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ABSTRACT

A technique for the solution of the Helmholtz equation together with associated boundary conditions is described. This method is based on a generalization of that used for the solution of the Dirichlet problem of potential theory, in which a dipole distribution is introduced on the boundary of a region to generate the potential inside. In order that the boundary conditions be satisfied, the distribution must be found as the solution of an integral equation. If the boundary is smooth, the equation is of Fredholm type, but if it has a corner the equation is singular.

The method is illustrated by applying it to a circular boundary, in which case the treatment can be given analytically. Then the problem of a sharp corner is analyzed, and properties of the solution are developed using the theory of singular integral equations. A few results are given for the numerical evaluation of eigenvalues of the Laplacian for some polygons which can also be solved analytically.

I. INTRODUCTION

In dealing with a wide variety of physical wave phenomena, one is often faced with the problem of finding a wave motion in a medium which is inhomogeneous overall, or has finite boundaries, but in which the medium is locally homogeneous and has finite discontinuities across various boundaries. In such cases one must typically find a solution to the equation

$$(\nabla^2 + \kappa_i^2) \quad \psi(\vec{r}) = 0, \tag{1}$$

where κ_i will be a different constant in each region, i, together with certain matching conditions for ψ at the boundaries of the region.

Aside from a few special cases which can be treated analytically, such problems must be solved numerically, usually with the aid of a high-speed computer. Commonly applied techniques of wide applicability in such calculations are the finite difference and finite element techniques. The former directly approximates the derivatives in Eq. (1) by finite differences, and the latter is best based on a Lagrangian variational principle from which Eq. (1) can be deduced. These techniques are generally applicable, independently of whether subregions are homogeneous or not. On the other hand, although boundary conditions at finite distances are easily treated, boundary conditions "at infinity" which arise in scattering problems are difficult to impose.

In this paper we consider a different method for the solution of such problems which is closely related to the classical solution of the Neumann and Dirichlet problems of potential theory. In this technique, the solution of Eq. (1) in a given region is achieved by the introduction of a dipole distribution on the boundary of that region. The boundary conditions can then be determined in terms of that distribution and one is led to integral equations for the boundary condition which must be solved. The method has two immediate advantages over the previously noted methods: 1. boundary conditions "at infinity" are easily introduced; and, 2. it is only necessary to consider points on the boundary to obtain the solution, thereby reducing the dimensionality of the problem by one. The storage requirements for a computer can thereby be reduced significantly. On the other hand, the method does have the disadvantage that the matrices which are generated have relatively few nonzero elements as compared to the former techniques which can have a small "band-width," and the elements of the matrices typically require calculation of more complicated functions. It is also true, of course, that the boundary distribution method can only be applied if individual regions are homogeneous. Thus for differing problems different techniques may be most efficacious. Although we believe that the method can be developed for use in three-dimensional problems, in this paper we will only consider the two-dimensional case, so that we will deal with a one-dimensional distribution on the boundary.

In Section II of this paper we will develop the general integral equation to which the boundary method leads. In Section III, we apply the technique to a simple but illuminating example which can be treated analytically. This case, in which the boundary is a circle, leads to an integral equation in which the kernel is completely continuous. On the other hand, we would like to apply the method to boundaries which have sharp corners, and in this case the integral equation is singular. Thus in Section IV we develop the necessary mathematical analysis for dealing with such cases. Finally, in Section V to illustrate the method we give results for the numerical determination of eigenmodes for a few two-dimensional polygons which have analytically known eigenfunctions, though the boundary distributions are not so known. From these results, it is easily seen that application of the method is quite feasible and can give good results.

II. THE DIPOLE DISTRIBUTION INTEGRAL EQUATION

The famous Dirichlet problem of potential theory is the determination of a solution of Laplace's equation in a region in which the potential takes a given value on the boundary. This problem has been solved for the inside of a closed region by the introduction of a Green's function and a continuous dipole distribution on the boundary. Thus, one writes

$$\phi(\vec{r}) = \oint_{S_V} D(\vec{r}\,') \, \nabla' G(\vec{r}, \vec{r}\,') \cdot d\vec{\sigma}\,'. \qquad (2)$$

In this expression $D(\vec{r})$ is the dipole distribution at $\vec{r}$, $G(\vec{r},\vec{r}\,')$ is the potential at $\vec{r}$ owing to a unit charge at $\vec{r}\,'$, and $d\vec{\sigma}\,'$ is the surface element directed along the outward normal. The integral is carried over the surface S_V which encloses the volume V, and $\phi(\vec{r})$ is thereby determined throughout V. In the two-dimensional case, in which we shall be interested in this paper,

$$G(\vec{r}, \vec{r}\,') = \frac{1}{2\pi} \log |\vec{r} - \vec{r}\,'| , \qquad (3)$$

and of course V becomes an area, and S_V its bounding contour. The solution of the Dirichlet problem is then reduced to finding the solution of an integral equation for $D(\vec{r})$.

If we introduce the $G(\vec{r}, \vec{r}\,')$ of Eq. (3) into Eq. (2), we find:

$$\phi(\vec{r}) = \frac{1}{2\pi} \oint_{S_V} D(\vec{r}\,') \frac{(\vec{r}\,' - \vec{r})\cdot d\sigma'}{|\vec{r}\,' - \vec{r}|^2} .$$

This expression would be useful for determining the potential at internal points of the region V, but if one wishes $\phi(\vec{r})$ on the boundary the limit must be taken from the inside, since the $\phi(\vec{r})$ obtained from this expression is discontinuous across the boundary. In the limit in which $\vec{r}$ approaches a point on the boundary where it is smooth, one can write

$$\lim_{\vec{r} \to S_V} \phi(\vec{r}) = \frac{D(\vec{r})}{2} + P \oint_{S_V} D(\vec{r}\,') \frac{(\vec{r}\,' - \vec{r})\cdot d\vec{\sigma}\,'}{2\pi |\vec{r}\,' - \vec{r}|^2} .$$

In this relation, the integrand is in general singular as $\vec{r}\,' \to \vec{r}$ but the integral can be defined as a principal value integral. If the side containing $\vec{r}$ is straight, the contribution from that side will vanish and the integral is then regular, but in any case, if the boundary satisfies a Liapunov smoothness condition, it can then be shown that the integral is in fact well-defined as a principal value integral as $\vec{r}\,' \to \vec{r}$.[1]

Thus, for a smooth boundary, a solution of the Dirichlet problem is obtained if one can solve the integral equation

$$f(\vec{r}) = \frac{D(\vec{r})}{2} + P \oint_{S_V} D(\vec{r}\,') \frac{(\vec{r}\,' - \vec{r})\cdot d\vec{\sigma}\,'}{2\pi |\vec{r}\,' - \vec{r}|^2} .$$

It can be shown[2] that this equation does in fact have a unique solution and so the problem is solved.

In our approach we use an extension of the preceding technique to the Helmholtz equation:

$$(\nabla^2 + \kappa^2)\, \psi(\vec{r}) = 0 . \tag{4}$$

In this case, we must choose $G(\vec{r}, \vec{r}\,')$ to be a solution of Eq. (4), with the result that:

$$G(\vec{r}, \vec{r}\,') = A\, J_0(\kappa |\vec{r} - \vec{r}\,'|) + B\, Y_0(\kappa |\vec{r} - \vec{r}\,'|) ,$$

where J_0 and Y_0 are the usual regular and irregular Bessel functions of order zero, and, A and B must be determined using the limiting condition as $\vec{r} \to \vec{r}\,'$, and, if applicable, the condition as $\vec{r} \to \infty$. If $\vec{r} \to \vec{r}\,'$, G will approach the same limit as for $\kappa = 0$, and so, since $Y_0(x) \sim (2/\pi) \log x$, as $x \to 0$, we find that $B = 1/4$. On the other hand, A will be determined for the specific problem considered: If one is dealing with the interior of a closed region, A can be chosen as zero. If, however, the region is open and $\vec{r}$ can approach infinity, A will then be chosen in such a way as to satisfy the asymptotic condition on $\psi(\vec{r})$.

In order to find the asymptotic condition, it is helpful to consider the time-dependent equation from which the Helmholtz equation typically arises. Since we are considering wave propagation, we would have

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(\vec{r}, t) = 0 , \qquad (5)$$

where c is the propagation velocity for waves in the region. If we assume $\phi(\vec{r}, t) = \psi(\vec{r})\exp(-i\omega t)$, we get Eq. (4), where $\kappa = \omega/c$. For a scattering situation, we consider as a typical case that a plane wave is incident on the scattering region, and write

$$\psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \psi_{sc}(\vec{r}) .$$

Here, $\psi_{sc}(\vec{r})$ is the scattered wave and we require that it must have only "outgoing" parts. Further, we introduce the distribution $D(\vec{r})$ on the boundaries where $\vec{r}$ is finite and they will then be used to generate ψ_{sc} only.

A simple way of determining A so that only "outgoing" scattered waves occur, with the assumed time dependence, is to require that

$$G(\vec{r}, \vec{r}') = -\frac{i}{4} H_0^{(1)}(\kappa|\vec{r} - \vec{r}'|) , \qquad (6)$$

where

$$H_0^{(1)}(x) \equiv J_0(x) + i Y_0$$

is the usual Hankel function.[3] This clearly gives the correct B, and if $\vec{r} \to \infty$:

$$G(\vec{r}, \vec{r}') \underset{r\to\infty}{\sim} -\frac{i}{4}\left(\frac{2}{\pi r}\right)^{1/2} e^{i(\kappa r - \pi/4)} ,$$

which clearly represents outgoing waves, since asymptotically

$$\phi_{sc}(\vec{r}, t) \underset{r \to \infty}{\sim} \exp[i(\kappa r - \omega t)].$$

An alternative more complete derivation of this form for G can be based on a requirement that time-dependent wave propagation be causal, that is, that the wave amplitude be zero at a point before the wave can get to that point with the given velocity, c .

If we now introduce Eq. (6) for $G(\vec{r}, \vec{r}')$, into Eq. (2), we have:

$$\psi_{sc}(\vec{r}) = -\frac{i}{4} \oint_{S_V} D(\vec{r}')\nabla' H_0^{(1)} (\kappa|\vec{r}' - \vec{r}|)\cdot d\vec{\sigma}'$$

or, since $H_0^{(1)}(x)' = -H_1^{(1)}(x)$,

$$\psi_{sc}(\vec{r}) = \frac{i\kappa}{4} \oint_{S_V} \frac{D(r')H_1^{(1)}(\kappa|\vec{r}' - \vec{r}|)(\vec{r}' - \vec{r})\cdot d\sigma'}{|\vec{r}' - \vec{r}|} . \tag{7}$$

Clearly the scattered wave given by Eq. (7) automatically gives only outgoing waves, and in fact we have

$$\psi_{sc}(\vec{r}) \underset{r \to \infty}{\sim} \frac{-ie^{i(\kappa r - 3\pi/4)}}{(8\pi r)^{1/2}} \oint_{S_V} D(\vec{r}')e^{-i\kappa\vec{r}'\cdot\hat{e}_r}\, \hat{e}_r\cdot d\vec{\sigma}' ,$$

where $\hat{e}_r \equiv \vec{r}/r$.

If we let $\vec{r}$ approach the boundary, the resulting equation has the same small $(\vec{r}' - \vec{r})$ behavior as in the Dirichlet case, so that we may write:

$$f(\vec{r}) = \frac{D(\vec{r})}{2} + \frac{i\kappa}{4} P \oint_{S_V} \frac{H_1^{(1)}(\kappa|\vec{r}\,' - \vec{r}|)}{|\vec{r}\,' - \vec{r}|} D(\vec{r}\,')(\vec{r}\,' - \vec{r})\cdot d\vec{\sigma}\,' . \tag{8}$$

The same considerations as in the Dirichlet case with regard to the singular nature of the equation apply. It is this integral equation which we propose to investigate.

There is one important difference between the potential problem and the Helmholtz problem that should be mentioned. Although the interior Dirichlet problem has a unique solution, the exterior problem does not, and in fact will only have a solution at all if

$$\oint f(\vec{r})\ |d\vec{\sigma}\,'| = 0 .$$

This follows from the fact that the homogeneous form of the exterior integral equation (in which $\vec{r}$ approaches the boundary from <u>outside</u>),

$$f(\vec{r}) = -\frac{D(\vec{r})}{2} + P \oint_{S_V} D(\vec{r}\,') \frac{(\vec{r}\,' - \vec{r})\cdot d\vec{\sigma}\,'}{2\pi|\vec{r}\,' - \vec{r}|^2} ,$$

has a solution. (It is easily seen that a constant satisfies the homogeneous equation.) Consequently, as follows from the Fredholm theory of this self-adjoint equation, only if $f(\vec{r})$ is orthogonal to the solution of the homogeneous equation is there a solution of the inhomogeneous equation. In the Helmholtz case, the homogeneous equation will not generally have a solution, so both the inner and outer problems have a unique solution.

III. CASE OF A CIRCULAR BOUNDARY

Although the boundary integral equation, Eq. (8), must generally be solved numerically, in the case of a circular boundary the equation can be treated analytically, and some insight can thereby be obtained. Before going on to more complicated cases, we will thus explore this case first.

If the radius of the circular boundary is R, Eq. (8) can be written as:

$$f(\theta) \equiv \frac{D(\theta)}{2} + \frac{i\kappa R}{4}\int_0^{2\pi} H_1^{(1)}\left(2\kappa R \sin\left|\frac{\theta' - \theta}{2}\right|\right) D(\theta')\sin\left|\frac{\theta' - \theta}{2}\right| d\theta' \;.$$

If $D(\theta)$ is expanded in a Fourier series,

$$D(\theta) = \sum_{n=-\infty}^{\infty} a_n \, e^{in\theta} \;, \tag{9}$$

we then have the integral, I_n, to evaluate:

$$I_n(\kappa R) = \int_0^{2\pi} H_1^{(1)}\left(2\kappa R \sin\left|\frac{\theta' - \theta}{2}\right|\right) \sin\left|\frac{\theta' - \theta}{2}\right| \, e^{in\theta'} \, d\theta' \;.$$

This integral may be evaluated[4] to give

$$I_n(\kappa R) = -\pi \frac{d}{d(\kappa R)} \left[H_n^{(1)}(\kappa R)\, J_n(\kappa R) \right] e^{in\theta} \;.$$

Thus we find:

$$f(\theta) = \frac{1}{2} \sum_{n=-\infty}^{\infty} a_n \left\{ 1 - \frac{i\pi\kappa R}{2} \left[H_n^{(1)}(\kappa R)\, J_n(\kappa R) \right]' \right\} e^{in\theta} \;,$$

and, on making use of the Wronskian relation between $H_n^{(1)}$ and J_n, we get:

$$f(\theta) = -\frac{i\pi\kappa R}{2} \sum_{n=-\infty}^{\infty} a_n J_n(\kappa R) H_n^{(1)}(\kappa R)' e^{in\theta} .$$

This result can be used to solve specific problems. For example, if one wishes the eigenmodes for the interior of a circular region in which $\phi(r, \theta)$ is zero on the boundary, one immediately obtains the relation:

$$J_n(\kappa R) H_n^{(1)}(\kappa R)' = 0 .$$

The modes associated with $J_n(\kappa R)$ are well known, but the apparent modes for $H_n^{(1)}(\kappa R)'$ are not,[5] and we will now demonstrate that for such κ's, even though the dipole distribution does not vanish, the associated $\psi(\vec{r})$ is zero everywhere inside the circle, so such solutions of the integral equation in this case are not useful. On the other hand, such solutions could arise in a numerical calculation of the integral equation, and one must be careful not to confuse them with nontrivial solutions. The distinction between solutions ψ would only be noticeable away from the boundary.

To find $\psi(r, \theta)$ once $D(\vec{r})$ is known, we can use

$$\psi(r, \theta) = \frac{i\kappa}{4} \int_0^{2\pi} H_1^{(1)}(\kappa w)\, D(\theta') \cos \chi \, d\theta'$$

where

$$w = \left[r^2 + R^2 - 2rR\cos(\theta - \theta') \right]^{1/2} ,$$

and χ is the angle between the vector $(\vec{r}' - \vec{r})$ and $\hat{e}_{r'}$.

(See Fig. 1.) We again express $D(\theta)$ in a Fourier series, Eq. (9), and, for $r < R$, we use Graf's addition theorem for Bessel functions[6] to give:

$$H_1^{(1)}(\kappa w) \cos \chi = \sum_{m=-\infty}^{\infty} H_{m+1}^{(1)}(\kappa R)\, J_m(\kappa r) \cos m(\theta - \theta') .$$

Thus one easily finds:

$$\phi(r, \theta) = \frac{i\pi\kappa R}{4} \sum_{n=-\infty}^{\infty} a_n \left\{ H_{n+1}^{(1)}(\kappa R)\, J_n(\kappa R) + H_{-n+1}^{(1)}(\kappa R)\, J_{-n}(\kappa r) \right\} e^{in\theta} .$$

Using the relations:[7]

$$J_{-n}(z) = (-1)^n J_n(z), \qquad H_{-n}^{(1)}(z) = (-1)^n H_n^{(1)}(z), \quad \text{and}$$

$$H_{n+1}^{(1)}(z) - H_{n-1}^{(1)}(z) = -2 H_n^{(1)}(z)' ,$$

we find:

$$\psi(r, \theta) = -\frac{i\pi\kappa R}{2} \sum_{n=-\infty}^{\infty} a_n e^{in\theta} J_n(\kappa r) H_n^{(1)}(\kappa R)' .$$

Thus, for the eigenmodes, we see that

$$\psi(r, \theta) = -\frac{i\pi\kappa R}{2} (a_n e^{in\theta} + a_{-n} e^{-in\theta}) J_n(\kappa r) H_n^{(1)}(\kappa R)' .$$

The a_n, a_{-n} are arbitrary, and, as stated earlier, we see that if $J_n(\kappa R) = 0$ we get the well-known eigenmodes, whereas if $H_n^{(1)}(\kappa R)' = 0$, $\psi(r, \theta) = 0$ for all r .

We can also use the circular case to illustrate the application of the integral equation to a wave scattering problem. We consider the scattering of a plane wave incident on a circular scatterer for which $\psi(r, \theta) = 0$ on the boundary, $r = R$. We now set

$$\psi(r, \theta) = \psi_{inc}(r, \theta) + \psi_{sc}(r, \theta)$$

as in Section II, where ψ_{inc} is the incident plane wave, and ψ_{sc} is the outgoing scattered wave. In this case the wave amplitude is given as:

$$\psi(r, \theta) = e^{i\kappa z} + \frac{i\kappa}{4} \oint \frac{H_1^{(1)}(\kappa |\vec{r}' - \vec{r}|)}{|\vec{r}' - \vec{r}|} D(\vec{r}')(\vec{r}' - \vec{r})\cdot d\vec{\sigma}' ,$$

$$= e^{i\kappa z} + \frac{i\kappa}{4} \int_0^{2\pi} H_1^{(1)}(\kappa w)\, D(\theta')\cos X \, d\theta' . \qquad (10)$$

In the limit $r \to \infty$, the scattered wave is given explicitly as:

$$\psi_{sc}(r, \theta) \underset{r \to \infty}{\sim} i(\kappa/8\pi r)^{1/2} \, e^{i\kappa r} \int_0^{2\pi} e^{-i\kappa R\cos(\theta' - \theta)}$$

$$\times \cos(\theta' - \theta)\, D(\theta')\, d\theta' ,$$

since X, the angle in the (r, R, w) triangle opposite r, is $\pi - (\theta' - \theta)$, as in Fig. 2.

Equation (10) for $D(\theta)$ can be solved similarly to the previous case. Here we require that:

$$\psi(R, \theta) = e^{i\kappa z} + \psi_{sc}(R,\theta) = 0 ,$$

and since[8]

$$e^{i\kappa z} = e^{i\kappa r \cos \theta}$$

$$= J_0(\kappa r) + 2 \sum_{n=1}^{\infty} i^n J_n(\kappa r) \cos n\theta ,$$

we require as the boundary condition on $\psi_{sc}(R, \theta)$:

$$f(\theta) = - J_0(\kappa R) - 2 \sum_{n=1}^{\infty} i^n J_n(\kappa R) \cos n\theta .$$

The boundary integral equation, Eq. (8), must be modified because $r > R$. This requires that the sign of the term, $D(\vec{r})/2$ must be reversed. If we again use Eq. (9) to express $D(\theta)$, we find for the coefficients a_n:

$$-i^n J_n(\kappa R) = - \frac{a_n}{2} + \frac{i\kappa R}{4} \int_0^{2\pi} H_1^{(1)}\left(2\kappa R \sin\left| \frac{\theta' - \theta}{2} \right|\right)$$

$$\times a_n e^{in(\theta'-\theta)} \sin \left| \frac{\theta' - \theta}{2} \right| d\theta' .$$

The integral is the same as in the previous problem, and we get:

$$-i^n J_n(\kappa R) = \left\{ - \frac{1}{2} - \frac{i\pi\kappa R}{4} \left[H_n^{(1)}(\kappa R) J_n(\kappa R) \right]' \right\} a_n .$$

Once more using the Wronskian relation, we find

$$a_n = \frac{2i^{n-1}}{\pi\kappa R} \frac{J_n(\kappa R)}{J_n(\kappa R)' H_n^{(1)}(\kappa R)} ,$$

and

$$D(\theta) = \frac{2}{\pi\kappa R} \sum_{n=-\infty}^{\infty} \frac{i^{n-1} J_n(\kappa R)}{J_n(\kappa R)' H_n^{(1)}(\kappa R)} e^{in\theta}$$

$$= \frac{-2i}{\pi\kappa R} \left\{ \frac{J_0(\kappa R)}{J_0(\kappa R)' H_0^{(1)}(\kappa R)} + \sum_{n=1}^{\infty} \frac{i^n J_n(\kappa R)}{J_n(\kappa R)' H_n^{(1)}(\kappa R)} \cos n\theta \right\}.$$

Having obtained $D(\theta)$ we can calculate $\psi_{sc}(r, \theta)$ by integration over the boundary. Thus, as in the interior problem, we have:

$$\psi_{sc}(r, \theta) = \frac{i\kappa R}{2} \int_0^{2\pi} H_1^{(1)}(\kappa w) \cos \chi \, D(\theta') \, d\theta' .$$

As before, we can use Graf's addition theorem, but since the angle which enters that theorem is the angle opposite the smaller of the sides r and R we must first express $\cos \chi = \cos \eta \cos(\theta' - \theta) - \sin \eta \sin (\theta' - \theta)$, where η is the angle needed for Graf's theorem (see Fig. 2). (In the interior problem, since $r < R$, χ was the correct angle.) If we now choose a particular term in $D(\theta) \sim \cos n\theta$, we find

$$I_n \equiv \frac{i\kappa R}{2} \int_0^{2\pi} H_1^{(1)}(\kappa w) \left[\cos\eta \cos(\theta' - \theta) - \sin\eta \sin(\theta' - \theta)\right] \times \cos n\theta' \, d\theta'$$

$$= - i\pi\kappa R \, H_n^{(1)}(\kappa r) \, J_n(\kappa R)' \cos n\theta \, .$$

If this is combined with the result for a_n, we obtain

$$\psi_{sc}(r, \theta) = - \frac{J_0(\kappa R)}{H_0^{(1)}(\kappa R)} H_0^{(1)}(\kappa r) - \sum_{n=1}^{\infty} 2 \frac{i^n J_n(\kappa R)}{H_n^{(1)}(\kappa R)} H_n^{(1)}(\kappa r) \cos n\theta,$$

a result which is easily obtained by using the Helmholtz equation directly in the usual fashion.

IV. ANALYSIS OF THE PROBLEM OF BOUNDARY WITH SHARP CORNERS

Although the boundary distribution technique can be applied directly to cases in which the boundary is smooth, i.e., satisfies a Liapunov condition, some additional analysis must be given if the boundary has sharp corners. In the former case, the kernel of the equation can be shown to be completely continuous and so the usual Fredholm theorems apply. On the other hand, if there are corners the kernel is singular.

To deal with this situation, we will consider a corner in a boundary and for simplicity we will assume that the sides of the corner are straight. The angle between these two sides will be called α. Further, since the singular nature of the equation comes about because of the small-distance behavior of the kernel, we divide the kernel into a leading term which includes the most singular part, and a remainder which is completely continuous. Thus we write:

$$H_1^{(1)}(x) \equiv -\frac{2i}{(\pi x)} + R(x) , \tag{11}$$

and we will focus attention principally on the first term.

If Eq. (11) is now introduced into Eq. (8), we find:

$$f(\vec{r}) = \frac{D(\vec{r})}{2} + P\oint \frac{D(\vec{r}\,')\ (\vec{r}\,' - \vec{r})\cdot d\vec{\sigma}\,'}{2\pi|\vec{r}\,' - \vec{r}|^2}$$

$$+ \frac{i\kappa}{4} P \oint \frac{R(\kappa|\vec{r}\,' - \vec{r}|)}{|\vec{r}\,' - \vec{r}|} D(\vec{r}\,')\ (\vec{r}\,' - \vec{r})\cdot d\vec{\sigma}\,' .$$

Let us now introduce the notation that $D_1(s)$ is $D(\vec{r})$ on the side 1 of the corner, where s is the distance from the corner, and $D_2(s)$ is $D(\vec{r})$ on side 2. With this notation, the equation is divided into pieces, and we find

$$f_1(s) = \frac{D_1(s)}{2} + \frac{s \sin\alpha}{2\pi} \int_0^{\ell_2} \frac{D_2(s')\, ds'}{(s'^2 - 2s'\, s\cos\alpha + s^2)}$$

$$+ \frac{i\kappa}{4}\, s \sin\alpha \int_0^{\ell_2} \frac{R\left[\kappa(s'^2 - 2s'\, s\cos\alpha + s^2)^{\frac{1}{2}}\right]}{(s'^2 - 2s'\, s\cos\alpha + s^2)^{\frac{1}{2}}} D_2(s')\, ds'$$

$$+ \frac{i\kappa}{4} \int_{C'} \frac{H_1^{(1)}(\kappa|\vec{r}\,' - \vec{r}|)}{|\vec{r}\,' - \vec{r}|} D(\vec{r}\,')\ (\vec{r}\,' - \vec{r})\cdot d\vec{\sigma}\,' ,$$

where $f_1(s)$ is the boundary value on side 1. For a straight side there is no contribution from the distribution $D_1(s)$ to the potential on that side except for the term $D_1(s)/2$, because the vector $\vec{r}' - \vec{r}$ is perpendicular to the surface element. The length of side 2 is ℓ_2. The integral over C' is the contribution from the distribution other than the part on sides 1 and 2. This last integral is analytic as a function of s, since it is a finite integral and $|\vec{r}' - \vec{r}| > 0$ for $\vec{r}'$ on C' and $\vec{r}$ on side 1.

Similarly, for side 2 we have:

$$f_2(s) = \frac{D_2(s)}{2} + \frac{s \sin \alpha}{2\pi} \int_0^{\ell_1} \frac{D_1(s')ds'}{(s'^2 - 2s' s \cos \alpha + s^2)} + \cdots ,$$

where the $\cdots$ indicates terms similar to the R, C' terms for f_1.

To analyze the corner singularity, we introduce $D_\pm(s) = D_1(s) \pm D_2(s)$, and we then obtain

$$\frac{D_\pm(s)}{2} \pm \frac{s \sin \alpha}{2\pi} \int_0^{\ell_m} \frac{D_\pm(s')ds'}{(s'^2 - 2s' s \cos \alpha + s^2)} = F_\pm(s) ,$$

where ℓ_m is the lesser of ℓ_1 and ℓ_2, and $F_\pm(s)$ includes the contributions of $f_i(s)$ and the remainder of the equations coming from R, C', and the integral for the larger ℓ_i beyond ℓ_m. Obviously, these integral equations have a singular kernel as $s, s' \to 0$, and so some care must be used in dealing with them, either for analytic or numerical purposes.

To proceed, we make a Mellin transformation of the equations to obtain

$$\frac{\Delta_\pm(\xi)}{2} \pm \frac{\sin\alpha}{(2\pi)^2 i} \int_{c-i\infty}^{c+i\infty} d\xi' \, \Delta_\pm(\xi') \int_0^{\ell_m} (s')^{-\xi'} ds'$$

$$\times \int_0^\infty \frac{s^\xi \, ds}{s'^2 - 2s\, s' \cos\alpha + s'^2} = \not{\phi}_\pm(\xi) \, . \tag{12}$$

In this equation, $\Delta(\xi) \equiv \int_0^\infty D(s) s^{\xi-1} ds$. To obtain Eq. (12), we have made the direct Mellin integration and have used the inverse relations:

$$D(s) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \Delta(\xi) s^{-\xi} ds.$$

The choice of the constant c will be discussed later. The transform of the function $F_\pm(s)$ is $\not{\phi}_\pm(\xi)$. In arriving at this equation we have interchanged the order of integrations, but this can be justified à posteriori. Next we can evaluate[9] the right-most integral in Eq.(12):

$$\int_0^\infty \frac{s^\xi \, ds}{s^2 - 2s\, s' \cos\alpha + s'^2} = \pi (s')^{\xi-1} \frac{\sin[(\pi - \alpha)\xi]}{\sin\alpha \; \sin(\pi\xi)} \, ,$$

where $0 < \alpha < 2\pi$, and $-1 < \mathrm{Re}\, \xi < 1$. Then we can carry out the next integral:

$$\int_0^{\ell_m} (s')^{\xi-\xi'-1} ds' = \frac{(\ell_m)^{\xi-\xi'}}{\xi - \xi'} \, ,$$

where we require that $\mathrm{Re}(\xi - \xi') > 0$. Thus the equations become:

$$\frac{\Delta_\pm(\xi)}{2} \pm \frac{r(\xi)}{4\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(\ell_m)^{\xi-\xi'} \Delta_\pm(\xi')d\xi'}{(\xi - \xi')} = \not{I}_\pm(\xi) , \qquad (13)$$

where

$$r(\xi) \equiv \frac{\sin(\pi - \alpha)\xi}{\sin \pi \xi} .$$

This equation is in standard singular integral equation form, and thus may be dealt with using known techniques.[10] We begin by considering the homogeneous equation, and introduce a function

$$H(\xi) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(\ell_m)^{\xi-\xi'} \Delta^{(0)}(\xi')d\xi'}{(\xi - \xi')} ,$$

where $\Delta^{(0)}$ is a solution of the homogeneous equation. Clearly $H(\xi)$ is an analytic function in the finite half-planes defined by $Re(\xi) \gtrless c$, and it has a discontinuity in crossing the contour of integration. If we define $H^{(\pm)}(\xi)$ to be the functions obtained from the integral in which $Re(\xi) \gtrless c$, respectively, together with their analytic continuations, we then easily find that

$$\Delta^{(0)}(\xi) = H^{(+)}(\xi) - H^{(-)}(\xi) ,$$

and so

$$\frac{1}{2} \left[H_\pm^{(+)}(\xi) - H_\pm^{(-)}(\xi) \pm r(\xi) H_\pm^{(+)}(\xi)\right] = 0 . \qquad (14)$$

This equation can be used to deduce the analytic structure of $\Delta_\pm^{(0)}(\xi)$.

We eventually wish to deduce the analytic structure of $D(s)$, which will require using the inverse transform on $\triangle(\xi)$. For the latter step, in the limit $s \to 0$, the contour in the inverse transform can be closed on the left, and so the behavior of $D(s)$ is determined by singularities on the left of the contour. In this region $H^{(-)}(\xi)$ is analytic, and so we can solve for $H^{(+)}(\xi)$ in terms of $H^{(-)}(\xi)$ using Eq. (14) to analytically continue $H^{(+)}(\xi)$ to the left of the contour. Thus we find:

$$H_{\pm}^{(+)}(\xi) = (1 \pm r(\xi))^{-1} H_{\pm}^{(-)}(\xi) .$$

A solution of this equation can be obtained by taking the logarithm of the equation and then noting that $\log H(\xi)$ is a function with a given discontinuity on the contour. The solution of this problem (the "Hilbert problem") then can be written:[11]

$$H_{\pm}(\xi) = \exp\left\{ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\ell n[1 \pm r(\xi')]}{\xi' - \xi} \, d\xi' \right\} .$$

We then see that $H_{\pm}^{(-)}(\xi)$ is analytic and nonzero on the left of the contour, and if we use Eq. (14) to analytically continue $H_{\pm}^{(+)}(\xi)$, it is evident that $H_{\pm}^{(+)}(\xi)$ will also be analytic unless

$$1 \pm r(\xi) = 0 .$$

At such points, $H_{\pm}^{(+)}(\xi)$ will generally have poles. Thus $\triangle_{\pm}^{(0)}(\xi)$ also has poles at such points.

The solution of Eq. (13) may now be obtained by introducing

$$\mathcal{H}(\xi) \equiv \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(\ell_m)^{\xi-\xi'} \Delta(\xi')d\xi'}{(\xi' - \xi)} ,$$

to get

$$(1 \pm r(\xi)) \mathcal{H}_{\pm}^{(+)}(\xi) - \mathcal{H}_{\pm}^{(-)}(\xi) = 2 \Phi_{\pm}(\xi) . \qquad (15)$$

Using

$$(1 \pm r(\xi)) = H_{\pm}^{(-)}(\xi)/H_{\pm}^{(+)}(\xi) ,$$

this equation can be written:

$$\frac{\mathcal{H}_{\pm}^{(+)}(\xi)}{H_{\pm}^{(+)}(\xi)} - \frac{\mathcal{H}_{\pm}^{(-)}(\xi)}{H_{\pm}^{(-)}(\xi)} = \frac{2 \Phi_{\pm}(\xi)}{H_{\pm}^{(-)}(\xi)} .$$

Again we have a discontinuity equation to satisfy and we obtain as the solution:

$$\mathcal{H}_{\pm}(\xi) = \frac{H_{\pm}(\xi)}{i\pi} \int_{c-i\infty}^{c+i\infty} \frac{\Phi_{\pm}(\xi')}{H_{\pm}^{(-)}(\xi')} \frac{d\xi'}{\xi' - \xi} .$$

Hence $\mathcal{H}_{\pm}^{(-)}(\xi)$ is analytic on the left of the contour, and if we use Eq. (15) to obtain the analytic continuation of $\mathcal{H}_{\pm}^{(+)}(\xi)$, we finally find that

$$\Delta(\xi) = \frac{\mp r(\xi) \mathcal{H}_{\pm}^{(-)}(\xi) + 2 \Phi(\xi)}{1 \pm r(\xi)} .$$

Thus, we can generally expect poles in $\Delta(\xi)$ in the left half plane wherever $1 \pm r(\xi) = 0$ on the left of the contour.

To complete the discussion, it is necessary to specify the contour; i.e., to determine c. In the first place, from the restriction on $\mathrm{Re}(\xi)$, we require that $-1 < c < 1$. In addition, the preceding development will only give a meaningful expression for $H(\xi)$ if $\ell n[1 \pm r(\xi)] \to 0$ as $|\mathrm{Im}\ \xi| \to \infty$. It is easily seen that $r(\xi) \sim \exp[(|\pi - \alpha| - \pi)|\mathrm{Im}\ \xi|]$ as $|\mathrm{Im}\ \xi| \to \infty$, so $r(\xi) \to 0$. Thus the logarithm will approach zero at ∞, unless it has an imaginary part of the form $i\pi n$. To guarantee that this does not happen, we can choose $c = 0$, since $r(\xi)$ is real and nonzero on the imaginary axis. Any other c satisfying the limit restriction is equally acceptable as long as the contour would not thereby be distorted from the imaginary axis by going past a zero of $1 \pm r(\xi)$, since in such a case the logarithm would acquire an imaginary part at ∞.

We now can conclude that $D(s)$ will behave as $\sim s^{-\xi_n}$ as $s \to 0$, where ξ_n is a pole in the transform, $\Delta(\xi)$. Such poles will appear if

$$1 = \mp \frac{\sin(\pi - \alpha)\xi_n}{\sin \pi\ \xi_n} ,$$

or

$$\sin \pi\ \xi_n = \mp \sin(\pi - \alpha)\xi_n$$

if $\xi \neq 0$. In the case of Δ_+, the solutions of this equation are

$$\xi_n^{(+)} = -\frac{(2n - 1)\pi}{\alpha}, \qquad -\frac{2n\pi}{2\pi - \alpha} ;$$

and in the case of Δ_-,

$$\xi_n^{(-)} = -\frac{2n\pi}{\alpha}, \qquad -\frac{(2n - 1)\pi}{2\pi - \alpha} ,$$

where n is any positive integer.

In addition to these poles, we must consider other possible singularities in $\Delta(\xi)$. Since $\mathcal{H}^{(-)}(\xi)$ is analytic, the only other possibility would be singularities in $\Phi(\xi)$. In fact, $\Phi(\xi)$ in part comes from contributions to $f(s)$ arising from distributions on the other boundaries, C', and since these contributions will be analytic near $s = 0$, this part of $\Phi_{\pm}(\xi)$ will be the transform of functions which have power series expansions; i.e., they have poles at the negative integers.[12] There will also be a pole in $\Phi_+(\xi)$ at $\xi = 0$, but because $f_-(0) = 0$, $\Phi_-(\xi)$ has no such pole. Thus, to the poles already given, we have additional poles at the integers.

Finally, we must consider singularities related to the remainder from $H_1^{(1)}(x)$, aside from the most singular part which has already been treated. For this we assume that $D(s) \sim s^{\xi}$, and then deduce the form of

$$I_{\xi}(s) = \frac{i\kappa s \sin\alpha}{4} \int_0^{\ell} \frac{H_1^{(1)}(\kappa w)}{w} (s')^{\xi} \, ds' ,$$

where, as in Sec. III, $w = (s'^2 - 2s' s \cos\alpha + s^2)^{\frac{1}{2}}$. This integral can be evaluated using Gegenbauer's addition theorem:[13]

$$I_{\xi}(s) = 2 \sin\alpha \sum_{m=0}^{\infty} (m+1)\, C_m^{(1)}(\cos\alpha) \Bigg\{ H_{m+1}^{(1)}(\kappa s)$$

$$\times \int_0^{s} (s')^{\xi-1} J_{m+1}(\kappa s') ds' + J_{m+1}(\kappa s) \int_s^{\ell} (s')^{\xi-1} H_{m+1}^{(1)}(\kappa s') ds' \Bigg\} ,$$

where $C_m^{(1)}(\cos\alpha)$ is a Gegenbauer polynomial. The Hankel function can be divided into two parts:

$$H^{(1)}_{m+1}(\kappa s) = \frac{2i}{\pi} \log s - J_{m+1}(\kappa s) + \psi_{m+1}(s) ,$$

where

$$\psi_{m+1}(s) = \sum_{n=0}^{\infty} a_n s^{-m-1+2n} .$$

If the series for $\psi_{m+1}(s)$ and $J_{m+1}(\kappa s)$ are introduced into the expression for $I_\xi(s)$, it is then easily found that $\log s$ does not occur in $I_\xi(s)$, and that the powers of s which occur are $\xi + 2n$, and $n + 1$, where n is an integer ≥ 0. Thus each of the poles, ξ_i, generates a series of poles spaced at even integers from ξ_i. This completes the determination of the analytic form of the solutions of the boundary integral equation at a sharp corner.

A few comments are appropriate at this point: In deducing the analytic form of the solution, we have assumed that the unknown functions on the remainder of the boundary away from the corner of interest can be treated as if they were known. The legitimacy of this approach can be rigorously established following the complete treatment of singular equations, but we did not feel that such an approach, which mainly only increases the complexity of notation and the bulk of the equations, was particularly illuminating and so we have chosen the more heuristic approach given above. We refer the interested reader to the rigorous treatments for a full discussion.[10]

In the above analysis, we have assumed also that the poles which appear are simple. While this is generally the case, in special cases, poles may come together. For example, if $\alpha = 2\pi/3$ we find that two poles occur at $\xi = -\frac{3}{2}$ for $\Delta^{(+)}$. In such a situation, the s-space function then has a term of the form $s^{-\xi} \log s$ as well as the usual $s^{-\xi}$.

V. NUMERICAL EXAMPLES

To illustrate the effectiveness of the boundary distribution method, we have used it to find approximate eigenvalues for a number of polygons in which the eigensolutions for $\psi(\vec{r})$ and κ are known. Thus we look for solutions of the integral equation in which $f(\vec{r}) = 0$. To our knowledge, the corresponding distributions, $D(\vec{r})$, cannot be obtained analytically in these cases so a direct comparison of the numerical results for them cannot be made.

At the outset, it should be noted that we do not feel that the boundary method is necessarily the best choice for finding such eigenvalues, and it is not for such problems that we ultimately wish to use it. A distinct disadvantage as compared with the finite element method, for example, is that it does not seem to satisfy an extremum condition, and, for the lowest eigenvalue, a minimum principle. Thus, by changing certain parameters in the calculation it is possible to find values for κ which change from below the analytic value to above it, and for a suitable choice one can get as accurate a result as desired. In the calculations to be described, this happens as the balance between the corner regions and the central regions is varied, even with the total number of points fixed. Thus the accuracy of the calculated κ is not completely satisfying as an indicator of the overall accuracy of the calculation.

Another disadvantage of the method for finding eigenvalues is that κ occurs in the kernel of the integral equation so that the approximating matrix must be recalculated for each choice of κ. In the finite element method, such iterative complexity is not necessary. In addition, the kernel is a complicated function, and the ensuing matrix has few, if any, zero elements. Thus, although fewer points need be used in the boundary method, it is not clear that overall efficiency is obtained. There is generally a trade-off between storage requirements and the complexity involved. On the other hand, for scattering problems it is not necessary to iterate the matrices, and the automatic satisfying of the outgoing scattered wave boundary condition seems to us a great advantage.

We have used the method of this paper to obtain the eigen-solutions for a square, for an equilateral triangle, for a 45° isosceles triangle, and for a 30°-60°-90° triangle. In each case the eigensolution for κ and $\psi(\vec{r})$ can be obtained analytically.

In reducing the integral equation to an approximate finite form we have approximated the integrals in the integral equation in two ways: For $\vec{r}'$ near a corner, we have assumed that $D(s)$ can be expanded in a finite series of terms of the form s^ξ, in which the ξ's chosen are the lowest values in the set of allowed ξ's. Then the kernel was broken into two parts, of which the first included the most singular terms as $\kappa|\vec{r}' - \vec{r}| \to 0$, and the second was the remainder. The first part together with the various (s^ξ)'s was integrated analytically using various rapidly convergent series, while the second part of the kernel was assumed to be approximated by a quadratic form, and this was then integrated exactly in conjunction with the factors s^ξ. The method used for this part of the kernel was quite analogous to that used in obtaining Simpson's rule. On the other hand, for $\vec{r}'$ away from a corner the entire kernel times $D(s')$ was assumed to be approximately a quadratic form in s', and then this function was integrated exactly, again analogously to Simpson's rule.

In the calculations reported here, we are dealing with a closed region, and hence in the kernel no asymptotic condition as $r \to \infty$ is needed. Thus we have chosen the Neumann function Y_1 instead of the Hankel function $H_1^{(1)}$ in the kernel. This has the advantage that the kernel and $D(\vec{r})$ become real. Further, we have chosen to reflect the distribution about one of the sides. This automatically satisfies the $\psi(\vec{r}) = 0$ boundary condition on that side, and no distribution is needed along it. We also find that the results for κ depend on which side is used for reflection in the 30°-60°-90°, and 45° isosceles triangles, and so the differences between solutions gives an indication of the accuracy of the numerical calculations.

A few results are presented in Table 1,[14] but a fuller discussion of these and other calculations will be published elsewhere. The analytic lowest eigenvalues in the various cases are:[15]

$$\kappa\ (\text{square}) = \sqrt{2}\,\pi\ ,$$

$$\kappa\ (\text{equilateral triangle}) = 4\pi/\sqrt{3}\ ,$$

$$\kappa\ (45^\circ\text{-isosceles triangle}) = \sqrt{5}\,\pi\ ,$$

and

$$\kappa\ (30^\circ\text{-}60^\circ\text{-}90^\circ\ \text{triangle}) = 4\pi\sqrt{7}/3\ ,$$

where in each case the longest side was chosen of unit length. In view of the lack of some variational principle, it is perhaps surprising that the eigenvalues are found as accurately as they are. This is the more surprising when the solutions for $\psi(\vec{r})$ are considered. In the cases given in the table, we have used the distribution found to calculate $\psi(\vec{r})$ at points inside the boundary using essentially the same integration approximations as were used in the integral equation itself. At points far from the boundary, the calculated $\psi(\vec{r})$'s agree quite well with the analytic values which are given in the Appendix. On the other hand, for points $\vec{r}$ near a boundary, the calculated errors were found to be $\sim 10^{-3}$-10^{-4}. These errors varied from point to point, of course, and decreased as the number of points per side was increased, but in all cases the errors in $\psi(\vec{r})$ compared to those in κ seemed more what one might expect if κ satisfied a variational principle than if it did not.

At any rate, the results clearly show that it is feasible to solve problems numerically by making use of distributions on the boundary, and hence an alternative to the finite-element or finite-difference methods is available.

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APPENDIX

In the numerical analysis we have made comparisons with various analytic eigensolutions of Helmholtz' equation. The solutions for the square are well known. For the various triangles they are:

Equilateral triangle

$$\psi(x,y) = \sin\frac{2\pi}{\sqrt{3}}(\sqrt{3}\,x + y) - \sin\frac{2\pi}{\sqrt{3}}(\sqrt{3}\,x - y) - \sin\frac{4\pi}{\sqrt{3}}y,$$

Isosceles right triangle

$$\psi(x,y) = \sin m\pi x \sin n\pi y - (-1)^{m+n} \sin n\pi x \sin m\pi y ,$$

and

$(30^\circ$-60°-$90^\circ)$ triangle

$$\psi(x,y) = \cos\frac{2\pi}{3}(5x + \sqrt{3}\,y) - \cos\frac{2\pi}{3}(5x - \sqrt{3}\,y)$$

$$+ \cos\frac{2\pi}{3}(-x - 3\sqrt{3}\,y) - \cos\frac{2\pi}{3}(-x + 3\sqrt{3}\,y)$$

$$+ \cos\frac{2\pi}{3}(-4x + 2\sqrt{3}\,y) - \cos\frac{2\pi}{3}(-4x - 2\sqrt{3}\,y).$$

FOOTNOTES AND REFERENCES

* This work was supported by the U. S. Atomic Energy Commission.

1. See, for example, W. Pogorzelski, Integral Equations and Their Applications (Pergamon, Long Island City, N. Y., 1966), p. 230, ff.

2. Ref. 1, p. 239 ff.

3. See, for example, M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions (U. S. Government Printing Office, Washington, D. C., 1964), p. 358, Eq. (9.1.3).

4. To obtain the result, we use the relation $H_0^{(1)}(x)' = -H_1^{(1)}(x)$ and the integrals in Ref. 3, p. 485, Eqs. (11.4.8), (11.4.9).

5. We have chosen $H_1^{(1)}$ in the kernel, and so the spurious κ's are not real. On the other hand, for a closed region it is more convenient to use Y_1 instead, and the analysis goes through as before except that the spurious modes occur then for $Y_1(\kappa R) = 0$. In this case the κ's are real.

6. See Ref. 3, p. 363, Eq. (9.1.79).

7. See Ref. 3, p. 358, Eqs. (9.1.5), (9.1.6); p. 361, Eq. (9.1.27).

8. The result is easily obtained from the generating function for Bessel functions. Cf. Ref. 3, p. 361, Eq. (9.1.41).

9. A. Erdélyi et al., Tables of Integral Transforms, vol. 1 (McGraw-Hill Book Co., Inc., New York, N. Y. 1954), p. 309, Sec. 6.2, Eq. (12).

10. See Ref. 1, Chapter XVI.

11. See Ref. 1, Chapter XVII.

12. At first sight, one might expect that $f_+(s)$ should only have even terms, and $f_-(s)$ only odd, but except for cases with special symmetries both functions have even and odd powers. The The only restriction is that $f_-(0) = 0$.

13. See Ref. 3, p. 363, Eq. (9.1.80).

14. In Table I, the number of points used per side is given in the column "PPS".

15. Grove C. Nooney (Dissertation), On the Vibrations of Triangular Membranes, Dept. of Mathematics, Stanford, University, Stanford, California, Oct. 22, 1953.

TABLE 1

Case	PPS	Refl. Side	κ	Error
Square	26	Any	4.442853185	-2.9753×10^{-5}
	38		4.442881485	-1.453×10^{-6}
	46		4.442882851	-8.7×10^{-8}
Equilateral Triangle	26	Any	7.255218367	2.091×10^{-5}
	36		7.255198910	1.453×10^{-6}
	46		7.255197164	2.93×10^{-7}
	66		7.255197276	1.81×10^{-7}
45°-45°-90°	26	Long side	7.024773683	-4.1047×10^{-5}
	26	Short side	7.024231683	-5.83047×10^{-4}
	46	Long side	7.024815759	1.029×10^{-6}
	46	Short side	7.024783389	-3.1341×10^{-5}
30°-60°-90°	26	Short side	11.08190852	-5.8866×10^{-4}
	26	Middle side	11.08296905	4.7187×10^{-4}
	26	Long side	11.08244263	-5.455×10^{-5}
	46	Short side	11.08253880	4.162×10^{-5}
	46	Middle side	11.08245430	-4.288×10^{-5}
	46	Long side	11.08240105	-9.613×10^{-5}

FIGURE CAPTIONS

Fig. 1. Illustration of variables used for points inside a circle.

Fig. 2. Illustration of variables used for points outside a circle.

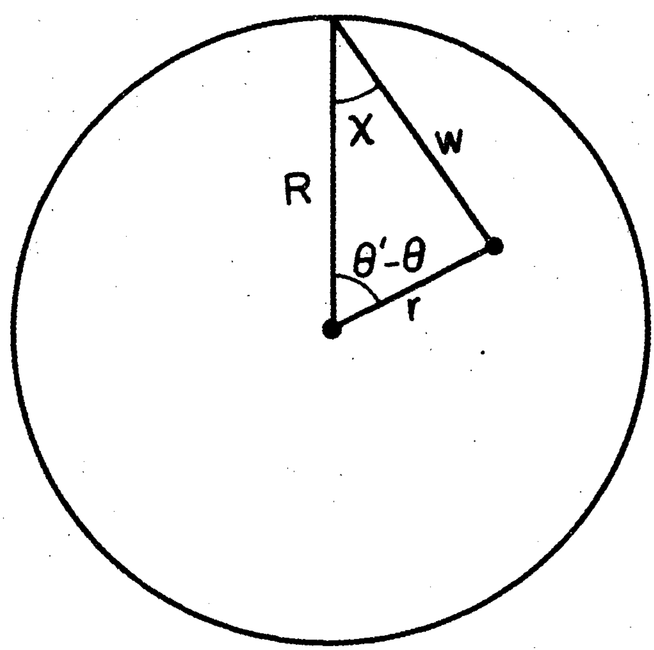

Fig. 1.

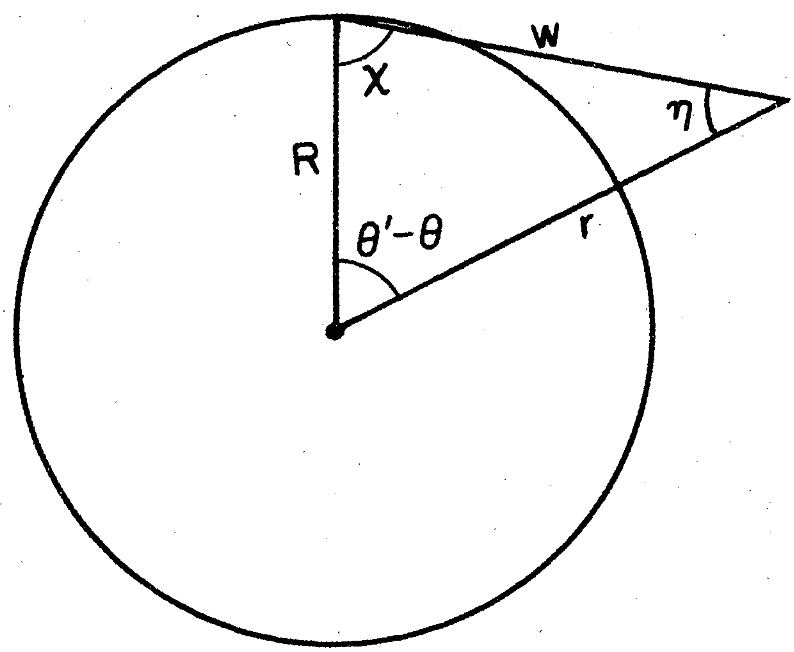

Fig. 2.

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