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PARABOLIC HILBERT SCHEMES VIA THE DUNKL-OPDAM SUBALGEBRA

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Abstract. In this note we explicitly construct an action of the rational Cherednik algebra $H_{1,m/n}(\mathcal{S}_n, \mathbb{C}^n)$ corresponding to the permutation representation of $\mathcal{S}_n$ on the $\mathbb{C}^*$-equivariant homology of parabolic Hilbert schemes of points on the plane curve singularity $\{x^m = y^n\}$ for coprime m and n. We use this to construct actions of quantized Gieseker algebras on parabolic Hilbert schemes on the same plane curve singularity, and actions of the Cherednik algebra at $t = 0$ on the equivariant homology of parabolic Hilbert schemes on the non-reduced curve $\{y^n = 0\}$. Our main tool is the study of the combinatorial representation theory of the rational Cherednik algebra via the subalgebra generated by Dunkl-Opdam elements.

1. Introduction

1.1. **Hilbert schemes on singular curves.** It is well-known and classical that, if C is a smooth algebraic curve, then the Hilbert scheme $\mathrm{Hilb}_k(C)$ of k points on C is smooth and, in fact, isomorphic to the symmetric product $\mathrm{Sym}^k(C)$. On the contrary, much less is known in the case where C is a singular curve. In particular, Maulik [36] proved a conjecture of Oblomkov and Shende [40] relating the Euler characteristics of Hilbert schemes of points on a plane curve singularity to the HOMFLY-PT homology of its link. A more general conjecture of Oblomkov, Rasmussen and Shende [41, 24] relates the homology of these Hilbert schemes to the HOMFLY-PT homology of the link.

One possible approach to understanding the homology of $\mathrm{Hilb}_k(C)$ is by constructing the action of interesting algebras on these. Rennemo [47] constructed an action of the two-dimensional Weyl algebra for an integral locally planar curve C (see also [38, 37]), and Kivinen [30] generalized this action to reduced locally planar curves with several components.

In this paper, we relate the geometry of (parabolic) Hilbert schemes on singular curves to the representation theory of the type A rational Cherednik algebra and other related algebras.

More precisely, consider coprime positive integers m and n, and let $C := \{x^m = y^n\}$ be a plane curve singularity in $\mathbb{C}^2$. Note that for every ideal $I \subseteq \mathcal{O}_C$ we have that $\dim(I/xI) = n$. We consider the *parabolic Hilbert scheme* $\mathrm{PHilb}_{k,n+k}(C)$ that is the following moduli space of flags

$$\mathrm{PHilb}_{k,n+k}(C) := \{\mathcal{O}_C \supset I_k \supset I_{k+1} \supset \cdots \supset I_{k+n} = xI_k\} \tag{1}$$

where I_s is an ideal in the ring of functions $\mathcal{O}_C$ of codimension s. Moreover, we set $\mathrm{PHilb}^x(C) := \sqcup_k \mathrm{PHilb}_{k,n+k}(C)$. The natural $\mathbb{C}^*$ action on C naturally lifts to $\mathrm{PHilb}^x(C)$. Since m and n are coprime, the fixed points are precisely the flags of monomial ideals. In particular, the classes of these fixed points form a basis for the localized equivariant cohomology. The first main result of this paper is the following.

Theorem 1.1. *There is a geometric action of the rational Cherednik algebra $H_{1,m/n}(\mathcal{S}_n, \mathbb{C}^n)$ on the localized $\mathbb{C}^*$-equivariant homology of $\mathrm{PHilb}^x(C)$. Moreover, with this action $H_*^{\mathbb{C}^*}(\mathrm{PHilb}^x(C))$ gets identified with the simple highest weight module $L_{m/n}(\mathrm{triv})$.*

Recall that the rational Cherednik algebra $H_{t,c} := H_{t,c}(\mathcal{S}_n, \mathbb{C}^n)$ contains the trivial idempotent $e := \frac{1}{n!}\sum_{p\in\mathcal{S}_n} p$, and we can form the spherical subalgebra $eH_{t,c}e$. As a consequence of Theorem

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1.1 we get that the spherical subalgebra acts on the equivariant homology of the Hilbert scheme $\mathrm{Hilb}(C) := \sqcup_k \mathrm{Hilb}_k(C)$.

Corollary 1.2. *There is an action of the spherical rational Cherednik algebra* $eH_{1,m/n}(\mathcal{S}_n, \mathbb{C}^n)e$ *on the localized* $\mathbb{C}^*$*-equivariant homology of* $\mathrm{Hilb}(C)$*. Moreover, with this action* $H_*^{\mathbb{C}^*}(\mathrm{Hilb}(C))$ *gets identified with* $eL_{m/n}(\mathrm{triv})$*.*

Remark 1.3. By [37, 38] the homology of the Hilbert schemes of singular curves is closely related to the homology of the corresponding compactified Jacobian, equipped with a certain "perverse" filtration. By [24, 42, 43, 49] the latter homology carries an action of the spherical trigonometric Cherednik algebra. Furthermore by [42, 43] the associated graded space admits a natural action of the spherical rational Cherednik algebra corresponding to the *reflection representation* of $\mathcal{S}_n$ (also known as spherical rational Cherednik algebra of $\mathfrak{sl}_n$). The construction of this action uses global Springer theory developed by Yun [52].

The main advantage of our proof of Theorem 1.1 is that it does not use compactified Jacobians or perverse filtration at all. The generators of $H_{1,m/n}(\mathcal{S}_n, \mathbb{C}^n)$ are identified with certain explicit operators in the homology of $\mathrm{PHilb}^x(C)$.

We explore some ramifications of this result. In the theory of rational Cherednik algebras there is a "$t = 0$" and "$t = 1$" dichotomy, see Section 2.1, and in the statement of Theorem 1.1 we assume that $t = 1$. While the representation theory of the Cherednik algebra is very sensitive to this dichotomy, we have a version of Theorem 1.1 in the $t = 0$ case.

To this end, consider the *non-reduced curve* $C_0 := \{y^n = 0\}$. The punctual Hilbert scheme on C_0 is the moduli space of finite-codimensional ideals in the local ring $\mathcal{O}_{C_0,0} = \mathbb{C}[[x, y]]/(y^n)$, and we may define the parabolic (punctual) Hilbert scheme $\mathrm{PHilb}_{k,n+k}(C_0)$ analogously to (1). Again we set $\mathrm{PHilb}^x(C_0) := \sqcup_k \mathrm{PHilb}_{k,n+k}(C_0)$. We show the following "$t = 0$" (or "$m = \infty$") analogue of Theorem 1.1.

Theorem 1.4. *There is a geometric action of* $H_{0,1}(\mathcal{S}_n, \mathbb{C}^n)$ *on the localized* $\mathbb{C}^*$*-equivariant cohomology of* $\mathrm{PHilb}^x(C_0)$*, where* C_0 *is the non-reduced curve* $\{y^n = 0\}$*. Moreover, with this action* $H_*^{\mathbb{C}^*}(\mathrm{PHilb}^x(C_0))$ *gets identified with the polynomial representation* $\Delta_{0,1}(\mathrm{triv})$*.*

Similarly to Corollary 1.2 we get an action of the spherical subalgebra $eH_{0,1}e$ on the equivariant homology of $\mathrm{Hilb}(C_0)$, and under this action $H_*^{\mathbb{C}^*}(\mathrm{Hilb}(C_0))$ gets identified with the polynomial representation of $eH_{0,1}e$.

1.2. **Quantized Gieseker varieties.** Another ramification of Theorem 1.1 connects parabolic Hilbert schemes to the representation theory of quantized Gieseker varieties. These are quantizations of the moduli space $\mathcal{M}(n, r)$ of rank r torsion-free sheaves on $\mathbb{P}^2$ with fixed trivialization at infinity and second Chern class $c_2 = n$. The quantization, denoted $\mathcal{A}_c(n, r)$, depends on a parameter $c \in \mathbb{C}$, see Section 8 for a precise definition. For example, when $r = 1$, $\mathcal{M}(n, 1)$ is simply the Hilbert scheme of n points in $\mathbb{C}^2$ and $\mathcal{A}_c(n, 1)$ is the spherical rational Cherednik algebra, see [19].

There is currently no presentation of the algebra $\mathcal{A}_c(n, r)$ by generators and relations. Nevertheless, Losev [33] managed to classify all finite-dimensional representations for a slightly smaller algebra $\overline{\mathcal{A}_c}(n, r)$ such that $\mathcal{A}_c(n, r) = \mathcal{D}(\mathbb{C}) \otimes \overline{\mathcal{A}_c}(n, r)$. Namely, if $c = m/n$, $\gcd(m, n) = 1$ and c is not in the interval $(-r, 0)$ then $\overline{\mathcal{A}_c}(n, r)$ has a unique irreducible finite-dimensional representation $\overline{\mathcal{L}}_{\frac{m}{n}}(n, r)$, otherwise there are none. Furthermore, the action of $GL(r)$ on $\mathcal{M}(n, r)$ induces quantum comoment map $\mathfrak{gl}(r) \to \mathcal{A}_c(n, r)$ and hence defines an action of $\mathfrak{gl}(r)$ on $\overline{\mathcal{L}}_{\frac{m}{n}}(n, r)$. In [15] Etingof, Krylov, Losev and the second author computed the dimension and graded $\mathfrak{gl}(r)$ character of $\overline{\mathcal{L}}_{\frac{m}{n}}(n, r)$.

In this paper we give a geometric construction of this representation for $m, n > 0$.

Fix an integer $r > 0$, and denote by $\mathcal{C}_r(n) \subseteq \mathbb{Z}^r_{\geq 0}$ the set of r-tuples of non-negative integers that add up to n. For $\gamma = (\gamma_1, \dots, \gamma_r) \in \mathcal{C}_r(n)$ we consider the following parabolic Hilbert scheme

(2)
$$\mathrm{PHilb}^{\gamma,x}(C) := \left\{\mathcal{O}_C \supseteq J^0 \supseteq J^1 \supseteq \cdots \supseteq J^r = xJ^0 : \dim(\mathcal{O}_C/J^0) < \infty \text{ and } \dim(J^{i-1}/J^i) = \gamma_i\right\}.$$

For example, $\mathrm{PHilb}^x(C) = \mathrm{PHilb}^{(1,\dots,1),x}(C)$, where $(1, 1, \dots, 1) \in \mathcal{C}_n(n)$ and $\mathrm{Hilb}(C) := \sqcup_{k\geq 0}\mathrm{Hilb}_k(C)$ is $\mathrm{PHilb}^{(n),x}(C)$, where $(n) \in \mathcal{C}_1(n)$. We define the *compositional parabolic Hilbert scheme* of C to be

$$\mathrm{CPHilb}^{r,x}(C) := \bigsqcup_{\gamma\in\mathcal{C}_r(n)} \mathrm{PHilb}^{\gamma,x}(C). \tag{3}$$

Remark 1.5. Note that if $\gamma_i \leq 1$ for every i then we have a natural isomorphism $\mathrm{PHilb}^{\gamma,x}(C) = \mathrm{PHilb}^x(C)$. In particular, $\mathrm{PHilb}^x(C)^{\times\binom{r}{n}} \subseteq \mathrm{CPHilb}^{r,x}(C)$. Similarly, if there exists i such that $\gamma_i = n$ and $\gamma_j = 0$ for $j \neq i$ then we have a natural isomorphism $\mathrm{PHilb}^{\gamma,x}(C) = \mathrm{Hilb}(C)$, so that $\mathrm{Hilb}(C)^{\times r} \subseteq \mathrm{CPHilb}^{r,x}(C)$.

Remark 1.6. Note that we have chosen one projection to define our parabolic Hilbert schemes. We could have instead chosen the other projection so that, for $\gamma \in \mathcal{C}_r(m)$ we have the parabolic Hilbert scheme $\mathrm{PHilb}^{\gamma,y}(C)$, where the condition $J^r = xJ^0$ in (2) is now replaced by $J^r = yJ^0$. With this, we have

$$\mathrm{CPHilb}^{r,y}(C) := \bigsqcup_{\gamma\in\mathcal{C}_r(m)} \mathrm{PHilb}^{\gamma,y}(C)$$

This distinction will be important for us below.

Theorem 1.7. *There is an action of the quantized Gieseker variety $\mathcal{A}_{m/n}(n, r)$ on the localized $\mathbb{C}^*$-equivariant cohomology of $\mathrm{CPHilb}^{r,y}(C)$. Moreover, with this action $H^{\mathbb{C}^*}_*(\mathrm{CPHilb}^{r,y}(C))$ gets identified with the unique irreducible $\mathcal{A}_{m/n}(n, r)$-module*

$$\mathcal{L}_{\frac{m}{n}}(n, r) = \mathcal{O}(\mathbb{C}) \otimes \overline{\mathcal{L}}_{\frac{m}{n}}(n, r)$$

with Gelfand-Kirillov dimension 1 where elements of negative degree act locally nilpotently. The homology of $\mathrm{PHilb}^{\gamma,y}(C)$ is identified with γ-weight space for $\mathfrak{gl}(r)$ action on $\mathcal{L}_{\frac{m}{n}}(n, r)$.

Example 1.8. When $r = 1$, $\mathcal{C}_r(m) = \{(m)\}$ and $\mathrm{CPHilb}^{1,y}(C) = \mathrm{Hilb}(C)$. Since in this case $\mathcal{A}_{m/n}(n, 1) = eH_{1,m/n}(\mathcal{S}_n, \mathbb{C}^n)e$, we see that Corollary 1.2 is a special case of Theorem 1.7.

Example 1.9. When $n = 1$, the curve C is smooth, and all the spaces $\mathrm{PHilb}^{\gamma}(C)$ are disjoint unions of $\mathbb{Z}_{\geq 0}$ copies of contractible spaces (labeled by $\dim \mathcal{O}_C/J^0$). Therefore the homology of $\mathrm{CPHilb}^{r,y}(C)$ can be naturally identified with

$$H^{\mathbb{C}^*}_*(\mathrm{CPHilb}^{r,y}(C)) = H^{\mathbb{C}^*}_*(\mathrm{pt}) \otimes \bigoplus_{\mathcal{C}_r(m)} \mathbb{C}[X] \simeq H^{\mathbb{C}^*}_*(\mathrm{pt}) \otimes S^m(\mathbb{C}^r) \otimes \mathbb{C}[X].$$

On the other hand, $\overline{\mathcal{A}_c}(1, r)$ is isomorphic to a certain quotient of $\mathcal{U}(\mathfrak{sl}(r))$, and $\overline{\mathcal{L}}_{\frac{m}{n}}(n, r) \simeq S^m(\mathbb{C}^r)$.

1.3. **Coulomb branches and generalized affine Springer fibers.** From the action of a reductive group G on a vector space N, Braverman, Finkelberg and Nakajima [3] construct an associative algebra called the Coulomb branch algebra, which is modeled after the equivariant homology of the affine grassmannian of G, where multiplication is given by convolution. This algebra admits a natural quantization that appears when we take the loop rotation into account for the equivariance. Webster in [50] generalized their construction by introducing a category of line defects, where the BFN quantized Coulomb branch algebra appears as the endomorphism algebra of an object. Roughly speaking, a line defect consists of the choice of a parahoric subgroup $P \subseteq G_{\mathbb{K}}$ and a subspace $L \subseteq N_{\mathbb{K}}$ preserved by P. The BFN quantized Coulomb branch algebra corresponds to the choice $P = G_{\mathbb{O}}$ and $L = N_{\mathbb{O}}$. It turns out that all of the algebras we work with in this paper appear as BFN quantized Coulomb branches or their generalizations:

- The spherical Cherednik algebra $eH_{1,c}(\mathcal{S}_n, \mathbb{C}^n)e$ is the BFN quantized Coulomb branch for $G = \mathrm{GL}_n$ and $N = \mathbb{C}^n \oplus \mathfrak{gl}_n$, [31, 51].
- The full Cherednik algebra $H_{1,c}(\mathcal{S}_n, \mathbb{C}^n)$ appears in the same setting as above, but choosing a nontrivial line defect associated to $P = I$, the standard Iwahori subgroup, and $L = \mathbb{O}^n \oplus \mathfrak{i}$, where $\mathfrak{i}$ is the Lie algebra of the standard Iwahori, [51, 32].
- The quantized Gieseker variety $\mathcal{A}_c(n, r)$ is the BFN quantized Coulomb branch for $G = \mathrm{GL}_n^{\times r}$ and $N = \mathbb{C}^n \oplus \mathfrak{gl}_n^{\oplus r}$. This follows from results of [39] and [34]. This is an example of *symplectic duality* [50] since $\mathcal{A}_c(n, r)$ appears both as the quantized Higgs branch for the Jordan quiver and the quantized Coulomb branch for the cyclic quiver with r nodes.

The recent paper [28] constructs an action of the quantized Coulomb branch in the cohomology of generalized affine Springer fibers in the sense of [21], again by certain convolution diagrams. This has been extended to the parahoric setting in [20]. We identify the different parabolic Hilbert schemes we consider with generalized affine Springer fibers.

- For $\mathrm{Hilb}(C)$, this is [20, Theorem 3.5].
- For $\mathrm{PHilb}^x(C)$, see Proposition 7.19.
- For $\mathrm{CPHilb}^{r,y}(C)$, see Proposition 8.9.

While we take this as a motivation for Theorems 1.1 and 1.7, our proofs do not use any of these technologies, in particular we do not obtain the action via convolution diagrams. The proofs of Theorems 1.1 and 1.4 are based on the study of the combinatorics of the various Hilbert schemes we consider, as well as the combinatorial representation theory of the rational Cherednik algebra. The development of this depends on a suitable presentation of this algebra, and we use work of Webster [51], and more recent work of LePage-Webster [32] to verify, in the case of the scheme $\mathrm{PHilb}^x(C)$, that our action coincides with the one constructed in [20] via convolution diagrams, see Section 7.5.

On the contrary, there is no known set of generators and relations for the algebra $\mathcal{A}_c(n, r)$. However, we use Theorem 1.1 together with [15, Theorem 2.17] that constructs representations of $\mathcal{A}_{m/n}(n, r)$ starting from representations of $H_{n/m}(m)$ to prove Theorem 1.7.

1.4. **Rational Cherednik algebras.** The main idea behind the proof of Theorem 1.1 is to identify a basis in $L_{m/n}(\mathrm{triv})$ that corresponds to the fixed-point basis in $H_*^{\mathbb{C}^*}(\sqcup_k \mathrm{PHilb}_{k,n+k}(C))$. Our main tool to construct this basis is a presentation of the rational Cherednik algebra $H_{t,c}(\mathcal{S}_n, \mathbb{C}^n)$ that is better-suited for this purpose than the usual presentation. To lighten notation, we write $H_c = H_{t,c}(\mathcal{S}_n, \mathbb{C}^n)$ below. Recall that, in its usual presentation, the algebra H_c has generators x_i, y_i $(i = 1, \dots, n)$ and $\mathcal{S}_n$. It is naturally graded, with x_i of degree 1, y_i of degree -1 and $\mathcal{S}_n$ in degree zero. Dunkl and Opdam [12] constructed a family of commuting operators $u_1, \dots, u_n$ of degree 0 in H_c. The algebra H_c is, in fact, generated by u_i, the group algebra of $\mathcal{S}_n$ and two additional generators

$$\tau := x_1(12\cdots n), \lambda := (12\cdots n)^{-1}y_1.$$

It is clear that τ, λ and $\mathcal{S}_n$ already generate the algebra since one can obtain x_1 and y_1 (and hence all x_i and y_i) using them. In Theorem 3.4 we give a complete list of relations between τ, λ, u_i and the generators of $\mathcal{S}_n$. This presentation of the algebra H_c has already appeared in the more complicated cyclotomic setting in the work of Griffeth [27] and Webster [51]. Since some relations become more transparent in the type A setting, we present it in detail. The generators u_i can be, in principle, eliminated, and the remaining relations are listed in Proposition 3.5.

We use the presentation of the algebra H_c via the Dunkl-Opdam operators to, in the case where c is a rational number with denominator precisely n, simultaneously diagonalize the operators u_i on the polynomial representation $\Delta_c(\mathrm{triv})$ and give an explicit combinatorial description of the eigenvalues. We prove that the action of the operators τ and λ sends an eigenvector to a multiple of another eigenvector, and describe the action of $\mathcal{S}_n$ on an eigenbasis explicitly. We remark that this has already appeared in work of Griffeth, see [25, 26, 27] and Remark 1.12, but we reprove these results with combinatorics that are more amenable to our geometric goal.

Theorem 1.10. *Let $c = m/n$, where $m, n \in \mathbb{Z}_{>0}, \gcd(m, n) = 1$. Then the following holds:*

(a) $\Delta_c(\mathrm{triv})$ *has a basis $v_{\mathbf{a}}$ labeled by nonnegative integer sequences $\mathbf{a} = (a_1, \ldots, a_n)$. The action of u_i, τ and λ in this basis is given by*

$$u_i v_{\mathbf{a}} = (a_i - (g_{\mathbf{a}}(i) - 1)c)v_{\mathbf{a}},\ \tau v_{\mathbf{a}} = v_{\pi\cdot\mathbf{a}},\ \lambda v_{\mathbf{a}} = (a_1 - (g_{\mathbf{a}}(1) - 1)c)v_{\pi^{-1}\cdot\mathbf{a}}$$

where $\pi(a_1, \ldots, a_n) = (a_n+1, a_1, \ldots, a_{n-1})$ and $g_{\mathbf{a}}$ is the minimal length permutation sorting the sequence $\mathbf{a}$ to be non-decreasing.

(b) $L_c(\mathrm{triv})$ *has a basis $v_{\mathbf{a}}$ labeled by sequences $(a_1, \ldots, a_n)$ such that $|a_i - a_j| \leq m$ for all i, j and if $a_i - a_j = m$ then $i < j$.*

The action of $\mathcal{S}_n$ in the basis $v_{\mathbf{a}}$ is given in Theorem 4.15.

Remark 1.11. Note that $\pi^{-1} \cdot \mathbf{a}$ is well defined unless $a_1 = 0$. In this case $a_1 - (g_{\mathbf{a}}(1) - 1)c = 0$, so $\lambda \cdot v_{\mathbf{a}}$ is well defined.

The proof of Theorem 1.1 is based, roughly speaking, on the comparison of the basis of fixed points in $H^*(\sqcup_k \mathrm{PHilb}_{k,n+k}(C))$ with the basis given by Theorem 1.10(b).

We give two proofs of Theorem 1.10. One of them is completely explicit, using intertwining operators to construct the basis $v_{\mathbf{a}}$ inductively. This is done in Section 4. The other one uses a Mackey-type result for the algebra H_c. The algebra $H_n(\mathbf{u})$ generated by $u_1, \ldots, u_n$ and $\mathcal{S}_n$ is isomorphic to the degenerate affine Hecke algebra of rank n. In Theorem 5.9 we construct a filtration of $\mathrm{Res}^{H_c}_{H_n(\mathbf{u})} \Delta_c(\mu)$ by $H_n(\mathbf{u})$-modules and explicitly describe the subquotients. As a consequence, we are able to give a combinatorial basis of all standard modules $\Delta_c(\mu)$.

Remark 1.12. In [26, Theorem 5.1] Griffeth constructs, for generic values of the parameter (t, c) an eigenbasis of every standard module, and in [25] he considers the case of the polynomial representation. Both Theorem 1.10 and the construction of a combinatorial basis for standard modules are a consequence of this and [26, Theorem 7.5] after specializing parameters. Our proof and construction of eigenbasis, using a Mackey-type formalism, is more conceptual and its combinatorics seem better-suited for geometric applications.

As further application of the combinatorics of the Dunkl-Opdam presentation of the algebra H_c we are able to give an explicit combinatorial construction of all the maps appearing in the BGG resolution of the module $L_c(\mathrm{triv})$ for $c = m/n$, and we show that the complex formed by these maps is indeed exact. In particular, we give a new construction of this resolution that avoids appealing to the representation theory of finite Hecke algebras at roots of unity via the Knizhnik-Zamolodchikov functor, which uses techniques of complex analysis. Moreover, we are able to give a combinatorial basis in the spirit of that of Theorem 1.10 for *every* simple module $L_c(\mu)$, see Corollary 6.22.

Remark 1.13. More concretely, the standard modules $\Delta_c(n - \ell, 1^\ell)$ and $\Delta_c(n - \ell + 1, 1^{\ell-1})$ have bases labeled by pairs $(\mathbf{a}, T)$ and $(\mathbf{a}', T')$ where T and T' are standard tableaux of the corresponding hook shapes. We explicitly compute matrix elements of the map between standard modules in this basis in the case $c = m/n$. As a consequence, we give two labelings of the basis in $L_c(n - \ell, 1^\ell)$ presented either as a simple quotient of $\Delta_c(1^\ell, n - \ell)$, or as the radical of $\Delta_c(n - \ell - 1, 1^{\ell+1})$, and an explicit bijection between them.

1.5. **Relation to other work.** Finally, we would like to comment on the relations of our work to the existing literature. As we have mentioned above, the Dunkl-Opdam presentation of the Cherednik algebra has already appeared in work of Griffeth and Webster, [26, 27, 51], where it has been used for different purposes. In particular, Griffeth [26, 27] uses the fact that the operators u_i are self-adjoint with respect to the Shapovalov form to compute the norm of elements in standard modules, see also [11], while Webster [51] uses the Dunkl-Opdam subalgebra to give a concrete equivalence between the category $\mathcal{O}$ and modules over the quiver Hecke algebra.

By [37, 38] the homology of the Hilbert schemes of singular curves is closely related to the homology of the corresponding compactified Jacobian which is isomorphic to the Springer fiber in the affine Grassmannian. One would expect a similar connection between our parabolic Hilbert schemes and affine Springer fibers in the affine flag variety. These affine Springer fibers do admit affine pavings, and the combinatorics of the affine cells was studied in detail in [35, 41, 23].

The (co)homology of the affine Springer fibers in affine flag variety was studied in [24, 42, 43, 49] where it was proved that it carries an action of the trigonometric Cherednik algebra. Furthermore, this (co)homology has certain "perverse" filtration, and the associated graded space admits a natural action of the rational Cherednik algebra corresponding to the *reflection representation* of $\mathcal{S}_n$ (also known as rational Cherednik algebra of $\mathfrak{sl}_n$). The construction of this action uses global Springer theory developed by Yun [52]. The combinatorics of finite dimensional representations of the rational Cherednik algebra for $\mathfrak{sl}_n$ was studied by Shin [44].

On the contrary, we find our construction to be more elementary than [42, 43]. Indeed, in our construction of geometric operators τ and λ we use neither perverse filtration nor global Springer theory. The combinatorial presentation of the algebra is easier in the $\mathfrak{gl}_n$ setup. Still, we make an explicit comparison with the results of [23] in Section 4.6, see Remark 4.27.

1.6. **Structure of the paper.** The main body of the paper follows a reverse structure from the introduction. First we study the representation theory of rational Cherednik algebras and then we move on to Hilbert schemes. Section 2.1 is devoted to the usual presentation of the rational Cherednik algebra, as well as background on combinatorics of the extended affine symmetric group. In Section 3.1 we give the Dunkl-Opdam presentation of H_c. Theorem 1.10 is proved in Section 4 and in Section 5 we prove a Mackey-type formula for representations of the algebra H_c. In Section 6 we study standard modules other than $\Delta(\text{triv})$. In particular, we give a combinatorial construction of all the maps in the BGG resolution of $L_{m/n}(\text{triv})$.

We turn to Hilbert schemes in Section 7. First, we examine the case of the reduced curve $C = \{x^m = y^n\}$ and prove Theorem 1.1, see Theorem 7.14. In this section, we also compare the parabolic Hilbert scheme with generalized affine Springer fibers, in particular proving that they admit paving by affine cells, see Section 7.4. Section 8 is devoted to the scheme $\mathrm{CPHilb}^{r,y}(C)$, we prove Theorem 1.7 as Theorem 8.6 and also realize this Hilbert scheme as a generalized affine Springer fiber. Finally, we study the case of the non-reduced curve $C = \{y^n = 0\}$ in Section 9 where we prove Theorem 1.4, see Theorem 9.8.

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2. Background

Throughout the rest of the paper we take $m, n \in \mathbb{Z}_{>0}$ and $\gcd(m,n) = 1$ unless otherwise stated.

2.1. **Rational Cherednik algebra.** We work with the rational Cherednik algebra $H_{t,c} := H_{t,c}(\mathcal{S}_n, \mathbb{C}^n)$ of $\mathcal{S}_n$ acting on $\mathbb{C}^n$ by permuting the coordinates. Let us recall that this is the quotient of the semidirect product algebra $\mathbb{C}\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle \rtimes \mathcal{S}_n$ by the relations

$$[x_i, x_j] = 0, \qquad [y_i, y_j] = 0,$$
$$[y_i, x_j] = c(ij), \qquad [y_i, x_i] = t - c\sum_{j\neq i}(ij)$$

Here t and c are complex parameters. Clearly, for a nonzero complex number $a \in \mathbb{C}^*$, $H_{at,ac} \cong H_{t,c}$. So we have the dichotomy $t = 0$ or $t = 1$. For most of the paper, we will assume that $t = 1$ and write $H_c := H_{1,c}$.

We recall some basic facts about $H_{t,c}$ following [1]. The algebra $H_{t,c}$ is graded, with x_i of degree 1, y_i of degree (-1), and $\mathcal{S}_n$ in degree zero. When $t = 1$, the grading on H_c is internal and defined by the Euler element

$$h = \frac{1}{2}\sum_i (x_iy_i + y_ix_i) = \sum_i x_iy_i + \frac{n}{2} - c\sum_{i<j}(ij).$$

Let us emphasize that the grading on $H_{0,c}$ is not internal. The algebra $H_{t,c}$ is also filtered, with x_i and y_i of filtration level 1 and $\mathcal{S}_n$ of filtration level 0. An important *PBW theorem* states that

$$\operatorname{gr} H_{t,c} = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \rtimes \mathcal{S}_n,$$

where gr denotes associated graded with respect to this filtration. This implies that a basis of $H_{t,c}$ as a $\mathbb{C}$-vector space is given by $x^{\mathbf{a}}\omega y^{\mathbf{b}}$, where $\omega \in \mathcal{S}_n$, $x^{\mathbf{a}} := x_1^{a_1} \cdots x_n^{a_n}$, $\mathbf{a} \in \mathbb{Z}^n_{\geq 0}$ and similarly for $y^{\mathbf{b}}$, $\mathbf{b} \in \mathbb{Z}^n_{\geq 0}$. We will refer to this basis as the *PBW basis* of $H_{t,c}$. Another easy consequence of the PBW theorem is that $H_{t,c}$ contains the following subalgebras:

$$H_n(\mathbf{x}) := \mathbb{C}[x_1, \ldots, x_n] \rtimes \mathcal{S}_n, \qquad H_n(\mathbf{y}) := \mathbb{C}[y_1, \ldots, y_n] \rtimes \mathcal{S}_n.$$

Next we need to consider some modules for $H_{t,c}$. We have the *standard modules*

$$\Delta_{t,c}(\mu) = \operatorname{Ind}_{H_n(\mathbf{y})}^{H_{t,c}} V_\mu \simeq V_\mu \otimes_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_n],$$

where V_μ is an irreducible representation of $\mathcal{S}_n$ corresponding to the Young diagram μ of size n, and y_i annihilate V_μ. In particular, for $\mu = (n)$ the representation V_μ is trivial, and we get the *polynomial representation* $\Delta_{t,c}(\mathrm{triv}) \simeq \mathbb{C}[x_1, \ldots, x_n]$.

When $t = 1$, it is not hard to see using the Euler element h that $\Delta_c(\mu) := \Delta_{1,c}(\mu)$ has a unique irreducible quotient that we denote by $L_c(\mu)$. In fact, these are the simple objects of the category $\mathcal{O}_c$, which is defined as the category of H_c-modules which are finitely generated over $\mathbb{C}[x_1, \ldots, x_n]$ and where y_i act locally nilpotently. For example, the standard modules $\Delta_c(\mu)$ belong to $\mathcal{O}_c$. We have the following facts about the category $\mathcal{O}_c$.

Theorem 2.1 ([1])**.** *a) If $c \in \mathbb{C} \setminus \mathbb{Q}$ then the category $\mathcal{O}_c$ is semisimple, and all standard modules $\Delta_c(\mu)$ are irreducible. The same is true if c is rational but its denominator is greater than n.*

b) Suppose that $c = m/n$ where $m, n \in \mathbb{Z}_{>0}, \gcd(m, n) = 1$. Then $\Delta_c(\mu)$ is irreducible, unless μ is a hook.

c) Suppose that $c = m/n$ where $m, n \in \mathbb{Z}_{>0}, \gcd(m, n) = 1$, and let $\mu_\ell = (n - \ell, 1^\ell)$ be a hook partition. Then the morphisms between standard modules have the following form:

$$\operatorname{Hom}_{H_c}(\Delta_c(\mu), \Delta_c(\mu')) = \begin{cases} \mathbb{C} & \text{if } \mu = \mu_\ell, \mu' = \mu_{\ell-1} \text{ for some } \ell, \\ 0 & \text{otherwise.} \end{cases}$$

This theorem is proved in [1] using the *Knizhnik-Zamolodchikov functor* which compares the representation theory of H_c to that of the type A finite Hecke algebra. In this paper we give an alternative combinatorial proof. The "otherwise" case of (b) is proved in Lemma 6.7, and Lemma

6.10, while the interesting morphism $\Delta_c(\mu_\ell) \to \Delta_c(\mu_{\ell-1})$ is constructed in Proposition 6.12. Part (b) easily follows from (c), see Corollary 6.9.

The representation theory of the algebra $H_{0,1}$ is very different from that of H_c. For example, it is no longer true that the standard module $\Delta_{0,1}(\mu)$ has a unique irreducible quotient, moreover, the algebra $H_{0,1}$ is finite over its center so every irreducible $H_{0,1}$-module is finite-dimensional, [13]. Still, in Section 9 we will use our results in the $t = 1$ case and a limiting procedure to give a combinatorial basis of $\Delta_{0,1}(\mu)$.

We will also consider the spherical subalgebra of $H_{t,c}$. Let $e := \frac{1}{n!}\sum_{p\in\mathcal{S}_n} p \in \mathbb{C}\mathcal{S}_n \subseteq H_{t,c}$ be the trivial idempotent for $\mathcal{S}_n$. The spherical rational Cherednik algebra is the corner algebra $eH_{t,c}e$. We have an obvious functor $H_{t,c}$-mod $\to eH_{t,c}e$-mod, given by $M \mapsto eM = M^{\mathcal{S}_n}$. When $t = 0$ or $t = 1$ and c is not a negative real number, this functor is known to be an equivalence.

2.2. **The extended affine symmetric group.**

Definition 2.2. The *extended affine symmetric group* is[1]

$$\widetilde{\mathcal{S}_n} = \left\langle \pi, s_i,\ 1 \le i < n \;\middle|\; \begin{array}{ll} s_is_{i+1}s_i = s_{i+1}s_is_{i+1} & \text{for } 1 \le i < n-1, \\ s_is_j = s_js_i & \text{for } j \ne i \pm 1, \\ \pi s_i = s_{i+1}\pi & \text{for } 1 \le i < n-1, \\ s_i^2 = 1 & \text{for } i \in \mathbb{Z}/n\mathbb{Z} \end{array} \right\rangle.$$

Letting $s_0 = \pi^{-1}s_1\pi$, we could consider generators s_i for $i \in \mathbb{Z}/n\mathbb{Z}$. In this case $\widetilde{\mathcal{S}_n}$ has as a subgroup the affine symmetric group $\widehat{\mathcal{S}_n} = \langle s_i \mid i \in \mathbb{Z}/n\mathbb{Z}\rangle$. However for the purposes of this paper, we rarely take this point of view. Further, we will refer to elements $p \in \widetilde{\mathcal{S}_n}$ as affine permutations, dropping the adjective "extended."

We recall that $\widetilde{\mathcal{S}_n}$ acts faithfully on $\mathbb{Z}$ by n-periodic permutations, i.e. bijections $p : \mathbb{Z} \to \mathbb{Z}$ such that $p(i + n) = p(i) + n$. For this action $\pi(i) = i + 1$. It also acts on the set $\mathbb{C}^n$ via:

$$\begin{aligned} s_i \cdot (\mathtt{w}_1, \dots, \mathtt{w}_i, \mathtt{w}_{i+1}, \dots \mathtt{w}_n) &= (\mathtt{w}_1, \dots, \mathtt{w}_{i+1}, \mathtt{w}_i, \dots, \mathtt{w}_n) \\ \pi \cdot (\mathtt{w}_1, \dots, \mathtt{w}_n) &= (\mathtt{w}_n + t, \mathtt{w}_1, \mathtt{w}_2, \dots, \mathtt{w}_{n-1}) \end{aligned} \tag{4}$$

To align the two actions, it is often convenient to extend an n-tuple to having coordinates indexed by all of $\mathbb{Z}$ via $\mathtt{w}_{i+kn} = \mathtt{w}_i - kt$. This is consistent with our conventions taken in Remark 3.3 below.

Just as with the finite symmmetric group, it is convenient to use window notation for affine permutations. The window notation of p is given by $[p(1), p(2), \dots, p(n)]$, which determines p by periodicity.

Definition 2.3. Let us define the *degree* of $p \in \widetilde{\mathcal{S}_n}$ to be $\frac{1}{n}\sum_{i=1}^n (p(i) - i)$.
Let $\widetilde{\mathcal{S}_n}^+$ denote the submonoid of affine permutations p such that $i > 0 \implies p(i) > 0$, i.e., the entries of p in window notation are all positive.

Note $p \in \widehat{\mathcal{S}_n}$ iff it has degree 0. The only permutations in $\widetilde{\mathcal{S}_n}^+$ of degree 0 are those in the finite symmetric group $\mathcal{S}_n$.

Let $\mathtt{t}_\mathbf{a} \in \widetilde{\mathcal{S}_n}$ denote *translation* by $\mathbf{a} \in \mathbb{Z}^n$. In other words, its window notation is $\mathtt{t}_\mathbf{a} = [1 + na_1, 2 + na_2, \dots, n + na_n]$.

Lemma 2.4. *Any permutation $\omega \in \widetilde{\mathcal{S}_n}$ can be uniquely written as $\omega = \mathtt{t}_\mathbf{a} g$ for $g \in \mathcal{S}_n$, $\mathbf{a} \in \mathbb{Z}^n$. Furthermore, $\omega \in \widetilde{\mathcal{S}_n}^+$ if and only if $\omega = \mathtt{t}_\mathbf{a} g$ and $a_i \ge 0$ for all i.*

Proof. By definition,

$$\mathtt{t}_\mathbf{a} g = [g(1) + na_{g(1)}, \dots, g(n) + na_{g(n)}].$$

[1]We drop the first relation when $n = 2$.

Given ω, we can uniquely write $\omega(i) = r_i + nq_i$ where $1 \leq r_i \leq n$, so $g(i) = r_i$ and $a_{g(i)} = q_i$. Since $\omega(i)$ for $1 \leq i \leq n$ all have different remainders modulo n, we have $g \in \mathcal{S}_n$. Finally, $\omega(i) > 0$ if and only if $a_{g(i)} \geq 0$. $\square$

Let sort($\mathbf{a}$) denote the non-decreasing ordering of $\mathbf{a}$, and $g_{\mathbf{a}} \in \mathcal{S}_n$ the shortest element such that $g_{\mathbf{a}} \cdot \mathbf{a} = \mathrm{sort}(\mathbf{a})$. Note that the element $g_{\mathbf{a}}$ is given by

$$g_{\mathbf{a}}(i) = \sharp\{j : a_j < a_i\} + \sharp\{j : j \leq i \,\text{and}\, a_i = a_j\}. \tag{5}$$

We denote

$$\omega_{\mathbf{a}} := \mathtt{t}_{\mathbf{a}} g_{\mathbf{a}}^{-1}.$$

Remark 2.5. Note that the element $g_{\mathbf{a}} \in \mathcal{S}_n$ is uniquely specified by the requirement that the window notation of $\mathtt{t}_{\mathbf{a}} g_{\mathbf{a}}^{-1}$ is increasing.

Lemma 2.6. *Assume $\omega_{\mathbf{a}}(i) = \omega_{\mathbf{b}}(i)$ for some $i \in \{1, \dots, n\}$. Then $g_{\mathbf{a}}^{-1}(i) = g_{\mathbf{b}}^{-1}(i)$ and $a_{g_{\mathbf{a}}^{-1}(i)} = b_{g_{\mathbf{b}}^{-1}(i)}$.*

Proof. If $\omega_{\mathbf{a}}(i) = \omega_{\mathbf{b}}(i)$ then $g_{\mathbf{a}}^{-1}(i) - g_{\mathbf{b}}^{-1}(i) = n(b_{g_{\mathbf{b}}^{-1}(i)} - a_{g_{\mathbf{a}}^{-1}(i)})$. But $g_{\mathbf{a}}^{-1}(i), g_{\mathbf{b}}^{-1}(i) \in \{1, \dots, n\}$ so their difference is only divisible by n if it is in fact 0. The result follows. $\square$

Corollary 2.7. *If $\omega_{\mathbf{a}} = \omega_{\mathbf{b}}$ then $\mathbf{a} = \mathbf{b}$.*

Let $\mathtt{L}^+_{\min}(n)$ denote the set $\mathtt{L}^+_{\min}(n) = \{\omega_{\mathbf{a}} \in \widetilde{\mathcal{S}_n} \mid \mathbf{a} \in \mathbb{Z}^n_{\geq 0}\}$. Then $\omega = \omega_{\mathbf{a}} \in \widetilde{\mathcal{S}_n}^+$ is a minimal length (left) coset representative of $\widetilde{\mathcal{S}_n}/\mathcal{S}_n$, i.e., we have $0 < \omega(1) < \omega(2) < \cdots < \omega(n)$. Further note that the degree of $\omega_{\mathbf{a}}$ as well as that of $\mathtt{t}_{\mathbf{a}}$ agrees with $||\mathbf{a}||$.

It is easy to see the following holds.

Lemma 2.8. *Let $\omega \in \mathtt{L}^+_{\min}(n)$ be of degree $r > 0$. Then there is a unique expression of the form*

$$\omega = (s_{\nu_r} \cdots s_2 s_1)\pi \cdots (s_{\nu_2} \cdots s_2 s_1)\pi(s_{\nu_1} \cdots s_2 s_1)\pi,$$

where $0 \leq \nu_{i+1} \leq \nu_i$. In other words ν is a partition with $\nu_1 < n$ and r parts, and we allow parts to be zero.

Proof. Let us induct on r, noting we exclude the case $r = 0$ from the hypotheses. This corresponds to $\omega = \mathrm{id}$. For $r = 1$ consider the window notation $\omega = [\omega(1), \cdots, \omega(n)]$. Recall $0 < \omega(1) < \omega(2) < \cdots < \omega(n)$. In particular $0 \leq \omega(i) - i$ but $n < \omega(n)$ since $\omega \neq \mathrm{id}$. Since the degree of ω is 1, $n = \sum_{i=1}^n (\omega(i) - i)$ which forces $\omega(n) \leq 2n$ and hence $0 < \omega(n) - n \leq n$. Then $\omega\pi^{-1} = [\omega(n) - n, \omega(1), \cdots, \omega(n-1)] \in \widetilde{\mathcal{S}_n}^+$ has degree 0 and so $\omega\pi^{-1} \in \mathcal{S}_n$. Let k be maximal such that $\omega(k) < \omega(n) - n$ and 0 otherwise, in which case we have $\omega = \pi$. Then $\omega\pi^{-1}s_1s_2\cdots s_k \in \mathcal{S}_n \cap \mathtt{L}^+_{\min}(n) = \{\mathrm{id}\}$ which implies $\omega = s_k \cdots s_2 s_1 \pi$. This proves the base case.

Next assume the claim holds for all affine permutations in $\mathtt{L}^+_{\min}(n)$ of degree $< r$. Suppose ω has degree r. Choose k exactly as above, and note $p = \omega\pi^{-1}s_1s_2\cdots s_k \in \mathtt{L}^+_{\min}(n)$ is of degree $r-1$. By the inductive hypothesis, the claim holds for p with respect to a partition with $r-1$ parts we renumber as $n > \nu_2 \geq \nu_3 \geq \cdots \geq \nu_r \geq 0$. Thus

$$\omega = \underbrace{(s_{\nu_r} \cdots s_2 s_1)\pi \cdots (s_{\nu_2} \cdots s_2 s_1)\pi}_{p}(s_k \cdots s_2 s_1)\pi.$$

We need only show $k \geq \nu_2$ and then set $\nu_1 = k$. Recall ν_2 is maximal such that $p(\nu_2) < p(n) - n$ and recall k is maximal such that $\omega(k) < \omega(n) - n$. By choice of k we have $p(k) = \omega(k) < \omega(n) - n$ and $p(k+1) = \omega(n) - n$. If $\nu_2 = k+1$ then $p(\nu_2) = p(k+1) = \omega(n) - n \geq p(n) - n$ and if $\nu_2 > k+1$ then $p(\nu_2) = \omega(\nu_2 - 1) > \omega(n) - n \geq p_n - n$, both of which are contradictions. $\square$

Remark 2.9. Given $\omega \in \mathtt{L}^+_{\min}(n)$, the partition ν can easily be obtained from the inversions of ω as follows. For the transposed partition ν^T which has $n-1$ parts, $\nu^T_i = \#\{k < 1 \mid \omega(k) > i\}$. Observe the length $\ell(\omega) = |\nu|$.

Example 2.10. Let $n = 5$, $\mathbf{a} = (0, 2, 0, 0, 1)$. Thus $g_{\mathbf{a}} = [1, 5, 2, 3, 4]$, $g_{\mathbf{a}}^{-1} = [1, 3, 4, 5, 2]$, $\mathtt{t}_{\mathbf{a}} = [1, 12, 3, 4, 10]$ and $\omega_{\mathbf{a}} = \mathtt{t}_{\mathbf{a}} g_{\mathbf{a}}^{-1} = [1, 3, 4, 10, 12] = s_1\pi s_3 s_2 s_1 \pi s_3 s_2 s_1 \pi$, and so $\nu = (3, 3, 1)$, $\nu^T = (3, 2, 2)$. Note $\omega(0) = 7, \omega(-1) = 5, \omega(-5) = 2$ and

$$
\begin{aligned}
&\{k < 1 \mid \omega(k) > 1\} &&= \{0, -1, -5\} && \nu^T_1 = 3\\
&\{k < 1 \mid \omega(k) > 2\} &&= \{0, -1\} && \nu^T_2 = 2\\
&\{k < 1 \mid \omega(k) > 3\} &&= \{0, -1\} && \nu^T_3 = 2\\
&\{k < 1 \mid \omega(k) > 4\} &&= \emptyset && \nu^T_4 = 0\\
&\{k < 1 \mid \omega(k) > 5 = n\} &&= \emptyset && \nu^T_5 = 0.
\end{aligned}
$$

There are other ways to obtain ν from $\mathbf{a}$, but discussing them is beyond the scope of this paper. We will merely mention without proof one such way. Given $\mathbf{a}$ construct its n-abacus (with beads at heights determined by $\mathbf{a}$) and then its corresponding n-core partition. Next, following Lapointe-Morse, remove all boxes from the n-core with hooklength $> n$ and left-justify the remaining boxes. For the $\mathbf{a}$ given above its 5-core is $(4, 3, 1)$, from which we remove its box in the upper left corner with hooklength 6 leaving us with $\nu = (3, 3, 1)$.

2.3. **m-stable and m-restricted permutations.** Here we recall some facts on m-stable and m-restricted affine permutations from [23].

Definition 2.11. ([23]) We call an affine permutation ω *m-stable* if the inequality $\omega(x+m) > \omega(x)$ holds for all x. We call an affine permutation ω *m-restricted* if for all $j < i$ one has $\omega(j) - \omega(i) \neq m$.

It is clear that ω is m-stable if and only if ω^{-1} is m-restricted. Also, ω is m-stable if and only if

$$\omega(\omega^{-1}(i) + m) > i \text{ for } i = 1, \ldots, n.$$

Definition 2.12. We call a subset $\mathbb{M} \subset \mathbb{Z}$ *(m, n)-invariant* if $\mathbb{M} + n \subset \mathbb{M}$ and $\mathbb{M} + m \subset \mathbb{M}$.

If $\omega \in \widetilde{\mathcal{S}_n}$ is an m-stable permutation then for all i the set

$$\mathbb{M}^i_\omega = \{x \in \mathbb{Z} : \omega(x) \geq i\} = \omega^{-1}[i, +\infty).$$

is (m, n)-invariant. Indeed, if $\omega(x) \geq i$ then $\omega(x + n) = \omega(x) + n > i$ by definition of affine permutation and $\omega(x + m) > \omega(x) \geq i$ because ω is m-stable.

Clearly, $\mathbb{M}^{i+n}_\omega = \mathbb{M}^i_\omega + n$ and ω is m-stable if and only if $\mathbb{M}^i_\omega$ is (m, n)-invariant for all i. This implies the following useful proposition.

Proposition 2.13. *An affine permutation ω is m-stable if and only if the sets $\mathbb{M}^i_\omega$ are (m, n)-invariant for $i = 1, \ldots, n$.*

Next, we would like to characterize m-stable and m-restricted permutations using window notation, assuming $\gcd(m, n) = 1$. As in [23], we use the affine permutation

$$p_m := [0, m, \ldots, (n-1)m].$$

Lemma 2.14. *Let $\omega p_m = [x_1, \ldots, x_n]$. Then ω is m-stable if and only if*

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq x_1 + mn.$$

Proof. It is sufficient to check the condition $\omega(x + m) > \omega(x)$ for a single choice of x in each remainder modulo n, in particular, for $x = 0, m, 2m, \ldots, (n-1)m$. Now for $1 \leq i \leq n$ we have $x_i = \omega(p_m(i)) = \omega((i-1)m)$, so ω is m-stable if $x_1 < \ldots < x_n$ and

$$x_n = \omega((n-1)m) < \omega(nm) = \omega(0) + nm = x_1 + mn.$$

□

The condition $x_1 < \ldots < x_n$ implies that we can write

$$\omega p_m = \mathtt{t}_{\mathbf{a}} g_{\mathbf{a}}^{-1},\ \omega = \mathtt{t}_{\mathbf{a}} g_{\mathbf{a}}^{-1} p_m^{-1},\ \omega^{-1} = p_m g_{\mathbf{a}} \mathtt{t}_{-\mathbf{a}}$$

for some vector $\mathbf{a} \in \mathbb{Z}^n$, and $g_{\mathbf{a}}$ as above. We can write

$$\omega^{-1}(g_{\mathbf{a}}^{-1}(i)) = p_m(-na_{g_{\mathbf{a}}^{-1}(i)} + i) = -na_{g_{\mathbf{a}}^{-1}(i)} + m(i-1),$$

so

$$\omega^{-1}(i) = -na_i + m(g_{\mathbf{a}}(i) - 1),\ i = 1, \ldots, n. \tag{6}$$

Hence, in window notation $\omega^{-1} = [-na_1 + m(g_{\mathbf{a}}(1) - 1), \ldots, -na_n + m(g_{\mathbf{a}}(n) - 1)]$.

We get the following result:

Lemma 2.15. *Let* $\gcd(m,n) = 1$*. A permutation* ω *is* m*-stable if and only if* ω^{-1} *can be written in the form* (6) *for some vector* $\mathbf{a} \in \mathbb{Z}^n$ *such that:*

- $a_i - a_j \le m$ *for all* i, j
- *if* $a_i - a_j = m$ *then* $i < j$*.*

Proof. Since $\omega p_m = \mathtt{t}_{\mathbf{a}} g_{\mathbf{a}}^{-1} = [x_1, \ldots, x_n]$, we get $x_1 < \ldots < x_n$. We need to check the last condition $x_n < x_1 + mn$ in terms of the vector $\mathbf{a}$.

Observe $x_i = na_{g_{\mathbf{a}}^{-1}(i)} + g_{\mathbf{a}}^{-1}(i)$, so $x_n < x_1 + mn$ if and only if either $a_{g_{\mathbf{a}}^{-1}(1)} + m > a_{g_{\mathbf{a}}^{-1}(n)}$ or $a_{g_{\mathbf{a}}^{-1}(1)} + m = a_{g_{\mathbf{a}}^{-1}(n)}$ and $g_{\mathbf{a}}^{-1}(1) > g_{\mathbf{a}}^{-1}(n)$.

Now $a_{g_{\mathbf{a}}^{-1}(1)} = \min(\mathbf{a}), a_{g_{\mathbf{a}}^{-1}(n)} = \max(\mathbf{a})$, so either $\max(\mathbf{a}) - \min(\mathbf{a}) < m$ or $\max(\mathbf{a}) - \min(\mathbf{a}) = m$ and all appearances of $\max(\mathbf{a})$ are to the left of all appearances of $\min(\mathbf{a})$ in $\mathbf{a}$. □

Remark 2.16. The above results were stated in [23] for the affine symmetric group $\widehat{\mathcal{S}_n}$ (as opposed to extended affine $\widetilde{\mathcal{S}_n}$), but are equivalent to them after imposing the balancing condition for all affine permutations. In particular, p_m should be replaced by the degree 0 affine permutation $\widehat{p}_m = [0 - \kappa, m - \kappa, \ldots, (n-1)m - \kappa]$ where $\kappa = \frac{1}{2}(mn - m - n - 1)$. In particular, Lemma 2.14 can be rephrased by saying that $\widehat{p}_m$ establishes a bijection between the set of m-stable affine permutations and the dilated fundamental alcove.

Example 2.17. Let $n = 5, m = 3, \mathbf{a} = (0, 1, 0, 0, 2)$. Thus $g_{\mathbf{a}}^{-1} = [1, 3, 4, 2, 5]$, $\omega_{\mathbf{a}} = [1, 3, 4, 7, 15]$, with inverses $\omega_{\mathbf{a}}^{-1} = [1, -1, 2, 3, -5]$ and $g_{\mathbf{a}} = [1, 4, 2, 3, 5]$. Note

$$\omega^{-1} = p_m \omega_{\mathbf{a}}^{-1} = [0, 3, 6, 9, 12] \circ [1, -1, 2, 3, -5] = [0, 4, 3, 6, 2]$$

is 3-restricted. Using (6) we can also check $\omega^{-1}(i) = -5a_i + 3(g_{\mathbf{a}}(i) - 1)$ as

$$(0, 4, 3, 6, 2) = -5(0, 1, 0, 0, 2) + 3(0, 3, 1, 2, 4) = -5(0, 1, 0, 0, 2) + 3\left((1, 4, 2, 3, 5) - (1, 1, 1, 1, 1)\right).$$

3. An alternative presentation of $H_{t,c}$

3.1. **Presentation of the algebra.** It will be convenient to work with a trigonometric presentation of the algebra $H_{t,c}$ that has already appeared in the work of Griffeth and Webster in the cyclotomic setting, [27, 51]. Since some relations become more transparent in the type A setting, we recall this presentation in detail. First, we have the Dunkl-Opdam elements in $H_{t,c}$:

$$u_i := x_i y_i - c \sum_{j<i} (ij).$$

Lemma 3.1. *The Dunkl-Opdam elements generate a polynomial subalgebra of* $H_{t,c}$*.*

Proof. It is straightforward to see that $u_i u_j = u_j u_i$. Since the leading term of u_i is $x_i y_i$, the leading terms of u_i in $\operatorname{gr} H_{t,c}$ are algebraically independent, and hence u_i are algebraically independent in $H_{t,c}$. □

We will denote this polynomial subalgebra by $\mathcal{A}$.

We remark that we have the following relations where, as usual, $s_i = (i, i+1) \in \mathcal{S}_n$ is a simple transposition.

$$s_i u_i = u_{i+1} s_i + c \tag{7}$$

$$s_j u_i = u_i s_j \text{ if } j \neq i, i-1 \tag{8}$$

Remark 3.2. These equations imply that u_i and s_j form a subalgebra in $H_{t,c}$ isomorphic to the degenerate affine Hecke algebra. We will denote this algebra by $H_n(\mathbf{u})$.

We will also need the following shift operators. Let $\tau := x_1(12\cdots n)$, $\lambda := (12\cdots n)^{-1} y_1$. Note that, for every i, we have $\tau = (1\cdots i)x_i(i\cdots n)$, $\lambda = (n\cdots i)y_i(i\cdots n)$. The following relations are straightforward to check.

$$\tau u_i = u_{i+1}\tau, i \neq n \tag{9}$$

$$\tau u_n = (u_1 - t)\tau \tag{10}$$

$$\lambda u_i = u_{i-1}\lambda, i \neq 1 \tag{11}$$

$$\lambda u_1 = (u_n + t)\lambda \tag{12}$$

$$s_i \tau = \tau s_{i-1}, i \neq 1 \tag{13}$$

$$s_1 \tau^2 = \tau^2 s_{n-1} \tag{14}$$

$$s_i \lambda = \lambda s_{i+1}, i \neq n-1 \tag{15}$$

$$s_{n-1}\lambda^2 = \lambda^2 s_1 \tag{16}$$

$$\tau\lambda = u_1 \tag{17}$$

$$\lambda\tau = u_n + t \tag{18}$$

$$\lambda s_1 \tau = \tau s_{n-1}\lambda + c \tag{19}$$

It is clear that the elements $s_1, \dots, s_{n-1}, \tau$ and λ generate the algebra $H_{t,c}$. It turns out that, together with the u_i, they give a presentation of this algebra.

Remark 3.3. Given relations (12) and (10), it is convenient to define u_i for $i \in \mathbb{Z}$ by setting $u_{i+nk} = u_i - kt$ for $1 \leq i \leq n$.

Theorem 3.4. *Let $\mathsf{H}_{t,c}$ be the algebra generated by elements $u_1, \dots, u_n, \tau, \lambda$ and the symmetric group $\mathcal{S}_n$, subject to the relations that $[u_i, u_j] = 0$ and (7)–(19). Then, $\mathsf{H}_{t,c} \cong H_{t,c}$.*

Proof. It is clear that we have a morphism $\mathsf{H}_{t,c} \to H_{t,c}$. We have to show that it is an isomorphism. To do so, we provide the inverse. Define $\mathsf{x}_i := s_{i-1}\cdots s_1 \tau s_{n-1}\cdots s_i \in \mathsf{H}_{t,c}$, $\mathsf{y}_i := s_i \cdots s_{n-1}\lambda s_1 \cdots s_{i-1} \in \mathsf{H}_{t,c}$. We claim that the map $x_i \mapsto \mathsf{x}_i, y_i \mapsto \mathsf{y}_i$ and that is the identity on $\mathcal{S}_n$ defines a morphism $H_{t,c} \to \mathsf{H}_{t,c}$. So we have to check that these elements satisfy the relations in $H_{t,c}$. It is straightforward to check the commutation relations between $\mathcal{S}_n$ and x_i, as well as between $\mathcal{S}_n$ and y_i.

We check that $\mathsf{x}_i\mathsf{x}_j = \mathsf{x}_j\mathsf{x}_i$. Assume $j < i$:

$$\begin{aligned}
\mathsf{x}_i\mathsf{x}_j &= (s_{i-1}\cdots s_1)\tau(s_{n-1}\cdots s_i)(s_{j-1}\cdots s_1)\tau(s_{n-1}\cdots s_j) \\
&= (s_{i-1}\cdots s_1)\tau(s_{j-1}\cdots s_1)(s_{n-1}\cdots s_i)\tau(s_{n-1}\cdots s_j) \\
&= (s_{i-1}\cdots s_1)(s_j\cdots s_2)\tau^2(s_{n-2}\cdots s_{i-1})(s_{n-1}\cdots s_j) \\
&= (s_{i-1}\cdots s_1)(s_j\cdots s_2)(s_1\tau^2 s_{n-1})(s_{n-2}\cdots s_{i-1})(s_{n-1}\cdots s_j) \\
&= (s_{i-1}\cdots s_2)(s_j\cdots s_2 s_1 s_2)\tau\tau(s_{n-2}s_{n-1}s_{n-2}\cdots s_{i-1})(s_{n-2}\cdots s_j) \\
&= (s_{i-1}\cdots s_2)(s_j\cdots s_2 s_1)(\tau s_1)(s_{n-1}\ \tau)(s_{n-1}\cdots s_{i-1})(s_{n-2}\cdots s_j) \\
&= (s_{j-1}\cdots s_2 s_1)\tau(s_{i-2}\cdots s_1)(s_{n-1}\cdots s_{j+1})\tau(s_{n-1}\cdots s_i) \\
&= (s_{j-1}\cdots s_2 s_1)\tau(s_{n-1}\cdots s_j)(s_{i-1}\cdots s_1)\tau(s_{n-1}\cdots s_i) \\
&= \mathsf{x}_j\mathsf{x}_i
\end{aligned}$$

where we have used the relation (14) in the form $\tau^2 = s_1\tau^2 s_{n-1}$, and we make repeated use of the relation (13). The proof that $\mathsf{y}_i\mathsf{y}_j = \mathsf{y}_j\mathsf{y}_i$ is similar.

Let us now compute the commutator $[\mathsf{y}_i, \mathsf{x}_i]$.

$$\begin{aligned}
[\mathsf{y}_i, \mathsf{x}_i] &= (s_i \cdots s_{n-1})\lambda\tau(s_{n-1}\cdots s_i) - (s_{i-1}\cdots s_1)\tau\lambda(s_1\cdots s_{i-1})\\
&= (s_i \cdots s_{n-1})(u_n + t)(s_{n-1}\cdots s_i) - (s_{i-1}\cdots s_1)u_1(s_1\cdots s_{i-1})\\
&= t + (s_i \cdots s_{n-1})u_n(s_{n-1}\cdots s_i) - (s_{i-1}\cdots s_1)u_1(s_1\cdots s_{i-1})\\
&= t + (s_i \cdots s_{n-2})(u_{n-1} - cs_{n-1})(s_{n-2}\cdots s_i) - (s_{i-1}\cdots s_2)(u_2 + cs_1)(s_2\cdots s_{i-1})\\
&= t + u_i - c\textstyle\sum_{j>i}(ij) - \Big(u_i + c\sum_{j<i}(ij)\Big) = t - c\sum_{j\neq i}(ij)
\end{aligned}$$

Finally, we have to compute $[\mathsf{y}_i, \mathsf{x}_j]$. We first assume $i = 1$, $j = 2$. So $\mathsf{y}_1 = s_1\cdots s_{n-1}\lambda$, $\mathsf{x}_2 = s_1\tau s_{n-1}\cdots s_2$. We have

$$\begin{aligned}
[\mathsf{y}_1, \mathsf{x}_2] &= s_1\cdots s_{n-1}(\lambda s_1\tau)s_{n-1}\cdots s_2 - (s_1\tau s_{n-1}\cdots s_2)(s_1 s_2\cdots s_{n-1}\lambda)\\
&= s_1\cdots s_{n-1}(c + \tau s_{n-1}\lambda)s_{n-1}\cdots s_2 - (s_1\tau s_{n-1}\cdots s_2)(s_1 s_2\cdots s_{n-1}\lambda)\\
&= cs_1 + s_1\cdots s_{n-1}\tau s_{n-1}\lambda s_{n-1}\cdots s_2 - s_1\tau(s_{n-1}\cdots s_2 s_1 s_2\cdots s_{n-1})\lambda\\
&= cs_1 + s_1\cdots s_{n-1}\tau s_{n-1}\lambda s_{n-1}\cdots s_2 - s_1\tau(s_1\cdots s_{n-2}s_{n-1}s_{n-2}\cdots s_1)\lambda\\
&= cs_1 + s_1\cdots s_{n-1}\tau s_{n-1}\lambda s_{n-1}\cdots s_2 - s_1 s_2\cdots s_{n-1}\tau s_{n-1}\lambda s_{n-1}\cdots s_2\\
&= cs_1
\end{aligned}$$

so the relation $[\mathsf{y}_1, \mathsf{x}_2] = cs_1 = c(12)$ holds. Now the result follows for arbitrary $i \neq j$ by the relations between $\mathcal{S}_n$ and $\mathsf{x}_i, \mathsf{y}_j$. So we have a morphism $H_{t,c} \to \mathsf{H}_{t,c}$, which is an inverse of the morphism mentioned at the beginning of this proof. □

We can also eliminate u_i from this presentation:

Proposition 3.5. *The algebra $H_{t,c}$ is generated by s_i, λ and τ subject to the equations* (13), (14), (15), (16), (19) *and one more equation*

$$\lambda\tau = t + s_1\cdots s_{n-1}\tau\lambda s_{n-1}\cdots s_1 - c\sum_{i=1}^{n-1} s_1\cdots s_i\cdots s_1 \tag{20}$$

Proof. We define $u_1 := \tau\lambda$ and

$$u_k := s_{k-1}\cdots s_1 u_1 s_1\cdots s_{k-1} - c\sum_{i=1}^{k-1} s_{k-1}\cdots s_i\cdots s_{k-1}.$$

Then the relations $s_i u_i = u_{i+1}s_i + c$ and $\lambda\tau = u_n + t$ are automatic. Let us prove that all other relations involving u_i follow. We have:

$$u_2\tau = s_1\tau\lambda s_1\tau - cs_1\tau = s_1\tau(\tau s_{n-1}\lambda + c) - cs_1\tau = s_1\tau^2 s_{n-1}\lambda = \tau^2\lambda = \tau u_1.$$

Now for all $1 < k < n$ we have

$$\begin{aligned}
\tau u_k = \tau s_{k-1}\cdots s_1 u_1 s_1\cdots s_{k-1} - c\textstyle\sum_{i=1}^{k-1}\tau s_{k-1}\cdots s_i\cdots s_{k-1} =\\
s_k\cdots s_2\tau u_1 s_1\cdots s_{k-1} - c\textstyle\sum_{i=1}^{k-1} s_k\cdots s_{i+1}\cdots s_k\tau =\\
s_k\cdots s_2 u_2\tau s_1\cdots s_{k-1} - c\textstyle\sum_{i=1}^{k-1} s_k\cdots s_{i+1}\cdots s_k\tau =\\
s_k\cdots s_2 u_2 s_2\cdots s_k\tau - c\textstyle\sum_{i=2}^{k} s_k\cdots s_i\cdots s_k\tau = u_{k+1}\tau.
\end{aligned}$$

Also,

$$\tau u_n = \tau(\lambda\tau - t) = (\tau\lambda - t)\tau = (u_1 - t)\tau.$$

The relations between λ and u_i can be checked similarly. This implies that u_i commute, for example,

$$u_1 u_k = \tau\lambda u_k = \tau u_{k-1}\lambda = u_k\tau\lambda = u_k u_1 \qquad (k \neq 1).$$

Also, for $i > 1$ we have

$$s_i u_1 = s_i\tau\lambda = \tau s_{i-1}\lambda = \tau\lambda s_i$$

and similarly $s_i u_j = u_j s_i$ for $j \neq i, i-1$. □

The following lemma relates the nonnegative part of $H_{t,c}$ to affine permutations.

Lemma 3.6. *Let $\mathcal{X}$ denote the monoid of monomials in s_i and τ (or, equivalently, in s_i and x_j). Then there is an isomorphism of monoids $F_{\mathcal{X}} : \mathcal{X} \to \widetilde{\mathcal{S}_n}^+$ such that*

$$F_{\mathcal{X}}(s_i) = s_i,\ F_{\mathcal{X}}(\tau) = \pi,\ F_{\mathcal{X}}(x_1^{a_1} \cdots x_n^{a_n}) = \mathtt{t}_{\mathbf{a}}$$

for $a_i \geq 0$.

Proof. We can define $F_{\mathcal{X}}$ by $F_{\mathcal{X}}(s_i) = s_i$, $F_{\mathcal{X}}(\tau) = \pi$. Since the relations $s_i\pi = \pi s_{i-1}$ and $s_1\pi^2 = \pi^2 s_{n-1}$ hold in $\widetilde{\mathcal{S}_n}$, $F_{\mathcal{X}}$ is a homomorphism. Considering the window notation for s_i and π, it is easy to see the image is $\widetilde{\mathcal{S}_n}^+$.

Now

$$F_{\mathcal{X}}(x_i) = s_{i-1} \cdots s_1 \pi s_{n-1} \cdots s_i = [1, \ldots, i-1, i+n, i+1, \ldots, n] = \mathtt{t}_{(0,\ldots,0,1,0,\ldots,0)}$$

so $F_{\mathcal{X}}(x_1^{a_1}, \cdots, x_n^{a_n}) = \mathtt{t}_{\mathbf{a}}$ for all $\mathbf{a} \in \mathbb{Z}^n_{\geq 0}$.

Finally, by Lemma 2.4 any element of $\widetilde{\mathcal{S}_n}^+$ can be uniquely written as $\omega = \mathtt{t}_{\mathbf{a}} g$ for $g \in \mathcal{S}_n$, $\mathbf{a} \in \mathbb{Z}^n_{\geq 0}$, while any element of $\mathcal{X}$ can be uniquely written as $x_1^{a_1} \cdots x_n^{a_n} g$ for $g \in \mathcal{S}_n$. Therefore $F_{\mathcal{X}}$ is a bijection. □

Corollary 3.7. *The monoid $\widetilde{\mathcal{S}_n}^+$ is generated by s_i, π modulo relations in $\mathcal{S}_n$ and*

$$s_i\pi = \pi s_{i-1},\ s_1\pi^2 = \pi^2 s_{n-1}.$$

Similarly, we have the following.

Lemma 3.8. *Let $\mathcal{Y}$ denote the monoid of monomials in s_i and λ (or, equivalently, in s_i and y_j). Let $\widetilde{\mathcal{S}_n}^-$ be the monoid generated by inverses of elements in $\widetilde{\mathcal{S}_n}^+$. Then there is an isomorphism of monoids $F_{\mathcal{Y}} : \mathcal{Y} \to \widetilde{\mathcal{S}_n}^-$ such that*

$$F_{\mathcal{Y}}(s_i) = s_i,\ F_{\mathcal{Y}}(\lambda) = \pi^{-1},\ F_{\mathcal{Y}}(y_1^{a_1} \cdots y_n^{a_n}) = \mathtt{t}_{-\mathbf{a}}.$$

Remark 3.9. The two isomorphisms $F_{\mathcal{X}}$ and $F_{\mathcal{Y}}$ are not compatible in the group $\widetilde{\mathcal{S}_n}$ in the sense that relations between elements in the two monoids may not hold for their preimages in $H_{t,c}$. For instance $\pi\pi^{-1} = \mathrm{id} = \pi^{-1}\pi$ in $\widetilde{\mathcal{S}_n}$ whereas $\tau\lambda \neq \lambda\tau$. See equations (17) and (18).

3.2. **Generalized eigenspaces and intertwining operators.** As above, we will denote by $\mathcal{A} \subseteq H_{t,c}$ the polynomial subalgebra generated by the Dunkl-Opdam elements $u_1, \ldots, u_n$.

For an $H_{t,c}$-module M and $\mathtt{w} \in \mathbb{C}^n$, let $M^{\mathrm{gen}}_{\mathtt{w}}$ denote the generalized eigenspace with weight $\mathtt{w}$, that is, $(u_i - \mathtt{w}_i)$ acts locally nilpotently on $M^{\mathrm{gen}}_{\mathtt{w}}$ for every i. We also denote by $M_{\mathtt{w}} \subseteq M^{\mathrm{gen}}_{\mathtt{w}}$ the subspace of honest simultaneous eigenvectors. At $t = 1$ the Euler element is $h = \sum u_i + n/2$ and it is therefore easy to see that every module in category $\mathcal{O}_c$ is locally finite for the $\mathcal{A}$-action, so that it decomposes as the direct sum of its generalized weight spaces, and each such space is finite-dimensional.

We are interested in the spectrum of $\mathcal{A}$ on the standard module $M := \Delta_{t,c}(\mathrm{triv})$. To study it, we will make use of the following intertwining operators, cf. [26, Section 4]:

$$\sigma_i := s_i - \frac{c}{u_i - u_{i+1}}, i = 1, \ldots, n-1, \qquad \tau = x_1(12\cdots n).$$

Note that $\tau \in H_{t,c}$, while the σ_i are elements of the localization $H_{t,c}[(u_i - u_j)^{-1} : i \neq j]$. Alternatively, given a representation M, we may think of τ as an operator which is defined globally on M, while σ_i is only defined on those generalized eigenspaces $M^{\mathrm{gen}}_{\mathtt{w}}$ for which $\mathtt{w}_i - \mathtt{w}_{i+1} \neq 0$, i.e., $s_i \cdot \mathtt{w} \neq \mathtt{w}$.

Lemma 3.10. *[[26, (4.13)]] We have $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ and, if $i < n$, $\tau\sigma_i = \sigma_{i+1}\tau$. Furthermore,*

$$\sigma_i^2 = \frac{(u_i - u_{i+1} - c)(u_i - u_{i+1} + c)}{(u_i - u_{i+1})^2}. \tag{21}$$

and $\lambda\sigma_i = \sigma_{i-1}\lambda$ if $i > 1$, also $\lambda\sigma_1\tau = \tau\sigma_{n-1}\lambda - \frac{u_0}{u_0-u_1}c = \tau\sigma_{n-1}\lambda - \frac{u_n+t}{u_n+t-u_1}c$.

It is not hard to see that we have

$$\sigma_i : M^{\mathrm{gen}}_{\mathtt{w}} \to M^{\mathrm{gen}}_{s_i\cdot\mathtt{w}}, \qquad \tau : M^{\mathrm{gen}}_{\mathtt{w}} \to M^{\mathrm{gen}}_{\pi\cdot\mathtt{w}}$$

where the symmetric group $\mathcal{S}_n$ acts on $\mathbb{C}^n$ by permuting the coordinates, and $\pi \cdot (\mathtt{w}_1, \dots, \mathtt{w}_n) = (\mathtt{w}_n + t, \mathtt{w}_1, \dots, \mathtt{w}_{n-1})$ as in Equation (4). In the first case we assume $s_i \cdot \mathtt{w} \neq \mathtt{w}$ so σ_i is well-defined.

Remark 3.11. Note that, if $\sigma_i|_{M^{\mathrm{gen}}_{\mathtt{w}}} = 0$, then $\mathtt{w}_i - \mathtt{w}_{i+1} = \pm c$. If M is free as a $\mathbb{C}[x_1, \dots, x_n]$-module (for example, a standard module) then $\tau|_{M^{\mathrm{gen}}_{\mathtt{w}}} \neq 0$ provided $M^{\mathrm{gen}}_{\mathtt{w}} \neq 0$.

Remark 3.12. Note that $\sigma_i = (s_iu_i - u_is_i)/(u_i - u_{i+1})$. These operators are well-defined on any simple $H_{t,c}$-module on which $\mathcal{A}$ acts semisimply. It is sometimes convenient to instead consider

$$\tilde{\sigma}_i := (s_iu_i - u_is_i)/(u_i - u_{i+1} + c).$$

These also satisy the braid relations and their quadratic relation becomes $\tilde{\sigma}_i^2 = 1$. We will see below (see Section 4.5) that these operators are well-defined on $L_c(\mathrm{triv})$.

Using intertwiners to construct and parameterize an $\mathcal{A}$-weight basis for an $\mathcal{A}$-semisimple (or calibrated) module, as well as giving the action of generators on that basis, follows ideas developed by Ram in [46] or Cherednik in [9]. In [46] the role of $\mathcal{A}$ was instead played by an appropriate commutative subalgebra of the affine Hecke algebra, but the constructions apply in our context as well.

4. The polynomial representation of H_c

4.1. **Combinatorics of integer sequences.** Recall that for $\mathbf{a} \in \mathbb{Z}^n_{\geq 0}$, we denote $||\mathbf{a}|| := \sum_i a_i$. As in Section 2.2, we denote by $g_{\mathbf{a}} \in \mathcal{S}_n$ the shortest element such that $g_{\mathbf{a}} \cdot \mathbf{a} = \mathrm{sort}(\mathbf{a})$.

Lemma 4.1. *For every $\mathbf{a} \in \mathbb{Z}^n_{\geq 0}$ we have $g_{\pi\cdot\mathbf{a}} = g_{\mathbf{a}}(12\cdots n)^{-1}$. If $a_i \neq a_{i+1}$, then we have $g_{s_i\cdot\mathbf{a}} = g_{\mathbf{a}}s_i$.*

Proof. We use the explicit equation (5) for $g_{\mathbf{a}}$. Assume $i \neq 1$. Denote $X_\pi := \{j : (\pi\cdot\mathbf{a})_j < (\pi\cdot\mathbf{a})_i\}$ and $Y_\pi := \{j : (\pi\cdot\mathbf{a})_j = (\pi\cdot\mathbf{a})_i \text{ and } j \leq i\}$. Similarly, denote $X := \{j : a_j < a_{i-1}\}$ and $Y := \{j : a_j = a_{i-1} \text{ and } j \leq i-1\}$, so that $g_{\pi\cdot\mathbf{a}}(i) = \sharp X_\pi + \sharp Y_\pi$ and $g_{\mathbf{a}}(i-1) = \sharp X + \sharp Y$. Note that, if $j \neq 1$, then $j \in X_\pi$ (resp. $j \in Y_\pi$) if and only if $j-1 \in X$ (resp. $j-1 \in Y$). Note also that we cannot have $n \in Y$ because $i-1 < n$. Moreover, we have that $1 \in X_\pi \cup Y_\pi$ if and only if $n \in X$ and, by the previous sentence, this happens if and only if $n \in X \cup Y$. This shows that $g_{\pi\cdot\mathbf{a}}(i) = g_{\mathbf{a}}(i-1)$. Note that this forces $g_{\pi\cdot\mathbf{a}}(1) = g_{\mathbf{a}}(n)$. So $g_{\pi\cdot\mathbf{a}} = g_{\mathbf{a}}(12\cdots n)^{-1}$, as needed. The other equality is clear. □

We denote by $\mathcal{P}_k(n) := \{\mathbf{a} \in \mathbb{Z}^n_{\geq 0} : ||\mathbf{a}|| = k\}$. There is a clear bijection between $\mathcal{P}_k(n)$ and the set of monomials of degree k in n variables. We will define a partial order on $\mathcal{P}_k(n)$ inductively. For $n = 2$ and even $k = 2\ell$, we have

$$(\ell, \ell) \prec (\ell+1, \ell-1) \prec (\ell-1, \ell+1) \prec \cdots \prec (2\ell, 0) \prec (0, 2\ell)$$

and for $k = 2\ell + 1$ odd we have

$$(\ell+1, \ell) \prec (\ell, \ell+1) \prec (\ell+2, \ell-1) \prec (\ell-1, \ell+2) \prec \cdots \prec (2\ell+1, 0) \prec (0, 2\ell+1).$$

Now assume we have defined partial orders on $\mathcal{P}_{k'}(n)$ for every k'. Let us define partial orders on $\mathcal{P}_k(n+1)$. The set $\mathcal{P}_0(n+1)$ is a singleton so there is nothing to do. On $\mathcal{P}_1(n+1)$ we have

$k=0$: id, $[1,2,3]$, $(0,0,0)$

$k=1$:

π	$s_1\pi$	$s_2s_1\pi$
$[2,3,4]$	$[1,3,5]$	$[1,2,6]$
$(1,0,0)\prec$	$(0,1,0)$	$\prec(0,0,1)$

$\underbrace{(1,0,0)}_{\mathcal{P}^\circ}\prec\underbrace{(0,1,0)\prec(0,0,1)}_{\mathcal{P}'}$

$k=2$:

π^2	$\pi s_1\pi$	$\pi s_2s_1\pi$	$s_1\pi s_1\pi$	$s_1\pi s_2s_1\pi$	$s_2s_1\pi s_2s_1\pi$
$[3,4,5]$	$[2,4,6]$	$[2,3,7]$	$[1,5,6]$	$[1,3,8]$	$[1,2,9]$
$(1,1,0)$	$\prec(1,0,1)$	$\prec(2,0,0)$	$\prec(0,1,1)$	$\prec(0,2,0)$	$\prec(0,0,2)$

$\underbrace{(1,1,0)\prec(1,0,1)\prec(2,0,0)}_{\mathcal{P}^\circ}\prec\underbrace{(0,1,1)\prec(0,2,0)\prec(0,0,2)}_{\mathcal{P}'}$

$k=3$:

π^3	$\pi^2s_1\pi$	$\pi^2s_2s_1\pi$	$\pi s_1\pi s_1\pi$	$\pi s_1\pi s_2s_1\pi$	$\pi s_2s_1\pi s_2s_1\pi$
$[4,5,6]$	$[3,5,7]$	$[3,4,8]$	$[2,6,7]$	$[2,4,9]$	$[2,3,10]$
$(1,1,1)$	$\prec(2,1,0)$	$\prec(1,2,0)$	$\prec(2,0,1)$	$\prec(1,0,2)$	$\prec(3,0,0)$

$s_1\pi s_1\pi s_1\pi$	$s_1\pi s_1\pi s_2s_1\pi$	$s_1\pi s_2s_1\pi s_2s_1\pi$	$s_2s_1\pi s_2s_1\pi s_2s_1\pi$
$[1,6,8]$	$[1,5,9]$	$[1,3,11]$	$[1,2,12]$
$\prec(0,2,1)$	$\prec(0,1,2)$	$\prec(0,3,0)$	$\prec(0,0,3)$

$\underbrace{(1,1,1)\prec(2,1,0)\prec(1,2,0)\prec(2,0,1)\prec(1,0,2)\prec(3,0,0)}_{\mathcal{P}^\circ}\prec\underbrace{(0,2,1)\prec(0,1,2)\prec(0,3,0)\prec(0,0,3)}_{\mathcal{P}'}$

FIGURE 1. The partial order on $\mathbf{a}\in\mathcal{P}_k(3)$ for degrees $k\leq 3$. So that one may compare $\prec$ to $>_{1ex}$ and to Bruhat order, above each $\mathbf{a}$ is the corresponding $\omega_{\mathbf{a}}$ both in window notation and it expression from Lemma 2.8.

$(1,0,\ldots,0)\prec(0,1,\ldots,0)\prec\cdots\prec(0,0,\ldots,1)$. Assume that we have defined a partial order on $\mathcal{P}_k(n+1)$. To define a partial order on $\mathcal{P}_{k+1}(n+1)$, we decompose

$$\mathcal{P}_{k+1}(n+1):=\mathcal{P}^\circ_{k+1}(n+1)\sqcup\mathcal{P}'_{k+1}(n+1)$$

where $\mathcal{P}^\circ_{k+1}(n+1):=\{\mathbf{a}\in\mathcal{P}_{k+1}(n+1): a_1\neq 0\}$ and $\mathcal{P}'_{k+1}(n+1):=\mathcal{P}_{k+1}(n+1)\setminus\mathcal{P}^\circ_{k+1}(n+1)$. The map π gives a bijection $\pi:\mathcal{P}_k(n+1)\to\mathcal{P}^\circ_{k+1}(n+1)$, and this gives a partial order on the set $\mathcal{P}^\circ_{k+1}(n+1)$. By forgetting $a_1=0$, $\mathcal{P}'_{k+1}(n+1)$ is identified with $\mathcal{P}_{k+1}(n)$, and this gives a partial order on the set $\mathcal{P}'_{k+1}(n+1)$. Finally, we declare every element in $\mathcal{P}^\circ_{k+1}(n+1)$ to be smaller than every element of $\mathcal{P}'_{k+1}(n+1)$ This gives a partial order on $\mathcal{P}_{k+1}(n+1)$. Figure 1 below gives some examples on how these partial orders look when $n=3$. For each $\mathbf{a}\in\mathbb{Z}^3_{\geq 0}$ listed, we also include $\omega_{\mathbf{a}}$, both in window notation and in its decomposition given by Lemma 2.8 for reference.

For another example, when $n=4$, $k=2$ we have $(1,1,0,0)\prec(1,0,1,0)\prec(1,0,0,1)\prec(2,0,0,0)\prec(0,1,1,0)\prec(0,1,0,1)\prec(0,2,0,0)\prec(0,0,1,1)\prec(0,0,2,0)\prec(0,0,0,2)$.

The following lemma gives properties of this partial order that will be important for us.

Lemma 4.2. *With the partial order defined above, $\mathcal{P}_k(n)$ is linearly ordered. Moreover, the following properties are satisfied.*

(1) *If $\mathbf{a}\prec\mathbf{b}$ in $\mathcal{P}_k(n)$, then $\pi\cdot\mathbf{a}\prec\pi\cdot\mathbf{b}$ in $\mathcal{P}_{k+1}(n)$.*
(2) *If $a_n\geq a_1>0$, then $(a_1,a_2,\ldots,a_{n-1},a_n)\prec(a_n+1,a_2,\ldots,a_{n-1},a_1-1)$,*

(3) *If* $a_i > a_{i+1}$*, then* $\mathbf{a} \prec s_i \cdot \mathbf{a}$.
(4) *If* $a_i > a_{i+1}$ *and* $\mathbf{b} \prec \mathbf{a}$*, then* $s_i \cdot \mathbf{b} \prec s_i \cdot \mathbf{a}$.

Proof. By induction, it follows easily that $\mathcal{P}_k(n)$ is linearly ordered, as it is defined to be the concatenation of two linearly ordered sets. Property (1) is obvious from the definition. It remains to show (2), (3) and (4). Note that when $n = 2$ or $k = 0, 1$, (2), (3) and (4) are easy to check from the explicit definition of the partial order on $\mathcal{P}_k(2)$ or on $\mathcal{P}_1(n)$. So we may use an inductive procedure. We assume that (2), (3) and (4) are valid for $\mathcal{P}_{k'}(n)$ for every k' and for $\mathcal{P}_0(n+1), \dots, \mathcal{P}_k(n+1)$, and we show that they are valid for $\mathcal{P}_{k+1}(n+1)$. Recall that $\mathcal{P}^{\circ}_{k+1}(n+1) = \pi(\mathcal{P}_k(n+1))$ and $\mathcal{P}'_{k+1}(n+1) = \mathcal{P}_{k+1}(n+1) \setminus \mathcal{P}^{\circ}_{k+1}(n+1)$.

We start with (2). Note that if $\mathbf{a}$ is as in (2), then $\mathbf{a} \in \mathcal{P}^{\circ}_{k+1}(n+1)$. Then (2) happens if and only if in $\mathcal{P}_k(n+1)$ we have $(a_2, \dots, a_n, a_{n+1}, a_1 - 1) \prec (a_2, \dots, a_n, a_1 - 1, a_{n+1})$. But this is clear because $\mathcal{P}_k(n+1)$ satisfies (3).

Now we move on to (3). Thanks to (1) and our inductive assumption, the only problem can arise with s_1: indeed, for $i > 1$ we can either go to $\mathcal{P}_k(n)$, if $a_1 > 0$; or to $\mathcal{P}_{k+1}(n-1)$ if $a_1 = 0$ and the result follows by induction. So we assume $i = 1$. If $\mathbf{a} \in \mathcal{P}'_{k+1}(n+1)$, then we can never have $a_1 > a_2$, so we may assume that $\mathbf{a} \in \mathcal{P}^{\circ}_{k+1}(n+1)$. If $a_2 = 0$, then $s_1 \cdot \mathbf{a} \in \mathcal{P}'_{k+1}(n+1)$, and we have $\mathbf{a} \prec s_1 \cdot \mathbf{a}$ by definition. Otherwise, we may assume that $a_1 > a_2 \geq 1$. Then $s_1 \cdot \mathbf{a} \succ \mathbf{a}$ is equivalent to, in $\mathcal{P}_k(n+1)$, having $(a_2, \dots, a_{n+1}, a_1 - 1) \prec (a_1, a_3, \dots, a_{n+1}, a_2 - 1)$. This is clear because $\mathcal{P}_k(n+1)$ satisfies (2).

Finally, we check (4). Note that we canot have $i = 1$ and $\mathbf{a} \in \mathcal{P}'_{k+1}(n+1)$ simultaneously. We also cannot have $\mathbf{a} \in \mathcal{P}^{\circ}_{k+1}(n+1)$ and $\mathbf{b} \in \mathcal{P}'_{k+1}(n+1)$ simultaneously. If both $\mathbf{a}$ and $\mathbf{b}$ belong to $\mathcal{P}'_{k+1}(n+1)$, the result follows by forgetting the initial 0 and using induction. If $\mathbf{a} \in \mathcal{P}'_{k+1}$ and $\mathbf{b} \in \mathcal{P}^{\circ}_{k+1}$ then, since $i \neq 1$, s_i preserves both $\mathcal{P}'_{k+1}(n+1)$ and $\mathcal{P}^{\circ}(n+1)$ so the result is also clear. The result is also clear if both $\mathbf{a}, \mathbf{b} \in \mathcal{P}^{\circ}_{k+1}(n+1)$ and $i \neq 1$. So it remains to check the case $\mathbf{a}, \mathbf{b} \in \mathcal{P}^{\circ}$, $i = 1$. If $a_2 = 0$, $b_2 \neq 0$, the result is clear.

If $a_2, b_2 = 0$, then we have that $\mathbf{a} \succ \mathbf{b}$ if and only if $(0, a_3, \dots, a_{n+1}, a_1 - 1) \succ (0, b_3, \dots, b_{n+1}, b_1 - 1)$. This happens if and only if in $\mathcal{P}_k(n)$ we have $(a_3, \dots, a_{n+1}, a_1 - 1) \succ (b_3, \dots, b_{n+1}, b_1 - 1)$. By (1), this implies that $(a_1, a_3, \dots, a_n) \succ (b_1, b_3, \dots, b_n)$ in $\mathcal{P}_{k+1}(n)$. But then $(0, a_1, a_3, \dots, a_n) \succ (0, b_1, b_3, \dots, b_n)$, which is what we wanted to show.

If $a_2, b_2 \neq 0$, we need to show that $(a_1, \dots, a_{n+1}, a_2 - 1) \succ (b_1, \dots, b_{n+1}, b_2 - 1)$. This happens if and only if $(a_3, \dots, a_{n+1}, a_2 - 1, a_1 - 1) \succ (b_3, \dots, b_2 - 1, b_1 - 1)$. But by assumption, $(a_3, \dots, a_{n+1}, a_1 - 1, a_2 - 1) \succ (b_3, \dots, b_1 - 1, b_2 - 1)$ and $a_1 - 1 > a_2 - 1$. Since $\mathcal{P}_{k-1}(n+1)$ satisfies (4), the result follows. ☐

4.2. **Comparing with lexicographic ordering.** We would like to elaborate on the partial order $\prec$ defined in Section 4.1. To do so, we will compare it to the lexicographic ordering on the window notation of an affine permutation $\omega_{\mathbf{a}} \in \mathtt{L}^{+}_{\min}(n)$ associated to $\mathbf{a} \in \mathbb{Z}^n_{\geq 0}$, cf. Section 2.2

The following result is easy to see (see also Lemma 2.6).

Lemma 4.3. *The assignment* $\mathbf{a} \mapsto \omega_{\mathbf{a}}$ *gives a bijection between* $\mathbb{Z}^n_{\geq 0}$ *and the set* $\mathtt{L}^{+}_{\min}(n)$*. Moreover,*

(a) *the set* $\mathcal{P}_k(n)$ *gets identified with*
$$\mathtt{L}^{+}_{\min}(n)_k := \{\omega \in \mathtt{L}^{+}_{\min}(n) : \deg \omega = k\}.$$
(b) $\mathcal{P}'_k$ *gets identified with* $\left\{\omega \in \mathtt{L}^{+}_{\min}(n)_k : \omega(1) = 1\right\}$.
(c) $\mathcal{P}^{\circ}_k$ *with* $\left\{\omega \in \mathtt{L}^{+}_{\min}(n)_k : \omega(1) > 1\right\}$.

For affine permutations $\omega, \omega' \in \widetilde{\mathcal{S}_n}$, we say that $\omega >_{\mathtt{lex}} \omega'$ if the window notation of ω is greater than that of ω' in lexicographic ordering. More explicitly, $\omega >_{\mathtt{lex}} \omega'$ if there exists $i \in \{1, \dots, n\}$ such that $\omega(j) = \omega'(j)$ for $j = 1, \dots, i-1$ and $\omega(i) > \omega'(i)$.

Lemma 4.4. *Let* $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n_{\geq 0}$ *and assume* $\omega_{\mathbf{a}} >_{\mathtt{lex}} \omega_{\mathbf{b}}$*. Then,* $\omega_{\pi \cdot \mathbf{a}} >_{\mathtt{lex}} \omega_{\pi \cdot \mathbf{b}}$.

Proof. If the window notation of ω is $[\omega(1), \omega(2), \cdots \omega(n)]$ then the window notation of $\pi\omega$ is $[\omega(1)+1, \omega(2)+1, \cdots \omega(n)+1]$. Hence it is obvious that $\omega >_{\mathtt{lex}} \omega' \implies \pi\omega >_{\mathtt{lex}} \pi\omega'$. Next, observe $\omega_{\pi\cdot\mathbf{a}} = \pi\omega_{\mathbf{a}}$ from which the lemma follows. Indeed the entries of the window notation for $\omega_{\mathbf{a}}$ sort those of $\mathtt{t}_{\mathbf{a}}$, and these have the form $i + na_i$. On the other hand, the entries of $\mathtt{t}_{\pi\cdot\mathbf{a}}$ are $i + na_{\pi^{-1}(i)} = i + na_{i-1}$ which we may reindex as $i+1+na_i$ as well as $1+n(a_n+1) = n+1+na_n$. ☐

The next result tells us that, even though the partial order $\prec$ looks complicated, it is in fact very natural when transported via the map $\mathbf{a} \mapsto \omega_{\mathbf{a}}$.

Lemma 4.5. *Let $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n_{\geq 0}$ be such that $||\mathbf{a}|| = ||\mathbf{b}||$. Then, $\mathbf{a} \prec \mathbf{b}$ if and only if $\omega_{\mathbf{a}} >_{\mathtt{lex}} \omega_{\mathbf{b}}$.*

Proof. Since for fixed degree we are dealing with linear orderings, by Lemma 4.3 we only need to check $\mathbf{a} \prec \mathbf{b}$ implies $\omega_{\mathbf{a}} >_{\mathtt{lex}} \omega_{\mathbf{b}}$. Let us denote $k := ||\mathbf{a}|| = ||\mathbf{b}||$. The case when $\mathbf{a} \in \mathcal{P}^\circ_k$ and $\mathbf{b} \in \mathcal{P}'_k$ follows from Lemma 4.3 (b) and (c). The case when $\mathbf{a}, \mathbf{b} \in \mathcal{P}^\circ_k$ follows from Lemma 4.4 and an inductive argument on k.

Finally, assume $\mathbf{a}, \mathbf{b} \in \mathcal{P}'_k$ so that $a_1 = b_1 = 0$ and therefore $\omega_{\mathbf{a}}(1) = \omega_{\mathbf{b}}(1) = 1$. We have to consider $\overline{\mathbf{a}} = (a_2, \ldots, a_n)$, $\overline{\mathbf{b}} = (b_2, \ldots, b_n)$. By induction on n we may assume $\omega_{\overline{\mathbf{a}}} >_{\mathtt{lex}} \omega_{\overline{\mathbf{b}}}$. Note that we have $g^{-1}_{\mathbf{a}}(i+1) = g^{-1}_{\overline{\mathbf{a}}}(i) + 1$ and that $\overline{a}_i = a_{i+1}$ for $i = 1, \ldots, n-1$, and

$$\omega_{\mathbf{a}} = \left[1, \omega_{\overline{\mathbf{a}}}(1) + 1 + a_{g_{\mathbf{a}}^{-1}(2)}, \ldots, \omega_{\overline{\mathbf{a}}}(n-1) + 1 + a_{g_{\mathbf{a}}^{-1}(n)}\right]$$

and similarly for $g_{\mathbf{b}}$, $\omega_{\mathbf{b}}$. By assumption, $\omega_{\overline{\mathbf{a}}} >_{\mathtt{lex}} \omega_{\overline{\mathbf{b}}}$. So there exists $i_0 \in \{1, \ldots, n-1\}$ such that $\omega_{\overline{\mathbf{a}}}(i) = \omega_{\overline{\mathbf{b}}}(i)$ for $i < i_0$ and $\omega_{\overline{\mathbf{a}}}(i_0) > \omega_{\overline{\mathbf{b}}}(i_0)$. If $i < i_0$ then by Lemma 2.6 we get $\omega_{\mathbf{a}}(i+1) = \omega_{\mathbf{b}}(i+1)$. Now, $\omega_{\overline{\mathbf{a}}}(i_0) > \omega_{\overline{\mathbf{b}}}(i_0)$ implies that $(n-1)(a_{g_{\mathbf{a}}^{-1}(i_0+1)} - b_{g_{\mathbf{b}}^{-1}(i_0+1)}) + g^{-1}_{\overline{\mathbf{a}}}(i_0) - g^{-1}_{\overline{\mathbf{b}}}(i_0) > 0$, from where we deduce that $a_{g_{\mathbf{a}}^{-1}(i_0+1)} \geq b_{g_{\mathbf{b}}^{-1}(i_0+1)}$. Finally,

$$\omega_{\mathbf{a}}(i_0+1) = \omega_{\overline{\mathbf{a}}}(i_0) + 1 + a_{g_{\mathbf{a}}^{-1}(i_0+1)} > \omega_{\overline{\mathbf{b}}}(i_0) + 1 + b_{g_{\mathbf{b}}^{-1}(i_0+1)} = \omega_{\mathbf{b}}(i_0+1)$$

and we conclude that $\omega_{\mathbf{a}} >_{\mathtt{lex}} \omega_{\mathbf{b}}$. ☐

One can check Figure 1 to see examples of the structure described in both Lemmas 4.4 and 4.5, as well as Lemma 4.6 below.

Now let $\omega \in \mathtt{L}^+_{\min}(n)$. Recall that by Lemma 2.8, that we may express ω in the form

$$\omega = (s_{\nu_r} \cdots s_2 s_1)\pi \cdots (s_{\nu_2} \cdots s_2 s_1)\pi(s_{\nu_1} \cdots s_2 s_1)\pi$$

where $0 \leq \nu_r \leq \cdots \leq \nu_1 < n$. We select $\ell \leq r$ and $j \leq \nu_\ell$ and consider the affine permutation

$$\widehat{\omega} := (s_{\nu_r} \cdots s_2 s_1)\pi \cdots (s_{\nu_\ell} \cdots s_{j+1}\widehat{s_j}s_{j-1} \cdots s_1)\pi \cdots (s_{\nu_1} \cdots s_2 s_1)\pi$$

where a hat over s_j means that we omit that transposition. Clearly, $\widehat{\omega}$ belongs to the monoid $\widetilde{\mathcal{S}_n}^+$, and $\deg(\omega) = \deg(\widehat{\omega})$. Let us denote by $\omega' \in \mathtt{L}^+_{\min}(n)$ the permutation whose window notation is the increasing arrangement of the window notation of $\widehat{\omega}$.

Lemma 4.6. *We have $\omega' >_{\mathtt{lex}} \omega$.*

Proof. Let us start with the easy observation that, in window notation:

$$s_k \cdots s_1 \pi = [1, 2, \ldots, k, k+2, \ldots, n, n+k+1]. \tag{22}$$

Now let us split ω as follows:

$$\omega = \underbrace{(s_{\nu_r} \cdots s_1)\pi \cdots (s_{\nu_{\ell+1}} \cdots s_1)\pi}_{\alpha} \underbrace{(s_{\nu_\ell} \cdots s_1)\pi}_{\beta} \underbrace{(s_{\nu_{\ell-1}} \cdots s_1)\pi \cdots (s_{\nu_1} \cdots s_1)\pi}_{\gamma}$$

Note that $\alpha, \beta, \gamma \in \mathtt{L}^+_{\min}(n)$. Moreover, letting $k := \nu_\ell$ since $\nu_\ell \le \nu_{\ell-1} \le \cdots \le \nu_1$ it follows from (22) that $\gamma = [1, 2, \ldots, k, \gamma(k+1), \ldots, \gamma(n)]$, where $k < \gamma(k+1) < \cdots < \gamma(n)$. If we write $\alpha = [\alpha(1), \ldots, \alpha(n)]$ we then have that

$$\omega = [\alpha(1), \ldots, \overset{j}{\alpha(j)}, \ldots, \alpha(k), \omega(k+1), \ldots, \underset{=\alpha(k+1)+zn}{\overset{p}{\omega(p)},} \ldots, \omega(n)],$$

where we let $1 \le p \le n$ be such that $\gamma(p) = zn$. Observe $z > 0$.

Let us now compute $\widehat{\omega}$. Let $\widehat{\beta} := s_{\nu_\ell} \cdots s_{j+1} \widehat{s_j} s_{j-1} \cdots s_1 \pi$, so that $\widehat{\omega} = \alpha \widehat{\beta} \gamma$. A straightforward computation shows that, in window notation, $\widehat{\beta} = [1, 2, \ldots, j-1, k+1, j+1, \ldots, k, k+2, \ldots, n, n+j]$. Then,

$$\widehat{\omega} = [\alpha(1), \ldots, \alpha(j-1), \underset{=\omega(p)-zn}{\overset{j}{\alpha(k+1)}}, \alpha(j+1), \ldots, \alpha(k), \omega(k+1), \ldots, \underset{=\omega(j)+zn}{\overset{p}{\alpha(j)+zn}}, \ldots, \omega(n)],$$

and in particular the window notation for ω and $\widehat{\omega}$ agree except for in the jth and pth entries. Since ω is already sorted so its entries increase, to show $\omega' >_{\mathtt{lex}} \omega$ it suffices to show $\widehat{\omega}(j) > \omega(j)$ and $\widehat{\omega}(p) > \omega(j)$. This will ensure that the first $j-1$ entries of ω and the sorted ω' agree, but the jth entry of ω' will be strictly larger than $\omega(j)$. We compute $\widehat{\omega}(j) = \alpha(k+1) > \alpha(j) = \omega(j)$ since $\alpha \in \mathtt{L}^+_{\min}(n)$, and $\widehat{\omega}(p) = \alpha(j) + zn > \alpha(j) = \omega(j)$.

□

Remark 4.7. From Lemma 4.6 we see that for $\omega, p \in \mathtt{L}^+_{\min}(n)$ of the same degree that

$$p <_{\mathtt{B}} \omega \text{ implies } p >_{\mathtt{lex}} \omega \tag{23}$$

in window notation. We could also deduce this from the characterization of Bruhat order for $\widehat{\mathcal{S}_n}$ given in Björner-Brenti [2, Theorem 8.3.7] (which one must extend appropriately to $\widetilde{\mathcal{S}_n}$; this is easy if only comparing permutations of the same degree). In particular, they characterize $p \le_{\mathtt{B}} \omega$ for $p, \omega \in \widehat{\mathcal{S}_n}$ if $p[i,j] \le \omega[i,j]$ for all $i, j \in \mathbb{Z}$ where

$$p[i,j] = \#\{a \le i \mid p(a) \ge j\}.$$

We can prove (23) as follows.

Let $p \ne \omega \in \mathtt{L}^+_{\min}(n)$ be of the same degree, which means $\sum_{i=1}^n p(i) = \sum_{i=1}^n \omega(i)$. (This sum is $n(n+1)/2$ in the case $p, \omega \in \widehat{\mathcal{S}_n}$.) Suppose that $p \not>_{\mathtt{lex}} \omega$ which means $p <_{\mathtt{lex}} \omega$. Hence for some $1 \le \ell \le n$ we have $p(1) = \omega(1), p(2) = \omega(2), \cdots, p(\ell-1) = \omega(\ell-1)$ but $p(\ell) < \omega(\ell)$. Since $\sum_{i=1}^n p(i) = \sum_{i=1}^n \omega(i)$, there must be some $\ell \le i \le n$ such that $p(i) > \omega(i)$. Let $i \le n$ be the largest such i. In other words for $n \ge r > i$ we have $p(r) \le \omega(r)$. Let $j := p(i)$. Let us compare $p[i,j]$ and $\omega[i,j]$.

Since $\omega \in \mathtt{L}^+_{\min}(n)$ we have $\omega(1) < \omega(2) < \cdots < \omega(n)$ and so given a such that $\omega(a) \ge j = p(i) > \omega(i)$ and $a \le i$ then $i+1 < a+kn < n$ for some $k > 0$. In particular $p(a) = p(a+kn) - kn \ge \omega(a+kn) - kn = \omega(a) \ge j$. Hence $\{a \le i \mid \omega(a) \ge j\} \subseteq \{a \le i \mid p(a) \ge j\}$. Further as $\omega(i) < p(i) = j$ the element i does not belong to the first set but does to the second. This shows $\omega[i,j] < p[i,j]$ and so $p \not<_{\mathtt{B}} \omega$. This proves (23).

We remark that even though $>_{\mathtt{lex}}$ is a total order on $\widetilde{\mathcal{S}_n}$, we only relate it to Bruhat order for two affine permutations of the same degree and that are both in $\mathtt{L}^+_{\min}(n)$.

4.3. **The action of the Dunkl-Opdam subalgebra.** For the rest of this section, we assume $t = 1$. The case $t = 0$ will be treated in Section 9.

For $\mathbf{a} \in \mathbb{Z}^n_{\ge 0}$ we let $\mathtt{w} := \mathtt{w}(\mathbf{a}) \in \mathbb{C}^n$ denote the weight whose ith component is $\mathtt{w}_i = a_i - (g_{\mathbf{a}}(i) - 1)c$. In other words, $\mathtt{w}(\mathbf{a}) = \omega_{\mathbf{a}} \cdot (0, -c, -2c, \cdots, (1-n)c)$ where, as mentioned above, we specialize $t = 1$. Now we are ready to describe the spectrum of $\mathcal{A}$ on $\Delta_c(\mathrm{triv})$, in the case where c is generic.

Proposition 4.8. *Assume that either $c \in \mathbb{C} \setminus \mathbb{Q}$ or that c is a rational number with denominator greater than n. Let $M = \Delta_c(\mathrm{triv})$. Then, $M^{\mathrm{gen}}_{\mathtt{w}(\mathbf{a})} \neq 0$ for every $\mathbf{a} \in \mathbb{Z}^n_{\geq 0}$, these are all the weight spaces of $\mathcal{A}$ on M, and each one of them is 1-dimensional (so that $M^{\mathrm{gen}}_{\mathtt{w}(\mathbf{a})} = M_{\mathtt{w}(\mathbf{a})}$.)*

Proof. Let $\mathbb{C}[x_1, \dots, x_n]_k \subseteq M$ denote the space of homogeneous degree k polynomials. We will show by induction on k that $\mathbb{C}[x_1, \dots, x_n]_k = \bigoplus_{||\mathbf{a}||=k} M^{\mathrm{gen}}_{\mathtt{w}(\mathbf{a})}$ and that each one of these weight spaces is nonzero.

The claim is clear for $k = 0$. Now let $\mathbf{a} \in \mathbb{Z}^n$ with $||\mathbf{a}|| = k+1$. Let i_0 be minimal such that $a_{i_0} > 0$. By the inductive hypothesis, the weight space $M_{\mathtt{w}(\mathbf{a}')}$ is nonzero, where $\mathbf{a}' = (a_1, \dots, \widehat{a_{i_0}}, \dots, a_n, a_{i_0} - 1)$. Since $\mathtt{w}(\mathbf{a})_{i_0} \neq \mathtt{w}(\mathbf{a})_j$ for $j < i_0$ and $\mathtt{w}(\mathbf{a})_{i_0} - \mathtt{w}(\mathbf{a})_j \neq \pm c$ for $j < i_0$ (here we use the genericity of c) we have that $\sigma_{i_0 - 1} \cdots \sigma_1 \tau : M_{w(a')} \to M_{\mathtt{w}(\mathbf{a})}$ is nonzero. Then, $M_{\mathtt{w}(\mathbf{a})} \neq 0$.

Now note that if $\mathtt{w}(\mathbf{a}) = \mathtt{w}(\mathbf{a}')$ then $\mathbf{a} = \mathbf{a}'$. Indeed, if $\mathtt{w}(\mathbf{a})_i = \mathtt{w}(\mathbf{a}')_i$ then $a_i - a'_i - (g_{\mathbf{a}}(i) - g_{\mathbf{a}'}(i))c = 0$. Since c is irrational, this can only happen when $a_i = a'_i$. By dimension reasons, it follows that each $M_{\mathtt{w}(\mathbf{a})}$ is 1-dimensional and $\mathbb{C}[x_1, \dots, x_n]_k = \bigoplus_{||\mathbf{a}||=k} M_{\mathtt{w}(\mathbf{a})}$. The result follows. ☐

In the proof above, we have used the fact that if c is generic, then for every $\mathbf{a} \in \mathbb{Z}^n_{\geq 0}$ we have $\mathtt{w}(\mathbf{a})_i - \mathtt{w}(\mathbf{a})_{i+1} \neq \pm c$ provided $a_i \neq a_{i+1}$. This is of course not true in the general case. However, we still have the following result.

Theorem 4.9. *Let $c = m/n > 0$ with $\gcd(m, n) = 1$ and $M = \Delta(\mathrm{triv})$. For any $\mathbf{a} \in \mathbb{Z}^n_{\geq 0}$, $M_{\mathtt{w}(\mathbf{a})} = M^{\mathrm{gen}}_{\mathtt{w}(\mathbf{a})} \neq 0$. Moreover, if $a_i > a_{i+1}$, then $\sigma_i|_{M_{\mathtt{w}(\mathbf{a})}} \neq 0$.*

Before starting the proof of Theorem 4.9 note that, for such $c = m/n$, we still have that $\mathtt{w}(\mathbf{a}) = \mathtt{w}(\mathbf{a}')$ only when $\mathbf{a} = \mathbf{a}'$, so we will use a similar strategy to that in the generic case. To properly implement this strategy, we will use the ordering on monomials defined in Section 4.1. From the formula $\sigma_i = s_i - \frac{c}{u_i - u_{i+1}}$ it follows that, if $M_{\mathtt{w}} \neq 0$ and σ_i is defined on $M_{\mathtt{w}}$, then $\sigma_i|_{M_{\mathtt{w}}} = 0$ if and only if $\mathtt{w}_i - \mathtt{w}_{i+1} = c$ (resp. $-c$) and every vector in $M_{\mathtt{w}}$ is symmetric (resp. antisymmetric) with respect to x_i and x_{i+1}.

Proof of Theorem 4.9. First we remark that, for $c = m/n$ as in Theorem 4.9 and every $a \in \mathbb{Z}^n$ we cannot have $\mathtt{w}(\mathbf{a})_i = \mathtt{w}(\mathbf{a})_{i+1}$. Indeed, $\mathtt{w}(\mathbf{a})_i - \mathtt{w}(\mathbf{a})_{i+1} = a_i - a_{i+1} - (g_{\mathbf{a}}(i) - g_{\mathbf{a}}(i+1))m/n$. Since $0 < |g_{\mathbf{a}}(i) - g_{\mathbf{a}}(i+1)| < n$, this last expression cannot be zero.

We will prove by induction on $||\mathbf{a}||$ that $M^{\mathrm{gen}}_{\mathtt{w}(\mathbf{a})}$ contains a nonzero element of the form

$$v_{\mathbf{a}} := x^{\mathbf{a}} + \sum_{\mathbf{a}' \prec \mathbf{a}} k_{\mathbf{a},\mathbf{a}'} x^{\mathbf{a}'}$$

where $x^{\mathbf{a}} := x_1^{a_1} \cdots x_n^{a_n}$. This is obvious for $||\mathbf{a}|| = 0$. Before we proceed further, we have the folllowing easy but important remark.

Remark 4.10. Note that, if $a_i > a_{i+1}$ then, thanks to Lemma 4.2(3), we have $s_i \cdot \mathbf{a} \not\prec \mathbf{a}$, so $v_{\mathbf{a}}$ cannot be symmetric nor antisymmetric on the variables x_i, x_{i+1}.

Now we reason inductively similarly to the proof of Proposition 4.8. Assume $||\mathbf{a}|| > 0$ and let i_0 be minimal such that $a_{i_0} \neq 0$. Define $\mathbf{a}^* = (a_1, \dots, \widehat{a_{i_0}}, \dots, a_n, a_{i_0} - 1)$. By the inductive assumption, $M_{\mathtt{w}(\mathbf{a}^*)} \neq 0$ and it contains $v_{\mathbf{a}^*}$ of the form specified above. Now $\tau(v_{\mathbf{a}^*}) \neq 0$, and thanks to Lemma 4.2(1), $v_{\pi \cdot \mathbf{a}^*} = \tau(v_{\mathbf{a}^*})$. By the first paragraph of this proof, the vector $\sigma_{i_0-1} \cdots \sigma_1(v_{\pi\cdot\mathbf{a}^*})$ is well-defined, by Remark 4.10 it is nonzero and by (3) and (4) of Lemma 4.2, $v_{\mathbf{a}} = \sigma_{i_0-1} \cdots \sigma_1(v_{\pi\cdot\mathbf{a}^*})$. The result follows. ☐

Remark 4.11. The same proof applies, mutatis mutandis, for the $\mathfrak{sl}_n$ RCA, with the operators defined as in [44]. In particular, we can find a basis $\{v_{\mathbf{a}} : \mathbf{a} \in \mathbb{Z}^{n-1}_{\geq 0}\}$ of simultaneous eigenvectors for the $\mathfrak{sl}_n$-Dunkl-Opdam operators.

Note that the action of τ is injective on $\Delta_c(\mathrm{triv})$. Combinatorially, the action of π on the set of nonnegative sequences $(a_1, \ldots, a_n)$ is injective, and any such sequence can be uniquely written as

$$(a_1, \ldots, a_n) = \pi^k \cdot (0, b_2, \ldots, b_{n-1}).$$

where $(0, b_1, \ldots, b_{n-1}) \in \mathbb{Z}^n_{\geq 0}$. That is, the generating set for this action consists of sequences with $a_1 = 0$, and it is in bijection with the basis in the polynomial representation of $\mathfrak{sl}_n$ RCA.

Remark 4.12. Note that Theorem 4.9 can also be obtained as a consequence of Theorem 5.9 below. However, we present a separate proof since it gives an explicit $\mathcal{A}$-eigenbasis of $\Delta_c(\mathrm{triv})$ from which it is easy to reconstruct the action of the entire algebra H_c, see e.g. Lemma 4.14 below.

4.4. **Recovering the action of H_c.** From now on, we assume that $c = m/n > 0$ with $\gcd(m, n) = 1$. We keep assuming $t = 1$. The goal of this section is to show that it is enough to know the weights of the subalgebra $\mathcal{A}$ on $\Delta_c(\mathrm{triv})$ in order to recover the action of the entire Cherednik algebra H_c. Thanks to Theorem 3.4 it is enough to know the action of the operators $u_1, \ldots, u_n, s_1, \ldots, s_{n-1}, \tau$ and λ. Recall that we have the basis $\{v_{\mathbf{a}} : \mathbf{a} \in \mathbb{Z}^n_{\geq 0}\}$ of $\Delta_c(\mathrm{triv})$.

Obviously, $u_i v_{\mathbf{a}} = \mathtt{w}_i v_{\mathbf{a}}$. Also note that, by construction, $\tau v_{\mathbf{a}} = v_{\pi\cdot\mathbf{a}}$. Note that the operator $\tau : \Delta_c(\mathrm{triv}) \to \Delta_c(\mathrm{triv})$ is injective and recall that $u_1 = \tau\lambda$. It is clear from the relations that $\lambda : \Delta_c(\mathrm{triv})_{\mathtt{w}(\mathbf{a})} \to \Delta_c(\mathrm{triv})_{\pi^{-1}\cdot\mathtt{w}(\mathbf{a})}$. It easily follows that $\lambda v_{\mathbf{a}} = \mathtt{w}_1 v_{\pi^{-1}\cdot\mathbf{a}}$. Note that $\pi^{-1}\cdot\mathbf{a}$ does not belong to $\mathcal{P}(n)$ if and only if $a_1 = 0$. The following easy lemma makes sure that we do not find a contradiction.

Lemma 4.13. *Let $\mathbf{a} \in \mathbb{Z}^n_{\geq 0}$. Then $\mathtt{w}_1 = 0$ if and only if $a_1 = 0$.*

Proof. If $a_1 \neq 0$ then, since $(g_{\mathbf{a}}(1) - 1)c$ cannot be a nonzero integer, we have

$$\mathtt{w}_1 = a_1 - (g_{\mathbf{a}}(1) - 1)c \neq 0.$$

On the other hand, if $a_1 = 0$ then, since all other coordinates of a are non-negative, we have $g_{\mathbf{a}}(1) = 1$. So $a_1 - (g_{\mathbf{a}}(1) - 1)c = 0$. □

Finally, we need to find the action of the operators s_i, $i = 1, \ldots, n-1$. To do this, we employ the intertwining operators $\sigma_i = s_i - \frac{c}{u_i - u_{i+1}}$.

Lemma 4.14. *Assume $a_i > a_{i+1}$. Then, $\sigma_i v_{\mathbf{a}} = v_{s_i\cdot\mathbf{a}}$.*

Proof. Clearly, $\sigma_i v_{\mathbf{a}}$ is an eigenvector for u_i with eigenvalue $\mathtt{w}(s_i\cdot\mathbf{a})$, so it is a multiple of $v_{s_i\cdot\mathbf{a}}$. Recall that $v_{\mathbf{a}}$ has the form $v_{\mathbf{a}} = x^{\mathbf{a}} + \sum_{\mathbf{b}\prec\mathbf{a}} k_{\mathbf{a},\mathbf{b}} x^{\mathbf{b}}$. Thanks to Lemma 4.2 (3) and (4), the largest (w.r.t. $\preceq$) monomial appearing in $\sigma_i v_{\mathbf{a}}$ is $x^{s_i\cdot\mathbf{a}}$, and it appears with coefficient 1. The result follows. □

Note that it follows that, if $a_i > a_{i+1}$:

$$s_i v_{\mathbf{a}} = v_{s_i\cdot\mathbf{a}} + \frac{c}{\mathtt{w}_i - \mathtt{w}_{i+1}} v_{\mathbf{a}}$$

where $\mathtt{w} = \mathtt{w}(\mathbf{a})$. Using (21) one can deduce that if $a_i < a_{i+1}$,

$$s_i v_{\mathbf{a}} = \frac{(\mathtt{w}_{i+1} - \mathtt{w}_i - c)(\mathtt{w}_{i+1} - \mathtt{w}_i + c)}{(\mathtt{w}_{i+1} - \mathtt{w}_i)^2} v_{s_i\cdot\mathbf{a}} + \frac{c}{\mathtt{w}_i - \mathtt{w}_{i+1}} v_{\mathbf{a}}$$

Finally, if $a_i = a_{i+1}$ note that $s_i \cdot \mathtt{w}(\mathbf{a})$ is *not* of the form $\mathtt{w}(\mathbf{b})$ for $\mathbf{b} \in \mathbb{Z}^n_{\geq 0}$. So $\sigma_i(v_{\mathbf{a}}) = 0$, and it follows that $s_i v_{\mathbf{a}} = v_{\mathbf{a}}$. Summarizing,

Theorem 4.15. *The module $\Delta_c(\mathrm{triv})$ has a basis given by $\{v_{\mathbf{a}} : \mathbf{a} \in \mathbb{Z}^n_{\geq 0}\}$, and the action of the algebra H_c on $\Delta_c(\mathrm{triv})$ is given by the following operators:*

$$\begin{aligned}
u_i v_{\mathbf{a}} &= \mathtt{w}_i v_{\mathbf{a}} \\
\tau v_{\mathbf{a}} &= v_{\pi\cdot\mathbf{a}} \\
\lambda v_{\mathbf{a}} &= \mathtt{w}_1 v_{\pi^{-1}\cdot\mathbf{a}} \\
s_i v_{\mathbf{a}} &= \begin{cases} v_{s_i\cdot\mathbf{a}} + \frac{c}{\mathtt{w}_i - \mathtt{w}_{i+1}} v_{\mathbf{a}} & a_i > a_{i+1} \\ \frac{(\mathtt{w}_{i+1}-\mathtt{w}_i-c)(\mathtt{w}_{i+1}-\mathtt{w}_i+c)}{(\mathtt{w}_i-\mathtt{w}_{i+1})^2} v_{s_i\cdot\mathbf{a}} + \frac{c}{\mathtt{w}_i-\mathtt{w}_{i+1}} v_{\mathbf{a}} & a_i < a_{i+1} \\ v_{\mathbf{a}} & a_i = a_{i+1} \end{cases}
\end{aligned}$$

where we denote $\mathtt{w}_i = a_i - (g_{\mathbf{a}}(i) - 1)c$.

4.5. **Renormalized basis.** For geometric applications, we will need a different basis of $\Delta_c(\mathrm{triv})$ that gives nicer formulas for the action of the operators s_i. This basis is a renormalization of the basis $v_{\mathbf{a}}$, but we have to be careful with the renormalization factor. The main result of this section is the following.

Proposition 4.16. *There exists a function* $\varphi : \mathbb{Z}^n_{\geq 0} \to \mathbb{C}^\times$ *such that, defining* $\widetilde{v}_{\mathbf{a}} := \varphi(\mathbf{a})v_{\mathbf{a}}$ *we have*

$$(1 - s_i)\widetilde{v}_{\mathbf{a}} = \frac{\mathtt{w}_i - \mathtt{w}_{i+1} - c}{\mathtt{w}_i - \mathtt{w}_{i+1}} (\widetilde{v}_{\mathbf{a}} - \widetilde{v}_{s_i\cdot\mathbf{a}})$$

for every $\mathbf{a} \in \mathbb{Z}^n_{\geq 0}$ *and* $i = 1, \ldots, n-1$.

Note that, the formula for $(1 - s_i)v_{\mathbf{a}}$ in our current basis $v_{\mathbf{a}}$ is

$$(1 - s_i)v_{\mathbf{a}} = \begin{cases} -v_{s_i\cdot\mathbf{a}} + \frac{\mathtt{w}_i-\mathtt{w}_{i+1}-c}{\mathtt{w}_i-\mathtt{w}_{i+1}} v_{\mathbf{a}}, & a_i > a_{i+1} \\ \frac{-(\mathtt{w}_{i+1}-\mathtt{w}_i-c)(\mathtt{w}_{i+1}-\mathtt{w}_i+c)}{(\mathtt{w}_{i+1}-\mathtt{w}_i)^2} v_{s_i\cdot\mathbf{a}} + \frac{\mathtt{w}_i-\mathtt{w}_{i+1}-c}{\mathtt{w}_i-\mathtt{w}_{i+1}} v_{\mathbf{a}}, & a_i < a_{i+1} \\ 0, & a_i = a_{i+1} \end{cases}$$

thus, if a function φ with the properties of that in the statement of Proposition 4.16 exists we must have

$$\frac{\varphi(\mathbf{a})}{\varphi(s_i\cdot\mathbf{a})} = \begin{cases} \frac{\mathtt{w}_i-\mathtt{w}_{i+1}-c}{\mathtt{w}_i-\mathtt{w}_{i+1}}, & a_i > a_{i+1} \\ \frac{\mathtt{w}_{i+1}-\mathtt{w}_i}{\mathtt{w}_{i+1}-\mathtt{w}_i-c} & a_i < a_{i+1}. \end{cases} \tag{24}$$

Moreover, if $a_i = a_{i+1}$ then $\mathtt{w}_i - \mathtt{w}_{i+1} = c$ and it follows that, no matter what the value of $\varphi(\mathbf{a})$ is, we have $(1 - s_i)\widetilde{v}_{\mathbf{a}} = 0 = \frac{\mathtt{w}_i-\mathtt{w}_{i+1}-c}{\mathtt{w}_i-\mathtt{w}_{i+1}}(\widetilde{v}_{\mathbf{a}} - \widetilde{v}_{s_i\cdot\mathbf{a}})$. Thus, we see that Proposition 4.16 is equivalent to the existence of a function satisfying (24).

Proof of Proposition 4.16. We show the existence of a function φ by induction on $||\mathbf{a}||$. For $\mathbf{a} = (0, \ldots, 0)$, we may simply define $\varphi(\mathbf{a}) = 1$. Now assume a function φ satisfying the conditions of the statement has been defined on the sets $\mathcal{P}_0(n), \ldots, \mathcal{P}_k(n)$ where, recall from Section 4.1, $\mathcal{P}_i(n) = \{(a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0} : a_1 + \cdots + a_n = i\}$. Our job is now to define $\varphi : \mathcal{P}_{k+1}(n) \to \mathbb{C}^\times$.

Let $\mathbf{a} \in \mathcal{P}_{k+1}(n)$. If $\pi^{-1}\cdot\mathbf{a} \in \mathcal{P}_k(n)$ (equivalently, if $a_1 > 0$) then we define $\varphi(\mathbf{a}) := \varphi(\pi^{-1}\cdot\mathbf{a})$. Otherwise, let $j > 1$ be minimal such that $a_j > 0$. We may assume that $\varphi(s_{j-1}\cdot\mathbf{a})$ has already been defined, and define $\varphi(\mathbf{a})$ using (24):

$$\varphi(\mathbf{a}) = \frac{\mathtt{w}_j - \mathtt{w}_{j-1}}{\mathtt{w}_j - \mathtt{w}_{j-1} - c}\varphi(s_{j-1}\cdot\mathbf{a}). \tag{25}$$

We need to show that, as defined, φ satisfies the statement of the proposition. As observed above, this is equivalent to φ satisfying (24). We consider several cases.

Case 1. $\pi^{-1}\cdot\mathbf{a}, \pi^{-1}s_i\cdot\mathbf{a} \in \mathcal{P}_k(n)$. In this case by definition

$$\frac{\varphi(\mathbf{a})}{\varphi(s_i\cdot\mathbf{a})} = \frac{\varphi(\pi^{-1}\cdot\mathbf{a})}{\varphi(\pi^{-1}s_i\cdot\mathbf{a})}$$

If $i > 1$, then $\pi^{-1}s_i\cdot\mathbf{a} = s_{i-1}\pi^{-1}\cdot\mathbf{a}$ and (24) follows by induction. If $i = 1$ note that the condition $\pi^{-1}\cdot\mathbf{a}, \pi^{-1}s_1\cdot\mathbf{a} \in \mathcal{P}_k(n)$ is equivalent to saying $a_1, a_2 > 0$. So $\pi^{-2}\cdot\mathbf{a}, \pi^{-2}s_1\cdot\mathbf{a} \in \mathcal{P}_{k-1}(n)$. We may assume that the function φ on $\mathcal{P}_k(n)$ is constructed from that on $\mathcal{P}_{k-1}(n)$ in a similar way to how we constructed φ on $\mathcal{P}_{k+1}(n)$. Thus,

$$\frac{\varphi(\mathbf{a})}{\varphi(s_1\cdot\mathbf{a})} = \frac{\varphi(\pi^{-2}\cdot\mathbf{a})}{\varphi(\pi^{-2}s_1\cdot\mathbf{a})} = \frac{\varphi(\pi^{-2}\cdot\mathbf{a})}{\varphi(s_{n-1}\pi^{-2}\cdot\mathbf{a})}$$

and (24) again follows by induction.

Case 2. $\pi^{-1}\cdot\mathbf{a} \notin \mathcal{P}_k(n), \pi^{-1}s_i\cdot\mathbf{a} \in \mathcal{P}_k(n)$. In this case, we are forced to have $i = 1$, $a_1 = 0$ and $a_2 > 0$, and $\varphi(\mathbf{a})$ is defined using (25) with $j = 2$. Thus, (24) follows by the construction of φ.

Case 3. $\pi^{-1}\cdot\mathbf{a} \in \mathcal{P}_k(n), \pi^{-1}s_i\cdot\mathbf{a} \notin \mathcal{P}_k(n)$. In this case, we are again forced to have $i = 1$, but now we have $a_1 > 0, a_2 = 0$. This case now follows similarly to Case 2.

Case 4. $\pi^{-1}\cdot\mathbf{a} \notin \mathcal{P}_k(n), \pi^{-1}s_i\cdot\mathbf{a} \notin \mathcal{P}_k(n)$. Let us denote by $i_0 := \min\{i : a_i > 0\}$, so that $i_0 > 1$. If $i < i_0 - 1$, then $\mathbf{a} = s_i\cdot\mathbf{a}$ and there is nothing to check. If $i = i_0 - 1$, then (24) follows by construction from (25). Finally, if $i > i_0 - 1$ then $i_0 \leq \min\{i : s(a)_i > 0\}$ and we may argue by induction.

We have considered all cases, and the statement follows. □

Remark 4.17. The proof of Proposition 4.16 shows that, if we further require $\tau(\widetilde{v}_{\mathbf{a}}) = \widetilde{v}_{\pi\cdot\mathbf{a}}$, then the function φ is determined up to multiplication by a nonzero complex number. In particular, the vectors $\widetilde{v}_{\mathbf{a}}$ are uniquely determined after specifying $\widetilde{v}_0$.

Remark 4.18. We could instead have defined the $\widetilde{v}_{\mathbf{a}}$ using the renormalized intertwiners $\widetilde{\sigma}_i$ of Remark 3.12 via the analogous formula to Lemma 4.14. The existence of the function φ is then obvious and $\varphi(\mathbf{a})$ can be given as a product formula with terms indexed by the inversions of $\omega_{\mathbf{a}}$.

Corollary 4.19. *The action of the operators u_i, τ and λ in the renormalized basis $\widetilde{v}_{\mathbf{a}}$ is given by the same equations as in Theorem 4.15:*

$$u_i\widetilde{v}_{\mathbf{a}} = \mathtt{w}_i\widetilde{v}_{\mathbf{a}}, \quad \tau\widetilde{v}_{\mathbf{a}} = \widetilde{v}_{\pi\cdot\mathbf{a}}, \quad \lambda\widetilde{v}_{\mathbf{a}} = \mathtt{w}_1\widetilde{v}_{\pi^{-1}\cdot\mathbf{a}}. \tag{26}$$

4.6. **The radical of $\Delta_c(\mathrm{triv})$.** Theorem 4.15 allows us to explicitly describe the weights appearing on the radical of the polynomial representation $\Delta_c(\mathrm{triv})$. We continue working with the parameter $c = m/n$, $\gcd(m;n) = 1$, $t = 1$. Define the set

$$\mathcal{S} := \{(a_1,\ldots,a_n) \in \mathbb{Z}^n_{\geq 0} : \max(a_i - a_j) > m, \text{ or } \max(a_i - a_j) = m \text{ and } a_i - a_j = m \text{ for some } j < i\}$$

Proposition 4.20. *The space*

$$S := \bigoplus_{\mathbf{a}\in\mathcal{S}} \mathbb{C}v_{\mathbf{a}}$$

is an H_c-submodule of $\Delta_c(\mathrm{triv})$.

Proof. To prove the claim, we need to check that S is closed under the action of the operators s_j, λ and τ. Let us first check that S is closed under the action of τ. For this, it is enough to check that $\mathcal{S}$ is closed under the action of π. So let us take $\mathbf{a} \in \mathcal{S}$. If $\max((\pi \cdot \mathbf{a})_i - (\pi \cdot \mathbf{a})_j) > m$, we are done. Else, we have two cases.

Case 1. If $\max((\pi \cdot \mathbf{a})_i - (\pi \cdot \mathbf{a})_j) < m$ then $\max((\pi \cdot \mathbf{a})_i - (\pi \cdot \mathbf{a})_j) < \max(a_i - a_j) = m$ and it is easy to see that this implies that $\min(a_i) = a_n$ and, moreover, $a_i > a_n$ for $i < n$. But then $\mathbf{a}$ cannot satisfy the condition $a_i - a_j = m$ for some $j < i$, a contradiction.

Case 2. The only remaining case is $\max((\pi \cdot \mathbf{a})_i - (\pi \cdot \mathbf{a})_j) = m$. If $\max(a_i - a_j) = m$, since $\mathbf{a} \in \mathcal{S}$, we can find i_0, j_0 with $a_{i_0} - a_{j_0} = m$ and $i_0 > j_0$. Obviously, we cannot have $j_0 = n$ and if $i_0 = n$ then $(\pi \cdot \mathbf{a})_1 - (\pi \cdot \mathbf{a})_{j_0+1} = a_n + 1 - a_{j_0} = m+1$, a contradiction. So $(\pi \cdot \mathbf{a})_{i_0+1} - (\pi \cdot \mathbf{a})_{j_0+1} = m$ and $\pi \cdot \mathbf{a} \in \mathcal{S}$. If $\max(a_i - a_j) = m+1$, then $\min((\pi \cdot \mathbf{a})_i) = (\pi \cdot \mathbf{a})_1$, so any j satisfying $(\pi \cdot \mathbf{a})_j - (\pi \cdot \mathbf{a})_1 = m$ sees that $\pi \cdot \mathbf{a} \in \mathcal{S}$.

Now we need to show that S is closed under the action of λ. So assume that $\mathbf{a} \in \mathcal{S}$ is such that $a_1 \neq 0$ (we do not need to worry about the case $a_1 = 0$ thanks to Lemma 4.13). We need to show that $\pi^{-1} \cdot \mathbf{a} \in \mathcal{S}$. Again, we have two cases.

Case 1. Assume that $\max((\pi^{-1} \cdot \mathbf{a})_i - (\pi^{-1} \cdot \mathbf{a})_j) < m$. Then, $\max(a_i - a_j) = m$, and $a_1 > a_j$ for every $j > 1$. But then, the only way we can get $a_i - a_j = m$ is with $i = 1$, so $\mathbf{a} \notin \mathcal{S}$, a contradiction.

Case 2. Assume that $\max((\pi^{-1} \cdot \mathbf{a})_i - (\pi^{-1} \cdot \mathbf{a})_j) = m$. If $\max(a_i - a_j) = m$, then we can find $i_0 > j_0$ with $a_{i_0} - a_{j_0} = m$. Note that $j_0 > 1$ as, otherwise, $(\pi^{-1} \cdot \mathbf{a})_{i_0-1} - (\pi^{-1} \cdot \mathbf{a})_n = m+1$, a contradiction. But then $(\pi^{-1} \cdot \mathbf{a})_{i_0-1} - (\pi^{-1} \cdot \mathbf{b})_{i_0-1} = m$, so $(\pi^{-1} \cdot \mathbf{a}) \in \mathcal{S}$. If $\max(a_i - a_j) = m+1$, then $\max(a_i) = a_1$ and $\max((\pi^{-1} \cdot \mathbf{a})_i) = (\pi^{-1} \cdot \mathbf{a})_n$. But then any j satisfying $(\pi^{-1} \cdot \mathbf{a})_n - (\pi^{-1} \cdot \mathbf{a})_j = m$ sees that $\pi^{-1} \cdot \mathbf{a} \in \mathcal{S}$.

Finally, we need to show that S is closed under the action of s_i. Assume that $\mathbf{a} \in \mathcal{S}$. Note that applying s_i does not change $\max(a_i - a_j)$, so we can see that the only possible way for $s_i \cdot \mathbf{a} \notin \mathcal{S}$ is if the following three conditions are satisfied:

(1) $a_{i+1} = a_i + m$
(2) $a_i < a_j \leq a_{i+1}$ for $j = 1, \ldots, i-1$ and
(3) $a_i \leq a_j < a_{i+1}$ for $j = i+1, \ldots, n$.

but in this case $\mathtt{w}_i - \mathtt{w}_{i+1} = c$, so $s_i v_{\mathbf{a}} = -v_{\mathbf{a}}$. We are done. □

Now let $\mathcal{T} := \mathbb{Z}^n_{\geq 0} \setminus \mathcal{S}$. More explicitly,

$$\mathcal{T} = \{\mathbf{a} \in \mathbb{Z}^n_{\geq 0} : a_i - a_j \leq m \text{ for every } i, j; \text{moreover, if } a_i - a_j = m \text{ then } j > i\}. \tag{27}$$

So that the quotient module $\Delta_c(\mathrm{triv})/S$ has an $\mathcal{A}$-eigenbasis $\{v_{\mathbf{a}} : \mathbf{a} \in \mathcal{T}\}$.

Corollary 4.21. *The submodule S is the maximal proper submodule of* $\Delta_c(\mathrm{triv})$.

Remark 4.22. It is well known (cf. [16]) that at this parameter $c = m/n$, the polynomial representation $\Delta_c(\mathrm{triv})$ has a unique proper submodule. Here we give a direct combinatorial proof of this fact which does not use Knizhnik-Zamolodchikov functor.

Proof. We need to prove that $\Delta_c(\mathrm{triv})/S$ is simple. Suppose that R is a submodule of $\Delta_c(\mathrm{triv})/S$, then it contains an eigenvector $v_{\mathbf{a}}$ for some $\mathbf{a} \in \mathcal{T}$. Let us prove that R in fact contains $v_{(0,\ldots,0)} = 1$ and hence $R = \Delta_c(\mathrm{triv})/S$.

Indeed, if $a_1 \neq 0$ then R contains $\lambda(v_{\mathbf{a}})$ which is a nonzero multiple of $v_{\pi^{-1} \cdot \mathbf{a}}$. It is easy to see that $\pi^{-1} \cdot \mathbf{a} \in \mathcal{T}$. If $a_1 = 0$ and $\mathbf{a} \in \mathcal{T}$ then $a_i < m$ for all i, so $|a_i - a_j| < m$ for all i, j and $\mathtt{w}_i(\mathbf{a}) - \mathtt{w}_j(\mathbf{a}) \neq \pm c$. Therefore for all i the intertwining operator σ_i sends $v_{\mathbf{a}}$ to a nonzero multiple of $v_{s_i \cdot \mathbf{a}}$, and $s_i \cdot \mathbf{a} \in \mathcal{T}$.

We conclude that by applying σ_i and λ, we can get from any vector $v_{\mathbf{a}}$, $\mathbf{a} \in \mathcal{T}$ to a nonzero multiple of $v_{(0,\ldots,0)} = 1$ such that all intermediate vectors are nonzero multiples of $v_{\mathbf{b}}, \mathbf{b} \in \mathcal{T}$.

□

Corollary 4.23. *The module $L_c(\mathrm{triv})$ has a basis $\{v_{\mathbf{a}} : \mathbf{a} \in \mathcal{T}\}$. The action of H_c on $L_c(\mathrm{triv})$ is given by the same formulas as in Theorem 4.15, with the understanding that we set $v_{\mathbf{a}} = 0$ if $\mathbf{a} \notin \mathcal{T}$.*

Remark 4.24. Of course, we also have the basis $\{\widetilde{v}_{\mathbf{a}} : \mathbf{a} \in \mathcal{T}\}$ of $L_c(\mathrm{triv})$, that was constructed in Section 4.5.

Remark 4.25. The proof of Proposition 4.20 can be easily adapted to the $\mathfrak{sl}_n$-setting, cf. Remark 4.11. In that case, we have $\mathcal{S}^{\mathfrak{sl}_n} := \{(a_1, \ldots, a_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1} : \max(a_i) \geq m\}$. In particular, $\mathcal{T}^{\mathfrak{sl}_n} = \{(a_1, \ldots, a_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1} : a_i < m \text{ for every } i\}$ and we recover the formula $\dim(L_c^{\mathfrak{sl}_n}(\mathrm{triv})) = m^{n-1}$.

As in Remark 4.11, we can prove that the action of π is injective on the basis in $\mathcal{T}$. The generating set for this action consists of sequences $(0, a_2, \ldots, a_n)$, and it is easy to see that such sequence is in $\mathcal{T}$ if and only if $a_i < m$ for all i.

The following lemma provides an interpretation of the indexing set $\mathcal{T}$ in terms of affine permutations.

Lemma 4.26. *Let $\mathtt{w}_i(\mathbf{a}) := \mathtt{w}(\mathbf{a})_i = a_i - \frac{m}{n}(g_a(i) - 1)$ be the weights of u_i as above. Consider the affine permutation*

$$\omega := [-n\mathtt{w}_1(\mathbf{a}), \ldots, -n\mathtt{w}_n(\mathbf{a})]^{-1}.$$

Then the following statements hold:

(a) $(a_1, \ldots, a_n) \in \mathcal{T}$ if and only if ω is m-stable.

(b) $a_i \geq 0$ for all i if and only if $\omega p_m \in \mathtt{L}^+_{\mathrm{min}}(n)$, where $p_m = [0, m, \ldots, (n-1)m]$.

Proof. We have

$$\omega^{-1}(i) = -n\mathtt{w}_i(\mathbf{a}) = -na_i + m(g_{\mathbf{a}}(i) - 1),$$

so by (6) we have $\omega p_m = \mathtt{t}_{\mathbf{a}} g_{\mathbf{a}}^{-1}$. By Lemma 2.15 ω is m-stable if and only if $\mathbf{a} \in \mathcal{T}$. Finally, $a_i \geq 0$ for all i if and only if $\mathtt{t}_{\mathbf{a}} g_{\mathbf{a}}^{-1} \in \mathtt{L}^+_{\mathrm{min}}(n)$. □

Remark 4.27. The action of π on $(a_1, \ldots, a_n)$ corresponds to the conjugation of ω_a by $\pi \in \widetilde{\mathcal{S}_n}$ which effectively slides the window in $\omega_{\mathbf{a}}$. Remark 4.25 gives a choice of a representative in each π-orbit with $a_1 = 0$ and $m > a_i \geq 0$ for $i > 1$.

From the viewpoint of affine permutations, a more natural choice of a representative is given by the balancing condition $\sum_{i=1}^n \omega(i) = \frac{n(n+1)}{2}$. The corresponding permutations will be still m-stable, and by Remark 2.16 they are in bijection with the alcoves insider the m-dilated fundamental alcove.

Therefore we get an explicit bijection between the alcoves insider the m-dilated fundamental alcove, m-stable balanced affine permutations and vectors $(0, a_2, \ldots, a_n)$ with $0 \leq a_i < m$.

5. A Mackey formula for $H_{t,c}$

5.1. **Basis in $H_{t,c}$.** In this section we present a basis in the algebra $H_{t,c}$ using the generators from Section 3.1, it is an analogue of the PBW basis from Section 2.1. Recall that $H_n(\mathbf{y})$ is the subalgebra generated by $\mathcal{S}_n$ and y_i (or, equivalently, by $\mathcal{S}_n$ and λ), and that $H_n(\mathbf{u})$ denotes the subalgebra generated by $\mathcal{S}_n$ and u_i.

Lemma 5.1. *(a) The algebra $H_n(\mathbf{y})$ has a basis*

$$g\lambda(s_1 s_2 \cdots s_{\mu_1}) \cdot \lambda(s_1 s_2 \cdots s_{\mu_2}) \cdots \lambda(s_1 s_2 \cdots s_{\mu_{r'}})$$

for $g \in \mathcal{S}_n$ and $0 \leq \mu_{r'} \leq \ldots \leq \mu_1$.

(b) The algebra $H_{t,c}$ has a basis

$$(s_{\nu_r} \cdots s_2 s_1)\tau \cdots (s_{\nu_1} \cdots s_2 s_1)\tau \cdot g \cdot \lambda(s_1 s_2 \cdots s_{\mu_1}) \cdot \lambda(s_1 s_2 \cdots s_{\mu_2}) \cdots \lambda(s_1 s_2 \cdots s_{\mu_{r'}}),$$

for $g \in \mathcal{S}_n$ and $0 \leq \mu_{r'} \leq \ldots \leq \mu_1, 0 \leq \nu_r \leq \cdots \leq \nu_1$.

(c) The algebra $H_{t,c}$ is free as a right $H_n(\mathbf{y})$-module with the basis

$$(s_{\nu_r} \cdots s_2 s_1)\tau \cdots (s_{\nu_1} \cdots s_2 s_1)\tau.$$

Proof. By Lemma 3.6 any monomial in x_i of degree r can be written as $F_{\mathcal{X}}^{-1}(\omega)$ for $\omega \in \widetilde{\mathcal{S}_n}^+$ also of degree r. By Lemma 2.8 we can write

$$\omega = (s_{\nu_r} \cdots s_2 s_1)\pi \cdots (s_{\nu_1} \cdots s_2 s_1)\pi \cdot g,\ 0 \leq \nu_r \leq \cdots \leq \nu_1.$$

so that

$$F_{\mathcal{X}}^{-1}(\omega) = (s_{\nu_r} \cdots s_2 s_1)\tau \cdots (s_{\nu_1} \cdots s_2 s_1)\tau \cdot g.$$

Similarly we can write

$$F_{\mathcal{Y}}^{-1}(\omega^{-1}) = g^{-1}\lambda(s_1 s_2 \cdots s_{\nu_1}) \cdot \lambda(s_1 s_2 \cdots s_{\nu_2}) \cdots \lambda(s_1 s_2 \cdots s_{\nu_r}).$$

The algebra $H_{t,c}$ has PBW basis $x_1^{a_1} \cdots x_n^{a_n} g' y_1^{b_1} \cdots y_n^{b_n}$, and we can rewrite $x_1^{a_1} \cdots x_n^{a_n}$ and $y_1^{b_1} \cdots y_n^{b_n}$ as above independently. Finally, (c) is obvious from (a) and (b). □

Remark 5.2. We can also write this basis in more compact form $F_{\mathcal{X}}^{-1}(\omega_1) g F_{\mathcal{Y}}^{-1}(\omega_2^{-1})$, where ω_1 and ω_2 are minimal length coset representatives in $\mathtt{L}^+_{\min}(n)$.

Corollary 5.3. *Let V be a finite dimensional representation of $\mathcal{S}_n$ with basis v_T, $T \in \mathcal{T}$. We can regard it as a representation of $H_n(\mathbf{y})$ where y_i act by 0. Then the induced representation $\Delta_{t,c}(V) := \mathrm{Ind}_{H_n(\mathbf{y})}^{H_{t,c}}(V)$ has the basis*

$$v_T(\omega) := F_{\mathcal{X}}^{-1}(\omega) v_T,$$

where ω is a minimal length coset representative in $\mathtt{L}^+_{\min}(n)$ and $T \in \mathcal{T}$.

We define a partial order on the basis elements of $\Delta_{t,c}(V)$ in Corollary 5.3 as follows: $v_T(\omega) < v_{T'}(\omega')$ if $\deg \omega = \deg \omega'$ and $\omega >_{\mathtt{lex}} \omega'$.

Example 5.4. If V is the trivial representation of $\mathcal{S}_n$ then $\Delta_{t,c}(V)$ is just the polynomial representation. By Lemma 3.6 the basis $v_T(\omega)$ matches the monomial basis in $\mathbb{C}[x_1, \ldots, x_n]$ and by Lemma 4.5 the partial order we have defined coincides with the partial order $\prec$ defined in Section 4.1.

Next, we want to understand the action of the degenerate affine Hecke algeba $H_n(\mathbf{u})$ in this basis. Via the homomorphism $\mathrm{ev}_0 : H_n(\mathbf{u}) \to \mathcal{S}_n$ that sends $u_1 \mapsto 0$ and $s_i \mapsto s_i$, the u_i act on V as Jucys-Murphy elements, and they can be simultaneously diagonalized.

Lemma 5.5. *Suppose that $v_T \in V$ has weight $\mathtt{w}$, i.e., it is a common eigenvector for the u_i with eigenvalues $\mathtt{w}_i$. Then*

$$\begin{aligned} u_i(v_T(\omega)) &= (\omega \cdot \mathtt{w})_i v_T(\omega) + \ell.\text{o.t} \\ &= \mathtt{w}_{\omega^{-1}(i)} v_T(\omega) + \ell.\text{o.t} \end{aligned}$$

where ω is a a minimal length coset representative in $\mathtt{L}^+_{\min}(n)$, $\omega \cdot \mathtt{w}$ is defined using the action (4) *and* ℓ.o.t *denotes lower order terms.*

Proof. As in Remark 3.3, to simplify notation, we define u_i for all $i \in \mathbb{Z}$ by $u_{i+n} = u_i - t$, and likewise for $\mathtt{w}_{i+n}$. Now the relations between u_i, s_j and τ get the following form:

$$s_j u_i = u_{s_j(i)} s_j + \begin{cases} c & \text{if } i \equiv j \bmod n \\ -c & \text{if } i \equiv j+1 \bmod n \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\tau u_i = u_{\pi(i)} \tau.$$

Overall, we can write

$$F_{\mathcal{X}}^{-1}(\omega) u_i = u_{\omega(i)} F_{\mathcal{X}}^{-1}(\omega) + \ldots$$

so

$$u_i F_{\mathcal{X}}^{-1}(\omega) v_T = F_{\mathcal{X}}^{-1}(\omega) u_{\omega^{-1}(i)} v_T + \ldots = (\omega \cdot \mathtt{w})_i F_{\mathcal{X}}^{-1}(\omega) v_T + \ldots,$$

and by Lemma 4.6 all extra terms are less than $F_{\mathcal{X}}^{-1}(\omega)v_T$ in our order.

□

Corollary 5.6. *The generalized eigenvalues of u_i on $\Delta_{t,c}(V)$ are expressed as $\mathtt{w} = \omega \cdot \kappa_T$ where κ_T are eigenvalues of Jucys-Murphy operators for the basis v_T and $\omega \in \mathtt{L}^+_{\text{min}}(n)$.*

5.2. **Decomposition into $H_n(\mathbf{u})$-modules.** In this section we give a more precise presentation of induced modules. First, we recall a useful construction of $H_n(\mathbf{u})$-modules which are induced from parabolic subgroups.

Let

$$\mathcal{P}^+(n) = \{\mathbf{a} \in \mathbb{Z}^n_{\geq 0} \mid a_1 \leq a_2 \leq \cdots \leq a_n\}$$
$$\mathcal{P}^-(n) = \{\mathbf{d} \in \mathbb{Z}^n_{\geq 0} \mid d_1 \geq d_2 \geq \cdots \geq d_n\}.$$

Let $\mathbf{a} \in \mathbb{Z}^n$ and let $\mathcal{S}(\mathbf{a})$ be its stabilizer in $\mathcal{S}_n$. In the special case $\mathbf{d} \in \mathcal{P}^-(n)$, the stabilizer $\mathcal{S}(\mathbf{d})$ is a standard parabolic subgroup. If $(k_1, \ldots, k_s)$ is the composition of n that gives the multiplicities of the entries of $\mathbf{d}$, then $\mathcal{S}(\mathbf{d}) = \langle s_i \mid d_i = d_{i+1}\rangle \simeq \mathcal{S}_{k_1} \times \cdots \times \mathcal{S}_{k_n}$. Note that this subgroup is conjugate to any such parabolic with the k_i reordered. Recall $\omega_0 = [n, \ldots, 2, 1]$ is the longest element of $\mathcal{S}_n$. Let

$$\mathbf{d}^{\text{rev}} = \omega_0 \cdot \mathbf{d} = (d_n, \ldots, d_2, d_1)$$

so in particular $\mathcal{S}(\mathbf{d})$ and $\mathcal{S}(\mathbf{d}^{\text{rev}})$ are conjugate; and $\mathbf{d} \in \mathcal{P}^-(n) \iff \mathbf{d}^{\text{rev}} \in \mathcal{P}^+(n)$. Let $\omega_0^{\mathbf{d}}$ be the longest element of $\mathcal{S}(\mathbf{d})$. Conjugation by $\omega_0\omega_0^{\mathbf{d}}$ induces an isomorphism $\mathcal{S}(\mathbf{d}) \to \mathcal{S}(\mathbf{d}^{\text{rev}})$ we will denote $\text{rev}_{\mathbf{d}}$. Observe $\omega_0\omega_0^{\mathbf{d}} = g_{\mathbf{d}}$ as it sorts $\mathbf{d}$ to $\mathbf{d}^{\text{rev}} = \text{sort}(\mathbf{d})$. In fact, we would have produced the same isomorphism conjugating by $\omega_{\mathbf{d}}^{-1}$, where we recall $\omega_{\mathbf{d}} \in \mathtt{L}^+_{\text{min}}(n) \subseteq \widetilde{\mathcal{S}_n}$. Similarly, we have a corresponding parabolic subalgebra of $H_n(\mathbf{u})$ we will denote

$$H(\mathbf{d}, \mathbf{u}) := \langle s_i, u_j \mid d_i = d_{i+1}, 1 \leq j \leq n\rangle = \mathbb{C}[u_1, \ldots, u_n] \rtimes \mathcal{S}(\mathbf{d}).$$

Just as we have an algebra automorphism $\mathtt{shift} : H_n(\mathbf{u}) \to H_n(\mathbf{u})$ that sends $s_i \mapsto s_i$ but does a constant shift $u_i \mapsto u_i + t$, $H(\mathbf{d}, \mathbf{u})$ has a "finer" automorphism that is the identity on $\mathcal{S}(\mathbf{d})$ and shifts $u_i \mapsto u_i + d_i t$. Using this $\mathtt{shift}$ map, we can extend

$$\begin{aligned} \text{rev}_{\mathbf{d}} : H(\mathbf{d}, \mathbf{u}) &\to H(\mathbf{d}^{\text{rev}}, \mathbf{u}) \\ s_i &\mapsto s_{g_{\mathbf{d}}(i)} = g_{\mathbf{d}} s_i g_{\mathbf{d}}^{-1} \\ u_j &\mapsto u_{g_{\mathbf{d}}(j)} + d_j t. \end{aligned}$$

Given an $H(\mathbf{d}^{\text{rev}}, \mathbf{u})$-module M, via the above algebra isomorphism we can turn it into the "twisted" $H(\mathbf{d}, \mathbf{u})$-module we denote $M^{\text{rev}_{\mathbf{d}}}$. Note that $\text{rev}_{\mathbf{d}}(s_i) = \omega_{\mathbf{d}}^{-1} s_i \omega_{\mathbf{d}}$ and $\text{rev}_{\mathbf{d}}(u_j) = u_{\omega_{\mathbf{d}}^{-1}(j)}$, when we extend the notion of the $u_j = u_{j+kn} + kt$ to be indexed by $j \in \mathbb{Z}$ as in Remark 3.3. However, the map $\text{rev}_{\mathbf{d}}$ does not agree with conjugation by $\omega_{\mathbf{d}}^{-1}$, but under some lens it does up to "lower order terms" in a sense that will be made more precise in Remark 5.10 below. This is consistent with the observation that in $H_n(\mathbf{u})$ we have $u_j F_{\mathcal{X}}^{-1}(\omega) = F_{\mathcal{X}}^{-1}(\omega) u_{\omega^{-1}(j)} + \ell.\text{o.t}$, where the latter are $F_{\mathcal{X}}^{-1}$ applied to terms lower than $\omega \in \widetilde{\mathcal{S}_n}$ in Bruhat order.

Example 5.7. Let $n = 5$, $\mathbf{d} = (2, 2, 0, 0, 0)$, so $\mathbf{d}^{\text{rev}} = (0, 0, 0, 2, 2)$. Note $\omega_{\mathbf{d}} = [3, 4, 5, 11, 12]$, $\omega_{\mathbf{d}}^{-1} = [-6, -5, 1, 2, 3]$, $\mathcal{S}(\mathbf{d}) \simeq \mathcal{S}_2 \times \mathcal{S}_3$ and $\mathcal{S}(\mathbf{d}^{\text{rev}}) \simeq \mathcal{S}_3 \times \mathcal{S}_2$.

The permutations $\omega_0 = [5, 4, 3, 2, 1]$, $\omega_0^{\mathbf{d}} = [2, 1, 5, 4, 3]$, $\omega_0\omega_0^{\mathbf{d}} = [4, 5, 1, 2, 3] = g_{\mathbf{d}}$. The restricted module $\text{Res}_{H(\mathbf{d}^{\text{rev}}, \mathbf{u})} \text{triv} = M \boxtimes N$ where M, N are one-dimensional spanned by weight vectors with u-weight $(0, -c, -2c)$ and $(-3c, -4c)$ repectively. The twisted $H(\mathbf{d}, \mathbf{u})$-module $[\text{Res}_{H(\mathbf{d}^{\text{rev}}, \mathbf{u})} \text{triv}]^{\text{rev}_{\mathbf{d}}}$

is one-dimensional, now spanned by weight vectors with u-weight $(-3c+2t, -4c+2t)$ and $(0, -c, -2c)$ repectively. It has the form $N^{\mathtt{shift}\times 2} \boxtimes M$. The map $\mathrm{rev}_{\mathbf{d}}$ sends

$$\begin{aligned} s_1 &\mapsto s_4 & u_1 &\mapsto u_{-6} = u_4 + 2t \\ s_3 &\mapsto s_1 & u_2 &\mapsto u_{-5} = u_5 + 2t \\ s_4 &\mapsto s_2 & u_3 &\mapsto u_1 \\ & & u_4 &\mapsto u_2 \\ & & u_5 &\mapsto u_3 \end{aligned}$$

Just as the minimal length left coset representatives $\{\omega_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{P}(n)\} = \mathtt{L}^+_{\min}(n) \subseteq \widetilde{\mathcal{S}_n}^+$ are those affine permutations whose window notation have positive increasing entries, the minimal length *double* coset representatives with respect to $\mathcal{S}_n$ are those $\omega_{\mathbf{a}}$ whose inverses' window notation have increasing entries. These are exactly the $\omega_{\mathbf{d}}$ for $\mathbf{d} \in \mathcal{P}^-(n)$. See Example 5.7.

Example 5.8. Let $n = 3$, $\mathbf{d} = (4, 1, 1)$, so $\omega_{\mathbf{d}} = [5, 6, 13]$ is a minimal length double coset representative as $\mathbf{d} \in \mathcal{P}^-(n)$. Note the double coset decomposes into left cosets as

$$\begin{aligned} \mathcal{S}_3[5,6,13]\mathcal{S}_3 &= [5,6,13]\mathcal{S}_3 \sqcup [4,6,14]\mathcal{S}_3 \sqcup [4,5,15]\mathcal{S}_3 \\ \text{i.e., } \mathcal{S}_3\omega_{\mathbf{d}}\mathcal{S}_3 &= \omega_{\mathbf{d}}\mathcal{S}_3 \sqcup \omega_{s_1\mathbf{d}}\mathcal{S}_3 \sqcup \omega_{s_2s_1\mathbf{d}}\mathcal{S}_3 \\ &= \omega_{(4,1,1)}\mathcal{S}_3 \sqcup \omega_{(1,4,1)}\mathcal{S}_3 \sqcup \omega_{(1,1,4)}\mathcal{S}_3. \end{aligned}$$

$\mathcal{S}_3/\mathcal{S}(\mathbf{d})$ has minimal length left coset representatives $\{\mathrm{id}, s_1, s_2s_1\}$.

It is well known that $\mathbb{C}\mathcal{S}_n$ is a free right module over $\mathbb{C}\mathcal{S}(\mathbf{d})$ of rank $\frac{n!}{k_1!\cdots k_s!}$. Given a representation M of $\mathcal{S}(\mathbf{d})$, we can consider the induced representation $\mathrm{Ind}_{\mathcal{S}(\mathbf{d})}^{\mathcal{S}_n} M$ which has dimension $\frac{n!}{k_1!\cdots k_s!}\dim M$. Note that if M is a $H(\mathbf{d},\mathbf{u})$-module, then $\mathrm{Ind}_{\mathcal{S}(\mathbf{d})}^{\mathcal{S}_n} M$ naturally has a structure of $H_n(\mathbf{u})$-module, which agrees with $\mathrm{Ind}_{H(\mathbf{d},\mathbf{u})}^{H_n(\mathbf{u})} M$.

Theorem 5.9. *Let V be a representation of $\mathcal{S}_n$, inflated to a representation of $H_n(\mathbf{y})$ by setting y_i to act as 0. The induced module $\Delta_{t,c}(V) = \mathrm{Ind}_{H_n(\mathbf{y})}^{H_{t,c}}(V)$ has a filtration such that subquotients are isomorphic as $H_n(\mathbf{u})$-modules to the induced representations*

$$V_{\mathbf{d}} := \mathrm{Ind}_{H(\mathbf{d},\mathbf{u})}^{H_n(\mathbf{u})}\left[\mathrm{Res}_{H(\mathbf{d}^{\mathrm{rev}},\mathbf{u})}^{H_n(\mathbf{u})} V\right]^{\mathrm{rev}_{\mathbf{d}}},\ \mathbf{d} \in \mathcal{P}^-(n),$$

where here we inflate V along the homomorphism ev_0.

Proof. Let $\mathbf{d} \in \mathcal{P}^-(n)$. By Lemma 5.1 and Lemma 2.8 $H_{t,c}$ has filtrations

$$\mathcal{B}_{\leq\mathbf{d}} = \bigoplus_{\substack{\mathbf{a}\in\mathcal{P}^-(n) \\ ||\mathbf{a}||\leq||\mathbf{d}|| \\ ||\mathbf{a}||=||\mathbf{d}|| \implies \mathbf{a}\geq_{1ex}\mathbf{d}}} \mathbb{C}\mathcal{S}_n F_{\mathcal{X}}^{-1}(\omega_{\mathbf{a}})H_n(\mathbf{y}) \qquad \mathcal{B}_{<\mathbf{d}} = \bigoplus_{\substack{\mathbf{a}\in\mathcal{P}^-(n) \\ ||\mathbf{a}||\leq||\mathbf{d}|| \\ ||\mathbf{a}||=||\mathbf{d}|| \implies \mathbf{a}>_{1ex}\mathbf{d}}} \mathbb{C}\mathcal{S}_n F_{\mathcal{X}}^{-1}(\omega_{\mathbf{a}})H_n(\mathbf{y})$$

clearly preserved by $\mathbb{C}\mathcal{S}_n$. By Lemma 4.6 the filtrations are also preserved by $H_n(\mathbf{u})$. These induce filtrations on $\Delta(V)$ with subquotients

$$V_{\mathbf{d}} = \mathcal{B}_{\leq\mathbf{d}}\Delta_{t,c}(V) \;/\; \mathcal{B}_{<\mathbf{d}}\Delta_{t,c}(V).$$

In the following argument we lighten notation, writing $\widehat{p}$ for $F_{\mathcal{X}}^{-1}(p)$, so for instance the above expressions would become $\mathbb{C}\mathcal{S}_n\widehat{\omega_{\mathbf{a}}}H_n(\mathbf{y})$.

Because $\mathcal{S}_n\omega_{\mathbf{d}}\mathcal{S}_n = \bigsqcup_{g\in\mathcal{S}_n/\mathcal{S}(\mathbf{d})} g\omega_{\mathbf{d}}\mathcal{S}_n = \bigsqcup_{g\in\mathcal{S}_n/\mathcal{S}(\mathbf{d})} \omega_{g\cdot\mathbf{d}}\mathcal{S}_n$, the following spaces are isomorphic not just as vector spaces, but as $\mathbb{C}\mathcal{S}_n$-modules, $V_{\mathbf{d}} \simeq \mathbb{C}\mathcal{S}_n\widehat{\omega_{\mathbf{d}}} \otimes_{\mathbb{C}\mathcal{S}_n} V$. In particular, as a $\mathbb{C}\mathcal{S}_n$-module, $V_{\mathbf{d}}$ is generated by $\widehat{\omega_{\mathbf{d}}} \otimes V$, and is spanned by the independent spaces $\widehat{\omega_{g\cdot\mathbf{d}}} \otimes_{\mathbb{C}} V$, for $g \in \mathcal{S}_n/\mathcal{S}(\mathbf{d})$. Note that if $s_i \in \mathcal{S}(\mathbf{d})$ then $\widehat{s_i\omega_{\mathbf{d}}} \otimes V = \widehat{\omega_{\mathbf{d}}(\omega_{\mathbf{d}}^{-1}s_i\omega_{\mathbf{d}})} \otimes V = \widehat{\omega_{\mathbf{d}}\,\mathrm{rev}_{\mathbf{d}}(s_i)} \otimes V = \widehat{\omega_{\mathbf{d}}} \otimes \mathrm{rev}_{\mathbf{d}}(s_i)V$. Further as $u_j\widehat{\omega_{\mathbf{d}}} = (\widehat{\omega_{\mathbf{d}}}\,\mathrm{rev}_{\mathbf{d}}(u_j) + \ell.\text{o.t})$, we have $u_j\widehat{\omega_{\mathbf{d}}} \otimes V = (\widehat{\omega_{\mathbf{d}}}\,\mathrm{rev}_{\mathbf{d}}(u_j) + \ell.\text{o.t}) \otimes V \equiv \widehat{\omega_{\mathbf{d}}} \otimes \mathrm{rev}_{\mathbf{d}}(u_j)V$ since the lower order terms here involve $\widehat{\omega_{\mathbf{a}}}$ with $\mathbf{a} >_{\mathtt{lex}} \mathbf{d}$ by Lemma 4.6, and these are killed in $V_{\mathbf{d}}$. Thus we see that as an $H_n(\mathbf{u})$-module $V_{\mathbf{d}} \simeq \mathrm{Ind}_{H(\mathbf{d},\mathbf{u})}^{H_n(\mathbf{u})}\left[\mathrm{Res}_{H(\mathbf{d}^{\mathrm{rev}},\mathbf{u})}^{H_n(\mathbf{u})} V\right]^{\mathrm{rev}_{\mathbf{d}}}$.

□

Remark 5.10. One can regard this theorem as a version of the classical Mackey formula:

$$\mathrm{Res}_K \mathrm{Ind}_H^G(\rho) = \bigoplus_{\omega \in K\backslash G/H} \mathrm{Ind}^K_{\omega H\omega^{-1}\cap K}(\rho^\omega) = \bigoplus_{\omega \in K\backslash G/H} \mathrm{Ind}^K_{\omega H\omega^{-1}\cap K}(\mathrm{Res}^H_{H\cap\omega^{-1}K\omega}\,\rho)^\omega,$$

where G is a finite group, H, K are its subgroups, ρ is a representation of H and $\rho^\omega(x) = \rho(\omega^{-1}x\omega)$.

Our setting shares many features with classical Mackey for the case $G = \widetilde{\mathcal{S}_n}$, $H = K = \mathcal{S}_n$, where the minimal length double coset representatives are $\{\omega_{\mathbf{d}} \mid \mathbf{d} \in \mathbb{Z}^n, d_1 \geq \cdots \geq d_n\}$. In that case $\mathcal{S}(\mathbf{d}) = \omega_{\mathbf{d}}\mathcal{S}_n\omega_{\mathbf{d}}^{-1} \cap \mathcal{S}_n$, $\mathcal{S}(\mathbf{d}^{\mathrm{rev}}) = \mathcal{S}_n \cap \omega_{\mathbf{d}}^{-1}\mathcal{S}_n\omega_{\mathbf{d}}$ and one computes the action on an induced module via $p\,(\omega_{\mathbf{d}} \otimes V) = \omega_{\mathbf{d}} \otimes (\omega_{\mathbf{d}}^{-1}p\omega_{\mathbf{d}})V = \omega_{\mathbf{d}} \otimes pV^{\omega_{\mathbf{d}}}$, which is also equal to $\omega_{\mathbf{d}} \otimes pV^{\mathrm{rev}_{\mathbf{d}}} = \omega_{\mathbf{d}} \otimes \mathrm{rev}_{\mathbf{d}}(p)V$ for $p \in \mathcal{S}(\mathbf{d})$.

In our setting $H(\mathbf{d},\mathbf{u})$ plays the role of $\omega_{\mathbf{d}}H\omega_{\mathbf{d}}^{-1} \cap K$; $H(\mathbf{d}^{\mathrm{rev}},\mathbf{u})$ the role of $H \cap \omega_{\mathbf{d}}^{-1}K\omega_{\mathbf{d}}$. $F_{\mathcal{X}}^{-1}(\omega_{\mathbf{d}}^{-1}p\omega_{\mathbf{d}}) =: \widehat{\omega_{\mathbf{d}}^{-1}p\omega_{\mathbf{d}}}$ makes sense for $p \in \mathcal{S}(\mathbf{d})$. On the other hand, $\omega_{\mathbf{d}}^{-1}u_i\omega_{\mathbf{d}}$ is problematic on many levels. This is in part why we must work with the isomorphism $\mathrm{rev}_{\mathbf{d}}$ above.

While conjugation by $\omega_{\mathbf{d}}^{-1}$ or by $g_{\mathbf{d}}$ gives us in isomorphism from $\mathcal{S}(\mathbf{d})$ to $\mathcal{S}(\mathbf{d}^{\mathrm{rev}})$ when working inside of $\widetilde{\mathcal{S}_n}$, the most natural way to extend the notion of conjugation by $\omega_{\mathbf{d}}^{-1}$ to $H_{t,c}$ does *not* send $H(\mathbf{d},\mathbf{u})$ to $H(\mathbf{d}^{\mathrm{rev}},\mathbf{u})$. While $F_{\mathcal{X}}^{-1}(\omega_{\mathbf{d}}) =: \widehat{\omega_{\mathbf{d}}}$ is not invertible, this is not the main obstruction; one can localize and invert the x_i (as one does with the trigonometric Cherednik algebra [48]).This essentially replaces τ with π and adjoins π^{-1}, so would enlarge our algebra and embed a copy of $\widetilde{\mathcal{S}_n}$. We can define $\pi u_i \pi^{-1} = u_{\pi(i)} = u_{i+1}$ and $\pi^{-1}u_i\pi = u_{i-1}$ using the convention in Remark 3.3, and this is compatible with relation (9). This allows us to define conjugation by $\omega_{\mathbf{d}}^{-1}$. It will still send $\mathcal{S}(\mathbf{d}) \to \mathcal{S}(\mathbf{d}^{\mathrm{rev}})$ but will not send $H(\mathbf{d},\mathbf{u}) \to H(\mathbf{d}^{\mathrm{rev}},\mathbf{u})$ as conjugation by $\mathcal{S}_n$ does not preserve $\mathcal{A}$ (even though conjugation by π does preserve $\mathcal{A}$). For $g \in \mathcal{S}_n$ recall that $H_n(\mathbf{u}) \ni g^{-1}u_ig = u_{g^{-1}(i)} + \ell.\text{o.t}$, where here lower is with respect to u degree. More specifically, $u_ig = gu_{g^{-1}(i)} +$ terms $<_{\mathtt{B}} g$ in Bruhat order. These are the lower order terms we throw away when considering $V_{\mathbf{d}}$ or $\mathcal{B}_{\leq\mathbf{d}}/\mathcal{B}_{<\mathbf{d}}$. Throwing away these lower order terms agrees with replacing conjugation by $\omega_{\mathbf{d}}^{-1}$ with the isomorphism $\mathrm{rev}_{\mathbf{d}} : H(\mathbf{d},\mathbf{u}) \to H(\mathbf{d}^{\mathrm{rev}},\mathbf{u})$ when describing the Mackey filtration.

As a corollary to Theorem 5.9 we have the following.

Corollary 5.11. *Let V be a $\mathbb{C}\mathcal{S}_n$-module such that when inflated along* ev_0 *to be an $H_n(\mathbf{u})$-module it has u-weight basis $\{v_T \mid T \in \mathcal{T}\}$. Let $\mathtt{w}_T$ denote the weight of v_T. If we assume $t \neq 0$, then the $H_{t,c}$-module $\mathrm{Ind}_{H_n(\mathbf{y})}^{H_{t,c}} V$ has finite dimensional generalized u-weight spaces and a generalized u-weight basis indexed by $\mathcal{P}(n) \times \mathcal{T}$. Its generalized weights are*

$$\{\omega_{\mathbf{a}} \cdot \mathtt{w}_T \mid \mathbf{a} \in \mathcal{P}(n), T \in \mathcal{T}\} = \{g\omega_{\mathbf{d}} \cdot \mathtt{w}_T \mid \mathbf{d} \in \mathcal{P}^-(n), g \in \mathcal{S}_n/\mathcal{S}(\mathbf{d}), T \in \mathcal{T}\}.$$

When $t = 0$, the weights are still given by the formula above but the u-weight spaces are no longer finite-dimensional. We study this case in detail in Section 9.

It is worth noting that given fixed $\mathtt{w} = (\mathtt{w}_1, \mathtt{w}_2, \dots \mathtt{w}_n) \in \mathbb{C}^n$, $\mathbf{d} = (d_1, d_2, \dots, d_n) \in \mathcal{P}^-(n)$ the set

$$\{g\omega_{\mathbf{d}} \cdot \mathtt{w} \mid g \in \mathcal{S}_n/\mathcal{S}(\mathbf{d})\} = \{f \cdot (\mathtt{w}_1 + d_nt, \mathtt{w}_2 + d_{n-1}t, \dots, \mathtt{w}_n + d_1t) \mid f \in \mathcal{S}_n/\mathcal{S}(\mathbf{d}^{\mathrm{rev}})\}.$$

Remark 5.12. Note that Proposition 4.8, Theorem 4.9 and Corollary 5.6, Theorem 4.9, Proposition 4.8, follow as corollaries to Theorem 5.9, see also Remark 4.12.

Example 5.13. Let us consider the Mackey formula for $M = \Delta_c(\mathrm{triv})$ in the case $n = 2, c = 2, t = 1$. As we shall see, it is not $\mathcal{A}$-semisimple. For weights of the form $\mathtt{w} = (d,d)$, $\dim M^{\mathrm{gen}}_{(d,d)} = 2 > 1 = \dim M_{(d,d)}$. For all other weights $\mathtt{w} = (\mathtt{w}_1, \mathtt{w}_2)$ with $\mathtt{w}_1 \neq \mathtt{w}_2$ that occur, its generalized weight spaces $M^{\mathrm{gen}}_{\mathtt{w}} = M_{\mathtt{w}}$ are 1-dimensional.

We have a single $T = \boxed{1}\boxed{2}$ and v_T has weight $\mathtt{w} = (0, -c) = (0, -2)$. For $\mathbf{d} \in \mathcal{P}^-(n)$ we split into two cases according to $\mathcal{S}(\mathbf{d})$.

Case 1: $\mathbf{d} = (d,d) = \mathbf{d}^{\mathrm{rev}}$, $\mathcal{S}(\mathbf{d}) = \mathcal{S}_2$. Thus our induction and restriction functors are trivial and $\mathrm{Ind}^{H_2(\mathbf{u})}_{H(\mathbf{d},\mathbf{u})}[\mathrm{Res}^{H_2(\mathbf{u})}_{H(\mathbf{d}^{\mathrm{rev}},\mathbf{u})} \mathrm{triv}]^{\mathrm{rev}_\mathbf{d}} = \mathrm{triv}^{\mathrm{rev}_\mathbf{d}}$. The module $\mathrm{triv}^{\mathrm{rev}_\mathbf{d}}$ still carries the trivial action of $\mathcal{S}_2$, but $u_1 = 0 + d, u_2 = -c + d$. In other words it corresponds to a weight vector $v_\mathbf{d} = v_{(d,d)} \in \Delta_c(\mathrm{triv})$ of weight $\mathtt{w} = (d, d-2)$. Recall we require $d \geq 0$.

Case 2: $\mathbf{d} = (d_1 > d_2)$, $\mathbf{d}^{\mathrm{rev}} = (d_2, d_1)$, $g_\mathbf{d} = s_1$, and $\mathcal{S}(\mathbf{d}) = \{\mathrm{id}\} = \mathcal{S}(\mathbf{d}^{\mathrm{rev}})$. Now $\mathrm{Res}^{H_2(\mathbf{u})}_{H(\mathbf{d}^{\mathrm{rev}},\mathbf{u})} \mathrm{triv} = \mathrm{Res}^{H_2(\mathbf{u})}_{\mathcal{A}} \mathrm{triv} = (0) \boxtimes (-c)$ where we write the one-dimensional $\mathcal{A} = \mathbb{C}[u_1] \otimes \mathbb{C}[u_2]$-module on which $u_1 - \alpha$ and $u_2 - \beta$ vanish as $(\alpha) \boxtimes (\beta)$. The twisted module is

$$[(0) \boxtimes (-c)]^{\mathrm{rev}_\mathbf{d}} = (-c + d_1) \boxtimes (0 + d_2) = (d_1 - 2) \boxtimes (d_2).$$

Finally

$$\mathrm{Ind}^{H_2(\mathbf{u})}_{H(\mathbf{d},\mathbf{u})}[\mathrm{Res}^{H_2(\mathbf{u})}_{H(\mathbf{d}^{\mathrm{rev}},\mathbf{u})} \mathrm{triv}]^{\mathrm{rev}_\mathbf{d}} = \mathrm{Ind}^{H_2(\mathbf{u})}_{\mathcal{A}} (d_1 - 2) \boxtimes (d_2).$$

This is an irreducible 2-dimensional $H_2(\mathbf{u})$-module.
In the special case $d_2 = d_1 - 2$ it is not A-semisimple. In other words the u_i act with Jordan blocks of size 2. The generalized $\mathtt{w} = (d_1 - 2, d_2) = (d_2, d_2)$-weight space is 2-dimensional and corresponds to the basis vectors in $\Delta_c(\mathrm{triv})$ which by abuse of notation we can still call $v_\mathbf{d} = v_{(d_2+2,d_2)}$ and $v_{s_1\mathbf{d}} = v_{(d_2,d_2+2)}$.
When $d_2 \neq d_1 - 2$ we get one-dimensional weight spaces spanned by $v_\mathbf{d} = v_{(d_1,d_2)}$ of weight $\mathtt{w} = (d_1 - 2, d_2)$ and $v_{s_1\mathbf{d}} = v_{(d_2,d_1)}$ of weight $s_1 \cdot \mathtt{w} = (d_2, d_1 - 2)$. Because these (generalized) weights occur with multiplicity one, the Mackey filtration tells us these generalized weight spaces are true weight spaces.

6. The standard modules

6.1. **Other standard modules.** We continue to assume the parameter c has the form $c = m/n > 0$ with $\gcd(m,n) = 1$ and $t = 1$. In this section, we will analyze the action of the Dunkl-Opdam subalgebra on a standard module $\Delta_c(\mu)$ where μ is not necessarily the trivial partition of n. We will denote by $\mathrm{SYT}(\mu)$ the set of standard tableaux on μ. For $T \in \mathrm{SYT}(\mu)$, T_i denotes the box of μ labeled by i under T, and $\mathrm{ct}_T(i)$ is the content of this box.

Definition 6.1. Let $(\mathbf{a}, T) \in \mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\mu)$. Denote by $\mathtt{w}(\mathbf{a}, T) \in \mathbb{C}^n$ the weight whose i-th component is $\mathtt{w}_i(\mathbf{a}, T) = a_i - \mathrm{ct}_T(g_\mathbf{a}(i))c$ where, recall, $g_\mathbf{a} \in \mathcal{S}_n$ is the minimal permutation that sorts $\mathbf{a}$ increasingly.

From Lemma 4.1, it is clear that we have that the intertwining operators send $\tau : \Delta_c(\mu)_{\mathtt{w}(\mathbf{a},T)} \to \Delta_c(\mu)_{\mathtt{w}(\pi\cdot\mathbf{a},T)}$ and, if $a_i \neq a_{i+1}$, $\sigma_i : \Delta_c(\mu)_{\mathtt{w}(\mathbf{a},T)} \to \Delta_c(\mu)_{\mathtt{w}(s_i\mathbf{a},T)}$. The following result generalizes Theorem 4.9.

Theorem 6.2. *Let $c = m/n > 0$ with $\gcd(m,n) = 1$ and $M = \Delta_c(\mu)$. Then, for any $(\mathbf{a}, T) \in \mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\mu)$ we have $M_{\mathtt{w}(\mathbf{a},T)} = M^{\mathrm{gen}}_{\mathtt{w}(\mathbf{a},T)}$ is 1-dimensional. Moreover, if $a_i > a_{i+1}$, then $\sigma_i|_{M_{\mathtt{w}(\mathbf{a},T)}} \neq 0$, and the action of the Dunkl-Opdam subalgebra on M is diagonalizable with eigenvalues given by $\mathtt{w}(\mathbf{a}, T)$.*

Proof. The operators u_i act on V_μ as classical Jucys-Murphy operators, and have spectrum $-\mathrm{ct}_T(i)c$ for $T \in \mathrm{SYT}(\mu)$. In other words the vector v_T has weight $\mathtt{w}_T = (-\mathrm{ct}_T(1)c, \ldots, -\mathrm{ct}_T(n)c)$. By

Corollary 5.11 the generalized u-weights of u_i on $\Delta_c(\mu)$ are given by $\omega_{\mathbf{a}}\mathtt{w}_T = \mathtt{w}(\mathbf{a},T)$. Now the theorem follows from Lemma 6.3 below. ☐

Lemma 6.3. *Let* $(\mathbf{a},T),(\mathbf{b},S)\in\mathbb{Z}^n_{\geq 0}\times\mathrm{SYT}(\mu)$. *If* $\mathtt{w}(\mathbf{a},T)=\mathtt{w}(\mathbf{b},S)$ *then* $\mathbf{a}=\mathbf{b}$ *and* $T=S$.

Proof. Assume $\mathtt{w}(\mathbf{a},T)=\mathtt{w}(\mathbf{b},S)$. Then, for every $i=1,\dots,n$,

$$a_i-b_i=c(\mathrm{ct}_T(g_{\mathbf{a}}(i))-\mathrm{ct}_S(g_{\mathbf{b}}(i))).$$

But T and S have the same shape μ, so

$$\mathrm{ct}_T(g_{\mathbf{a}}(i))-\mathrm{ct}_S(g_{\mathbf{b}}(i))\in\{-n+1,-n+1,\dots,0,\dots,n-2,n-1\}$$

so, by our assumption on $c=m/n$, we must have $a_i-b_i=0$. From here, we have $\mathbf{a}=\mathbf{b}$ and $\mathrm{ct}_S(i)=\mathrm{ct}_T(i)$ for every $i=1,\dots,n$, which implies $S=T$. ☐

Remark 6.4. For arbitrary t,c, Corollary 5.11 still applies and the same proof shows that the generalized eigenvalues of u_i on $\Delta_{t,c}(\mu)$ are given by $\mathtt{w}(\mathbf{a},T)$.

Let us now see that for $c=m/n,\gcd(m,n)=1$ the action of H_c on the standard module $\Delta_c(\mu)$ is completely determined by the spectrum of the Dunkl-Opdam subalgebra. Fix an eigenbasis $\{v_T : T\in\mathrm{SYT}(\mu)\}$ of V_μ for the Jucys-Murphy operators. Note that we have the basis $v_T(\omega)$, $\omega\in\mathtt{L}^+_{\min}(n)$ of $\Delta_c(\mu)$, cf. Corollary 5.3. For each $\mathbf{a}\in\mathbb{Z}^n_{\geq 0}$ and every standard Young tableau T on μ, denote by $v(\mathbf{a},T)\in\Delta_c(\mu)_{\mathtt{w}(\mathbf{a},T)}$ a nonzero vector, normalized so that $v_T(\omega_{\mathbf{a}})$ appears with coefficient 1 in $v(\mathbf{a},T)$.Then, as in the proof of Theorem 4.15 we have

$$\begin{aligned}
u_iv(\mathbf{a},T)&=\mathtt{w}_iv(\mathbf{a},T)\\
\tau v(\mathbf{a},T)&=v(\pi\cdot\mathbf{a},T)\\
\lambda v(\mathbf{a},T)&=\mathtt{w}_1v(\pi^{-1}\cdot\mathbf{a},T)\\
s_iv(\mathbf{a},T)&=\begin{cases}v(s_i\cdot\mathbf{a},T)+\frac{c}{\mathtt{w}_i-\mathtt{w}_{i+1}}v(\mathbf{a},T) & a_i>a_{i+1}\\ \frac{(\mathtt{w}_{i+1}-\mathtt{w}_i-c)(\mathtt{w}_{i+1}-\mathtt{w}_i+c)}{(\mathtt{w}_i-\mathtt{w}_{i+1})^2}v(s_i\cdot\mathbf{a},T)+\frac{c}{\mathtt{w}_{i+1}-\mathtt{w}_i}v(\mathbf{a},T) & a_i<a_{i+1}\end{cases}
\end{aligned}$$

where $\mathtt{w}=\mathtt{w}(\mathbf{a},T)$. Finding $s_iv(\mathbf{a},T)$ when $a_i=a_{i+1}$ is subtler. The weight of $s_iv(\mathbf{a},T)$ is $s_i\cdot\mathtt{w}$. Note that, in this case, $g_{\mathbf{a}}(i+1)=g_{\mathbf{a}}(i)+1$. Let us denote by $s_{g_{\mathbf{a}}(i)}(T)$ the tableau that is obtained from T by permuting the entries $g_{\mathbf{a}}(i)$ and $g_{\mathbf{a}}(i+1)$. Note that $s_{g_{\mathbf{a}}(i)}(T)$ may not be standard, and this is the case precisely when in the tableau T we have

(1) $T_{g_{\mathbf{a}}(i)}=(R,C)$ and $T_{g_{\mathbf{a}}(i)+1}=(R+1,C)$ for some box (R,C) in μ, or
(2) $T_{g_{\mathbf{a}}(i)}=(R,C)$ and $T_{g_{\mathbf{a}}(i)+1}=(R,C+1)$ for some box (R,C) in μ.

In these cases, $s_i\cdot\mathtt{w}(a,T)$ is not of the form $\mathtt{w}(\mathbf{a}',T')$ for a standard tableau T', so we must have $\sigma_iv(\mathbf{a},T)=0$ and therefore $s_iv(\mathbf{a},T)=\pm v(\mathbf{a},T)$. Moreover, using the explicit formula $\sigma_i=s_i-\frac{c}{u_i-u_{i+1}}$ we see that

$$s_iv(\mathbf{a},T)=\begin{cases}v(\mathbf{a},T) & a_i=a_{i+1}\text{ and }g_{\mathbf{a}}(i),g_{\mathbf{a}}(i+1)\text{ belong to the same row in }T\\ -v(\mathbf{a},T) & a_i=a_{i+1}\text{ and }g_{\mathbf{a}}(i),g_{\mathbf{a}}(i+1)\text{ belong to the same column in }T\end{cases}$$

Finally, if $s_{g_{\mathbf{a}}(i)}T$ is a standard tableau, then $\sigma_iv(\mathbf{a},T)=v(\mathbf{a},s_{g_{\mathbf{a}}(i)}(T))$ and we get

$$s_iv(\mathbf{a},T)=v\left(\mathbf{a},s_{g_{\mathbf{a}}(i)}(T)\right)+\frac{c}{\mathtt{w}_i-\mathtt{w}_{i+1}}v(\mathbf{a},T)$$

so we have recovered the action of H_c on $\Delta_c(\mu)$.

6.2. **Maps between standards.** In this section and the next one, we study maps between standard modules.

Lemma 6.5. *Suppose that* $c = m/n, \gcd(m,n) = 1$. *Let* $d_i(\mu)$ *be the number of boxes in* μ *with content* $i \bmod n$. *Then for all* $(\mathbf{a}, T) \in \mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\mu)$ *one has*

$$\sharp\{j : n\mathtt{w}_j(\mathbf{a}, T) \equiv -mi \bmod n\} = d_i(\mu).$$

Proof. We have

$$n\mathtt{w}_j(\mathbf{a}, T) = na_j - m\,\mathrm{ct}_T(g_{\mathbf{a}}(j)) \equiv -m\,\mathrm{ct}_T(g_{\mathbf{a}}(j)) \bmod n.$$

Since $g_{\mathbf{a}}$ is a permutation, $T_{g_{\mathbf{a}}(j)}$ runs over all boxes in μ and the vector $\mathrm{ct}_T(g_{\mathbf{a}}(j))$ has exactly $d_i(\mu)$ entries equal to $i \bmod n$. □

Remark 6.6. A similar argument and Remark 6.4 show that for $c = m/\ell,\ \gcd(m, \ell) = 1$ one has

$$\sharp\{j : \ell\mathtt{w}_j(\mathbf{a}, T) \equiv -mi \bmod \ell\} = d_i^{(\ell)}(\mu),$$

where $d_i^{(\ell)}(\mu)$ is the number of boxes in μ with content $i \bmod \ell$.

Lemma 6.7. *Suppose that* $c = m/n, \gcd(m,n) = 1$. *Let* $\mu \neq \mu'$ *be two partitions of* n. *Then* $\mathrm{Hom}_{H_c}(\Delta_c(\mu), \Delta_c(\mu')) = 0$ *unless both* μ *and* μ' *are hook partitions.*

Proof. Suppose that $\mathrm{Hom}_{H_c}(\Delta_c(\mu), \Delta_c(\mu')) \neq 0$, then $\mathtt{w}(\mathbf{a}, T) = \mathtt{w}(\mathbf{a}', T')$ for some $(\mathbf{a}, T) \in \mathbb{Z}_{\geq 0} \times \mathrm{SYT}(\mu)$ and $(\mathbf{a}', T') \in \mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\mu')$. By Lemma 6.5 we get $d_i(\mu) = d_i(\mu')$ for all i, which implies that μ and μ' have the same n-core [29].

Since μ has size n, either its n-core is empty and μ is a hook, or μ is an n-core itself. The same applies to μ', so they can share an n-core only if both partitions are hooks. □

Remark 6.8. A similar argument shows that for $c = m/\ell,\ \gcd(m, \ell) = 1$ one could possibly have $\mathrm{Hom}_{H_c}(\Delta_c(\mu), \Delta_c(\mu')) \neq 0$ only if μ, μ' have the same ℓ-core. This is known via the KZ functor, cf. [1] and we have obtained a purely combinatorial proof.

Corollary 6.9. *Let* $c = m/n, \gcd(m,n) = 1$. *If* μ *is not a hook partition then* $\Delta_c(\mu)$ *is irreducible.*

Proof. The proof is standard but we include it here for completeness. If R is a submodule of $\Delta(\mu)$, then (since the action of $y_1, \ldots, y_n$ is locally nilpotent) there is a vector $v \in R$ such that $y_1 v = \cdots = y_n v = 0$. It spans a finite-dimensional subspace U under the action of $\mathcal{S}_n$, and $\lambda(U) = 0$, it contains an irreducible representation of $\mathcal{S}_n$ isomorphic to $V_{\mu'}$. Then there is a notrivial morphism H_c-modules $\Delta(\mu') \to \Delta(\mu)$ which sends $V_{\mu'}$ to this subspace. □

We determine the morphisms between $\Delta_c(\mu)$ for hook partitions μ in the next subsection.

6.3. **The BGG resolution.** Throughout this section we assume $c = m/n, \gcd(m,n) = 1$.

Let us denote by $V_{\mu_\ell} := \wedge^\ell \mathbb{C}^{n-1}$ the hook representation of $\mathcal{S}_n$, so that μ_ℓ is the partition $(n-\ell, 1^\ell)$, $\ell = 0, \ldots, n-1$. In particular, V_{μ_0} is the trivial representation and $V_{\mu_{n-1}}$ the sign representation. It is known [1] that the representation $L_{m/n} := L_{m/n}(\mathrm{triv})$ admits a resolution

$$0 \to \Delta_c(\mu_{n-1}) \to \cdots \to \Delta_c(\mu_1) \to \Delta_c(\mu_0) \to 0 \tag{28}$$

that in fact coincides with the Koszul resolution of $L_{m/n}$ when considering a standard module as a $\mathbb{C}[x_1, \ldots, x_n]$-module. In this section, we will construct the resolution (28) in a purely combinatorial manner. We remark that this has been recently generalized in [17] to some other BGG resolutions.

Let us set up some notation. For each collection $1 < i_1 < \cdots < i_\ell \leq n$, let $T_{i_1 < i_2 < \cdots < i_\ell}$ be the tableau on μ_ℓ that has the numbers $1, i_1, \ldots, i_\ell$ on its leg. Clearly, every tableau on μ_ℓ is of this form.

Recall that for each element $(\mathbf{a}, T) \in \mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\mu)$ we have a nonzero vector $v(\mathbf{a}, T) \in \Delta_c(\mu)_{\mathtt{w}(\mathbf{a},T)}$. Clearly, every map on $\Delta_c(\mu)$ is completely determined by the image of the vectors $v_T = v(0, T)$, $T \in \mathrm{SYT}(\mu)$.

Lemma 6.10. *Suppose that $\ell \neq j, j+1$. Then* $\mathrm{Hom}_{H_c}(\Delta_c(\mu_\ell), \Delta_c(\mu_j)) = 0$.

Proof. Let $T = T_{i_1<i_2<\cdots<i_\ell}$ be a standard tableau of shape μ_ℓ, we have

$$n\mathtt{w}_{i_1}(0,T) = m, \ldots, n\mathtt{w}_{i_\ell}(0,T) = m\ell,\ n\mathtt{w}_i(0,T) < 0 \text{ for } i \neq 1, i_1, \ldots, i_\ell.$$

Now suppose that $\mathtt{w}(0,T) = \mathtt{w}(\mathbf{a}, T')$ for some $(\mathbf{a}, T') \in \mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\mu_j)$. One has

$$0 > n\mathtt{w}_i(\mathbf{a}, T') = na_i - m \cdot \mathrm{ct}_{T'}(g_{\mathbf{a}}(i)) \geq -m \cdot \mathrm{ct}_{T'}(g_{\mathbf{a}}(i)) \text{ for } i \neq 1, i_1, \ldots, i_\ell,$$

so μ_j has at least $n - \ell - 1$ boxes with positive contents, and $\ell \geq j$.

Suppose that $\ell \geq j+2$. It is easy to see that the equation

$$-m\,\mathrm{ct}_T(i) = na_i - m\,\mathrm{ct}_{T'}(g_{\mathbf{a}}(i)),$$

implies

$$\begin{cases} a_i = m,\ \mathrm{ct}_T(i) + n = \mathrm{ct}_{T'}(g_{\mathbf{a}}(i)) & \text{if } i = i_{j+1}, \ldots, i_\ell, \\ a_i = 0,\ \mathrm{ct}_T(i) = \mathrm{ct}_{T'}(g_{\mathbf{a}}(i)) & \text{otherwise.} \end{cases}$$

By definition, this implies $g_{\mathbf{a}}(i_{j+1}) = n - \ell + 1, \ldots, g_{\mathbf{a}}(i_\ell) = n$, so

$$\mathrm{ct}_{T'}(n - \ell + 1) = n - (j+1), \ldots, \mathrm{ct}_{T'}(n) = n - \ell.$$

But this means that the first row of T' contains the numbers $n, n-1, \ldots, n-\ell+1$ in decreasing order, contradiction. ☐

Remark 6.11. Note that if a simple $L_c(\mu)$ appears as a composition factor inside a standard module $\Delta_c(\mu')$ then all weights $\mathtt{w}(0,T)$ have to appear as weights of $\Delta_c(\mu')$, where T is a standard Young tableau on μ. Thus, the proof of Lemma 6.10 shows that the only composition factors of $\Delta_c(\mu_j)$ can be $L_c(\mu_j)$ and $L_c(\mu_{j+1})$.

Moreover, the multiplicity of $L_c(\mu)$ as a composition factor of $\Delta_c(\mu')$ is bounded above by the dimension of the (generalized) weight space $\Delta_c(\mu')_{\mathtt{w}(0,T)}$ where T is any standard Young tableau on μ. Thus, $[\Delta_c(\mu_j) : L_c(\mu_{j+1})] \leq 1$. We will see in the next proposition that this multiplicity is always equal to 1.

Proposition 6.12. *For $\ell = 1, \ldots, n-1$, the homomorphism space* $\mathrm{Hom}_{H_c}(\Delta_c(\mu_\ell), \Delta_c(\mu_{\ell-1}))$ *is 1-dimensional. Up to a nonzero scalar, the unique homomorphism $\phi_\ell : \Delta_c(\mu_\ell) \to \Delta_c(\mu_{\ell-1})$ is determined by $\phi_\ell(v(0, T_{i_1<\cdots<i_\ell})) = v(me_{i_\ell}, T_{i_1<\cdots<i_{\ell-1}})$.*

Proof. That $\mathrm{Hom}_{H_c}(\Delta_c(\mu_\ell), \Delta_c(\mu_{\ell-1}))$ is at most 1-dimensional follows because weight-spaces are 1-dimensional and $\Delta_c(\mu_\ell)$ is cyclic. Now let $\phi_\ell : V_{\mu_\ell} \to \Delta_c(\mu_{\ell-1})$ be the $\mathbb{C}$-linear homomorphism given in the statement of the proposition, where we identify V_{μ_ℓ} with the span of $\{v(0,T) : T \in \mathrm{SYT}(\mu_\ell)\}$. Fix a tableau $T = T_{i_1<\cdots<i_\ell}$ on μ_ℓ and let $T' := T_{i_1<\cdots<i_{\ell-1}}$, a tableau on $\mu_{\ell-1}$. First, we will check that

$$\mathtt{w}(0, T_{i_1<\cdots<i_\ell}) = \mathtt{w}(me_{i_\ell}, T_{i_1<\cdots<i_{\ell-1}})$$

Let us denote the left-hand side of this equation by $\mathtt{w}$, and the right-hand side by $\mathtt{w}'$. We have that $\mathtt{w}_{i_j} = cj$ for $j = 1, \ldots, \ell$. Let us compute $\mathtt{w}'_{i_j}$. First, note that $g_{me_{i_\ell}} = (i_\ell, i_\ell{+}1, \ldots, n)^{-1}$. So we have that $\mathtt{w}'_{i_j} = 0 - (-j)c = jc$ if $j < \ell$. For $j = \ell$, we have that $\mathtt{w}'_{i_\ell} = m - \mathrm{ct}_{T'}(n)c = m - (n-\ell)c = \ell c$.

Now set $i_0 := 0$ and $i_{\ell+1} := n+1$. Assume $i \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_\ell\}$, so there exists a unique $j = 0, \ldots, \ell$ such that $i_j < i < i_{j+1}$. Note that it follows that $\mathtt{w}_i = -(i-j-1)c$. If $j < \ell$, we have that $\mathtt{w}'_i = 0 - \mathrm{ct}_{T'}(i)c = -(i-j-1)c$. If $j = \ell$, then $\mathtt{w}'_i = 0 - \mathrm{ct}_{T'}(i-1)c = -((i-1)-\ell)c = -(i-\ell-1)c$. So we have that $\mathtt{w} = \mathtt{w}'$ as desired. Using this, we will show that ϕ_ℓ intertwines the $\mathcal{S}_n$-action. Obviously, $g_0 = \mathrm{id} \in \mathcal{S}_n$, which we will use below without further mention. Let $j \in \{1, \ldots, n-1\}$. We have several cases.

Case 1. $j \in \{i_1, \ldots, i_\ell\}, j+1 \notin \{i_1, \ldots, i_\ell\}$. Then $s_j(T)$ is a standard Young tableau, and

$$s_j v(0,T) = v(0, s_jT) + \frac{c}{\mathtt{w}_j - \mathtt{w}_{j+1}} v(0,T).$$

Now we have to compute $s_j v(me_{i_\ell}, T')$. We have two subcases.
Case 1.1. $j \neq i_\ell$. Since $j+1 \neq i_\ell$, we have that the j and $(j+1)$-st entries of me_{i_ℓ} are both 0, and $s_j T'$ is a standard Young tableau. Thus,

$$s_j v(me_{i_\ell}, T') = v(me_{i_\ell}, s_j T') + \frac{c}{\mathtt{w}'_j - \mathtt{w}'_{j+1}} v(me_{i_\ell}, T').$$

Now $s_j \phi_\ell(v(0,T)) = \phi_\ell(s_j v(0,T))$ follows because $\mathtt{w} = \mathtt{w}'$, as we have checked.
Case 1.2. $j = i_\ell$. Here we have

$$s_j v(me_{i_\ell}, T') = s_j v(me_j, T') = v(me_{j+1}, T') + \frac{c}{\mathtt{w}'_j - \mathtt{w}'_{j+1}} v(me_j, T').$$

Note that $\phi(v(0, s_j T)) = v(me_{j+1}, T')$, so we again have $s_j \phi(v(0,T)) = \phi(s_j v(0,T))$.
Case 2. $j \notin \{i_1, \dots, i_\ell\}, j+1 \in \{i_1, \dots, i_\ell\}$. This is similar to Case 1.
Case 3. $j, j+1 \in \{i_1, \dots, i_\ell\}$. Here we have that $s_j v(0,T) = -v(0,T)$. So we have to compare $-v(e_{i_\ell}, T')$ with $s_j v(e_{i_\ell}, T')$. Again we have two subcases.
Case 3.1. $j+1 \neq i_\ell$. Here it is very easy to see that $s_j v(e_{i_\ell}, T') = -v(e_{i_\ell}, T')$, as wanted.
Case 3.2. $j+1 = i_\ell$. Note that here we have

$$\mathtt{w}_{j+1} - \mathtt{w}_j = \mathtt{w}'_{j+1} - \mathtt{w}'_j = -c$$

and therefore

$$s_j v(me_{i_\ell}, T') = \frac{(\mathtt{w}_{j+1} - \mathtt{w}_j - c)(\mathtt{w}_{j+1} - \mathtt{w}_j + c)}{(\mathtt{w}_j - \mathtt{w}_{j+1})^2} v(s_j(me_{i_\ell}), T') + \frac{c}{\mathtt{w}_{j+1} - \mathtt{w}_j} v(me_{i_\ell}, T') = -v(me_{i_\ell}, T')$$

as wanted.
Case 4. $j, j+1 \notin \{i_1, \dots, i_\ell\}$. This is similar to Case 3.

In any case, we have $\phi_\ell(s_j v(0,T)) = s_j \phi_\ell(v(0,T))$, so $\phi_\ell : V_{\mu_\ell} \to \Delta_c(\mu_{\ell-1})$ intertwines the $\mathcal{S}_n$-action. To show that ϕ_ℓ does define a morphism of H_c-modules it therefore suffices to check that $y_1, \dots, y_n$ act by 0 on $\phi_\ell(V_{\mu_\ell})$. Note that $\lambda = (12\cdots n)^{-1} y_1$ acts by 0 on $\phi_\ell(V_{\mu_\ell})$. Now,

$$y_i \phi_\ell(v(0,T)) = s_i \cdots s_{n-1} \lambda s_1 \cdots s_{i-1} \phi_\ell(v(0,T)) = s_{n-1} \cdots s_{n-1} \lambda \phi_\ell(s_1 \cdots s_{i-1} v(0,T)) = 0$$

where the last equality follows because $\lambda \phi_\ell(V_{\mu_\ell}) = 0$. This finishes the proof. □

Corollary 6.13. *For any $\ell = 0, \dots, n-1$, the standard module $\Delta_c(\mu_\ell)$ has a unique composition series $0 \subsetneq I_\ell \subsetneq \Delta_c(\mu_\ell)$. Moreover, $I_\ell \cong L_c(\mu_{\ell+1})$ and $\Delta_c(\mu_\ell)/I_\ell = L_c(\mu_\ell)$.*

Proof. From Remark 6.11 and Proposition 6.12 it follows that

$$[\Delta_c(\mu_\ell) : L_c(\mu)] = \begin{cases} 1, & \mu = \mu_\ell, \mu_{\ell+1} \\ 0 & \text{else.} \end{cases}$$

moreover, $L_c(\mu_{\ell+1})$ cannot appear as a quotient of $\Delta_c(\mu_\ell)$. So defining $I_\ell := \phi_{\ell+1}(\Delta_c(\mu_{\ell+1}))$ the result follows. □

Corollary 6.14. *We have* $\operatorname{im}(\phi_{\ell+1}) = \ker(\phi_\ell)$*. In other words, the complex* $\Delta_c(\mu_{\ell+1}) \xrightarrow{\phi_{\ell+1}} \Delta_c(\mu_\ell)$ *is exact outside of degree* 0 *and coincides with* (28)*.*

Proof. It is enough to see that $\ker(\phi_\ell) = I_\ell$. For this, it is enough to see that ϕ_ℓ is neither zero nor injective. That it is nonzero is obvious. Thanks to Lemma 6.10 we must have $\phi_{\ell+1} \circ \phi_\ell = 0$. So $\phi_\ell(I_\ell) = 0$ and ϕ_ℓ is not injective. □

6.4. **Alternate proof of existence and description of ϕ_ℓ.** In this subsection we present an alternative construction of the maps ϕ_ℓ using the results in Section 5.

Lemma 6.15. *Let $\mathcal{D} = \tau(\sigma_{n-1}\cdots\sigma_2\sigma_1\tau)^{m-1}$. Then for $1 < i < n$ $\sigma_i\mathcal{D} = \mathcal{D}\sigma_{i-1}$ and $u_i\mathcal{D} = \mathcal{D}u_{i-1}$, but $u_1\mathcal{D} = \mathcal{D}(u_n + mt)$.*

The proof is an easy computation we leave to the reader. Recall for $\mathbf{e}_1 = (1, 0, \ldots, 0)$ that $H(\mathbf{e_1}, \mathbf{u}) = H_1(\mathbf{u}) \otimes H_{n-1}(\mathbf{u})$.

Lemma 6.16. *Let $U \subseteq V_{(n-\ell+1,1^{\ell-1})}$ be the $\mathcal{S}_{n-1}\times\mathcal{S}_1$-submodule spanned by all v_T where $T = T_{i_1<i_2<\cdots<i_{\ell-1}}$ with $i_{\ell-1}\neq n$. In particular $U \simeq V_{(n-\ell,1^{\ell-1})}$ as an $\mathcal{S}_{n-1}$-module.*

1. *Let t, c be such that $\Delta_{t,c}(n-\ell+1,1^{\ell-1})$ is $\mathcal{A}$-semisimple. Then $H(\mathbf{e_1},\mathbf{u})\mathcal{D}U \subseteq \Delta_{t,c}(n-\ell+1,1^{\ell-1})$ is an $H(\mathbf{e_1},\mathbf{u})$-submodule which is isomorphic to $V_{(n-\ell,1^{\ell-1})}$ as an $H_{n-1}(\mathbf{u})$-module on which u_1 acts identically as $c(\ell-n)+mt$.*
2. *In the case $t = 1, c = \frac{m}{n}, \gcd(m,n) = 1$, then u_1 acts as $c\ell$ and $H_n(\mathbf{u})\mathcal{D}U \simeq \mathrm{Ind}_{H(\mathbf{e_1},\mathbf{u})}^{H_n(\mathbf{u})}(c\ell)\boxtimes V_{(n-\ell,1^{\ell-1})}$.*

Proof. The first statement follows from Lemma 6.15. Since $\Delta_{t,c}(n-\ell+1,1^{\ell-1})$ is $\mathcal{A}$-semisimple the σ_i act triangularly with respect to the s_i. So the action of the σ_i on the inflation via ev_0 of an $\mathcal{S}_{n-1}$-module completely determines the $\mathcal{S}_{n-1}$ structure. Recall that via ev_0 the u_i will act as Jucys-Murphy operators.
For the second statement, we use Lemma 6.15 to determine the action of u_1. Because $F_{\mathcal{X}}^{-1}(\omega_{m\mathbf{e}_n}) = \tau(s_{n-1}\cdots s_2s_1\tau)^{m-1}$ is a minimal length double coset representative we get the second statement. □

Lemma 6.17. $\mathrm{Ind}_{H(\mathbf{e_1},\mathbf{u})}^{H_n(\mathbf{u})}(c\ell)\boxtimes V_{(n-\ell,1^{\ell-1})}$ *has an $H_n(\mathbf{u})$-submodule isomorphic to $V_{(n-\ell,1^{\ell})}$ (inflated along* ev_0*). In particular u_1 is identically zero on this submodule.*

The proof is a standard result for the degenerate affine Hecke algebra.

Lemma 6.18. *Let $M = \Delta_{t,c}(V)$ be an $H_{t,c}$-module which has a $H_n(\mathbf{u})$-submodule N on which u_1 acts identically as zero. Then for $1 \leq i \leq n$ the y_i act as zero on N.*

Proof. Recall $u_1 = x_1y_1$. Since $\Delta_{t,c}(V)$ is free as a $\mathbb{C}[x_1,\ldots,x_n]$-module, x_1 has no torsion so in particular y_1 is zero on N. As N is $\mathcal{S}_n$-invariant and $y_i = (1,2,\ldots,i)y_1(i,\ldots,2,1)$, all the y_i must act as zero. □

As a consequence we get that $\Delta(n-\ell+1,1^{\ell-1})$ has a $\mathcal{S}_n$-submodule isomorphic to $V_{(n-\ell,1^\ell)}$ on which all y_i vanish. Thus Frobenius Reciprocity gives us a nonzero $H_{t,c}$ homomorphism

$$\Delta_c(n-\ell,1^\ell) \xrightarrow{\phi_\ell} \Delta_c(n-\ell+1,1^{\ell-1}).$$

This yields an alternate proof of Proposition 6.12.

More concretely, we can normalize the basis $\{v_T \mid T \in \mathrm{SYT}(\mu_\ell)\}$ of $V_{(n-\ell+1,1^{\ell-1})}$ so that we fix $v_{\mathbf{T}}$ for $\mathbf{T} = T_{2<3<\cdots<\ell+1}$ and take the other basis vectors to be $\sigma_\omega v_{\mathbf{T}} =: v_{\omega\cdot\mathbf{T}}$ for $\mathrm{id} \leq \omega \leq [1, n-\ell+1, \ldots, n-1, n, 2, 3, \ldots, n-\ell]$ in weak Bruhat order. (Recall as the σ_i satisfy the braid relations, σ_ω makes sense.) Then ϕ_ℓ is determined by

$$v_{\mathbf{T}} \mapsto \sigma_\ell\cdots\sigma_2\sigma_1\mathcal{D}v_{T_{2<3<\cdots<\ell}}$$

where all tableau on the left of $\mapsto$ have shape $(n-\ell,1^\ell)$ but all tableau on the right have shape $(n-\ell+1,1^{\ell-1})$ More generally (noting $i_{\ell-1}\geq\ell$) we have

$$v_{T_{i_1<i_2<\cdots<i_\ell}} \mapsto \sigma_{i_{\ell-1}}\cdots\sigma_2\sigma_1\mathcal{D}v_{T_{i_1<i_2<\cdots<i_{\ell-1}}}.$$

In particular when $i_\ell = n$ we get

$$v_{T_{i_1<i_2<\cdots<n}} \mapsto (\sigma_{n-1}\cdots\sigma_2\sigma_1\tau)^m v_{T_{i_1<i_2<\cdots<i_{\ell-1}}}.$$

Recall $(s_{n-1}\cdots s_2 s_1 \tau)^m = \mathtt{t}_{m\mathbf{e}_n}$. One can easily check the above vectors' u-weights are preserved by ϕ_ℓ. It is only slightly more work to check with the above assignment that ϕ_ℓ intertwines the σ_i acting on the $v_T, T \in \mathrm{SYT}(n-\ell, 1^\ell)$.

6.5. **Weight basis of simples.** We continue assuming $c = m/n$ with m and n coprime positive integers. In this section, we generalize Proposition 4.20 and we describe weights belonging to the maximal proper submodule of every standard module $\Delta_c(\mu)$. Thanks to Corollary 6.9, this question is only interesting when $\mu = \mu_\ell$ is a hook partition. Moreover, since $\Delta_c(\mu_{n-1})$ is simple, we may and will assume throughout this section that $0 \leq \ell < n-1$.

Lemma 6.19. *Let $(\mathbf{a}, T) \in \mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\mu_\ell)$. Then, there exists $(\mathbf{b}, T') \in \mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\mu_{\ell+1})$ such that $\mathtt{w}(\mathbf{a}, T) = \mathtt{w}(\mathbf{b}, T')$ if and only if either*

- $a_{g_{\mathbf{a}}^{-1}(n)} - m > a_{g_{\mathbf{a}}^{-1}(i_\ell)}$ *or*
- $a_{g_{\mathbf{a}}^{-1}(n)} - m = a_{g_{\mathbf{a}}^{-1}(i_\ell)}$ *and* $g_{\mathbf{a}}^{-1}(n) > g_{\mathbf{a}}^{-1}(i_\ell)$

where i_ℓ is the number labeling the box with smallest content of μ_ℓ on the tableau T. Moreover, if this is the case, then $(\mathbf{b}, T')$ is uniquely determined.

Proof. Following the notation of Section 6.3, let us denote $T = T_{i_1 < \cdots < i_\ell}$. We will, first, see that there is a unique $\mathbf{b} \in \mathbb{Z}^n$ (possibly with negative entries) and T' a tableau on $\mu_{\ell+1}$ (possibly non-standard) such that $\mathtt{w}(\mathbf{a}, T) = \mathtt{w}(\mathbf{b}, T')$. Indeed, if such pair $(\mathbf{b}, T')$ exists we must have

$$n(a_i - b_i) = m(\mathrm{ct}_T(g_{\mathbf{a}}(i)) - \mathrm{ct}_{T'}(g_{\mathbf{b}}(i)))$$

for every $i = 1, \ldots, n$. Since m and n are coprime and $\mu_\ell, \mu_{\ell+1}$ are adjacent hooks, we must have that either

(i) $a_i = b_i$ and $T_{g_{\mathbf{a}}(i)} = T'_{g_{\mathbf{b}}(i)}$ (meaning that this box is in $\mu_\ell \cap \mu_{\ell+1}$) or
(ii) $a_i - b_i = m$, $T_{g_{\mathbf{a}}(i)}$ is the box of highest content in μ_ℓ, and $T'_{g_{\mathbf{b}}(i)}$ is the box of lowest content in $\mu_{\ell+1}$.

Let $k \in \{1, \ldots, n\}$ be such that T_k is the box of highest content of μ_ℓ. From (i) and (ii), the vector b is uniquely specified: $b_i = a_i$ if $i \neq g_{\mathbf{a}}^{-1}(k)$, and $b_{g_{\mathbf{a}}^{-1}(k)} = a_{g_{\mathbf{a}}^{-1}(k)} - m$. Moreover, the tableau T' is also uniquely specified: $T'_{g_{\mathbf{b}}(i)} = T_{g_{\mathbf{a}}(i)}$ if $i \neq g_{\mathbf{a}}^{-1}(k)$, and $T'_{g_{\mathbf{b}}(g_{\mathbf{a}}^{-1}(k))}$ is the box with lowest content in $\mu_{\ell+1}$. Our job now is to check that all coordinates of $\mathbf{b}$ are non-negative and T' is standard if and only if the conditions of the lemma are satisfied. Clearly, $\mathbf{b}$ is non-negative if and only if $a_{g_{\mathbf{a}}^{-1}(k)} \geq m$, so we will focus on the condition that T' is standard.

Let us first verify that T' is standard on $\mu_\ell \cap \mu_{\ell+1}$. Indeed, consider two consecutive boxes in $\mu_\ell \cap \mu_{\ell+1}$ and let $j_1 < j_2$ be their labels under T. Note that $j_1, j_2 \neq k$. By definition of $g_{\mathbf{a}}$ and $\mathbf{b}$ we have

$$b_{g_{\mathbf{a}}^{-1}(j_1)} = a_{g_{\mathbf{a}}^{-1}(j_1)} \leq a_{g_{\mathbf{a}}^{-1}(j_2)} = b_{g_{\mathbf{a}}^{-1}(j_2)}$$

and, if we have an equality, $g_{\mathbf{a}}^{-1}(j_1) < g_{\mathbf{a}}^{-1}(j_2)$. From the definition of $g_{\mathbf{b}}$ it follows that $g_{\mathbf{b}} g_{\mathbf{a}}^{-1}(j_1) < g_{\mathbf{b}} g_{\mathbf{a}}^{-1}(j_2)$, as wanted.

So T' is standard if and only if $g_{\mathbf{b}} g_{\mathbf{a}}^{-1}(i_\ell) < g_{\mathbf{b}} g_{\mathbf{a}}^{-1}(k)$. If $i_\ell = n$, we have $b_{g_{\mathbf{a}}^{-1}(i_\ell)} = a_{g_{\mathbf{a}}^{-1}(n)} > a_{g_{\mathbf{a}}^{-1}(k)} - m = b_{g_{\mathbf{a}}^{-1}(k)}$ and therefore $g_{\mathbf{b}} g_{\mathbf{a}}^{-1}(i_\ell) > g_{\mathbf{b}} g_{\mathbf{a}}^{-1}(k)$. Thus, we must have $k = n$ and $i_\ell < n$. It follows now that the tableau T' is standard if and only if either $b_{g_{\mathbf{a}}^{-1}(n)} > b_{g_{\mathbf{a}}^{-1}(i_\ell)}$ or $b_{g_{\mathbf{a}}^{-1}(n)} = b_{g_{\mathbf{a}}^{-1}(i_\ell)}$ and $g_{\mathbf{a}}^{-1}(n) > g_{\mathbf{a}}^{-1}(i_\ell)$, which translates precisely into the conditions of the statement of the lemma. Finally, note that $a_{g_{\mathbf{a}}^{-1}(n)} - m \geq a_{g_{\mathbf{a}}^{-1}(i_\ell)}$ automatically implies $a_{g_{\mathbf{a}}^{-1}(n)} - m \geq 0$. We are done. ☐

Remark 6.20. Note that for $\ell = 0$ there is a unique tableau T on μ_0 and $i_\ell = 1$. In this case, $a_{g_{\mathbf{a}}^{-1}(1)} = \min \mathbf{a}$ and $a_{g_{\mathbf{a}}^{-1}(n)} = \max \mathbf{a}$, so we recover the conditions defining the set $\mathcal{S}$ in Section 4.6.

Corollary 6.21. *Let $(\mathbf{a}, T) \in \mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\mu_\ell)$. Then, $v(\mathbf{a}, T) \in I_\ell$ if and only if there exists $(\mathbf{b}, T') \in \mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\mu_{\ell+1})$ such that $\mathtt{w}(\mathbf{a}, T) = \mathtt{w}(\mathbf{b}, T')$.*

Proof. Since $I_\ell = \phi_{\ell+1}(\Delta_c(\mu_{\ell+1}))$, the necessity is clear. For sufficiency, assume that such $(\mathbf{b}, T')$ exists. It is enough to see that $v(\mathbf{b}, T') \notin I_{\ell+1}$ and to see this we can check that there does not exist $(\mathbf{d}, T'') \in \mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\mu_{\ell+2})$ such that $\mathtt{w}(\mathbf{b}, T') = \mathtt{w}(\mathbf{d}, T'')$. So we have to check that $(\mathbf{b}, T')$ does not satisfy the conditions of Lemma 6.19. Let $i_{\ell+1} := g_{\mathbf{b}} g_{\mathbf{a}}^{-1}(n)$, and note that $T'_{i_{\ell+1}}$ is precisely the box with lowest content in $\mu_{\ell+1}$. Now,

$$b_{g_{\mathbf{b}}^{-1}(n)} - m = a_{g_{\mathbf{b}}^{-1}(n)} - m \leq a_{g_{\mathbf{a}}^{-1}(n)} - m = b_{g_{\mathbf{a}}^{-1}(n)} = b_{g_{\mathbf{b}}^{-1}(i_{\ell+1})}$$

If the inequality is strict, we are done. Else, we need to show that $g_{\mathbf{b}}^{-1}(n) < g_{\mathbf{b}}^{-1}(i_{\ell+1}) = g_{\mathbf{a}}^{-1}(n)$. But in this case we have $a_{g_{\mathbf{b}}^{-1}(n)} = a_{g_{\mathbf{a}}^{-1}(n)}$ and the result now follows by the definition of $g_{\mathbf{a}}$. ☐

Corollary 6.22. *Assume $0 \leq \ell < n-1$ and let $(\mathbf{a}, T) \in \mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\mu_\ell)$. Let us denote by i_ℓ the label of the box with smallest content of μ_ℓ under T. Then, $v(\mathbf{a}, T) \in I_\ell$ if and only if either*

- $a_{g_{\mathbf{a}}^{-1}(n)} - m > a_{g_{\mathbf{a}}^{-1}(i_\ell)}$ *or*
- $a_{g_{\mathbf{a}}^{-1}(n)} - m = a_{g_{\mathbf{a}}^{-1}(i_\ell)}$ *and* $g_{\mathbf{a}}^{-1}(n) > g_{\mathbf{a}}^{-1}(i_\ell)$

It follows that $L_c(\mu_\ell) = \Delta_c(\mu_\ell)/I_\ell$ has a weight basis indexed by pairs $(\mathbf{a}, T) \in \mathbb{Z}_{\geq 0} \times \mathrm{SYT}(\mu_\ell)$ such that

- ○ $a_{g_{\mathbf{a}}^{-1}(n)} - m < a_{g_{\mathbf{a}}^{-1}(i_\ell)}$ *or*
- ○ $a_{g_{\mathbf{a}}^{-1}(n)} - m = a_{g_{\mathbf{a}}^{-1}(i_\ell)}$ *and* $g_{\mathbf{a}}^{-1}(n) < g_{\mathbf{a}}^{-1}(i_\ell)$.

Remark 6.23. Note that, if $0 < \ell < n-1$, then $L_c(\mu_\ell) \cong I_{\ell-1}$. Thus, there is a weight-preserving bijection between pairs $(\mathbf{a}, T) \in \mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\mu_{\ell-1})$ satisfying the condition marked with • in Corollary 6.22 and those pairs $(\mathbf{b}, T') \in \mathbb{Z}^n_{\geq 0} \times \mathrm{SYT}(\mu_\ell)$ satisfying the conditions market with ○. This bijection is described in the proof of Lemma 6.19.

7. Singular curves

For coprime $m, n \geq 1$ we consider the plane curve singularity $C = \{x^m = y^n\}$ at the origin. It has an action of $\mathbb{C}^*$ given by $(x, y) \mapsto (s^n x, s^m y)$. This action extends to the local ring of functions on C which is isomorphic to $\mathcal{O}_C = \mathbb{C}[[x, y]]/(x^m - y^n)$. A homogeneous basis in $\mathcal{O}_C$ can be described as follows:

$$\mathcal{O}_C = \mathbb{C}[[x]]\langle 1, \ldots, y^{n-1}\rangle \tag{29}$$

This presentation shows that $\mathcal{O}_C$ is a free module over $\mathbb{C}[[x]]$ of rank n, and the multiplication by y is given by the matrix

$$Y = \begin{pmatrix} 0 & 0 & \cdots & 0 & x^m \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{30}$$

7.1. **Hilbert schemes on singular curves.** By definition, the Hilbert scheme of k points on C is the moduli space of codimension k ideals

$$\mathrm{Hilb}_k(C) = \{I \subset \mathcal{O}_C : I \text{ ideal}, \dim \mathcal{O}_C/I = k\}.$$

The action of $\mathbb{C}^*$ on C extends to an action on $\mathrm{Hilb}_k(C)$ for all k. The fixed points of this action are monomial ideals. In terms of the identification (29) such a monomial ideal is generated over $\mathbb{C}[[x]]$ by monomials of the form $\langle x^{c_1}, yx^{c_2}, \ldots, y^{n-1}x^{c_n}\rangle$. Since it is invariant under the multiplication of y (or the matrix Y above), we get a system of inequalities

$$c_1 \geq c_2 \geq \ldots \geq c_n \geq c_1 - m. \tag{31}$$

y^2	xy^2	x^2y^2	x^3y^2	x^4y^2	x^5y^2	x^6y^2	x^7y^2	
y	xy	x^2y	x^3y	x^4y	x^5y	x^6y	x^7y	
1	x	x^2	x^3	x^4	x^5	x^6	x^7	

FIGURE 2. An ideal on $C = \{x^4 = y^3\}$ generated by x^3y^2 and x^5y. Note that $y \cdot x^3y^2 = x^7$. The codimension of the ideal is $15 = 7+5+3 = c_1+c_2+c_3$ which is also the number of boxes under the staircase.

Note that $\dim \mathcal{O}_C/I = k = \sum c_i$. In the notation of [40], such ideals can be represented by staircases of height n and width at most m. See Figure 2.

Lemma 7.1. *Suppose that $I \subset \mathcal{O}_C$ is spanned over $\mathbb{C}[[x]]$ by $y^{\alpha_1}x^{c_1}, \ldots, y^{\alpha_n}x^{c_n}$ where $\{\alpha_1, \ldots, \alpha_n\} = \{0, \ldots, n-1\}$. Then the following holds:*

(a) *If $\max(c_i) - \min(c_i) > m$ then I is not an ideal in $\mathcal{O}_C$ for any choice of α_i.*
(b) *If $\max(c_i) - \min(c_i) \leq m$ then there exists a unique ideal I of this form.*

Proof. Assume that $I = \mathbb{C}[[x]]\langle y^{\alpha_1}x^{c_1}, \ldots y^{\alpha_n}x^{c_n}\rangle$ is an ideal in $\mathcal{O}_C$. Let $\widetilde{g} \in \mathcal{S}_n$ be the permutation in $\mathcal{S}_n$ which sorts the α_i in increasing order. Then by (31) $c_{\widetilde{g}^{-1}(1)} \geq c_{\widetilde{g}^{-1}(2)} \geq \ldots \geq c_{\widetilde{g}^{-1}(n)}$. Observe that $c_{\widetilde{g}^{-1}(1)} = \max(c_i)$ and $c_{\widetilde{g}^{-1}(n)} = \min(c_i)$. Therefore the condition $c_{\widetilde{g}^{-1}(n)} \geq c_{\widetilde{g}^{-1}(1)} - m$ in (31) is equivalent to $\max(c_i) - \min(c_i) \leq m$.

For part (b), the uniqueness is clear. □

Let $I = \mathbb{C}[[x]]\langle x^{c_1}, yx^{c_2}, \ldots, y^{n-1}x^{c_n}\rangle$ be a monomial ideal in $\mathcal{O}_C$ where c_i satisfy (31). We define a composition $\widetilde{\lambda} = (\widetilde{\lambda_1}, \ldots, \widetilde{\lambda_\ell}), \sum \widetilde{\lambda_i} = n$ by looking at vertical runs of the staircase defined by c_i:

$$c_1 = \ldots = c_{\widetilde{\lambda_1}} > c_{\widetilde{\lambda_1}+1} = \ldots = c_{\widetilde{\lambda_1}+\widetilde{\lambda_2}} > \ldots > c_{n-\widetilde{\lambda_\ell}+1} = \ldots = c_n$$

We also define a composition λ as follows:

$$\lambda = \begin{cases} \widetilde{\lambda} & \text{if } c_1 - c_n < m, \\ (\widetilde{\lambda_1} + \widetilde{\lambda_\ell}, \widetilde{\lambda_2}, \ldots, \widetilde{\lambda_{\ell-1}}) & \text{if } c_1 - c_n = m. \end{cases}$$

Lemma 7.2. *The operator Y acting on the space I/xI has Jordan blocks of sizes λ_i.*

Proof. The space I/xI is spanned (over $\mathbb{C}$) by $\langle v_1 = x^{c_1}, v_2 = yx^{c_2}, \ldots, v_n = y^{n-1}x^{c_n}\rangle$. Clearly, if $c_1 = c_2$ then $Y(v_1) = v_2$, otherwise $Y(v_1)$ vanishes in I/xI . Similarly, we see chains of vectors

$$v_1 \xrightarrow{Y} \ldots \xrightarrow{Y} v_{\widetilde{\lambda_1}} \xrightarrow{Y} 0, \qquad v_{\widetilde{\lambda_1}+1} \xrightarrow{Y} \ldots \xrightarrow{Y} v_{\widetilde{\lambda_1}+\widetilde{\lambda_2}} \xrightarrow{Y} 0, \qquad \ldots, \qquad v_{n-\widetilde{\lambda_\ell}+1} \xrightarrow{Y} \ldots \xrightarrow{Y} v_n.$$

Finally, $Y(v_n) = x^{c_n+m}$, so if $c_n+m > c_1$ then $Y(v_n) = 0$, otherwise $c_n+m = c_1$ and $Y(v_n) = v_1$. □

7.2. **Parabolic Hilbert schemes on singular curves.** Observe that for any ideal I we have $\dim I/xI = n$. So we can define the parabolic Hilbert scheme as the moduli space of flags of ideals

$$\mathrm{PHilb}_{k,n+k}(C) := \{\mathcal{O}_C \supset I_k \supset I_{k+1} \supset \cdots \supset I_{k+n} = xI_k : I_s \text{ ideal}, \ \dim \mathcal{O}_C/I_s = s\}$$

and we define

$$\mathrm{PHilb}^x(C) := \bigsqcup_{k \geq 0} \mathrm{PHilb}_{k,n+k}(C).$$

Again, we would like to describe the fixed points of the $\mathbb{C}^*$ action on this variety explicitly. These are described by flags where all I_s are monomial ideals. As above, for $i = 1, \ldots, n$ we

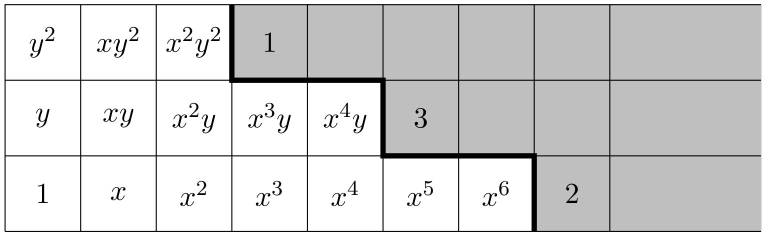


FIGURE 3. A flag of monomial ideals in $\mathrm{PHilb}_{15,15+3}(x^4 = y^3)$:
$I_{15} = \langle x^3y^2, x^5y\rangle, I_{16} = \langle x^4y^2, x^5y, x^7\rangle, I_{17} = \langle x^4y^2, x^5y\rangle$.
Here $y^{\alpha_1}x^{c_1} = x^3y^2,\ y^{\alpha_2}x^{c_2} = x^7,\ y^{\alpha_3}x^{c_3} = x^5y$.

can assume that the one-dimensional space I_{k+i-1}/I_{k+i} is spanned by the monomial $y^{\alpha_i}x^{c_i}$ where $0 \le \alpha_i \le n-1$. In particular,

$$I_k = \mathbb{C}[[x]]\langle y^{\alpha_1}x^{c_1}, \ldots y^{\alpha_n}x^{c_n}\rangle.$$

Note that if $c_i = c_j$ and $i < j$ then $\alpha_i < \alpha_j$, so by Lemma 7.1 α_i are uniquely determined by c_i.

Furthermore, we can extend this construction by defining $I_{i+n} = xI_i$ for all integers $i \ge k$. Note that it follows that $\alpha_{i+n} = \alpha_i$ and $c_{i+n} = c_i + 1$ for $i \ge 1$.

Lemma 7.3. *The vector* $\mathbf{c} = (c_1, \ldots, c_n)$ *determines a fixed point in* $\mathrm{PHilb}_{k,n+k}$ *if and only if either of the two equivalent conditions hold:*

(a) *For all* $t > 0$ *one has*

$$\max_{i=t}^{t+n-1}(c_i) - \min_{i=t}^{t+n-1}(c_i) \le m \tag{32}$$

(b) *One has* $\max_{i=1}^{n}(c_i) - \min_{i=1}^{n}(c_i) \le m$ *and whenever* $c_j + m = c_i$ *then* $j < i$.

Proof. By construction, for all $t > 0$ the subspace I_{k-1+t} is spanned over $\mathbb{C}[[x]]$ by the monomials $\langle y^{\alpha_t}x^{c_t}, \ldots, y^{\alpha_{t+n-1}}x^{c_{t+n-1}}\rangle$, so by Lemma 7.1 it is an ideal if and only if (32) holds. This proves (a).

Now let us prove that (a) and (b) are equivalent. Indeed, the left hand side of (32) is n-periodic, so it is sufficient to consider $t \le n$. Assume that $\max_{i=1}^{n}(c_i) - \min_{i=1}^{n}(c_i) \le m$, then (32) does not hold if and only if there exists $i < t$ and $j \ge t$ such that $c_i = c_j + m$ and $c_{i+n} = c_i + 1 = c_j + m + 1$. ☐

Remark 7.4. The proof of Lemma 7.3 is very similar to the proof of Lemma 2.15. Indeed, this is not a coincidence: let us parametrize the curve C by $(x, y) = (z^n, z^m)$, then any monomial in x and y corresponds to a monomial in z. A monomial ideal in $\mathcal{O}_C$ then corresponds to an (m, n)-invariant subset in $\mathbb{Z}_{\ge 0}$, and a flag of monomial ideals corresponds to a flag of (m, n)-invariant subsets. By Proposition 2.13 such flag determines an m-stable affine permutation. We conclude that fixed points on parabolic Hilbert scheme are in bijection with m-stable affine permutations ω such that $\omega p_m \in \mathtt{L}^+_{\min}(n)$.

We define the line bundles $\mathcal{L}_i$, $1 \le i \le n$ on the parabolic flag Hilbert scheme as follows. The fiber of $\mathcal{L}_i$ over the flag $I_k \supset I_{k+1} \supset \cdots \supset I_{k+n} = xI_k$ is I_{k+i-1}/I_{k+i}. Then we have the following:

Lemma 7.5. *There is a bijection between the eigenbasis* $v_{\mathbf{a}}$ *in* $L_{m/n}(\mathrm{triv})$ *(defined in Corollary 4.23) and the set of* $\mathbb{C}^*$ *fixed points in* $\mathrm{PHilb}^x(C)$*. Under this bijection, the weight of* $\mathcal{L}_i$ *at a fixed point corresponds to the eigenvalue* $n\mathtt{w}_{n+1-i}(\mathbf{a}) + m(n-1)$ *of the operator* $nu_{n+1-i} + m(n-1)$ *on* $v_{\mathbf{a}}$.

Proof. Recall that by Corollary 4.23 the basis $v_{\mathbf{a}}$ in $L_{m/n}(\mathrm{triv})$ is parametrized by sequences of nonnegative integers $\mathbf{a} = (a_1, \ldots, a_n)$ such that $a_i - a_j \le m$ for every i, j, and if $a_i - a_j = m$ then $j > i$. The eigenvalues of u_i are given by $\mathtt{w}_i = a_i - (g_{\mathbf{a}}(i) - 1)\frac{m}{n}$ where $g_{\mathbf{a}}$ is the permutation which sorts $\mathbf{a}$ in non-decreasing order (here we substituted $c = \frac{m}{n}$).

On the other hand, the fixed points in $\mathrm{PHilb}_{k,n+k}$ are determined by sequences of monomials $(y^{\alpha_i}x^{c_i})$ where $\max_{i=1}^n(c_i) - \min_{i=1}^n(c_i) \leq m$ and whenever $c_j + m = c_i$ then $j < i$. We remark that since $y^{\alpha_i}x^{c_i}$ spans the quotient I_{k+i-1}/I_{k+i} it follows that when $c_i = c_j$ with $i < j$ we have $\alpha_i < \alpha_j$. Clearly, the assignment $c_i = a_{n+1-i}$ is a bijection intertwining the restrictions on a_i and on c_i. Note $k = \sum c_i = \sum a_i = ||\mathbf{a}||$.

Finally, the line bundle $\mathcal{L}_i$ has the equivariant weight $m\alpha_i + nc_i$. We have $\alpha_i = \widetilde{g}(i) - 1$, where $\widetilde{g}$ is the permutation defined in the proof of Lemma 7.1 which sorts the α_i in increasing order. Clearly, $\widetilde{g}(i) = n + 1 - g_{\mathbf{a}}(n + 1 - i)$, hence

$$m\alpha_i + nc_i = m(n + 1 - g_{\mathbf{a}}(n + 1 - i) - 1) + na_{n+1-i} =$$
$$m(n - 1) + m(1 - g_{\mathbf{a}}(n + 1 - i)) + na_{n+1-i} = n\mathtt{w}_{n+1-i} + m(n - 1).$$

□

Example 7.6. For $\mathbf{a} = (0, \dots, 0)$ we get $\mathtt{w}_i(\mathbf{a}) = -(i - 1)\frac{m}{n}$ while the corresponding fixed point in $\mathrm{PHilb}_{0,n}$ corresponds to the flag $\mathcal{O}_C \supseteq y\mathcal{O}_C \supseteq \cdots \supseteq y^{n-1}\mathcal{O}_C$. The section of $\mathcal{L}_i$ is given by monomial y^{i-1} which has weight $m(i - 1)$. Now

$$n\mathtt{w}_{n+1-i}(\mathbf{a}) + m(n - 1) = -m(n + 1 - i - 1) + m(n - 1) = m(i - 1).$$

7.3. **Geometric operators.** There is a natural projection $\pi : \mathrm{PHilb}_{k,n+k} \to \mathrm{Hilb}_k$ which sends a flag $I_k \supset I_{k+1} \supset \cdots \supset I_{k+n} = xI_k$ to I_k. The fibers of this projection are just the classical Springer fibers consisting of complete flags in I_k/xI_k invariant under the action of Y. In particular, Lemma 7.2 immediately implies the following.

Lemma 7.7. *Given an ideal $I = \mathbb{C}[[x]]\langle y^{\alpha_1}x^{c_1}, \dots y^{\alpha_n}x^{c_n}\rangle$ in Hilb_k there are $\binom{n}{\lambda_1,\dots,\lambda_\ell}$ fixed points in $\mathrm{PHilb}_{k,n+k}$ projecting to I. There is a Springer action of $\mathcal{S}_n$ on these fixed points, in which they span the induced representation from $\mathcal{S}_{\lambda_1} \times \cdots \times \mathcal{S}_{\lambda_\ell}$ to $\mathcal{S}_n$.*

Here λ is determined by c_i as in Lemma 7.2, and ℓ is the length of λ.

In what follows we will need a more explicit description of this action in the fixed point basis. For this, we can also give a more explicit geometric description. Let $\mathrm{PHilb}_{k,n+k}^{(i)}$ denote the moduli space of flags of ideals

$$\mathrm{PHilb}_{k,n+k}^{(i)} = \{I_k \supset I_{k+1} \supset \cdots \supset I_{k+i} \supset I_{k+i+2} \supset \cdots \supset I_{k+n} = xI_k\}.$$

There is a natural projection $\pi_i : \mathrm{PHilb}_{k,n+k} \to \mathrm{PHilb}_{k,n+k}^{(i)}$. Let $Z_i \subset \mathrm{PHilb}_{k,n+k}^{(i)}$ denote the locus where $yI_{k+i} \subset I_{k+i+2}$. The key properties of π_i are captured by the following lemma:

Lemma 7.8. *(a) The map π_i is an isomorphism outside Z_i and a $\mathbb{P}^1$-fibration over Z_i.*

(b) The preimage $\pi_i^{-1}(Z_i)$ is cut out by a section of the line bundle $\mathcal{L}_i^{-1}\mathcal{L}_{i+1}$.

(c) A fixed point corresponding to $v_{\mathbf{a}}$ is not in $\pi_i^{-1}(Z_i)$ if and only if $\mathtt{w}_{n+1-i}(\mathbf{a}) = \mathtt{w}_{n-i}(\mathbf{a}) - \frac{m}{n}$.

(d) The tangent bundle to the fiber of π_i over Z_i is isomorphic to $\mathcal{L}_i\mathcal{L}_{i+1}^{-1}$.

Proof. (a) The fiber of π_i naturally corresponds to the space of y-invariant lines in two-dimensional space I_{k+i}/I_{k+i+2}. Since y is nilpotent on I_{k+i}/I_{k+i+2}, it is either identically zero and every line is y-invariant, or it is a Jordan block and has unique y-invariant line.

(b) A flag $I_k \supset I_{k+1} \supset \cdots \supset I_{k+n} = xI_k$ is in $\pi_i^{-1}(Z_i)$ if and only if $yI_{k+i} \subset I_{k+i+2}$. Since $yI_{k+i} \subset I_{k+i+1}$, we have a map $s_y : \mathcal{L}_i \to \mathcal{L}_{i+1}$ which is equivalent to a section of $\mathcal{L}_i^{-1}\mathcal{L}_{i+1}$.

(c) A fixed point is not in $\pi_i^{-1}(Z_i)$ if and only if the weight of $\mathcal{L}_{i+1}$ differs from the weight of $\mathcal{L}_i$ by m. By Lemma 7.5 we get

$$n\mathtt{w}_{n-i} + m(n - 1) = n\mathtt{w}_{n+1-i} + m(n - 1) + m, \;\; \mathtt{w}_{n-i} = \mathtt{w}_{n+1-i} + \frac{m}{n}.$$

(d) Recall that the tangent space to $\mathbb{P}^1 = \mathbb{P}(V)$ at a line ℓ is canonically isomorphic to $\mathrm{Hom}(\ell, V/\ell)$. In our case $\ell \simeq I_{k+i+1}/I_{k+i+2} = \mathcal{L}_{i+1}$ and $V/\ell \simeq I_{k+i}/I_{k+i+1} = \mathcal{L}_i$. So the tangent bundle to the fiber is isomorphic to $\mathrm{Hom}(\mathcal{L}_{i+1}, \mathcal{L}_i) \simeq \mathcal{L}_i\mathcal{L}_{i+1}^{-1}$. □

We can use the maps π_i to define the Springer action of $\mathcal{S}_n$ on the homology of $\mathrm{PHilb}_{k,n+k}$. Let $\gamma_i : \pi_i^{-1}(Z_i) \hookrightarrow \mathrm{PHilb}_{k,n+k}$ denote the natural inclusion map. By Lemma 7.8 we have well-defined Gysin maps $\gamma_i^* : H_*(\mathrm{PHilb}_{k,n+k}) \to H_*(\pi_i^{-1}(Z_i))$ and $\pi_i^* : H_*(Z_i) \to H_*(\pi_i^{-1}(Z_i))$. Consider the composition

$$(33) \qquad B_i : H_*(\mathrm{PHilb}_{k,n+k}) \xrightarrow{\gamma_i^*} H_*(\pi_i^{-1}(Z_i)) \xrightarrow{\pi_{i*}} H_*(Z_i) \xrightarrow{\pi_i^*} H_*(\pi_i^{-1}(Z_i)) \xrightarrow{\gamma_{i*}} H_*(\mathrm{PHilb}_{k,n+k}).$$

By Lemma 7.5 we can identify the fixed point basis in the equivariant cohomology of $\sqcup_k \mathrm{PHilb}_{k,n+k}$ with the basis $v_{\mathbf{a}}$ in the representation $L_{m/n} = L_{m/n}(\mathrm{triv})$. In fact, it is more natural to identify it with the renormalized basis $\widetilde{v}_{\mathbf{a}}$.

Lemma 7.9. *The action of B_i in the equivariant cohomology of $\sqcup_k \mathrm{PHilb}_{k,n+k}$ agrees with the action of $1 - s_{n-i}$ on $L_{m/n}$, if we identify the fixed point basis in the former with $\widetilde{v_{\mathbf{a}}}$.*

Proof. We just need to compute the matrix elements of all the operators involved in the definition of B_i. By Lemma 7.5 (b) the subvariety $\pi_i^{-1}(Z_i)$ is cut out by a section of $\mathcal{L}_i^{-1}\mathcal{L}_{i+1}$ corresponding to the map $s_y : \mathcal{L}_i \to \mathcal{L}_{i+1}$. This map has weight m, and so the Gysin map γ_i^* correspond to the multiplication by $c_1(\mathcal{L}_{i+1}) - c_1(\mathcal{L}_i) - m$ which at a fixed point corresponds to the multiplication by $(n\mathtt{w}_{n-i} - n\mathtt{w}_{n+1-i} - m)$. Note that by Lemma 7.5 (c) this annihilates the classes of all fixed points outside $\pi_i^{-1}(Z_i)$.

The map π_{i*} just maps the class of the fixed point in $\mathrm{PHilb}_{k,n+k}$ to the class of the corresponding class in $\mathrm{PHilb}_{k,n+k}^{(i)}$. The map π_i^*, however, amounts to dividing by the cotangent weight of the fiber computed in Lemma 7.8 (d).

By combining these factors, it is now easy to compare the matrix elements of B_i with the ones in Proposition 4.16 and observing that for $c = m/n$ one gets:

$$(1 - s_i)\widetilde{v}_{\mathbf{a}} = \frac{n\mathtt{w}_i - n\mathtt{w}_{i+1} - m}{n\mathtt{w}_i - n\mathtt{w}_{i+1}} \left(\widetilde{v}_{\mathbf{a}} - \widetilde{v}_{s_i \cdot \mathbf{a}}\right)$$

where $\mathtt{w} = \mathtt{w}(\mathbf{a})$. □

We also have a geometric analogue of the shift operator τ. Given a flag $I_k \supset I_{k+1} \supset \cdots \supset I_{k+n} = xI_k$, we can consider the flag $I_{k+1} \supset \cdots \supset I_{k+n} = xI_k \supset I_{k+n+1} = xI_{k+1}$. This defines a map $T : \mathrm{PHilb}_{k,n+k} \to \mathrm{PHilb}_{k+1,n+k+1}$.

Definition 7.10. We define $W_{k,n+k} \subset \mathrm{PHilb}_{k,n+k}$ as the set of flags $I_k \supset I_{k+1} \supset \cdots \supset I_{k+n} = xI_k$ such that $I_{k+n-1} \subset x\mathcal{O}_C$.

It is easy to see that $W_{k,n+k}$ is a closed subvariety in $\mathrm{PHilb}_{k,n+k}$.

Lemma 7.11. *The map $T : \mathrm{PHilb}_{k,n+k} \to \mathrm{PHilb}_{k+1,n+k+1}$ is injective and its image coincides with $W_{k+1,n+k+1}$. In particular, $\mathrm{PHilb}_{k,n+k}$ and $W_{k+1,n+k+1}$ are isomorphic.*

Proof. The image of T is contained in $W_{k+1,n+k+1}$ by construction. Given a flag $I_{k+1} \supset I_{k+2} \supset \cdots \supset I_{k+n+1} = xI_{k+1}$ in $W_{k+1,n+k+1}$, we have $I_{k+n} \subset x\mathcal{O}_c$, so we can define an ideal $I_k := x^{-1}I_{k+n}$. Since $I_{k+n} \supset xI_{k+1}$, we have $I_k \supset I_{k+1}$. Therefore $I_k \supset I_{k+1} \supset \cdots \supset I_{k+n} = xI_k$ is a well defined point in $\mathrm{PHilb}_{k,n+k}$ sent to the original flag by T. □

Recall that the $\mathcal{L}_n$ has fibers $I_{k+n-1}/I_{k+n} = I_{k+n-1}/xI_n$. The inclusion $I_{k+n-1} \hookrightarrow \mathcal{O}_C$ induces a map $i : \mathcal{L}_n \to \mathcal{O}_C/x\mathcal{O}_C$.

Lemma 7.12. *Define the covector $\eta : \mathcal{O}_C/x\mathcal{O}_C \to \mathbb{C}$ by the equation $\eta(y^{n-1}) = 1, \eta(y^k) = 0$ for $0 \le k < n-1$. Then $W_{k,n+k}$ is the zero locus of the composition*

$$s : \mathcal{L}_n \xrightarrow{i} \mathcal{O}_C/x\mathcal{O}_C \xrightarrow{\eta} \mathbb{C} \tag{34}$$

or, equivalently, the zero locus of the section $s : \mathbb{C} \to \mathcal{L}_n^{-1}$.

Proof. Recall that $W_{k,n+k}$ is cut out by condition $I_{k+n-1} \subset x\mathcal{O}_C$ which is equivalent to vanishing of $i(\mathcal{L}_n)$. Since $i(\mathcal{L}_n)$ is a y-invariant subspace of $\mathcal{O}_C/x\mathcal{O}_C$ of dimension at most 1, either $i(\mathcal{L}_n) = 0$ or $i(\mathcal{L}_n) = \langle y^{n-1}\rangle$. Therefore $i(\mathcal{L}_n) = 0$ if and only if $\eta(i(\mathcal{L}_n)) = 0$. □

Note that $\mathrm{PHilb}_{k,n+k}$ is in general very singular and has several irreducible components. The section s might vanish on some of these components identically. Still, by Lemma 7.12 we can define Gysin map[18]

$$j^* : H_*(\mathrm{PHilb}_{k,n+k}) \to H_{*-2}(W_{k,n+k}).$$

where $j = j_k$ is the inclusion $j : W_{k,n+k} \hookrightarrow \mathrm{PHilb}_{k,n+k}$. We define Λ as the composition

$$\Lambda : H_*(\mathrm{PHilb}_{k+1,n+k+1}) \xrightarrow{j^*} H_{*-2}(W_{k+1,n+k+1}) \xrightarrow{\simeq} H_{*-2}(\mathrm{PHilb}_{k,n+k}).$$

Lemma 7.13. *We have $T_* \circ \Lambda(-) = c_1(\mathcal{L}_n) \cap (-)$.*

Proof. Indeed, if $j : W_{k,n+k} \hookrightarrow \mathrm{PHilb}_{k,n+k}$ is the inclusion, then

$$T_* \circ \Lambda(-) = j_* j^*(-) = c_1(\mathcal{L}_n) \cap (-)$$

by Lemma 7.12. □

Theorem 7.14. *(a) The total localized equivariant homology*

$$U = \bigoplus_{k=0}^{\infty} H_*^{\mathbb{C}^*}(\mathrm{PHilb}_{k,n+k})$$

has an action of the rational Cherednik algebra $H_{n,m}$. The action of $\mathcal{S}_n$ is the Springer action described above, $u_{n+1-i} + m(n-1)$ correspond to capping with $c_1(\mathcal{L}_i)$ and the operators T and Λ on U correspond to the action of τ and λ.

(b) The representation U is irreducible and isomorphic to $L_{n,m}(\mathrm{triv})$. Under this isomorphism, fixed points of $\mathbb{C}^*$ action correspond to the eigenbasis $\widetilde{v}_{\mathbf{a}}$.

Proof. Let $\overline{s}_i, \overline{u_i}, \overline{\tau}$ and $\overline{\lambda}$ be the generators of $H_{1,c}$ where $c = m/n$. Recall that $H_{n,m}$ is isomorphic to $H_{1,m/n}$, under this isomorphism the generators s_i, u_i, τ and λ of $H_{n,m}$ are mapped to $\overline{s}_i, n\overline{u_i}, \overline{\tau}$ and $n\overline{\lambda}$ respectively. Below, we will use this isomorphism to identify $L_{n,m}(\mathrm{triv})$ with $L_{m/n}$.

By localization theorem [4, 22] U is spanned by classes of fixed points. By Lemma 7.5 these are in bijection with the basis $v_{\mathbf{a}}$ (or, equivalently, $\widetilde{v}_{\mathbf{a}}$) in $L_{m/n}$. This defines an isomorphism between U and $L_{m/n}$ as vector spaces.

Next, we prove that the geometrically defined actions of u_i, s_i, T and Λ agree with the corresponding actions on $L_{m/n}$. This is done by explicitly comparing their matrix elements. For u_i this follows from Lemma 7.5. For s_i this follows from Lemma 7.9. For T and τ it is easy to see from equation (26).

The action of Λ is uniquely determined by Lemma 7.13. More precisely, the map η in Lemma 7.12 has equivariant weight $-m(n-1)$ (since the weight of y^{n-1} equals $m(n-1)$), while by Lemma 7.5 $\mathcal{L}_n$ has weight $n\mathtt{w}_1 + m(n-1)$. Therefore section s in Lemma 7.12 has weight $n\mathtt{w}$. By Lemma 7.13 we conclude that $T_* \circ \Lambda = n\overline{u_1} = u_1$.

Finally, the operators u_i, s_i, T and Λ satisfy the relations in $H_{n,m}$ since their counterparts on $L_{n,m}(\mathrm{triv})$ do. Therefore there is indeed an action of $H_{n,m}$ on U and it is an irreducible representation. □

Remark 7.15. In principle, one can check all the relations between the geometric operators directly (similarly to the computations in [8]), but the above proof seems to be more transparent.

Remark 7.16. Note that the grading of $L_{m/n}$ by eigenvalues of the Euler operator, where $\deg(v_{\mathbf{a}}) = ||\mathbf{a}|| = \sum a_i$ corresponds to the grading by k in $\bigoplus_k H_*^{\mathbb{C}^*}(\mathrm{PHilb}_{k,n+k})$.

Consider now the Hilbert scheme $\mathrm{Hilb}(C) := \sqcup_k \mathrm{Hilb}_k(C)$, and recall that we have defined $\mathrm{PHilb}^x(C) := \sqcup_k \mathrm{PHilb}_{k,n+k}(C)$. We have a $\mathbb{C}^*$-equivariant projection $\Pi : \mathrm{PHilb}^x(C) \to \mathrm{Hilb}(C)$, $(I_k \supset \cdots \supset I_{k+n} = xI_k) \mapsto I_k$, that induces an $\mathcal{S}_n$-invariant map on (localized) equivariant homology

$$\Pi_* : H_*^{\mathbb{C}^*}(\mathrm{PHilb}^x(C)) \to H_*^{\mathbb{C}^*}(\mathrm{Hilb}(C)).$$

Now let $I_k \in \mathrm{Hilb}_k(C)$ be a monomial ideal. Thanks to Lemma 7.7, and using the notation there, the span of the elements in $H_*^{\mathbb{C}^*}(\mathrm{PHilb}_{k,n+k}(C))$ mapping to $[I_k]$ is the induced representation $\mathrm{Ind}_{\mathcal{S}_{\lambda_1} \times \cdots \times \mathcal{S}_{\lambda_\ell}}^{\mathcal{S}_n} \mathrm{triv}$. Now by adjunction

$$\mathrm{Hom}_{\mathcal{S}_n}(\mathrm{triv}, \mathrm{Ind}_{\mathcal{S}_{\lambda_1} \times \cdots \times \mathcal{S}_{\lambda_\ell}}^{\mathcal{S}_n} \mathrm{triv}) = \mathrm{Hom}_{\mathcal{S}_{\lambda_1} \times \cdots \times \mathcal{S}_{\lambda_\ell}}(\mathrm{Res}_{\mathcal{S}_{\lambda_1} \times \cdots \times \mathcal{S}_{\lambda_\ell}}^{\mathcal{S}_n} \mathrm{triv}, \mathrm{triv}) = \mathbb{C}$$

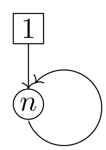
so up to scalars there is a unique $\mathcal{S}_n$-equivariant section to the projection $\Pi_* : \Pi_*^{-1}(\mathbb{C}[I_k]) \cap H_*^{\mathbb{C}^*}(\mathrm{PHilb}_{k,n+k}(C)) \to \mathbb{C}[I_k]$. As a consequence we get the following result.

Proposition 7.17. *There is a natural identification* $H_*^{\mathbb{C}^*}(\mathrm{Hilb}(C)) = H_*^{\mathbb{C}^*}(\mathrm{PHilb}^x(C))^{\mathcal{S}_n}$. *In particular, we obtain a geometric action of the* **spherical** *rational Cherednik algebra* $eH_{1,m/n}e$ *on* $H_*^{\mathbb{C}^*}(\mathrm{Hilb}(C))$, *that makes it an irreducible module isomorphic to* $eL_{m/n} = L_{m/n}^{\mathcal{S}_n}$, *where* $e := \frac{1}{n!}\sum_{p \in \mathcal{S}_n} p$ *is the trivial idempotent in* $\mathbb{C}\mathcal{S}_n$.

Remark 7.18. In [20], Garner and Kivinen study an action of the spherical rational Cherednik algebra on the homology of $\mathrm{Hilb}(C)$ using the Coulomb branch perspective. They identify $\mathrm{Hilb}(C)$ with a generalized affine Springer fiber and use the realization of $eH_{1,m/n}e$ as a quantized Coulomb branch algebra [31, 51] to define an action via convolution diagrams. We will compare their construction to ours, in the parabolic setting, in Section 7.5.

7.4. **Parabolic Hilbert schemes as generalized affine Springer fibers.** The goal of this section is to show that $\mathrm{PHilb}^x(C) = \bigsqcup_k \mathrm{PHilb}_{k,n+k}$ can be realized as a generalized affine Springer fiber. Thanks to [21], a consequence of this is that $\mathrm{PHilb}_{k,n+k}$ admits a paving by affine cells and therefore its cohomology is equivariantly formal.

Let us set $G := \mathrm{GL}_n$, acting on the vector space $N := \mathbb{C}^n \oplus \mathfrak{gl}_n$, so that N is the representation space of the framed Jordan quiver:



We will denote $\mathbb{K} := \mathbb{C}((\epsilon))$ and $\mathbb{O} := \mathbb{C}[[\epsilon]]$. We consider the groups $G_{\mathbb{O}} \subseteq G_{\mathbb{K}}$ of invertible $\mathbb{O}$-linear (resp. $\mathbb{K}$-linear) transformations on $\mathbb{O}^n$ (resp. $\mathbb{K}^n$).

We choose an $\mathbb{O}$-basis $\{b_1, \ldots, b_n\}$ of $\mathbb{O}^n$. We define b_i for $i \in \mathbb{Z}$ by setting $b_{i+n} := \epsilon b_i$. The *standard flag* is the flag of $\mathbb{O}$-lattices in $\mathbb{K}^n$

$$\cdots \supseteq \mathcal{I}_{j-1} \supseteq \mathcal{I}_j \supseteq \mathcal{I}_{j+1} \supseteq \cdots$$

where $\mathcal{I}_j$ is the $\mathbb{O}$-span of $\{b_j, b_{j+1}, \ldots, b_{j+n-1}\}$. We denote by $I \subseteq G_{\mathbb{K}}$ the standard Iwahori subgroup, that is, the stabilizer of the standard flag. The quotient space $\mathcal{F}l := G_{\mathbb{K}}/I$ is known as the affine flag variety. This is an ind-scheme parametrizing flags of $\mathbb{O}$-lattices $\cdots \supseteq \mathcal{J}_{j-1} \supseteq \mathcal{J}_j \supseteq \mathcal{J}_{j+1} \supseteq \cdots$ in $\mathbb{K}^n$ subject to the condition $\mathcal{J}_{j+n} = \epsilon \mathcal{J}_j$ for every integer $j \in \mathbb{Z}$.

The group $G_{\mathbb{K}}$ acts on the module $N_{\mathbb{K}} := \mathbb{K}\otimes N = \mathbb{K}^n \oplus \mathfrak{gl}_n(\mathbb{K})$ in the natural way, and the subgroup $G_{\mathbb{O}} \subseteq G_{\mathbb{K}}$ preserves the $\mathbb{O}$-submodule $N_{\mathbb{O}} := \mathbb{O}\otimes N \subseteq N_{\mathbb{K}}$. Now we consider the element $Y \in \mathfrak{gl}_n(\mathbb{O})$ that in the basis $\{b_1,\dots,b_n\}$ is represented by the matrix (30), with x replaced by ϵ, and the element $(b_1, Y) \in N_{\mathbb{O}}$. We will consider the *generalized affine Springer fiber*, cf. [3, 20, 21]

$$\mathrm{Spr}(b_1, Y) := \{[g] \in \mathcal{F}l \mid (gb_1, gYg^{-1}) \in \mathbb{O}^n \oplus \mathfrak{i}\} \subseteq \mathcal{F}l$$

where $\mathfrak{i}$ is the Lie algebra of the Iwahori subgroup I. More concretely, $\mathfrak{i} := \{X \in \mathfrak{gl}_n(\mathbb{O}) \mid X|_{\epsilon=0} \text{ is lower triangular}\}$.

Proposition 7.19. *We have an isomorphism*

$$\mathrm{Spr}(b_1, Y) \cong \bigsqcup_k \mathrm{PHilb}_{k,n+k}$$

Proof. We use the presentation of $\mathcal{O}_C$ at the beginning of this section as a free $\mathbb{C}[[x]]$-module of rank n. In this presentation, an ideal of $\mathcal{O}_C$ corresponds to a $\mathbb{C}[[x]]$-submodule $I \subseteq \mathbb{C}[[x]]^n$ closed under the action of the matrix Y in (30). Similarly, an element of $\bigsqcup_k \mathrm{PHilb}_{k,n+k}$ corresponds to a flag of $\mathbb{C}[[x]]$-submodules $\mathbb{C}[[x]]^n \supseteq I_k \supseteq \cdots \supseteq I_{k+n-1} \supseteq xI_k$ such that $\dim \mathbb{C}[[x]]^n/I_j = j < \infty$, each ideal I_j is stable under the action of Y and $\dim I_j/I_{j+1} = 1$. Now, we identify $[g] \in \mathcal{F}l$ with the flag $g\mathcal{I}_1 \supseteq \cdots \supseteq g\mathcal{I}_{n-1} \supseteq g\mathcal{I}_n = \epsilon g\mathcal{I}_1$ where, as above, $\mathcal{I}_1 \supseteq \mathcal{I}_2 \supseteq \cdots$ is the standard flag. Identifying $x = \epsilon$, we see that to prove the proposition we have to check that the following conditions are equivalent:

(1) $g\mathcal{I}_1 \subseteq \mathbb{O}^n$ and $g\mathcal{I}_j$ is closed under the action of Y for every $j \geq 1$.
(2) $gb_1 \in \mathbb{O}^n$ and $gYg^{-1} \in \mathfrak{i}$.

Since $g\mathcal{I}_j$ is the $\mathbb{O}$-span of $\{gb_j, \dots, gb_{j+n-1}\}$ and $b_{j+n} = \epsilon b_j$ for every j, it is easy to see that (1) $\Rightarrow$ (2). Let us check that (2) $\Rightarrow$ (1). First, we need to check that $gb_1, \dots, gb_n \in \mathbb{O}^n$. We do this by induction, the base of induction being one of the conditions in (2). Now, for $i = 1, \dots, n-1$, $gb_{i+1} = gYb_i = gYg^{-1}(gb_i)$. By induction hypothesis $gb_i \in \mathbb{O}^n$ and, by (2), $gYg^{-1} \in \mathfrak{i} \subseteq \mathfrak{gl}_n(\mathbb{O})$. So $g\mathcal{I}_1 \subseteq \mathbb{O}^n$.

Now we need to show that $g\mathcal{I}_j$ is closed under the action of Y for every $j \geq 1$. It is clearly enough to do this for $j = 1, \dots, n-1$. The condition $gYg^{-1} \in \mathfrak{i}$ is equivalent to $Ygb_i \in \mathbb{O}\text{-span}\{gb_i, \dots, gb_{i+n-1}\}$ for every $i = 1, \dots, n$. This clearly implies that $g\mathcal{I}_j$ is closed under Y. ∎

Remark 7.20. Proposition 7.19 is a special case of a flag version of the main result of [20], which the authors kindly provided a preliminary version of.

Now we would like to verify that the generalized affine Springer fiber $\mathrm{Spr}(b_1, Y)$ satisfies the conditions of [21, (3.2)]. Following that paper, let us denote by $\mathfrak{a} := X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$, where $A \subseteq G = \mathrm{GL}_n$ is a maximal torus, that we identify with the set of diagonal invertible matrices. For each weight $\xi \in \mathfrak{a}^*$, let us denote by $N_\xi \subseteq N$ the corresponding weight space. For $a \in \mathfrak{a}$ and $t \in \mathbb{R}$, we denote

$$N_{\mathbb{K},a,t} := \prod_{\substack{\xi\in\mathfrak{a}^*, d\in\mathbb{Z} \\ \langle \xi, a\rangle + d \geq t}} N_\xi \epsilon^d \subseteq N_{\mathbb{K}}.$$

For $a \in \mathfrak{a}$, let $\mathfrak{g}_a := \mathfrak{g}_{\mathbb{K},a,0} \cap \mathfrak{g}_{\mathbb{O}}$. This is a Lie subalgebra of $\mathfrak{g}_{\mathbb{O}}$ and we let $G_a \subseteq G_{\mathbb{O}}$ be the corresponding subgroup, which is an Iwahori subgroup.

Lemma 7.21. *There exist $a \in \mathfrak{a}$ and $t \in \mathbb{R}$ such that $G_a = I$ is the Iwahori subgroup, and $N_{\mathbb{K},a,t} = \mathbb{O}^n \oplus \mathfrak{i}$.*

Proof. Take any $a = \mathrm{diag}(a_1, \dots, a_n) \in \mathfrak{a}$ with $0 < a_1 < \cdots < a_n < 1$ and $t = 0$. It is straightforward to verify the result. □

Note that Lemma 7.21 tells us that $\mathrm{Spr}(b_1, Y)$ is one of the varieties considered in [21, Section 3]. In the notation of that paper, we have

$$\mathrm{Spr}(b_1, Y) = \mathcal{F}_a(t, (b_1, Y))$$

In [21, Section 3.2] it was proved that $\mathcal{F}_a(t, (b_1, Y))$ has affine paving provided that there exist $b \in \mathfrak{a}$ and $c \in \mathbb{R}$ such that the following conditions are satisfied:

- $c \geq t$
- $(b_1, Y) \in N_{\mathbb{K},b,c}$
- The projection $\overline{(b_1, Y)}$ is G-good (in the sense of [21]), that is, that no nonzero G-unstable covector in N^* vanishes on the $\mathfrak{gl}_n$-orbit of $\overline{(b_1, Y)}$

To verify these conditions, we consider $b = \mathrm{diag}(c, 2c, \dots, nc) \in \mathfrak{a}$, where $c := m/n$. Obviously $c > t = 0$, and is easy to check that $(b_1, Y) \in N_{\mathbb{K},b,c}$. For the last condition, we need to verify that the element

$$\overline{(b_1, Y)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \in N$$

is G-good. This is a consequence of the following result.

Proposition 7.22. *Let X be a regular semisimple matrix and v a cyclic vector for X. Then $(v, X) \in N$ is G-good.*

To prove Proposition 7.22, we first give a necessary condition for a vector in the adjoint representation $\mathfrak{gl}_n$ to be unstable.

Lemma 7.23. *Assume $B \in \mathfrak{gl}_n$ is G-unstable. Then, B is nilpotent.*

Proof. By definition, cf. [21], B is unstable if and only if there exists a semisimple matrix y and $t_1, \dots, t_k > 0$ such that $B = B_1 + \cdots + B_k$, with $[y, B_k] = t_k B_k$. Since the t_i are strictly positive and the filtration given by y is bounded above, the result follows. □

Returning to the setting of Proposition 7.22, we may assume that X is already in diagonal form. So $X = \mathrm{diag}(x_i)$ with $x_i \neq x_j$ for $i \neq j$ and $v = (v_i)$, the cyclicity condition is equivalent to $v_i \neq 0$ for every i.

Lemma 7.24. *Let $(w, B) \in N$ be such that $\mathrm{tr}(B[\xi, X]) + w \cdot \xi v = 0$ for every $\xi \in \mathfrak{gl}_n$. Then*

(1) *$w_i = 0$ for every $i = 1, \dots, n$.*
(2) *$b_{ij} = 0$ for $i \neq j$.*

Proof. The proof is straightforward, but let us give it for the sake of completion. We have $[\xi, X] = (\xi_{ij}(x_i - x_j))_{ij}$ and $\xi.v = (\sum_j \xi_{ij} v_j)_i$. Thus,

$$\mathrm{tr}(B[\xi, X]) + w \cdot \xi v = \sum_{i,j=1}^{n} b_{ij}\xi_{ji}(x_j - x_i) + \xi_{ij} v_i w_j = 0$$

for every matrix $\xi \in \mathfrak{gl}_n$. Taking the matrix ξ with $\xi_{ii} = 1$ and all other coordinates 0 we see, using $v_i \neq 0$, that $w_i = 0$. Now take $i \neq j$. Taking the matrix ξ with $\xi_{ij} \neq 0$ and all other coordinates 0 we see, using $x_i - x_j \neq 0$, that $b_{ji} = 0$. The result follows. □

Proof of Proposition 7.22. Let $(w, B) \in N^* \cong N$ be an unstable covector vanishing on $\mathfrak{gl}_n \cdot (v, X)$, where we use the trace form to identify $N^* \cong N$. Thanks to Lemma 7.24 (1) we have that $w = 0$. It follows now from Lemma 7.23 that B is nilpotent. But Lemma 7.24 (2) implies that B is semisimple as well. So $B = 0$, and it follows that (v, X) is G-good. $\square$

From [21], we obtain the following result.

Corollary 7.25. *The generalized affine Springer fiber* $\mathrm{Spr}(b_1, Y) = \bigsqcup_k \mathrm{PHilb}_{k,n+k}$ *is paved by affine spaces. Thus, its cohomology is equivariantly formal.*

Remark 7.26. Classical affine Springer fiber $\mathrm{Spr}(Y)$ can be obtained by similar construction for $N = \mathfrak{gl}_n$. Similar to Proposition 7.19, it can be defined as the space of Y-invariant flags

$$g\mathcal{I}_1 \supseteq \cdots \supseteq g\mathcal{I}_{n-1} \supseteq g\mathcal{I}_n = \epsilon g\mathcal{I}_1$$

in $\mathbb{K}^n$, but these flags are no longer required to be contained in $\mathbb{O}^n$. It was proved in [35, 21] that for the same matrix Y given by (30) the classical affine Springer fiber $\mathrm{Spr}(Y)$ is paved by affine spaces, and the combinatorics of this paving was studied e.g. in [35, 23].

The Springer action of $\mathcal{S}_n$ and the operator T in cohomology of $\mathrm{Spr}(Y)$ were considered in [52, 42, 43, 49]. They were shown to generate the extended affine symmetric group, in particular, T is invertible. Indeed,

$$T\left[g\mathcal{I}_1 \supseteq \cdots \supseteq g\mathcal{I}_{n-1} \supseteq g\mathcal{I}_n = \epsilon g\mathcal{I}_1\right] = \left[g\mathcal{I}_2 \supseteq \cdots \supseteq g\mathcal{I}_{n-1} \supseteq \epsilon g\mathcal{I}_1 \supseteq g\mathcal{I}_2\right]$$

while

$$T^{-1}\left[g\mathcal{I}_1 \supseteq \cdots \supseteq g\mathcal{I}_{n-1} \supseteq g\mathcal{I}_n = \epsilon g\mathcal{I}_1\right] = \left[\epsilon^{-1} g\mathcal{I}_{n-1} \supseteq g\mathcal{I}_1 \supseteq \cdots \supseteq g\mathcal{I}_{n-1}\right].$$

Furthermore, $\mathcal{S}_n$, T and line bundles $\mathcal{L}_i$ were used in [42, 43] to construct the action of the **trigonometric** Cherednik algebra on the equivariant homology of the affine Springer fiber.

In our setting, the failure of T to be invertible gives rise to a new operator Λ and together they generate the **rational** Cherednik algebra. This shows both the similarity and a subtle distinction between the trigonometric and rational setup.

7.5. **Comparison to action by convolution diagrams.** The main result of [28] constructs an action of the Coulomb branch algebra for (G, N) in the equivariant homology of any generalized affine Springer fiber for (G, N) satisfying some mild assumptions. If the affine Springer fiber is invariant under the loop rotation, then the action extends to the equivariant homology. The main result of [20] identifies the Hilbert schemes of points on arbitrary plane curve singularities with the generalized affine Springer fibers for $(G, N) = (\mathrm{GL}_n, \mathbb{C}^n \oplus \mathfrak{gl}_n)$, as in Section 7.4. By combining these results, [20] defines an action of the rational Cherednik algebra in the (equivariant) homology of Hilbert schemes of points on arbitrary plane curve singularities. The goal of this section is to compare their action with ours for the singularity $\{x^m = y^n\}$, see also [20, Section 4.3.2].

Let $\mathbf{t}, \mathbf{c}$ be formal variables and consider the $\mathbb{C}[\mathbf{t}, \mathbf{c}]$-algebra $H_{\mathbf{t},\mathbf{c}}(\mathcal{S}_n, \mathbb{C}^n)$ defined by the same relations as the usual Cherednik algebra but with the parameters t, c replaced by the variables $\mathbf{t}, \mathbf{c}$. Thanks to work of Webster, see [32, 51], $H_{\mathbf{t},\mathbf{c}} := H_{\mathbf{t},\mathbf{c}}(\mathcal{S}_n, \mathbb{C}^n)$ is a generalized BFN Coulomb branch algebra.

Recall that if we have a reductive group G acting on a vector space N, the BFN Coulomb branch algebra is defined as the equivariant Borel-Moore homology $H_*^{G_{\mathbb{O}} \rtimes \mathbb{C}^*_{\mathrm{rot}}}(\mathcal{R}_{G,N})$ where $\mathcal{R}_{G,N}$ is a space modeled after the affine Grassmannian and $\mathbb{C}^*_{\mathrm{rot}}$ is the torus acting by loop rotations, see [3] for details. When $G = \mathrm{GL}_n$ and $N = \mathbb{C}^n \oplus \mathfrak{gl}_n$ we get precisely the spherical rational Cherednik algebra. To get the full Cherednik algebra, we need to replace $\mathcal{R}_{G,N}$ with a larger space $\mathcal{R}'_{G,N}$ that is rather modeled after the affine flag variety, so we have $H_{\mathbf{t},\mathbf{c}} \cong H_*^{(I \rtimes \mathbb{C}^*_{\mathrm{rot}}) \times \mathbb{C}^*_{\mathrm{fl}}}(\mathcal{R}'_{G,N})$ where $I \subseteq G_{\mathbb{K}}$ is the standard Iwahori and the action of $\mathbb{C}^*_{\mathrm{fl}}$ comes from the framing vector. The parameter $\mathbf{t}$ is the $\mathbb{C}^*_{\mathrm{rot}}$-equivariant parameter, and the parameter $\mathbf{c}$ is the $\mathbb{C}^*_{\mathrm{fl}}$-equivariant parameter. See [32, 51] for details.

To compare the actions we look at the isomorphism $H_{\mathbf{t},\mathbf{c}} \cong H_*^{(I\rtimes\mathbb{C}^*_{\mathrm{rot}})\times\mathbb{C}^*_{\mathrm{fl}}}(\mathcal{R}'_{G,N})$ constructed by Webster in [51, Lemma 4.2]. First, we have both algebras acting on a polynomial algebra $\mathbb{C}[\mathbf{t},\mathbf{c}][U_1,\ldots,U_n]$. On the Cherednik algebra side, this comes from identifying U_i with the Dunkl-Opdam elements u_i, and we remark that this is *not* the usual polynomial representation of $H_{\mathbf{t},\mathbf{c}}$, see [51, (2.17)–(2.22)]. On the Coulomb side, this comes from identifying $\mathbb{C}[\mathbf{t},\mathbf{c}][U_1,\ldots,U_n] \cong H_*^{(I\rtimes\mathbb{C}^*_{\mathrm{rot}})\times\mathbb{C}^*_{\mathrm{fl}}}(\mathrm{pt})$, where the U_i are the Chern classes of the tautological line bundles on the affine flag variety. Both representations are faithful, and we need to identify the operators on $\mathbb{C}[\mathbf{t},\mathbf{c}][U_1,\ldots,U_n]$ corresponding to τ,λ and $\mathcal{S}_n$.

According to [51, Lemma 4.2], the action of τ corresponds to the action of the correspondence: [2]

$$\mathtt{T} := \{(F_\bullet, F'_\bullet) \in \mathcal{F}l \times \mathcal{F}l : F_i = F'_{i-1}\},$$

while the action of λ corresponds to the action of the correspondence:

$$\mathtt{L} := \{(F_\bullet, F'_\bullet) \in \mathcal{F}l \times \mathcal{F}l : F_i = F'_{i+1}\}.$$

Remark 7.27. Note that the rational Cherednik algebra $H_{\mathbf{t},\mathbf{c}}$ admits a *Fourier transform*, that is, a $\mathbb{C}[\mathbf{t},\mathbf{c}]$-involution sending $y_i \mapsto x_i$, $x_i \mapsto -y_i$ and $s_i \mapsto s_i$. On the Coulomb branch setting, this automorphism interchanges the correspondences T and L. So there is a choice of isomorphism $H_{\mathbf{t},\mathbf{c}} \to H_*^{(I\rtimes\mathbb{C}^*_{\mathrm{rot}})\times\mathbb{C}^*_{\mathrm{fl}}}(\mathcal{R}'_{G,N})$. To resolve this, we note that according to [20, Proposition 1.4] the action of $H_{\mathbf{t},\mathbf{c}}$ on $H_*^{\mathbb{C}^*}(\mathrm{PHilb}^x(C))$ coming from a $\mathbb{C}[\mathbf{t},\mathbf{c}]$-isomorphism $H_{\mathbf{t},\mathbf{c}} \to H_*^{(I\rtimes\mathbb{C}^*_{\mathrm{rot}})\times\mathbb{C}^*_{\mathrm{fl}}}(\mathcal{R}'_{G,N})$ factors through $H_{m/n} = H_{\mathbf{t},\mathbf{c}}/(\mathbf{t}-1, \mathbf{c}-m/n)$ and we choose the isomorphism that sends the module constructed in [20, Theorem 4.9] to the category $\mathcal{O}_{m/n}$.

It follows from the comparison of convolution diagrams to correspondences in [20, Section 4.2.1] that the actions of τ,λ that we defined coincide with those defined by [20, Theorem 4.9 and Corollary 4.16]. The action of $\mathcal{S}_n$ that we defined comes from projections to partial flag varieties, cf. Section 7.3 while that in [20] comes from the usual Springer action of $\mathcal{S}_n$ on the homology of Springer fibers. The coincidence of these is well-known. Since the algebra $H_{t,c}$ is generated by τ,λ and $\mathcal{S}_n$, Proposition 3.5, we obtain the following result, see also [20, Theorem 4.29].

Proposition 7.28. *The action of $H_{m/n}$ on $H_*^{\mathbb{C}^*}(\mathrm{PHilb}^x(C))$ defined in Theorem 7.14 coincides with that constructed by Garner and Kivinen in* [20, Proposition 1.4].

Corollary 7.29. *There is an action of $H_{m/n}$ on the non-localized equivariant homology $H_*^{\mathbb{C}^*}(\mathrm{PHilb}^x(C))$ lifting the action from Theorem 7.14.*

Remark 7.30. Let C be a plane curve singularity and assume that the x-projection $C \to \mathbb{C}$ has degree n. Garner and Kivinen in [20] construct an action of $H_{0,0} = \mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n] \rtimes \mathcal{S}_n = H_{\mathbf{t},\mathbf{c}}/(\mathbf{t},\mathbf{c})$ on the *non-equivariant* homology $H_*(\mathrm{PHilb}^x(C))$, see also [28].

Remark 7.31. If $C = \{x^m = y^n\}$ and $\gcd(m,n) = d > 1$ then the curve C has d irreducible components. There is a $\mathbb{C}^*$ action on C and on Hilbert schemes on C, and the results of [20] still apply, so one gets an interesting representation of the rational Cherednik algebra $H_{m/n}$ in the equivariant homology of $\sqcup_k \mathrm{PHilb}^{k,n+k}(C)$. It would be very interesting to study this representation.

Note that the $\mathbb{C}^*$ action on the Hilbert schemes no longer has isolated fixed points, so even computing the character of this representation is a nontrivial problem. Nevertheless, we expect the representation to have minimal support in the sense of [14]. Indeed, the conjectures of [41] relate the homology of $\mathrm{Hilb}(C)$ to the HOMFLY-PT invariant of the (m,n) torus link. On the other hand, by [14, Theorem 4.11] the same invariant can be obtained as a character of a certain explicit minimally supported representation of the spherical rational Cherednik algebra with parameter m/n.

[2] Note that our τ is Webster's σ, while our λ is denoted τ by Webster.

8. Parabolic Hilbert schemes and quantized Gieseker varieties

In this section, we use Theorem 7.14 together with [15] to study the geometric representation theory of quantized Gieseker varieties.

8.1. **Quantized Gieseker varieties.** Fix positive integers $n, r > 0$ and consider the vector space

$$R := \mathfrak{gl}_n \oplus \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n).$$

We have a natural action of the group GL_n on R, so every element $\xi \in \mathfrak{gl}_n$ induces a vector field on R, that we denote by ξ_R. In particular, $\xi_R \in D(R)$, the algebra of polynomial differential operators on R. Note that GL_n acts on $D(R)$. Let $c \in \mathbb{C}$. It is straightforward to see that the following space is in fact an associative algebra,

$$\mathcal{A}_c(n,r) := \left[\frac{D(R)}{D(R)\{\xi_R - c\,\mathrm{tr}(\xi) : \xi \in \mathfrak{gl}_n\}}\right]^{\mathrm{GL}_n}$$

we call $\mathcal{A}_c(n,r)$ a *quantized Gieseker variety.*

Example 8.1. When $r = 1$ then $\mathcal{A}_c(n,r) = eH_ce$, the spherical subalgebra in the type $\mathfrak{gl}_n$-Cherednik algebra. This follows from the main result of [19].

Let us now deal with the representation theory of $\mathcal{A}_c(n,r)$. We follow [15, Section 3]. Let $T_0 \subseteq \mathrm{GL}_r$ be a maximal torus, and $T := \mathbb{C}^* \times T_0$. For each co-character $\nu : \mathbb{C}^* \to T$ we can define a category $\mathcal{O}_\nu(\mathcal{A}_c(n,r))$ of highest-weight $\mathcal{A}_c(n,r)$-modules. The co-character ν has the form $t \mapsto (t^{\nu_0}, \nu'(t))$ for some co-character ν' of GL_r. If $\nu_0 \neq 0$, then $\mathcal{O}_\nu(\mathcal{A}_c(n,r))$ admits a module of Gelfand-Kirillov (GK)-dimension 1 if and only if $c = m/n$, where $\gcd(m,n) = 1$ and $c \notin (-r, 0)$. In this case, $\mathcal{O}_\nu(\mathcal{A}_c(n,r))$ admits a unique irreducible representation of GK-dimension 1, that we denote $\mathcal{L}^\nu_{m/n}(n,r)$. Moreover, $\mathcal{L}^\nu_{m/n}(n,r)$ depends only on the sign of ν_0, so we have two cases: $\mathcal{L}^+_{m/n}(n,r)$ and $\mathcal{L}^-_{m/n}(n,r)$. We denote $\mathcal{L}_{m/n} := \mathcal{L}^-_{m/n}(n,r)$. Our goal is to give a geometric description of this representation.

The next proposition follows from [15].

Proposition 8.2. *Assume $m, n > 0$. We have a vector space isomorphism*

$$\mathcal{L}_{m/n}(n,r) = (L_{n/m}(\mathrm{triv}) \otimes (\mathbb{C}^r)^{\otimes m})^{\mathcal{S}_m}$$

where $L_{n/m}(\mathrm{triv})$ is the simple highest weight representation of $H_{n/m}(\mathcal{S}_m, \mathbb{C}^m)$ and the action of $\mathcal{S}_m$ on $L_{n/m}(\mathrm{triv}) \otimes (\mathbb{C}^r)^{\otimes m}$ is diagonal.

Proof. The $\mathfrak{sl}_n$-version of this result is [15, Corollary 2.18]. The $\mathfrak{gl}_n$-version is proved identically. Alternatively, it follows from the $\mathfrak{sl}_n$-version by multiplying both sides of [15, Corollary 2.18] by a polynomial algebra in one variable. □

We would like to emphasize that in the statement of Proposition 8.2 there is a swap in the parameters n, m.

Let us elaborate on the statement of Proposition 8.2. A priori, it is only a vector space identification. However, we can recover the action of $\mathcal{A}_{m/n}(n,r)$ on the space $(L_{n/m}(\mathrm{triv}) \otimes (\mathbb{C}^r)^{\otimes m})^{\mathcal{S}_m}$ as follows. First, we construct a matrix version of the rational Cherednik algebra.

Definition 8.3. Let $t, c \in \mathbb{C}$ and $m, r \in \mathbb{Z}_{>0}$. We define the algebra $H_{t,c}(m,r)$ as the quotient of the semidirect product $(\mathbb{C}\langle x_1, \dots, x_m, y_1, \dots, y_m\rangle \otimes (\mathrm{End}(\mathbb{C}^r))^{\otimes m}) \rtimes \mathcal{S}_m$ by the relations

- $[y_\ell, y_N] = 0 = [x_\ell, x_N]$ for any $\ell, N = 1, \dots, m$.
- $[y_\ell, x_N] = c\left(\sum_{i,j=1}^r (E_{ij})_\ell (E_{ji})_N\right)(\ell, N)$ if $\ell \neq N$.
- $[y_\ell, x_\ell] = t - c\sum_{N \neq \ell}\left(\sum_{i,j=1}^r (E_{ij})_\ell (E_{ji})_N\right)(\ell, N)$

x^2	x^2y	x^2y^2	4					
x	xy	xy^2	xy^3	xy^4	2			
1	y	y^2	y^3	y^4	y^5	y^6	4	

FIGURE 4. An element of $\mathrm{CPHilb}^{6,y}(\{x^3 = y^4\})$. Here, $J^0 = J^1 = \langle x^2y^3, xy^5\rangle$, $J^2 = J^3 = \langle x^2y^3, xy^6\rangle$ and $J^4 = J^5 = J^6 = yJ^0 = \langle x^2y^4, xy^6\rangle$. Also $\gamma = (0, 1, 0, 2, 0, 0) \in \mathcal{C}_6(3)$ which corresponds to 2^14^2. Note that the roles of m and n, as well as those of x and y are different from those in Figures 2 and 3.

where E_{ij} is the $r \times r$ matrix that has a 1 in the (i, j)-th position and zeroes everywhere else, and $(E_{ij})_\ell \in \mathrm{End}(\mathbb{C}^r)^{\otimes m}$ is $\mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes E_{ij} \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id}$, where E_{ij} is in the ℓ-th position.

For example, when $r = 1$ we simply recover the rational Cherednik algebra $H_{t,c}(\mathcal{S}_m, \mathbb{C}^m)$. To lighten notation but still emphasize the role of m over n, we will write $H_{t,c}(m)$ in place of $H_{t,c}(\mathcal{S}_m, \mathbb{C}^m)$ or $H_{t,c}$ below.

It is clear from the relations that if M is an $H_{t,c}(m)$-module, then $M \otimes (\mathbb{C}^r)^{\otimes m}$ becomes an $H_{t,c}(m, r)$-module, where the elements $x_1, \ldots, x_m, y_1, \ldots, y_m$ act only on the M tensor factor, the elements from $\mathrm{End}(\mathbb{C}^r)^{\otimes m}$ act only on the $(\mathbb{C}^r)^{\otimes m}$ tensor factor, and $\mathcal{S}_m$ acts diagonally. In fact, this defines a category equivalence $H_{t,c}(m)\text{-mod} \to H_{t,c}(m, r)\text{-mod}$, see [15]. Thus, the algebra $H_{1,n/m}(m, r)$ acts on $L_{n/m}(\mathrm{triv}) \otimes (\mathbb{C}^r)^{\otimes m}$.

Now we can form the spherical subalgebra $eH_{1,n/m}(m, r)e$, where $e = \frac{1}{m!}\sum_{p\in\mathcal{S}_m} p$, that acts on the space $(L_{n/m}(\mathrm{triv}) \otimes (\mathbb{C}^r)^{\otimes m})^{\mathcal{S}_m}$. Upon the identification $\mathcal{L}_{m/n}(n, r) = (L_{n/m}(\mathrm{triv}) \otimes (\mathbb{C}^r)^{\otimes m})^{\mathcal{S}_m}$ of Proposition 8.2, the actions of $\mathcal{A}_{m/n}(n, r)$ and $eH_{1,n/m}(m, r)e$ on their respective spaces get identified. This follows from [15, Section 2] after minor modifications.

8.2. **Compositional parabolic Hilbert schemes, combinatorially.** We consider the curve $C = \{x^m = y^n\}$. Let us consider the scheme

$$\mathrm{CPHilb}^{r,y} := \{\mathcal{O}_C \supseteq J^0 \supseteq \cdots \supseteq J^{r-1} \supseteq J^r = yJ^0\}$$

where J^k are ideals in $\mathcal{O}_C$ of finite codimension (not necessarily k). We have an action of $\mathbb{C}^*$ on $\mathrm{CPHilb}^{r,y}$, and the fixed points can be identified with chains of monomial ideals. We can encode these as follows. Start with the monomial ideal $J^0 = \mathbb{C}[[y]]\langle y^{c_1}, xy^{c_2}, \ldots, x^{m-1}y^{c_m}\rangle \subseteq \mathcal{O}_C$. For $k = 1, \ldots, r$ let $\gamma_k := \dim(J^{k-1}/J^k) \geq 0$. Note that $\sum_{k=1}^{r} \gamma_k = m$. The space J^{k-1}/J^k is spanned by the monomials $x^{\alpha_{k,1}}y^{c_{\alpha_{k,1}}}, \ldots, x^{\alpha_{k,\gamma_k}}y^{c_{\alpha_{k,\gamma_k}}}$ where $\alpha_{k,1} < \cdots < \alpha_{k,\gamma_k}$. Note that if $c_{\alpha_{k,i}} = c_{\alpha_{k',j}}$ for some $k < k'$ then $\alpha_{k,i} < \alpha_{k',j}$. Moreover, if $c_{\alpha_{k,i}} - c_{\alpha_{k',j}} = n$ then $k' \leq k$.

Pictorially, we consider the staircase diagram defined by the ideal J^0 and we fill in the box corresponding to the monomial $x^{\alpha_{k,i}}y^{c_{\alpha_{k,i}}}$ with the number k. In particular, the number of boxes labeled by k is precisely γ_k. See Figure 4. Note that the labels of these boxes are weakly increasing along each vertical run of the staircase diagram, where we read bottom-to-top. Moreover, if two labeled boxes are n horizontal steps apart, then the label of the top box is no greater than that of the bottom box.

The localized equivariant homology $H_*^{\mathbb{C}^*}(\mathrm{CPHilb}^{r,y}(C))$ then admits a basis indexed by classes of fixed points. As in Section 7.1, see in particular Lemma 7.2 for a monomial ideal $J^0 = \mathbb{C}[[y]]\langle y^{c_1}, xy^{c_2}, \ldots, x^{m-1}y^{c_m}\rangle$ we can define a composition $(\lambda_1, \ldots, \lambda_\ell)$ of m. Thanks to the discussion above, the flags of monomial ideals that start with J^0 can be labeled by ℓ-tuples of monomials $(m_1, \ldots, m_\ell)$, where m_i is a monomial of degree λ_i in r variables.

On the other hand, it follows from Lemma 7.7 that, as a $\mathcal{S}_m$-module we have

$$L_{n/m}(\mathrm{triv}) = \bigoplus_{\substack{J^0 \subseteq \mathcal{O}_C \\ J^0 \text{ monomial ideal}}} \mathrm{Ind}_{\mathcal{S}_{\lambda_1}\times\cdots\times\mathcal{S}_{\lambda_\ell}}^{\mathcal{S}_m} \mathrm{triv}$$

So that

$$\begin{aligned} \mathcal{L}_{m/n}(n,r) =& (L_{n/m}(\mathrm{triv}) \otimes (\mathbb{C}^r)^{\otimes m})^{\mathcal{S}_m} \\ =& \bigoplus_{\substack{J^0 \subseteq \mathcal{O}_C \\ J^0 \text{ monomial ideal}}} (\mathrm{Ind}_{\mathcal{S}_{\lambda_1}\times\cdots\times\mathcal{S}_{\lambda_\ell}}^{\mathcal{S}_m} \mathrm{triv} \otimes (\mathbb{C}^r)^{\otimes m})^{\mathcal{S}_m} \\ =& \bigoplus_{\substack{J^0 \subseteq \mathcal{O}_C \\ J^0 \text{ monomial ideal}}} \mathrm{Hom}_{\mathcal{S}_m}(\mathrm{Ind}_{\mathcal{S}_{\lambda_1}\times\cdots\times\mathcal{S}_{\lambda_\ell}}^{\mathcal{S}_m} \mathrm{triv}, (\mathbb{C}^r)^{\otimes m}) \\ =& \bigoplus_{\substack{J^0 \subseteq \mathcal{O}_C \\ J^0 \text{ monomial ideal}}} \mathrm{Hom}_{\mathcal{S}_{\lambda_1}\times\cdots\times\mathcal{S}_{\lambda_\ell}}(\mathrm{triv}, \mathrm{Res}_{\mathcal{S}_{\lambda_1}\times\cdots\times\mathcal{S}_{\lambda_\ell}}^{\mathcal{S}_m} (\mathbb{C}^r)^{\otimes m}) \\ =& \bigoplus_{\substack{J^0 \subseteq \mathcal{O}_C \\ J^0 \text{ monomial ideal}}} \mathrm{Sym}^{\lambda_1}(\mathbb{C}^r) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_\ell}(\mathbb{C}^r) \end{aligned} \tag{35}$$

This suggests that we have an identification $\mathcal{L}_{m/n}(n,r) \cong H_*^{\mathbb{C}^*}(\mathrm{CPHilb}^{r,y}(C))$. In the next section, we are going to realize this identification geometrically.

8.3. **Compositional parabolic Hilbert schemes, geometrically.** Let us recall that we have the decomposition

$$\mathrm{CPHilb}^{r,y}(C) = \bigsqcup_{\gamma \in \mathcal{C}_r(m)} \mathrm{PHilb}^{\gamma,y}(C)$$

where $\mathrm{PHilb}^{\gamma,y}(C) = \{\mathcal{O}_C \supseteq J^0 \supseteq \cdots \supseteq J^r = yJ^0 \mid \dim(J^{k-1}/J^k) = \gamma_k\}$ and $\mathcal{C}_r(m)$ are weak compositions of m with r parts. In particular, $H_*^{\mathbb{C}^*}(\mathrm{CPHilb}^{r,y}(C)) = \bigoplus_{\gamma\in\mathcal{C}_r(m)} H_*^{\mathbb{C}^*}(\mathrm{PHilb}^{\gamma,y}(C))$.

Now, for each $\gamma \in \mathcal{C}_r(m)$ we have a map

$$\begin{aligned} \Pi^\gamma &: \mathrm{PHilb}^y(C) \to \mathrm{PHilb}^{\gamma,y}(C) \\ (I_k \supseteq I_{k+1} \supseteq \cdots \supseteq I_{k+m} = yI_k) &\mapsto (J^0 \supseteq J^1 \supseteq \cdots \supseteq J^r = yJ^0) \end{aligned}$$

where $J^\ell := I_{k+\sum_{i=1}^\ell \gamma_i}$.

Lemma 8.4. *Let $\gamma \in \mathcal{C}_r(m)$ and consider the standard parabolic subgroup $\mathcal{S}_{\gamma^{\mathrm{rev}}} = \mathcal{S}_{\gamma_r} \times \cdots \times \mathcal{S}_{\gamma_1} \subseteq \mathcal{S}_m$. The map $\Pi_*^\gamma : H_*^{\mathbb{C}^*}(\mathrm{PHilb}^y(C)) \to H_*^{\mathbb{C}^*}(\mathrm{PHilb}^{\gamma,y}(C))$ induces an identification*

$$H_*^{\mathbb{C}^*}(\mathrm{PHilb}^y(C))^{\mathcal{S}_{\gamma^{\mathrm{rev}}}} = H_*^{\mathbb{C}^*}(\mathrm{PHilb}^{\gamma,y}(C)).$$

Proof. First, we verify that Π_*^γ is $\mathcal{S}_{\gamma^{\mathrm{rev}}}$-invariant, that is, it is constant on $\mathcal{S}_{\gamma^{\mathrm{rev}}}$-orbits. The group $\mathcal{S}_{\gamma^{\mathrm{rev}}}$ is generated by simple reflections s_i, $i \notin \{\gamma_r, \gamma_r + \gamma_{r-1}, \ldots, \gamma_r + \cdots + \gamma_2\}$. It is enough to verify that Π_*^γ is invariant under each of these simple reflections.

By definition, Π^γ sends an element $(I_k \supseteq I_{k+1} \supseteq \cdots \supseteq I_{k+m} = yI_k)$ to a flag involving only the ideals $I_k, I_{k+\gamma_1}, I_{k+\gamma_1+\gamma_2}, \ldots, I_{k+\gamma_1+\cdots+\gamma_{r-1}}$, and each one of these ideals has a multiplicity determined by the zeroes in γ. The invariance now follows from the explicit form of the action of s_i obtained in Lemma 7.9.

Now we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{PHilb}^y(C) & \xrightarrow{\Pi^\gamma} & \mathrm{PHilb}^{\gamma,y}(C) \\ & {}_{\Pi}\searrow \quad \swarrow_{\widetilde{\Pi}} & \\ & \mathrm{Hilb}(C) & \end{array}$$

The fiber of an ideal I over Π is precisely the Springer fiber $\mathrm{Spr}(x) \subseteq \mathcal{F}l(I/yI)$ consisting of *full* flags of subspaces in $I/yI \cong \mathbb{C}^m$ that are stable under the action of the nilpotent operator x. Likewise, the fiber of I over $\widetilde{\Pi}$ is the Spaltenstein variety $\mathrm{Spr}^\gamma(x) \subseteq \mathcal{F}l^\gamma(I/yI)$, consisting of *partial* flags of subspaces in I/yI that are stable under the action of x. It is a standard result from Springer theory, see e.g. [6] or [52, Section 2.6] that

$$H_*(\mathrm{Spr}(x))^{\mathcal{S}_\gamma} = H_*(\mathrm{Spr}^\gamma(x))$$

from which the result follows. □

Thanks to the previous lemma and observing that $\gamma \mapsto \gamma^{\mathrm{rev}}$ is an involution on $\mathcal{C}_r(m)$ we get

$$H_*^{\mathbb{C}^*}(\mathrm{CPHilb}^{r,y}(C)) = \bigoplus_{\gamma\in\mathcal{C}_r(m)} H_*^{\mathbb{C}^*}(\mathrm{PHilb}^{\gamma,y}(C)) = \bigoplus_{\gamma\in\mathcal{C}_r(m)} H^{\mathbb{C}^*}(\mathrm{PHilb}^{y}(C))^{\mathcal{S}_\gamma}$$

on the other hand, we have the following well-known result.

Lemma 8.5. *Let V be a representation of $\mathcal{S}_m$ and $r > 0$. Then*

$$(V \otimes (\mathbb{C}^r)^{\otimes m})^{\mathcal{S}_m} = \bigoplus_{\gamma\in\mathcal{C}_r(m)} V^{\mathcal{S}_\gamma}.$$

Moreover, $V^{\mathcal{S}_\gamma}$ is the γ-weight space for the $\mathfrak{gl}(r)$ action on the left hand side.

Proof. Fix a basis $e_1, \ldots, e_r$ of $\mathbb{C}^r$. For $\gamma \in \mathcal{C}_r(m)$, let $(\mathbb{C}^r)^{\otimes m}_\gamma$ be the span of those tensors $e_{i_1} \otimes \cdots \otimes e_{i_m}$ such that $\gamma_j = \sharp\{k : i_k = j\}$ (this is γ-weight subspace in $(\mathbb{C}^r)^{\otimes m}$). It follows by definition that $(\mathbb{C}^r)^{\otimes m}_\gamma$ is stable under the action of $\mathcal{S}_m$ and moreover that $(\mathbb{C}^r)^{\otimes m}_\gamma = \mathrm{Ind}_{\mathcal{S}_\gamma}^{\mathcal{S}_m}\mathrm{triv}$. Thus, we get $(\mathbb{C}^r)^{\otimes m} = \oplus_{\gamma\in\mathcal{C}_r(m)} \mathrm{Ind}_{\mathcal{S}_\gamma}^{\mathcal{S}_m}\mathrm{triv}$ and the result now follows by adjunction. □

Theorem 8.6. *Let m and n be coprime positive integers, and $r > 0$. There is an action of the algebra $\mathcal{A}_{m/n}(n, r)$ on the (localized) equivariant homology $H_*^{\mathbb{C}^*}(\mathrm{CPHilb}^{r,y})(C)$, where C is the singular curve $\{x^m = y^n\}$, and with this action we have $H_*^{\mathbb{C}^*}(\mathrm{CPHilb}^{r,y}(C)) \cong \mathcal{L}_{m/n}(n, r)$.*

Proof. We have a natural action of the spherical subalgebra $eH_{1,n/m}(m, r)e$ on

$$(L_{n/m}(\mathrm{triv}) \otimes (\mathbb{C}^r)^{\otimes m})^{\mathcal{S}_m}.$$

Thanks to Theorem 7.14 the latter space can be identified with $(H_*^{\mathbb{C}^*}(\mathrm{PHilb}^y(C)) \otimes (\mathbb{C}^r)^{\otimes m})^{\mathcal{S}_m}$ which in turn, by Lemmas 8.4 and 8.5 is naturally identified with $H_*^{\mathbb{C}^*}(\mathrm{CPHilb}^{r,y}(C))$. The result now follows from Proposition 8.2. □

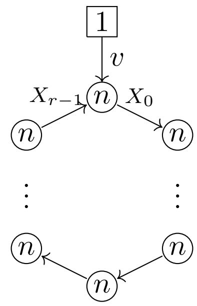
Example 8.7. When $r = 1$, we have $\mathrm{CPHilb}^{1,y}(C) = \mathrm{Hilb}(C)$ and, up to [7, Proposition 9.5], we recover Proposition 7.17.

Remark 8.8. We can realize the generators E_i, F_i of $\mathfrak{gl}(r)$ by explicit correspondences between $\mathrm{PHilb}^{\gamma,y}$ and $\mathrm{PHilb}^{\gamma',y}$ similar to [5, Theorem 3.4].

8.4. **Compositional parabolic Hilbert schemes as generalized affine Springer fibers.** Just as with parabolic Hilbert schemes, the compositional parabolic scheme $\mathrm{CPHilb}^{r,y}(C)$ admits an interpretation as a generalized affine Springer fiber. In this setting, we let the group $G := \mathrm{GL}_n^{\times r}$ act on the vector space $N := \mathbb{C}^n \oplus \mathfrak{gl}_n^{\oplus r}$ in the following way:

$$(g_0, g_1, \dots, g_{r-1}).(v, X_0, \dots, X_{r-1}) = (g_0 v, g_1 X_0 g_0^{-1}, \dots, g_0 X_{r-1} g_{r-1}^{-1}).$$

We can visualize N in terms of representations of the following cyclic quiver:



As in Section 7.4 we consider the groups $G_{\mathbb{O}} \subseteq G_{\mathbb{K}}$. We will consider the *affine Grassmannian*

$$\mathcal{G}r_G := G_{\mathbb{K}}/G_{\mathbb{O}} = (\mathrm{GL}_{n,\mathbb{K}} / \mathrm{GL}_{n,\mathbb{O}})^{\times r}$$

that parametrizes r-tuples of $\mathbb{O}$-lattices inside $\mathbb{K}^n$. The group $G_{\mathbb{K}}$ acts on $N_{\mathbb{K}} := N \otimes \mathbb{K}$, and $G_{\mathbb{O}}$ preserves $N_{\mathbb{O}}$. Recall the definition of $b_1 \in \mathbb{O}^n$ and $Y \in \mathfrak{gl}_n(\mathbb{O})$ from Section 7.4. Here, we will consider the following generalized affine Springer fiber

$$\mathrm{Spr}(b_1, \mathrm{Id}, \mathrm{Id}, \dots, Y) := \{[g] \in \mathcal{G}r_G \mid g.(b_1, \mathrm{Id}, \dots, Y) \in N_{\mathbb{O}}\} \subseteq \mathcal{G}r_G.$$

Proposition 8.9. *We have an isomorphism*

$$\mathrm{Spr}(b_1, \mathrm{Id}, \mathrm{Id}, \dots, Y) \cong \mathrm{CPHilb}^{r,y}(C).$$

Proof. By definition, an element $[g] = [g_0, \dots, g_{r-1}] \in \mathcal{G}r_G$ belongs to $\mathrm{Spr}(b_1, \mathrm{Id}, \mathrm{Id}, \dots, Y)$ if and only if $g_0 b_1 \in \mathbb{O}^n$, $g_{i+1} g_i^{-1} \in \mathfrak{gl}_n(\mathbb{O})$ for $i = 0, \dots, r-2$ and $g_0 Y g_{r-1}^{-1} \in \mathfrak{gl}_n(\mathbb{O})$. It easily follows from here that $g_i b_1 \in \mathbb{O}^n$ and $g_i Y g_i^{-1} \in \mathbb{O}^n$ for every $i = 0, \dots, r-1$. Thanks to [20, Theorem 3.3] this implies that $\mathrm{Spr}(b_1, \mathrm{Id}, \dots, Y) \subseteq \mathrm{Hilb}(C)^{\times r}$. Let $(J^0, \dots, J^{r-1}) \in \mathrm{Hilb}(C)^{\times r}$ be the point corresponding to $[g_0, \dots, g_{r-1}]$. The condition $g_{i+1} g_i^{-1} \in \mathfrak{gl}_n(\mathbb{O})$ for $i = 0, \dots, r-2$ translates to $J^0 \supseteq J^1 \supseteq \dots \supseteq J^{r-1}$, while the condition $g_0 Y g_{r-1}^{-1} \in \mathfrak{gl}_n(\mathbb{O})$ translates to $J^{r-1} \supseteq y J^0$. The result follows. $\square$

Remark 8.10. Similar to [20], the same proof shows that for an arbitrary plane curve singularity C such that the x-projection has degree n, the scheme $\mathrm{CPHilb}^{r,y}(C)$ can be presented as a generalized affine Springer fiber for $G = \mathrm{GL}_n^{\times r}$ and $N := \mathbb{C}^n \oplus \mathfrak{gl}_n^{\oplus r}$.

Just as in Section 7.4, $\mathrm{Spr}(b_1, \mathrm{Id}, \mathrm{Id}, \dots, Y)$ can be realized as one of the varieties considered by [21]. Indeed, it is straightforward to verify that

$$\mathrm{Spr}(b_1, \mathrm{Id}, \dots, \mathrm{Id}, Y) = \mathcal{F}_a(t, (b_1, \mathrm{Id}, \dots, \mathrm{Id}, Y))$$

where $t = 0$ and $a \in \mathfrak{a} := X_*(A) \otimes \mathbb{R}$ is also 0 where, recall, $A \subseteq G$ is a maximal torus. We can verify that $\mathrm{Spr}(b_1, \mathrm{Id}, \dots, Y)$ admits an affine paving as follows. Recall that we need to find $b \in \mathfrak{a}$ and $c \in \mathbb{R}$ satisfying the three conditions of Section 7.4. We can take $c = m/n > t = 0$ and $b = (b^0, b^1, \dots, b^{r-1})$, where $b^0 = b^1 = \dots = b^{r-2} = \mathrm{diag}(0, 0, \dots, 0)$ and $b^{r-1} = \mathrm{diag}(c, c, \dots, c)$. We need to verify that the element

$$\overline{(b_1, \mathrm{Id}, \dots, Y)} = (b_1, \mathrm{Id}, \dots, Y|_{\epsilon=1}) \in N$$

is G-good. This follows because the element

$$\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0 & 0 & \cdots & 0 & Y|_{\epsilon=1}\\ \mathrm{Id} & 0 & \cdots & 0 & 0\\ 0 & \mathrm{Id} & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & 0\\ 0 & 0 & \cdots & \mathrm{Id} & 0\end{pmatrix} \in \mathbb{C}^{rn} \oplus \mathfrak{gl}_{nr}$$

is GL_{nr}-good, which in turn is a consequence of Proposition 7.22. Thus, thanks to [21] we get the following.

Proposition 8.11. *The Hilbert scheme* $\mathrm{CPHilb}^{r,y}(C)$ *is paved by affine spaces. Thus, its cohomology is equivariantly formal.*

Remark 8.12. Similarly to what is done in Section 7.4 one can show that for a composition $\gamma \in \mathcal{C}_r(m)$ the variety $\mathrm{PHilb}^{\gamma,y}(C)$ admits a paving by affine spaces. This gives another proof of Proposition 8.11.

Remark 8.13. The algebra of functions on the Gieseker variety $\mathcal{M}(n,r)$ is known, thanks to results of Nakajima-Takayama [39], see also [10], to be the (non-quantized) Coulomb branch algebra for the gauge theory with gauge group $G = \mathrm{GL}_n^{\times r}$ and matter representation $N = \mathbb{C}^n \oplus \mathfrak{gl}_n^{\oplus r}$ as defined in this section. Uniqueness of quantizations proved by Losev [34, Theorem 3.4] then shows that the algebra $\mathcal{A}_c(n,r)$ is the corresponding quantized Coulomb branch algebra. It would be interesting to compare the action of $\mathcal{A}_c(n,r)$ on $H_*^{\mathbb{C}^*}(\mathrm{CPHilb}^{r,y}(C))$ we have constructed here with an action by convolution diagrams as in [20, 28].

Remark 8.14. Let C be a plane curve singularity such that the x-projection $C \to \mathbb{C}$ has degree n. One can use the techniques developed by Hilburn-Kamnitzer-Weekes in [28] and Garner-Kivinen in [20] to show that there is an action of the algebra of functions $\mathbb{C}[\mathcal{M}(n,r)]$ on the non-equivariant homology $H_*(\mathrm{CPHilb}^y(C))$, cf. Section 7.5 and Remark 8.10.

Remark 8.15. As in Remark 7.31, we can consider the case $C = \{x^m = y^n\}$ for $\gcd(m,n) = d > 1$. In this case by [20, 28] there is an action of the quantum Gieseker algebra $\mathcal{A}_{\frac{m}{n}}(n,r)$ on $H_*^{\mathbb{C}^*}(\mathrm{CPHilb}^{r,y}(C))$. We expect this representation to have minimal support in the sense of [15]. Note that by [15, Theorem 2.17, Lemma 4.1] minimally supported representations of $\mathcal{A}_{\frac{m}{n}}(n,r)$ are related to the minimally supported representations of $H_{\frac{n}{m}}$ in a way similar to Proposition 8.2.

9. Limit $m \to \infty$

In this section, we will see that, in the limit $m \to \infty$, the action of the Dunkl-Opdam subalgebra on $\Delta(\mathrm{triv}) = \mathbb{C}[x_1, \dots, x_n]$ is still diagonalizable, and we will provide an explicit basis of $\Delta(\mathrm{triv})$ completely analogous to that of Theorem 4.15. Since $H_c = H_{1,c} \cong H_{1/c,1}$, having $c \to \infty$ will yield an action of the algebra $H_{0,1}$.

9.1. The polynomial representation.

Recall that, for generic c or for c having denominator precisely n, the action of the Dunkl-Opdam subalgebra on the polynomial representation $\Delta_c(\mathrm{triv})$ is diagonalizable. This is, of course, not true for every c, as an easy calculation in the case $n = 2$, $c = 1$ shows. However, we have the following result.

Proposition 9.1. *For any* $c \in \mathbb{C}$*, the action of the Dunkl-Opdam subalgebra on* $\Delta_c(\mathrm{triv}) = \mathbb{C}[x_1, \dots, x_n]$ *is diagonalizable up to degree* $\lfloor |c(n-1)| \rfloor$*. Moreover, up to this degree, the action of the algebra* H_c *is given by the same operators as in Theorem 4.15.*

Proof. Following the strategy of the proof of Theorem 4.9, we need to construct the eigenvectors $v_{\mathbf{a}}$ for $||\mathbf{a}|| < \lfloor |c(n-1)| \rfloor$. The only obstruction to constructing these eigenvectors is that the intertwining operator σ_i may not be well-defined on the eigenspace $M_{\mathtt{w}(\mathbf{a})}$. But this is only the case when $\mathtt{w}(\mathbf{a})_i = \mathtt{w}(\mathbf{a})_{i+1}$. Recall that $\mathtt{w}(\mathbf{a})_i - \mathtt{w}(\mathbf{a})_{i+1} = a_i - a_{i+1} - (g_{\mathbf{a}}(i) - g_{\mathbf{a}}(i+1))c$ where $g_{\mathbf{a}}$ is the shortest permutation that sorts $\mathbf{a}$. Since $g_{\mathbf{a}}(i) - g_{\mathbf{a}}(i+1) \in \{\pm 1, \dots, \pm(n-1)\}$, the result follows. □

Thanks to the previous proposition, letting $c \to \infty$ and appropriately rescaling, we get the following "$t = 0$" analogue of Theorem 4.15.

Theorem 9.2. *The $H_{0,1}$-module $\Delta_{0,1}(\mathrm{triv}) := H_{0,1} \otimes_{\mathbb{C}[y_1,\dots,y_n]\rtimes\mathcal{S}_n} \mathrm{triv}$ has a basis given by $\{v_{\mathbf{a}} : \mathbf{a} \in \mathbb{Z}^n_{\geq 0}\}$, and the action of the algebra $H_{0,1}$ on $\Delta_{0,1}(\mathrm{triv})$ is given by the following operators.*

$$\begin{aligned} u_i v_{\mathbf{a}} &= \mathtt{w}_i v_{\mathbf{a}} \\ \tau v_{\mathbf{a}} &= v_{\pi\cdot\mathbf{a}} \\ \lambda v_{\mathbf{a}} &= \mathtt{w}_1 v_{\pi^{-1}\cdot\mathbf{a}} \\ s_i v_{\mathbf{a}} &= \begin{cases} v_{s_i\cdot\mathbf{a}} + \frac{1}{g_{\mathbf{a}}(i+1)-g_{\mathbf{a}}(i)} v_{\mathbf{a}} & a_i > a_{i+1} \\ \frac{(g_{\mathbf{a}}(i)-g_{\mathbf{a}}(i+1)-1)(g_{\mathbf{a}}(i)-g_{\mathbf{a}}(i+1)+1)}{(g_{\mathbf{a}}(i)-g_{\mathbf{a}}(i+1))^2} v_{s_i\cdot\mathbf{a}} + \frac{1}{g_{\mathbf{a}}(i)-g_{\mathbf{a}}(i+1)} v_{\mathbf{a}} & a_i < a_{i+1} \\ v_{\mathbf{a}} & a_i = a_{i+1} \end{cases} \end{aligned}$$

where $\mathtt{w}_i := \mathtt{w}_i(\mathbf{a}) = (1 - g_{\mathbf{a}}(i))$ and, as before, $g_{\mathbf{a}}$ is the minimal-length permutation that sorts $\mathbf{a}$.

Remark 9.3. As above, one can also define the renormalized basis $\widetilde{v}_{\mathbf{a}}$ such that

$$(1+s_i)\widetilde{v}_{\mathbf{a}} = \frac{g_{\mathbf{a}}(i+1) - g_{\mathbf{a}}(i) - 1}{g_{\mathbf{a}}(i+1) - g_{\mathbf{a}}(i)} \widetilde{v}_{s_i\cdot\mathbf{a}} + \frac{g_{\mathbf{a}}(i+1) - g_{\mathbf{a}}(i) + 1}{g_{\mathbf{a}}(i+1) - g_{\mathbf{a}}(i)} \widetilde{v}_{\mathbf{a}}$$

Remark 9.4. We remark that, unlike the $t = 1$ case, the module $\Delta_{0,1}(\mathrm{triv})$ is never irreducible. Its unique irreducible *graded* quotient is $L_{0,1}(\mathrm{triv}) = \mathbb{C}[x_1, \dots, x_n]/(\mathbb{C}[x_1, \dots, x_n]^{\mathcal{S}_n}_+)$.

Note that the proof of Theorem 9.2 can be extended to any Verma module $\Delta_{0,1}(\mu) := H_{0,1} \otimes_{\mathbb{C}[\mathbf{y}]\rtimes\mathcal{S}_n} V_\mu$. In particular, we get that $\Delta_{0,1}(\mu)$ has a basis given by $v(\mathbf{a}, T)$, where $\mathbf{a} \in \mathbb{Z}^n_{\geq 0}$ and $T \in \mathrm{SYT}(\mu)$. The action of $H_{0,1}$ on $\Delta_{0,1}(\mu)$ is given by

$$\begin{aligned} u_i v(\mathbf{a}, T) &= \mathtt{w}_i(\mathbf{a}, T) v(\mathbf{a}, T) \\ \tau v(\mathbf{a}, T) &= v(\pi\cdot\mathbf{a}, T) \\ \lambda v(\mathbf{a}, T) &= \mathtt{w}_1(\mathbf{a}, T) v(\pi^{-1}\cdot\mathbf{a}, T) \\ s_i v(\mathbf{a}, T) &= \begin{cases} v(s_i\cdot\mathbf{a}, T) - A_2 v(\mathbf{a}, T) & a_i > a_{i+1} \\ A_1 v(s_i\cdot\mathbf{a}, T) + A_2 v(\mathbf{a}, T) & a_i < a_{i+1} \\ (\mathrm{ct}_T(g_{\mathbf{a}}(i+1)) - \mathrm{ct}_T(g_{\mathbf{a}}(i)))v(\mathbf{a}, T) & a_i = a_{i+1} \text{ and } s_{g_{\mathbf{a}}(i)}(T) \notin \mathrm{SYT}(\mu) \\ v(\mathbf{a}, s_{g_{\mathbf{a}}(i)}(T)) - A_2 v(\mathbf{a}, T) & a_i = a_{i+1} \text{ and } s_{g_{\mathbf{a}}(i)}(T) \in \mathrm{SYT}(\mu) \end{cases} \end{aligned}$$

where $\mathtt{w}_i(\mathbf{a}, T) = -\mathrm{ct}_T(g_{\mathbf{a}}(i))$,

$$A_1 = \frac{(\mathrm{ct}_T(g_{\mathbf{a}}(i)) - \mathrm{ct}_T(g_{\mathbf{a}}(i+1)) - 1)(\mathrm{ct}_T(g_{\mathbf{a}}(i)) - \mathrm{ct}_T(g_{\mathbf{a}}(i+1)) + 1)}{(\mathrm{ct}_T(g_{\mathbf{a}}(i)) - \mathrm{ct}_T(g_{\mathbf{a}}(i+1)))^2}$$

and

$$A_2 = \frac{1}{\mathrm{ct}_T(g_{\mathbf{a}}(i)) - \mathrm{ct}_T(g_{\mathbf{a}}(i+1))}.$$

9.2. **Hilbert scheme of the non-reduced line.** On the geometric side, the curve $\{x^m = y^n\}$ has a natural limit at $m \to \infty$, namely, the non-reduced line $\{y^n = 0\}$. The ring of functions on $C_0 = \{y^n = 0\}$ has a basis $x^i y^j$ for $i \geq 0, n-1 \geq j \geq 0$, as above.

The Hilbert scheme of points on $\{y^n = 0\}$ is the moduli space of ideals in the local ring

$$\mathcal{O}_{C_0,0} = \mathbb{C}[[x,y]]/y^n = \mathbb{C}[[x]]\langle 1, \ldots, y^{n-1}\rangle.$$

The multiplication by y is given by the matrix similar to (30):

$$Y = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

We consider the $\mathbb{C}^*$ action on C_0 and on $\mathcal{O}_{C_0,0}$ such that y has weight 1 and x has weight 0. It naturally extends to the action on the punctual Hilbert scheme $\mathrm{Hilb}_k(C_0, 0)$.

Lemma 9.5. *The fixed points of this action are isolated and correspond to monomial ideals.*

Proof. An ideal I in $\mathcal{O}_{C_0}$ is fixed under this $\mathbb{C}^*$ action if and only it it is generated by functions $y^{\alpha_i} p_i(x)$ which are homogeneous in y but not necessary in x. On the other hand, in the ring of formal power series $p_i(x)$ is proportional to x^{c_i} up to a unit, and hence I is the monomial ideal generated by $y^{\alpha_i} x^{c_i}$ for $1 \leq i \leq n$. □

Remark 9.6. It is important for the above proof that we work with the punctual Hilbert scheme of ideals supported at the origin, rather than with the full Hilbert scheme.

Remark 9.7. Unlike the curve $\{x^m = y^n\}$, the curve C_0 has an action of another $\mathbb{C}^*$ such that y has weight 0 and x has weight 1. The weight of this action on a monomial ideal I generated by the $y^{\alpha_i} x^{c_i}$ equals $\sum c_i = \dim \mathcal{O}_C/I = k$.

Similarly, one can define the parabolic Hilbert scheme $\mathrm{PHilb}_{k,n+k}(C_0)$ as the space of flags of ideals $I_k \supset I_{k+1} \supset \cdots \supset xI_k = I_{k+n}$ in $\mathcal{O}_{C_0}$, and $\mathrm{PHilb}^x(C_0) := \sqcup_k \mathrm{PHilb}_{k,n+k}(C_0)$. The fixed points in $\mathrm{PHilb}^x(C_0)$ are determined by sequences of monomials $(y^{\alpha_i} x^{c_i})$ with no restrictions on c_i. As in Lemma 7.5, we have $\alpha_i = \widetilde{g}_{\mathbf{c}}(i) - 1$, where $\widetilde{g}_{\mathbf{c}}$ is the permutation which sorts c_i in non-increasing order (recall that when $c_i = c_j$ with $i < j$ we have $\alpha_i < \alpha_j$) . We can write $c_i = a_{n+1-i}$ and $\widetilde{g}_{\mathbf{c}}(i) = n + 1 - g_{\mathbf{a}}(n+1-i)$.

The construction of geometric operators corresponding to u_i, s_i, τ and λ extends verbatim to this case, however, one needs to be careful with the equivariant weights. Now $\mathcal{L}_i$ has the weight of the monomial $(y^{\alpha_i} x^{c_i})$, that is

$$c_1(\mathcal{L}_i) = \alpha_i = \widetilde{g}_{\mathbf{c}}(i) - 1 = n - g_{\mathbf{a}}(n+1-i) = (n-1) + \mathtt{w}_{n+1-i}.$$

The operators T and Λ can be defined as in Section 7.3, and their matrix elements can be computed similarly. Observe that $\mathcal{O}_{C_0}/x\mathcal{O}_{C_0}$ still has a unique Y-invariant one dimensional subspace generated by y^{n-1} which has weight $(n-1)$. The computation in Theorem 7.14 then implies $T \circ \Lambda = u_1$. We conclude the following:

Theorem 9.8. *Consider the non-reduced curve $C_0 = \{y^n = 0\}$ with the $\mathbb{C}^*$ action $(x,y) \mapsto (x, sy)$. Then the $\mathbb{C}^*$ equivariant cohomology*

$$U_\infty = \bigoplus_{k=0}^{\infty} H_*^{\mathbb{C}^*}(\mathrm{PHilb}_{k,n+k}(C_0))$$

has an action of the rational Cherednik algebra $H_{0,1}$ defined by the same operators in Theorem 7.14. This representation is isomorphic to the polynomial representation of $H_{0,1}$.

Finally, we would like to mention that the constructions of Section 8 can be extended to this setup, and we get the following result.

Theorem 9.9. *With the same notation as in Theorem 9.8 the $\mathbb{C}^*$-equivariant cohomology*

$$H_*^{\mathbb{C}^*}(\mathrm{CPHilb}^{r,x}(C_0))$$

has an action of the spherical algebra $eH_{0,1}(n,r)e$, where $H_{0,1}(n,r)$ is the matrix version of the Cherednik algebra defined in Definition 8.3. This representation is isomorphic to the representation $(\mathbb{C}[x_1,\dots,x_n]\otimes(\mathbb{C}^r)^{\otimes n})^{\mathcal{S}_n}$ defined in a natural way.

Remark 9.10. From its interpretation as a generalized affine Springer fiber, see Section 8.4, it follows that the homology $H_*^{\mathbb{C}^*}(\mathrm{CPHilb}^{r,x}(C_0))$ admits an action of a flavor deformation of the algebra of functions on the Gieseker variety $\mathcal{M}(n,r)$. When $r=1$, this flavor deformation is precisely $eH_{0,1}(n,1)e$, which is known to be commutative and it is in fact the algebra of functions on the Calogero-Moser space, [13]. It is unclear the relationship that the flavor deformation bears to $eH_{0,1}(n,r)e$ when $r>1$.

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