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Z-Graded Maximal Orders of GK 3

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

James William Berglund

Committee in charge:

Professor Daniel Rogalski, Chair
Professor Ronald Evans
Professor Lance Small
Professor Alexander Vardy
Professor Kenneth Zeger

2012

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Chair

University of California, San Diego

2012

DEDICATION

To my parents.

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ABSTRACT OF THE DISSERTATION

Z-Graded Maximal Orders of GK 3

by

James William Berglund

Doctor of Philosophy in Mathematics

University of California, San Diego, 2012

Professor Daniel Rogalski, Chair

The first Weyl Algebra can be viewed to have \mathbb{Z} -graded quotient ring $Q = k(u)[t, t^{-1}; \sigma]$, and Bell and Rogalski have classified all simple \mathbb{Z} -graded subrings of this quotient ring with Gelfand-Kirillov (GK) dimension 2. In this paper, we seek to understand maximal orders of this quotient ring with GK dimension 3. We start by examining a representative example, $k\langle \frac{1}{u}t, t^{-1} \rangle \subset Q$, and then move on to show that any \mathbb{Z} -graded maximal order $A \subset Q$ must have A_0 be a localization of $k[u]$, or a ring in the form $k[S]$, where S is a σ -closed set of rational functions of the form $\frac{1}{u-a}$. Finally, we completely classify the possible \mathbb{Z} -graded maximal orders inside $k(u)[t, t^{-1}; \sigma]$.

Chapter 1

Introduction

Throughout this paper, let k be an algebraically closed field of characteristic zero.

The Weyl algebra is a celebrated ring in noncommutative ring theory. The first Weyl Algebra, which has presentation

$$A_1 = k\langle x, y \rangle / (yx - xy + 1), \quad (1.1)$$

has especially nice properties. We can view it to be the free algebra over two non-commuting variables x and y over some field k , with the relationship between the x and y being exactly that $yx = xy - 1$. From another perspective, it is the realization of a very natural ring of operators on $k[t]$. In this viewpoint, we view y to correspond to multiplication by t and x to correspond to d/dt . Moreover, it is a simple ring that is not a division ring, which is a property that never happens in commutative algebra, and it also is a simple Noetherian domain, which means that ascending chains of right or left ideals eventually stabilize. While this is a much studied ring, even still people are examining this ring from new perspectives.

A graded ring is a ring R that can be written as a sum of abelian groups $\oplus R_i$, such that $R_i R_j \subset R_{i+j}$. Note that we can consider A_1 as a \mathbb{Z} -graded ring, by assigning x to have degree 1, and y to have degree -1 . When viewed in this way, we can start looking at the graded ring-theoretic structure of the ring. S. Sierra in particular was able to classify all rings that were graded-Morita equivalent to A_1 ; that is, she was able to find all rings that had essentially the same graded-module

structure [11]. This result inspired J. Bell and D. Rogalski to also examine the graded structure of rings similar to the Weyl algebra. The first thing they noted was that if we set $xy = t$, we obtain a different point of view of A_1 ; the graded quotient ring is isomorphic to a skew-Laurent ring

$$Q(A_1) \cong k(u)[t, t^{-1}; \sigma], \quad (1.2)$$

where σ is the automorphism that has $\sigma(u) = u + 1$. In other words, we can view this ring to be all the Laurent polynomials with coefficients in $k(u)$, with the relationship that $tf(u) = \sigma(f(u))t$. They then endeavored to try to classify all \mathbb{Z} -graded GK 2 simple rings that were subrings of $k(u)[t, t^{-1}; \sigma]$. GK dimension is a noncommutative analogue of classical Krull dimension, and gives the asymptotic growth of a generating set of the algebra. So in other words, they wanted to find all comparatively small subrings of $k(u)[t, t^{-1}; \sigma]$, which they were able to do.

Theorem 1.3 (Bell-Rogalski [3]). *A ring R is a \mathbb{Z} -graded simple subring of $k(u)[t, t^{-1}; \sigma]$ that has GK dimension 2, where k has characteristic zero, if and only if it is isomorphic to one of the following forms: Let $R_0 = k[u]$, and choose $c \in k(u)$ with $c\sigma(c) \cdots \sigma^n(c) \in k[u]$ for sufficiently large n . Then if we set $s = c^{-1}t$, then*

$$R = k[u][t, t^{-1}; \sigma] \cap k[u][s, s^{-1}; \sigma]. \quad (1.4)$$

So with this result, in this thesis the goal is now to try to relax the simple GK 2 restrictions on the ring, and simply classify all \mathbb{Z} -graded maximal orders in $k(u)[t, t^{-1}; \sigma]$. Maximal orders are the noncommutative analogue to integrally closed rings, which correspond in dimension one to smooth curves in algebraic geometry, so this is a natural restriction to make. Now, to begin with, in this paper we will examine an example that arose in the paper of Bell and Rogalski: $A = k\langle \frac{1}{u}t, t^{-1} \rangle \subset k(u)[t, t^{-1}; \sigma]$, the k -algebra generated by the elements $\frac{1}{u}t$ and t^{-1} . The ring A has some interesting properties:

Theorem 1.5. *Let $A = k\langle \frac{1}{u}t, t^{-1} \rangle$. Then the zeroth degree piece A_0 is not finitely generated as a k -algebra. Additionally, A is Noetherian, has classical Krull dimension 2, and is a maximal order.*

Moreover, we will be able to specifically describe the structure of the prime ideals and simple graded modules of A . It turns out that A_0 in this example is special. We will show that:

Theorem 1.6. *Let $R \subset k(u)$. Then if R has field of fraction $k(u)$ and is both σ -closed and integrally closed, then R is the localization of some polynomial ring. In particular, either $R = k[u]S^{-1}$, for some σ -closed set S , or*

$$R = \left\{ \frac{f}{g} \mid \deg(f) \leq \deg(g), g \text{ only has factors in some } \sigma\text{-closed set } S \right\}. \quad (1.7)$$

Now, we want to restrict our search for subrings of $k(u)[t, t^{-1}; \sigma]$ that are maximal orders. The integrally closed subrings of $k(u)$ turn out to be critical in the classification of these type of rings.

Theorem 1.8. *If $R \subset k(u)[t, t^{-1}; \sigma]$ is a maximal order, then R_0 is integrally closed and σ -closed.*

So we can restrict our search for maximal orders A to two cases: one where A_0 is some localization of a polynomial ring, and the other where A_0 is a particular ring of rational functions as in 1.7. The first case is similar to the case considered by the paper of Bell and Rogalski [3], but the second case is quite different, as extra ring structure develops. Since A_0 is a PID, we can write $A = \bigoplus_{n=-\infty}^{\infty} a_n A_0 t^n$, for some choice of $a_n \in k(u)$, and we will refer to the a_n as structure constants. We can view each $a_n = b_n c_n$, where $\sigma(c_n)A_0 = c_n A_0$, and $\sigma(b_n)A_0 \neq b_n A_0$. It turns out after suitably adjusting t , we can adjust the b_n so that they are eventually constant for sufficiently high n . The c_n portion of each structure constant behaves in a regular, periodic manner. Putting these together, we get the following main theorem.

Theorem 1.9. *Let A be a finitely-generated \mathbb{Z} -graded algebra that is an order of $k(u)[t, t^{-1}; \sigma]$, that has A_0 being a ring of all rational functions of degree at most zero, with poles at some set of points that is closed under σ , as in 1.7. Then after an appropriate choice of t and s , there exists some choice of n, x, y that induce a sequence m_i , where $m_i = \lfloor -\frac{xi}{n} \rfloor$ when $i < 0$ and $m_i = \lfloor \frac{yi}{n} \rfloor$ for $i > 0$. Consider*

$$R = \left(\bigoplus u^{m_i} A_0 t^i \right) \cap \left(\bigoplus u^{m_i} A_0 s^i \right). \quad (1.10)$$

Then R is a maximal order for A .

This problem has many natural questions arising from it. For example, in the Bell and Rogalski paper [3], they showed there were two fundamental automorphisms that arose in this situation: one that sends u to $u + 1$, which is the case we studied, and one that is the automorphism that sends u to cu , where c is not a root of unity. So examining the situation with the new automorphism might yield interesting results.

We can also more dramatically change the base field as well. Instead of taking the quotient field to be $k(u)[t, t^{-1}; \sigma]$, instead we can take it to be $k(X)[t, t^{-1}; \sigma]$ for some elliptic curve X . Elliptic curves have a natural translation operation and reasonable divisor structure, so similar methods as used in this paper should be applicable in order to classify the maximal orders in this quotient ring. Furthermore, instead of considering $k(u)$ as the base field, we can consider rational function fields of higher transcendence degree. In the Bell and Rogalski paper, they were able to classify simple rings of minimal GK dimension over fields with any transcendence degree, so it is possible their results could extend to classify the maximal orders of these quotient rings as well.

Finally, it would be interesting to consider the module structure of this class of rings. In a paper by P. Smith [12], he was able to find a commutative graded ring that had that had the same module structure as the first Weyl algebra. So, it might be possible to find a similar commutative graded ring that has the same module structure as $k\langle \frac{1}{u}t, t^{-1} \rangle$.

Chapter 2

Background

2.1 Localization

An important tool for studying rings in general is the idea of localization.

Definition 2.1. A set S is a multiplicative subset of R if for all $s_1, s_2 \in S$, $s_1 s_2 \in S$.

Let R be a commutative domain, and let S be a multiplicative set of R not containing 0. The idea is we want to construct a ring that R embeds into that has inverses at exactly the elements of S .

Definition 2.2. The localization RS^{-1} is the ring $(R \times S)/\sim$, where \sim is the equivalence relation $(r_1, s_1) \sim (r_2, s_2)$ if $r_1 s_2 - s_1 r_2 = 0$. This ring has multiplication defined by $(r_1, s_1)(r_2, s_2) = (r_1 r_2, s_1 s_2)$ and $(r_1, s_1) + (r_2, s_2) = (r_1 s_2 + r_2 s_1, s_1 s_2)$. We denote the element (r, s) in this ring as rs^{-1} .

The most important property of localization is that it preserves the ideal structure of the ring in a strong fashion.

Theorem 2.3. *Let I be an ideal of a commutative domain R such that $I \cap S = \emptyset$. Then there is a corresponding ideal IS^{-1} in RS^{-1} , and moreover this correspondence respects the inclusion of ideals. Furthermore, all ideals of RS^{-1} have form IS^{-1} for some $I < R$.*

Proof. See [6, III.4.7], and [6, III.4.9]. □

We want to also do this in the non-commutative case, however the process is much trickier. The concern is that if we have $r_1 s_1^{-1} r_2 s_2^{-1}$, moving the r_2 past the s_1^{-1} might be impossible. In other words, for all $s \in S$ and $r \in R$, we need there to exist some r' and s' such that $s^{-1}r = r's'^{-1}$, or in other words

$$rs' = sr'. \quad (2.4)$$

Definition 2.5. If a multiplicative set $S \subset R$ has the property that for all $r \in R$ and $s \in S$, there exists some $r' \in R$ and $s' \in S$ that satisfies the condition found in 2.4, or in other words that $rS \cap sR \neq \emptyset$, we say S satisfies the right Ore condition.

Theorem 2.6. *If $S \subset R$ consists only of regular elements and satisfies the Ore condition, we can form the localization RS^{-1} , similar to the commutative case. If RS^{-1} is right Noetherian, then for all ideals $I < R$ that have $I \cap S = \emptyset$, there is a corresponding ideal $IS^{-1} < RS^{-1}$. Additionally, all ideals in RS^{-1} have form IS^{-1} for some $I < R$.*

Proof. See [8, 2.1.12] and [8, 2.1.16]. □

It turns out that wide classes of rings automatically satisfy this property.

Theorem 2.7. *Let S be the set of all nonzero elements of R . If R is a right Noetherian domain, then S satisfies the Ore condition.*

Proof. See [8, 2.1.15]. □

Definition 2.8. If in a domain the set S of all nonzero elements satisfies the Ore condition, we call RS^{-1} the right quotient ring of R .

2.2 Orders in Rings

In algebraic geometry, the notion of integral closure is very important, since integrally closed rings correspond to normal geometric objects, and smooth geometric objects in dimension 1. Thus, we are motivated to use a similar notion for noncommutative rings. We will develop a notion of orders in quotient rings, and then maximal orders, following the presentation of McConnell and Robson [8]. Let R be a subring of Q , its quotient ring.

Definition 2.9. The ring R is considered a right order in Q if each $q \in Q$ has form rs^{-1} for some $r, s \in R$. R is a left order if each $q \in Q$ has the form $s^{-1}r$. If a ring R is both a left and right order, it is considered an order.

It turns out there is a natural way we can define an equivalence relation on the right orders of a quotient ring.

Lemma 2.10. *Let R be a right order in a quotient ring Q and let S be a subring of Q . Suppose there are units a and b of Q such that $aRb \subset S$. Then S is a right order in Q .*

Proof. Consider for each $q \in Q$ the element $a^{-1}qa$. Since R is a right order, there are $r, t \in R$ with $a^{-1}qa = rt^{-1}$. But now

$$q = art^{-1}a^{-1} = arbb^{-1}t^{-1}a^{-1} = (arb)(atb)^{-1}.$$

Therefore, every element of q can be written in the form $s_1s_2^{-1}$, and thus S is a right order of Q . \square

This leads us to make a natural equivalence relation on the right orders of Q .

Definition 2.11. R and S are considered equivalent right orders of Q if there are units $a_i, b_i \in Q$ such that $a_1Rb_1 \subset S$, and $a_2Sb_2 \subset R$.

So now, in each equivalence class of right orders, there potentially are elements that are maximal with respect to inclusion. We will call these orders maximal (right) orders. This is an important concept since the restriction to the commutative case returns the definition of an integrally closed domain.

Lemma 2.12. *Let R be a commutative ring that is a right order of Q . Then R is a maximal order if and only if R is completely integrally closed, that is, if there are $a, q \in Q$ with $aq^n \in R$ for all n , then $q \in R$.*

Proof. Let R be a maximal order. Then for any a, q that has $aq^n \in R$ for all n , then note $aR[q] \subset R$. So R and $R[q]$ are equivalent orders, and since $R \subset R[q]$, this implies that $R = R[q]$. But then $q \in R$, as needed. For the other direction, say

R is completely integrally closed, and let $R \subset R'$ be an equivalent order. So there is some $a \in Q$ with $aR' \subset R$. Take any $q \in R'$, and note that $aR[q] \subset aR' \subset R$. But now $aq^n \in R$ for all n . But since R is completely integrally closed, this means $q \in R$. So thus $R = R'$, and R is a maximal order. \square

Lemma 2.13. *For a commutative Noetherian ring, R is integrally closed if and only if R is completely integrally closed.*

Proof. As a reminder, R is integrally closed if for any element q in the quotient ring of R that is a zero of some monic polynomial in $R[x]$, then $q \in R$.

Say R is completely integrally closed. Note for all $q \in Q$ with q that is a root of some monic polynomial $p(x) \in R[x]$, that $R[q]$ is a finitely-generated module. Say $p(x)$ has degree m . Then, $q \in Q(R)$, so $q = \frac{r_1}{r_2}$, with $r_i \in R$. Consider $R[q]$ as a R -module. Now $r_2^m R[q] \subset R$, since $\{1, q, \dots, q^{m-1}\}$ generate $R[q]$. Thus $r_2^m q^n \in R$ for all n , so since R is completely integrally closed then $q \in R$. Therefore, R is integrally closed. Say R was integrally closed, and let $a, q \in Q$ be such that $aq^n \in R$ for all n . Note $R[q] \subset a^{-1}R$ as R -modules, and since R is Noetherian, $a^{-1}R$ is also Noetherian as an R -module, so thus $R[q]$ is finitely generated. But then for sufficiently high n , q^n can be written as a R -linear sum of lesser powers, so thus there is a monic polynomial that has q as a root. Thus $q \in R$, as needed. \square

So thus the theory of maximal orders is the noncommutative analogue of integral closure for commutative Noetherian rings.

It will be useful to have a way to check if an order is maximal, so the next few propositions will lead to a result that we can use to check to see if an order is maximal. First, the following result is given without proof:

Lemma 2.14. *Suppose R and S are equivalent right orders in Q with $R \subset S$. Then there are equivalent right orders T and T' in Q such that $R \subset T, T' \subset S$, and units r_1, r_2 of Q in R such that $r_1 S \subset T$, $T r_2 \subset R$, $S r_2 \subset T'$, and $r_1 T' \subset R$. Thus, $r_1 S r_2 \subset R$.*

Definition 2.15. The right order of an ideal $I < R$ is the set

$$O_r(I) = \{q \in Q \mid Iq \subset I\}. \quad (2.16)$$

Analogously, the left order of the ideal is the set

$$O_l(I) = \{q \in Q \mid qI \subset I\}. \quad (2.17)$$

Lemma 2.18. *$O_r(I)$ and $O_l(I)$ are right orders in Q equivalent to R .*

Proof. Note that for any $x \in I$, that $xO_r(I) \subset I \subset R$, and $R \subset O_r(I)$ by property of R being an ideal of I . So thus R and $O_r(I)$ are equivalent orders. Also, note that $O_l(I)x \subset I \subset R$, and again $R \subset O_l(I)$, so R and $O_l(I)$ are equivalent orders. \square

So with this machinery, we can develop this method to check for maximal orders:

Proposition 2.19. *Let R be an order in Q . Then R is a maximal order if and only if $O_r(I) = O_l(I) = R$ for all ideals I of R .*

Proof. For the forward direction, if R is a maximal order, note by 2.18 that $O_r(I)$ and $O_l(I)$ are equivalent orders to R that contain R . So necessarily $O_r(I) = O_l(I) = R$.

For the backwards direction, say there is some S such that $R \subset S$, and R and S are equivalent orders. So by Lemma 2.14 there exists some T such that $R \subset T \subset S$, and some $r_1, r_2 \in R$ with $r_1S \subset T$ and $Tr_2 \subset R$. Let $I = \{x \in R \mid Tx \subset R\}$. Then I is a non-empty ideal of R , and moreover $T \subset O_l(I)$, as for any $x \in I, t \in T$, we have $tx \in R$ by definition of I , and moreover, $tx \in I$ as $Ttx \subset Tx \subset R$. So thus $T = R$. Likewise, let $J = \{x \in R \mid xS \subset T\}$. Again, J is a non-empty ideal of R , and $S \subset O_r(I)$, as for any $x \in I, s \in S$ we have $xs \in T = R$, and then $xsS = xS \subset T$. So $S = R$, as needed, and thus R is a maximal right order. \square

2.3 Grading in Rings and Modules

Let R be a ring, and G be an Abelian group, which for our purposes will be \mathbb{Z} under addition.

Definition 2.20. The ring R is G -graded if R can be written as a direct sum of Abelian groups R_g in the form $R = \bigoplus_{g \in G} R_g$, where for all $g_1, g_2 \in G$, $R_{g_1} R_{g_2} \subset R_{g_1 + g_2}$.

Definition 2.21. A \mathbb{Z} -graded ring R is considered \mathbb{N} -graded if $R_i = 0$ for all $i < 0$.

Example 2.22. For example, $\mathbb{Z}[x]$ is a \mathbb{N} -graded ring, as $\mathbb{Z}[x] = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}x^n$. Generally, any polynomial ring is \mathbb{N} -graded, with multiple choices for the grading. For example, $\mathbb{Z}[x, y] = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}[x]y^n$, in which case we create our grading in terms of the degree in y . However, we can also grade $\mathbb{Z}[x, y] = \bigoplus_{n \in \mathbb{N}} \sum_{i=0}^n \mathbb{Z}x^i y^{n-i}$, in which case we graded according to the total degree in each monomial.

Example 2.23. $A_1 = k\langle x, y \rangle / (yx - xy - 1)$ is \mathbb{Z} -graded, with y having degree 1, and x having degree -1 .

Definition 2.24. If R is G -graded, $r \in R$ is a *homogeneous element* if there is some $g \in G$ such that $r \in R_g$.

Example 2.25. In the second grading discussed above for $\mathbb{Z}[x, y]$, $x^2y + 3y^3$ is a homogeneous element. However, it is not homogeneous in the first grading.

Definition 2.26. In a graded ring, an ideal is homogeneous if it is generated by only homogeneous elements. If an ideal is homogeneous, we can write it as $I = \bigoplus_{g \in G} I_g$.

Example 2.27. In $\mathbb{Z}[x]$, the ideal $I = \{f | f(0) = 0\}$, is a homogeneous ideal, since $I = (x)$. However the ideal $I = (x + 2)$ is not homogeneous.

Lemma 2.28. *If I is a homogenous ideal in a graded ring R , then R/I is a graded ring as well, with $(R/I)_g = R_g/I_g$.*

We can associate graded modules to graded rings.

Definition 2.29. Let R be G -graded. A G -graded right module $M = \bigoplus_{g \in G} M_g$ is a R -module such that for all $g_1, g_2 \in G$, $M_{g_1} R_{g_2} \subset M_{g_1 g_2}$.

We can also define a graded quotient ring to a graded domain, if the set of homogeneous elements satisfies the Ore condition.

Definition 2.30. Let S be the set of all nonzero homogeneous elements of a domain R . If S satisfies the Ore condition, then we say $Q_{gr}(R) = RS^{-1}$, and we refer to this ring as the right graded quotient ring of R . It is the smallest ring that has inverses to all homogeneous elements.

It turns out now to check for maximal orders in graded rings, it is sufficient only to check graded ideals and look in the graded quotient ring, as this next result adapted from a paper of Rogalski shows [10, 9.1]:

Lemma 2.31. *Let R be a \mathbb{Z} -graded graded domain which has graded quotient ring D and right quotient ring Q . Then R is a maximal order if and only if $O_r^g(I) = R = O_l^g(I)$ holds for all homogenous nonzero ideals I of A , where $O_r^g(I) = \{q \in D \mid Iq \subset I\}$, and $O_l^g(I) = \{q \in D \mid qI \subset I\}$.*

Proof. Write $D = K[t, t^{-1}; \sigma]$, for some division ring K and automorphism σ on K . We know K is a maximal order in itself, so thus it follows that D is a maximal order in Q [7, IV.2.1, V.2.3].

So now, for the nontrivial direction, say $O_r^g(I) = R = O_l^g(I)$ for all homogeneous ideals I . Take some ideal $I < R$, and say there is $q \in Q$ with $qI \subset I$. By ideal correspondence, ID is a two-sided ideal of D , thus we have $qID \subset ID$. But D is a maximal order, so this forces $q \in D$. So we can write $q = \sum_{i=m}^n d_i$, with $d_i \in D_i$.

Let \tilde{I} be the homogeneous ideal generated by all leading coefficients in I . Note we must have $d_n \tilde{I} \subset \tilde{I}$, so thus $d_n \in O_l^g(\tilde{I})$. Therefore, $d_n \in R$. But since $qI \subset I$ and $d_n I \subset I$ now, we see that $(q - d_n)I \subset I$, so $q - d_n \in O_l(I)$. We can repeat this argument though, and see that all the $d_i \in A$. So $q \in A$, as needed. Thus, $O_l(I) = A$. We can repeat this argument switching the direction of everything, to get that $O_r(I) = A$ as well, as needed. \square

2.4 Skew-Polynomial Rings

Let R be a ring with 1. There is an easy way to make a polynomial-like ring off R that shares several properties with the polynomial ring.

Definition 2.32. Let R be a ring and σ be an automorphism of R . The skew-polynomial ring $R[u; \sigma]$ is the ring that consists of elements of the form $a_n u^n + \cdots + a_0$, along with the property that $ua = \sigma(a)u$.

Example 2.33. Let $R = \mathbb{C}$, and σ send z to \bar{z} . Then $R[u; \sigma]$ is a skew-polynomial ring, and as an example

$$(1 + 3i)u^3(1 + 2i)u = (1 + 3i)\sigma^3(1 + 2i)u^4 = (1 + 3i)(1 - 2i)u^4 = (7 + i)u^4. \quad (2.34)$$

First note that skew-polynomial rings have a natural notion of degree, similar to polynomials, and thus have a natural \mathbb{N} -grading. Also like a polynomial ring, the properties of R help determine the properties of $R[u, \sigma]$.

Proposition 2.35. *If R is a integral domain, $R[u; \sigma]$ is an integral domain. If R is a Noetherian ring, so is $R[u; \sigma]$.*

Proof. If R is an integral domain, note that

$$(a_n u^n + \cdots + a_0)(b_m u^m + \cdots + b_0) = a_n \sigma^{-n}(b_m) u^{n+m} + \cdots + a_0 b_0 \quad (2.36)$$

and since σ is an automorphism, $b_m \neq 0$, so thus this is nonzero as the highest degree term has a nonzero coefficient. For the proof of the second statement, see [8, 1.2.9]. \square

Finally, it will be nice to know if the graded quotient ring of a skew-polynomial ring $R[u; \sigma]$ exists, and it does.

Proposition 2.37. *Let R be a commutative ring, and let S be the set of all nonzero homogeneous elements in $A = R[u; \sigma]$. Then S satisfies the Ore condition.*

Proof. First, we claim that we only have to consider the Ore condition for homogeneous elements in R . Suppose we knew that for any choice of homogeneous $r_1, r_2 \in R$ and $s \in S$, that there exist $t, v \in S$ and $x, w \in R$ such that $r_1 t = sw$ and $r_2 v = sx$. Find homogeneous a and b such that $ua = vb$. Thus we have that $r_1(ua) = s(wa)$ and $r_2(vb) = s(xb)$. Therefore, we have $(r_1 + r_2)(ua) = s(wa + xb)$,

so $r_1 + r_2$ and s satisfy the Ore condition as well. Now, fix some $rt^n \in A_n$ and $st^m \in S$, and note that $\sigma^{-n}(s)t^m \neq 0$, as σ is an automorphism, and thus,

$$rt^n \sigma^{-n}(s)t^m = st^m \sigma^{-m}(r)t^n. \quad (2.38)$$

Therefore, $rS \cap sR \neq \emptyset$, as needed. So S satisfies the Ore condition. \square

2.5 Dimension in Rings

In commutative ring theory, a natural notion to study the size of a k -algebra is Krull Dimension.

Definition 2.39. The classical Krull dimension of a ring R is the supremum of lengths n of chains of prime ideals $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$.

Example 2.40. $k[x_1, x_2, \dots, x_n]$ has Krull dimension n , while \mathbb{Z} has Krull dimension 1.

This definition is not quite as appropriate in non-commutative rings though, since prime ideals might be sufficiently scarce to not say anything meaningful for the ring. So the wish is to generalize this notion into something that coincides with Krull dimension in commutative rings, but also has meaning in non-commutative rings. This motivation lead to the creation of Gelfand-Kirillov dimension (GK dimension).

Definition 2.41. Let R be a finitely-generated k -algebra, with a finite dimensional generating k -subspace V that contains 1. The GK dimension of R is

$$\limsup_{n \rightarrow \infty} \log(\dim_k V^n) / \log(n). \quad (2.42)$$

This definition is fairly hard to follow initially, so some examples are in order:

Example 2.43. Let $R = k[x, y]$. A generating set of R is $V = k\{1, x, y\}$, and V^n has a basis of $\frac{(n+1)(n+2)}{2}$ elements in it, as there are exactly $n + 1$ elements of

exactly degree n , and V_n contains all elements of degree less than or equal to n .

So as $n \rightarrow \infty$,

$$\frac{\log(\dim V^n)}{\log(n)} \rightarrow 2, \quad (2.44)$$

and thus the GK dimension of R is 2.

Example 2.45. Let R be the first Weyl Algebra: $R = k[x, y]/(yx - xy - 1)$. Take $V = k\{1, x, y\}$. Then it can be shown that a basis for V^n can be given by elements of the form $\{x^i y^j \mid i + j \leq n\}$. So the previous analysis holds, and R again has GK dimension 2.

Generally, a ring having GK dimension n corresponds to the growth of V^n being asymptotically polynomial with exponent n . A natural question to ask regarding GK dimension is whether it is invariant no matter the generating set, and it turns out it is.

Theorem 2.46. *GK dimension is well defined.*

Proof. See [8, 8.1.10]. □

Chapter 3

The Major Example

The major example we first examine is $A = k\langle \frac{1}{u}t, t^{-1} \rangle \subset k(u)[t, t^{-1}; \sigma]$, with σ the automorphism sending u to $u + 1$, and it turns out this is a representative example. In this section we will explore all the ring-theoretic properties of this ring, and see that many of the same techniques can be used in the more general case to classify maximal orders. To begin with, just to define some notation that will be used:

Definition 3.1. Let $A^+ = \bigoplus_{n=0}^{\infty} A_n$, and $A^- = \bigoplus_{n=-\infty}^0 A_n$. That is, A^+ is the \mathbb{N} -graded subring of A that consists of the non-negative degree graded pieces of A , and A^- is the \mathbb{N} -graded subring that consists of all the non-positive degree pieces of A .

First off, A turns out to have a very nice characterization:

Proposition 3.2. A_0 is the ring of rational functions of degree at most 0 with poles at the integers. Moreover, A^+ has the form $A_0[w; \sigma]$, where $w = \frac{1}{u}t$, and A^- has the form $A_0[v; \sigma^{-1}]$, where $v = t^{-1}$.

Proof. Note that $(\frac{1}{u}t)^n t^{-n}$ is in A_0 . Expanding this out, we get a rational function f_n with

$$f_n = \left(\frac{1}{u}t\right)^n t^{-n} = \frac{1}{u} \frac{1}{u+1} \cdots \frac{1}{u+(n-1)}.$$

First, let us establish that $k[\{\frac{1}{u-c}\}_{c \in \mathbb{Z}}] \subset A_0$. Consider the set $\{f_0, f_1, \dots, f_n\}$. If we bring this set of polynomials to their common denominator $u(u+1) \cdots (u+(n-1))$,

note that the numerator of f_0 will be a degree n polynomial, the numerator of f_1 will be a degree $n - 1$ polynomial, and so on. So thus we easily see that the numerators of these rational functions span the space of polynomials with degree at most n . Thus, by considering the span of $\{f_0, \dots, f_n\}$, we can obtain elements in A_0 of the form $\frac{p(u)}{u(u+1)\dots(u+n-1)}$, with p being an arbitrary polynomial of degree at most n . In particular, we can get $\frac{1}{u+c}$ for any choice of $c \in \mathbb{Z}^{\geq 0}$. For $c \in \mathbb{Z}^-$, we repeat this construction, but we look at elements of $t^{-n} \left(\frac{1}{u}\right)^n$ to obtain the rational functions with denominator $u - c$. Note that naturally $A_0 \subset k[\{\frac{1}{u-c}\}_{c \in \mathbb{Z}}]$, as A_0 is generated by $\frac{1}{u}t$ and t^{-1} , and σ sends integer poles to other integer poles. Finally, this ring has the description as given by virtue of the theory of partial fractions.

For the description of the full ring, we can proceed by induction on the length of word of the generators. Say we knew that all words of length n in $\frac{1}{u}t$ and t^{-1} can be written either in the form A_0 , $A_0 \left(\frac{1}{u}t\right)^m$ or $A_0(t^{-1})^m$, for some $m \geq 1$. So consider a word of length $n + 1$. Set aside the left-most term of the word, and note the other terms form a word of length n . If our initial term was $\frac{1}{u}t$, our word can be rewritten either in the form $\frac{1}{u}tA_0$, $\frac{1}{u}tA_0 \left(\frac{1}{u}t\right)^m$ or $\frac{1}{u}tA_0(t^{-1})^m$. All of these can be changed to the correct form though: the first can be written $\sigma(A_0)\frac{1}{u}t$, the second $\sigma(A_0) \left(\frac{1}{u}t\right)^{m+1}$, and the third as $\sigma(A_0)\frac{1}{u}(t^{-1})^{m-1}$. As $\sigma(A_0) \subset A_0$, and $\frac{1}{u}A_0 \subset A_0$, we see in all the cases we have an element of the correct form. If the initial term was t^{-1} , the analysis is essentially the same. Thus we see that $A^+ = A_0[w; \sigma]$, and $A^{-1} = A_0[v; \sigma^{-1}]$, as needed. \square

With the characterization in hand, we can proceed to classify the prime ideals of A_0 , and we get the following.

Proposition 3.3. *The prime ideals of A_0 are exactly the ideals of the form $(\frac{1}{u} + c)$, where $c \in k - \{\frac{1}{n} \mid n \in \mathbb{Z} \setminus \{0\}\}$, and the zero ideal. Moreover, these ideals are all distinct, depending on the choice of c .*

Proof. Let P be our prime ideal, and take a element $\frac{f}{g} \in P$ with minimal degree in the denominator. First, we rule out that $\deg(g) = 0$, as that would force f to be a unit. We claim that $\deg(g) = 1$. Suppose not, so that we can factor

$g = g_1g_2$, where $\deg(g_1) = 1$, and factor $f = f_1f_2$, with $\deg(f_1) \leq \deg(g_1)$ and $\deg(f_2) \leq \deg(g_2)$. This procedure is possible since k is algebraically closed. Now, $\frac{f_1}{g_1}, \frac{f_2}{g_2} \in A_0$, and since P is prime, one of these elements must be in P . This contradicts the minimality of g . So thus our ideal has an element of the form $\frac{a+bu}{c+du}$, which after division and multiplication by d , we can view to have the form $a' + \frac{b'}{u-c'}$. Now, say our ideal has two elements of this form, call them $a_1 + \frac{b_1}{u-c_1}$ and $a_2 + \frac{b_2}{u-c_2}$. We can multiply each element by $\frac{u-c_i}{u}$, and we get $a_i + \frac{b_i - a_i c_i}{u}$. If $b_i - a_i c_i = 0$, then our ideal has a unit, so say not. We can scale both of the elements by the $b_i - a_i c_i$, so now we have two elements of the form $a'_i + \frac{1}{u}$. So if $a'_1 = a'_2$, they were originally scalar multiples, so we're done. If $a'_1 \neq a'_2$, we can subtract $a'_1 + \frac{1}{u}$ from $a'_2 + \frac{1}{u}$ and get an element of the form $a'_1 - a'_2 \neq 0$, which is a unit. The ideal would then be the entire ring, so thus would not be prime. We need to make sure $a + \frac{1}{u}$ isn't a unit. Rewriting, we see that $a + \frac{1}{u} = a \frac{u+1}{u}$. So this is a unit if $u + \frac{1}{a}$ is a function with a zero at an integer, thus $\frac{1}{a}$ cannot be in \mathbb{Z} . Finally, note that we can do this analysis with every rational function in the ideal, and if there wasn't a common factor of the form $a + \frac{1}{u}$ between them, our analysis shows that the ideal must necessarily be a unit ideal by virtue of being able to generate two distinct elements of this form. Thus the prime ideal must be principal, generated by the common factor. \square

With this, we get a nice corollary:

Proposition 3.4. *A_0 is a PID, and A is Noetherian.*

Proof. By a theorem in Hungerford [6, 8.2.4], if a ring has all prime ideals being principal, the ring itself is principal. So A_0 is a PID.

Now, let $w = \frac{1}{u}t$, and $v = t^{-1}$. Say we have an ascending chain of right ideals in A , $I_0 \subset I_1 \subset \dots \subset I_n \subset \dots$. First, for each I_n , let

$$J_n = \{a \mid aw^k + a_{k-1}w^{k-1} + \dots \in I_n\} \quad (3.5)$$

be the ideal of leading coefficients of elements with positive degree in I_n . First, we need to check that J_n are ideals of A_0 . Given two elements $a, b \in J_n$ with $a + b \neq 0$, note that there are corresponding functions $f_a = aw^n + a_{n-1}w^{n-1} + \dots$

and $f_b = bw^m + b_{m-1}w^{m-1} + \dots$. Assuming $n > m$, multiplying f_b by w^{n-m} on the right and summing, we get an element of I_n with leading coefficient $a + b$, as needed. Also, for any $r \in A_0$, there is a polynomial with leading coefficient ar , indeed, multiply f_a by $\sigma^{-n}(r)$, and we get $f_a\sigma^{-n}(r) = arw^n + a_{n-1}\sigma^{-1}(r)w^{n-1} + \dots$, as needed.

We have $J_0 \subset J_1 \subset \dots \subset J_n \subset \dots$. A_0 is a PID, so this sequence eventually stabilizes to some J , and there is some f in some I_N such that the leading coefficient of f generates this ideal. Let $\deg(f) = M$. Now, for all $0 < m < M$, we will make a similar argument. For each m , let

$$J_{n,m} = \{a \mid aw^k + a_{k-1}w^{k-1} + \dots \in I_n, k \leq m\} \quad (3.6)$$

be the set of leading coefficients of elements of I_n with with degree at most m , having at least one term of positive degree. Using the same argument as above, we generate ideals $J_{n,m}$ of A_0 , and from these ideals we can find $f_m \in I_{N_m}$ such that f_m has an appropriate leading coefficient to generate $J_{n,m}$. Now, say our original chain of ideals is infinitely ascending. So for $k \gg N, N_1, \dots, N_m$, we have I_{k+1} strictly containing I_k . Take a term g with minimal positive degree that is in I_{k+1} but not I_k , and say $\deg(g) = d$. If $d > M$, note that the leading of coefficient of g is some A_0 multiple of the leading coefficient in f , so there is some $r \in A_0$ such that $g - rf w^{d-M}$ has degree strictly less then the degree of g . If $\deg(g) < \deg(f)$, then the leading coefficient of g is some A_0 multiple of the leading coefficient in $f_{\deg(g)}$, so $g - rf_d w^{d-\deg(f_d)}$ has degree strictly less then g . In either case, we get a contradiction violating the minimality of our choice of g . So thus if g exists, it cannot have any terms of positive degree.

However, we can repeat this argument, this time considering the most negative coefficients of elements with at least one term of negative degree in I_n , viewing our indeterminate variable to be v , instead of w . So g cannot have any terms of negative degree. Thus $g \in A_0$. Now the $I_n \cap A_0$ generate a ascending chain of ideals in A_0 . By the Noetherian property in A_0 , however, this must stabilize as well. So if we choose sufficiently high K , we get $I_k = I_{k+1}$ for all $k > K$ as needed. Thus A satisfies the Noetherian property for right ideals. For left ideals, we can do a similar argument, but we have to consider all our ring elements to be written

$w^n a_n + \dots + wa_1 + a_0 + va_{-1} + \dots v^m a_{-m}$. Our ideals of leading coefficients are different, but the argument is essentially the same. \square

From here, we are able to characterize the prime spectrum of A . To begin with, we need a lemma adapted from a paper of Bell, Rogalski, and Sierra [2].

Lemma 3.7. *Nonzero prime ideals in A contain a nonzero homogeneous prime ideal.*

Proof. Note that the graded quotient ring of A is $Q = k(u)[t, t^{-1}; \sigma]$. Since $k(u)$ has no σ -fixed ideals, and powers of σ never are an inner automorphism of $k(u)$, this forces Q to be simple [8, 1.8.5]. So take a nonzero prime ideal $P \subset A$. We claim that P must contain a homogeneous element. For sake of contradiction, suppose not. Let S be the set of all homogeneous elements in A , and note Q is the localization of A at S . As $P \subset A$ is a proper ideal that has no homogeneous elements, PS^{-1} is therefore a proper ideal of Q , via the ideal correspondence in localization. Q is simple though, so this is impossible. Thus P must have some nonzero homogeneous element in the ideal. Now, let \tilde{P} be the ideal generated by all homogeneous elements in P , which is not the 0 ideal by before. Note if homogeneous ideals \tilde{I} and \tilde{J} have $\tilde{I}\tilde{J} \subset \tilde{P}$, we necessarily have $\tilde{I}\tilde{J} \subset P$ as well. Since P is prime, this forces one to be in the ideal, say \tilde{I} . But by the construction of \tilde{P} , $\tilde{I} \subset \tilde{P}$ as needed. \square

Also, knowing the σ -fixed ideals of A_0 will be very helpful.

Lemma 3.8. *The only σ -fixed ideals of A_0 are $\{(\frac{1}{u})^n \mid n \geq 0\}$, and (0) .*

Proof. Say $(\frac{f}{g}) \neq 0$ is σ -fixed, with n being the difference in degrees between the denominator and numerator. Then $(\frac{f}{g}) = (\frac{\sigma(f)}{\sigma(g)})$. So there is some $a \in A_0$ such that $\frac{f}{g} = a \frac{\sigma(f)}{\sigma(g)}$. So we get that $\frac{f\sigma(g)}{\sigma(f)g} \in A_0$. This forces f to have all integer roots, as if it did not, there would be a largest such root z . Now $\sigma(f)$ has a root $z + 1$, and $f\sigma(g)$ has no corresponding root to be able to cancel it, as g only has roots at integers, and the non-integer roots of f are at most z by construction, so thus cannot be $z + 1$. If f has only integer roots though, we can find an appropriate unit $\frac{g}{u^n f} \in A_0$ so our ideal would have the desired form. \square

So now, we can prove our main result:

Theorem 3.9. *A has classical Krull dimension 2. Let $I = \left(\frac{1}{u}\right) < A_0$. The nonzero prime ideals of A are either of the form*

1. $P_i = I \left(\frac{1}{u}t\right)^i$, for $i \geq 0$, and $P_i = A_i$, for $i < 0$,
2. $Q_i = A_i$ for $i > 0$, and $Q_i = It^i$ for $i \leq 0$,
3. $S_0 = I$, $S_i = A_i$ for $i \neq 0$, or
4. *Lifts of primes found in the quotient ring A/P and A/Q , which are both isomorphic to polynomial rings.*

Proof. So the first goal is to classify the graded-ideal structure for A . Let I be our graded ideal, and for now focus on $I^+ = \bigoplus_{j=0}^{\infty} V_j \left(\frac{1}{u}t\right)^j$. Note since $\frac{1}{u}t \in A$, we get $V_j \subset V_{j+1}$. We also get $\sigma(V_j) \subset V_{j+1}$, via multiplication of $\frac{1}{u}t$ from the opposite side. All the V_j are ideals of A_0 , and since A_0 is a PID, it is Noetherian, so eventually for sufficiently large $n > N$, we have $V_n = V_{n+1}$, and moreover this shows that $V_n = \sigma(V_n)$. So these ideals are forced to be σ -fixed. Moreover, a similar result is true for $I^- = \bigoplus_{j=0}^{\infty} V_{-j}t^{-j}$, except we merely have to multiply by t^{-1} now, and we find that for sufficiently large $n > N$, $V_{-n} = V_{-(n+1)}$, and these V_{-n} are σ -fixed.

So, returning to the analysis of the A^+ portion of the ideal, choose some n high enough that $V_n = V_m$ for all $m > n$. The ideal V_n is σ -fixed by above, so thus it has the form $\left(\frac{1}{u}t^k\right)$, or 0. Now consider the ideal V_{n-1} . We know that $\frac{1}{u}V_n \subset V_{n-1}$, by virtue of multiplying I_n by t^{-1} , and $V_{n-1} \subset V_n$, via multiplying I_{n-1} by $\frac{1}{u}t$. Thus we have

$$\left(\frac{1}{u}t^{k+1}\right) \subset V_{n-1} \subset \left(\frac{1}{u}t^k\right), \quad (3.10)$$

and we can conclude that V_{n-1} is either $\left(\frac{1}{u}t^{k+1}\right)$ or $\left(\frac{1}{u}t^k\right)$. Therefore, to each I_n we can associate a positive integer a_n such that $V_n = \left(\frac{1}{u}t^{a_n}\right)$, and the a_n are a non-increasing series from $n = 0$ to infinity. The same analysis can be done on the negative portion of the ideal as well, and we see that the a_{-n} form a non-increasing series from $n = 0$ to infinity at well.

So now suppose $V_0 = (\frac{1}{u}^{a_0})$, where $a_0 > 1$. When we consider A/I , note that elements of negative degree in $(A/I)_0$ are nilpotent. To see this, we examine the structure of V_0 . We claim that we can view V_0 to be the ideal of all elements with degree less than or equal to a_0 . Say we have an element $\frac{f}{g}$, with $\deg(f) = n$, and $\deg(g) = a_0 + n$, for some $n \geq 0$. Then we have:

$$\frac{f}{(u - c_1) \cdots (u - c_n)} = \frac{f}{u^n} \frac{1}{u^{a_0}} \frac{u^n}{(u - c_1) \cdots (u - c_n)}. \quad (3.11)$$

Both $\frac{f}{u^n}$ and $\frac{u^n}{(u - c_1) \cdots (u - c_n)}$ are in A_0 , so thus this belongs to V_0 . Clearly V_0 only consists of elements with degree at most a_0 , so the claim is established. Now consider the ideal $J/I < A/I$, where $J/I = (\frac{1}{u})$. Note J/I is nonzero, since $a_0 > 1$, so at least $\frac{1}{u} \in J/I$. However, $(J/I)^{a_0} = 0$, as V_0 contains everything that has degree at most $-n$, and as $V_0 \subset V_i$ for all i , all the V_i have this property as well. Thus, A/I has a nonzero nilpotent ideal, so thus is not prime. Therefore, $a_0 = 1$ by force, and $A/I_0 \cong k$. Letting $w = \frac{1}{u}t$ and $v = t^{-1}$, we can see that A/I has either the form $(A/I)_n^+ = kw^n$ or $(A/I)_n^+ = 0$, with the latter case happening for all larger n once it appears. Likewise, $(A/I)_n^- = kv^n$ or $(A/I)_n^- = 0$, with the same property that once a graded piece is zero all further graded pieces are zero. Note that $wv = 0$ in A/I , as $wv = \frac{1}{u} \in I_0$. Thus, we can see by checking the multiplication of homogeneous elements that this ring is now commutative, and therefore we simply need to ensure this ring is an integral domain. Clearly, we cannot have elements of both v and w in the ring, so without loss of generality, assume $(A/I)_{-n} = 0$ for $n \geq 0$. If we have $(A/I)_1 = 0$, we simply will have k , and if $(A/I)_1 \neq 0$, note that $w^n \neq 0$ for all $n > 0$, so thus $(A/I)_n \neq 0$ for all $n > 0$.

Thus, once we mod out by the graded homogenous ideals we either get a k alone, or a polynomial ring. We know by Lemma 3.7 that all height one primes are homogeneous, and that polynomial rings have classical Krull dimension 1, so thus our ring has classical Krull dimension 2, as needed. Additionally, we have our classification of the prime ideals. \square

Moreover, using similar techniques, we can characterize all simple graded modules of A .

Theorem 3.12. *A has two families of right simple graded modules M : The first has exactly one nonzero graded component $M_i \cong A_0/(\frac{1}{u})$, and the other has the form A/I , where $I_n = (\frac{1}{u} + c) (\frac{1}{u}t)^n$ for $n \geq 0$, and $I_n = (\frac{1}{u} + c) t^n$, for $n < 0$, and c is not the inverse of a nonzero integer.*

Proof. Let $M = \bigoplus M_i$ be a simple module of A , $w = \frac{1}{u}t$, and $v = t^{-1}$. First note that each M_i is a simple right A_0 module. For sake of a contradiction, say that nonzero $N_0 < M_0$ as A_0 modules. Now consider the right A module N_0A . $N_0A < M$, so thus $N_0A = M$. In particular, $(N_0A)_0 = M_0$. But $(N_0A)_0 = N_0A_0 = N_0$, so $N_0 = M_0$ as needed.

Now, since A_0 is a PID, its simple modules have the form A_0/m , where m is some maximal ideal of A_0 . Say, up to a shift, that M_0 is a nonzero component of M . Then either $m = (1/u)$ or $m = (1/u - c_0)$, where c_0 is not the reciprocal of any integer. Now, to find simple graded modules, we need to find graded maximal right ideals I of A , since necessarily if M is simple then $M \cong A/I$. So say $I_n = V_n (\frac{1}{u}t)^n$ if $n \geq 0$, and $I_n = V_n t^n$ if $n < 0$. Note that

$$V_0 \subset V_1 \subset \cdots \subset V_n \subset \cdots \quad (3.13)$$

is the inclusion induced by multiplication by $\frac{1}{u}t$ on the right, and likewise

$$V_0 \subset V_{-1} \subset \cdots \subset V_{-n} \subset \cdots \quad (3.14)$$

is the inclusion induced by multiplication by t^{-1} on the right. Since M_0 must be simple, we then know I_0 is either $(\frac{1}{u})$ or $(\frac{1}{u} + c)$ for some c . So let us do the $(\frac{1}{u})$ case first.

In this case, I is contained in the right ideal J , where $J_0 = (\frac{1}{u})$, and $J_i = A_i$ for $i \neq 0$. This is an ideal, as $A_i A_{-i} \subset \frac{1}{u}A_0$ for $i \neq 0$. Note that the induced module has the form $M_0 = k$, and $M_i = 0$ for $i \neq 0$.

In the other case, we claim if $V_0 = (\frac{1}{u} + c)$, then all other $V_i = (\frac{1}{u} + c)$ as well. So by the inclusions already discussed, and since $(\frac{1}{u} + c)$ is maximal, for each direction, either all the V_i have this form, or there is some choice of N such that $V_i = A_0$ for all $i \geq N$. We claim the latter is impossible though. If this is the case, there is a smallest possible choice of N . Now, note that $A_0 (\frac{1}{u}t)^N t^{-1} \subset I$, so in

particular $\frac{1}{u} \left(\frac{1}{u}t\right)^{N-1} \in I$, so thus $\frac{1}{u} \in V_{N-1}$. However, this violates the choice of N , since if $\frac{1}{u} + c$ and $\frac{1}{u}$ are both in V_{N-1} , then $V_{N-1} = A_0$. So thus $V_0 = \left(\frac{1}{u} + c\right)$, then all $V_i = \left(\frac{1}{u} + c\right)$, and this choice of V_i gives a I such that A/I is the other class of simple modules.

Note that in the first case, the shift will move where the non-zero graded piece is, while in the second case, the shift keeps the form of the module, possibly changing the choice of c . \square

Now, we will show that A is a maximal order.

Proposition 3.15. *A is a maximal order.*

Proof. By Lemma 2.31, it is sufficient only to consider the graded quotient ring and graded ideals of A . So take any nonzero graded ideal $I \subset A$, and say homogeneous $q \in k(u)[t, t^{-1}; \sigma]$ has $Iq \subset I$. Assume q has positive degree, so that $q = rw^n$, with $n \geq 0$, and $I^+ = \bigoplus_{k=0}^{\infty} V_k w^k$. By Theorem 3.9, the graded ideal structure of A is classified. In particular, we have for sufficiently large m that $V_m = V_{m+n}$, and all these ideals are σ -fixed. Thus we have

$$V_m w^m r w^n = V_m \sigma^{-m}(r) w^{m+n} \subset V_{m+n} w^{m+n} = V_m w^{m+n}. \quad (3.16)$$

Now, if r has positive degree as a rational function, we would get an immediate contradiction, as V_m must have a maximal degree element, and multiplication by r would only increase the degree of this element. So thus r has negative degree as a rational function. Suppose that $r \notin A_0$, so thus r must have poles not at the integers, call them $\{z_1, \dots, z_n\}$. Moreover, since $V_m \sigma^{-m}(r) \subset V_m$, every element in V_m must have zeros at $\{z_1 - m, \dots, z_n - m\}$ in order to have V_m remain an ideal of A_0 . However, we can repeat this argument, now using V_{m+n} and V_{m+2n} as our ideals, and conclude that every element of V_m must have zeros at $\{z_1 - 2m, \dots, z_n - 2m\}$ as well. Thus, by repeating this argument, we force elements of V_m to have zeros at infinitely many locations, which is clearly impossible. So thus $r \in A_0$, as needed, so $q \in A$ and A is a maximal order. \square

Finally, we will show A does indeed have GK dimension 3.

Proposition 3.17. *A has GK dimension 3.*

Proof. Take a generating subspace for A of the form $k + k\left(\frac{1}{u}t\right) + kt^{-1}$. Consider an element of the form $a_{n,x,y} = \left(\frac{1}{u}t\right)^n \left(\left(\frac{1}{u}t\right)(t^{-1})\right)^{x-1} \left(\frac{1}{u}t\right) \left(\left(\frac{1}{u}t\right)(t^{-1})\right)^y (t^{-1})^{n+1}$. After expanding, we can see that

$$a_{n,x,y} = \frac{1}{u} \cdots \frac{1}{u+n-1} \left(\frac{1}{u+n}\right)^x \left(\frac{1}{u+n+1}\right)^y. \quad (3.18)$$

Additionally, $a_{n,x,y}$ is made up of $2(n+x+y)$ generators. Now, consider the set

$$\{a_{0,n,0}, a_{0,n-1,1}, \dots, a_{0,2,n-2}, a_{1,n-2,1}, \dots, a_{n-2,1,1}\}. \quad (3.19)$$

We claim this set is linearly independent. First, observe that $a_{0,n,0}$ has a pole of order n at zero, and no other element on the list has such a pole of that order, so thus $a_{0,n,0}$ is linearly independent with the rest of the set. In general, an element $a_{n,x,y}$ has a pole of order x at n , and a pole of order one at all $m < n$. Thus, as we go down the list, there is always some pole in each element such that the rest of the elements to the right of it in the set do not have a pole of that order or higher. So thus, this is a linearly independent set, and moreover, it has $\frac{n(n-1)}{2} + 1$ elements. However, this is just the elements that have net zero degree. If we allow the degree to be positive or negative, we can do the same construction as before, except we initially start by setting aside excess factors of $\frac{1}{u}t$ or t^{-1} as appropriate.

Now, say that we are computing the number of linearly independent elements in V^n . Taking n to be even, we have $\frac{n}{2}$ generators to form elements of degree zero, $\frac{n}{2} - 1$ generators to form elements of degree ± 2 , and so on. Thus we have at least

$$\sum_{k=1}^{n/2} \frac{k(k-1)}{2} + 1 = \frac{1}{48}(n^3 + 20n) \quad (3.20)$$

linearly independent elements. Therefore A has at least GK dimension 3. However, by a paper of Rogalski and Zhang [9, 1.6], we know that $k(u)[t, t^{-1}; \sigma]$ has GK dimension 3, so A has at most GK dimension 3. So thus the GK dimension of A is exactly 3, as needed. \square

Chapter 4

Integrally closed σ -closed subrings of $K(u)$

The next goal is to investigate graded maximal orders of $k(u)[t, t^{-1}; \sigma]$, and this is very closely related to finding all integrally closed subrings of $k(u)$. Let $R \subset k(u)$ be a subring. First, we need a way to reduce to the finitely-generated case:

Lemma 4.1. *Let $R \subset k(u)$ be a subring of $k(u)$ such that $Q(R) = k(u)$. Then R has a finitely-generated integrally-closed subring R' with $Q(R') = k(u)$.*

Proof. First we want to find a finitely-generated subring $S \subset R$ such that $Q(S) = k(u)$. Assume this is not possible, for sake of contradiction. So take an element r_1 and form $Q_1 = Q(k[r_1])$. Since $Q_1 \neq k(u)$ by assumption, take some $r_2 \in R - Q_1$, and form $Q_2 = Q(k[r_1, r_2])$. Again, this cannot be $k(u)$ by assumption, so we can find some $r_3 \in R - Q_2$. Repeating this process, we get a chain of quotient rings

$$k \subsetneq Q_1 \subsetneq Q_2 \subsetneq \cdots \subsetneq Q_n \subsetneq \cdots$$

that never terminates. Since k is algebraically closed, all the Q_i must have transcendence degree at least one, and by virtue of all these being subsets of $k(u)$, they all have transcendence degree at most one. So thus all Q_i have transcendence degree one. However, by a theorem in Hungerford [6, 5.6.11], these type of chains cannot go on infinitely. Thus there is some finitely generated subring of S

of R that has $Q(S) = k(u)$. However, taking the integral closure of S , we get a finitely-generated integral closure R' , by a theorem in Eisenbud [4, 13.13]. \square

Theorem 4.2. *Let $R \subset k(u)$. Then if R is both σ -closed and integrally closed, then R is the localization of some polynomial ring. In particular, either $R = k[u]S^{-1}$, for some σ -closed set S , or*

$$R = k[S^{-1}] = \left\{ \frac{f}{g} \mid \deg(f) \leq \deg(g), g \text{ only has factors in some } \sigma\text{-closed set } S \right\}. \quad (4.3)$$

Proof. To begin, by Lemma 4.1 we may take a finitely-generated subring $R' \subset R$ that has $Q(R') = k(u)$. Now R , due to being an integral domain, has a corresponding affine variety $V = \text{Spec } R$. This variety has dimension one, since its corresponding quotient ring is $k(u)$, and moreover, it is smooth, since the local ring of any point is integrally closed, and the integral closure of the entire ring passes to the localization. Now by a result in Hartshorne [5, 6.2A], an integrally closed Noetherian local domain of dimension one is equivalent to being a regular local ring, so V is smooth by definition. But now by a corollary in Hartshorne [5, 6.10], every smooth affine curve is isomorphic to a open subset of a nonsingular projective curve, and that projective curve necessarily must be \mathbb{P}^1 , as there is a 1-1 correspondence between smooth projective varieties and their function fields, and we know the function field of V must be $k(u)$. So thus, we have $V \cong \mathbb{P}^1 - S$, for some finite non-empty subset of points S in \mathbb{P}^1 . However, since \mathbb{P}^1 minus a single point is isomorphic to \mathbb{A}^1 , we have $V \cong \mathbb{A}^1 - S'$, where S' has one fewer element than S . The coordinate ring of $\mathbb{A}^1 - S'$ is known though, and by the correspondence of coordinate rings and ideals, we induce an isomorphism $R \cong k[v]T'^{-1}$, where T' corresponds to the set of points S' that we remove. Note that $T' \subset \{v - a \mid a \in k\}$, by this correspondence.

This isomorphism of rings must induce an isomorphism on the function fields though, so $k(v) = Q(R) = k(u)$. By an exercise in Hungerford [6, V.2 ex. 6], we know that $v = \frac{a+ub}{c+ud}$ for some choice of $a, b, c, d \in k$ with $ad - bc \neq 0$. However, by long division and appropriate scaling, we see we either have $u \in R$ (if $d = 0$), or we have an element of the form $\frac{1}{c+u} \in R$. Now there are two cases, either u is in our ring or u is not in our ring.

In the first case, say that R contains $k[u]$. First suppose R contains an element with some denominator, say $\frac{f(u)}{(u-a)g(u)}$. We can multiply through by $g(u)$ to get an element of the form $\frac{f(u)}{(u-a)}$. Now, after polynomial long division, we get an element of the form $p(u) + \frac{c}{u-a}$. But R contains $k[u]$, so thus we get R having an element of the form $\frac{1}{u-a}$. We can add $u-a$ to the localization set though and account for this element. Now we iterate through all the possible factors in the denominator, and we see R is a localization of $k[u]$, at a set S corresponding to all linear factors appearing in any denominator. Note that S is σ -closed as we demand R to be a σ -closed ring.

In the other case, note that $k[\frac{1}{u+c}]S^{-1} = R'$. However, S must correspond to localization at specific points in A^1 , based on our construction of the ring, so thus functions of the form $\frac{1}{u+c} - a$. Thus the elements of the localization have the form $r = \frac{1}{\frac{1}{u+c} - a}$, with $a \neq 0$ by virtue of u not being in R . Multiplying r by $\frac{1}{u+c}$, we get an element of the form $\frac{1}{1-a_0u-ca_0}$, which after appropriate scaling has the form $\frac{1}{u-a'}$. So to get a factor in the denominator, it has to come from some negative one degree term in the localization, which is what we need. So thus, R' is generated by elements of the form $\frac{1}{u-a}$, and by using an argument very similar to the one in Proposition 3.2, we can show that R' has the form described by 4.3. To finish the proof, note that we can repeat this argument, only using elements in $R - R'$. Since $u \notin R$, we always arrive in this case, which consequently makes our set S larger. So thus, R has the form described by 4.3 as well. Note that S is σ -closed, as R is.

Note that these rings are both in some sense localizations of polynomial rings, but the difference is the interaction with σ : $k[u]$ is σ -fixed, while $k[\frac{1}{u-a}]$ is not. \square

The reason that integrally closed, σ -closed subrings of $k(u)$ are so important is that they are the only choices for the degree 0 pieces of maximal orders in $k(u)[t, t^{-1}; \sigma]$.

Theorem 4.4. *If $R = \bigoplus_{i=-\infty}^{\infty} V_i t^i$ is a maximal order in $k(u)[t, t^{-1}; \sigma]$, then R_0 is a σ -closed integrally-closed subring of $k(u)$.*

Proof. Assume, for sake of contradiction, that R_0 is not integrally closed but R is a maximal order.

To begin with, note $V_1tV_{-1}t^{-1} = V_1\sigma(V_{-1}) \subset R_0$. Since $R_0 \subset k(u)$ is commutative, note $V_1\sigma(V_{-1})$ is both a R_0 module, since $R_0R_1t \subset R_1t$, and a $\sigma(R_0)$ module as well, using the same reasoning. Thus, there exists some $i_1 \in R_1\sigma(R_{-1})$ such that $\sigma(R_0)i_1 \subset R_1\sigma(R_{-1}) \subset R_0$. Thus, $k\langle R_0, \sigma(R_0) \rangle$ is an equivalent order to R_0 . In general, $V_nt^nV_{-n}t^{-n} = V_n\sigma^n(V_{-n})$ and this is both a R_0 and $\sigma^n(R_0)$ module, so there is some nonzero $r_n \in R_0$ such that $\sigma^n(R_0)r_n \subset R_0$, by using the same argument as above. Thus, we can see that R_0 and $k\langle \sigma^{-n}(R_0), \dots, \sigma^n(R_0) \rangle$ are equivalent orders, as multiplying the second ring by $r_{-n} \cdots r_n$ sends the ring into R_0 . The Krull-Akizuki Theorem asserts that subrings of $k(u)$ are Noetherian [4, 11.13], and in Noetherian commutative rings, maximal orders and integrally closed domains are equivalent concepts [8, 5.1.3]. So let \tilde{R}_0 be the integral closure of R_0 . It is also the integral closure of $k\langle \sigma^{-n}(R_0), \dots, \sigma^n(R_0) \rangle$, for each n . So thus \tilde{R}_0 is σ -closed. Indeed, for any $r \in \tilde{R}_0$, there are $a_n \in R_0$ with $r^n + a_{n-1}r^{n-1} + \cdots + a_0 = 0$. But now $\sigma(r)^n + \sigma(a_{n-1})\sigma(r)^{n-1} + \cdots + \sigma(a_0) = 0$, and \tilde{R}_0 is the integral closure of $k\langle R_0, \sigma(R_0) \rangle$ as well, so $\sigma(r) \in \tilde{R}_0$ as needed.

Finally, consider $\tilde{R} = \bigoplus_{i=-\infty}^{\infty} \tilde{R}_0 R_i t^i$. This is a ring, since \tilde{R}_0 is σ -closed, it passes freely through the t^i . Moreover, $R_0 \subset \tilde{R}_0$, and from Theorem 4.2, we know that \tilde{R}_0 is a localization of $k[v]$. So now consider, $R_0[v]$: this contains $k[v]$, and thus must be a localization of $k[v]$. So $R_0[v] = \tilde{R}_0$, as localizations of polynomial rings are integrally closed. However, $R_0[v]$ is a finite R_0 -module, as $v \in \tilde{R}_0$. Writing $v = rs^{-1}$, with $r, s \in R$, and supposing that the minimal polynomial of v in $R[x]$ has degree n , $s^{n-1}\tilde{R}_0 \subset R_0$. Therefore \tilde{R} is an equivalent order of R , but since $R \subset \tilde{R}$, this contradicts our initial assumption that R is a maximal order. So thus R_0 is integrally closed. \square

It will be useful for later to note some properties of the rings described in Theorem 4.2.

Lemma 4.5. *Let $R = k[S^{-1}]$ be a subring of $k(u)$ that consists of all rational functions with degree at most zero with poles at a σ -fixed set S , as in 4.3. Then R*

is a PID, and the σ -fixed ideals of R are exactly the ideals $I_n = \{r : r \in R, \deg(r) \leq -n\}$.

Proof. The first statement immediately follows since R is a localization of some polynomial ring over k , and the localization of a PID is again a PID. Let I be a σ -fixed ideal of R . So $I = \left(\frac{f}{g}\right)$. If the zeros of f solely belong to the allowable poles in the ring, note that for any f' and g' , with $\deg(f') = \deg(f)$ and $\deg(g') = \deg(g)$, we have $\frac{f}{g} \frac{f'}{f} \frac{g}{g'} = \frac{f'}{g'} \in I$. If there is a zero of f that does not belong to the allowable poles, note that $\frac{\sigma(f)}{\sigma(g)} \in I$, so there must exist $r \in R$ such that $r \frac{f}{g} = \frac{\sigma(f)}{\sigma(g)}$. However, there must be one such of these zeros that is not a σ -iterate of any other zero in f , as σ has infinite order. So thus there is a zero of f that is not a zero of $\sigma(f)$, so thus r must have a pole at that zero. But this is impossible by construction. So thus $I = I_n$ for some n , as needed. \square

Chapter 5

The classification of \mathbb{Z} -graded rings with A_0 in the zeroth degree piece

Let $k[S^{-1}]$ be a subring of $k(u)$ that consists of all rational functions with degree at most zero and with poles only at some set S that is σ -closed. The goal is to find a classification of all finitely-generated \mathbb{Z} -graded GK 3 k -algebras $B \subset k(u)[t, t^{-1}; \sigma]$, that have $k[S^{-1}]$ in their zeroth degree piece. So the first goal is to classify the structure of $B_{\geq 0}$ and $B_{\leq 0}$, then glue them together. Without loss of generality, assume that $\frac{1}{u} \in A_0$, which we can do by replacing u with $u + a$ for some a , so we can denote by $(\frac{1}{u})$ the ideal I_1 described in Lemma 4.5.

So first off, note $B_i = a_i A_0 t^i$, for some choice of $a_i \in A_0$, due to each B_i being an A_0 module contained in $k(u)$, and thus a principal A_0 module. Since B is finitely generated, we can assume the generating elements lie in the graded pieces B_{k_1}, \dots, B_{k_m} . Thus, for all n , we must have

$$B_n \subset \sum_{a_i \in \{k_1, \dots, k_m\}, \sum a_i = n} B_{a_1} \cdots B_{a_x}. \quad (5.1)$$

We claim that for $n \gg 0$ each summand on the right hand side can be rewritten as the product of graded pieces with non-negative degree. To begin with, we can view the product lying in $B_+ B_-$, where B_- is the graded piece with all the negative graded pieces multiplied together, and B_+ is the graded piece arising after all the

non-negative pieces are multiplied together. If there is a negative graded piece in the product, call it B_m , take the largest graded piece appearing in the process, B_M , and note $B_m B_M \subset B_{m+M}$. Note if n is large enough, $M < n$, so $m + M < n$ as well. Consider a new product with B_M and B_m replaced by B_{M+m} . After this substitution, the new product, when written as $B'_- B'_+$, has B'_- lying in higher degree than B_- . In other words, the total degree of the negative components is getting closer to zero. Thus, by iterating this process, we see that we can force $B_- = 0$, so that the entire product of graded pieces has components that only live in positive degrees. So thus, we see that, for sufficiently large n ,

$$B_n \subset \sum_{i=1}^{n-1} B_i B_{n-i}. \quad (5.2)$$

This in turn, leads to the condition on the structure constants that

$$a_n \in \sum_{i=1}^{n-1} a_i \sigma^i(a_{n-i}) A_0. \quad (5.3)$$

Note that in negative degrees, we can do the same argument as above, but this time aim to eliminate the B_+ , and see that we get an analogous equation. In this case though the structure constants in 5.3 all correspond to negative degrees.

Let $X = \text{Spec } A_0$, and let σ be the automorphism that σ induces on X . Note that from arguments above, σ has exactly one fixed point, the ideal $(1/u)$: the rest of the prime ideals get permuted.

The idea of the argument is we can transform the grading variable t to a more convenient form in order to simplify the long-term behavior of the constant a_i . To do this, we need to translate this problem to algebraic-geometric language, then follow the results of Artin-Stafford. So assume $t \in B_1$, and note that $a_n A_0$ generates a subsheaf \mathcal{F}_n of \mathcal{K} on X , where \mathcal{K} is the constant sheaf of rational functions. Note each \mathcal{F}_n is free of rank 1, and thus is of the form $\mathcal{O}_X(D_n)$, for some Weil Divisor D_n on X . This divisor is principal, and it can be worked out that the divisor has form (a_n^{-1}) . So the condition on the structure constants now can be expressed as:

$$D_n = \bigcup_{i=1}^r D_i + \sigma^{-i}(D_{n-i}), \quad (5.4)$$

where $E = \bigcup D_i$ represents the smallest divisor that has $E - D_i \geq 0$ for all i . We call a sequence of divisors that has the property in 5.4 a σ -divisor sequence. Now we follow results in the Artin-Stafford paper.

So now there are two cases. The first case is covered by [1, 2.7]:

Lemma 5.5. *Let D_n be a σ -divisor sequence, and let \mathcal{J} be a subset of X consisting entirely of fixed points of σ and set $E_n = D_n|_{\mathcal{J}}$. Then there exists l such that $E_{nl} = nE_l$ for all $n \geq 1$.*

In our ring, there is exactly one point that is fixed under the σ -action, and that is the point corresponding to the $(1/u)$ ideal. So thus by this lemma, after taking an appropriate Veronese subring, we can make the contribution by the fixed point have the form given by the lemma - it goes up by a fixed amount as we increase the degree. Now, we have pretty fine control over the contribution of the fixed ideal to the ring.

Theorem 5.6. *Let $A \subset k(u)[t, t^{-1}; \sigma]$ be a finitely-generated \mathbb{Z} -graded k -algebra that has $A_0 = k[S^{-1}]$, and whose structure constants all have form u^k . Then there exists n , x , and y , with $n, x, y \in \mathbb{Z}$, $n \geq 1$, $x + y \leq 0$, that generate an algebra B described by $B_k = u^{\lfloor -\frac{xk}{n} \rfloor} A_0 t^k$, for $k < 0$, and $B_k = u^{\lfloor \frac{yk}{n} \rfloor} A_0 t^k$ for $k > 0$, such that B is a maximal order for A .*

Proof. Let $A = \bigoplus_{-\infty}^{\infty} u^{a_k} A_0 t^k$. First note that

$$A_i A_j = u^{a_i} \sigma^{-i}(u^{a_j}) A_0 t^{i+j} = u^{a_i} (u - i)^{a_j} A_0 = u^{a_i + a_j} A_0, \quad (5.7)$$

as $\frac{u^{a_j}}{(u-i)^{a_j}}$ is a unit in A_0 . Thus, we can focus on the behavior of the exponents. We know by Lemma 5.5 that there exists some n , x , and y such that $a_{kn} = -xk$ for $k < 0$ and $a_{kn} = yk$ for $k > 0$. Since $A_{-n} A_n \subset A_0$, we have that $x + y \leq 0$. Now fix some $i > 0$, and let l be the least common multiple between i and n . Note that $(A_i)^{l/i} \subset A_l$, and that $u^n A_0 \subset u^m A_0$ forces $n \leq m$, because $u^n A_0$ is the set of all rational functions with poles at S with degree at most n , so the smaller A_0 -module must have a smaller power of u corresponding to it. So we have $a_i \frac{l}{i} \leq y \frac{l}{n}$, which after rearrangement gives us $a_i \leq y \frac{i}{n}$. We want the maximal order to correspond to the largest possible ring fitting our parameters, so thus set $b_i = \lfloor y \frac{i}{n} \rfloor$. Similarly,

for $i < 0$, we set $b_i = \lfloor -x \frac{i}{n} \rfloor$. We claim that this choice of structure constants does make a graded ring. Consider the multiplication of $B_i B_j$. If i and j are both positive, we have $b_i + b_j = \lfloor y \frac{i}{n} \rfloor + \lfloor y \frac{j}{n} \rfloor \leq \lfloor y \frac{i+j}{n} \rfloor = b_{i+j}$. A similar thing happens if i and j are both negative. Now say i is positive and j is negative. Then we have

$$b_i + b_j = \left\lfloor y \frac{i}{n} \right\rfloor + \left\lfloor -x \frac{j}{n} \right\rfloor \leq \left\lfloor \frac{yi - xj}{n} \right\rfloor. \quad (5.8)$$

If $i + j > 0$, we can rewrite this expression as $\left\lfloor \frac{y(i+j) - j(x+y)}{n} \right\rfloor$, and since $x + y \leq 0$, this expression is at largest $\left\lfloor \frac{y(i+j)}{n} \right\rfloor = b_{i+j}$, as needed. Similarly, if $i + j < 0$, then we have this expression rewritten as $\left\lfloor \frac{i(x+y) - x(i+j)}{n} \right\rfloor$, and again since $x + y \leq 0$, we gave this is at most $\left\lfloor \frac{-x(i+j)}{n} \right\rfloor = b_{i+j}$. So this choice of b_i indeed induces a graded ring, as claimed.

Next, we show that the constructed ring B is an equivalent order of A . Note that since $A_{kn} A_c \subset A_{kn+c}$, we have that $a_{kn} + a_c \leq a_{kn+c}$. Furthermore, B was chosen to have maximal graded pieces with respect to the constraint that $a_{kn} = b_{kn} = -xk$ for $k < 0$ and $a_{kn} = b_{kn} = yk$, so we have $a_{kn+c} \leq b_{kn+c}$. Finally, note that $b_{kn+c} = b_{kn} + b_c$, since floor functions naturally split over addition if one of the summands is an integer. Putting this all together, we see that $a_{kn} + a_c = b_{kn+c} - b_c + a_c$ is a lower bound for a_{kn+c} , and a_{kn+c} is bounded by above by b_{kn+c} . In other words, we have

$$a_c - b_c \leq a_{kn+c} - b_{kn+c} \leq 0. \quad (5.9)$$

Iterating c from 0 to $n - 1$, we see that the structure constants of a and b differ at most by the maximum value δ that $b_c - a_c$ takes. Thus there exists an element in $k(u)$, $u^{-\delta}$, that has $u^{-\delta} B \subset A$. Clearly, we have $A \subset B$, so A and B are equivalent orders. We will show B is a maximal order in a later result. \square

Now, it would be nice to know given some choice of n, x, y whether the associated ring corresponding to these constants is finitely generated. First, we will need a technical lemma.

Lemma 5.10. *Suppose R contains all rational functions of degree at most zero, with denominators taken from some set of linear functions S . Then for any func-*

tion of the form $\frac{f}{p(u-a)}$, where $p \in k[u]$ and whose factorization lies completely in S , $(u-a) \notin S$, and $\deg(f) < \deg(p)+1$ there exists $r \in R$ such that $r + \frac{f}{p(u-a)} = \frac{1}{(u-a)}$.

Proof. Since p and $(u-a)$ share no roots, by the theory of partial fractions, we can write $\frac{f}{p(u-a)} = \frac{f'}{p} + \frac{c}{u-a}$ for some $f' \in k[u]$ with $\deg(f') < \deg(p)$. However, now $\frac{f}{p(u-a)} - \frac{f'}{p} = \frac{c}{u-a}$ as needed. \square

Proposition 5.11. *Let B be a maximal order as described by Theorem 5.6, with some n , x and y associated to it. Then B is a finitely generated k -algebra if and only if the denominators in A_0 are taken from finitely many σ -orbits.*

Proof. Clearly, if B is finitely generated, then the denominators in A_0 are taken from finitely many σ -orbits. So without loss of generality, assume the σ -orbit we are examining is the one generated by u . Let $f = (u-1)^{b_{-1}}t^{-1}$, and let $g = u^{b_1+1}(u+1)^{-1}t$, and let us consider $R = \langle f, g, \frac{1}{u} \rangle$. We claim that we can generate all $\frac{1}{u+c}$ from this generating set, and we will proceed by induction on c . Note $\frac{1}{u} \in R$ by definition, and assume that $\frac{1}{u}, \dots, \frac{1}{u+n-1} \in R$. Note

$$\begin{aligned} g^n f^n &= u^{b_1+1}(u+1)^{-1} \cdots (u+(n-1))^{b_1+1}(u+n)^{-1}(u+(n-1))^{b_{-1}} \cdots u^{b_{-1}} \\ &= u^{b_{-1}+b_1+1}(u+1)^{b_{-1}+b_1} \cdots (u+(n-1))^{b_{-1}+b_1}(u+n)^{-1}. \end{aligned} \tag{5.12}$$

First, note that $b_{-1} + b_1 \leq 0$. If $b_{-1} + b_1 = 0$, we get $\frac{u}{u+n}$ in our ring, so thus have $\frac{1}{u+n}$ in our ring as well. If $b_{-1} + b_1 < 0$, note that 5.12 has the form required by Lemma 5.10 as there is only one factor of $(u+n)$ and the other factors in the denominator are $(u+i)$ with $i < n$, so thus we know this element, along with the elements already in R , can give $\frac{1}{u+n}$ as needed. Also note to get negative factors, if we let $f' = u^{b_{-1}+1}(u-1)^{-1}t^{-1}$, and $g' = (u+1)^{b_1}t$,

$$\begin{aligned} f'^m g'^m &= u^{b_{-1}+1}(u-1)^{-1} \cdots u^{b_{-1}+1}(u-n)^{-1}(u-(n-1))^{b_1} \cdots u^{b_1} \\ &= u^{b_{-1}+b_1+1}(u-1)^{b_{-1}+b_1} \cdots (u-(n-1))^{b_{-1}+b_1}(u-n)^{-1}. \end{aligned} \tag{5.13}$$

Again, either $b_{-1} + b_1 = 0$, and we are done, or else 5.13 has the form required by Lemma 5.10, as there is only one factor of $(u-n)$, and the rest of the factors are $(u+i)$ for $i > -n$. Therefore, we get $\frac{1}{u-n} \in R$ for all n . Thus $\{\frac{1}{u}, f, g, f', g'\}$

generates the portion of A_0 corresponding to that σ -orbit. Since there are finitely many σ -orbits, we can generate A_0 with finitely many elements.

To finish the proof, note that if $\frac{ix}{n}$ is an integer, then $\left\lfloor \frac{(ai+k)x}{n} \right\rfloor = a\frac{ix}{n} + \left\lfloor \frac{kx}{n} \right\rfloor$. Let i be the least integer such that $\frac{ix}{n}$ is an integer, and note that $b_{ai+k} - b_{ai} = \left\lfloor \frac{kx}{n} \right\rfloor$, where $0 \leq k < i$. So thus we can generate the negative degree portion of the ring with finitely many elements; $u^{b-n}t^{-n}$, and then a collection of elements $u^{\left\lfloor \frac{kx}{n} \right\rfloor}t^{-k}$, with $0 \leq k < i$. We can do an identical method for the positive degree portion of the ring as well. Thus, we obtained a finite generating set for our ring, as needed. \square

Now, we need to consider the case where the structure constants do not simply have the form u^n . To start with, we will summarize a result from the paper by Bell and Rogalski.

Theorem 5.14 (Bell-Rogalski [3]). *Let $T = k[u]S^{-1}$, and consider a finitely-generated k -algebra of the form $R = \bigoplus_{i=-\infty}^{\infty} a_i T^i$, with all the $a_i \in k(u)$. Then, after a change of t , for $i \gg 0$, we have $R = aT^i$ with $a \in T$, and $a_i \in T$ for $i \geq 0$. Moreover, there exists some $s = (\sigma(c))^{-1}t$, with $c\sigma(c) \cdots \sigma^n(c) \in T$ for $n \gg 0$, such that for $i \ll 0$, we have $R_i = bTs^i$ with $b \in T$, and $b_i \in T$ for $i \leq 0$.*

Using this result, we present our main theorem.

Theorem 5.15. *Let A be a finitely-generated \mathbb{Z} -graded algebra that is an order of $k(u)[t, t^{-1}; \sigma]$, and let $A_0 = k[S^{-1}]$. Then after appropriate choice of t , there exists $a \in A_0$ such that for $i \gg 0$ $A_i = au^{n_i}A_0t^i$. Moreover, there exists some $s = (\sigma(c))^{-1}t$, with $c\sigma(c) \cdots \sigma^n(c) \in A_0$ and $b \in A_0$ such that for $i \ll 0$, $A_i = bu^{n_i}A_0s^i$. From the n_i sequence, we can find n, x, y from Theorem 5.6, and have them induce a sequence m_i , where $m_i = \left\lfloor -\frac{xi}{n} \right\rfloor$ when $i < 0$ and $m_i = \left\lfloor \frac{yi}{n} \right\rfloor$ for $i > 0$. Consider*

$$R = \left(\bigoplus u^{m_i} A_0 t^i \right) \cap \left(\bigoplus u^{m_i} A_0 s^i \right). \quad (5.16)$$

Then R is a maximal order for A .

Proof. Take $T = k[u]$, and as T is σ -fixed, we can consider the graded algebra TA . Note that TA_0 is just the localization of a polynomial ring TS^{-1} , so thus by

Theorem 5.14, after adjusting t , we have $TA_i = aTA_0t^i$ for $i \gg 0$, with $a \in TA_0$. Say that $A_i = a_iA_0t^i$. Then we have that $a_iTA_0 = aTA_0$. The units of TA_0 are exactly the rational functions whose numerators and denominators have factors that appear exactly in S , so we have $a_i = a\frac{f}{g}$, where f and g have factors that only appear in S . Take a such that a has only zeroes that do not appear in S , and enough factors of u^{-1} to make a have net degree zero. Now, $a_iA_0 = a\frac{f}{g}A_0$, and for each factor f' in f we have $\frac{u}{f'}$ a unit, and for each factor g' in g we have $\frac{g'}{u}$ a unit, so thus $a_iA_0 = au^{n_i}A_0$ for some n_i . So thus we have in large degree that $A_i = au^{n_i}A_0t^i$. Note in smaller positive degree, we can repeat this process to see that $A_i = a_iu^{n_i}A_0t^i$, with $a_i \in A_0$ with degree zero, as the structure constant in TA belongs to T . Likewise, there exists an $s = (\sigma(c))^{-1}t$ such that for $i \ll 0$, we have $A_i = bu^{n_i}A_0s^i$, and $A_i = b_iu^{n_i}A_0s^i$ for $i < 0$, with $b_i \in A_0$ with degree zero. Note that this process that adjusts t to s is only changing factors not in u^iA_0 , so thus the u^{n_i} sequence is the same in both expressions of the ring.

Say dt^z in our ring. Then for sufficiently large i we have

$$au^{n_i}t^i dt^z = a\sigma^i(d)u^{n_i}t^{i+z} \subset au^{n_i+z}A_0t^{i+z}. \quad (5.17)$$

So in particular, we have $\sigma^i(d) \in A_0u^{n_i+z-n_i} = A_0u^c$, for some $c \geq 0$. Thus, to each a_i in A , we can associate some c_i such that $a_iu^{c_i} \in A_0$. Likewise, to each b_i we can associate some d_i such that $b_iu^{d_i} \in A_0$.

Now, we will construct the ring R . Let $R = (\bigoplus u^{m_i}A_0t^i) \cap (\bigoplus u^{m_i}A_0s^i)$, for the m_i sequence induced by the n_i . We claim R is an equivalent order for A . By Theorem 5.6, we know that $m_i - n_i$ is at most some constant δ . Suppose we know $A_n = au^{n_i}t^i$ for $i > N_+$, and $A_n = bu^{n_i}s^i$ for $s < N_-$. Then, multiplying R by

$$q = u^{-\delta} \prod_{i=N_-}^{N_+} a_iu^{c_i}b_iu^{d_i} \quad (5.18)$$

we see that $qR \subset A$. From above we showed that $A \subset R$, so they are equivalent orders.

Also note that by the theorem that $c\sigma(c) \cdots \sigma^n(c) \in TA_0$ for $n \gg 0$. We have freedom to change c by a unit in TA_0 , and also have knowledge that c only

permutes the factors in each structure constant which are not σ -fixed. So in particular, we can adjust c so $\deg(c) = 0$, by adding an appropriate factor of u^k , so that c is a product of irreducibles in A_0 that do not include $\frac{1}{u}$. Now we have that $c\sigma(c)\cdots\sigma^n(c)$ has degree zero and is in TA_0 , so necessarily it must be in A_0 . Furthermore, this shows that for $i \gg 0$, $R_n = u^{mi}A_0t^i$, and for $i \ll 0$, $R_n = u^{mi}A_0s^i$.

Now consider a homogeneous ideal $J < R$. Consider $i \gg 0$, so that $R_i = u^{mi}A_0t^i$. Then $J_i = H_i u^{mi}A_0t^i$, for some ideal $H_i < A_0$. Take $i = nk$, where n is the periodic constant induced by the sequence. Note that since $m_{nk} = [ky] = ky$, $m_{nk+c} = m_{nk} + m_c$. Now, we have that $u^{m_{nk}}t^{nk} \in R_{nk}$, so we get that $H_{nk}u^{m_{nk}}A_0t^{nk}u^{m_{nk}}t^{nk} \subset J_{2nk}$. Noting that $u^{m_{nk}}\sigma^{-nk}(u^{m_{nk}}) = \epsilon u^{m_{2nk}}$, for some unit $\epsilon \in A_0$, we get $H_{nk} < H_{2nk}$. Repeating this argument, we get a sequence of A_0 ideals:

$$H_{nk} < H_{2nk} < \cdots < H_{mnk} < \cdots \quad (5.19)$$

and since A_0 is Noetherian, this sequence eventually stabilizes to an ideal I . Now note that $u^{m_{nk}}t^{nk}Iu^{m_{nk}}t^{nk} = \sigma^{nk}(I)u^{m_{2nk}} \subset Iu^{m_{2nk}}$, so I is σ^{nk} closed. However, by 3.8, we see I must have form $(\frac{1}{u^c})$, for some $c \geq 0$. Next, consider a homogeneous element $q = dt^z \in \text{End}_A(J, J)$, with $d \in k(u)$. Note that if we have $Iq \subset I$, $Iq^n \subset I$ as well. So for now, take z to be a multiple of n . We know that for some $l \gg 0$, we have

$$\frac{1}{u^c}A_0u^{m_{ln}}t^{ln}dt^z \subset \frac{1}{u^c}A_0u^{m_{ln+z}}t^{ln+z}. \quad (5.20)$$

Note we know $H_{ln+z} = I$ as z is taken to be a multiple of n . So thus we have,

$$-c + m_{ln} + \deg(d) \leq -c + m_{ln+z} = -c + m_{ln} + m_z. \quad (5.21)$$

We know $m_{ln+z} = m_{ln} + m_z$ as the floor function splits cleanly when at least one argument is an integer, as it is here. But now we see that $\deg(d) \leq m_k$. Moreover d must have poles only in S , else the containment in 5.20 could not work. Indeed, $u^{m_{ln}-c}\sigma^{-ln}(d) \in u^{m_{ln+z}-c}A_0$, so if d has some pole not in S , so does $\sigma^{-ln}(d)$, but no element in $u^{m_{ln+z}-c}A_0$ has this property. So thus $dt^z \in u^{m_z}A_0t^z$, if k is a multiple of n . So now, we have shown that if $Iq \subset I$, then $q^n \in R$.

Now, in the general case, if $Iq \subset I$, note that by the above argument, q must have denominators in the permissible set. Thus, $Iq \subset I$ is only dependent on degree

considerations. We know that $n \deg(d) \leq m_{nz}$. But $m_{nz} = yz$, so $\deg(d) \leq \frac{yz}{n}$. However, $\deg(d)$ is an integer, so we can freely write $\deg(d) \leq \lfloor \frac{yz}{n} \rfloor = m_z$. Thus we have $q \in u^{m_z} A_0 t^z$ in general, as needed. Repeating this entire argument in negative degrees, writing everything in terms of $u^{m_i} A_0 s^i$ for $i \ll 0$, we arrive at exactly the same conclusions, and we get that our homogeneous element $q = d' s^z$ also belongs to $\bigoplus u^{m_i} A_0 s^i$. Thus, this element is in the intersection, as needed. So by Lemma 2.31, as $\text{End}_A(J, J) \subset R$, R is a maximal order. \square

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