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ONLINE CHANGE-POINT DETECTION FOR FUNCTIONAL DATA

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Abstract

In this paper, we propose a CUSUM-type online change-point detection procedure for monitoring a change in the mean function of dependent functional data. Our method is fully nonparametric and does not require dimension reduction for the functional observations. For the proposed sequential monitoring scheme, we provide the limiting distribution of the CUSUM monitoring statistic under the null hypothesis of no change, which yields the threshold to control the global false alarm rate asymptotically. Furthermore, we show that the proposed sequential test has an asymptotic power one. The method is illustrated by means of Monte Carlo simulation studies and an application to a real data set.

Keywords: Change-point detection, CUSUM statistic, functional data, online sequential monitoring.

1. Introduction

With modern technology related to data collection and storage, functional data have become increasingly available in many scientific fields, such as meteorology, biomedicine, chemistry and neuroimaging (Yao et al. (2005),Zhu et al. (2014), Yu et al. (2016), Zhu et al. (2022). Functional data analysis studies theory and analysis of data sampling from infinite-dimensional random objects in the form of functions, curves or images, or other general objects (Dubey and Müller (2020), Yang and Yao (2023)), and has attracted increasing interest in the statistical research community. The monographs of Ramsay and Silverman (2005), Ferraty (2011), Hsing and Eubank (2015) and the review article of Wang et al. (2016) provide a comprehensive introduction of functional data analysis methods.

An important problem in statistical modeling of functional data is the problem of testing for structural stability as changes in the data generating process may have a substantial impact on statistical inference developed under the assumption of stationarity. Because of its importance, there exists a lot of literature about change-point detection in functional observations. Berkes et al. (2009) proposed a CUSUM-type test for change-point detection in a sequence of independent functional data, and the asymptotic theory for the change-point estimator was studied by Aue et al. (2011). Zhang et al. (2011) and Aston and Kirch (2012) considered the detection and estimation of the change-point in functional time series data. Aue et al. (2014) studied change-point detection in the context of functional linear models with spatially distributed functional data. Gromenko et al. (2017) developed a statistical method for inferring whether the pattern of a spatio-temporal process has changed over a specific time period, and their statistic reduces to that of Berkes et al. (2009) when there is only a single spatial location. Aue et al. (2018) proposed a fully functional change-point detection procedure, which does not require dimension reduction for functional observations. Chiou et al. (2019) introduced a two-stage procedure for identifying multiple changes in the mean function of a sequence of functional data, and Chen et al. (2023) further studied the change-point estimator proposed in Chiou et al. (2019) and presented a new multiple change-point detection criterion. Horváth et al. (2023) developed a new method to test the mean stability of functional data observed irregularly and subject to changes on observation frequency.

The aforementioned works focused on a-posteriori or offline change-point analysis for functional data. This means that the full data set is available for analysis at the outset of the investigation, and can be utilized to test for change-points and estimate their locations. However, with the rapid improvement of data acquisition technology, data are collected more and more automatically, and functional data streams involving continuous sequential observations appear frequently in modern financial and biomedical industries (Aue et al., 2014; Horváth et al., 2023), which require online monitoring procedures to sequentially make a decision after each new observation whether a break is likely to have occurred. Wald introduced online sequential monitoring procedures in 1943 (Wald, 2004), which are of great interest wherever data need to be monitored and changes need to be detected as soon as they have occurred. For example, monitoring medical data of patients requires quick interventions in case of critical changes (Fried and Imhoff, 2004). In finance, a steady control of accounting processes enables a fast detection of fraudulent activities in order to avert further damage (Berkes et al., 2004). However, Wald's original approach can not ensure the probability of having a false alarm before some specified time to be controlled, and relies on some strong parametric assumptions about the pre-change and post-change distributions. Some important works along the line of Wald include Wald (1945), Page (1954), Lorden (1971), Lai (1995) and Lai (2001).

Chu et al. (1996) modified Wald's approach and introduced a new way of online sequential testing, which allows to control the asymptotic error (Type-I error) if no changes occur while having asymptotic power one under alternatives, and does not require parametric assumptions on the distributions of data. Their method is based on the so-called "non-contamination assumption" which means that there exists a historic data set without a change. The existence of a training data set is not an unreasonable assumption as one would normally have observed a stable process for some time before starting to monitor for changes. Under the framework of Chu et al. (1996), the false alarm rate of the monitoring scheme is controlled. Hence, the scheme performs best in applications in which the costs associated with false alarms are high. In addition, in some engineering applications, the punctuality of the service and the cost of maintenance are crucial to the operation of a company, and both are adversely affected by false alarms. In practice, frequent false alarms will result in a waste of resources in terms of repairing and replacement. Therefore, a controlled Type-I error of the monitoring scheme is important. The work of Chu et al. (1996) has been pursued by others, such as Aue and Horváth (2004), Aue et al. (2009), Zhou et al. (2015), Kirch and Kamgaing (2015), Dette and Gösmann (2020), Horváth et al. (2020, 2021), Wu et al. (2022), Horváth et al. (2022) and Aue and Kirch (2023), to name a few.

To the best of our knowledge, there is no online sequential procedure for monitoring of functional data stream with theoretical guarantees. In this paper, following the framework of Chu et al. (1996), we propose a sequential monitoring scheme for online change-point detection in the mean function of dependent functional data stream. Our methodology is fully nonparametric and does not require dimension reduction for functional observations. For the proposed sequential monitoring scheme, we provide the limiting distribution of the CUSUM monitoring statistic under the null hypothesis of no change, which yields the threshold to control the global false alarm rate asymptotically. Furthermore, we show that the proposed sequential test has an asymptotic power one. Our numerical results show that the proposed method performs well in practical situations.

The rest of this paper is organized as follows. In Section 2, we present the problem setting, the proposed online sequential monitoring scheme, and the general regularity assumptions, and establish the asymptotic behavior of the sequential test statistic under the null and alternative hypotheses. Section 3 provides simulation studies that examine the finite-sample performance of the proposed sequential monitoring statistic. In Section 4, we apply the monitoring scheme to the central England temperature data set. Some concluding remarks are provided in Section 5. All detailed proofs are provided in the Appendix.

2. Methodology and Asymptotic Properties

Let $\{X_i(t), i = 1, 2, \ldots\}$ be a sequence of continuous random functions in $\mathcal{L}^2(\mathcal{T})$, a Hilbert space of square integrable functions on a closed interval $\mathcal{T}$, with the inner product $\langle f, g\rangle = \int_{\mathcal{T}} f(t)g(t)dt$ for any f and g in $\mathcal{L}^2(\mathcal{T})$ and the induced norm $\|f\|^2 = \langle f, f\rangle$. Without loss of generality, we consider $\mathcal{T} = [0, 1]$. Assume that the functional data follow the model

$$X_i(t) = \mu_i(t) + \varepsilon_i(t),\ \ i = 1, 2, \ldots,$$

where $\mu_i(t) = \mathbb{E}\{X_i(t)\}$, and the innovation $\varepsilon_i(t)$ satisfies $\mathbb{E}\{\varepsilon_i(t)\} = 0$ and $\mathbb{E}\left(\|\varepsilon_i\|^2\right) < \infty$. Throughout, the following Assumption 1 is made, roughly entailing that the innovations $\{\varepsilon_i, i = 1, 2, \ldots\}$ are weakly dependent, stationary functional time series.

Assumption 1. *There is a measurable function $f : S^\infty \to \mathcal{L}^2[0, 1]$, where S is a measurable space, and i.i.d. innovations $\{\epsilon_i, i \in \mathbb{Z}\}$ taking values in S such that $\varepsilon_i(t) = f(\epsilon_i, \epsilon_{i-1}, \ldots)(t)$ for $i \geq 1$. Moreover we assume that there exist $\mathfrak{m}$-dependent sequences $\{\varepsilon_{i,\mathfrak{m}}(t), i = 1, 2, \ldots\}$ such that for some $p > 2$*

$$\sum_{\mathfrak{m}=0}^{\infty} \{\mathbb{E}\left(\|\varepsilon_i - \varepsilon_{i,\mathfrak{m}}\|^p\right)\}^{1/p} < \infty,$$

where $\varepsilon_{i,\mathfrak{m}}(t) = f\left(\epsilon_i, \ldots, \epsilon_{i-\mathfrak{m}+1}, \epsilon^*_{i-\mathfrak{m}}, \epsilon^*_{i-\mathfrak{m}-1}, \ldots\right)(t)$ with ϵ^*_i being independent copies of $\varepsilon_{i,0}$ independent of $\{\varepsilon_i, i \in \mathbb{Z}\}$.

Remark 1. *A process that satisfies Assumption 1 is termed $\mathcal{L}^p-\mathfrak{m}$ approximable (Hörmann and Kokoszka, 2010). This is a mild assumption which is satisfied by many processes, such* *as functional auto-regressive and auto-regressive moving average (Bosq, 2000; Aue et al., 2015) and functional generalized auto-regressive conditional heteroscedastic processes (Aue et al., 2017). Aue and van Delft (2020) introduced a new stationarity test for functional time series based on frequency domain methods.*

In this paper we are interested in sequentially monitoring whether or not the mean functions $\{\mu_i(t), i = 1, 2, \ldots\}$ remain stable over time, and follow Chu et al. (1996) we assume that there is a non-contaminated training data of length m for which

$$\mu_i(t) = \mu_0(t), \quad i = 1, \ldots, m. \tag{2.1}$$

The existence of a training data set is not an unreasonable assumption as one would normally have observed a stable process for some time before starting to monitor for changes. We can think of this as a situation where the data have already been observed in a controlled environment (Horváth et al., 2021). Because the mean functions remain stable until observation m, (2.1) can be used for comparisons with future functional observations to find out if the stability assumption still holds. In addition, the size m of the training data set allows for rigorous asymptotic theory by letting m grow to infinity, while still allowing the monitoring to continue possibly forever. We shall introduce a monitoring procedure to discriminate between the null hypothesis of no change in the mean function

$$H_0 : \mu_i(t) = \mu_0(t), \quad i = m + 1, m + 2, \ldots, \tag{2.2}$$

and the alternative hypothesis of a change at an unknown time

$$\begin{aligned} H_1 : &\text{ There is a } k^* \geq 1 \text{ such that } \mu_i(t) = \mu_0(t), i \leq m + k^*, \\ &\text{but } \mu_i(t) = \mu_0(t) + \delta_m(t), \quad i > m + k^*, \end{aligned} \tag{2.3}$$

where the change $\delta_m(t) \neq 0$ is allowed to depend on the size m of the training data to include local changes for which $\|\delta_m\| \to 0$ as $m \to \infty$, and k^* is an unknown change-point time.

To sequentially test the null hypothesis (2.2) against its alternative (2.3), we need to define a stopping rule. Usually, these monitoring procedures are given in terms of first excess times of suitably constructed detectors and threshold functions; see Aue et al. (2009), Horváth et al. (2020), Aue and Kirch (2023), among others. Our sequential method is based on the classical sequential CUSUM procedure, which has been studied in Chu et al. (1996). We propose the following CUSUM monitoring statistic

$$\Gamma(m,k) = \sum_{i=m+1}^{m+k} \left(X_i - \bar{X}_m\right) = k\left(\frac{1}{k}\sum_{i=m+1}^{m+k} X_i - \bar{X}_m\right),$$

where $\bar{X}_m = \frac{1}{m}\sum_{i=1}^{m} X_i$. The quantity $\Gamma(m,k)$ is referred to as a detector. The right-hand side of the equation shows that after each new observation the arithmetic mean of the historic observations and the arithmetic mean of the new observations up to the current monitoring time point are compared. Therefore, it is natural to raise an alarm as soon as $||\Gamma(m,k)||$ crosses a threshold.

Since the monitoring continues as long as no change is detected, the monitoring horizon might be infinite such that a weight function $w(m,k)$ is required in order to control the asymptotic size of the detection procedure. The null hypothesis is rejected and an alarm is raised as soon as

$$w^2(m,k)||\Gamma(m,k)||^2 > c_\alpha,$$

where the critical value c_α is chosen such that the testing procedure holds the level α asymptotically. As long as

$$w^2(m,k)||\Gamma(m,k)||^2 \leq c_\alpha,$$

monitoring continues indefinitely. The stopping time τ_m is thus defined as follows:

$$\tau_m = \begin{cases} \inf\left\{k \geq 1 : w^2(m,k)||\Gamma(m,k)||^2 > c_\alpha\right\} \\ \infty, \quad \text{if } w^2(m,k)||\Gamma(m,k)||^2 \leq c_\alpha \text{ for all } k \geq 1. \end{cases}$$

In our sequential detection approach, the key is to control the rate of false alarms. This is expressed as the following asymptotic condition:

$$\lim_{m\to\infty} \mathbb{P}_{H_0}(\tau_m < \infty) = \lim_{m\to\infty} \mathbb{P}_{H_0}\left(\sup_{k\geq 1} w^2(m,k)||\Gamma(m,k)||^2 > c_\alpha\right) = \alpha, \tag{2.4}$$

where $\mathbb{P}_{H_0}$ denotes the probability under the null hypothesis (2.2), and $0 < \alpha < 1$ is a prescribed significance level. According to (2.4), the probability of false stopping (rejection of the null hypothesis when it is correct) is α, if m is large. We will show that if the weight function is chosen appropriately (2.4) is met and the proposed sequential test has an asymptotic power one, that is

$$\lim_{m\to\infty} \mathbb{P}_{H_1}(\tau_m < \infty) = 1. \tag{2.5}$$

If we can derive the limiting distribution of $\sup_{k\geq 1} w^2(m,k)||\Gamma(m,k)||^2$ under the null hypothesis and some mild conditions, choosing the critical value c_α as the $(1-\alpha)$-quantile of this limiting distribution achieves (2.4). To this end, we impose the following regularity conditions on the weight function:

Assumption 2. *Let the weight function satisfy*

(i) $w(m,k) = m^{-1/2}\rho\left(\frac{k}{m}\right)$ *for* $k > l_m$ *with* $\frac{l_m}{m} \to 0$ *and* $w(m,k) = 0$ *for* $k \leq l_m$*, where* $\rho : [0,\infty) \to \mathbb{R}^+$ *is a positive and continuous function.*

(ii) $\lim_{t\to 0} t^\gamma \rho(t) < \infty$ *for some* $0 \leq \gamma < \frac{1}{2}$.

(iii) $\lim_{t\to\infty} t\rho(t) < \infty$.

Remark 2. *Part (i) allows to start the monitoring only after some observations have been collected in order to avoid false alarms at the very beginning of the monitoring period which can easily be caused as the first values of the monitoring statistic are quite volatile due to the small monitoring sample. Parts (ii) and (iii) show that the behavior of the weight function is controlled at the beginning and the infinite end of the monitoring period. The weight function depends on the tuning parameter* $\gamma \in [0, 1/2)$*, which adjusts the sensitivity of the testing procedure. Note that the right endpoint 1/2 is excluded, since* H_0 *would else, due to* *the law of the iterated logarithm for Brownian motions at zero, be rejected with probability one regardless whether it is true or not (Aue and Kirch, 2023). In this paper we use the weight function*

$$w(m,k) = m^{-1/2}\left(1+\frac{k}{m}\right)^{-1}\left(\frac{k}{m+k}\right)^{-\gamma},\ 0 \le \gamma < \frac{1}{2},$$

that is, $\rho(t) = (1+t)^{-1}\left(\frac{t}{1+t}\right)^{-\gamma}$, *which is popular in the literature as it leads to nice limiting distributions and it fulfills all necessary assumptions for the theoretical results to follow (Horváth et al., 2021).*

The following Assumption 3 is required to control the behavior at the beginning and at the infinite end of the monitoring period.

Assumption 3. *Assume that the following Hájek-Rényi-type inequalities hold:*

(i) For all $0 \le \beta < \frac{1}{2}$,

$$\sup_{1\le k\le m}\left\|\frac{1}{m^{\frac{1}{2}-\beta}k^{\beta}}\sum_{i=1}^{k}\varepsilon_i\right\| = O_P(1) \qquad \textit{as } m \to \infty.$$

(ii) For any sequence $k_m > 0$,

$$\sup_{k\ge k_m}\left\|\frac{1}{k}\sum_{i=1}^{k}\varepsilon_i\right\| = O_P\left(\frac{1}{\sqrt{k_m}}\right) \textit{ uniformly in } m \textit{ as } k_m \to \infty.$$

Denote the long-run covariance function of the error sequence $\{\varepsilon_i, i = 1, 2, \ldots\}$ as $C_\varepsilon(t,s) = \sum_{l=-\infty}^{\infty} \mathrm{cov}\{\varepsilon_0(t), \varepsilon_l(s)\}$. It constitutes the limiting covariance kernel of $\sqrt{n}$ times the centred sample mean under H_0. This kernel was considered initially in Hörmann and Kokoszka (2010). More details about it can be found in Panaretos and Tavakoli (2013) and Horváth et al. (2013). The Hilbert-Schmidt integral operator with the kernel $C_\varepsilon(t,s)$ is

$$\mathcal{C}(f)(s) = \int_0^1 C_\varepsilon(t,s)f(t)\mathrm{d}t,\ f \in \mathcal{L}^2[0,1],$$

which further defines a non-increasing sequence of non-negative eigenvalues $\{\lambda_l, l \ge 1\}$ and a corresponding orthonormal basis of eigenfunctions $\{\phi_l, l \ge 1\}$ satisfying

$$\mathcal{C}(\phi_l)(s) = \int_0^1 C_\varepsilon(t,s)\phi_l(t)\mathrm{d}t = \lambda_l\phi_l(s). \tag{2.6}$$

We estimate C_ε using a kernel long-run covariance estimator computed from $\left\{X_i - \bar{X}_m, 1 \le i \le m\right\}$. Let $\mathcal{K}$ be a continuous and bounded kernel function satisfying

$$\mathcal{K}(0) = 1, \mathcal{K}(u) = \mathcal{K}(-u), \mathcal{K}(u) = 0 \text{ if } |u| > c \text{ for some } c > 0. \tag{2.7}$$

The estimator for C_ε is given by

$$\begin{aligned}\widehat{C}_\varepsilon(t,s) &= \sum_{j=-(m-1)}^{m-1} \mathcal{K}\left(\frac{j}{h}\right) \widehat{\eta}_j(t,s) \\ &= \widehat{\eta}_0(t,s) + \sum_{j=1}^{m-1} \mathcal{K}\left(\frac{j}{h}\right) \left\{\widehat{\eta}_j(t,s) + \widehat{\eta}_j(s,t)\right\},\end{aligned}$$

where

$$\widehat{\eta}_j(t,s) = \begin{cases} \frac{1}{m}\sum_{i=1}^{m-j} \left\{X_i(t) - \bar{X}_m(t)\right\}\left\{X_{i+j}(s) - \bar{X}_m(s)\right\}, & 0 \le j < m, \\ \frac{1}{m}\sum_{i=1-j}^{m} \left\{X_i(t) - \bar{X}_m(t)\right\}\left\{X_{i+j}(s) - \bar{X}_m(s)\right\}, & -m < j < 0, \end{cases}$$

and $h = h(m)$ is a smoothing parameter satisfying

$$h(m) \to \infty, \quad h(m)/m \to 0, \text{ as } m \to \infty. \tag{2.8}$$

We assume that

$$\mathfrak{m}\left\{\mathbb{E}\left(\|\varepsilon_i - \varepsilon_{i,\mathfrak{m}}\|^p\right)\right\}^{1/p} \to 0, \text{ as } \mathfrak{m} \to \infty. \tag{2.9}$$

Condition (2.9) holds if, in Assumption 1, $\left\{\mathbb{E}\left(\|\varepsilon_i - \varepsilon_{i,\mathfrak{m}}\|^p\right)\right\}^{1/p} = O\left(\mathfrak{m}^{-\nu}\right)$ for some $\nu > 1$, which is a fairly mild assumption (Horváth et al., 2013; Berkes et al., 2016). Through $\widehat{C}_\varepsilon$, eigenvalue estimates $\widehat{\lambda}_l$ of λ_l for $l \ge 1$ are defined via the integral operator

$$\int \widehat{C}_\varepsilon(t,s)\widehat{\phi}_l(t)\mathrm{d}t = \widehat{\lambda}_l\widehat{\phi}_l(s). \tag{2.10}$$

Theorem 2 in Horváth et al. (2013) shows that under Assumption 1 and Conditions (2.7)-(2.9)

$$\iint \left\{\widehat{C}_\epsilon(t,s) - C_\varepsilon(t,s)\right\}^2 \mathrm{d}t\mathrm{d}s \xrightarrow{\mathbb{P}} 0. \tag{2.11}$$

We obtain from (2.11) using Lemma 2.2 in Horváth and Kokoszka (2012) that for any $d \ge 1$,

$$\max_{1\le l\le d} \left|\widehat{\lambda}_l - \lambda_l\right| \xrightarrow{\mathbb{P}} 0. \tag{2.12}$$

The eigenvalues of $\mathcal{C}$ determine the limiting distribution of $\sup_{k\geq 1} w^2(m,k) \left\|\Gamma(m,k)\right\|^2$ as detailed in the following theorem.

Theorem 1. *Under Assumptions 1-3 and H_0, we have*

$$\sup_{k\geq 1} w^2(m,k) \left\|\Gamma(m,k)\right\|^2 \xrightarrow{\mathcal{D}} \sup_{0<s<1} s^{-2\gamma} \sum_{l=1}^{\infty} \lambda_l B_l^2(s),$$

where B_l for $l \geq 1$ are i.i.d. standard Brownian motions defined on [0, 1].

Remark 3. *Theorem 1 gives the weak convergence of the proposed monitoring statistic. The result can be used to derive the decision boundary c_α with an asymptotically correct size α, such that we can control the Type-I error of the sequential monitoring procedure. As the limiting distribution depends on the unknown eigenvalues $\{\lambda_l, l \geq 1\}$ and standard Brownian motions, according to* (2.12), *Monte Carlo simulation can be used to approximate this limiting distribution by using estimated eigenvalues (Aue et al., 2018). Implementation details are provided in Section 3.*

Under the alternative, we need to impose some additional but still mild conditions on the weight function. These conditions guarantee that the monitoring still continues after a change occurs. Assumption 4 (i) guarantees that the weight function is not too small at a large t so that a late change can be detected; Assumption 4 (ii) ensures that the weight function is strictly positive after an early change occurs.

Assumption 4. *Assume that*

(i) If $\frac{k^*}{m} \to \infty$ as $m \to \infty$, we have $\liminf_{t\to\infty} t\rho(t) > 0$.

(ii) If $\frac{k^*}{m} \leq t_1$ for all $m \geq 1$ for some $t_1 > 0$, we have $\inf_{t\in[t_2,t_3]} \rho(t) \geqslant c$ for some $t_1 < t_2 < t_3$, where c is a positive number.

The following theorem shows that the monitoring will eventually stop and the null hypothesis will be rejected with an asymptotic probability one under the alternative that a change occurs.

Theorem 2. *Let Assumptions 1-4 be satisfied. Furthermore assume that $\sqrt{m}\,\|\delta_m\| \to \infty$. Then, it holds under H_1*

$$\sup_{k\geq 1} w^2(m,k)\,\|\Gamma(m,k)\|^2 \xrightarrow{\mathbb{P}} \infty.$$

Remark 4. *The quantity $\sqrt{m}\,\|\delta_m\|$ can be thought of as the signal strength, where we require $\sqrt{m}\,\|\delta_m\| \to \infty$ to obtain an asymptotic power one. This shows that the magnitude of the change must be larger for good power behavior if the length of the training data set is smaller. Theorem 2 shows that, under H_1, the asymptotic power of the proposed CUSUM monitoring statistic converges to one as $m \to \infty$, and* (2.5) *holds. Hence, when there is a change-point, the proposed sequential monitoring scheme declares a change in the mean function with a probability approaching one.*

3. Simulation Studies

In this section we present simulation results to examine the finite-sample performance of the proposed sequential monitoring scheme. For the kernel long-run covariance estimator $\widehat{C}_\varepsilon$ in Section 2, we follow the recommendation of Aue et al. (2018) to use the Bartlett kernel function $\mathcal{K}(u) = (1-|u|)\mathbb{1}\{|u|<1\}$ and the bandwidth h is obtained using the data-driven bandwidth selection procedure proposed by Rice and Shang (2017). For more methods on bandwidth selection, please refer to Horváth et al. (2016) and Shang (2018). Based on the estimator $\widehat{C}_\varepsilon$, the first D empirical eigenvalues $\widehat{\lambda}_l$ for $1 \leq l \leq D$ satisfying equation (2.10) were computed. By then simulating D independent standard Brownian motions $B_l(s)$ for $1 \leq l \leq D$ on $[0,1]$, a realization of the limiting distribution given in Theorem 1 is estimated by $\sup_{0<s<1} s^{-2\gamma}\sum_{l=1}^{D}\widehat{\lambda}_l B_l^2(s)$. To select D, we use the popular method based on the cumulative percentage of total variability (CPV) (Ramsay and Silverman, 2005) calculated as

$$\mathrm{CPV}(D) = \frac{\sum_{l=1}^{D}\widehat{\lambda}_l}{\sum_{l=1}^{\infty}\widehat{\lambda}_l},$$

the number of eigenvalues, D, is chosen as the smallest number, D, such that $\mathrm{CPV}(D) \geq 0.95$. We consider data generating process:

$$\varepsilon_i(t) = \sum_{l=1}^{20} \xi_{i,l}\varphi_l(t),$$

where $\varphi_l(t) = 2^{1/2}\cos((l-1)\pi t)$, $\boldsymbol{\xi}_i = (\xi_{i,1}, \ldots, \xi_{i,20})^\top = \eta\boldsymbol{\Psi}\boldsymbol{\xi}_{i-1} + \mathbf{N}_i$, $\boldsymbol{\xi}_0$ and $\mathbf{N}_i = (N_{i,1}, \ldots, N_{i,20})^\top$ are independent and identically distributed (i.i.d.), $N_{i,l}$ for $i \geq 1$ and $1 \leq l \leq 20$ are independent $\mathcal{N}\left(0, 2^{-2l}\right)$, η is set at four levels: 0 (i.i.d. case), 0.25 (low correlation), 0.5 (moderate correlation), and 0.75 (high correlation), and $\boldsymbol{\Psi}$ is a 20×20 diagonal matrix whose diagonal elements consist of independent, centred normal random variables with standard deviations $\left\{2^{-2l} : l = 1, \ldots, 20\right\}$. A scaling was applied to achieve $\|\boldsymbol{\Psi}\|_F = 1$, where $\|\boldsymbol{\Psi}\|_F$ is the Frobenius norm of $\boldsymbol{\Psi}$. Without loss of generality, we set $\mu_0(t) = 0$. The simulation results of $\eta = 0$ and 0.25 are placed in the supplementary material of our paper.

The critical values of the limiting distribution are reported in Table 1. The simulation results in Table 1 are based on 5,000 repetitions of $\sup_{0<s<1} s^{-2\gamma}\sum_{l=1}^{D}\widehat{\lambda}_l B_l^2(s)$. The standard Brownian motions $B_l(s)$ for $1 \leq l \leq D$ are approximated on a grid of 10,000 equally-spaced points in [0,1]. Let K denote the length of monitoring sample. We repeat the Monte Carlo simulation 500 times and report the empirical sizes at the significance levels of 1%, 5% and 10% for the moderate and high correlation cases in Tables 2 and 3, respectively. It can be observed that the empirical sizes are generally close to their theoretical levels, though there are some slight deviations. Overall, the proposed sequential test has reasonably good empirical sizes close to the nominal sizes as suggested by the theory developed in Section 2.

To investigate the empirical power of the sequential test, we consider $\delta_m(t) = \frac{C}{\sqrt{20}}\sum_{l=1}^{20}\varphi_l(t)$, where the constant C controls the magnitude of the change. The Monte Carlo simulation is repeated 500 times and Tables 4 and 5 present the empirical power for $k^* = 2$, 20 and 50 at significance level 5%. There are four major findings. First, the sequential detection

Table 1: Critical value c_α of (2.4).

	$\eta = 0.5$			$\eta = 0.75$		
$\gamma\backslash\alpha$	0.01	0.05	0.10	0.01	0.05	0.10
0.00	3.970	2.645	2.030	6.343	3.875	3.057
0.15	4.089	2.824	2.155	6.512	4.112	3.205
0.25	4.488	2.993	2.361	6.581	4.427	3.445
0.35	4.752	3.246	2.686	7.004	4.736	3.776
0.45	5.624	3.972	3.328	8.126	5.867	4.953

Table 2: Empirical size for the moderate correlation case ($\eta = 0.5$).

	$m = 100, K = 200$			$m = 200, K = 400$			$m = 400, K = 800$		
$\gamma\backslash\alpha$	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
0.00	0.012	0.058	0.082	0.016	0.048	0.090	0.016	0.052	0.094
0.15	0.020	0.064	0.108	0.010	0.068	0.120	0.012	0.054	0.114
0.25	0.010	0.064	0.110	0.012	0.048	0.118	0.019	0.048	0.108
0.35	0.012	0.068	0.096	0.016	0.052	0.108	0.020	0.068	0.112
0.45	0.016	0.052	0.090	0.020	0.058	0.106	0.014	0.056	0.092

Table 3: Empirical size for the high correlation case ($\eta = 0.75$).

	$m = 100, K = 200$			$m = 200, K = 400$			$m = 400, K = 800$		
$\gamma\backslash\alpha$	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
0.00	0.016	0.066	0.112	0.018	0.068	0.112	0.006	0.066	0.128
0.15	0.018	0.076	0.124	0.018	0.064	0.108	0.008	0.058	0.124
0.25	0.019	0.072	0.122	0.015	0.068	0.122	0.014	0.062	0.116
0.35	0.020	0.052	0.098	0.016	0.046	0.100	0.014	0.060	0.110
0.45	0.008	0.050	0.086	0.008	0.038	0.088	0.013	0.062	0.098

works well when k^* is small, that is the change happens immediately after the training sample. Second, our sequential test has higher power when the change occurs relatively early in the monitoring sample since the rejection rates of $k^* = 2$ are generally higher than those of $k^* = 50$. Third, the power tends to one as the magnitude of the change or the historical training sample $m \to \infty$, as predicted by the theory. Lastly, the choice of γ has a minor impact on the power; it usually peaks if $\gamma = 0.25$ or 0.35.

Figure 1 shows the estimated densities of the distributions of the stopping time τ_m for $C = 0.5$, $m = 100$, $K = 200$ and the low correlation case focusing $k^* = 2$ (left panel) and $k^* = 50$ (right panel) at significance level 5%. The Monte Carlo simulation is also repeated 500 times. When a change occurs immediately ($k^* = 2$) in the monitoring sample, the change-point detectors with γ close to 0.5 have the shortest delay in detection, and the change-point detector with $\gamma = 0$ is unlikely to trigger an alarm until approximately after time 20. For a late change like $k^* = 50$, using a large γ can slightly decrease the detection delay, but at the cost of a higher probability of a false alarm, and the detector with $\gamma = 0.35$ or 0.45 is not recommended for the practical application due to the findings in the right panel of Figure 1, where we observe some spurious detections even before a change has occurred. Overall, $\gamma = 0.25$ is recommended because it gives a good balance between the short delay and the negligible proportion of false early alarms.

4. Real Data Analysis

In this section we apply the proposed method to the central England temperature data (Parker et al., 1992), which are available from the website www.metoffice.gov.uk/hadobs/hadcet/. This data set represents the longest continuous thermometer-based temperature record on earth, and contains average daily temperatures in central England from 1772 to 2023. Following Berkes et al. (2009), we treat the annual temperature curves as functional data, with 365 measurements on each curve. As such, the data become a functional data sequence of

Table 4: Empirical power (in %) for the moderate correlation case ($\eta = 0.5$) at significance level 5%.

		$m = 100, K = 200$			$m = 200, K = 400$			$m = 400, K = 800$		
C	$\gamma \backslash k^*$	2	20	50	2	20	50	2	20	50
	0.00	26.8	21.6	14.4	49.2	39.8	35.8	100	100	100
	0.15	29.6	22.2	19.8	58.4	49.4	39.8	100	100	100
0.15	0.25	33.0	29.6	19.4	65.0	54.4	45.2	100	100	100
	0.35	37.2	31.4	27.4	67.0	60.8	53.0	100	100	100
	0.45	36.4	30.6	30.6	58.8	55.2	50.0	100	100	100
	0.00	41.8	32.6	22.6	100	100	93.2	100	100	100
	0.15	51.8	38.8	21.8	100	100	98.6	100	100	100
0.20	0.25	51.8	42.8	30.0	100	100	99.8	100	100	100
	0.35	56.4	44.6	38.4	100	100	100	100	100	100
	0.45	51.2	44.8	34.6	100	100	88.6	100	100	100
	0.00	100	100	61.2	100	100	100	100	100	100
	0.15	100	100	69.4	100	100	100	100	100	100
0.30	0.25	100	100	76.2	100	100	100	100	100	100
	0.35	100	100	74.8	100	100	100	100	100	100
	0.45	100	100	60.0	100	100	100	100	100	100
	0.00	100	100	100	100	100	100	100	100	100
	0.15	100	100	100	100	100	100	100	100	100
0.40	0.25	100	100	100	100	100	100	100	100	100
	0.35	100	100	100	100	100	100	100	100	100
	0.45	100	100	100	100	100	100	100	100	100

Table 5: Empirical power (in %) for the high correlation case ($\eta = 0.75$) at significance level 5%.

		$m = 100, K = 200$			$m = 200, K = 400$			$m = 400, K = 800$		
C	$\gamma \backslash k^*$	2	20	50	2	20	50	2	20	50
	0.00	24.0	18.8	20.0	39.0	34.6	28.2	95.2	86.8	74.6
	0.15	34.4	27.4	23.0	49.0	38.4	32.8	99.6	96.4	87.2
0.15	0.25	27.4	29.6	29.0	46.2	42.2	35.0	100	99.6	90.8
	0.35	38.6	35.0	31.4	56.4	50.2	45.4	100	99.0	90.0
	0.45	38.8	34.4	32.2	53.2	51.2	38.6	93.4	88.2	74.4
	0.00	33.8	27.0	22.6	69.6	59.4	49.8	100	100	100
	0.15	37.6	31.8	26.4	85.2	70.2	54.4	100	100	100
0.20	0.25	46.8	38.2	26.6	89.4	76.6	58.8	100	100	100
	0.35	50.8	38.4	34.8	90.2	80.4	61.2	100	100	100
	0.45	48.4	38.6	34.6	75.2	67.6	59.0	100	100	100
	0.00	90.6	60.8	34.4	100	100	100	100	100	100
	0.15	99.4	70.4	43.6	100	100	100	100	100	100
0.30	0.25	99.2	74.8	45.6	100	100	100	100	100	100
	0.35	99.8	78.2	49.8	100	100	100	100	100	100
	0.45	87.8	62.4	46.6	100	100	100	100	100	100
	0.00	100	100	94.8	100	100	100	100	100	100
	0.15	100	100	99.8	100	100	100	100	100	100
0.40	0.25	100	100	99.8	100	100	100	100	100	100
	0.35	100	100	100	100	100	100	100	100	100
	0.45	100	100	86.8	100	100	100	100	100	100

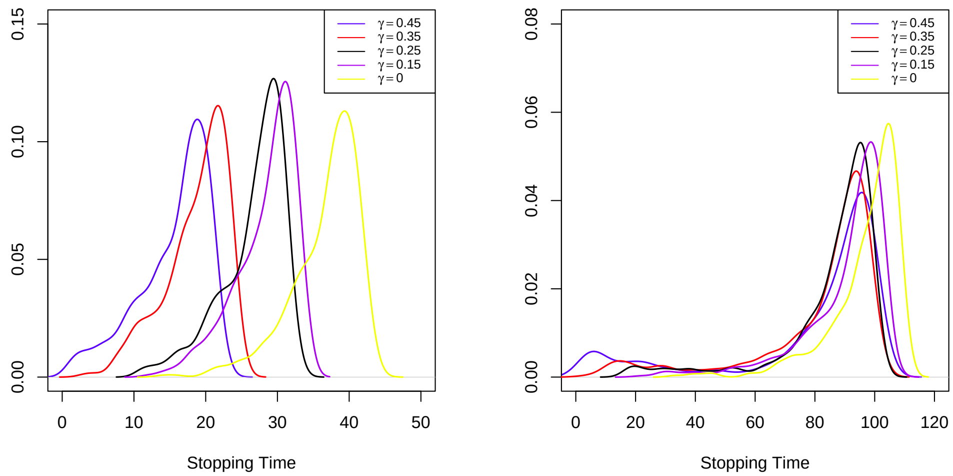

Figure 1: Estimated densities of the stopping time at significance level 5%. Left panel: $k^* = 2$. Right panel: $k^* = 50$.

length 252. We also presmoothed the discrete observations using 12 B-spline basis functions before applying our monitoring procedure proposed in Section 2. We use the significance level $\alpha = 0.05, \gamma = 0.25$ and the first eight eigenvalues proposed by Berkes et al. (2009).

We conduct two change detection experiments using this data set. For the first experiment, we choose the training period of 1782-1846 and the monitoring period of 1847-1986. The length of the training period, m, was chosen in such a way that it does not contain any obvious events that might violate the assumption of a constant mean function of the central England temperature curves. The p-value of the proposed sequential test is 0.008 and the alarm time given by the sequential test is 1934. Once a change is detected, the monitoring procedure is terminated and can be restarted after a long training sample is available. Hence, for the second experiment, we choose the training period of 1935-1984 and the monitoring period of 1985-2023. The p-value of the sequential test is 2×10^{-4} and the alarm time given by the test is 2004. In addition, based on the long-run covariance function estimation method proposed by Rice and Shang (2017), we obtain two alarm times 1928 and 2000. The two change-points, 1927 and 1993, detected by Zhang et al. (2011) are fairly close to the change-points, 1925 and 1992, identified by Berkes et al. (2009). These two change-points roughly correspond to the beginning of mass industralization and the advent of rapid global warming. From the above results, we can see that our sequential monitoring scheme has a short detection delay, which is common for CUSUM-type test procedures. Therefore, the sequential test does well in locating the change-point.

5. Conclusions

In this paper, we propose a CUSUM-type sequential change-point detection method to monitor a change in the mean function of dependent functional data within the framework of Chu et al. (1996). Our methodology is fully nonparametric and does not require dimension reduction for functional observations. The monitoring scheme is shown to have an asymptotically zero Type-II error for any prescribed level of Type-I error. Therefore, the false alarm rate of the monitoring scheme is controlled. The numerical results show that the proposed sequential monitoring scheme performs well.

The limiting distribution of the proposed sequential monitoring statistic under the null hypothesis depends on the unknown eigenvalues of the long-run covariance function of the errors. However, consistent estimation of the long-run covariance function is a thorny issue in practice, as the choice of bandwidth is notoriously difficult. Thus, it is of interest to develop a sequential monitoring procedure that can bypass the estimation of the long-run covariance function. In addition, the behavior of the stopping time for the sequential monitoring scheme is also of interest. These issues are left for future research.

Supplementary Materials

The supplementary material contains all proofs of the theoretical results.

Disclosure Statement

The authors report there are no competing interests to declare.

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