Model-Independent Extraction of $|V_{cb}|$ from $\bar{B} \to D^* \ell \nu$

Benjamín Grinstein and Andrew Kobach

Physics Department, University of California, San Diego, La Jolla, CA 92093, USA

(Dated: June 9, 2017)

We fit the unfolded data of $\bar{B}^0 \to D^{*+} \ell \nu$ from the Belle experiment, where $\ell \equiv e, \mu$, using a method independent of heavy quark symmetry to extrapolate to zero-recoil and extract the value of $|V_{cb}|$. This results in $|V_{cb}| = (41.9^{+2.0}_{-1.9}) \times 10^{-3}$, which is robust to changes in the theoretical inputs and very consistent with the value extracted from inclusive semileptonic $B$ decays.

I. INTRODUCTION

The discrepancy between the measured values of the CKM matrix element $|V_{cb}|$ from inclusive versus exclusive semileptonic $B$ decays has been an ongoing dilemma for a few decades; for a recent review, see Ref. [1]. Currently, the world averages for $|V_{cb}|$ are $[2, 3]$:

\begin{align}
|V_{cb}| &= (39.18 \pm 0.99) \times 10^{-3} \quad (\bar{B} \to D(\ell \nu)) \\
|V_{cb}| &= (38.71 \pm 0.75) \times 10^{-3} \quad (\bar{B} \to D^* \ell \nu) \\
|V_{cb}| &= (42.19 \pm 0.78) \times 10^{-3} \quad (\bar{B} \to X_c \ell \nu, \text{kinetic scheme}) \\
|V_{cb}| &= (41.98 \pm 0.45) \times 10^{-3} \quad (\bar{B} \to X_c \ell \nu, \text{1S scheme})
\end{align}

The extracted values of $|V_{cb}|$ from exclusive decays, especially the value of $|V_{cb}|$ measured in $\bar{B} \to D^* \ell \nu$, are systematically lower than those from inclusive decays. When measuring $|V_{cb}|$ in exclusive $\bar{B} \to D^{(*)} \ell \nu$ decays, it has been commonplace to use the parameterization developed by Caprini, Lellouch, and Neubert (CLN) [4], which utilizes not only dispersion relations to bound the hadronic form factors, but also relations at $1/m_Q$ in the heavy quark expansion, and claims to describe the full differential decay to within a few percent given only 4 parameters. Interestingly, it has now become clear that the CLN parameterization is no longer a good fit to the $\bar{B} \to D \ell \nu$ data from current experiments [6]. One would naively expect the same to be true, in general, for $\bar{B} \to D^* \ell \nu$ as well.

A recent analysis by Belle [7] used the CLN parameterization to extract a value for $|V_{cb}|$ at zero recoil given a dataset of $\bar{B}^0 \to D^{*+} \ell \nu$, resulting in $|V_{cb}| = (37.4 \pm 1.3) \times 10^{-3}$. This is one of the first times an experiment with a large dataset has published unfolded data for $\bar{B} \to D^* \ell \nu$, so one can easily explore if a different parameterization of the hadronic form factors gives rise to a different value of $|V_{cb}|$. We use the method developed by Boyd, Grinstein, and Lebed (BGL) in Refs. [8–12], from which the original CLN parameterization was based, but which does not utilize any assumptions from heavy quark symmetry. We find that while both the CLN and BGL parameterizations give a good fit to the Belle data, the BGL parameterization gives a larger value for $|V_{cb}|$ when extrapolating to zero recoil, consistent with the value of $|V_{cb}|$ measured in inclusive analyses [2, 3]. As we prepared this manuscript, a similar analysis was made public, using the same BGL parameterization [13]. We are pleased to see the results match nearly identically. Another analysis was recently made public, which highlighted the importance of including theoretical uncertainties from $\Lambda_{QCD}/m_Q$ corrections in the relations between the form factors in the heavy quark expansion [13], which suggests that the uncertainties associated with the extracted value of $|V_{cb}|$ using the CLN method may be underestimated. Moreover, in their analysis of $B \to D \ell \nu$ lattice data, the Fermilab Lattice and MILC collaborations decide not to quote results of their CLN fits because they are more confident in the errors obtained from the BGL parameterization which, they state, can be used to obtain $|V_{cb}|$ even as the uncertainties become arbitrarily more precise [16].

In all, the current evidence points to the strong possibility that the tension between inclusive and exclusive measurements of $|V_{cb}|$ in semileptonic $B$ decays may be due to the use of the CLN parameterization of $\bar{B} \to D^{(*)} \ell \nu$, and a different parameterization should be employed.

1 A brief history of the developments that led to the BGL parametrization and a more complete list of references can be found in Ref. [13].
II. DIFFERENTIAL DECAY RATE AND BGL PARAMETERIZATION

Using the notation in Ref. [12], the $\bar{B} \to D^*$ matrix elements are defined as

$$
\langle D^*(\varepsilon, p') | \bar{c} \gamma^\mu b | B(p) \rangle = i g_\mu \epsilon^{\mu\alpha\beta} \varepsilon_\alpha \gamma_\beta \beta',
$$

$$
\langle D^*(\varepsilon, p') | \bar{c} \gamma^\mu \gamma^5 b | B(p) \rangle = f \varepsilon^{\mu} + (\varepsilon^\mu \cdot p) [a_+ (p + p')^\mu + a_-(p - p')^\mu],
$$

where $\varepsilon^\mu$ is the polarization tensor of the vector $D^*$ meson. In the limit when the final-state leptons are massless, the full differential decay rate for $\bar{B} \to D^* \ell \bar{\nu}$ is

$$
\frac{d\Gamma(\bar{B} \to D^* \ell \bar{\nu})}{dw \, d\cos \theta_L \, d\cos \theta_E \, d\chi} = \frac{3\alpha^2 \epsilon_{cW}^2 G_F^2 |V_{cb}|^2}{1024 \pi^4} |p_{D^*}|^2 q^2 r \left[ (1 - \cos \theta_L)^2 \sin^2 \theta_E H_+^2 + (1 + \cos \theta_L)^2 \sin^2 \theta_E H_-^2 

+ 4 \sin^2 \theta_L \cos^2 \theta_E H_0^2 - 2 \sin^2 \theta_L \sin^2 \theta_E \cos 2\chi H_+ H_-

- 4 \sin \theta_L (1 - \cos \theta_L) \sin \theta_E \cos \chi H_+ H_0

+ 4 \sin \theta_L (1 + \cos \theta_L) \sin \theta_E \cos \chi H_- H_0 \right],
$$

(7)

where $q^\mu$ is the 4-momentum of the lepton system, $r \equiv m_{D^*} / m_B$, and $|p_{D^*}|$ is the magnitude of the $D^*$ 3-momentum in the rest frame of the $\bar{B}$:

$$
w \equiv \frac{m_B^2 + m_{D^*}^2 - q^2}{2m_B m_{D^*}}, \quad q^2 = m_B^2 + m_{D^*}^2 - 2m_B m_{D^*} w, \quad |p_{D^*}| = m_{D^*} \sqrt{w^2 - 1}.
$$

Here, $H_+$, $H_-$, and $H_0$ are form factors associated with each of the three helicity states of the $D^*$, all of which are functions of $q^2$. Also, $\theta_L$ is the angle between the anti-neutrino and the direction antiparallel to the $D^*$ in the rest frame of the leptonic system, $\theta_E$ is the angle between the $D^*$ momentum and its daughter $D$ meson, and $\chi$ is the angle between the planes defined by the the leptonic system and the $D^*$ system. The factor $\eta_{cW}$ incorporates the leading electroweak corrections [17], $\eta_{cW} = 1 + \alpha / \pi \ln(M_Z/m_B) \approx 1.0066$. In terms of the form factors in Eqs. (5) and (6),

$$
H_+ = f - m_B |p_{D^*}| g,
$$

$$
H_- = f + m_B |p_{D^*}| g,
$$

$$
H_0 = \frac{1}{m_{D^*} \sqrt{q^2}} \left[ 2m_B^2 |p_{D^*}|^2 a_+ - \frac{1}{2} (q^2 - m_B^2 + m_{D^*}^2) f \right] \equiv \mathcal{F}_1 \sqrt{q^2},
$$

(11)

A detailed discussion about the BGL method for parameterizing the form factors $f$, $g$, and $\mathcal{F}_1$ can be found in Ref. [12]. The final result gives a parametrization of each form factor in terms of $N + 1$ coefficients:

$$
g(z) = \frac{1}{P_g(z) \phi_g(z)} \sum_{n=0}^{N} a_n z^n, \quad f(z) = \frac{1}{P_f(z) \phi_f(z)} \sum_{n=0}^{N} b_n z^n, \quad \mathcal{F}_1(z) = \frac{1}{P_{\mathcal{F}_1}(z) \phi_{\mathcal{F}_1}(z)} \sum_{n=0}^{N} c_n z^n,
$$

(12)

where the conformal variable $z$ is defined as

$$
z = \frac{w + 1 - \sqrt{2a}}{w + 1 + \sqrt{2a}}.
$$

(13)

Here, $a = 1$ can be chosen such that $z = 0$ corresponds to zero recoil, and the coefficients $a_n$, $b_n$, and $c_n$ are bounded by unitarity [10],

$$
\sum_{n=0}^{N} |a_n|^2 \leq 1, \quad \text{and} \quad \sum_{n=0}^{N} (|b_n|^2 + |c_n|^2) \leq 1.
$$

(14)

From Eq. (11), $\mathcal{F}_1(0) = (m_B - m_{D^*}) f(0)$; hence $b_0$ and $c_0$ are not independent, i.e.,

$$
c_0 = \frac{(m_B - m_{D^*}) \phi_{\mathcal{F}_1}(0)}{\phi_f(0)} b_0.
$$

(15)
The Blaschke factors \( P(z) \) remove poles for \( q^2 < (m_B + m_{D^*})^2 \) associated with on-shell production of \( B_c^* \) bound states:

\[
P_g(z) = \prod_i \frac{z - z_{P_i}}{1 - z z_{P_i}}, \quad P_f(z) = P_{F_1}(z) = \prod_i \frac{z - z_{P_i}}{1 - z z_{P_i}},
\]

(16)

\[
z_p \equiv \frac{\sqrt{(m_B + m_{D^*})^2 - m_P^2} - \sqrt{(m_B + m_{D^*})^2 - (m_B - m_{D^*})^2}}{\sqrt{(m_B + m_{D^*})^2 - m_P^2} + \sqrt{(m_B + m_{D^*})^2 - (m_B - m_{D^*})^2}}
\]

(17)

For the form factor \( g \), the index \( i \) runs over the 4 vector \( B_c^* \) states, and for the \( f \) and \( F_1 \), it runs over the 4 axial vector states, as listed in Table II. The outer functions \( \phi \) are defined as:

\[
\phi_g(z) = \sqrt{\frac{256 n_I}{3 \pi \chi^T(\mp u)}} \frac{r(1 + z)(1 - z)^{3/2}}{(1 + r)(1 - z) + 2 \sqrt{r}(1 + z)} \]  

\[
\phi_f(z) = \frac{1}{m_B^2} \sqrt{\frac{16 n_I}{3 \pi \chi^T(-u)}} \frac{r(1 + z)(1 - z)^{3/2}}{(1 + r)(1 - z) + 2 \sqrt{r}(1 + z)} \]

\[
\phi_{F_1}(z) = \frac{1}{m_B^2} \sqrt{\frac{8 n_I}{3 \pi \chi^T(-u)}} \frac{r(1 + z)(1 - z)^{5/2}}{(1 + r)(1 - z) + 2 \sqrt{r}(1 + z)} \]

(18)  

(19)  

(20)

where \( n_I \) is the effective number of light quarks, \( u \equiv m_c/m_b \), and \( \chi^T(\pm u) \) is related to a perturbative calculation at \( q^2 = 0 \). We use the result for \( \chi^T(\pm u) \) in the pole mass scheme, and take numerical values to be those listed in Table II as done in Ref. [12], ignoring the small contributions from condensates:

\[
\chi^T(+0.33) = 5.28 \times 10^{-4} \text{ GeV}^{-2}, \quad \chi^T(-0.33) = 3.07 \times 10^{-4} \text{ GeV}^{-2}
\]

(21)

<table>
<thead>
<tr>
<th>Type</th>
<th>Mass [GeV]</th>
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<td>vector</td>
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</tr>
<tr>
<td>axial vector</td>
<td>6.730, 6.736, 7.135, 7.142</td>
</tr>
</tbody>
</table>

TABLE I: \( B_c^* \) masses used in this analysis from [18]. Only states that have a mass less than \( m_B + m_{D^*} \) are included.

| \( m_B \) | 5.279 GeV |
| \( m_{D^*} \) | 2.010 GeV |
| \( \alpha_s \) | 0.22 |
| \( m_b \) | 4.9 GeV |
| \( m_c/m_b \) | 0.33 |
| \( n_I \) | 2.6 |

TABLE II: Numerical inputs for this analysis. The meson masses are taken from Ref. [19]. See the text at the end of Section II for a discussion regarding the choices of the other parameters.

We do not incorporate any uncertainties associated with the numerical inputs to the BGL theory parameterization. Specifically, the uncertainties associated with \( m_B \) and \( m_{D^*} \) are negligible in this analysis. Also, varying the inputs that only affect the overall normalization of the outer function, namely \( n_I \) and those used in the perturbative calculation of \( \chi^T \), amounts to changing the right-hand side of the bounds in Eq. (14) by a corresponding fractional amount. We impose the bounds Eq. (14) as constraints, however, the resultant best-fit values are orders of magnitude from saturating the unitarity bounds, and these constraints effectively play no role. Because of this, changing the values of, say, \( \alpha_s, m_b, \text{ etc.} \), will change the best-fit values for the free parameters in the theory, but will not change the extracted value \( |V_{cb}| \), to a very good approximation. The remaining inputs are \( B_c^* \) pole masses. They affect both the effectiveness of the unitarity bound and the shape of form factors. For example, for \( F \equiv g, f, F_1 \), the unitarity bound implies that the truncation error for finite \( N \) in Eq. (12) is bounded over the physical region.
Fig. 3. This results in:

Using the definition of the form factors in Eqs. (7), (9), and (12), the relationship between $V_{cb}|_{\text{ew}}F(1)$ and our fitting parameters is:

$$|V_{cb}|_{\text{ew}}F(1) = \frac{1}{2\sqrt{m_Bm_{D^*}}} \left( \frac{[\hat{b}_0]}{P_f(0)\phi_f(0)} \right)$$

One can estimate on the lattice the value of $F(1)$. Given the definition of the form factors in Eqs. (7), (9), and (12), the relationship between $|V_{cb}|_{\text{ew}}F(1)$ and our fitting parameters is:

$$|V_{cb}|_{\text{ew}}F(1) = \frac{1}{2\sqrt{m_Bm_{D^*}}} \left( \frac{[\hat{b}_0]}{P_f(0)\phi_f(0)} \right)$$

Given the statistical ensemble of $\hat{b}_0$ provided by the MCMC, we can estimate the likelihood, and thus the variance in the $\chi^2$ about its minimum, i.e., $\Delta \chi^2$, for the value of $|V_{cb}|_{\text{ew}}F(1)$ extracted from the Belle data, as shown in Fig. 3. This results in:

$$|V_{cb}|_{\text{ew}}F(1) = (38.2 \pm 1.7) \times 10^{-3}$$

Using the $\eta_{ew} = 1.0066$ and the FNAL/MILC value of $F(1) = 0.906 \pm 0.013$ from Ref. [20], the final result for $|V_{cb}|$ is

$$|V_{cb}| = (41.9 \pm 2.0) \times 10^{-3}$$

This result is nearly identical to that of the recent analysis by the authors of Ref. [14], and systematically larger than the value extracted by the Belle experiment using the CLN parameterization [7]. Our result is comparable to a recent analysis by the authors of Ref. [15], which reevaluate the CLN parameterization, using relations from heavy quark symmetry, but not ignoring important sources of theoretical uncertainty when fitting to the same data.

As explained above, extracting the value of $|V_{cb}|$ using the BGL parameterization is robust to changes in the numerical inputs.
FIG. 1: The result of the fit to the form factors defined in Eq. (12), where \( \tilde{a}_0 \), \( \tilde{a}_1 \), and \( \tilde{b}_0 \) are defined as \( \eta_{ew}|V_{cb}|a_i \), \( \eta_{ew}|V_{cb}|b_i \), and \( \eta_{ew}|V_{cb}|c_i \), respectively. The blue, yellow, and red lines correspond to the 68%, 95%, and 99% CL, respectively. Any jaggedness in the contours is from the finite size of the MCMC ensemble.

We further explore the effect of Blaschke factors on the extraction of \( |V_{cb}| \) by squaring the Blaschke factors for \( f \) and \( F_1 \). One might, a priori, expect to produce radical variations in the \( z \) dependence of the form factors. Surprisingly, while we fit very different values for the free parameters, the extracted value of \( |V_{cb}||\eta_{ew}F'(1) \) is nearly unchanged (after updating the expression in Eq. (23) to account for the different definition of the form factors). This gives further evidence that the BGL parameterization is robust when extrapolating to zero recoil. On the other hand, the precise values of the theoretical inputs may be essential when extracting the functional dependence of the form factors over the entire kinematic range.
FIG. 2: The Belle data (black points) compared to the results of our fit using the BGL parameterization (blue), and the results of the Belle analysis using the CLN parameterization (red).

IV. SUMMARY AND CONCLUSIONS

The long-standing tension between the exclusive and inclusive determinations of $|V_{cb}|$ in semileptonic $B$ decays could have been exacerbated by underestimated uncertainties in the exclusive analyses when using the CLN parameterization \[6, 14, 15\]. As it stands, the data has become too precise to use the original CLN proposal in the case of $\bar{B} \to D\ell\nu$, as discussed in Ref. \[6\], and one might expect the same could be true for $\bar{B} \to D^*\ell\nu$.

The Belle experiment recently released an unfolded dataset of $\bar{B}_0 \to D^*\ell\nu$, which resulted in the following value of $|V_{cb}|$ using the CLN parameterization \[7\]:

$$|V_{cb}| = (37.4 \pm 1.3) \times 10^{-3} \quad \text{(CLN)}$$  \(26\)

Using this data, we fit to the BGL parameterization, which does not rely on any assumptions regarding heavy quark symmetry, and obtain:

$$|V_{cb}| = (41.9^{+2.0}_{-1.9}) \times 10^{-3} \quad \text{(BGL)}$$  \(27\)
This result is consistent with the value of $|V_{cb}|$ measured using inclusive semileptonic $B$ decays. We are able to corroborate the results of the analysis in Ref. [14], which appeared as our manuscript was in preparation. One can see by eye in Fig. 2 that the BGL parametrization gives a different fit to the data, and a systematically larger value of the $B \rightarrow D^* \ell \nu$ rate near zero-recoil, than when using the CLN parameterization. Furthermore, we find that the BGL method is robust when extrapolating to zero-recoil, i.e., large changes in the inputs associated with the form factors in Eq. (12) result in almost no change in the extracted value of $|V_{cb}|$.

Acknowledgments

We thank Aneesh Manohar for useful conversations and feedback, Florian Bernlochner for providing the unfolded data from Belle, and Kevin Kelly for advice on graphics and using Monte Carlo. This work is supported in part by DOE grant #de-sc0009919.