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# ON THE DUALITY CONDITION FOR A HERMITIAN SCALAR FIELD



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# On the duality condition for a Hermitian scalar field



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A general Hermitian scalar field, assumed to be an operator-valued tempered distribution, is considered. A theorem which relates certain complex Lorentz transformations to the $TCP$ transformation is stated and proved. With reference to this theorem, duality conditions are considered, and it is shown that such conditions hold under various physically reasonable assumptions about the field. A theorem analogous to Borchers' theorem on relatively local fields is stated and proved. Local internal symmetries are discussed, and it is shown that any such symmetry commutes with the Poincaré group and with the $TCP$ transformation.

## I. INTRODUCTION AND OUTLINE

The so-called duality condition in quantum field theory and in the theory of algebras of local observables has been discussed by many authors.[1-8] From these studies it appears that it would be a desirable, if not essential, feature of a local theory that such a condition holds. *Very* roughly stated the duality condition for a region $R$ in spacetime says that the set of all operators which commute with all operators locally associated with $R$ is equal to the set of all operators locally associated with the causal complement of $R$. It was first shown by Araki[2] that conditions of this nature do hold for a class of suitably restricted regions $R$ in the case of a free Hermitian scalar field. It is the purpose of this paper to discuss the duality condition in quantum field theory in the general case, i.e., without making the assumption that the field is free.

Our considerations are within the framework of conventional quantum field theory, as formulated by Wightman and others.[9-11] We shall restrict our discussion to the case of a single local Hermitian scalar field, assumed to be an operator-valued tempered distribution. We will state the assumptions in some detail in Sec. II, in which we also explain the notation to be followed. Our discussion can readily be extended to more general cases, but, in order to avoid complications which might obscure the main line of argument, we present our ideas in what appears to us to be the simplest possible setting.

In Sec. III we consider some implications of the "spectral condition", i.e., the assumption that the spectrum of the 4-momentum operator $P$ associated with the translation subgroup of the Poincaré group is contained in the closed forward light cone. We here review some facts, by and large well known, which will be of interest in the subsequent discussion, and we consider a slightly modified version of a well-known theorem of Reeh and Schlieder.[12]

In Sec. IV we consider complex Lorentz transformations, and a connection between these and the antiunitary inversion transformation (TCP-operation). Since the Hilbert space of physical states carries a strongly continuous unitary representation of the Poincaré group, it follows that there exist dense sets of analytic vectors of the associated Lie algebra and of sub-Lie algebras of this Lie algebra. It is a characteristic feature of quantum field theory that such sets of analytic vectors can be constructed "naturally" in terms of suitable multilinear expressions in the fields and the vacuum state vector $\Omega$. We shall in particular consider the following issue. Let $W_R$ be the wedge-shaped region $W_R=\{x\,|\,x^3>|x^4|\}$ in Minkowski space, and let $\mathcal{P}_0(W_R)$ be the polynomial algebra generated by field operators averaged with test functions with support in $W_R$. Let $V(\mathbf{e}_3,t)$, $t$ real, denote the velocity transformation in the Poincaré group whose action on Minkowski space is described by the four × four matrix

$$V(\mathbf{e}_3,t)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&\cosh(t)&\sinh(t)\\0&0&\sinh(t)&\cosh(t)\end{bmatrix}. \qquad (1)$$

The set of all $V(\mathbf{e}_3,t)$ is thus a one-parameter Abelian group of velocity transformations in the 3-direction which maps the wedge region $W_R$ onto itself. To the element $V(\mathbf{e}_3,t)$ corresponds the unitary operator $U(V(\mathbf{e}_3,t),0)=\exp(-itK_3)$ on the Hilbert space, where $K_3$ is an (unbounded) self-adjoint operator. We shall show that every vector $X\Omega$, with $X\in\mathcal{P}_0(W_R)$, is in the domain of the normal operators $\exp(-izK_3)$ for the complex variable $z$ in the closed strip $\pi\geq \mathrm{Im}(z)\geq 0$. The vector-valued function $\exp(-izK_3)X\Omega$ is a strongly continuous function of $z$ on the above closed strip, and an analytic function of $z$ on the (open) interior of the strip. We shall furthermore show that for any such vector

$$\exp(\pi K_3)X\Omega=JX^*\Omega \qquad (2)$$

where $J$ is the antiunitary involution defined by

$$J=U(R(\mathbf{e}_3,\pi),0)\Theta_0 \qquad (3)$$

where $R(\mathbf{e}_3,\pi)$ is the rotation by angle $\pi$ about the 3-axis [and $U(R(\mathbf{e}_3,\pi),0)$ the corresponding unitary operator on the Hilbert space], and where $\Theta_0$ is the TCP-operator.

The relation (2) is the main result of Sec. IV. It holds, in fact, for a somewhat larger class of field operators, as stated precisely in Theorem 1.

  

Section V is devoted to a discussion of some mathematical questions relating to (2). We consider families of operators which satisfy the relation (2), and, in particular, we discuss the properties of any von Neumann algebra $\mathcal{A}_R$, of bounded operators $X$ which satisfy (2), and such that furthermore $J\mathcal{A}_R J = \mathcal{A}'_R$, where $\mathcal{A}'_R$ denotes the commutant of $\mathcal{A}_R$. The main results, relative to the subsequent discussion in Secs. VI and VII, are stated in Theorem 2 and Lemma 15. Our discussion is closely related to a theory of Tomita[13] on the structure of von Neumann algebras (and of modular Hilbert algebras), and we discuss the connection.

In Sec. VI we discuss a *particular* duality condition, for the wedge region $W_R$. Let $W_L$ be the causal complement of $\overline{W}_R$, i.e., the wedge region $W_L = \{x \mid x^3 < -|x^4|\}$, and let $\mathcal{P}_0(W_L)$ be the polynomial algebra generated by field operators averaged with test functions with support in $W_L$. We consider four particular conditions on the quantum field under which the polynomial algebras $\mathcal{P}_0(W_R)$, respectively $\mathcal{P}_0(W_L)$, of *unbounded* operators define von Neumann algebras $\mathcal{A}(W_R)$, respectively $\mathcal{A}(W_L)$, of *bounded* operators which can be regarded as locally associated with the wedge regions $W_R$ and $W_L$, and we prove that these von Neumann algebras satisfy the *duality condition* $\mathcal{A}(W_R)' = \mathcal{A}(W_L)$. We also show that the TCP-symmetry of the field carries over to the system of bounded local operators in the sense that $J\mathcal{A}(W_R)J = \mathcal{A}(W_L)$. These results are formulated in Theorems 3 and 4.

Theorem 3 includes in particular the following result, which holds *generally*, i.e., without any additional assumption about the quantum field beyond the minimum assumptions discussed in Sec. II. If $X$ is a bounded operator which commutes with all (linear) field operators averaged with test functions with support in $W_L$, and if $Y$ is a bounded operator which commutes with all field operators averaged with test functions with support in $W_R$, then $X$ commutes with $Y$. This statement is analogous to a well-known theorem of Borchers on the local nature of fields which are local relative to a local irreducible field.[14]

We have not solved the problem of whether the von Neumann algebras (of bounded operators) associated with wedge regions, or other regions, always exist, and we are thus forced to make additional assumptions, which, however, are not unreasonable physically. This question appears to be intimately related to the hitherto unsolved problem of whether a sufficiently large set of quantum field operators have *local* self-adjoint extensions (within the framework of the customary minimal assumptions of quantum field theory). We discuss the notion of a local self-adjoint extension of the field, and we show that it implies the existence of a system of local von Neumann algebras which satisfies the duality condition. We also show that the existence of such a system follows from other conditions which appear to be less restrictive than the condition that the field has a local self-adjoint extension.

In Sec. VII we discuss the duality condition for a particular set of bounded regions, namely the set of all so-called double cones. The von Neumann algebras associated with the bounded regions are constructed from the von Neumann algebras associated with the wedge regions. We describe the properties of these algebras in Theorems 5 and 6, and we show that the duality condition for the algebras associated with the wedge regions implies an appropriate duality condition for the algebras associated with double cones.

Finally, we consider the notion of a local internal symmetry, and we prove (Theorem 7) that if the duality condition holds for the wedge algebras, then every local internal symmetry commutes with the Poincaré group, and with the TCP-transformation.

## II. BASIC ASSUMPTIONS; DISCUSSION OF NOTATION

Minkowski space $\mathcal{M}$ is parametrized by the customary Cartesian coordinates $x = (x^1, x^2, x^3, x^4)$. The Lorentz "metric" is so defined that $x \cdot y = x^4y^4 - x^1y^1 - x^2y^2 - x^3y^3$. The elements $\Lambda = \Lambda(M, y)$ of the *proper* Poincaré group $\overline{L}_0$ are parametrized by a four-by-four Lorentz matrix $M$, and a real 4-vector $y$, such that the image $\Lambda x$ of a point $x \in \mathcal{M}$ under any $\Lambda \in \overline{L}_0$ is given by $\Lambda x = \Lambda(M, y)x = Mx + y$.

The Hilbert space $\mathcal{H}$ of physical states is assumed to be *separable*. It is assumed to carry a strongly continuous unitary representation $\Lambda \to U(\Lambda)$ of the Poincaré group $\overline{L}_0$. We write $U(\Lambda(M,x)) = U(M,x)$, and we employ the special notation $T(x) = U(I,x)$ for the representatives of the translation subgroup. The translations have the common spectral resolution

$$T(x) = U(I,x) = \int \exp(ix \cdot p)\, \mu(d^4p) \tag{4}$$

and it is assumed that the support of the spectral measure $\mu$ is contained in the closed forward light cone $\overline{V}_+$ (in momentum space). This assumption about the support of $\mu$ will be referred to as the "spectral condition" in what follows.

We assume the existence of a vacuum state, represented by the unit vector $\Omega$, *uniquely* characterized by its invariance under all Poincaré translations: thus $U(\Lambda)\,\Omega = \Omega$.

We denote by $\mathcal{D}(R^n)$ the set of all complex-valued infinitely differentiable function of compact support on $n$-dimensional Euclidean space $R^n$, and we denote by $\mathcal{S}(R^n)$ the space of test functions on $R^n$ in terms of which tempered distributions are defined. The space $\mathcal{S}(R^n)$ is regarded as endowed with the particular topology appropriate to the definition of tempered distributions,[15] and we employ the notation

$$\mathcal{S}\text{-}\lim_{\alpha\to\infty} f_\alpha = 0 \tag{5}$$

to state that a sequence of test functions $f_\alpha$ converges to zero relative to this topology. We shall be concerned with test functions on $R^{4n}$, where $R^{4n}$ is regarded as the direct sum of an ordered $n$-tuplet of replicas of Minkowski space, and the points of $R^{4n}$ are accordingly parametrized by an ordered $n$-tuplet $(x_1, x_2, \ldots, x_n)$ of 4-vectors $x_k$. A *specific* interpretation of $R^{4n}$ in this manner is always understood, as reflected in the above parametrization of the space. In accordance with the above we define an action of $\overline{L}_0$ on $\mathcal{S}(R^{4n})$ by

$$f(x_1, \ldots, x_n) \to \Lambda f(x_1, \ldots, x_n) = f(\Lambda^{-1}x_1, \ldots, \Lambda^{-1}x_n). \tag{6}$$

This mapping is continuous relative to the test function space topology, and

$$\mathcal{S}\text{-}\lim_{\Lambda\to I} \Lambda f = f. \tag{7}$$

Throughout this paper it will be important to keep track of the domains of unbounded operators. To deal effectively with such issues we shall frequently employ the unorthodox notation $(X, D)$ for an operator $X$ defined on a domain $D$. The adjoint of $(X, D)$ is denoted $(X, D)^*$ and if $D(X^*)$ is the domain of the adjoint we can write $(X, D)^* = (X^*, D(X^*))$. If $(X, D)$ is closable we write $(X, D)^{**} = (X^{**}, D(X^{**}))$ for the closure. This notation is never employed for manifestly bounded operators, which are regarded as defined on the entire Hilbert space.

We shall consider a theory of a single local Hermitian scalar field $\varphi(x)$, assumed to be an operator-valued tempered distribution.[9–11,16] Such a theory is characterized by the following features:

(a) There exists a linear manifold $D_1$, *dense* in the Hilbert space $\mathcal{H}$, and an algebra $\mathcal{P}(\mathcal{M})$ of operators $(X, D_1)$ defined on $D_1$. The domain $D_1$ contains the vacuum state vector $\Omega$. For each $n \geq 1$ there exists a linear mapping of $\mathcal{S}(R^{4n})$ into $\mathcal{P}(\mathcal{M})$. The image of any $f \in \mathcal{S}(R^{4n})$ under this mapping is denoted $\varphi\{f\}$. We note here that $\varphi\{f\}$ is the operator which is customarily defined symbolically by the integral at right in

$$\varphi\{f\} = \int_{(\infty)} d^4(x_1) \cdots d^4(x_n) f(x_1, \ldots, x_n)\, \varphi(x_1) \cdots \varphi(x_n). \tag{8}$$

The domain $D_1$ is *precisely* equal to $\mathcal{P}(\mathcal{M})\Omega$, and the algebra $\mathcal{P}(\mathcal{M})$ is *precisely* equal to the linear span of the identity operator I and the set of all operators $\varphi\{f\}$. If $f \in \mathcal{S}(R^{4n})$ and $g \in \mathcal{S}(R^{4m})$, and if $h \in \mathcal{S}(R^{4n+4m})$ is given by

$$h(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) = f(x_1, \ldots, x_n)\, g(x_{n+1}, \ldots, x_{n+m}), \tag{9}$$

then

$$\varphi\{f\}\varphi\{g\} = \varphi\{h\} \quad \text{on } D_1. \tag{10}$$

We note that this is consistent with the symbolic definition in (8).

(b) Let $(X, D_1) \to (X^\dagger, D_1)$ denote the *antilinear* involutory mapping of $\mathcal{P}(\mathcal{M})$ onto itself uniquely determined by

$$\mathrm{I}^\dagger = \mathrm{I}, \quad \varphi\{f\}^\dagger = \varphi\{f^\dagger\}, \tag{11}$$

where

$$f^\dagger(x_1, x_2, \ldots, x_n) = f^*(x_n, \ldots, x_2, x_1) \tag{12}$$

for any $f \in \mathcal{S}(R^{4n})$.

The domain $D_1$ is contained in the domain of the adjoint $(X, D_1)^*$ of every $(X, D_1) \in \mathcal{P}(\mathcal{M})$, and

$$(X^\dagger, D_1) = (X^*, D_1) \subset (X, D_1)^*. \tag{13a}$$

In particular,

$$(\varphi\{f^\dagger\}, D_1) \subset (\varphi\{f\}, D_1)^*. \tag{13b}$$

Every operator $(X, D_1) \in \mathcal{P}(\mathcal{M})$ is thus closable, and $(X^\dagger, D_1)$ is the Hermitian conjugate of $(X, D_1)$.

(c) The domain $D_1$ is invariant under the Poincaré group: $U(\Lambda) D_1 = D_1$ for all $\Lambda \in \overline{L}_0$. The action of $\overline{L}_0$ by conjugation on $\mathcal{P}(\mathcal{M})$ (and hence the action of $\overline{L}_0$ of the Hilbert space $\mathcal{H}$) is uniquely determined by the condition

$$U(\Lambda)(\varphi\{f\}, D_1)\, U(\Lambda)^{-1} = (\varphi\{\Lambda f\}, D_1) \tag{14}$$

(d) The mapping $f \to \varphi\{f\}$ is such that if $\{f_\alpha \mid f_\alpha \in \mathcal{S}(R^{4n}), \alpha = 1, \ldots, \infty\}$ is any sequence of test functions which tends to zero in the sense of the test function space topology, i. e., such that (5) holds, then

$$\text{s-}\lim_{\alpha\to\infty} X\varphi\{f_\alpha\}\psi = 0 \tag{15}$$

for any $(X, D_1) \in \mathcal{P}(\mathcal{M})$ and any $\psi \in D_1$.

(e) Let $R$ be any open subset of Minkowski space. Let $\mathcal{P}(R)$ denote the linear span of the identity operator I and all operators $(\varphi\{f\}, D_1)$, where $f \in \mathcal{S}(R^{4n})$ for some $n \geq 1$ and such that $\text{supp}(f) \subset \{(x_1, \ldots, x_n) \mid x_k \in R, k = 1, \ldots, n\}$.

Then, if $R_1$ and $R_2$ are any two open subsets of Minkowski space which are spacelike separated [i. e., $(x - y)\cdot(x - y) < 0$ for any $x \in R_1$, $y \in R_2$], we have

$$[X, Y]\psi = 0, \quad \text{all } \psi \in D_1, \tag{16}$$

for all $X \in \mathcal{P}(R_1)$ and all $Y \in \mathcal{P}(R_2)$.

Our purpose with the preceding account was to state precisely what we assume, and not to formulate a minimal set of postulates for field theory. It will be noted that the conditions which we have stated are in fact not all logically independent of each other. It should also be noted that we do not assume anything beyond what is implied by the usual *minimal* assumptions for quantum field theory.

Since operators linear in the field will be of particular interest, we employ a special notation for the case $f \in \mathcal{S}(R^4)$, namely,

$$\varphi[f] = \varphi\{f\} = \int_{(\infty)} d^4(x) f(x)\, \varphi(x). \tag{17}$$

For any open subset $R$ of Minkowski space we denote by $\mathcal{P}_0(R)$ the polynomial algebra generated by the identity I, and all operators $(\varphi[f], D_1)$ such that $\text{supp}(f) \subset R$. With reference to the definition of the algebra $\mathcal{P}(R)$ in (e) above, we then have $\mathcal{P}_0(R) \subset \mathcal{P}(R) \subset \mathcal{P}(\mathcal{M})$. We state some well-known properties of these algebras as follows.

*Lemma* 1: (a) (Theorem of Reeh and Schlieder[12]) Let $R$ be any open, nonempty subset of Minkowski space $\mathcal{M}$. Then $\mathcal{P}_0(R)\Omega$ is dense in the Hilbert space $\mathcal{H}$.

(b) Let $(X, D_1) \in \mathcal{P}(R)$. Then there exists a sequence of operators $\{(X_\alpha, D_1) \mid (X_\alpha, D_1) \in \mathcal{P}_0(R), \alpha = 1, \ldots, \infty\}$ such that

$$\text{s-}\lim_{\alpha\to\infty} YX_\alpha\psi = YX\psi \tag{18}$$

for every $Y \in \mathcal{P}(\mathcal{M})$ and every $\psi \in D_1$.

(c) The linear manifold $D_0 \subset D_1$ defined as $D_0 = \mathcal{P}_0(\mathcal{M})\Omega$ is dense in the Hilbert space, and

$$(X, D_0)^* = (X, D_1)^*, \quad (X, D_0)^{**} = (X, D_1)^{**} \tag{19}$$

for every $(X, D_1) \in \mathcal{P}(\mathcal{M})$.

The above is of interest with reference to other approaches to field theory, in which the initial object of interest is $\varphi[f]$, defined on $D_0$, and where the commutation relation (16) is at first assumed only for operators $X$ and $Y$ of this special form. After the appropriate extensions and constructions one arrives at the equivalent of our formulation. We preferred to introduce the domain $D_1$ immediately, and to regard all field operators as defined on precisely $D_1$. The symbols $X^*$, $X^{**}$, and $X^\dagger$, for $(X, D_1) \in \mathcal{P}(\mathcal{M})$, thus refer to the adjoint, closure and Hermitian conjugate defined relative to this domain.

Whereas the domains $D_0$ and $D_1$ are Poincaré invariant, this is, of course, in general not the case for the domain $D(X^*)$ of $(X, D_1)^*$ and the domain $D(X^{**})$ of $(X, D_1)^{**}$. We have the relations

$$(U(\Lambda) X U(\Lambda)^{-1}, D_1)^* = (U(\Lambda) X^* U(\Lambda)^{-1},\ U(\Lambda) D(X^*)) \tag{20a}$$

$$(U(\Lambda) X U(\Lambda)^{-1}, D_1)^{**} = (U(\Lambda) X^{**} U(\Lambda)^{-1}, U(\Lambda) D(X^{**})). \tag{20b}$$

We finally note that it trivially follows from (13a) that

$$(X^\dagger, D_1)^{**} = (X^{\dagger **}, D(X^{\dagger **})) \subset (X, D_1)^* = (X^*, D(X^*)). \tag{21}$$

For a particular operator $(X, D_1)$ equality obtains in (21) above if and only if $D_1$ is a core for $(X, D_1)^*$. [For a Hermitian operator this means that $(X, D_1)$ is essentially self-adjoint.] In general discussions of field theory no assumption is made about the possible existence of a set of field operators for which (21) might hold as an equality.

## III. ABOUT SOME CONSEQUENCES OF THE SPECTRAL CONDITION

It is well-known that the unitary representation $x \to T(x)$ of the translation group can be extended to a representation of the semigroup of all complex translations $z = x + iy$, with $x$ and $y$ real, $y \in \bar{V}_+$, by

$$T(z) = \int \exp(iz \cdot p) \mu(d^4p) = \exp(iz \cdot P) \tag{22}$$

where the operator-valued function $T(z)$ satisfies $\|T(z)\| = 1$ and is a strongly continuous function of $z$ on the closed forward imaginary tube $\bar{V}_{+i} = \{z \mid \mathrm{Im}(z) \in \bar{V}_+\}$. Furthermore, the function $T(z)$ is analytic in the sense of the uniform topology on the open forward imaginary tube $V_{+i}$, which implies in particular that the vector-valued function $T(z)\psi$ of $z$ is strongly analytic on $V_{+i}$ for any $\psi \in \mathcal{H}$.

Let $f \in \mathcal{S}(R^{4n})$. We define a Fourier transform $\tilde{f}$ of $f$ by

$$\tilde{f}(p_1, \ldots, p_n) = \int_{(\infty)} d^4(x_1) \cdots d^4(x_n) f(x_1, \ldots, x_n) \exp\left(i \sum_{r=1}^n x_r \cdot p_r\right). \tag{23}$$

We consider the following:

*Lemma* 2: Let $z \in \bar{V}_{+i}$, i.e., $z$ is any complex 4-vector in the closed forward imaginary tube. Then

$$T(z) D_1 \subset D_1. \tag{24}$$

If $f \in \mathcal{S}(R^{4n})$ there exists an $f_z \in \mathcal{S}(R^{4n})$ such that

$$\tilde{f}_z(p_1, \ldots, p_n) = \tilde{f}(p_1, \ldots, p_n) \exp\left(iz \cdot \sum_{r=1}^n p_r\right) \tag{25a}$$

for $(p_1, \ldots, p_n) \in V_n$, where $V_n$ is the subset of $R^{4n}$ defined by

$$V_n = \left\{(p_1, \ldots, p_n) \,\Big|\, \sum_{r=k}^n p_r \in \bar{V}_+,\ k = 1, \ldots, n\right\} \tag{25b}$$

and for every such $f_z$ we have

$$T(z)\varphi\{f\}\Omega = \varphi\{f_z\}\Omega. \tag{25c}$$

The above facts are well known, and we refer to the monograph by Jost[17] for a discussion of these and related issues. Here we only note the following. It is a consequence of the spectral condition that any vector $\varphi\{f\}\Omega$ only depends on the restriction of $\tilde{f}$ to the set $V_n$ defined in (25b), i.e., if $\tilde{f} = 0$ on $V_n$, then the vector vanishes. It is of interest to exhibit a particular function $f_z$ which satisfies (25a), and hence (25c). Let $u_0(t)$ be an infinitely differentiable function of $t$ on $R^1$ such that $u_0(t) = 1$ for $t \geq 0$ and $u_0(t) = 0$ for $t \leq -1$. We define a function $E(p; z)$ of the real 4-vector $p$ and the complex 4-vector $z$ by

$$E(p; z) = u_0(p \cdot p) u_0(p^4) \exp(iz \cdot p). \tag{26}$$

This function satisfies $E(p; z) = \exp(iz \cdot p)$ for $p \in \bar{V}_+$. It is easily seen that for any $z \in V_{+i}$ the function $E(p; z)$, as a function of $p$, is included in $\mathcal{S}(R^4)$. Furthermore, if $f \in \mathcal{S}(R^{4n})$, then the function $f_z$ with the Fourier transform

$$\tilde{f}_z(p_1, \ldots, p_n) = E(p; z) \tilde{f}(p_1, \ldots, p_n), \quad p = \sum_{r=1}^n p_r, \tag{27}$$

is, as a function of $(x_1, \ldots, x_n)$, included in $\mathcal{S}(R^{4n})$ for any $z \in \bar{V}_{+i}$. Now (25a) holds trivially, and it follows that (25c) holds.

The next lemma can be regarded as a generalization of the preceding lemma.

*Lemma* 3: Let $T_n$ be the open tube region in $4n$-dimensional complex space $C^{4n}$, regarded as the direct sum of $n$ replicas of complex Minkowski space, which is defined by

$$T_n = \{(z_1, \ldots, z_n) \mid z_k \in V_{+i},\ k = 1, \ldots, n\}. \tag{28}$$

Let $\{f_k \mid f_k \in \mathcal{S}(R^4),\ k = 1, \ldots, n\}$ be any $n$-tuplet of test functions. Then we have the following:

(a) The vector

$$\beta(z_1, \ldots, z_n) = T(z_1)\varphi[f_1] T(z_2)\varphi[f_2] \cdots T(z_n)\varphi[f_n]\Omega \tag{29}$$

is well defined (through successive left multiplications) for all $(z_1, \ldots, z_n) \in \bar{T}_n$, and

$$\beta(z_1, \ldots, z_n) = \varphi\{f\}\Omega, \tag{30a}$$

where $f = f(x_1, \ldots, x_n; z_1, \ldots, z_n)$ is the function whose Fourier transform with respect to the variables $(x_1, \ldots, x_n)$ is given by

$$\tilde{f}(p_1, \ldots, p_n; z_1, \ldots, z_n) = \prod_{k=1}^n \tilde{f}_k(p_k) E\left(\sum_{r=k}^n p_r; z_k\right) \tag{30b}$$

and where $E(p; z)$ is the function defined in (26).

(b) The vector-valued function $\beta(z_1, \ldots, z_n)$ of $(z_1, \ldots, z_n)$ is strongly continuous on the closed tube $\bar{T}_n$, and analytic on the open tube $T_n$.

*Proof*: (1) The assertions in part (a) follow trivially from Lemma 2, by induction on $n$.

(2) The proof that $\beta$ is strongly continuous on $\overline{T}_n$ requires an examination of the function $\tilde{f}$ given by (30b). We regard this function as a vector-valued function on $\overline{T}_n$, i.e., as a function of $(z_1, \ldots, z_n)$ with range in $\mathcal{S}(R^{4n})$. In view of the simple nature of the function $E(p;z)$, as given by (26), it is now easily shown that $\tilde{f}$ is continuous on $\overline{T}_n$ in the sense of the test function space topology; since this topology is invariant under the Fourier transform, the same holds for $f$, regarded as an $\mathcal{S}(R^{4n})$-valued function on $\overline{T}_n$. It follows, in view of the assumption expressed in (15), that $\beta$ is strongly continuous as asserted.

(3) Since $\beta$ is strongly continuous on $\overline{T}_n$ it follows that $\beta$ is bounded on any closed polydisc contained in $\overline{T}_n$. To show that $\beta$ is analytic on $T_n$ it therefore suffices to show that the function $\langle \eta | \beta(z_1, \ldots, z_n)\rangle$ is analytic in each complex 4-vector $z_k$ separately for each $\eta$ in a dense set of vectors in the Hilbert space. We select $D_1$ as the dense set and we then have, for $k = 1, \ldots, n$, $\langle \eta | \beta(z_1, \ldots, z_n)\rangle = \langle \xi_k | T(z_k)\zeta_k\rangle$, with $\zeta_k$, $\xi_k$ independent of $z_k$. This scalar product is trivially analytic for $z_k \in V_{+i}$, which establishes the second assertion in part (b).

We are specifically interested in vectors of the form shown in (29), but it is worth noting that the lemma has an obvious generalization, in which the operators $\varphi[f_k]$ in (29) are replaced by arbitrary operators $X_k \in \mathcal{P}(\mathcal{M})$.

We next consider an almost trivial extension of the theorem of Reeh and Schlieder,[12] which will be needed later.

*Lemma* 4: Let $\{R_n | n = 1, \ldots, \infty\}$ be any set of open, nonempty subsets of Minkowski space. For such a set, and for any $n \geq 1$, let $S_n$ denote the linear span of all vectors of the form

$$\psi = \varphi[f_1]\varphi[f_2]\cdots\varphi[f_n]\Omega \tag{31}$$

with $f_k \in \mathcal{S}(R^4)$, $\mathrm{supp}(f_k) \subset R_k$, for $k = 1, \ldots, n$.

Then the linear span of the vacuum vector $\Omega$ and the union of all the linear manifolds $S_n$ is dense in the Hilbert space $\mathcal{H}$.

This version differs from the original formulation only in the circumstance that the regions $R_k$ need not all be the same. We feel justified in omitting the proof since it requires only a very minor modification of the proof in the case of equal regions, as presented in the monograph of Streater and Wightman.[18] The lemma can also easily be proved on the basis of Lemma 3.

We next consider an interesting family of vector-valued functions on $T_n$ discussed by Jost.[19]

*Lemma* 5: (a) For each $n \geq 1$, let $E_n$ be the set of all functions $f(x_1, \ldots, x_n; z_1, \ldots, z_n)$ defined for $(x_1, \ldots, x_n) \in R^{4n}$ and $(z_1, \ldots, z_n) \in T_n$, and such that $f \in \mathcal{S}(R^{4n})$ and such that the Fourier transform $\tilde{f}$ of $f$ relative to the variables $(x_1, \ldots, x_n)$ satisfies the condition

$$\tilde{f}(p_1, \ldots, p_n; z_1, \ldots, z_n) = \exp\left(i \sum_{k=1}^{n} \sum_{r=k}^{n} z_k \cdot p_r\right) \tag{32a}$$

for all $(p_1, \ldots, p_n) \in V_n$, with $V_n$ defined as in (25b). The set $E_n$ is nonempty, and it contains in particular the function $f_0$ defined in terms of its Fourier transform by

$$\tilde{f}_0(p_1, \ldots, p_n; z_1, \ldots, z_n) = \prod_{k=1}^{n} E\left(\sum_{r=k}^{n} p_r; z_k\right) \tag{32b}$$

where the function $E(p;z)$ is defined as in (26).

To the set $E_n$ corresponds a *unique* vector-valued function $\phi(z_1, \ldots, z_n)$ on $T_n$, defined by

$$\phi(z_1, \ldots, z_n) = \varphi\{f\}\Omega \tag{32c}$$

where $f$ is any element of $E_n$.

(b) The vector-valued function $\phi(z_1, \ldots, z_n)$ is strongly continuous on $T_n$.

(c) Let $\{f_k | f_k \in \mathcal{D}(R^4),\ k = 1, \ldots, n\}$ be any $n$-tuplet of test functions of compact support. Then, for any $(z_1, \ldots, z_n) \in T_n$,

$$\int_{(\infty)} d^4(x_1)\cdots d^4(x_n) f_1(x_1)f_2(x_2)\cdots f_n(x_n)$$
$$\times \phi(z_1 + x_1, z_2 + x_2 - x_1, z_3 + x_3 - x_2, \ldots, z_n + x_n - x_{n-1})$$
$$= T(z_1)\varphi[f_1]T(z_2)\varphi[f_2]\cdots T(z_n)\varphi[f_n]\Omega \tag{33}$$

where the integral at left exists as a vector-valued Riemann integral relative to the strong topology for $\mathcal{H}$.

*Proof*: (1) The function $f_0$ trivially satisfies (32a). That it is included in $\mathcal{S}(R^{4n})$, as a function of $(x_1, \ldots, x_n)$, for any $(z_1, \ldots, z_n) \in T_n$, follows readily from the fact that $E(p;z) \in \mathcal{S}(R^4)$, for any $z \in V_{+i}$. That the vector at right in (32c) is the same for all $f \in E_n$ follows from the fact that this vector depends only on the restriction of $\tilde{f}$ to $V_n$.

(2) That the function $\phi$ is strongly continuous on $T_n$ is easily established through an examination of the properties of the function $f_0$, as defined in (32b). The considerations are the same as in the proof of the strong continuity of the vector $\beta$ in Lemma 3, and in fact somewhat simpler since $(z_1, \ldots, z_n)$ is now restricted to the open tube $T_n$.

(3) The assertion about the integral in (33) is now trivial, and the identity follows from a well-known convolution theorem for tempered distributions.[20] We note that the restriction that the functions $f_k$ be of compact support is in fact unnecessary, but since we shall only require the lemma as stated, we selected this version in order to make the matter completely trivial.

We conclude this section by a statement of some well-known facts about the vector-valued functions $\phi$, which will be of crucial importance in our subsequent discussion.

*Lemma* 6: (a) The vector-valued function $\phi(z_1, \ldots, z_n)$, defined as in Lemma 5, is an analytic function of $(z_1, \ldots, z_n)$ on $T_n$.

(b) For any element $\Lambda = \Lambda(M, x)$ of the Poincaré group $\overline{L}_0$,

$$U(\Lambda)\phi(z_1, \ldots, z_n) = \phi(Mz_1 + x, Mz_2, Mz_3, \ldots, Mz_n). \tag{34}$$

(c) For any $(z_1, \ldots, z_n) \in T_n$ the vector $\phi(z_1, \ldots, z_n)$ is an analytic vector for the Lie algebra of the group $U(\overline{L}_0)$.

*About the proof*: A detailed proof of the assertion (a) based on an examination of the properties of the function $f_0$ defined in (32b) is straightforward but somewhat cumbersome. For this reason it might be worthwhile to note that there is a simple proof based on Lemmas 3 and 5, as follows. Let $g(x)\in\mathcal{D}(R^4)$ be such that $\tilde{g}(0)=1$. Let $\lambda>1$. We construct the vector $\beta(z_1,\ldots,z_n;\lambda)$ as in (29), with $f_k(x)=\lambda^4 g(\lambda x)$, for $k=1,\ldots,n$. This vector-valued function of $(z_1,\ldots,z_n)$ is an analytic function of these variables on $T_n$, by Lemma 3. It is easily seen, in view of (33), and in view of the strong continuity of $\phi$ on $T_n$, that $\beta(z_1,\ldots,z_n;\lambda)$ tends to $\phi(z_1,\ldots,z_n)$ as $\lambda$ tends to infinity, *uniformly* on any closed polydisc contained in $T_n$, and hence $\phi$ is analytic on $T_n$.

The assertion (b) of the lemma is trivial, and the assertion (c) follows trivially from (a) and (b).

We finally note that the vector $\phi$ might be written as

$$\phi(z_1,\ldots,z_n)=\varphi(z_1)\varphi(z_1+z_2)\cdots\varphi(z_1+z_2+\cdots+z_n)\Omega. \quad (35)$$

This formula has a proper interpretation within distribution theory, but it is here offered for heuristic purposes only.

## IV. COMPLEX LORENTZ TRANSFORMATIONS AND THE INVERSION TRANSFORMATION

We define a "right wedge" $W_R$, and a "left wedge" $W_L$, as the following open subsets of Minkowski space:

$$W_R=\{x\,|\,x^3>|x^4|\},\quad W_L=\{x\,|\,x^3<-|x^4|\}. \quad (36)$$

These two regions are bounded by two characteristic planes whose intersection is the 2-plane $\{x\,|\,x^3=x^4=0\}$.

For *any* subset $R$ of Minkowski space $\mathcal{M}$ we define the *causal complement* $R^c$ of $R$ by

$$R^c=\{x\,|\,(x-y)\cdot(x-y)<0,\ \text{all } y\in R\}. \quad (37)$$

We note that with this definition $W_R{}^c=\overline{W}_L$ and $W_L{}^c=\overline{W}_R$, where the bar denotes the closure. We shall say that $W_R$ and $W_L$ form a complementary pair of wedges, despite the fact that $W_R$ is not precisely the causal complement of $W_L$ within our definition of this notion.[21]

To the pair of wedges $W_R$ and $W_L$ corresponds a four-dimensional subgroup $\overline{L}_0(W_R)=\overline{L}_0(W_L)$ of the group $\overline{L}_0$, namely, the group of all Poincaré transformations which map $W_R$ *onto* $W_R$, and $W_L$ *onto* $W_L$. It is easily seen that this subgroup contains, and is generated by, all translations in the 1- and 2-directions, all rotations about the 3-axis, and all velocity transformations $V(\mathbf{e}_3,t)$ in the 3-direction. We consider the one-parameter Abelian subgroup $\{V(\mathbf{e}_3,t)\,|\,t\in R^1\}$ of these velocity transformations, where $V(\mathbf{e}_3,t)$ is the four-by-four Lorentz matrix given in (1) in Sec. I. To $V(\mathbf{e}_3,t)$ corresponds the unitary operator $U(V(\mathbf{e}_3,t),0)$, which we shall also denote by the shorter symbol $V(t)$, since it will play an important role in our discussion. By Stone's theorem there exists a unique self-adjoint operator $(K_3,D_K)$ such that

$$V(t)=U(V(\mathbf{e}_3,t),0)=\exp(-itK_3),\quad \text{all real } t. \quad (38)$$

We shall consider the analytic continuation of the function $V(t)$ to the complex plane. It is well known that to any self-adjoint operator $(K_3,D_K)$ corresponds a representation $\tau\to\exp(-i\tau K_3)=V(\tau)$ of the additive group of all complex numbers $\tau$ by (in general unbounded) operators. These operators have the common spectral resolution

$$V(\tau)=\exp(-i\tau K_3)=\int\exp(-i\tau s)\mu_K(ds) \quad (39)$$

where $\mu_K$ is the spectral measure in the spectral resolution of the operator $(K_3,D_K)$. The domain of the closed operators $V(\tau)$ depends only on $\mathrm{Im}(\tau)$. Hence, for any $\tau=\rho+i\lambda$, with $\rho,\lambda$ real, let $D_V(\lambda)$ be the linear manifold such that the operator $(V(\tau),D_V(\lambda))$ is *closed* and *normal*. The domain $D_V(\lambda)$ is given by

$$D_V(\lambda)=(1+V(i\lambda))^{-1}\mathcal{H} \quad (40)$$

for any real $\lambda$.

Let $\lambda\neq 0$ be real. Then $D_V(\lambda)$ is a core for all operators $(V(\tau),D_V(\mathrm{Im}(\tau)))$ such that $0\le\mathrm{Im}(\tau)/\lambda\le 1$. If $\psi\in D_V(\lambda)$, then the vector-valued function $V(\tau)\psi$ of $\tau$ is well defined, strongly continuous and bounded on the closed strip $0\le\mathrm{Im}(\tau)/\lambda\le 1$, and an analytic function of $\tau$ on the interior of this strip.

Common cores exist for the operators $V(\tau)$. For later reference we state as a lemma some well-known facts about a particular family of such cores.

*Lemma* 7: (a) Let $c(s)\in\mathcal{D}(R^1)$, and let the bounded operator $c(K_3)$ be defined by

$$c(K_3)=\int c(s)\mu_K(ds). \quad (41)$$

Then $c(K_3)\mathcal{H}\subset D_V(\lambda)$ for all real $\lambda$. The function $\exp(-i\tau s)c(s)$ is also in $\mathcal{D}(R^1)$ for any complex $\tau$, and

$$V(\tau)c(K_3)=\int\exp(-i\tau s)c(s)\mu_K(ds). \quad (42)$$

The operator-valued function $V(\tau)c(K_3)$ is a bounded operator for every complex $\tau$, and it is an entire analytic function of $\tau$ in the sense of the uniform topology.

(b) Let $D$ be any dense linear manifold, and let the linear manifold $D_c$ be defined by

$$D_c=\mathrm{span}\{c(K_3)D\,|\,c(s)\in\mathcal{D}(R^1)\}. \quad (43a)$$

Then $D_c$ is dense, and a core for every operator $(V(\tau),D_V(\mathrm{Im}(\tau)))$, i.e., $D_c\subset D_V(\mathrm{Im}(\tau))$ and

$$(V(\tau),D_c)^{**}=(V(\tau),D_V(\mathrm{Im}(\tau))). \quad (43b)$$

(c) If $c(s)\in\mathcal{D}(R^1)$, then $c(K_3)$ is also given by

$$c(K_3)=\int_{-\infty}^{\infty}dt\,\hat{c}(t)V(t) \quad (44a)$$

where $\hat{c}(t)$ is the Fourier transform of $c(s)$ defined by

$$\hat{c}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}ds\,\exp(its)c(s). \quad (44b)$$

We shall next consider the action of the complex velocity transformation $V(\tau)$ on the vectors $\phi(z_1,\ldots,z_n)$ introduced in Lemma 5. We first note that the matrix-valued function $V(\mathbf{e}_3,t)$, defined in (1) in Sec. I, is an entire analytic function of $t$. Let $z=x+iy$, $x$ and $y$ real, be any complex 4-vector, and let $\tau$ be any complex number. We shall write

$$z(\tau)=V(\mathbf{e}_3,\tau)z \quad (45a)$$

and we then have, for $\tau=i\lambda$,

$$z^1(i\lambda)=x^1+iy^1,\quad z^2(i\lambda)=x^2+iy^2,$$

0 0 0 4 3 0 1 2 1 5

$$z^3(i\lambda) = (x^3\cos(\lambda) - y^4\sin(\lambda)) + i(y^3\cos(\lambda) + x^4\sin(\lambda)), \tag{45b}$$

$$z^4(i\lambda) = (x^4\cos(\lambda) - y^3\sin(\lambda)) + i(y^4\cos(\lambda) + x^3\sin(\lambda)).$$

We have written the explicit transformation formulas in the above form because we are particularly interested in the case of a real $\lambda$, i. e., the case of a pure imaginary velocity transformation. We can now state the following:

*Lemma* 8: Let $(z_1,\ldots,z_n)$ be an $n$-tuplet of complex 4-vectors $z_k = x_k + iy_k$, where $x_k, y_k$, real, $y_k{}^1 = y_k{}^2 = 0$, $y_k{}^4 > |y_k{}^3|$, for $k = 1,\ldots,n$.

(a) If $x_k \in W_R$ (i. e., $x_k{}^3 > |x_k{}^4|$), for $k = 1,\ldots,n$, then $(z_1(i\lambda),\ldots,z_n(i\lambda)) \in T_n$ for all $\lambda \in [0,\pi/2]$. The vector $\phi(z_1,\ldots,z_n)$ is in the domain $D_V(\pi/2)$, and

$$V(i\lambda)\phi(z_1,\ldots,z_n) = \phi(z_1(i\lambda),\ldots,z_n(i\lambda)) \tag{46}$$

for all $\lambda \in [0,\pi/2]$.

(b) If $x_k \in W_L$ (i. e., $x_k{}^3 < -|x_k{}^4|$), for $k = 1,\ldots,n$, then $(z_1(i\lambda),\ldots,z_n(i\lambda)) \in T_n$ for all $\lambda \in [-\pi/2, 0]$. The vector $\phi(z_1,\ldots,z_n)$ is in the domain $D_V(-\pi/2)$, and the relation (46) holds for all $\lambda \in [-\pi/2, 0]$.

*Proof*: (1) We consider the assertions in part (a). By inspection of the explicit formulas (45b), it is easily seen that if $z = x + iy$ is a complex four-vector such that $y^1 = y^2 = 0$, $y^4 > |y^3|$, and $x^3 > |x^4|$, then $\mathrm{Im}(z(i\lambda)) \in V_+$ for all $\lambda \in [0,\pi/2]$. Hence, in view of the stated conditions on $(z_1,\ldots,z_n)$, we have $(z_1(i\lambda),\ldots,z_n(i\lambda)) \in T_n$ for all $\lambda$ on the closed interval, with $T_n$ defined as in Lemma 3. Since $T_n$ is open there exists a connected open neighborhood $N$ (in the complex $\lambda$-plane) of the closed segment $[0,\pi/2]$ such that $(z_1(i\lambda),\ldots,z_n(i\lambda)) \in T_n$ for $\lambda \in N$, and hence the vector $\phi(z_1(i\lambda),\ldots,z_n(i\lambda))$ is well defined for $\lambda \in N$. By Lemma 6 this vector, regarded as a function of $\lambda$, is an analytic function on $N$.

(2) Let $D_c$ be defined as in (43a), with $D = \mathcal{H}$. For any $\eta \in D_c$ the function $f_1(\lambda) = \langle V(i\lambda)^*\eta \,|\, \phi(z_1,\ldots,z_n)\rangle$ is an entire analytic function of $\lambda$, by Lemma 7. We define the function $f_2(\lambda)$ on $N$ by $f_2(\lambda) = \langle \eta \,|\, \phi(z_1(i\lambda),\ldots,z_n(i\lambda))\rangle$. By Lemma 6 we have $f_1(\lambda) = f_2(\lambda)$ for $i\lambda$ in some *real* neighborhood of $\lambda = 0$, and it follows that $f_1(\lambda) = f_2(\lambda)$ on $N$. Since this holds for any $\eta \in D_c$, and since $D_c$ is a core for every $(V(\tau), D_V(\mathrm{Im}(\tau)))$, it follows that $\phi(z_1,\ldots,z_n) \in D_V(\mathrm{Im}(i\lambda))$ for $\lambda \in N$, and that (46) holds for all $\lambda \in N$. This proves the assertions in part (a).

(3) The assertions in part (b) are proved in an entirely analogous fashion.

We next consider an *involutory* mapping $x \to \mathcal{J}x$ of Minkowski space onto itself, defined by

$$\mathcal{J}x = -R(\mathrm{e}_3,\pi)x \quad \text{or} \quad \mathcal{J}(x^1,x^2,x^3,x^4) = (x^1,x^2,-x^3,-x^4) \tag{47}$$

where $R(\mathrm{e}_3,\pi)$ denotes the rotation by angle $\pi$ about the 3-axis. We see that $\mathcal{J}$ maps $W_R$ onto $W_L$, and the mapping can be described as a reflection in the common "edge" $\{x \,|\, x^3 = x^4 = 0\}$ of the pair of wedges $W_R$ and $W_L$.

By inspection of (45b) we see that

$$\mathcal{J} = V(\mathrm{e}_3, i\pi) \tag{48}$$

and this circumstance suggests the heuristic idea that something akin to $V(i\pi)\varphi(x)V(i\pi)^{-1} = \varphi(\mathcal{J}x)$ might hold. This formula is, of course, pure nonsense as it stands, but in the following we shall establish some facts which in a sense reflect the above heuristic idea.

*Lemma* 9: Let $(x_1,\ldots,x_n)$ be such that $x_k \in W_R$ for $k = 1,\ldots,n$. Let $v$ be the real forward timelike 4-vector with components $v = (0,0,0,1)$, and let $t$ be a real variable. Then

$$\begin{aligned} &\operatorname*{s-lim}_{t\to 0+} V(i\pi/2)\phi(x_1+itv, x_2+itv,\ldots,x_n+itv) \\ &= \operatorname*{s-lim}_{t\to 0+} V(-i\pi/2)\phi(\mathcal{J}x_1+itv, \mathcal{J}x_2+itv,\ldots,\mathcal{J}x_n+itv) \\ &= \phi(z_1,\ldots,z_n) \end{aligned} \tag{49}$$

where $z_k = (x_k{}^1, x_k{}^2, ix_k{}^4, ix_k{}^3)$, for $k = 1,\ldots,n$.

*Proof*: By Lemma 8, part (a), we have, for $t > 0$,

$$V(i\pi/2)\phi(x_1+itv,\ldots,x_n+itv) = \phi(z_1',\ldots,z_n') \tag{50a}$$

where

$$z_k' = z_k'(t) = z_k - (0,0,t,0), \quad \text{for } k = 1,\ldots,n. \tag{50b}$$

Since $\mathcal{J}x_k \in W_L$ if $x_k \in W_R$, we similarly have, by part (b) of Lemma 8, for any $t > 0$,

$$V(-i\pi/2)\phi(\mathcal{J}x_1+itv,\ldots,\mathcal{J}x_n+itv) = \phi(z_1'',\ldots,z_n'') \tag{50c}$$

with

$$z_k'' = z_k''(t) = z_k + (0,0,t,0), \quad \text{for } k = 1,\ldots,n. \tag{50d}$$

We note that $(z_1',\ldots,z_n') \in T_n$, and $(z_1'',\ldots,z_n'') \in T_n$, for all real $t$, and it follows from Lemma 5 that the vectors at right in (50a) and (50c) have well-defined strong limits as $t$ tends to zero. The equalities in (49) then follow from (50b) and (50d).

*Lemma* 10: Let $R_1$ be a bounded, open, nonempty subset of $W_R$, and let $x_0 \in W_R$ be such that $(x - x_0) \in W_L$ for all $x \in \bar{R}_1$. For any integer $n > 1$ we define the set $R_n$ by

$$R_n = \{x + (n-1)x_0 \,|\, x \in R_1\}. \tag{51}$$

(a) Then $R_n \subset W_R$ for all $n$, and if $n > k$, then $(x' - x'') \in W_R$ for all $x' \in R_n$, $x'' \in R_k$. In particular, $R_n$ is spacelike separated from $R_k$ (i. e., $R_n \subset R_k{}^c$) if $n \neq k$.

(b) Let $\{f_k \,|\, k = 1,\ldots,n\}$ be an $n$-tuplet of test functions such that $f_k \in \mathcal{S}(R^4)$ and $\mathrm{supp}(f_k) \subset R_k$, for $k = 1,\ldots,n$. Let $f_k{}^j$ denote the test function defined by $f_k{}^j(x) = f_k(\mathcal{J}x)$. Let $c(s) \in \mathcal{D}(R^1)$. Then

$$\begin{aligned} &V(i\pi)c(K_3)\varphi[f_1]\varphi[f_2]\cdots\varphi[f_n]\Omega \\ &\quad = c(K_3)\varphi[f_1{}^j]\varphi[f_2{}^j]\cdots\varphi[f_n{}^j]\Omega. \end{aligned} \tag{52}$$

*Proof*: (1) The assertions in part (a) are trivial, and need not be proved here.

(2) Let $v = (0,0,0,1)$. We consider the string of equalities:

$$\begin{aligned} &V(i\pi/2)c(K_3)\varphi[f_1]\varphi[f_2]\cdots\varphi[f_n]\Omega \\ &= \operatorname*{s-lim}_{t\to 0+} V(i\pi/2)c(K_3)T(itv)\varphi[f_1]T(itv)\varphi[f_2]\cdots T(itv) \\ &\quad \times \varphi[f_n]\Omega \end{aligned}$$

$$= \underset{t \to 0+}{\text{s-lim}}\, V(i\pi/2)c(K_3) \int_{(\infty)} d^4(x_1) \cdots d^4(x_n) f_1(x_1) f_2(x_2) \cdots f_n(x_n)$$

$$\times \phi(itv + x_1, itv + x_2 - x_1, itv + x_3 - x_2, \ldots, itv + x_n - x_{n-1})$$

$$= \int_{(\infty)} d^4(x_1) \cdots d^4(x_n) f_1(x_1) f_2(x_2) \cdots f_n(x_n)$$

$$\times \underset{t \to 0+}{\text{s-lim}}\, V(i\pi/2)c(K_3)$$

$$\times \phi(itv + x_1, itv + x_2 - x_1, itv + x_3 - x_2, \ldots, itv + x_n - x_{n-1})$$

$$= \int_{(\infty)} d^4(x_1) \cdots d^4(x_n) f_1{}^J(x_1) f_2{}^J(x_2) \cdots f_n{}^J(x_n)$$

$$\times \underset{t \to 0+}{\text{s-lim}}\, V(-i\pi/2)c(K_3)$$

$$\times \phi(itv + x_1, itv + x_2 - x_1, itv + x_3 - x_2, \ldots, itv + x_n - x_{n-1})$$

$$= \underset{t \to 0+}{\text{s-lim}}\, V(-i\pi/2)c(K_3)T(itv)$$

$$\times \varphi[f_1{}^J]T(itv)\varphi[f_2{}^J] \cdots T(itv)\varphi[f_n{}^J]\Omega$$

$$= V(-i\pi/2)c(K_3)\varphi[f_1{}^J]\varphi[f_2{}^J] \cdots \varphi[f_n{}^J]\Omega. \qquad (53)$$

That the first member in (53) is equal to the second member, and that the last member is equal to the next to the last member, follows from Lemma 3 (i.e., from the strong continuity of the function there denoted $\beta$), and from the fact that the operators $V(i\pi/2)c(K_3)$ and $V(-i\pi/2)c(K_3)$ are bounded. That the second member is equal to the third member follows from the formula (33) in Lemma 5. In view of the properties of the integrand in the third member which follow from the facts stated in Lemma 9, and from the nature of the functions $f_k$, it is permissible to let the bounded operator $V(i\pi/2)c(K_3)$ act on the integrand, and to take the strong limit before integration. We note that the relationships between the supports of the function $f_k$, as expressed in the assertions (a) of the present lemma, are essential at this step. Because of these relationships the arguments of the function $\phi$ appearing in the integrand satisfy the premises of Lemma 9, which is thus applicable. The third and the fourth members are thus equal. In a similar fashion we conclude that the fifth and the sixth members are equal. The equality of the fourth and the fifth members follows from Lemma 9. (Note the trivial change in integration variables).

(3) We finally note that the vector in (53) is in the domain of $(V(i\pi/2), D_V(\pi/2))$, and if we multiply the first and the last members in the string by this operator we obtain (52).

It should be noted that the condition that the field be *local* has played no role in our discussion so far, and in particular the formula (52) does not depend on the assumption of locality. We shall now consider some additional conclusions which can be drawn if we take into account the locality condition (16).

From the work of Jost[22] it is well known that in a local field theory based on our general assumptions there exists an antiunitary involution $\Theta_0$, which can be interpreted physically as an inversion transformation, or TCP-transformation (with respect to the origin in Minkowski space). This operator satisfies the conditions

$$\Theta_0{}^2 = I, \quad \Theta_0\Omega = \Omega, \quad \Theta_0 U(M, x)\Theta_0 = U(M, -x), \qquad (54a)$$

and

$$\Theta_0 \varphi(x)\Theta_0 = \varphi(-x), \qquad (54b)$$

where the last relation refers specifically to the case of a Hermitian scalar field.

We shall introduce another antiunitary involution $J$, defined by

$$J = U(R(e_3, \pi), 0)\Theta_0 = \Theta_0 U(R(e_3, \pi), 0) \qquad (55)$$

where, as before, $R(e_3, \pi)$ denotes the rotation by angle $\pi$ about the 3-axis. It is easily seen that

$$J^2 = I, \quad J\Omega = \Omega, \quad JU(M, x)J = U(\mathcal{J}M\mathcal{J}, \mathcal{J}x) \qquad (56a)$$

where $\mathcal{J}$ is defined in (47). Furthermore, $JD_1 = D_1$, and

$$J\varphi[f]J = \varphi[f^J]^* \quad \text{on } D_1 \qquad (56b)$$

for any $f \in \mathcal{S}(R^4)$, and where $f^J(x) = f(\mathcal{J}x)$.

We consider the third relation in (56a) for the case of a (real) velocity transformation in the 3-direction. We have

$$JV(t)J = V(t), \quad \text{all real } t. \qquad (57a)$$

From this relation, and from the fact that $J$ is an antiunitary involution, we readily conclude that

$$JD_K = D_K, \quad J(K_3, D_K)J = -(K_3, D_K), \qquad (57b)$$

$$JD_V(\lambda) = D_V(-\lambda), \quad J(V(\tau), D_V(\lambda))J = (V(\tau^*), D_V(-\lambda)) \qquad (57c)$$

for any complex $\tau = \rho + i\lambda$, $\rho$ and $\lambda$ real.

As the formula (52) suggests, the complex velocity transformations $V(i\pi)$ and $V(-i\pi)$ will be of particular interest. We shall employ the special notation

$$D_+ = D_V(\pi), \quad D_- = D_V(-\pi) \qquad (58)$$

for the domains of these operators, and $(V(i\pi), D_+)$ and $(V(-i\pi), D_-)$ are thus self-adjoint. We then have

$$D_+ = JD_- = V(-i\pi)D_-, \quad D_- = JD_+ = V(i\pi)D_+, \qquad (59a)$$

and

$$J(V(i\pi), D_+)J = (V(-i\pi), D_-),$$

$$J(V(-i\pi), D_-)J = (V(i\pi), D_+). \qquad (59b)$$

The antiunitary involution $J$ can be regarded as associated with the pair of wedges $W_R$ and $W_L$, or, if we like, with their common "edge," whereas the involution $\Theta_0$ is associated with a point, the origin of Minkowski space. $J$ is the Hilbert space object corresponding to the involution $\mathcal{J}$ on Minkowski space, as revealed by (56b). We note that if $\operatorname{supp}(f) \subset W_R$, then $\operatorname{supp}(f^J) \subset W_L$, and vice versa. Conjugation with $J$ thus maps operators locally associated with the right wedge $W_R$ into operators locally associated with the left wedge $W_L$. We also note that

$$JU(\Lambda)J = U(\Lambda), \quad \text{all } \Lambda \in \bar{L}_0(W_R), \qquad (60)$$

where $\bar{L}_0(W_R)$ is the group of all Poincaré transformations which map $W_R$ *onto* $W_R$.

We shall next consider an extension of Lemma 10 which incorporates the condition that the field be local.

*Lemma* 11: Let $\{R_n | n = 1, \ldots, \infty\}$ be a *fixed* set of bounded, open, nonempty subsets of $W_R$, constructed as in Lemma 10. Let $\mathcal{Q}$ be the linear span of the identity operator I and all operators $(Q, D_1)$ of the form

$$Q = \varphi[f_1]\varphi[f_2]\cdots\varphi[f_n] \quad (61)$$

where $\{f_k | k=1,\ldots,n\}$ is any $n$-tuplet of test functions such that $f_k \in \mathcal{S}(R^4)$ and $\text{supp}(f_k) \subset R_k$, for $k=1,\ldots,n$.

Then:

(a) The linear manifold $D_q = \mathcal{Q}\Omega$ is dense in the Hilbert space $\mathcal{H}$, and $D_{qc} = \text{span}\{c(K_3)D_q | c(s) \in \mathcal{D}(R^1)\}$ is a core for every operator $(V(\tau), D_V(\text{Im}(\tau)))$.

(b) $(Q^*, D_1) \in \mathcal{Q}$ if $(Q, D_1) \in \mathcal{Q}$.

(c) If $(Q, D_1) \in \mathcal{Q}$ and $c(s) \in \mathcal{D}(R^1)$, then

$$V(i\pi)c(K_3)Q\Omega = c(K_3)JQ^*\Omega. \quad (62)$$

*Proof*: (1) The assertions (a) follow directly from Lemmas 4 and 7.

(2) The assertion (b) is trivial if $Q$ is a multiple of I. If $Q$ is of the special form (61) we have

$$Q^\dagger = \varphi[f_n^\dagger]\cdots\varphi[f_2^\dagger]\varphi[f_1^\dagger]$$
$$= \varphi[f_1^\dagger]\varphi[f_2^\dagger]\cdots\varphi[f_n^\dagger], \quad (63)$$

where the second member is equal to the third in view of the locality condition (16), and in view of the relationships between the supports of the functions $f_k$, as stated in part (a) of Lemma 10. Since $(Q^*, D_1) = (Q^\dagger, D_1)$, we see that $(Q^*, D_1) \in \mathcal{Q}$.

(3) The relation (62) is trivial if $Q$ is a multiple of I. For $Q$ of the special form (61) we have, in view of (63),

$$JQ^\dagger J = \varphi[f_1^J]\varphi[f_2^J]\cdots\varphi[f_n^J]. \quad (64)$$

Since $Q^*\Omega = Q^\dagger\Omega$ the relation (62) then follows from (64) and from (52) in Lemma 10. This, in effect, proves the assertion (c).

To an $n$-tuplet $(x_1,\ldots,x_n)$ such that $x_k \in R_k$ for $k=1,\ldots,n$, corresponds the $n$-tuplet $(x_1, x_2-x_1, x_3-x_2,\ldots,x_n-x_{n-1})$, which is a so-called Jost point.[23] We note here that there is a very close connection between our considerations and Jost's beautiful proof of the TCP-theorem.[22] In a sense the key point is the fact that the complex Lorentz transformations $V(e_3, i\lambda)$, for $\lambda \in (0,\pi)$, map the wedge region $W_R$ into the forward imaginary tube $V_{+i}$. This fact, and the associated connection between complex Lorentz transformations and the inversion transformation, were discovered by Jost, and form the basis of his proof.

We are now in a position to state and prove the key theorem. For the definition of the algebras $\mathcal{P}(W_R)$ and $\mathcal{P}(W_L)$ we refer to our general definition [in Sec. II, immediately following Eq. (15)] of the algebra $\mathcal{P}(R)$, for any open $R \subset \mathcal{M}$. The algebra $\mathcal{P}(W_R)$, respectively the algebra $\mathcal{P}(W_L)$, can be regarded as consisting of field operators locally associated with the wedge region $W_R$, respectively the region $W_L$.

*Theorem* 1: (a) The algebras $\mathcal{P}(W_R)$ and $\mathcal{P}(W_L)$ are $*$-algebras with the antilinear involution $(X, D_1) \to (X^*, D_1)$. They commute on $D_1$, i.e.,

$$[X, Y]\psi = 0 \quad (65)$$

for all $\psi \in D_1$ and for all $X \in \mathcal{P}(W_R)$, $Y \in \mathcal{P}(W_L)$.

(b) The vacuum vector $\Omega$ is cyclic and separating for both $\mathcal{P}(W_R)$ and $\mathcal{P}(W_L)$.

(c) With $V(t) = U(V(e_3, t), 0)$ (a velocity transformation in the 3-direction),

$$V(t)\mathcal{P}(W_R)V(t)^{-1} = \mathcal{P}(W_R), \quad V(t)\mathcal{P}(W_L)V(t)^{-1} = \mathcal{P}(W_L) \quad (66)$$

for all real $t$, and with $J$ defined by (55),

$$J\mathcal{P}(W_R)J = \mathcal{P}(W_L). \quad (67)$$

(d) With $D_+$ and $D_-$ defined as in (58),

$$\mathcal{P}(W_R)\Omega \subset D_+, \quad \mathcal{P}(W_L)\Omega \subset D_-. \quad (68a)$$

For any $X \in \mathcal{P}(W_R)$

$$V(i\pi)X\Omega = JX^*\Omega \quad (68b)$$

and for any $Y \in \mathcal{P}(W_L)$

$$V(-i\pi)Y\Omega = JY^*\Omega. \quad (68c)$$

(e) The condition

$$C_R X\Omega = X^*\Omega, \quad \text{all } X \in \mathcal{P}(W_R), \quad (69a)$$

defines an *antilinear* operator $(C_R, \mathcal{P}(W_R)\Omega)$, and the condition

$$C_L Y\Omega = Y^*\Omega, \quad \text{all } Y \in \mathcal{P}(W_L), \quad (69b)$$

defines an *antilinear* operator $(C_L, \mathcal{P}(W_L)\Omega)$.

These two operators satisfy the relations

$$(C_R, \mathcal{P}(W_R)\Omega)^{**} = (C_L, \mathcal{P}(W_L)\Omega)^* = (JV(i\pi), D_+), \quad (69c)$$
$$(C_L, \mathcal{P}(W_L)\Omega)^{**} = (C_R, \mathcal{P}(W_R)\Omega)^* = (JV(-i\pi), D_-). \quad (69d)$$

*Proof*: (1) The assertions (a) and (c) are trivial. That $\Omega$ is a cyclic vector for the algebras follows from the Reeh–Schlieder theorem. That $\Omega$ is separating for $\mathcal{P}(W_R)$ follows readily from the commutation relation (65), and from the fact that $\Omega$ is cyclic for $\mathcal{P}(W_L)$. In a similar manner we conclude that $\Omega$ is separating for $\mathcal{P}(W_L)$.[24]

(2) We now consider the assertions (d) and (e). We note that our formulation is tautological in the sense that the assertions (d) are trivially implied by the assertions (e). We presented the matter in this manner because we wanted the relations (68b) and (68c) to stand out as clearly as possible.

For didactic reasons we shall first prove the assertions (d), independently of the considerations in (e). Let a set $\mathcal{Q}$ of operators, and a domain $D_{qc}$, be constructed exactly as in Lemma 11. We note that $\mathcal{Q} \subset \mathcal{P}(W_R)$.

Let $Q \in \mathcal{Q}$, $X \in \mathcal{P}(W_R)$, and $c(s) \in \mathcal{D}(R^1)$. We introduce the integral representation (44) of the operator $c(K_3)$, and we note that

$$c^*(-K_3) = \int_{-\infty}^{\infty} dt\, \hat{c}^*(t)V(t) \quad (70a)$$

where $\hat{c}(t)$ is given by (44b).

We consider the following string of equalities:

$$\langle X\Omega | V(i\pi)c(K_3)Q\Omega\rangle$$
$$= \langle X\Omega | c(K_3)JQ^*\Omega\rangle = \langle X\Omega | Jc^*(-K_3)Q^*\Omega\rangle$$
$$= \langle c^*(-K_3)Q^*\Omega | JX\Omega\rangle$$

$$= \int_{-\infty}^{\infty} dt \hat{c}(t) \langle V(t) Q^* V(t)^{-1} \Omega \,|\, (JXJ)\Omega \rangle$$

$$= \int_{-\infty}^{\infty} dt \hat{c}(t) \langle (JXJ)^* \Omega \,|\, V(t) Q V(t)^{-1} \Omega \rangle$$

$$= \langle JX^* \Omega \,|\, c(K_3) Q \Omega \rangle. \qquad (70b)$$

The first two members are equal in view of (62) in Lemma 11. The equality of the second and the third members follows from (57b), and since $J$ is an antiunitary involution these expressions are equal to the fourth member. The equality of the fourth and fifth members follows from (70a). The integrands in the fifth and sixth members are equal because the operator $V(t)Q^\dagger V(t)^{-1} \in \mathcal{P}(W_R)$ commutes with the operator $JXJ \in \mathcal{P}(W_L)$ on $D_1$. The equality of the last two members follows from (44a).

In view of the construction of the domain $D_{qc}$ we conclude from (70b) that if $\eta$ is any vector in $D_{qc}$, then

$$\langle X\Omega \,|\, V(i\pi)\eta \rangle = \langle JX^* \Omega \,|\, \eta \rangle. \qquad (70c)$$

Since $D_{qc}$ is a core for $(V(i\pi), D_+)$ (by Lemma 11), it follows from (70c) that $X\Omega \in D_+$, and that (68b) holds.

The relation (68c) and the second relation in (68a) then follows trivially from (67) and (59b).

(3) The assertions (e) involve antilinear operators, and since the theory of such operators might appear less familiar than the theory of linear operators we shall make a few remarks about the subject. Let $(A, D_a)$ be an antilinear operator, defined on a dense domain $D_a$. The adjoint $(A, D_a)^* = (A^*, D_a^*)$ of $(A, D_a)$ is defined as follows. A vector $\eta$ is in the domain $D_a^*$ of the adjoint if and only if there exists a vector $\zeta(\eta)$ such that $\langle \eta | A\xi \rangle = \langle \xi | \zeta(\eta) \rangle$ for every $\xi \in D_a$. The operator $A^*$ on $D_a^*$ is then defined by $A^*\eta = \zeta(\eta)$, and it is also antilinear. The operator $(A, D_a)$ is closable if and only if its adjoint is densely defined, and if it is closable its closure $(A, D_a)^{**}$ is the adjoint of the adjoint $(A^*, D_a^*)$. The properties of an antilinear operator $(A, D_a)$ can be conveniently studied in terms of the *linear* operator $(L, D_a) = (J_0 A, D_a) = J_0(A, D_a)$, where $J_0$ is an arbitrary *antiunitary* operator. We then have $(A, D_a)^* = (L^* J_0, J_0^{-1} D(L^*))$. The operator $(A, D_a)$ is closable if and only if $(L, D_a)$ is closable, and if it is closable, then $(A, D_a)^{**} = J_0^{-1}(L, D_a)^{**}$. The well-known polar decomposition theorem for linear operators has a counterpart for antilinear operators, as we easily see in view of the above. We note that the formulas (69c) and (69d) explicitly describe the polar decompositions of the adjoints and closures of the "adjointing operators" $C_R$ and $C_L$ defined by (69a) and (69b).

(4) After this digression we consider the assertions (e). It follows at once from the definition (69a), and from (68b) that

$$(JV(i\pi), D_+) \supset (C_R, \mathcal{P}(W_R)\Omega), \qquad (71a)$$

and if we take the closures of both members in (71a) we obtain

$$(JV(i\pi), D_+) \supset (C_R, \mathcal{P}(W_R)\Omega)^{**} \qquad (71b)$$

since $(V(i\pi), D_+)$ is self-adjoint and $(JV(i\pi), D_+)$ therefore is closed.

We shall now show that

$$(C_R, \mathcal{P}(W_R)\Omega)^{**} \supset (JV(i\pi), D_{qc}). \qquad (71c)$$

Let $\eta$ be any vector in the domain of $(C_R, \mathcal{P}(W_R)\Omega)^*$. Let $Q \in \mathcal{Q}$, and $c(s) \in \mathcal{D}(R^1)$. We again introduce the integral representation (44) for the operator $c(K_3)$, and we consider the string of equalities:

$$\langle C_R^* \eta \,|\, c(K_3) Q\Omega \rangle$$

$$= \int_{-\infty}^{\infty} dt\, \hat{c}(t) \langle C_R^* \eta \,|\, V(t)\, Q\, V(t)^{-1} \Omega \rangle$$

$$= \int_{-\infty}^{\infty} dt\, \hat{c}(t) \langle V(t)\, Q^*\, V(t)^{-1}\, \Omega \,|\, \eta \rangle$$

$$= \langle c^*(-K_3)\, Q^* \Omega \,|\, \eta \rangle = \langle JV(i\pi)\, c(K_3)\, Q\Omega \,|\, \eta \rangle. \qquad (71d)$$

The equality of the second and third members follows from the fact that $V(t)\,QV(t)^{-1}\Omega$ is in the domain of the antilinear operator $(C_R, \mathcal{P}(W_R)\Omega)$. The reasoning behind the other steps is similar to the reasoning in (2) above. In view of the construction of the domain $D_{qc}$ the equalities (71d) imply (71c).

Since $D_{qc}$ is a core for $(V(i\pi), D_+)$, we have

$$(JV(i\pi), D_+) = (JV(i\pi), D_{qc})^{**} \qquad (71e)$$

and it follows from (71b) and (71e) that

$$(C_R, \mathcal{P}(W_R)\Omega)^{**} = (JV(i\pi), D_+). \qquad (71f)$$

The analogous relation

$$(C_L, \mathcal{P}(W_L)\Omega)^{**} = (JV(-i\pi), D_-) \qquad (71g)$$

is most easily proved by considering the conjugation of both members in (71f) by $J$. The remaining relations in (69c) and (69d) follow trivially from (71f) and (71g), and from the relation

$$(JV(i\pi), D_+)^* = (JV(-i\pi), D_-). \qquad (71h)$$

This completes the proof of the theorem. We conclude this section with some remarks which we hope will further clarify the situation.

Concerning the relations (69c) and (69d) we note the following. If we are given two algebras, denoted $\tilde{\mathcal{P}}(W_R)$ and $\tilde{\mathcal{P}}(W_L)$, which satisfy the conditions (a) and (b), and the relation (67), of Theorem 1 (for some antiunitary involution $J$), and if we define the "adjointing operators" $\tilde{C}_R$ and $\tilde{C}_L$ by (69a) and (69b), then it can be shown that these antilinear operators are closable, and that

$$(\tilde{C}_L, \tilde{\mathcal{P}}(W_L)\Omega)^* \supset (\tilde{C}_R, \tilde{\mathcal{P}}(W_R)\Omega)^{**}. \qquad (72)$$

However, it *cannot* be concluded that the inclusion in (72) can be replaced by equality. We can see this as follows (within the framework of quantum field theory). Suppose that the two algebras had been defined "wrongly" in such a way that they were actually equal to two algebras which in our notation are written as $\mathcal{P}(W_R')$, respectively $\mathcal{P}(W_L')$, where $W_L' = \mathcal{J} W_R'$, and where $W_R'$ is a wedge properly included in $W_R$, and obtained from $W_R$ through a translation. The conditions (a) and (b), and the relation (67), of Theorem 1 would then be satisfied, and the relation (72) would hold. The two members in (72) are, however, not equal, because the "wrong" algebras are "too small." It is significant that the "wrong" algebras, constructed as above, also do not satisfy the relations (66), which say that the algebras are invariant under all velocity transformations $V(t)$.

As the above considerations indicate, it is easy to construct a large set of *distinct* closed extensions of $(C_R, \mathcal{P}(W_R)\Omega)$. Let $W_R''$ be any wedge obtained by a translation of $W_R$, and such that $W_R'' \supset W_R$. We define the operator $(C_R'', \mathcal{P}(W_R'')\Omega)$ in analogy with (69a), and we then have $(C_R'', \mathcal{P}(W_R'')\Omega) \supset (C_R, \mathcal{P}(W_R)\Omega)$, with a corresponding inclusion relation for the closures. It is easily seen that the closures are distinct if $W_R'' \neq W_R$.

Lemma 11 states facts about the field operators which are of crucial importance in the proof of Theorem 1. However, if we consider the role played by this lemma in the proof, it might seem miraculous that one can draw general conclusions about *all* the operators in $\mathcal{P}(W_R)$ from the properties of operators in a particular set $\mathcal{Q}$ which are locally associated with a family of regions $\{R_n \mid n=1,\ldots,\infty\}$ which does not cover $W_R$. Now it should be noted that the construction of the domain $D_{ac}$ involves operators in $V(t)\mathcal{Q}V(t)^{-1}$, for any real $t$, but it is still the case that the set of regions $\{V(e_3,t)R_n \mid n=1,\ldots,\infty, t\in R^1\}$ does not cover $W_R$ either. A closer examination of this issue reveals that the "potency" of the set $\mathcal{Q}$ ultimately depends on the geometrical fact that if $x$ is *any* point of $W_R$, then $\{V(e_3,t)x \mid t\in R^1\}^{cc} = W_R$, where the superscript cc denotes the causal complement of the causal complement.

Finally, we note that since $\mathcal{Q}\subset \mathcal{P}(W_R)$ it follows, in view of (68b) in Theorem 1, that the factor $c(K_3)$ in both members of (62) in Lemma 11 is in fact "unnecessary": The relation also makes sense if $c(K_3)$ is replaced by 1. We introduced this factor in order to have simple proofs of Lemmas 10 and 11.

## V. ON SOME ALGEBRAIC QUESTIONS CONNECTED WITH THEOREM 1.

This section is a mathematical preliminary to our discussion of physical duality conditions in the next section. The questions which we shall discuss are related to the issues of Theorem 1, although one might say that we are here more concerned with the properties of the triplet $(\Omega, J, K_3)$ than with the quantum fields.

We shall first be concerned with the characterization of operators in general (bounded or unbounded) which satisfy relations such as (68b) and (68c) in Theorem 1.

*Lemma* 12: Let $\mathcal{U}(W_R)$ be the set of all closable operators $(X, D(X))$ such that $\Omega\in D(X)\cap D(X^*)$, and such that $X\Omega\in D_+$ and

$$V(i\pi)X\Omega = JX^*\Omega. \tag{73a}$$

Let $\mathcal{U}(W_L)$ be the set of all closable operators $(Y, D(Y))$, such that $\Omega\in D(Y)\cap D(Y^*)$, and such that $Y\Omega\in D_-$ and

$$V(-i\pi)Y\Omega = JY^*\Omega. \tag{73b}$$

Then:

(a) $(X, D(X))^* = (X^*, D(X^*))\in \mathcal{U}(W_R)$ if $(X, D(X))\in\mathcal{U}(W_R)$ and $(Y, D(Y))^* = (Y^*, D(Y^*))\in\mathcal{U}(W_L)$ if $(Y, D(Y))\in\mathcal{U}(W_L)$.

(b)

$$J\mathcal{U}(W_R)J = \mathcal{U}(W_L),\quad J\mathcal{U}(W_L)J = \mathcal{U}(W_R), \tag{74}$$

i.e., $(X, D(X))\in\mathcal{U}(W_R)$ if and only if $(JXJ, JD(X))\in\mathcal{U}(W_L)$.

(c)

$$V(t)\mathcal{U}(W_R)V(t)^{-1} = \mathcal{U}(W_R),\quad V(t)\mathcal{U}(W_L)V(t)^{-1} = \mathcal{U}(W_L) \tag{75}$$

for all real $t$.

(d) Let $\mathcal{U}_b(W_R)$ denote the set of all *bounded* operators in $\mathcal{U}(W_R)$, and let $\mathcal{U}_b(W_L)$ denote the set of all *bounded* operators in $\mathcal{U}(W_L)$. Then

$$\mathcal{U}_b(W_R)\Omega = \mathcal{U}(W_R)\Omega = D_+,\quad \mathcal{U}_b(W_L)\Omega = \mathcal{U}(W_L)\Omega = D_-. \tag{76}$$

(e) The relation

$$\langle X^*\Omega \mid Y\Omega\rangle = \langle Y^*\Omega \mid X\Omega\rangle \tag{77}$$

holds for all operators $(X, D(X))\in\mathcal{U}(W_R)$, $(Y, D(Y))\in\mathcal{U}(W_L)$.

If a closable operator $(X, D(X))$ is such that $\Omega\in D(X)\cap D(X^*)$, then $(X, D(X))\in\mathcal{U}(W_R)$ if and only if (77) holds for all $(Y, D(Y))\in\mathcal{U}(W_L)$.

If a closable operator $(Y, D(Y))$ is such that $\Omega\in D(Y)\cap D(Y^*)$, then $(Y, D(Y))\in\mathcal{U}(W_L)$ if and only if (77) holds for all $(X, D(X))\in\mathcal{U}(W_R)$.

(f)

$$\mathcal{P}(W_R)\subset\mathcal{U}(W_R),\quad \mathcal{P}(W_L)\subset\mathcal{U}(W_L). \tag{78}$$

*Proof*: (1) The assertions (a) and (b) are trivial if we take into account the relations (59a) and (59b). The assertion (c) is completely trivial.

(2) We prove the assertions (d) by exhibiting explicit mappings of $D_+$ into $\mathcal{U}_b(W_R)$ and of $D_-$ into $\mathcal{U}_b(W_L)$. For any $\xi\in D_+$, let the bounded operator $Z_+(\xi)$ be defined by

$$Z_+(\xi) = |\xi\rangle\langle\Omega| + |\Omega\rangle\langle JV(i\pi)\xi| - \langle\Omega|\xi\rangle\,|\Omega\rangle\langle\Omega|. \tag{79a}$$

If we note that $\langle\Omega\mid\xi\rangle = \langle JV(i\pi)\xi\mid\Omega\rangle$, we easily see that the mapping $\xi\to Z_+(\xi)$ is a linear mapping of $D_+$ into $\mathcal{U}_b(W_R)$ such that

$$Z_+(\xi)\Omega = \xi,\quad Z_+(\xi)^*\Omega = JV(i\pi)\xi. \tag{79b}$$

This proves the equalities at left in (76). The equalities at right in (76) are proved in a similar manner, through a consideration of the mapping $\eta\to Z_-(\eta)$, where $\eta\in D_-$ and

$$Z_-(\eta) = |\eta\rangle\langle\Omega| + |\Omega\rangle\langle JV(-i\pi)\eta| - \langle\Omega|\eta\rangle\,|\Omega\rangle\langle\Omega|. \tag{79c}$$

(3) We next consider the assertions (e) in the lemma. Let $(X, D(X))\in\mathcal{U}(W_R)$ and $(Y, D(Y))\in\mathcal{U}(W_L)$. It follows from the relations (73) that

$$\begin{aligned}\langle X^*\Omega\mid Y\Omega\rangle &= \langle JV(i\pi)X\Omega\mid Y\Omega\rangle = \langle V(-i\pi)JX\Omega\mid Y\Omega\rangle\\ &= \langle JX\Omega\mid V(-i\pi)Y\Omega\rangle = \langle JX\Omega\mid JY^*\Omega\rangle\\ &= \langle Y^*\Omega\mid X\Omega\rangle\end{aligned} \tag{80}$$

which proves the formula (77).

(4) Now let $(X, D(X))$ be a closable operator such that $\Omega\in D(X)\cap D(X^*)$. The condition that (77) hold for all

$(Y, D(Y)) \in \mathcal{U}(W_L)$ is, in view of part (d) of the lemma, equivalent to the condition that

$$\langle X^*\Omega | \eta\rangle = \langle JV(-i\pi)\eta | X\Omega\rangle \tag{81}$$

for every $\eta \in D_-$. It is easily seen that Eq. (81) is equivalent to the equation

$$\langle J\eta | JX^*\Omega\rangle = \langle V(i\pi)J\eta | X\Omega\rangle. \tag{82}$$

Since $JD_- = D_+$, and since $(V(i\pi), D_+)$ is self-adjoint, we conclude that if (81), and hence (82), holds for every $\eta \in D_-$, then $X\Omega \in D_+$, and (73a) holds, i.e., $(X, D(X))$ is in the set $\mathcal{U}(W_R)$.

In the same manner we prove the last assertion in part (e).

(5) The assertion (f) in the lemma is a paraphrase of the assertions (d) in Theorem 1. This completes the proof.

It should be noted that the sets $\mathcal{U}(W_R)$ and $\mathcal{U}(W_L)$ are *not* algebras, and in fact not even linear manifolds. The sets $\mathcal{U}_b(W_R)$ and $\mathcal{U}_b(W_L)$ of bounded operators are not algebras either, but linear manifolds which are easily seen to be weakly closed. That an operator $X$ is included in one of the sets $\mathcal{U}(W_R)$ or $\mathcal{U}(W_L)$ is, in a sense, not a very restrictive condition: It is only a condition on the vectors $X\Omega$ and $X^*\Omega$. We found it convenient to introduce these sets since we will be dealing with operators which have properties such as those considered in the lemma.

We next consider some criteria for operators to be in these sets.

*Lemma* 13: (a) Let $(X, D(X))$ be closable, and such that $\Omega \in D(X) \cap D(X^*)$. Then $(X, D(X)) \in \mathcal{U}(W_R)$ if and only if there exists a set $\mathcal{C}_L \subset \mathcal{U}(W_L)$ such that span$\{\mathcal{C}_L\Omega\}$ is a core for $(V(-i\pi), D_-)$, and such that the relation

$$\langle X^*\Omega | Y\Omega\rangle = \langle Y^*\Omega | X\Omega\rangle \tag{83}$$

holds for all $(Y, D(Y)) \in \mathcal{C}_L$.

(b) Let $(Y, D(Y))$ be closable, and such that $\Omega \in D(Y) \cap D(Y^*)$. Then $(Y, D(Y)) \in \mathcal{U}(W_L)$ if and only if there exists a set $\mathcal{C}_R \subset \mathcal{U}(W_R)$ such that span $\{\mathcal{C}_R\Omega\}$ is a core for $(V(i\pi), D_+)$, and such that the relation (83) holds for all $(X, D(X)) \in \mathcal{C}_R$.

(c) Let $(X, D(X))$ be closable, and such that $\Omega \in D(X) \cap D(X^*)$. Then $(X, D(X)) \in \mathcal{U}(W_R)$ if and only if there exists a set $\mathcal{Q}_L \subset \mathcal{U}(W_L)$ such that span $\{\mathcal{Q}_L\Omega\}$ is dense in the Hilbert space $\mathcal{H}$, and

$$V(t)\mathcal{Q}_L V(t)^{-1} = \mathcal{Q}_L, \quad \text{all real } t, \tag{84a}$$

and such that the relation (83) holds for all $(Y, D(Y)) \in \mathcal{Q}_L$.

In particular, $(X, D(X)) \in \mathcal{U}(W_R)$ if and only if (83) holds for every $(Y, D_1) \in \mathcal{P}_0(W_L)$.

(d) Let $(Y, D(Y))$ be closable, and such that $\Omega \in D(Y) \cap D(Y^*)$. Then $(Y, D(Y)) \in \mathcal{U}(W_L)$ if and only if there exists a set $\mathcal{Q}_R \subset \mathcal{U}(W_R)$ such that span$\{\mathcal{Q}_R\Omega\}$ is dense in the Hilbert space $\mathcal{H}$, and

$$V(t)\mathcal{Q}_R V(t)^{-1} = \mathcal{Q}_R, \quad \text{all real } t, \tag{84b}$$

and such that the relation (83) holds for all $(X, D(X)) \in \mathcal{Q}_R$.

In particular, $(Y, D(Y)) \in \mathcal{U}(W_L)$ if and only if (83) holds for every $(X, D_1) \in \mathcal{P}_0(W_R)$.

*Proof*: (1) We consider the assertion (a). In view of the discussion in step (4) of the proof of the preceding lemma, we can restate the condition on $X$ as follows: The relation (82) holds for all $\eta$ in a core of $(V(-i\pi), D_-)$. Now, if $D'$ is a core for $(V(-i\pi), D_-)$, then $JD'$ is a core for $(V(i\pi), D_+)$, and we thus conclude, with reference to (82), that $X\Omega \in D_+$, and that (73a) holds. In an analogous manner we prove the assertion (b) in the lemma.

(2) The premises in part (c) of the lemma can be restated as follows: The relation

$$\langle JV(t)\eta | JX^*\Omega\rangle = \langle V(i\pi)JV(t)\eta | X\Omega\rangle \tag{85a}$$

holds for all real $t$, and all $\eta$ in the dense set $D'' = \text{span}\{\mathcal{Q}_L\Omega\}$. Let $c(s) \in \mathcal{D}(R^1)$. In view of (85a) and the relations (44a) and (44b) we then have

$$\begin{aligned}\langle Jc(K_3)\eta | JX^*\Omega\rangle &= \int_{-\infty}^{\infty} dt\hat{c}(t)\langle JV(t)\eta | JX^*\Omega\rangle \\ &= \int_{-\infty}^{\infty} dt\hat{c}(t)\langle V(i\pi)JV(t)\eta | X\Omega\rangle = \langle V(i\pi)Jc(K_3)\eta | X\Omega\rangle\end{aligned} \tag{85b}$$

for all $\eta \in D''$. In view of Lemma 7 the set $D''_c = \text{span}\{c(K_3)\eta | c(s) \in \mathcal{D}(R^1), \eta \in D''\}$ is a core for $(V(-i\pi), D_-)$, and the equality of the first and fourth members in (85b) then implies, and in step (1) above, that $(X, D(X)) \in \mathcal{U}(W_R)$.

In particular, these considerations hold for the case when $\mathcal{Q}_L = \mathcal{P}_0(W_L)$.

The assertions (d) are proved in an analogous manner.

We shall next consider the situation which arises when a subset of one of the sets $\mathcal{U}(W_R)$ or $\mathcal{U}(W_L)$ is an algebra. The following lemma is a preliminary for this study.

*Lemma* 14: Let $X_1, X_2 \in \mathcal{U}(W_R)$ be two *bounded* operators with the property that

$$X_1 V(t) X_2^* V(t)^{-1} \in \mathcal{U}(W_R), \quad \text{all real } t. \tag{86}$$

Then

$$X_1(JX_2J)\Omega = (JX_2J)X_1\Omega. \tag{87}$$

*Proof*: (1) Let $Y \in \mathcal{U}_b(W_L)$. The condition (86) then implies that

$$\langle Y\Omega | X_1 V(t) X_2^*\Omega\rangle = \langle V(t)X_2V(t)^{-1}X_1^*\Omega | Y^*\Omega\rangle \tag{88a}$$

for all real $t$. After a simple transformation of the right member, on the basis of the relations (73a) and (73b), we obtain from (88a) the relation

$$\langle Y\Omega | X_1 V(t) X_2^*\Omega\rangle = \langle V(-t-i\pi)Y\Omega | JX_2JV(i\pi - t)X_1\Omega\rangle. \tag{88b}$$

(2) In view of the properties of the exponential function $V(\tau) = \exp(-i\tau K_3)$ discussed in Sec. III (immediately preceding Lemma 7), we note that the three vector-valued functions of $\tau$ given by

$$X_1 V(\tau) X_2^*\Omega, \quad JX_2JV(i\pi - \tau)X_1\Omega, \tag{89a}$$

and

$$V(-\tau^* - i\pi)Y\Omega \qquad (89b)$$

are all well defined and strongly continuous on the closed strip $0 \le \mathrm{Im}(\tau) \le \pi$ in the complex $\tau$-plane. The functions in (89a) are strongly analytic functions of $\tau$ on the corresponding open strip, and the function in (89b) is a strongly analytic function of $\tau^*$ on the open strip $0 > \mathrm{Im}(\tau^*) > -\pi$. It follows that the function $f(\tau)$ defined by

$$f(\tau) = \langle Y\Omega | X_1 V(\tau) X_2^* \Omega\rangle - \langle V(-\tau^* - i\pi)Y\Omega | JX_2JV(i\pi - \tau)X_1\Omega\rangle \qquad (89c)$$

is continuous on the closed strip $0 \le \mathrm{Im}(\tau) \le \pi$ and an analytic function of $\tau$ on the open strip $0 < \mathrm{Im}(\tau) < \pi$. By (88b) we have $f(t) = 0$ for all real $t$, and it follows that $f(\tau) = 0$ throughout the closed strip. In particular, we have $f(i\pi) = 0$, which, in view of (89c) and the relation (73a), implies that

$$\langle Y\Omega | X_1 J X_2 \Omega\rangle = \langle Y\Omega | J X_2 J X_1 \Omega\rangle \qquad (89d)$$

for all $Y \in \mathcal{U}_b(W_L)$. Since $\mathcal{U}_b(W_L)\Omega$ is dense in the Hilbert space $\mathcal{H}$ by Lemma 12 the relation (87) follows.

We shall now consider von Neumann algebras of bounded operators. If $\mathcal{B}$ is any set of bounded operators we denote the commutant of $\mathcal{B}$ by $\mathcal{B}'$, and we write $\mathcal{B}''$ for $(\mathcal{B}')'$.

*Theorem* 2: Let $\mathcal{A}_R \subset \mathcal{U}(W_R)$ be a von Neumann algebra such that $\mathcal{A}_R\Omega$ is dense in the Hilbert space $\mathcal{H}$, and such that

$$V(t)\mathcal{A}_R V(t)^{-1} = \mathcal{A}_R, \quad \text{all real } t. \qquad (90)$$

Let the von Neumann algebra $\mathcal{A}_L$ be defined by $\mathcal{A}_L = J\mathcal{A}_R J$. Then:

(a)

$$\mathcal{A}'_R = J\mathcal{A}_R J = \mathcal{A}_L \subset \mathcal{U}(W_L),$$
$$\mathcal{A}'_L = J\mathcal{A}_L J = \mathcal{A}_R \subset \mathcal{U}(W_R). \qquad (91)$$

(b) The vector $\Omega$ is cyclic and separating for $\mathcal{A}_R$ and $\mathcal{A}_L$.

(c) For any real $t$,

$$V(t)\mathcal{A}_L V(t)^{-1} = \mathcal{A}_L. \qquad (92)$$

(d) The linear manifold $\mathcal{A}_R\Omega$ is a core for $(V(i\pi), D_+)$, and hence also for the antilinear operator $(JV(i\pi), D_+)$.

The linear manifold $\mathcal{A}_L\Omega$ is a core for $(V(-i\pi), D_-)$, and hence also for the antilinear operator $(JV(-i\pi), D_-)$.

The linear manifold $\{\mathcal{A}_R\Omega\} \cap \{\mathcal{A}_L\Omega\}$ is dense in the Hilbert space $\mathcal{H}$, and a core for the operators $(V(i\pi), D_+)$ and $(V(-i\pi), D_-)$.

(e) The von Neumann algebra $\mathcal{A}_R$ is "maximal" in the sense that if $\mathcal{A}$ is any von Neumann algebra with $\Omega$ as a *separating* vector, and such that $\mathcal{A}_R \subset \mathcal{A}$, and such that $V(t)\mathcal{A}V(t)^{-1} = \mathcal{A}$ for all real $t$, then $\mathcal{A} = \mathcal{A}_R$. The algebra $\mathcal{A}_R$ is "minimal" in the sense that if $\mathcal{A}$ is a von Neumann algebra with $\Omega$ as a *cyclic* vector, and such that $\mathcal{A} \subset \mathcal{A}_R$, and such that $V(t)\mathcal{A}V(t)^{-1} = \mathcal{A}$ for all real $t$, then $\mathcal{A} = \mathcal{A}_R$.

The algebra $\mathcal{A}_L$ is "maximal" and "minimal" in the same sense.

(f) The von Neumann algebra $\mathcal{A}_R$ is also "maximal within $\mathcal{U}(W_R)$" in the sense that if $\mathcal{A}$ is any von Neumann algebra such that $\mathcal{A}_R \subset \mathcal{A} \subset \mathcal{U}(W_R)$, then $\mathcal{A} = \mathcal{A}_R$.

The algebra $\mathcal{A}_L$ is "maximal within $\mathcal{U}(W_L)$" in the analogous sense.

*Proof*: (1) We note that the premises of Lemma 14 are satisfied by any two operators in $\mathcal{A}_R$. Let $X_1, X_2, X_3 \in \mathcal{A}_R$. In view of the lemma we have the following string of equalities:

$$JX_2JX_1X_3\Omega = X_1X_3JX_2\Omega = (X_1J)JX_3JX_2\Omega = X_1JX_2JX_3\Omega. \qquad (93a)$$

Since, by the premises of the theorem, the set $\{X_3\Omega | X_3 \in \mathcal{A}_R\}$ is dense in $\mathcal{H}$, we conclude that $[(JX_2J), X_1] = 0$, for any two $X_1, X_2 \in \mathcal{A}_R$, and hence we have $J\mathcal{A}_R J \subset \mathcal{A}'_R$.

(2) The premises of part (d) of Lemma 13 are satisfied for any $Y \in \mathcal{A}'_R$ with $\mathcal{Q}_R = \mathcal{A}_R$, and it follows that $\mathcal{A}'_R \subset \mathcal{U}(W_L)$. In view of the conclusion in step (1) above we thus have

$$\mathcal{A}_L = J\mathcal{A}_R J \subset \mathcal{A}'_R \subset \mathcal{U}(W_L). \qquad (93b)$$

(3) Since $\mathcal{A}_R\Omega$ is dense, the set $J\mathcal{A}'_R J\Omega$ is also dense, in view of (93b). The condition (90) implies that $V(t)\mathcal{A}'_R V(t)^{-1} = \mathcal{A}'_R$, and hence that $V(t)(J\mathcal{A}'_R J)V(t)^{-1} = J\mathcal{A}'_R J$, for all real $t$. Since it follows from (93b) that $J\mathcal{A}'_R J \subset \mathcal{U}(W_R)$, we conclude, by the same reasoning as in step (1) above, that

$$\mathcal{A}'_R = J(J\mathcal{A}'_R J)J \subset (J\mathcal{A}'_R J)' = J\mathcal{A}''_R J = J\mathcal{A}_R J. \qquad (93c)$$

The relations (91) then follow trivially from (93b) and (93c). From what has been said we also conclude that (92) holds.

(4) We prove the assertions (d) on the basis of (92) and (90). Let $c(s) \in \mathcal{D}(R^1)$, and let $X \in \mathcal{A}_R$. We define the operator $X_c$ by

$$X_c = \int_{-\infty}^{\infty} dt\, \hat{c}(t) V(t) X V(t)^{-1} \qquad (94a)$$

where $\hat{c}(t)$ is given in (44b). We obviously have $X_c \in \mathcal{A}_R$, and furthermore

$$X_c\Omega = c(K_3)X\Omega. \qquad (94b)$$

We then conclude, in view of Lemma 7, that the linear manifold $D_A = \{X_c\Omega | X \in \mathcal{A}_R, c(s) \in \mathcal{D}(R^1)\}$ is a core for *every* operator $(V(z), D_V(\mathrm{Im}(z)))$.

For every $Y \in \mathcal{A}_L$, and any $c(s) \in \mathcal{D}(R^1)$, we define $Y_c$ by the integral at right in (94a), with $X$ replaced by $Y$. We then have $Y_c \in \mathcal{A}_L$, and

$$Y_c\Omega = c(K_3)Y\Omega = (V(i\pi)c(K_3))(JY^*J)\Omega \qquad (94c)$$

where the second member is equal to the third in view of (73b). Since $JY^*J \in \mathcal{A}_R$, and since $\exp(s\pi)c(s) \in \mathcal{D}(R^1)$, we conclude that $D_A = \{Y_c\Omega | Y \in \mathcal{A}_L, c(s) \in \mathcal{D}(R^1)\}$. Since $\mathcal{A}_R\Omega \subset D_+$ and $\mathcal{A}_L\Omega \subset D_-$, the assertions (d) now follow trivially from the properties of the manifold $D_A$.

(5) The vector $\Omega$ is a cyclic vector for $\mathcal{A}_R$ by the premises, and also, trivially, a cyclic vector for $\mathcal{A}_L$. In view of (91) it follows that $\Omega$ is a separating vector for both $\mathcal{A}_R$ and $\mathcal{A}_L$.

(6) We next consider the assertion in part (e) of the theorem. If $\mathcal{A}$ is any von Neumann algebra with $\Omega$ as a separating vector, and such that $\mathcal{A}_R \subset \mathcal{A}$, and such that $V(t)\mathcal{A}V(t)^{-1} = \mathcal{A}$ for all real $t$, then $\mathcal{A}' \subset \mathcal{A}_R' \subset \mathcal{U}(W_L)$, and $\Omega$ is a cyclic vector for $\mathcal{A}'$, and hence for $J\mathcal{A}'J \subset \mathcal{U}(W_R)$. Furthermore, $V(t)(J\mathcal{A}'J)V(t)^{-1} = J\mathcal{A}'J$. The von Neumann algebra $J\mathcal{A}'J$ thus satisfies the premises of the present theorem, and it follows from the already established relations (91) that $J\mathcal{A}J = \mathcal{A}'$, and from this relation it readily follows that $\mathcal{A} = \mathcal{A}_R$, as asserted.

Suppose now that $\mathcal{A}$ is a von Neumann algebra with $\Omega$ as a *cyclic* vector, and such that $\mathcal{A} \subset \mathcal{A}_R$, and such that $V(t)\mathcal{A}V(t)^{-1} = \mathcal{A}$ for all real $t$. Then $\mathcal{A}$ satisfies the premises of the present theorem. In particular, $\mathcal{A}$ is "maximal," which implies that $\mathcal{A} = \mathcal{A}_R$.

In a similar fashion we show that $\mathcal{A}_L$ is "maximal" and "minimal."

(7) To prove the assertion (f) we consider the string of equalities (93a). Suppose that $X_1, X_3 \in \mathcal{A}_R$, and suppose that $X_2$ is an element of a von Neumann algebra $\mathcal{A}$ such that $\mathcal{A}_R \subset \mathcal{A} \subset \mathcal{U}(W_R)$. It is easily seen that the premises of Lemma 14 are satisfied by the pair of operators $(X_1X_3)$ and $X_2$, and also by the pair of operators $X_3$ and $X_2$. It follows that the equalities in (93a) also hold in the present case, and we conclude, as in step (1) of the proof, that $JX_2J \in \mathcal{A}_R'$, i.e., $J\mathcal{A}J \subset \mathcal{A}_R'$. It follows that $\mathcal{A} \subset J\mathcal{A}_R'J = \mathcal{A}_R$, and hence we have $\mathcal{A} = \mathcal{A}_R$, as asserted. This completes the proof of the theorem.

It should be noted that this theorem as such has little to do with the quantum field. It is of physical interest only if the algebra $\mathcal{A}_R$ is in some sense "generated" by field operators in $\mathcal{P}(W_R)$. We are not here asserting that such an algebra $\mathcal{A}_R$ actually exists. This issue will be discussed in the next section.

At this point we wish to discuss the relationship between our considerations and the Tomita–Takesaki theory of modular Hilbert algebras.[13,25] Within the framework of this theory one is able to draw some highly interesting conclusions about the structure of von Neumann algebras. The main theorem (from our point of view) is due to Tomita, and we shall state the facts in the following form.

Let $\mathcal{A}$ be a von Neumann algebra (of operators on a separable Hilbert space) which has a cyclic and separating vector $\Omega$, and let $\mathcal{A}'$ denote its commutant. Then there exists a unique antiunitary involution $J$, and a unique self-adjoint operator $(K, D_K)$, which satisfy the following conditions:

(a) $J\Omega = \Omega, \quad \Omega \in D_K, \quad K\Omega = 0;$ (95a)

(b) $J\mathcal{A}J = \mathcal{A}'$; (95b)

(c) $JD_K = D_K, \quad J(K, D_K)J = (-K, D_K);$ (95c)

(d) $\exp(-itK)\mathcal{A}\exp(itK) = \mathcal{A},$

$\exp(-itK)\mathcal{A}'\exp(itK) = \mathcal{A}',$ (95d)

for all real $t$, and the one-parameter group of unitary operators $\exp(-itK)$ is thus, acting by conjugation, a group of automorphisms of $\mathcal{A}$ and of $\mathcal{A}'$.

(e) If $(C, \mathcal{A}\Omega)$ is the antilinear operator defined by

$$CX\Omega = X^*\Omega, \quad \text{all } X \in \mathcal{A}, \tag{95e}$$

then

$$(J\exp(\pi K), D_+) = (C, \mathcal{A}\Omega)^{**} \tag{95f}$$

where $D_+$ is the linear manifold such that $(\exp(\pi K), D_+)$ is self-adjoint.

We note here that the operator $\exp(2\pi K)$ is traditionally denoted by $\Delta$ in papers on the subject: Our notation in terms of the operator $K$ is specific for this paper, and motivated by our physical considerations.

The existing proofs of Tomita's theorem can hardly be regarded as trivial. Given the von Neumann algebra $\mathcal{A}$ and the cyclic and separating vector $\Omega$, the operators $J$ and $\Delta$ [and also the operator $K$ by $2\pi K = \ln(\Delta)$] are in fact determined through (95f), which describes the polar decomposition of the closure of the antilinear operator $(C, \mathcal{A}\Omega)$. With this construction it is easily shown that the relations (95a) and (95c) hold, but the relations (95b) and (95d) are entirely nontrivial. In this paper we do not depend on Tomita's theorem, but we wanted to point out its relevance to our discussion. In particular our Theorem 2 is within the purview of the Tomita–Takesaki theory. In a sense this theorem contains nothing new, but we wanted to state the facts in this form for later reference, and also to prove these facts in an elementary way directly from the particular set of premises which arises naturally from our physical considerations. In our case the existence of $J$ and $K$ is not the issue since we are given the triplet $(\Omega, J, K_3)$ to start with. If we now compare the situation described in Theorem 2 with the situation described in Tomita's theorem we see that our operators $J$ and $K = K_3$ are precisely the operators which in Tomita's theorem are determined by the algebra $\mathcal{A} = \mathcal{A}_R$.

Let us also note here that there are similarities between our discussion of Lemma 14 and Theorem 2, and the work of Haag, Hugenholtz, and Winnink,[26] and the work of Kastler, Pool, and Thue Poulsen.[27]

If we consider Theorem 1 we note some further analogies with the Tomita–Takesaki theory, although it should be noted that Theorem 1 concerns unbounded operators, rather than bounded operators as in Tomita's theorem. The definition (69a) is thus analogous to the definition (95e) above, and the relation (69c) is analogous to (95f). The relation (67) has a tenuous connection with (95b), but it should be noted that it is not proper to regard the algebra $\mathcal{P}(W_L)$ as the "commutant" of $\mathcal{P}(W_R)$: These algebras are rather analogous to some pair of algebras which generate the algebras $\mathcal{A}$ and $\mathcal{A}'$.

The connection between the duality condition in quantum field theory and Tomita's theorem has been discussed previously by Eckmann and Osterwalder, in their discussion of the duality condition for a *free* field.[7] We shall comment further on this in Sec. VII.

We conclude this section with an addendum to Theorem 2.

*Lemma* 15: Let $\mathcal{A}_R$ be a von Neumann algebra which satisfies the premises of Theorem 2. Then $\mathcal{A}_R$ and $\mathcal{A}_L = J\mathcal{A}_RJ = \mathcal{A}_R'$ are factors.

*Proof*: That the algebras $\mathcal{A}_R$ and $\mathcal{A}_L$ are *factors* means that their centers are equal to the set $\{cI\}$ of all complex multiples of the identity. In the case at hand this condition is equivalent to the statement $\mathcal{A}_R \cap \mathcal{A}_L = \{cI\}$.

Let $Z \in \mathcal{A}_R \cap \mathcal{A}_L$. Since $Z$ is then an element of the set $\mathcal{U}(W_R) \cap \mathcal{U}(W_L)$, it follows from (73a) and (73b) that

$$V(i\pi)Z\Omega = JZ^*\Omega = V(-i\pi)Z\Omega. \tag{96a}$$

This implies that $V(i\pi)Z\Omega \in D_+$, and that

$$V(2\pi i)Z\Omega = \exp(2\pi K_3)Z\Omega = Z\Omega, \tag{96b}$$

which implies that $Z\Omega$ is an eigenvector of $K_3$, with eigenvalue 0. It is easily seen (and well known) that under our general assumptions about the nature of the representation of $\bar{L}_0$ carried by the Hilbert space $\mathcal{H}$, the only eigenvector of $K_3$ is the vacuum vector $\Omega$. It follows from the above that $Z\Omega = c\Omega$, for some complex number $c$, and hence that $Z = cI$. This proves the lemma.

## VI. THE DUALITY CONDITION FOR THE WEDGE REGIONS $W_R$ AND $W_L$

In this section we shall consider conditions under which the operators in $\mathcal{P}(W_R)$ "generate" a von Neumann algebra $\mathcal{A}_R$ which satisfies the premises of Theorem 2. The basic idea is very simple. We try to construct $\mathcal{A}_R$ as the "commutant" of a suitable subset of operators in $\mathcal{P}(W_L)$. The execution of this idea is, however, beset with "technical" difficulties which derive from the fact that the operator in $\mathcal{P}(W_L)$ are in general unbounded. Furthermore, we are faced with the unfortunate situation that practically nothing is known about the nature of these operators as mathematical objects. It is, for instance, not known at present whether the field operators $\varphi[f]$, with $f$ real, have any *local* self-adjoint extensions in a sense which will be discussed later. In our discussion we wish to avoid making assumptions which might later turn out to be too restrictive. For this reason we do not try to define the algebra $\mathcal{A}_R$ in terms of the commutant of *all* the operators in the set $\mathcal{P}(W_L)$, but instead in terms of the commutant of the field operators $\varphi[f]$, with $\operatorname{supp}(f) \subset W_L$.

We begin with some general considerations about the commutant of a subset of $\mathcal{P}(\mathcal{M})$.

*Lemma* 16: Let $\mathcal{J}$ be a subset of $\mathcal{P}(\mathcal{M})$, such that $(X^*, D_1) \in \mathcal{J}$ for all $(X, D_1) \in \mathcal{J}$. Let $\mathcal{K}_f$ be the set of all *bounded* operators $Q$ such that

$$QD_1 \subset D(X^{**}), \quad [Q, X^{**}]\psi = 0 \tag{97a}$$

for all $\psi \in D_1$, and all $(X, D_1) \in \mathcal{J}$. Then:

(a)

$$QD(X^{**}) \subset D(X^{**}), \quad [Q, X^{**}]\psi = 0 \quad \text{for all } \psi \in D(X^{**}), \tag{97b}$$

$$Q^*D(X^*) \subset D(X^*), \quad [Q^*, X^*]\phi = 0 \quad \text{for all } \phi \in D(X^*), \tag{97c}$$

for all $(X, D_1) \in \mathcal{J}$.

(b) The set $\mathcal{K}_f$ is a weakly closed algebra. The set $\mathcal{A}_f = \mathcal{K}_f \cap \mathcal{K}_f^* = \{Q \mid Q, Q^* \in \mathcal{K}_f\}$ is a von Neumann algebra. This algebra is precisely equal to the set of all bounded operators $Q$ such that

$$(X, D_1)^{**}Q \supset Q(X, D_1)^{**}, \quad (X, D_1)^*Q \supset Q(X, D_1)^* \tag{98}$$

for all $(X, D_1) \in \mathcal{J}$.

(c) If $G$ is any unitary operator such that $GD_1 = D_1$ and $G\mathcal{J}G^{-1} \subset \mathcal{J}$, then $G^{-1}\mathcal{A}_f G \subset \mathcal{A}_f$.

(d) Let $\mathcal{P}_f$ be the polynomial algebra (on $D_1$) generated by $\mathcal{J}$. Then

$$\langle X^*\phi \mid Q\psi\rangle = \langle Q^*\phi \mid X\psi\rangle \tag{99}$$

for any $X \in \mathcal{P}_f$, any $Q \in \mathcal{A}_f$, and any $\phi, \psi \in D_1$.

We omit the proofs since the above lemma is merely a summary of trivial and well-known facts. That $\mathcal{A}_f$ is a von Neumann algebra if all operators $Q$ in this set satisfies (98) was shown by von Neumann,[28] and the conditions (98) correspond to his conditions that the *bounded* operators $Q$ and $Q^*$ commute with the closable operator $(X, D_1)$. We note here that $\mathcal{K}_f$ need not be a von Neumann algebra, i.e., $Q^*$ is not necessarily included in $\mathcal{K}_f$ for every $Q \in \mathcal{K}_f$. This circumstance derives from the fact that the adjoints of the operators in $\mathcal{J}$ are not necessarily included in the set of all closures of the operators in $\mathcal{J}$. If it happens to be the case that $(X^\dagger, D_1)^* = (X, D_1)^{**}$ for all $(X, D_1) \in \mathcal{J}$, then $\mathcal{K}_f = \mathcal{K}_f^* = \mathcal{A}_f$.

We shall define the commutants of sets of field operators in terms of the conditions (98), and we are now prepared to state a somewhat lengthy theorem concerning the commutants of field operators associated with either one of the wedge regions $W_R$ and $W_L$.

*Theorem* 3: Let $\mathcal{A}_c(W_R)$ be the von Neumann algebra of all bounded operators $Q$ such that

$$\begin{aligned} &Q(\varphi[f], D_1)^{**} \subset (\varphi[f], D_1)^{**}Q, \\ &Q(\varphi[f], D_1)^{*} \subset (\varphi[f], D_1)^{*}Q \end{aligned} \tag{100}$$

for all $f \in \mathcal{S}(R^4)$ such that $\operatorname{supp}(f) \subset W_L$.

Similarly, let $\mathcal{A}_c(W_L)$ be the von Neumann algebra of all bounded operators $Q$ such that (100) holds for all $f \in \mathcal{S}(R^4)$ such that $\operatorname{supp}(f) \subset W_R$.

Then:

(a)

$$\mathcal{A}_c(W_R) \subset \mathcal{A}_c(W_L)', \quad \mathcal{A}_c(W_L) \subset \mathcal{A}_c(W_R)'. \tag{101}$$

(b)

$$\mathcal{A}_c(W_R) = U(R(\mathrm{e}_1, \pi), 0)\mathcal{A}_c(W_L)U(R(\mathrm{e}_1, \pi), 0)^{-1} \tag{102a}$$

where $R(\mathrm{e}_1, \pi)$ denotes the rotation by angle $\pi$ about the 1-axis.

Let $\sigma(W_R)$ be the semigroup of all elements in the Poincaré group $\bar{L}_0$ which map $W_R$ *into* $W_R$. Similarly, let $\sigma(W_L) = \{\Lambda^{-1} \mid \Lambda \in \sigma(W_R)\}$ be the semigroup of all elements in the group $\bar{L}_0$ which map $W_L$ *into* $W_L$. Then

$$U(\Lambda)\mathcal{A}_c(W_R)U(\Lambda)^{-1} \subset \mathcal{A}_c(W_R), \quad \text{all } \Lambda \in \sigma(W_R), \tag{102b}$$

and

$$U(\Lambda)\mathcal{A}_c(W_L)U(\Lambda)^{-1} \subset \mathcal{A}_c(W_L), \quad \text{all } \Lambda \in \sigma(W_L). \tag{102c}$$

The set $\bar{L}_0(W_R) = \sigma(W_R) \cap \sigma(W_L)$ is the group of all elements of $\overline{L}_0$ which map $W_R$ *onto* $W_R$ and $W_L$ *onto* $W_L$, and we have

$$U(\Lambda)\mathcal{A}_c(W_R)U(\Lambda)^{-1}=\mathcal{A}_c(W_R), \quad U(\Lambda)\mathcal{A}_c(W_L)U(\Lambda)^{-1}=\mathcal{A}_c(W_L) \tag{102d}$$

for all $\Lambda\in\overline{L}_0(W_R)$. In particular,

$$V(t)\mathcal{A}_c(W_R)V(t)^{-1}=\mathcal{A}_c(W_R), \quad V(t)\mathcal{A}_c(W_L)V(t)^{-1}=\mathcal{A}_c(W_L) \tag{102e}$$

for all real $t$.

(c)

$$\mathcal{A}_c(W_R)=J\mathcal{A}_c(W_L)J. \tag{102f}$$

(d) The relations

$$\langle X^*\phi\,|\,Y\psi\rangle=\langle Y^*\phi\,|\,X\psi\rangle, \quad \text{all } \phi,\psi\in D_1, \tag{103}$$

hold for all $X\in\mathcal{A}_c(W_R)$ and all $Y\in\mathcal{P}(W_L)$.

The relations (103) also hold for all $X\in\mathcal{P}(W_R)$ and all $Y\in\mathcal{A}_c(W_L)$.

(e) With the notation in Lemma 12 we have $\mathcal{A}_c(W_R)\subset\mathcal{U}_b(W_R)$ and $\mathcal{A}_c(W_L)\subset\mathcal{U}_b(W_L)$, and hence $\mathcal{A}_c(W_R)\Omega\subset D_+$, $\mathcal{A}_c(W_L)\Omega\subset D_-$, and

$$V(i\pi)X\Omega=JX^*\Omega, \quad \text{all } X\in\mathcal{A}_c(W_R), \tag{104a}$$

$$V(-i\pi)Y\Omega=JY^*\Omega, \quad \text{all } Y\in\mathcal{A}_c(W_L). \tag{104b}$$

(f) If it is the case, *in addition*, that $\mathcal{A}_c(W_R)\Omega$ is dense in the Hilbert space $\mathcal{H}$, then the algebra $\mathcal{A}_R=\mathcal{A}_c(W_R)$ satisfies all the premises of Theorem 2 and Lemma 15, and, with reference to the notation in Theorem 2, $\mathcal{A}_L=\mathcal{A}_c(W_L)$. In particular, the algebras $\mathcal{A}_c(W_R)$ and $\mathcal{A}_c(W_L)$ are *factors*, and they satisfy the *duality condition*

$$\mathcal{A}_c(W_R)=\mathcal{A}_c(W_L)'. \tag{105}$$

*Proof*: (1) That $\mathcal{A}_c(W_R)$ and $\mathcal{A}_c(W_L)$ are indeed von Neumann algebras follows from Lemma 16. We temporarily postpone the proof of the relations (101) (of which either one implies the other). The assertions (b) and (c) of the theorem are all trivial. We consider the assertions in part (d). From Lemma 16 it follows that (103) holds for all $X\in\mathcal{A}_c(W_R)$ and all $Y\in\mathcal{P}_0(W_L)$. In view of Lemma 1 these relations also hold for all $Y\in\mathcal{P}(W_L)$ and all $X\in\mathcal{A}_c(W_R)$, as asserted. Analogous considerations apply to the second assertion (d).

(2) The assertions (e) now follow trivially from Lemma 13 and part (d) of the theorem [setting $\phi=\psi=\Omega$ in (103)].

(3) Having established part (e) we conclude from (102e) and (102f), on the basis of Lemma 14, that

$$[X,Y]\Omega=0 \tag{106a}$$

for all $X\in\mathcal{A}_c(W_R)$ and all $Y\in\mathcal{A}_c(W_L)$.

Let $x\in W_R$, and let $X(x)=T(x)XT(x)^{-1}$. We then have $\Lambda(\mathrm{I},x)\in\sigma(W_R)$, i.e., $\Lambda(\mathrm{I},x)W_R\subset W_R$, and hence $X(x)\in\mathcal{A}_c(W_R)$ whenever $X\in\mathcal{A}_c(W_R)$. For any such $X(x)$ the relation (106a) thus holds for any $Y\in\mathcal{A}_c(W_L)$, with $X(x)$ substituted for $X$.

Let $R=W_R\cap\Lambda(\mathrm{I},x)W_L$. This region is open and nonempty for any $x\in W_R$. It is easily seen that if $Q=[X(x),Y]$, with $X(x)$ and $Y$ as above, then the conditions (100) hold for any $f\in\mathcal{S}(R^4)$ such that $\mathrm{supp}(f)\subset R$. By Lemma 16 we then conclude that

$$\langle Z_1\Omega\,|\,[X(x),Y]Z_2\Omega\rangle=\langle Z_2{}^*Z_1\Omega\,|\,[X(x),Y]\Omega\rangle=0 \tag{106b}$$

for any $Z_1,Z_2\in\mathcal{P}_0(R)$. Since $\mathcal{P}_0(R)\Omega$ is dense it follows that $[X(x),Y]=0$, for all $x\in W_R$. Since the point $x=0$ is on the boundary of $W_R$, and since $X(x)$ is a strongly continuous function of $x$ [in view of the strong continuity of the function $T(x)$] we conclude that $[X,Y]=0$. This proves the assertions (a) of the Theorem.

(4) The assertions (f) follow trivially from Theorem 2 and Lemma 15. This completes the proof of the theorem.

We note that the assertions (b) in the theorem correspond to geometrical conditions which obviously have to be satisfied if we wish to regard $\mathcal{A}_c(W_R)$ as locally associated with $W_R$ and $\mathcal{A}_c(W_L)$ as locally associated with $W_L$. In a theory in which a physical TCP-operator exists, as is the case here, the condition (102f) must also hold. The commutation relations implied by (101) correspond to a *minimal* condition of "physical independence" of the operators in $\mathcal{A}_c(W_R)$ from the operators in $\mathcal{A}_c(W_L)$. We note that the result (101) is analogous to a well-known theorem of Borchers concerning the local nature of a field which is local relative to a local irreducible field.[14] The relations (103) in part (d) are "commutation relations" between the bounded operators in the von Neumann algebras and the unbounded operators in $\mathcal{P}(\mathcal{M})$ in a sense which is weaker than the sense in which $Q$ commutes with $\varphi[f]$ in (100). The assertions (d) can be restated as follows[29]:

$$X(Y^*,D_1)\subset(Y,D_1)^*X \tag{107a}$$

for all $X\in\mathcal{A}_c(W_R)$ and all $Y\in\mathcal{P}(W_L)$, and

$$Y(X^*,D_1)\subset(X,D_1)^*Y \tag{107b}$$

for all $Y\in\mathcal{A}_c(W_L)$ and all $X\in\mathcal{P}(W_R)$.

In the following we shall call a pair of von Neumann algebras $\mathcal{A}(W_R)$ and $\mathcal{A}(W_L)$ a pair of *local wedge-algebras* if and only if they satisfy all the relations (101)—(103) which the algebras $\mathcal{A}_c(W_R)$ and $\mathcal{A}_c(W_L)$ satisfy. It follows that a pair of local wedge-algebras *also* satisfies the relations (104), by the same reasoning as in the proof of Theorem 3. Note that neither the duality condition (105), nor the commutation relations (100), are implied in the notion of a pair of local wedge-algebras.

With respect to the duality condition (105) the situation is as follows. The algebras $\mathcal{A}_c(W_R)$ and $\mathcal{A}_c(W_L)$ are uniquely determined by the field $\varphi(x)$, and it is then a matter of "checking" whether these algebras are sufficiently large in the sense that $\mathcal{A}_c(W_R)\Omega$ is dense in the Hilbert space $\mathcal{H}$. We do not know at this time whether $\mathcal{A}_c(W_R)\Omega$ is dense in general, i.e., with *no* additional assumptions about the field. It seems to us that in a physical theory described in terms of local observables and a local quantum field $\varphi(x)$ it must be the case that there exists a von Neumann algebra $\mathcal{A}(W_R)$, generated by the observables associated with the region $W_R$, and similarly an algebra $\mathcal{A}(W_L)$, and such that these algebras satisfy the conditions (a)—(d) in Theorem 3. In addition, we might require that the family of observables associated with $W_R$ is sufficiently large so that $\mathcal{A}(W_R)\Omega$ is dense in $\mathcal{H}$. As an example of the kind of considerations which are relevant here we refer to the work of Licht on "strict localization.[30]" If the algebra $\mathcal{A}(W_R)$ satisfies the above conditions, then $\mathcal{A}(W_R) \subset \mathcal{U}(W_R)$ and the relation (104a) holds because $\mathcal{A}(W_R)$ is a local wedge-algebra, and since $\mathcal{A}(W_R)\Omega$ is dense, it follows that the duality condition $\mathcal{A}(W_R)' = \mathcal{A}(W_L)$ holds.

If it is the case that $\mathcal{A}_c(W_R)\Omega$ is dense we would define the "algebra of observables" $\mathcal{A}(W_R)$ by $\mathcal{A}(W_R) = \mathcal{A}_c(W_R)$, with reference to the construction in Theorem 3. If $\mathcal{A}_c(W_R)\Omega$ is *not* dense, the algebra $\mathcal{A}(W_R)$, if it exists, would have to be defined differently. *One* possibility is the following. It might be the case that $\mathcal{A}(W_R)$ could be defined in a satisfactory manner as the commutant of some other subset of $\mathcal{P}(W_L)$ which is "better behaved" than the set of operators $\varphi[f]$ in $\mathcal{P}(W_L)$. Since we feel that we have no basis for a rational choice we shall not discuss this possibility. Another possibility is that there might exist, within the framework of the particular theory, natural extensions of the field operators $\varphi[f]$. We could then try to define $\mathcal{A}(W_R)$ as the commutant of the extensions of the operators $\varphi[f]$ in $\mathcal{P}(W_L)$, if it so happens that $\mathcal{A}(W_R)\Omega$ is dense for this choice. We shall consider a particular case of this situation below. The general problem of how to define algebras of bounded operators in terms of the unbounded field operators has been discussed by many authors, and what we say below is not particularly novel.[1,16,29–31]

We shall now consider four particular conditions on the quantum field which seem to us to be interesting to contemplate. Each one of these conditions guarantees the existence of local von Neumann algebras which satisfy the duality condition (105) (for the wedge regions $W_R$ and $W_L$).

*Condition I*: The linear manifold $\mathcal{A}_c(W_R)\Omega$ is dense in the Hilbert space $\mathcal{H}$, where $\mathcal{A}_c(W_R)$ is the von Neumann algebra constructed from the field as in Theorem 3.

*Condition II*: For any open nonempty subset $R$ of Minkowski space the linear manifold $\mathcal{C}(R)\Omega$ is dense in the Hilbert space $\mathcal{H}$, where $\mathcal{C}(R)$ is the von Neumann algebra of all bounded operators $Q$ such that

$$\begin{aligned} &Q(\varphi[f], D_1)^{**} \subset (\varphi[f], D_1)^{**}Q, \\ &Q(\varphi[f], D_1)^{*} \subset (\varphi[f], D_1)^{*}Q \end{aligned} \tag{108}$$

for all $f \in \mathcal{S}(R^4)$ such that $\text{supp}(f) \subset (\bar{R})^c$, where $(\bar{R})^c$ denotes the causal complement of the closure of $R$.

*Condition III*: The quantum field $\varphi(x)$ has a *local self-adjoint extention* in the following sense. To each $f \in \mathcal{S}(R^4)$ corresponds a *closed* operator $(\bar{\varphi}[f], D(f))$ such that:

(a)

$$(\bar{\varphi}[f], D(f))^* = (\bar{\varphi}[f^*], D(f^*)), \tag{109a}$$

$$(\bar{\varphi}[f], D(f)) \supset (\varphi[f], D_1) \tag{109b}$$

for all $f \in \mathcal{S}(R^4)$. The operator $(\bar{\varphi}[f], D(f))$ is thus self-adjoint if $f$ is real.

(b) If $r(x) \in \mathcal{S}(R^4)$ is *real*, and if $f(x) \in \mathcal{S}(R^4)$ such that $\text{supp}(r) \subset (\text{supp}(f))^c$, then

$$F(\bar{\varphi}[f], D(f)) \subset (\bar{\varphi}[f], D(f))F \tag{110}$$

for any spectral projection $F$ of the self-adjoint operator $(\bar{\varphi}[r], D(r))$.

(c) For any $f \in \mathcal{S}(R^4)$, $\Lambda \in \bar{L}_0$,

$$U(\Lambda)(\bar{\varphi}[f], D(f))U(\Lambda)^{-1} = (\bar{\varphi}[\Lambda f], D(\Lambda f)). \tag{111}$$

*Condition IV*: Condition III holds, with

$$(\bar{\varphi}[f], D(f)) = (\varphi[f], D_1)^{**} \tag{112}$$

for all $f \in \mathcal{S}(R^4)$.

The Condition II trivially implies the Condition I, and we have $\mathcal{C}(W_R) = \mathcal{A}_c(W_R)$, $\mathcal{C}(W_L) = \mathcal{A}_c(W_L)$. Both conditions thus imply the duality condition (105) for the wedge regions. We shall consider further implications of Condition II in the next section.

Condition III is (as far as we know) much stronger than the condition that every operator $(\varphi[f], D_1)$, with $f \in \mathcal{S}(R^4)$ and $f$ real, has a self-adjoint extension. The conditions (110) and (111) can be interpreted as the conditions that the extension of the field is also a local scalar field. Condition IV is the most restrictive of the conditions. It, in effect, states that the quantum field $\varphi(x)$ has a *unique* local, covariant, self-adjoint extension, given by (112).

*Theorem* 4: Condition III is assumed. Let $\mathcal{A}(W_R)$ be the set of all bounded operators $Q$ such that

$$Q(\bar{\varphi}[f], D(f)) \subset (\bar{\varphi}[f], D(f))Q \tag{113}$$

for all $f \in \mathcal{S}(R^4)$ such that $\text{supp}(f) \subset W_L$. Let $\mathcal{A}(W_L)$ be the set of all bounded operators $Q$ such that (113) holds for all $f \in \mathcal{S}(R^4)$ such that $\text{supp}(f) \subset W_R$. Then:

(a) $\mathcal{A}(W_R)$ and $\mathcal{A}(W_L)$ are von Neumann algebras with the vacuum vector $\Omega$ as a cyclic and separating vector. Both algebras are factors, and they satisfy the duality condition

$$\mathcal{A}(W_R)' = \mathcal{A}(W_L). \tag{114}$$

(b) If $\mathcal{A}_c(W_R)$ and $\mathcal{A}_c(W_L)$ are defined as in Theorem 3, then

$$\mathcal{A}_c(W_R) \subset \mathcal{A}(W_R), \quad \mathcal{A}_c(W_L) \subset \mathcal{A}(W_L), \tag{115}$$

and equality obtains if and only if $\mathcal{A}_c(W_R)\Omega$ is dense in $\mathcal{H}$.

(c) The algebras $\mathcal{A}(W_R)$ and $\mathcal{A}(W_L)$ form a pair of local wedge-algebras, i.e., they satisfy all the conditions (a)—(e) in Theorem 3 which the algebras $\mathcal{A}_c(W_R)$ and $\mathcal{A}_c(W_L)$ satisfy.

(d) Let $\mathcal{G}(W_R)$ be the set of all spectral projections of all operators $(\bar{\varphi}[f], D(f))$, with $f$ real, $f \in \mathcal{S}(R^4)$, and $\text{supp}(f) \subset W_R$. Similarly, let $\mathcal{G}(W_L)$ be the set of all spectral projections of all operators $(\bar{\varphi}[f], D(f))$, with $f$ real, $f \in \mathcal{S}(R^4)$, and $\text{supp}(f) \subset W_L$. Then

$$\mathcal{A}(W_R) = \mathcal{G}(W_R)'', \quad \mathcal{A}(W_L) = \mathcal{G}(W_L)''. \tag{116}$$

*Proof*: (1) We first note that in view of (109a) the set $\mathcal{A}(W_R)$, as defined in terms of (113), is the commutant of a set of operators which is closed under the formation of the adjoint. Hence $\mathcal{A}(W_R)$, and similarly $\mathcal{A}(W_L)$, are von Neumann algebras.

From the relation (111), which describes the action of the Poincaré group (by conjugation) on the extended field, it trivially follows that the algebras $\mathcal{A}(W_R)$ and $\mathcal{A}(W_L)$ satisfy all the relations (102a)—(102e) in Theorem 3, and, in particular,

$$V(t)\mathcal{A}(W_R)V(t)^{-1}=\mathcal{A}(W_R), \quad V(t)\mathcal{A}(W_L)V(t)^{-1}=\mathcal{A}(W_L) \tag{117}$$

for all real $t$. Note, however, that the relation (102f) in part (c) of Theorem 3 *does not* follow trivially from (111).

(2) Let $\psi, \phi \in D_1$, and let $f\in \mathcal{S}(R^4)$, $\mathrm{supp}(f)\subset W_L$. For any $X\in\mathcal{A}(W_R)$ we have

$$\langle\psi|X\varphi[f]\phi\rangle=\langle\psi|\overline{\varphi}[f]X\phi\rangle=\langle\psi|\overline{\varphi}[f^*]^*X\phi\rangle$$
$$=\langle\overline{\varphi}[f^*]\psi|X\phi\rangle=\langle\varphi[f^*]\psi|X\phi\rangle. \tag{118a}$$

From the equality of the first and last members of (118a) it readily follows that the relations

$$\langle X^*\psi|Y\phi\rangle=\langle Y^*\psi|X\phi\rangle, \quad \text{all } \phi,\psi\in D_1, \tag{118b}$$

hold for all $X\in\mathcal{A}(W_R)$ and all $Y\in \mathcal{P}(W_L)$. In a similar manner, we conclude that (118b) also hold for all $X\in \mathcal{P}(W_R)$ and all $Y\in\mathcal{A}(W_L)$. As in the proof of Theorem 3 we conclude that

$$\mathcal{A}(W_R)\subset\mathcal{U}_b(W_R), \quad \mathcal{A}(W_L)\subset\mathcal{U}_b(W_L). \tag{118c}$$

(3) Trivially we have $\mathcal{G}(W_R)''\subset\mathcal{A}(W_R)$ and $\mathcal{G}(W_L)''\subset\mathcal{A}(W_L)$. We shall show that $\Omega$ is a cyclic vector of the von Neumann algebra $\mathcal{G}(W_R)''$.

Let $\{R_n|n=1,\ldots,\infty\}$ be a set of subsets of $W_R$, constructed as in Lemma 10. Let $\{f_k|k=1,\ldots,n\}$ be an $n$-tuplet of *real* test functions such that $f_k\in\mathcal{S}(R^4)$ and $\mathrm{supp}(f_k)\subset R_k$, for $k=1,\ldots,n$. In view of the nature of the regions $R_k$ it follows that the self-adjoint operators $(\overline{\varphi}[f_k], D(f_k))$, $k=1,\ldots,n$, all commute with each other, in the sense that their spectral projections commute. Let $F_k(\lambda)$ be the spectral projection of $(\overline{\varphi}[f_k], D(f_k))$ corresponding to the interval $(-\lambda,\lambda)$, where $\lambda>0$, and let the bounded operator $Q_k(\lambda)$ be given by $Q_k(\lambda)=\overline{\varphi}[f_k]F_k(\lambda)$, for each $k=1,\ldots,n$. We then have

$$F_1(\lambda)F_2(\lambda)\cdots F_n(\lambda)\varphi[f_1]\varphi[f_2]\cdots\varphi[f_n]\Omega$$
$$=Q_1(\lambda)Q_2(\lambda)\cdots Q_n(\lambda)\Omega \tag{119a}$$

and hence

$$\underset{\lambda\to\infty}{\text{s-lim}}\, Q_1(\lambda)Q_2(\lambda)\cdots Q_n(\lambda)\Omega=\varphi[f_1]\varphi[f_2]\cdots\varphi[f_n]\Omega. \tag{119b}$$

The operators $Q_k(\lambda)$ are all included in $\mathcal{G}(W_R)''$, and since (119b) holds for any $n>0$, and any choice of real test functions, we conclude that $\overline{\mathcal{G}(W_R)''\Omega}=\overline{\mathcal{Q}}\Omega$, where $\mathcal{Q}$ is defined as in Lemma 11. By Lemma 11 it then follows that $\mathcal{G}(W_R)''\Omega$ is dense in $\mathcal{H}$, and hence $\mathcal{A}(W_R)\Omega$ is also dense.

(4) It is trivially the case that $V(t)\mathcal{G}(W_R)''V(t)^{-1}=\mathcal{G}(W_R)''$ for all real $t$. We now note that both $\mathcal{A}(W_R)$ and $\mathcal{G}(W_R)''$ satisfy the premises of Theorem 2, with $\mathcal{A}_R=\mathcal{A}(W_R)$, or with $\mathcal{A}_R=\mathcal{G}(W_R)''$. It follows from this theorem, in view of $\mathcal{G}(W_R)''\subset\mathcal{A}(W_R)$, that

$$\mathcal{G}(W_R)''=\mathcal{A}(W_R)=J\mathcal{A}(W_R)'J=J\mathcal{G}(W_R)'J. \tag{120a}$$

Similar considerations apply to $\mathcal{A}(W_L)$ and $\mathcal{G}(W_L)$, and we thus establish the relations (116).

We trivially have $\mathcal{G}(W_R)\subset\mathcal{G}(W_L)'$, and hence $\mathcal{G}(W_R)''\subset\mathcal{G}(W_L)'$. Similarly, $\mathcal{G}(W_L)''\subset\mathcal{G}(W_R)'$, and it follows, in view of (120a), that $\mathcal{G}(W_R)''=J\mathcal{G}(W_R)'J=\mathcal{G}(W_L)'$, i.e.,

$$\mathcal{A}(W_R)=J\mathcal{A}(W_L)J, \tag{120b}$$

which shows that $J$ acts as asserted (and as expected) on the algebras $\mathcal{A}(W_R)$ and $\mathcal{A}(W_L)$, which have now been shown to form a pair of local wedge-algebras. The duality condition (114) follows trivially from (120a) and (120b).

(5) It remains to prove the relations (115). Let $X\in\mathcal{A}(W_R)$, $X_c\in\mathcal{A}_c(W_R)$, and let $f\in\mathcal{S}(R^4)$, $\mathrm{supp}(f)\subset W_L$. For any vectors $\phi,\psi\in D_1$ we have

$$\langle\psi|XX_c\varphi[f]\phi\rangle=\langle\psi|X\varphi[f]^{**}X_c\phi\rangle=\langle\psi|X\overline{\varphi}[f]X_c\phi\rangle$$
$$=\langle\psi|\overline{\varphi}[f]XX_c\phi\rangle=\langle\psi|\overline{\varphi}[f^*]^*XX_c\phi\rangle$$
$$=\langle\overline{\varphi}[f^*]\psi|XX_c\phi\rangle=\langle\varphi[f^*]\psi|XX_c\phi\rangle. \tag{121a}$$

From the equality of the first and the last members of (121a) it readily follows that

$$\langle Y^*\Omega|XX_c\Omega\rangle=\langle\Omega|XX_cY\Omega\rangle \tag{121b}$$

for any $Y\in\mathcal{P}_0(W_L)$. By Lemma 13 we conclude that $XX_c\in\mathcal{U}(W_R)$.

Since $X$ and $X_c$ are arbitrary elements of $\mathcal{A}(W_R)$ and $\mathcal{A}_c(W_R)$, and since $V(t)\mathcal{A}_c(W_R)V(t)^{-1}=\mathcal{A}_c(W_R)$, we conclude that $XV(t)X_c^*V(t)^{-1}\in\mathcal{U}(W_R)$. The operators $X$ and $X_c$ then satisfy the premises of Lemma 14, and it follows that

$$X(JX_cJ)\Omega=(JX_cJ)X\Omega. \tag{121c}$$

for any $X\in\mathcal{A}(W_R)$ and any $X_c\in\mathcal{A}_c(W_R)$. Since $\mathcal{A}(W_R)\Omega$ is dense in the Hilbert space it follows, by the same kind of reasoning as in step (1) of the proof of Theorem 2, that $[(JX_cJ),X]=0$, which means that $J\mathcal{A}_c(W_R)J\subset\mathcal{A}(W_R)'$. In view of (120a) this implies the first relation (115). The second relation is obtained by conjugating the first by $J$.

This completes the proof of the theorem. We add a corollary which describes the situation under Condition IV. It is almost completely trivial in content.

*Corollary to Theorem 4*: Condition IV is assumed, and hence Condition III obtains. The quantum field has one and only one local self-adjoint extension $\overline{\varphi}(x)$, namely, $(\overline{\varphi}[f],D(f))=(\varphi[f],D_1)^{**}$ for all $f\in\mathcal{S}(R^4)$. The domains $D_0$ and $D_1$ are cores for all operators $(\varphi[f],D_1)^*$, and

$$(\varphi[f],D_1)^*=(\varphi[f^*],D_1)^{**}=(\overline{\varphi}[f^*),D(f^*)). \tag{122}$$

With the notation in Theorems 3 and 4,

$$\mathcal{A}_c(W_R)=\mathcal{A}(W_R), \quad \mathcal{A}_c(W_L)=\mathcal{A}(W_L), \tag{123}$$

and all the conclusions in these theorems hold for the above algebras.

If we are allowed to speculate about the results in this section, we wish to say that we are inclined to believe that in a satisfactory local theory there ought to exist at least one field which satisfies Condition III, although this does not seem to be necessary for the duality condition to hold. It is well known that the general conditions on the field which we stated in Sec. II have to be amended with some conditions which guarantee that the theory really describes physical particles. In particular, some kind of "dynamical principle" is sorely needed. It might, of course, be the case that Condition III is already implied by the minimal assumptions in Sec. II, but if this is not so we would like to believe that the condition at least holds in a properly amended theory. We can imagine a situation in which the local self-adjoint extension of the field is unique, without $D_1$ being a core for the extensions of the individual field operators $\varphi[f]$. Condition IV might thus be unduly restrictive. An even more restrictive condition, according to which $\Omega$ is an analytic vector for all Hermitian field operators $\varphi[f]$, has been discussed by Borchers and Zimmermann.[31] Such a condition cannot hold generally since it is violated by Wick polynomials of free fields, but it is conceivable that it could hold for one particular field in a particular theory. (It is well known that it does hold for a free field.)

Let us finally remark that most of our considerations up to this point also apply to a field theory in two-dimensional spacetime, in view of the special geometric properties of the wedge regions $W_R$ and $W_L$.

## VII. THE DUALITY CONDITION FOR A FAMILY OF BOUNDED REGIONS; LOCAL INTERNAL SYMMETRIES

The discussion in this section will be based on the assumption that there exists a pair of local wedge-algebras $\mathcal{A}(W_R)$ and $\mathcal{A}(W_L)$, which satisfy the duality condition $\mathcal{A}(W_R)' = \mathcal{A}(W_L)$.

These algebras thus in particular satisfy all the conditions (a)—(e) in Theorem 3, which the algebras $\mathcal{A}_c(W_R)$ and $\mathcal{A}_c(W_L)$ satisfy.

The operators in the von Neumann algebra $\mathcal{A}(W_R)$ can be regarded as "locally associated" with the region $W_R$. The existence of the wedge-algebras does not, however, guarantee (as far as we can see) that there exist nontrivial von Neumann algebras which can reasonably be regarded as associated with *bounded* regions in spacetime. In a satisfactory theory of local observables we would certainly require that there exists a sufficiently large set of bounded (self-adjoint) operators which correspond to measurements within some bounded regions in spacetime. Condition I on the field, discussed in the preceding section, would thus by itself appear too weak for a satisfactory theory, although it does guarantee the existence of the local wedge-algebras. As we shall see, either one of our Conditions II—IV does imply the existence of a set of truly "local" operators with reasonable properties. We note here that our particular conditions, although not physically unreasonable, are nevertheless quite arbitrary. We are not here asserting that anyone of these conditions *has* to hold, nor are we asserting that they guarantee that the theory has a physical interpretation which is satisfactory in *every* respect.

Let us now consider the definition of von Neumann algebras for other regions than the wedges $W_R$ and $W_L$.

For any subset $R$ of Minkowski space $\mathcal{M}$ we denote by $\Lambda R$ the image of $R$ under any element $\Lambda$ of the Poincaré group $\bar{L}_0$. We define $\mathcal{W}$ as the set of all (open) wedge regions bounded by two intersecting characteristic planes, i.e.,

$$\mathcal{W} = \{\Lambda W_R \mid \Lambda \in \bar{L}_0\}. \tag{124a}$$

For every $W \in \mathcal{W}$ we define the von Neumann algebra $\mathcal{A}(W)$ by

$$\mathcal{A}(\Lambda W_R) = U(\Lambda)\mathcal{A}(W_R)U(\Lambda)^{-1}, \quad \text{all } \Lambda \in \bar{L}_0. \tag{124b}$$

We note that this definition is consistent since we assumed that $\mathcal{A}(W_R)$ and $\mathcal{A}(W_L)$ satisfy the relations (102a)—(102e) in Theorem 3.

It is natural to define von Neumann algebras for a suitable family of bounded regions in terms of intersections of the von Neumann algebras $\mathcal{A}(W)$. Since we hope to discuss these issues elsewhere in greater detail, and within a more general framework, we shall here restrict our considerations to a set of particularly simple bounded regions, namely, the so-called double cones. For any two points $x_1$ and $x_2$ in Minkowski space such that $x_2 \in V_+(x_1)$ [where $V_+(x_1)$ is the forward light cone with $x_1$ as apex], we define the double cone $C = C(x_1, x_2)$ by

$$C(x_1, x_2) = V_+(x_1) \cap V_-(x_2), \tag{125a}$$

where $V_-(x_2)$ is the backward light cone with $x_2$ as apex. The double cones so defined are thus open and nonempty. We denote by $\mathcal{D}_c$ the set of all double cones.

For any double cone $C$ we define a von Neumann algebra $\mathcal{B}(\bar{C})$ by

$$\mathcal{B}(\bar{C}) = \cap\{\mathcal{A}(W) \mid W \in \mathcal{W},\ W \supset \bar{C}\}. \tag{125b}$$

Here $\bar{C}$ denotes the closure of $C$. We prefer to regard $\mathcal{B}(\bar{C})$ as associated with the closed set $\bar{C}$, and hence the above notation.

We shall next extend the domain of the mapping $W \to \mathcal{A}(W)$ to include all *open* regions $\bar{C}^c$ which are the causal complements of closed double cones $\bar{C}$. For any $C \in \mathcal{D}_c$ we define the von Neumann algebra $\mathcal{A}(\bar{C}^c)$ by

$$\mathcal{A}(\bar{C}^c) = \{\mathcal{A}(W) \mid W \in \mathcal{W},\ W \subset \bar{C}^c\}''. \tag{126}$$

We shall now state two theorems about the properties of the algebras which we have introduced above. The conclusions in the first of these do not depend on the duality condition, but follow fairly trivially from the relative locality of the wedge-algebras, and from the "geometrical" conditions in parts (b) and (c) of Theorem 3.

*Theorem* 5: Let $\mathcal{A}(W_R)$ and $\mathcal{A}(W_L)$ be a pair of von Neumann algebras such that

$$\mathcal{A}(W_R) \subset \mathcal{A}(W_L)' \tag{127}$$

and

$$\mathcal{A}(W_R) = J\mathcal{A}(W_L)J, \tag{128a}$$

$$\mathcal{A}(W_R) = U(R(\mathbf{e}_1, \pi), 0)\mathcal{A}(W_L)U(R(\mathbf{e}_1, \pi), 0)^{-1}, \tag{128b}$$

$$U(\Lambda)\mathcal{A}(W_R)U(\Lambda)^{-1} \subset \mathcal{A}(W_R), \quad \text{all } \Lambda \in \sigma(W_R), \tag{128c}$$

where $\sigma(W_R)$ is the semigroup of all Poincaré transformations which map $W_R$ *into* $W_R$.

Let $\mathcal{A}(W)$ be defined by (124b), for any $W \in \mathcal{W}$. Let $\mathcal{B}(\bar{C})$ be defined by (125b), and let $\mathcal{A}(\bar{C}^c)$ be defined by

(126), for any double cone $C$. Then:

(a)

$$\mathcal{A}(\Lambda W) = U(\Lambda)\mathcal{A}(W)U(\Lambda)^{-1} \quad (129a)$$

for all $W \in \mathcal{W}$, all $\Lambda \in \bar{L}_0$;

$$\mathcal{B}(\Lambda\bar{C}) = U(\Lambda)\mathcal{B}(\bar{C})U(\Lambda)^{-1}, \quad (129b)$$

$$\mathcal{A}(\Lambda\bar{C}^c) = U(\Lambda)\mathcal{A}(\bar{C}^c)U(\Lambda)^{-1}, \quad (129c)$$

for all $C \in \mathcal{D}_c$, all $\Lambda \in \bar{L}_0$.

(b)

$$\mathcal{A}(\mathcal{J} W) = J\mathcal{A}(W)J, \quad (130a)$$

$$\mathcal{B}(\mathcal{J}\bar{C}) = J\mathcal{B}(\bar{C})J, \quad \mathcal{A}(\mathcal{J}\bar{C}^c) = J\mathcal{A}(\bar{C}^c)J \quad (130b)$$

for all $W \in \mathcal{W}$, $C \in \mathcal{D}_c$, and where $\mathcal{J}$ is given by (47).

(c)

$$\mathcal{A}(W) \supset \mathcal{A}(W_1), \quad \text{if } W, W_1 \in \mathcal{W}, \ W \supset W_1, \quad (131a)$$

$$\mathcal{B}(\bar{C}) \supset \mathcal{B}(\bar{C}_1), \quad \mathcal{A}(\bar{C}^c) \subset \mathcal{A}(\bar{C}_1^c) \quad (131b)$$

for all $C, C_1 \in \mathcal{D}_c$ such that $C \supset C_1$ (and hence $\bar{C}^c \subset \bar{C}_1^c$), and

$$\mathcal{B}(\bar{C}_1) \subset \mathcal{A}(W) \subset \mathcal{A}(\bar{C}_2^c) \quad (131c)$$

for all $W \in \mathcal{W}$, $C_1, C_2 \in \mathcal{D}_c$, such that $C_1 \subset W \subset \bar{C}_2^c$.

(d) The algebras $\mathcal{B}(\bar{C})$ are *local*, in the sense that

$$\mathcal{B}(\bar{C}_1) \subset \mathcal{B}(\bar{C}_2)' \quad (132a)$$

for any $C_1, C_2 \in \mathcal{D}_c$, such that $C_1 \subset \bar{C}_2^c$. Furthermore,

$$\mathcal{B}(\bar{C})' \supset \mathcal{A}(\bar{C}^c) \quad (132b)$$

for any $C \in \mathcal{D}_c$.

(e) The mapping $W \to \mathcal{A}(W)$ is *continuous from the outside* in the sense that

$$\mathcal{A}(W) = \cap\{\mathcal{A}(W_0) \mid W_0 \in \mathcal{W}, W_0 \supset \bar{W}\} \quad (133a)$$

and it is *continuous from the inside* in the sense that

$$\mathcal{A}(W) = \{\mathcal{A}(W_1) \mid W_1 \in \mathcal{W}, \bar{W}_1 \subset W\}''. \quad (133b)$$

The mapping $\bar{C} \to \mathcal{B}(\bar{C})$ is continuous from the outside in the sense that

$$\mathcal{B}(\bar{C}) = \cap\{\mathcal{B}(\bar{C}_0) \mid C_0 \in \mathcal{D}_c, \bar{C} \subset C_0\}. \quad (133c)$$

The mapping $\bar{C}^c \to \mathcal{A}(\bar{C}^c)$ is continuous from the inside in the sense that

$$\mathcal{A}(\bar{C}^c) = \{\mathcal{A}(\bar{C}_1^c) \mid C_1 \in \mathcal{D}_c, C_1 \supset \bar{C}\}''. \quad (133d)$$

*Proof*: (1) The assertions (a) and (b) are trivial. The relation (131a) follows trivially from (128c) and the definition (124b). The relations (131b) follow directly from the definitions (125b) and (126).

(2) We next consider the assertions in part (e) of the theorem. To prove (133a) it clearly suffices to prove this relation for the special case of $W = W_R$. For this case, let $\mathcal{A}$ denote the von Neumann algebra defined by the right member in (133a). We obviously have $\mathcal{A}(W_R) \subset \mathcal{A}$. Let $x \in W_R$. We then have $T(x)\mathcal{A}T(x)^{-1} \subset \mathcal{A}(W_R)$. Since the function $T(x)$ is strongly continuous, and since the point $x = 0$ is included in $\bar{W}_R$, we conclude that $\mathcal{A} = \mathcal{A}(W_R)$. Hence (133a) holds.

The relation (133b) follows readily from (133a). The relation (133c) follows from the definition (125b), and the relation (133d) follows from (133b) and the definition (126).

(3) The relation (131c) in part (b) of the theorem now follows trivially, in view of (133a).

(4) It remains to prove the assertions (d). Let $C$ be a double cone, and let $W = \Lambda W_R$ be any wedge such that $W \subset \bar{C}^c$. Then $C \subset \Lambda W_L$, and it follows from (127) and (131c) that $\mathcal{B}(\bar{C})' \supset \mathcal{A}(\Lambda W_L)' \supset \mathcal{A}(W)$. In view of the definition (126) this implies the relation (132b). The relation (132a) then follows trivially from (132b) and (131c). This completes the proof of the theorem.

We note that the relations (131a) and (131b) are in fact implied by the relations (133b)—(133d), and our presentation is thus somewhat tautological. In view of the relation (133a), which says that the wedge-algebras are "continuous from the outside," we might well write $\mathcal{B}(\bar{W}) = \mathcal{A}(W)$ for any wedge $W$, corresponding to the idea that a wedge $W$ is a limiting case of a double cone. We note here that the algebra $\mathcal{A}(\bar{C}^c)$ need *not* be continuous from the outside, and that the algebra $\mathcal{B}(\bar{C})$ need not be continuous from the inside, for any double cone $C$.

*Theorem* 6: Let $\mathcal{A}(W_R)$ and $\mathcal{A}(W_L)$ be a pair of von Neumann algebras which satisfy all the premises of Theorem 5. It is assumed that these algebras satisfy the duality condition

$$\mathcal{A}(W_L) = \mathcal{A}(W_R)'. \quad (134)$$

Furthermore, it is assumed that $\Omega$ is a cyclic and separating vector for $\mathcal{A}(W_R)$, and that $\mathcal{A}(W_R) \subset \mathcal{U}(W_R)$, where $\mathcal{U}(W_R)$ is defined as in Lemma 12, and hence

$$V(i\pi)X\Omega = JX^*\Omega, \quad \text{all } X \in \mathcal{A}(W_R). \quad (135)$$

Let the von Neumann algebras $\mathcal{A}(W)$, $\mathcal{A}(\bar{C}^c)$, and $\mathcal{B}(\bar{C})$ be constructed as in Theorem 5. Then:

(a) The algebras $\mathcal{B}(\bar{C})$ and $\mathcal{A}(\bar{C}^c)$ satisfy the *duality condition*

$$\mathcal{B}(\bar{C})' = \mathcal{A}(\bar{C}^c). \quad (136)$$

(b) *If there exists* a double cone $C_0$ such that $\mathcal{B}(\bar{C}_0)\Omega$ is dense in the Hilbert space $\mathcal{H}$, then

$$\mathcal{A}(\bar{C}_1^c) = \{\mathcal{B}(\bar{C}) \mid C \in \mathcal{D}_c, \bar{C} \subset \bar{C}_1^c\}'' \quad (137a)$$

for every $C_1 \in \mathcal{D}_c$, and

$$\mathcal{A}(W) = \{\mathcal{B}(\Lambda\bar{C}_0) \mid \Lambda \in \bar{L}_0, \Lambda\bar{C}_0 \subset W\}'', \quad (137b)$$

$$\mathcal{A}(\bar{C}_1^c) = \{\mathcal{B}(\Lambda\bar{C}_0) \mid \Lambda \in \bar{L}_0, \Lambda\bar{C}_0 \subset \bar{C}_1^c\}'' \quad (137c)$$

for every $C_1 \in \mathcal{D}_c$, $W \in \mathcal{W}$. If, furthermore, $\bar{C}_0 \subset W_R$, then

$$\mathcal{A}(W_R) = \{V(t)\mathcal{B}(\bar{C}_0)V(t)^{-1} \mid t \in R^1\}''. \quad (137d)$$

(c) *If the quantum field satisfies Condition II*, and if $\mathcal{A}(W_R) = \mathcal{A}_c(W_R)$, with $\mathcal{A}_c(W_R)$ defined as in Theorem 3, then the pair of von Neumann algebras $\mathcal{A}(W_R)$ and $\mathcal{A}(W_L) = \mathcal{A}(W_R)'$ satisfies the premises of the present theorem. The vector $\Omega$ is a cyclic and separating vector for every algebra $\mathcal{B}(\bar{C})$, and for every algebra $\mathcal{A}(\bar{C}^c)$. The relation (137a) holds, and the relations (137b) and (137c) hold for every $C_0 \in \mathcal{D}_c$.

If $\mathcal{C}(R)$ is defined as in the statement of Condition II, then

$$\mathcal{B}(\bar{C}) \supset \mathcal{C}(C) \tag{138}$$

for all $C \in \mathcal{D}_c$.

(d) *If the quantum field satisfies Condition III, or Condition IV*, then the pair of algebras $\mathcal{A}(W_R)$ and $\mathcal{A}(W_L)$, defined as in Theorem 4, satisfies the premises of the present theorem, and $\Omega$ is a cyclic and separating vectors for every algebra $\mathcal{B}(\bar{C})$, and for every algebra $\mathcal{A}(\bar{C}^c)$. The relations (137a)–(137d) hold as in (b) above, for any $C_0 \in \mathcal{D}_c$.

Furthermore, if $\mathcal{G}(C)$ is the set of all spectral projections of all operators $(\bar{\varphi}[f], D(f))$, with *f real*, $f \in \mathcal{S}(R^4)$, and $\text{supp}(f) \subset C$, then,

$$\mathcal{G}(C)'' \subset \mathcal{B}(\bar{C}) \tag{139}$$

and, for any $C_1 \in \mathcal{D}_c$,

$$\mathcal{A}(\bar{C}_1^c) = \{\mathcal{G}(C) \mid C \in \mathcal{D}_c, \bar{C} \subset \bar{C}_1^c\}''. \tag{140}$$

*Proof*: (1) All the conclusions of Theorem 5 hold. The duality condition (136) follows easily from the duality condition $\mathcal{A}(W_L) = \mathcal{A}(W_R)'$ for the wedge-algebras, if we note that

$$\mathcal{A}(\bar{C}^c) = \{\mathcal{A}(\Lambda W_L) \mid \Lambda \in \bar{L}_0, \Lambda W_R \supset \bar{C}\}'' \\ = (\cap \{\mathcal{A}(\Lambda W_L)' \mid \Lambda \in \bar{L}_0, \Lambda W_R \supset \bar{C}\})' = \mathcal{B}(\bar{C})', \tag{141}$$

where the equality of the first and the second members follows from (133d) in Theorem 5.

(2) We next consider the assertions (b), assuming now that a $C_0$ in $\mathcal{D}_0$ exists, such that $\mathcal{B}(\bar{C}_0)\Omega$ is dense. Without loss of generality we can assume that $\bar{C}_0 \subset W_R$. Let $\mathcal{A}_R$ be equal to the *right* member in (137d). Then $\Omega$ is a cyclic vector for the von Neumann algebra $\mathcal{A}_R$, and it follows from the definition of this algebra that $V(t)\mathcal{A}_R V(t)^{-1} = \mathcal{A}_R$ for all real $t$. Since, obviously, $\mathcal{A}_R \subset \mathcal{A}(W_R) \subset \mathcal{U}(W_R)$, we conclude that $\mathcal{A}_R$ satisfies the premises of Theorem 2, and it follows from that theorem that $\mathcal{A}_R = \mathcal{A}(W_R)$. This proves the relation (137d). The relations (137a)–(137c) then follow trivially from (137d).

(3) The assertions (c) are completely trivial. We now consider the assertions (d). The crux of the matter is that $\mathcal{G}(C)''\Omega$ is dense for any double cone $C$. That this is so is established by the same kind of reasoning as in step (3) in the proof of Theorem 4, but with the modification that for any integer $n > 0$ the regions $R_k$, $k = 1, \ldots, n$, are selected as any set of $n$ nonempty open sets in $C$ such that the closures of any two of these regions are spacelike separated. Having thus shown that $\mathcal{G}(C)''\Omega$ is dense, we consider the case when the double cone $C$ satisfies $\bar{C} \subset W_R$, and we define a von Neumann algebra $\mathcal{A}_R$ by

$$\mathcal{A}_R = \{V(t)\mathcal{G}(C)V(t)^{-1} \mid t \in R^1\}''. \tag{142}$$

The relation (139) is trivial, and we can now apply the reasoning in step (2) above to $\mathcal{A}_R$. We conclude that $\mathcal{A}_R = \mathcal{A}(W_R)$, and from this the relation (140) follows readily. This completes the proof of the theorem.

We feel that it is entirely proper to call the condition (136) a "duality condition," at least in the case when there exists a double cone $C_0$ such that $\mathcal{B}(\bar{C}_0)\Omega$ is dense in the Hilbert space $\mathcal{H}$. In this case we have the following situation. There exists a family of truly local operators, namely, the set of all the operators in all the algebras $\mathcal{B}(\bar{C})$, which is sufficiently large such that the local operators generate the algebras $\mathcal{A}(W)$ and $\mathcal{A}(\bar{C}^c)$ in the sense of (137a) and (137b). The algebra $\mathcal{A}(\bar{C}^c)$ in (136), which is associated with the unbounded region $\bar{C}^c$, is thus itself generated by "local observables," and this circumstance, in our opinion, adds luster to the duality condition. As we have seen this situation obtains if the field satisfies either one of Conditions II, III, or IV.

It should be noted, however, that even if the field satisfies Condition IV it is in general *not* the case that $\mathcal{B}(\bar{C}) = \mathcal{G}(C)''$, i.e., the local algebra $\mathcal{B}(\bar{C})$ need not be generated by the spectral projections of the self-adjoint operators $(\bar{\varphi}[f], D(f))$, with *f real*, $f \in \mathcal{S}(R^4)$, and $\text{supp}(f) \subset C$. The duality condition in the case of a generalized free field has been studied by Landau,[8,32] and with reference to our discussion we can express the results as follows: For certain kinds of generalized free fields we have $\mathcal{B}(\bar{C}) \neq \mathcal{G}(C)''$. For a detailed discussion of this circumstance we refer to the work of Landau. The algebra $\mathcal{G}(C)''$ generated by the generalized field *alone* is thus "too small" to satisfy the duality condition. The situation is, however, entirely different if instead we consider the algebra generated (locally) by *all* the local generalized free fields which are local relative to the original field.

The duality condition for a free Hermitian scalar field was first proved by Araki,[2] by an entirely different method. The von Neumann algebras generated by a free field have been studied extensively.[6,7,28,33,34] It is well known that in this case the field operators $(\varphi[f], D_1)$, with $f$ real, $f \in \mathcal{S}(R^4)$, are all essentially self-adjoint, and our Condition IV obtains. Furthermore, it is the case that $\mathcal{B}(\bar{C}) = \mathcal{G}(C)''$, for all double cones $C$. It should here be noted that Araki's proof of the duality condition, as well as the subsequent modified proofs by Osterwalder,[6] Eckmann and Osterwalder,[7] and by Landau,[8] hold for more general regions than double cones and wedges. The discussion in the work of Eckmann and Osterwalder is based on Tomita's theorem, but also on the very special properties of a free field, and it is not clear to us how the discussion could be generalized to the case of an arbitrary field. We also do not know at this time whether there is any simple "physical—geometrical" interpretation of the Tomita operators $J$ and $V(i\pi)$ for a double cone, or for a more general region. The remarkably simple interpretation of these operators for the case of the wedge regions probably reflects the very special geometric properties of the pair $W_R$ and $W_L$.

We shall conclude the present study with a discussion by local internal symmetries. Such symmetries were discussed by Landau and Wichmann,[35] within the framework of quantum field theory, and within the framework of the theory of local systems of algebras, and it was shown that a local internal symmetry, as defined in that paper, commutes with all translations in the Poincaré group. It was shown by Landau,[36] and by Herbst,[37] that such symmetries also commute with the homogeneous Lorentz transformations under the additional assumption that asymptotic Fock spaces exist, i. e., that the theory has a sensible physical interpretation in terms of particle states.

The definition of a local internal symmetry $G$ in the paper of Landau and Wichmann can be stated as follows, for the case of wedge regions: $G$ is a unitary operator such that

$$G\Omega=\Omega,\quad G\mathcal{A}(W)G^{-1}\subset\mathcal{A}(\overline{W}^c)' \tag{143}$$

for all $W\in\mathcal{W}$. It should be noted that no duality condition was assumed in the quoted work, and it seems to us that the above definition can then be criticized: In particular, it could happen that the set of all symmetries so defined does not form a group. However, the above definition is satisfactory if the duality condition $\mathcal{A}(\overline{W}^c)'=\mathcal{A}(W)$ holds, because it is then easy to show that $G\mathcal{A}(W)G^{-1}=\mathcal{A}(W)$ for all $W\in\mathcal{W}$. In particular, it follows that the set of all local internal symmetries forms a group.

In view of the above we shall here define a local internal symmetry by replacing the second condition in (143) by the condition that $G\mathcal{A}(W)G^{-1}=\mathcal{A}(W)$, for all $W\in\mathcal{W}$.

*Theorem* 7: Let $\mathcal{A}(W_R)$ and $\mathcal{A}(W_L)$ be a pair of local wedge algebras, which satisfy the general premises of Theorem 6, and let $\mathcal{A}(W)$, $\mathcal{B}(\overline{C})$, and $\mathcal{A}(\overline{C}^c)$ be defined as in Theorems 5 and 6.

Let $G$ be a unitary operator such that

$$G\Omega=\Omega,\quad G\mathcal{A}(W)G^{-1}=\mathcal{A}(W),\quad \text{all } W\in\mathcal{W}. \tag{144}$$

Then:

(a) The operator $G$ commutes with the TCP-transformation, and with all Poincaré transformations, i. e.,

$$\Theta_0 G\Theta_0=G,\quad U(\Lambda)GU(\Lambda)^{-1}=G,\quad \text{all } \Lambda\in\overline{L}_0. \tag{145}$$

(b) For all double cones $C$,

$$G\mathcal{B}(\overline{C})G^{-1}=\mathcal{B}(\overline{C}),\quad G\mathcal{A}(\overline{C}^c)G^{-1}=\mathcal{A}(\overline{C}^c). \tag{146}$$

(c) The set of all unitary operators $G$ which satisfy the conditions (144) forms a group; the *group of all local internal symmetries*.

*Proof*: (1) The second condition (144) holds in particular for $W=W_R$. The algebra $\mathcal{A}_R=\mathcal{A}(W_R)$ satisfies the premises of Theorem 2, and in particular $\mathcal{A}(W_R)\Omega$ is a core for the self-adjoint operator $(V(i\pi), D_+)$. The conditions (144) trivially imply that $G^{-1}\mathcal{A}(W_R)\Omega=\mathcal{A}(W_R)\Omega$, and it follows that $\mathcal{A}(W_R)\Omega$ is also a core for the self-adjoint operator $(G^{-1}V(i\pi)G, G^{-1}D_+)$. Let $X\in\mathcal{A}(W_R)$. We then have

$$V(i\pi)GX\Omega=JGX^*\Omega=(JGJ)V(i\pi)X\Omega \tag{147a}$$

where the first two members are equal because $GXG^{-1}\in\mathcal{A}(W_R)$. We thus have

$$(G^{-1}V(i\pi)G,\mathcal{A}(W_R)\Omega)=(G^{-1}JGJ)(V(i\pi),\mathcal{A}(W_R)\Omega). \tag{147b}$$

Since $(G^{-1}V(i\pi)G,\mathcal{A}(W_R)\Omega)$ and $(V(i\pi),\mathcal{A}(W_R)\Omega)$ are essentially self-adjoint, and since $G^{-1}JGJ$ is unitary, it follows, by the polar decomposition theorem, that $G^{-1}D_+=D_+$, $(V(i\pi),D_+)=(G^{-1}V(i\pi)G,D_+)$, and[38]

$$JG=GJ. \tag{148a}$$

(2) The same considerations apply to the algebra $\mathcal{A}(W)$ associated with any other wedge $W=\Lambda W_R$. The Tomita operator "$J$" for the algebra $\mathcal{A}(\Lambda W_R)$ is $U(\Lambda)JU(\Lambda)^{-1}$, and thus we have

$$U(\Lambda)JU(\Lambda)^{-1}G=GU(\Lambda)JU(\Lambda)^{-1} \tag{148b}$$

for all $\Lambda\in\overline{L}_0$. In view of the third relation (56a) we then have, after multiplication of both members in (148b) by $J$ from the left,

$$U(\mathcal{J}\Lambda\mathcal{J}\Lambda^{-1})G=GU(\mathcal{J}\Lambda\mathcal{J}\Lambda^{-1}) \tag{148c}$$

for all $\Lambda\in\overline{L}_0$. It is easily seen that this implies that $G$ commutes with all $U(\Lambda)$, and it then follows from (148a) that $G$ also commutes with $\Theta_0$.

(3) The remaining statements in the theorem are completely trivial.

In conclusion let us state that the considerations in this section can be generalized to other families of bounded regions. We chose to discuss these issues for double cones only, in order to avoid geometrical complications which might obscure the basically very simple mainline of argument.

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[18]See Ref. 10, Chap. 4, p. 138.
[19]See Ref. 11, p. 73. It should be noted that our choice of

independent variables is different (in a trivial way) from the choice in Jost's book.

[20]See Ref. 10, Chap. 2, p. 41. The extension to vector-valued tempered distributions is trivial.

[21]The question arises whether our definition (37) is really the "best possible." If the notion of causal complement is to correspond to a physical notion of causal independence one would like to require that the causal complement of any *open* region $R$ contains the *interior* of $R^c$ as defined in (37). There is hardly any physical basis for a more specific statement, and how the boundaries are handled is then only a question of mathematical convenience. For the discussion in this paper this issue is not important, but we have considered generalizations, and with these in mind it seemed to us that the definition (37) is appropriate.

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[24]This is, of course, well-known: See Ref. 10, p. 139.

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