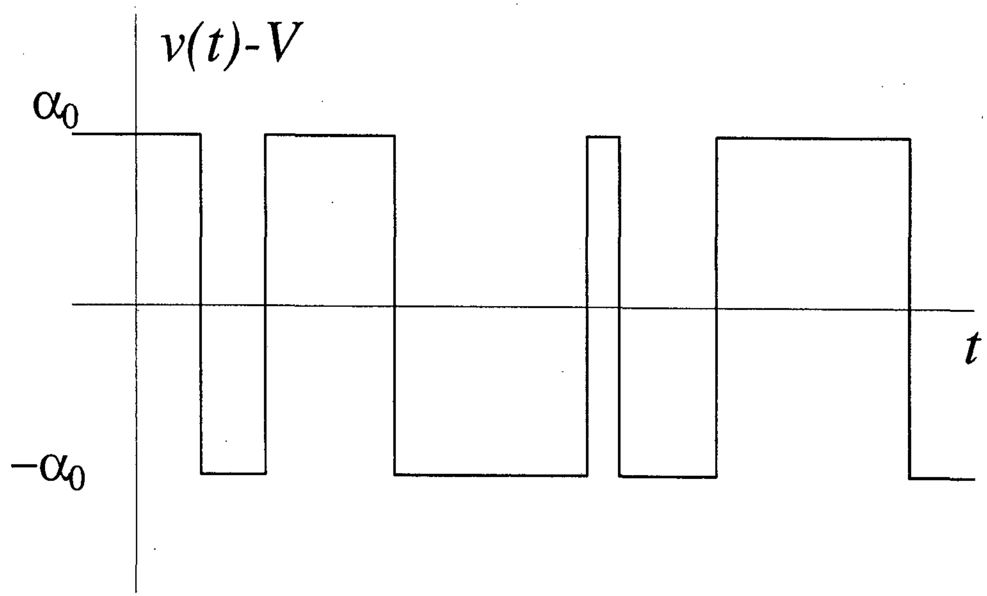

Figure 1. Typical realization of the fluctuation $v(t)-V$ in the random telegraph process.

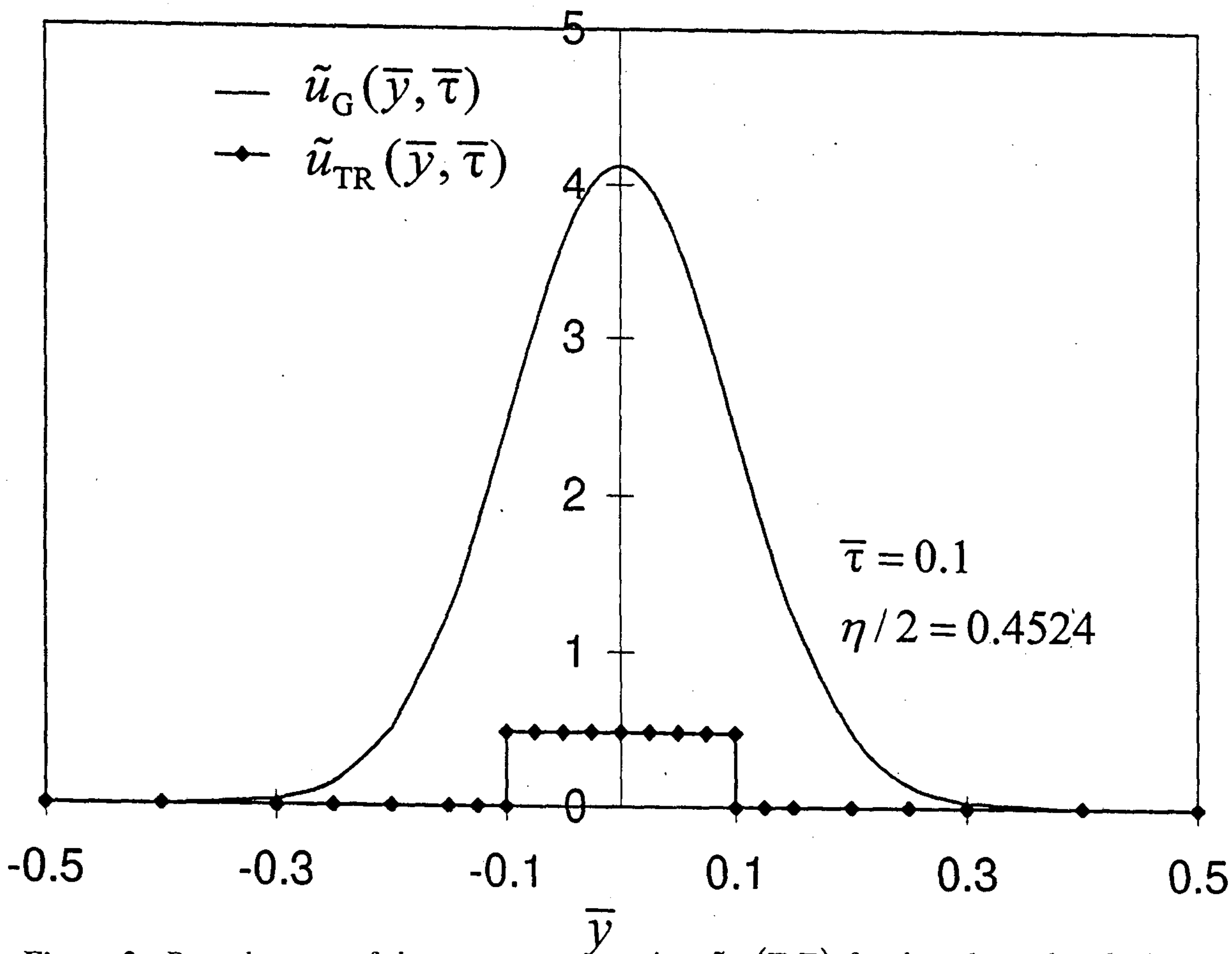

Figure 2. R egular part of the mean concentration $\tilde{u}_{TR}(\bar{y},\bar{\tau})$ for the telegraph velocity and the mean concentration $\tilde{u}_G(\bar{y},\bar{\tau})$ for the Gaussian velocity at $\bar{\tau}=0.1$. The initial concentration is the Dirac's δ function: $\tilde{u}_T(\bar{y},0)=\tilde{u}_G(\bar{y},0)=\delta(\bar{y})$.

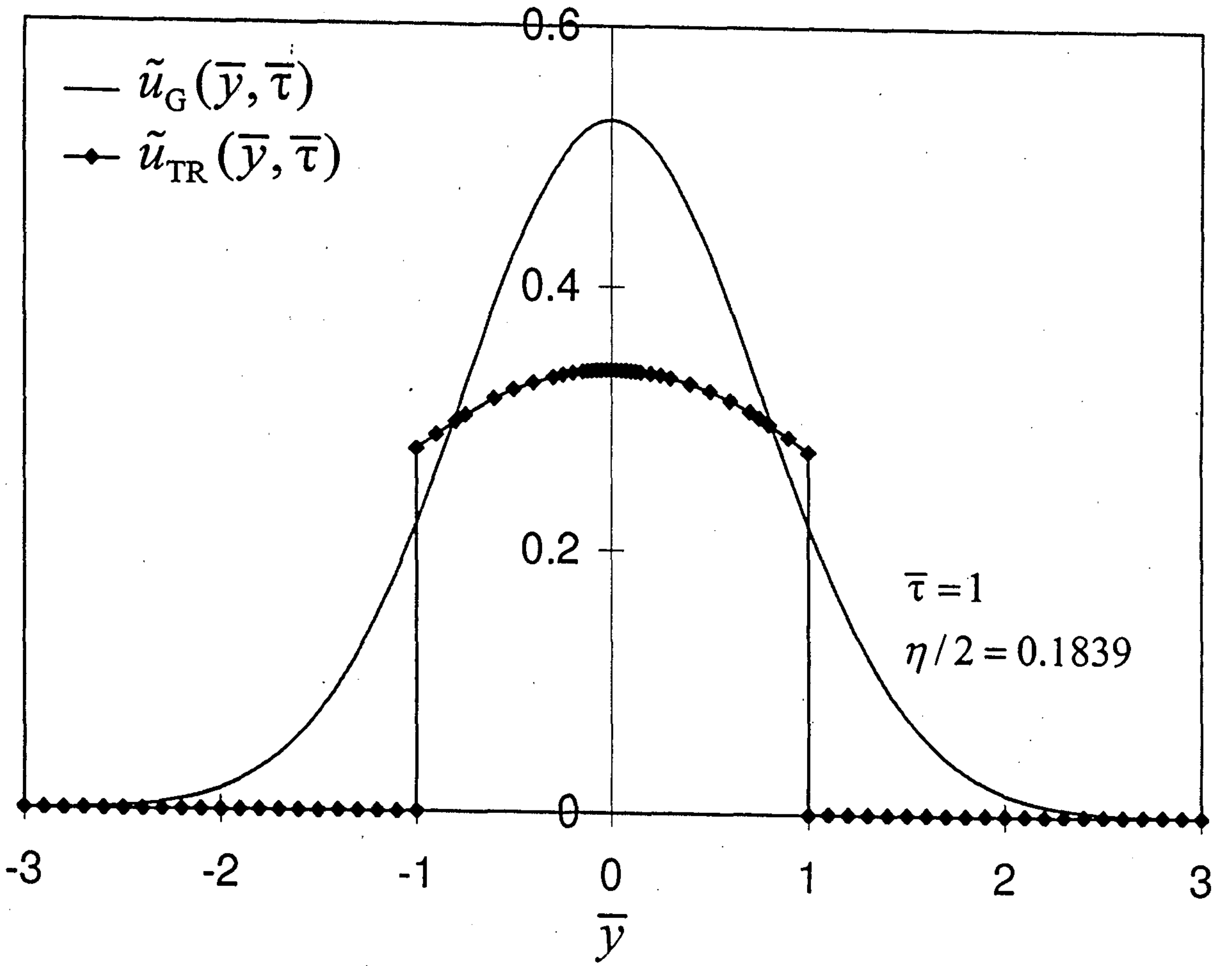

Figure 3. Regular part of the mean concentration $\tilde{u}_{TR}(\bar{y},\bar{\tau})$ for the telegraph velocity and the mean concentration $\tilde{u}_G(\bar{y},\bar{\tau})$ for the Gaussian velocity at $\bar{\tau}=1$. The initial concentration is the Dirac's δ function: $\tilde{u}_T(\bar{y},0)=\tilde{u}_G(\bar{y},0)=\delta(\bar{y})$.

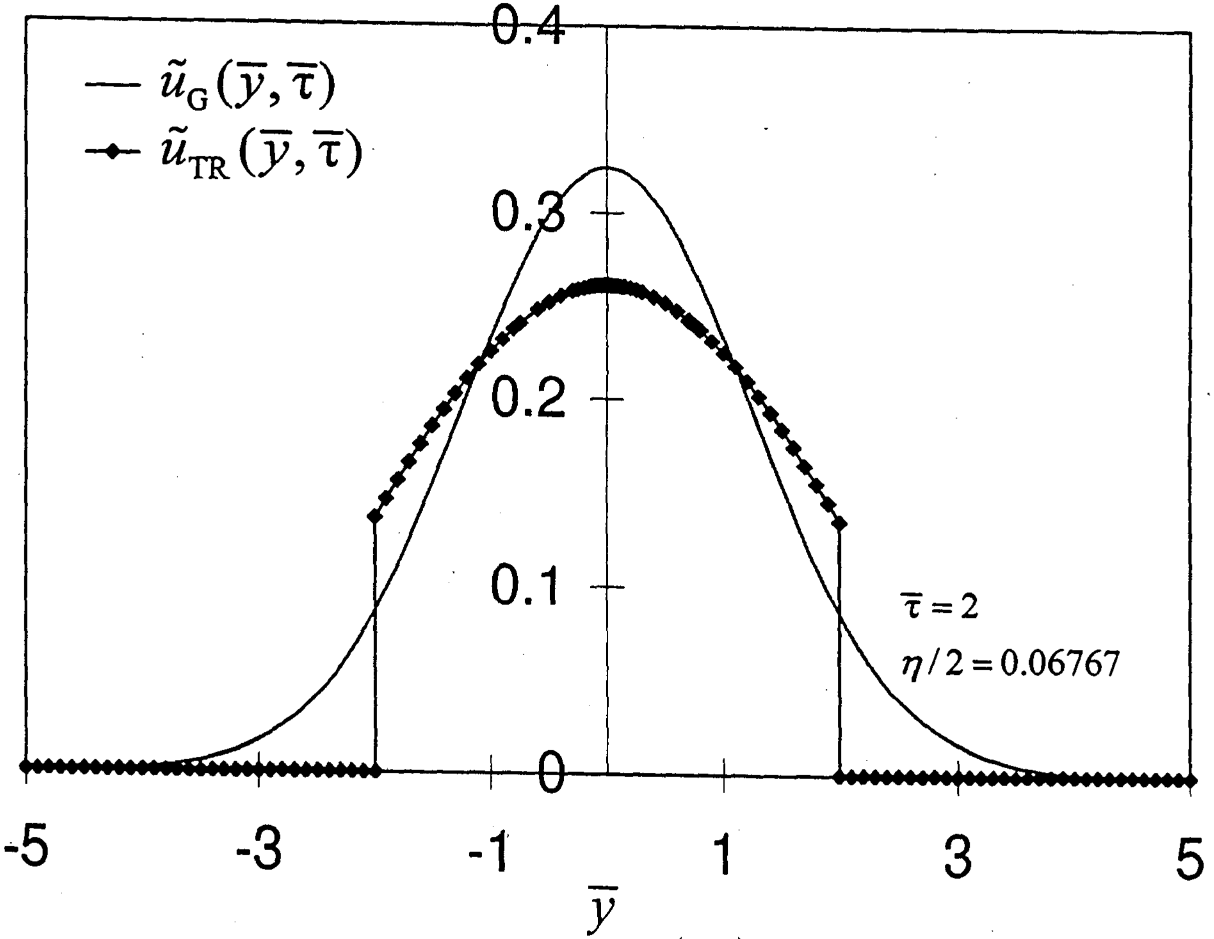

Figure 4. Regular part of the mean concentration $\tilde{u}_{TR}(\overline{y},\overline{\tau})$ for the telegraph velocity and the mean concentration $\tilde{u}_G(\overline{y},\overline{\tau})$ for the Gaussian velocity at $\overline{\tau}=2$. The initial concentration is the Dirac's δ function: $\tilde{u}_T(\overline{y},0)=\tilde{u}_G(\overline{y},0)=\delta(\overline{y})$.

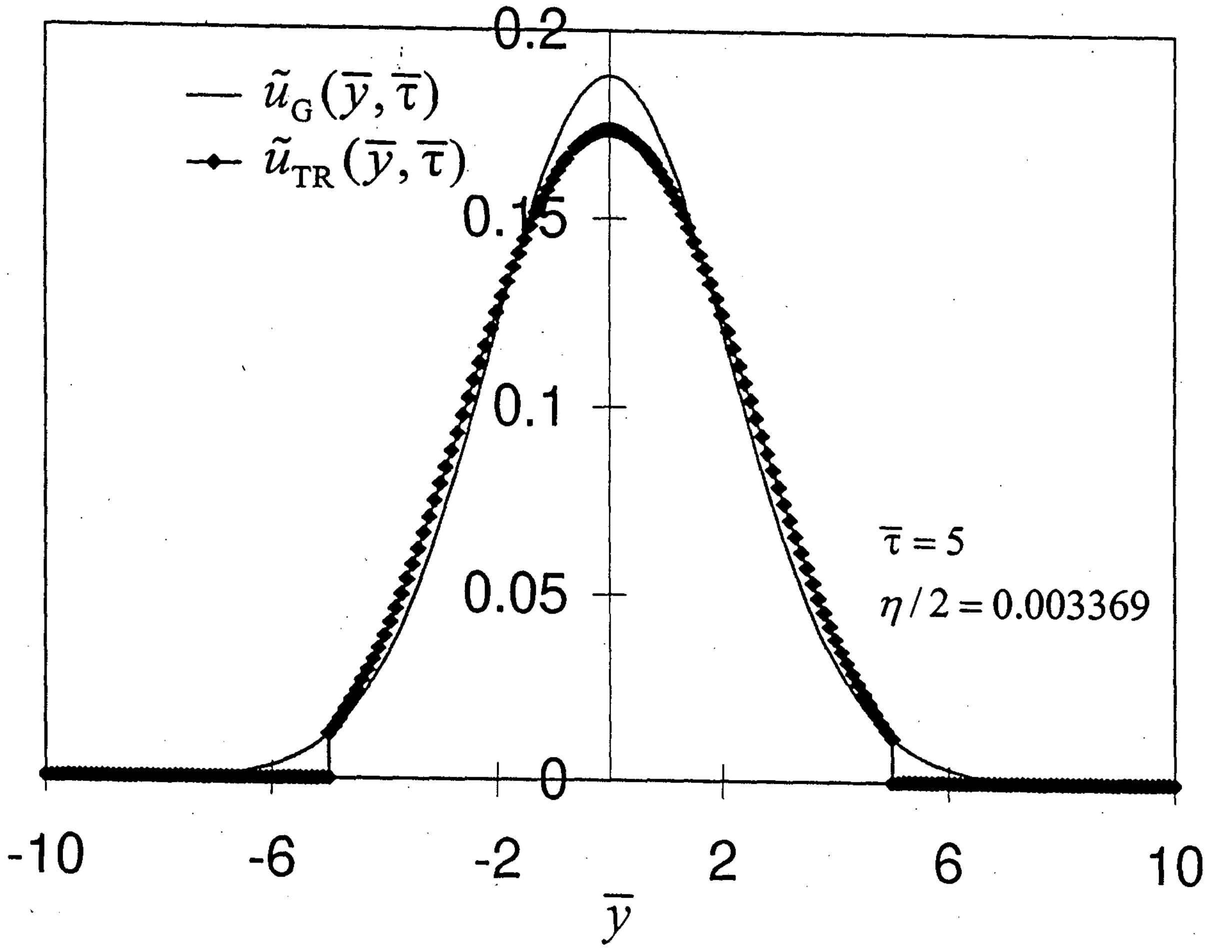

Figure 5. Regular part of the mean concentration $\tilde{u}_{TR}(\bar{y},\bar{\tau})$ for the telegraph velocity and the mean concentration $\tilde{u}_G(\bar{y},\bar{\tau})$ for the Gaussian velocity at $\bar{\tau}=5$. The initial concentration is the Dirac's δ function: $\tilde{u}_T(\bar{y},0)=\tilde{u}_G(\bar{y},0)=\delta(\bar{y})$.

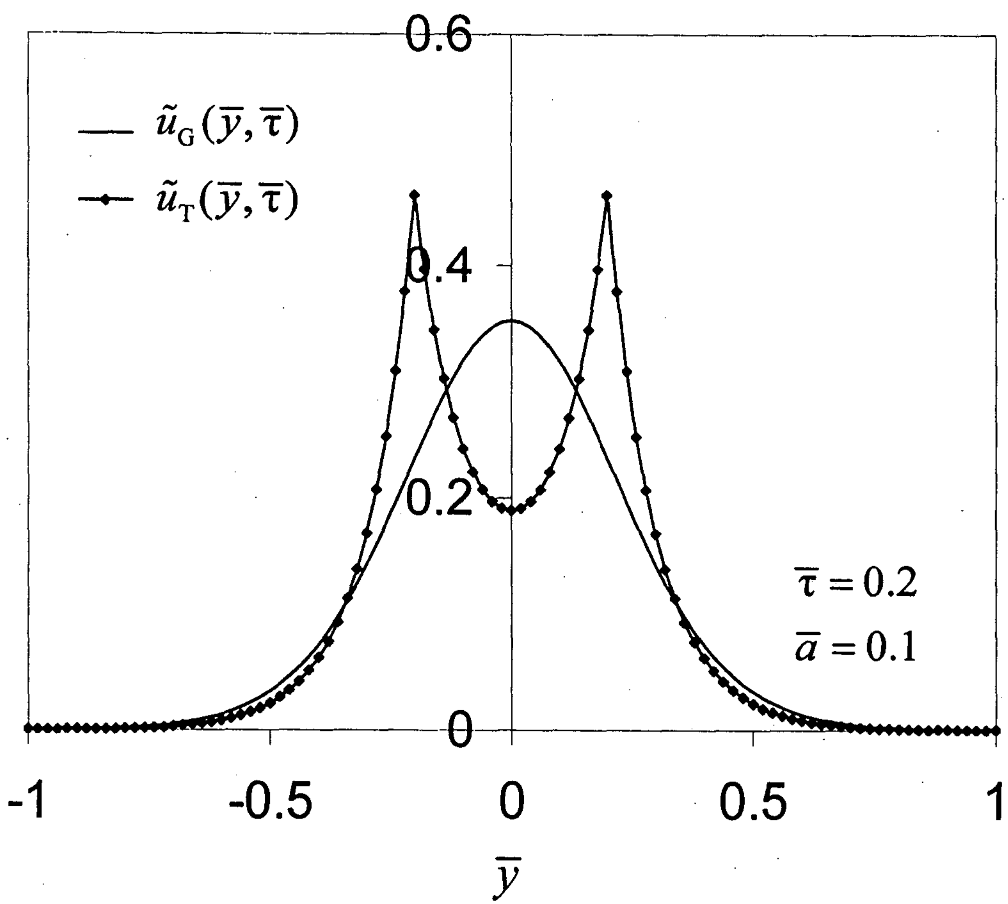

Figure 6. Mean concentration $\tilde{u}_T(\bar{y},\bar{\tau})$ for the telegraph velocity process and $\tilde{u}_G(\bar{y},\bar{\tau})$ for the Gaussian velocity process for $\bar{a} = 0.1$ at $\bar{\tau} = 0.2$. The initial concentrations are $\tilde{u}_T(\bar{y},0) = \tilde{u}_G(\bar{y},0) = \exp\left(-|\bar{y}/0.1|\right)$.

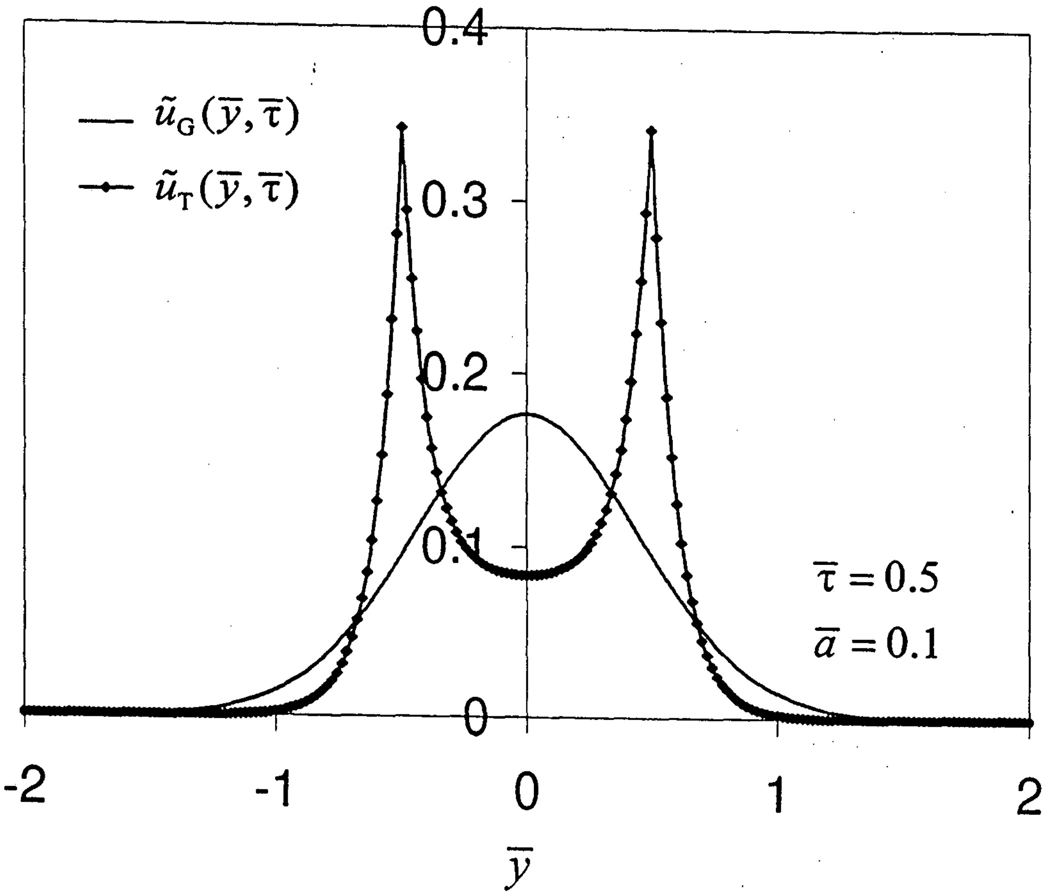

Figure 7. Mean concentration $\tilde{u}_T(\overline{y},\overline{\tau})$ for the telegraph velocity process and $\tilde{u}_G(\overline{y},\overline{\tau})$ for the Gaussian velocity process for $\overline{a}=0.1$ at $\overline{\tau}=0.5$. The initial concentrations are $\tilde{u}_T(\overline{y},0)=\tilde{u}_G(\overline{y},0)=\exp\left(-\left|\overline{y}/0.1\right|\right)$.

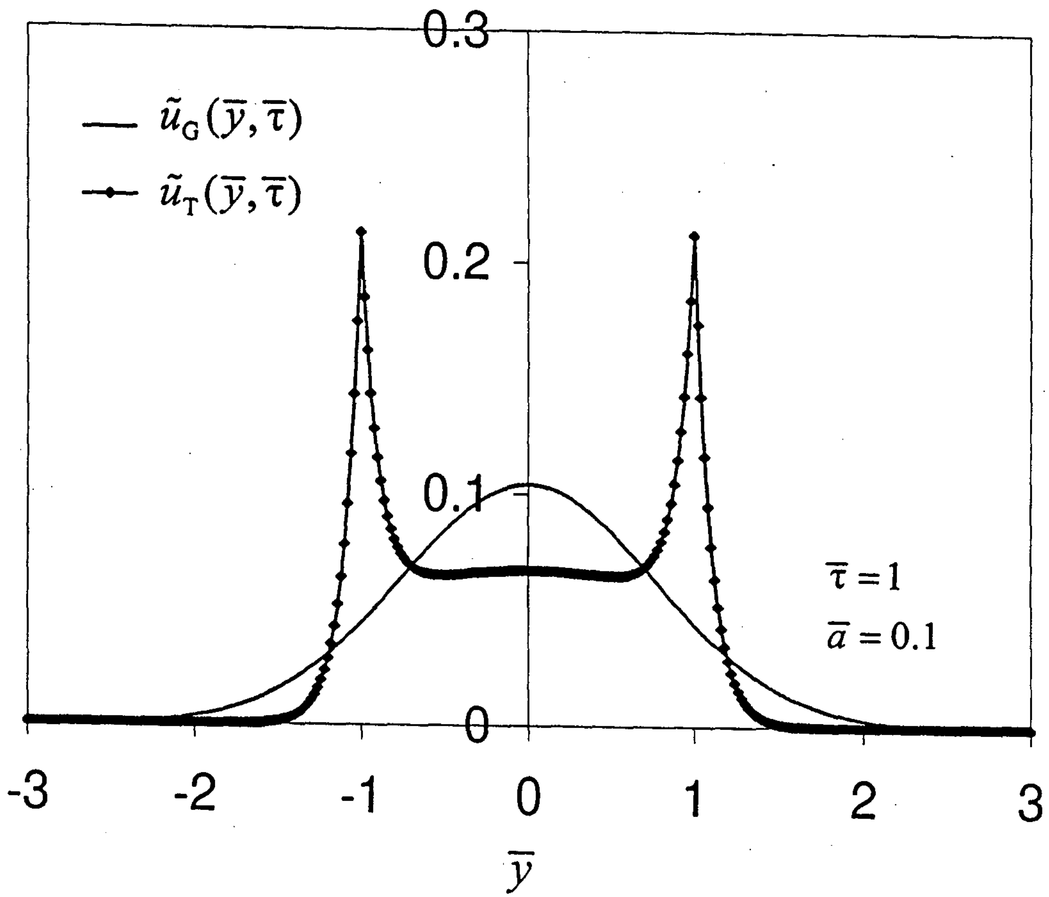

Figure 8. Mean concentration $\tilde{u}_T(\bar{y},\bar{\tau})$ for the telegraph velocity process and $\tilde{u}_G(\bar{y},\bar{\tau})$ for the Gaussian velocity process for $\bar{a}=0.1$ at $\bar{\tau}=1$. The initial concentrations are $\tilde{u}_T(\bar{y},0)=\tilde{u}_G(\bar{y},0)=\exp(-|\bar{y}/0.1|)$.

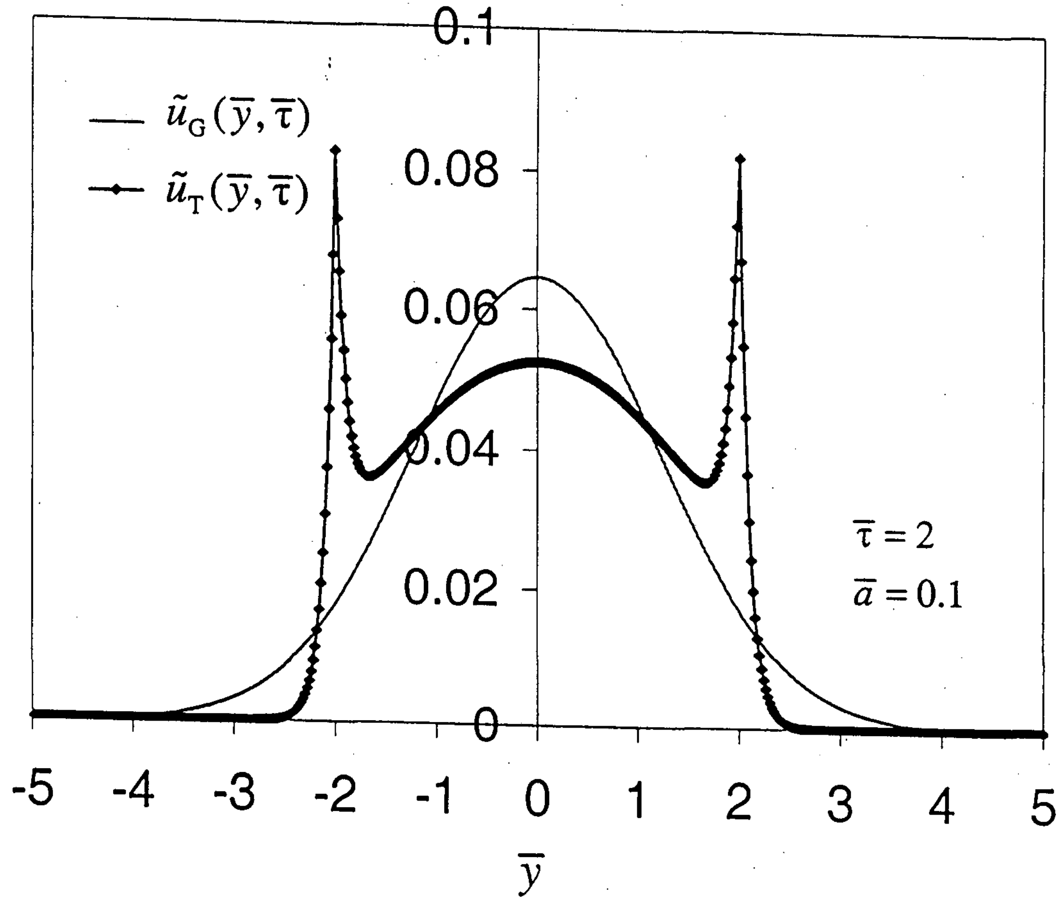

Figure 9. Mean concentration $\tilde{u}_T(\overline{y},\overline{\tau})$ for the telegraph velocity process and $\tilde{u}_G(\overline{y},\overline{\tau})$ for the Gaussian velocity process for $\overline{a}=0.1$ at $\overline{\tau}=2$. The initial concentrations are $\tilde{u}_T(\overline{y},0)=\tilde{u}_G(\overline{y},0)=\exp(-|\overline{y}/0.1|)$.

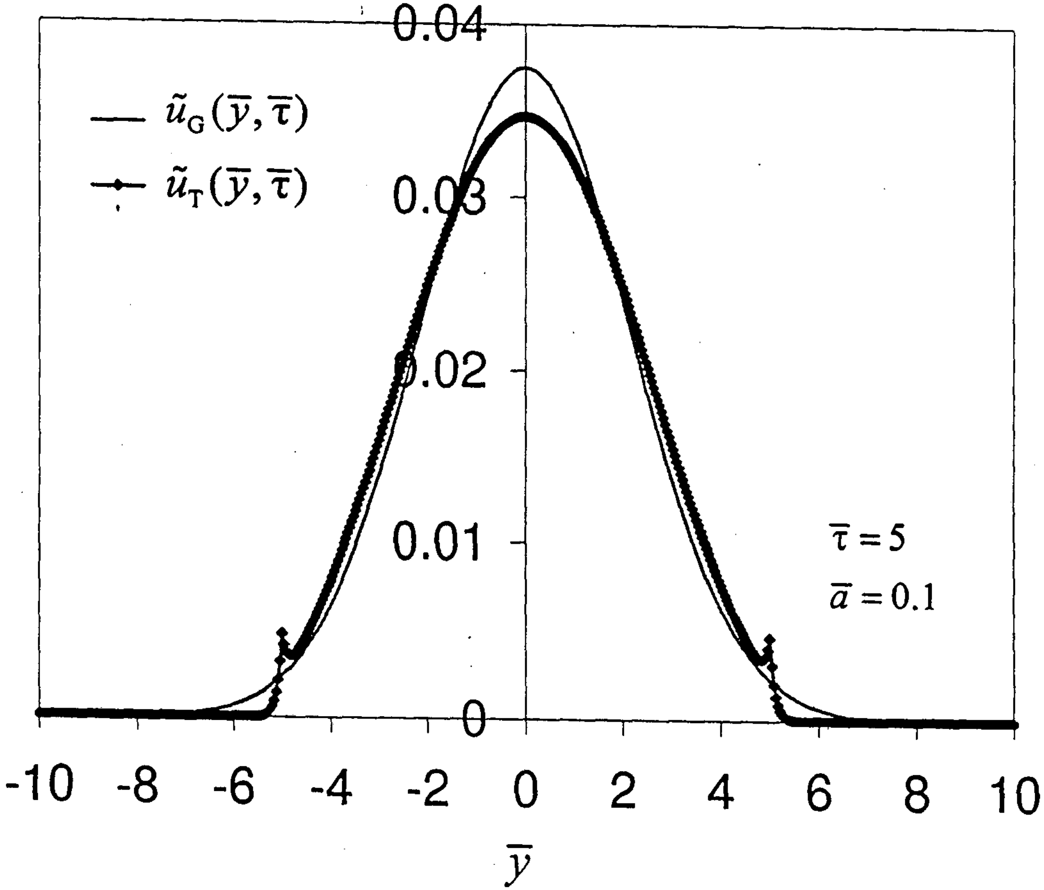

Figure 10 . Mean concentration $\tilde{u}_T(\overline{y},\overline{\tau})$ for the telegraph velocity process and $\tilde{u}_G(\overline{y},\overline{\tau})$ for the Gaussian velocity process for $\overline{a}=0.1$ at $\overline{\tau}=5$. The initial concentrations are $\tilde{u}_T(\overline{y},0)=\tilde{u}_G(\overline{y},0)=\exp\left(-\left|\overline{y}/0.1\right|\right)$.

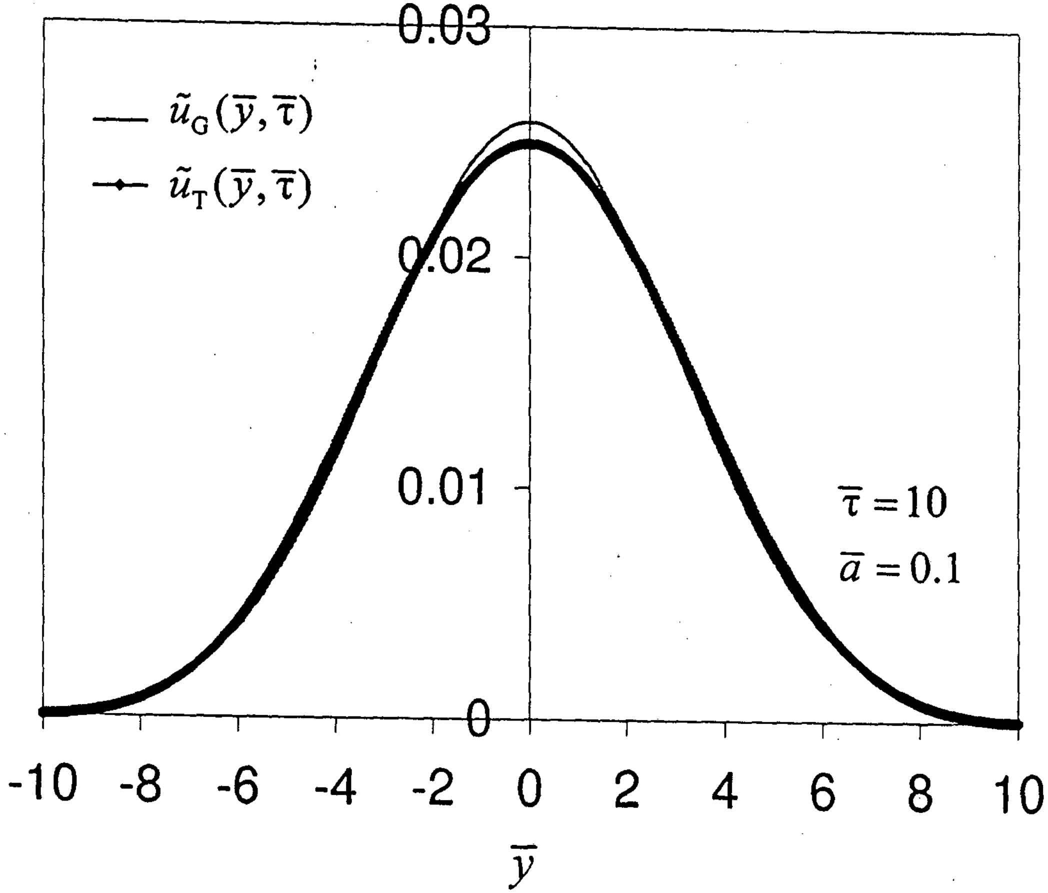

Figure 11. Mean concentration $\tilde{u}_T(\overline{y},\overline{\tau})$ for the telegraph velocity process and $\tilde{u}_G(\overline{y},\overline{\tau})$ for the Gaussian velocity process for $\overline{a}=0.1$ at $\overline{\tau}=10$. The initial concentrations are $\tilde{u}_T(\overline{y},0)=\tilde{u}_G(\overline{y},0)=\exp\left(-\left|\overline{y}/0.1\right|\right)$.

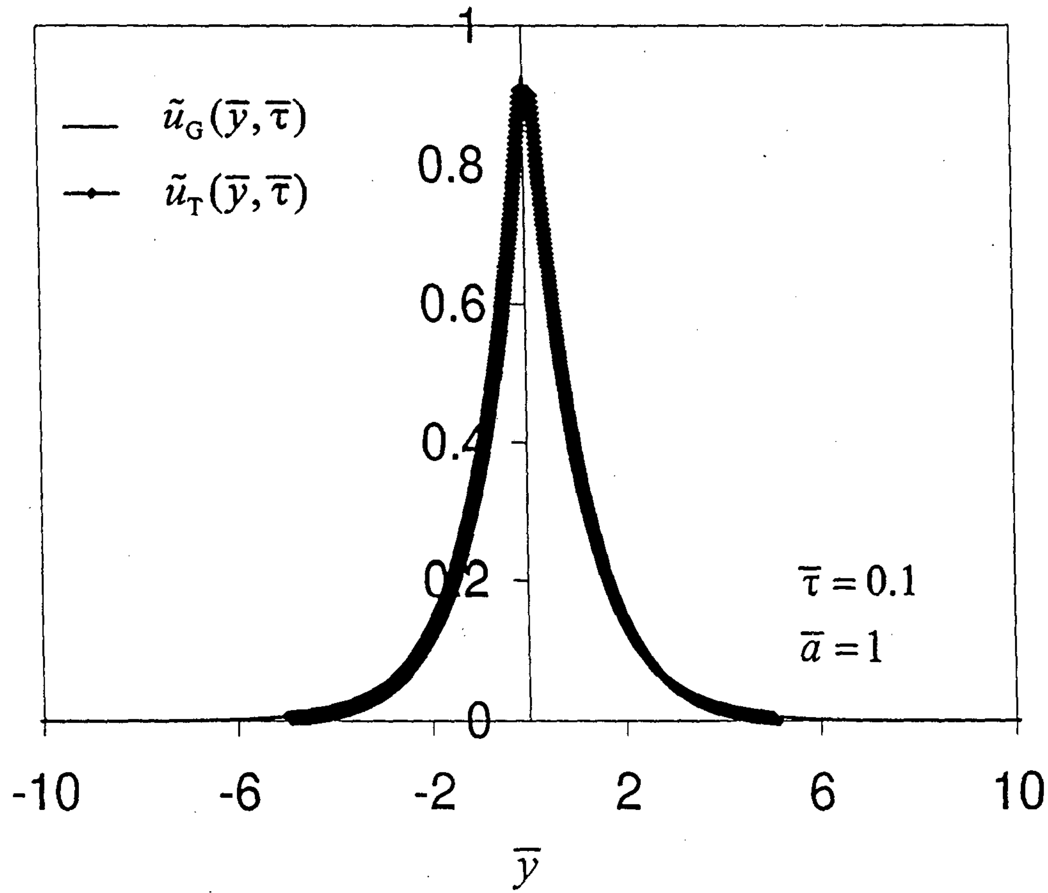

Figure 12. Mean concentration $\tilde{u}_T(\bar{y},\bar{\tau})$ for the telegraph velocity process and $\tilde{u}_G(\bar{y},\bar{\tau})$ for the Gaussian velocity process for $\bar{a}=1$ at $\bar{\tau}=0.1$. The initial concentrations are $\tilde{u}_T(\bar{y},0)=\tilde{u}_G(\bar{y},0)=\exp(-|\bar{y}|)$.

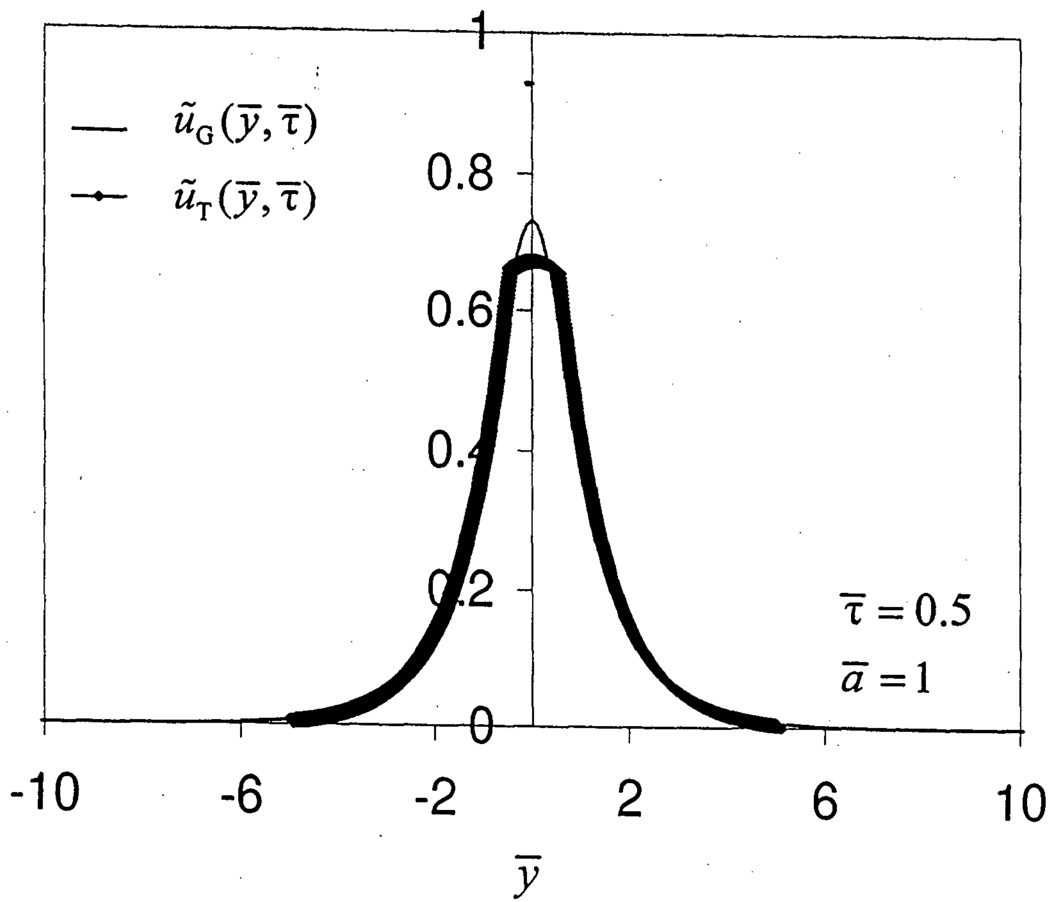

Figure 13. Mean concentration $\tilde{u}_T(\overline{y},\overline{\tau})$ for the telegraph velocity process and $\tilde{u}_G(\overline{y},\overline{\tau})$ for the Gaussian velocity process for $\overline{a}=1$ at $\overline{\tau}=0.5$. The initial concentrations are $\tilde{u}_T(\overline{y},0)=\tilde{u}_G(\overline{y},0)=\exp\left(-\left|\overline{y}\right|\right)$.

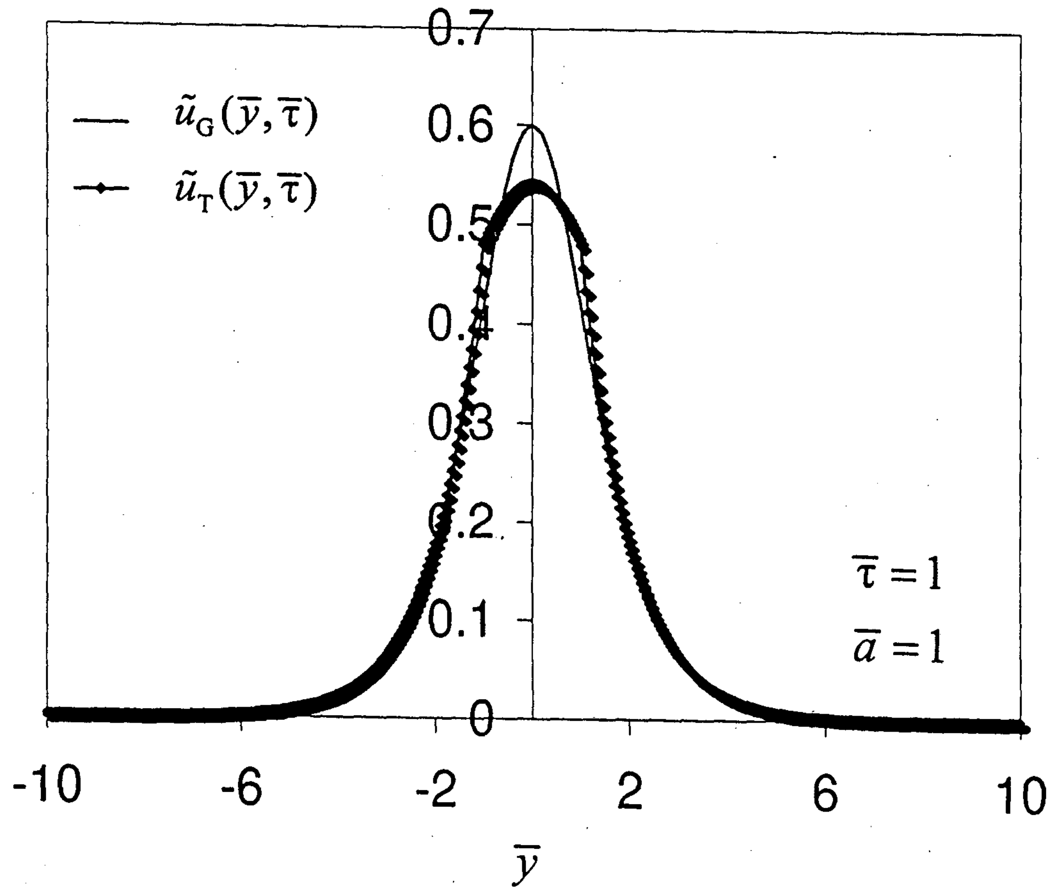

Figure 14. Mean concentration $\tilde{u}_T(\bar{y},\bar{\tau})$ for the telegraph velocity process and $\tilde{u}_G(\bar{y},\bar{\tau})$ for the Gaussian velocity process for $\bar{a}=1$ at $\bar{\tau}=1$. The initial concentrations are $\tilde{u}_T(\bar{y},0)=\tilde{u}_G(\bar{y},0)=\exp(-|\bar{y}|)$.

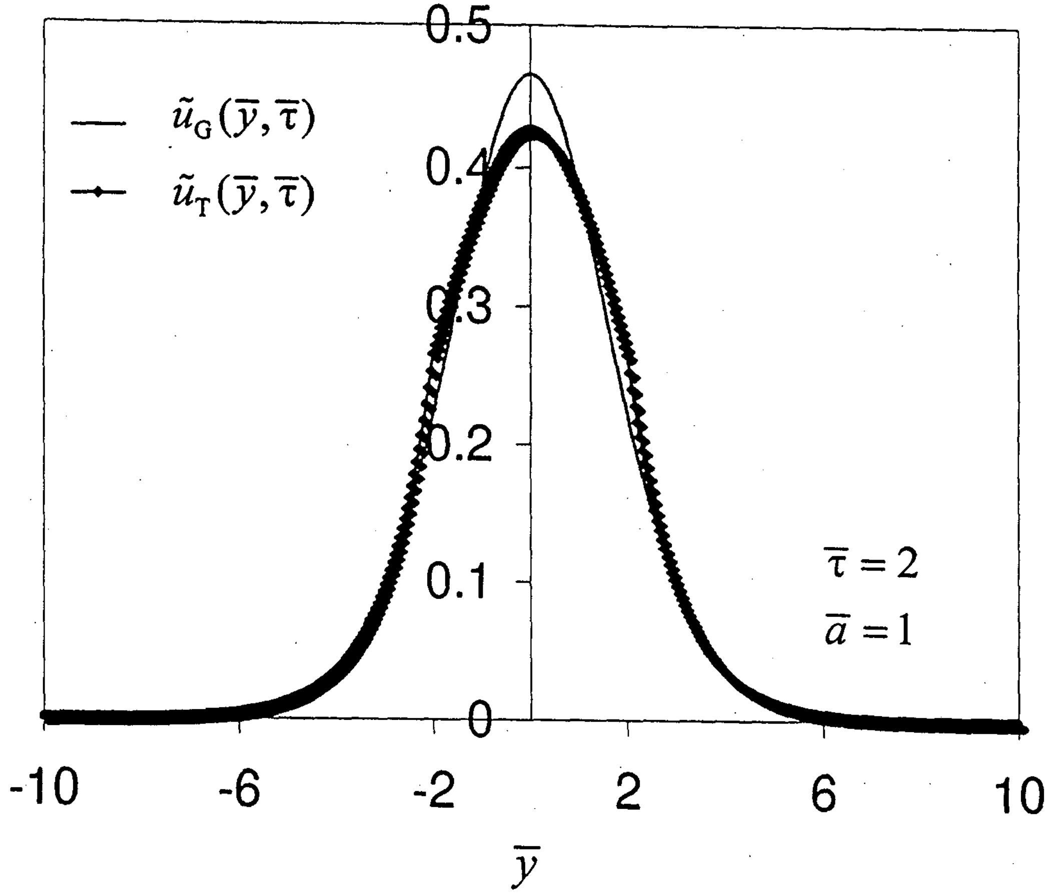

Figure 15. Mean concentration $\tilde{u}_T(\bar{y},\bar{\tau})$ for the telegraph velocity process and $\tilde{u}_G(\bar{y},\bar{\tau})$ for the Gaussian velocity process for $\bar{a}=1$ at $\bar{\tau}=2$. The initial concentrations are $\tilde{u}_T(\bar{y},0)=\tilde{u}_G(\bar{y},0)=\exp(-|\bar{y}|)$.

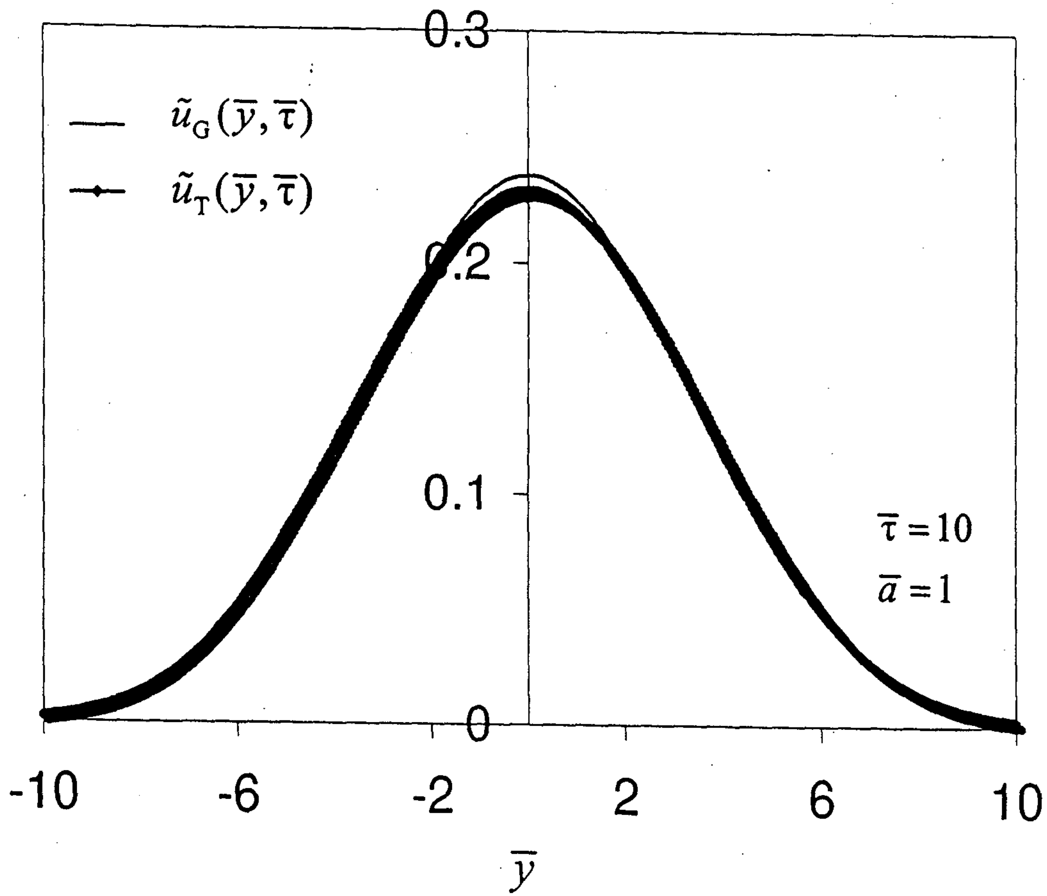

Figure 16. Mean concentration $\tilde{u}_T(\bar{y},\bar{\tau})$ for the telegraph velocity process and $\tilde{u}_G(\bar{y},\bar{\tau})$ for the Gaussian velocity process for $\bar{a}=1$ at $\bar{\tau}=10$. The initial concentrations are $\tilde{u}_T(\bar{y},0)=\tilde{u}_G(\bar{y},0)=\exp(-|\bar{y}|)$.

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Prepared for the U.S. Department of Energy under Contract No. DE-AC03-76SF00098