Global structure of Witten's 2+1 gravity on $\mathbf{R} \times T^2$

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Abstract

We investigate the space $\mathcal{M}$ of classical solutions to Witten's formulation of 2+1 gravity on the manifold $\mathbf{R} \times T^2$. $\mathcal{M}$ is connected, but neither Hausdorff nor a manifold. However, removing from $\mathcal{M}$ a set of measure zero yields a connected manifold which is naturally viewed as the cotangent bundle over a non-Hausdorff base space. Avenues towards quantizing the theory are discussed in view of the relation between spacetime metrics and the various parts of $\mathcal{M}$.

1 Introduction

The observation that vacuum Einstein gravity in 2+1 spacetime dimensions has no local dynamical degrees of freedom has stimulated interest in 2+1 gravity as an arena where quantum gravity can be investigated without many of the technical complications that are present in 3+1 spacetime dimensions. Of particular interest for the 3+1 theory is the relationship between the various 2+1 quantum theories that have been constructed in the metric, connection, and loop formulations. For a recent review, see [1].

In this contribution we shall consider Witten's formulation of 2+1 gravity [2, 3] and its relation to the conventional metric formulation. On manifolds of the form $\mathbf{R} \times \Sigma$, where Σ is a closed orientable surface of genus $g > 1$, the situation is well understood: the space of classical solutions to Witten's theory contains several disconnected components, one of which is a smooth manifold isomorphic to the solution space of the conventional metric formulation [3, 4, 5]. On the manifold $\mathbf{R} \times S^2$ the situation is trivial, in the sense that Witten's theory possesses only one classical solution and the conventional metric formulation possesses no solutions [4, 5]. Our aim is to describe the solution space to Witten's theory on the manifold $\mathbf{R} \times T^2$, and to explore the avenues that the global structure of this solution space offers for quantizing the theory. The details and more references to earlier work can be found in [6].

2 Outline of results

Recall that Witten's formulation [2, 3] of 2+1 gravity on an orientable manifold M can be derived from the action $S(\bar{e}, \bar{A}) = \int_M \text{Tr}(\bar{e} \wedge \bar{F})$. Here $\bar{e}$ is a co-triad taking values in the dual of the Lie algebra of $\text{SO}(2,1)$, $\bar{F}$ is the curvature of the $\text{SO}(2,1)$ connection $\bar{A}$, and the trace refers to a contraction in the Lie algebra indices. For $M = \mathbf{R} \times \Sigma$, where Σ is a

closed orientable surface, the pull-backs of $\bar{e}$ and $\bar{A}$ to Σ define an ISO(2, 1) connection $\mathcal{A}$ on Σ. Witten [3] observed that the equations of motion enforce $\mathcal{A}$ to be flat, and further that the dynamics consists entirely of gauge transformations of $\mathcal{A}$. Choosing $\mathcal{A}$ to live on the trivial principal bundle $\text{ISO}_0(2, 1) \times \Sigma$, where $\text{ISO}_0(2, 1)$ is the connected component of ISO(2, 1), one thus sees that the space of classical solutions is just the moduli space of flat $\text{ISO}_0(2, 1)$ connections on Σ, modulo $\text{ISO}_0(2, 1)$ gauge transformations. This space can be described as the space of group homomorphisms from the fundamental group of Σ to $\text{ISO}_0(2, 1)$, modulo overall conjugation by $\text{ISO}_0(2, 1)$ [1].

We now focus on the case where Σ is the torus T^2. Let $\mathcal{M}$ denote the space of classical solutions to Witten's theory. As the fundamental group of the torus is the abelian group $\mathbf{Z} \times \mathbf{Z}$, we see from the above that the points in $\mathcal{M}$ are just equivalence classes of pairs of commuting elements of $\text{ISO}_0(2, 1)$ modulo $\text{ISO}_0(2, 1)$ conjugation. We need to give a characterization of such equivalence classes.

Suppose for the moment that we were considering Euclidean rather than Lorentzian gravity. In this case $\text{ISO}_0(2, 1)$ would be replaced by ISO(3), which is the group of rigid body motions in three dimensional Euclidean space. Now, a classic result known as Euler's theorem says that an element of ISO(3) can always be written as a rotation about some axis followed by a translation along the same axis. If two elements of ISO(3), not both purely translational, are written in this fashion and then required to commute, one finds that the respective axes of rotation must be the same axis, and by conjugation this axis can be chosen to be (say) the z-axis. One thus recovers a space whose points are parametrized by the two rotation angles and the magnitudes of the two translations, modulo certain identifications which stem from further conjugation by a rotation by π about the x-axis. The special case where both ISO(3) elements are purely translational requires a separate consideration; such points are parametrized by the magnitudes of the two translations and the angle between them, again modulo certain identifications. The space is thus roughly speaking four dimensional, but not quite a manifold.

Return now to Lorentzian gravity. The crucial difference between ISO(3) and $\text{ISO}_0(2, 1)$ for us is, of course, that there are (apart from the identity rotation) three distinct types of Lorentz rotations in $\text{ISO}_0(2, 1)$: the boosts, which fix a spacelike axis, the rotations, which fix a timelike axis, and the null rotations, which fix a null axis. For rotations and boosts there are natural analogues of Euler's theorem, and the analysis proceeds fairly similarly to that in the Euclidean case. The rotational part $\mathcal{M}_t$ of $\mathcal{M}$ can be understood as the cotangent bundle over the punctured torus: the base space arises from the Lorentz components and the cotangent fibers arise from the translational components of the $\text{ISO}_0(2, 1)$ elements. The symplectic structure that allows the interpretation of this space as a cotangent bundle arises from the Hamiltonian decomposition of the action. The boost part $\mathcal{M}_s$ of $\mathcal{M}$ can be similarly understood as the cotangent bundle over the punctured plane. For null rotations, however, there is no direct analogue of Euler's theorem. The null part $\mathcal{M}_n$ of $\mathcal{M}$ turns out to be a three dimensional manifold with topology $S^1 \times \mathbf{R}^2$, the factor S^1 coming from the Lorentz components and the factor $\mathbf{R}^2$ coming from the translational components of the $\text{ISO}_0(2, 1)$ elements. The remaining part $\mathcal{M}_0$ of $\mathcal{M}$ consists of the case where both $\text{ISO}_0(2, 1)$ elements are purely translational. $\mathcal{M}_0$ is close to being a three dimensional manifold, but its structure is complicated by the different possibilities for the spacelike/timelike/null character of the plane or line (or point) which the two translation vectors span.

A first observation is that $\mathcal{M}$ is a *connected* space. This is in a striking contrast with the situation in Witten's theory on manifolds where the torus is replace by a higher genus

surface [3]. Indeed, one can view $\mathcal{M}$ as the two four dimensional manifolds $\mathcal{M}_t$ and $\mathcal{M}_s$ glued together by the three dimensional manifold $\mathcal{M}_n$ and the set $\mathcal{M}_0$. The gluing is not smooth, and $\mathcal{M}$ itself is neither Hausdorff nor a manifold. However, the connected set $\mathcal{M} \setminus \mathcal{M}_0 = \mathcal{M}_t \cup \mathcal{M}_n \cup \mathcal{M}_s$, which contains all of $\mathcal{M}$ except a set of measure zero, *is* a manifold: it can be viewed as the cotangent bundle whose base space consists of the base space of $\mathcal{M}_s$ (punctured plane) and the base space of $\mathcal{M}_t$ (punctured torus) glued together at the punctures; the circle which provides the glue is the S^1 factor of $\mathcal{M}_n$. The circle joins to the base space of $\mathcal{M}_t$ in a one-to-one fashion, but the joining of the base space of $\mathcal{M}_s$ to the circle is two-to-one. This makes $\mathcal{M} \setminus \mathcal{M}_0$ a non-Hausdorff manifold.

The above structure of $\mathcal{M}$ suggests various avenues for quantizing the theory. One possibility is to quantize $\mathcal{M}_s$ and $\mathcal{M}_t$ separately; this leads to the theories considered in [7, 8, 9]. However, it is also possible to perform a quantization on all of $\mathcal{M} \setminus \mathcal{M}_0$ at once. The resulting larger theory contains the theories of [7, 8, 9] as its parts, and in particular it contains operators that induce transitions between these smaller theories.

3   Discussion

Is there any physical interest in the large quantum theory constructed on all of $\mathcal{M} \setminus \mathcal{M}_0$? We would like to end by speculating that this question may be related to the role of closed timelike loops in quantum gravity.

Recall that in the conventional metric formulation of 2+1 gravity one assumes that the spacetime metric is nondegenerate, and in the Hamiltonian decomposition on the manifold $\mathbf{R} \times T^2$ one further assumes that the induced metric on T^2 is spacelike [4]. When such spacetimes are mapped to Witten's description, the image $\mathcal{M}_{\text{metric}}$ lies in $\mathcal{M}_s \cup \mathcal{M}_0$, filling most of $\mathcal{M}_s$ but only roughly half of $\mathcal{M}_0$ [4, 5]. However, it can be shown [6] that for any point in $\mathcal{M}$, with the exception of a set of measure zero, there exist corresponding nondegenerate spacetime metrics on $\mathbf{R} \times T^2$. For points in $\mathcal{M}$ that are not in $\mathcal{M}_{\text{metric}}$, such a nondegenerate spacetime will necessarily contain closed timelike or null loops. This suggests viewing the large quantum theory described above as a theory containing, in some rough sense, transitions between spacetimes with closed causal loops and spacetimes without closed causal loops. It would be of interest to understand whether this speculation could be augmented into a more concrete statement.

References

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