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Stochastic Groundwater Flow Analysis in the Presence of Trends in Heterogeneous Hydraulic Conductivity Fields[1]

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Due to changes in lithostatic pressure, differential fracturing across bedding planes and irregularities in depositional environments, hydraulic conductivity exhibits heterogeneities and trends at various spatial scales. Using spectral theory, we have examined the effect of trends in hydraulic conductivity on (1) the solution of the mean equation for hydraulic head, (2) the covariance of hydraulic head, (3) the cross-covariances of hydraulic head and log-hydraulic conductivity perturbations and their gradients, and (4) the effective hydraulic conductivity. It is shown that the field of hydraulic head is sensitive to the presence of trends in ways that cannot be predicted by the classical analysis based on stationary hydraulic conductivity fields. The controlling variables for the second moments of hydraulic head are the mean hydraulic gradient, the correlation scale of log-hydraulic conductivity and its variance, and the slope of the trend in log-hydraulic conductivity. The mean hydraulic gradient introduces complications in the analysis since it is, in general, spatially variable. In this respect, our results are approximate, yet indicative of the true role of spatially variable patterns of log-hydraulic conductivity on groundwater flow systems.

KEY WORDS: hydraulic conductivity, spectral theory, groundwater flow.

INTRODUCTION

The spectral analysis of stochastic groundwater flow has been, by and large, based on the assumption of stationary log-hydraulic conductivity fields. In other words, it is assumed that there exist heterogeneities that can be characterized statistically at the scale of the integral or correlation scale of log-hydraulic conductivity, and that there are no systematic patterns, or trends, of spatial log-

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hydraulic conductivity variability. The justification for the stationarity assumption goes beyond pure mathematical convenience: it is critical for establishing ergodic properties of aquifer variables, and to reconcile spatial averages, typified by empirical measurements, with theoretical averages. Much of the literature on the subject of stationary processes in the stochastic analysis of groundwater has been summarized in Dagan (1989). Important results on the behavior of groundwater processes and contaminant transport under stationary conditions have been reported in the pioneering works by Bakr et al. (1978) and Gelhar and Axness (1983), and more recent reviews by Gelhar (1986) and Dagan (1985, 1989). Most recently, Rajaram and McLaughlin (1990) have examined the problem of large-scale trend identification in hydrologic data, including trends present in hydraulic conductivity fields. Smith and Freeze (1979) examined the influence of a trend in the mean log-hydraulic conductivity in a two-dimensional stochastic analysis.

In view of the practical importance of understanding the role of spatial trends on the behavior of hydraulic head, we have attempted to characterize the fundamental differences between the stationary and nonstationary stochastic analysis of groundwater flow systems. The approach used in this work is to (1) approximate the nonstationary behavior using the tools of the classical (stationary) analysis while, at the same time, introducing the effect of trends in log-hydraulic conductivity fields, and (2) based on the approximate results, examine the conditions under which the approximate method of analysis is justified. It is shown in this paper that bounds can be established under which the approximate method of analysis is valid, and that a fundamental insight is gained about the effect of trends in log-hydraulic conductivity on (1) the first and second moments of hydraulic head, (2) the order of magnitude of nonlinear terms in the equation of groundwater flow, and (3) the effective hydraulic conductivity.

A MODEL FOR NONSTATIONARY HYDRAULIC CONDUCTIVITY FIELDS

Let Y denote the natural logarithm of hydraulic conductivity, K. The log-hydraulic conductivity is represented as the sum of a trend, T, constant level, F, and a zero-mean, statistically homogeneous perturbation, f. Specifically,

$$Y = T + F + f \tag{1}$$

The log-hydraulic conductivity is a random field; the trend, T, and the constant level, F, are deterministic variables. The trend, however, is variable in space, whereas the constant level is independent of the spatial location. Figure 1 illustrates the three-component model suggested by Eq. (1). Figure 2 (adapted from Farvolden et al., 1988), shows hydrogeologic logs at a deep well located in the Eye–Dashwa Lakes Pluton near Atitokan, Ontario. The logs show the profiles

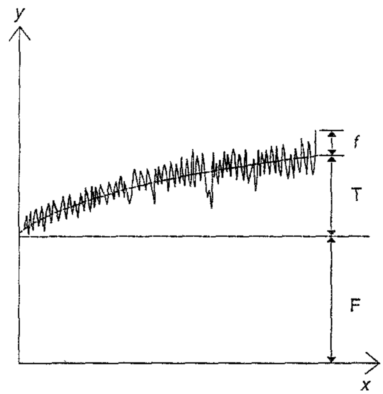

Fig. 1. Model for nonstationary log-hydraulic conductivity field.

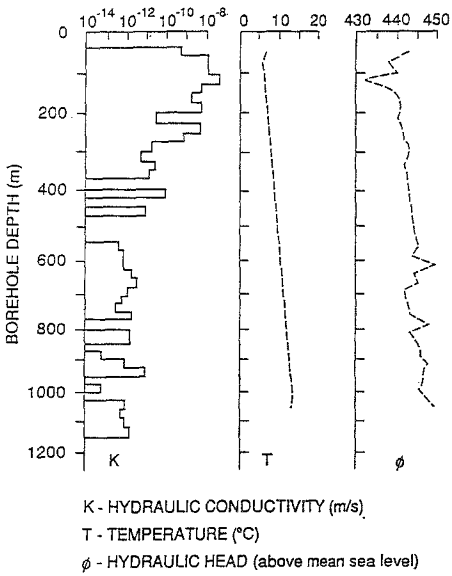

Fig. 2. Nonstationary hydraulic conductivity at a deep well (with corresponding head and temperature logs) with gradual trend (adapted from Farvolden et al., 1988).

of hydraulic conductivity, temperature, and hydraulic head in a groundwater flow system composed of a shallow (less than 150 m deep), intermediate (150–600 m) and deep (deeper than 600 m) flow regimes. It can be seen in Fig. 2 that the hydraulic conductivity shows a slight declining profile in the shallow zone, where hydraulic head also decreases with depth. Below a depth of approximately 150 m, in the intermediate zone, the hydraulic conductivity drops rapidly until reaching a depth of about 600 m. Thereafter, within the deep zone, the hydraulic conductivity is quasi-stable, with a tenuous decrease with depth, correlated with larger lithostatic pressure and smaller porosity at greater depths. The larger hydraulic conductivity near the surface is explained by a higher fracture density at shallow depths. Below the lower boundary of the shallow zone, water temperature increases at a steady rate governed by the geothermal gradient. Hydraulic head also shows an increasing trend below the shallow zone, albeit with some sharp fluctuations within the deep zone. In deep unconsolidated deposits the hydraulic conductivity also tends to decrease as a result of a greater lithostatic load. Figure 2 suggests a model of the type proposed in Eq. (1) for log-hydraulic conductivity: the sum of a spatially-variable trend, a constant level and local-scale, random, variability. The classical, small-perturbation analysis, based on stationary or statistically homogeneous log-hydraulic conductivity fields (see, e.g., Bakr et al., 1978), neglects the trend, T, and this feature constitutes the major difference between our approach and the classical, stationary analysis.

THE MEAN AND PERTURBATION EQUATIONS FOR STEADY-STATE AQUIFER FLOW

Consider the continuity equation for confined, steady-state, aquifer flow (tensorial index notation is used):

$$\frac{\partial}{\partial x_i}\left(K\frac{\partial \phi}{\partial x_i}\right) = 0 \tag{2}$$

in which K represents hydraulic conductivity and ϕ denotes hydraulic head. The logarithm of K, or log-hydraulic conductivity, is described by Eq. (1), whereas the hydraulic head, ϕ, is decomposed into a spatially variable but deterministic mean, H, and a stochastically homogeneous, zero-mean, perturbation, h:

$$\phi = H + h \tag{3}$$

It will be shown below that in view of the nonstationary behavior of log-hydraulic conductivity, embodied in the trend, T, in Eq. (1), the statistical homogeneity of the hydraulic head perturbation, h, is only an approximate assumption. How strong this assumption is, as will be seen below, depends on the nature of the spatial variability of the mean hydraulic-head gradient and on the structure of the trend.

Following Loaiciga and Mariño (1990), the equation governing the mean hydraulic head, H, is obtained after substituting Eqs. (1) and (3) into Eq. (2), and subsequently taking the expected value in the resulting equation to yield:

$$\frac{\partial^2 H}{\partial x_i \partial x_i} + \frac{\partial T}{\partial x_i}\frac{\partial H}{\partial x_i} + E\left\{\frac{\partial f}{\partial x_i}\frac{\partial h}{\partial x_i}\right\} = 0 \tag{4}$$

Equation (4) is the differential equation that governs the mean hydraulic head, H. The presence of the trend, T, introduces the second term in the left-hand side of Eq. (4). In solving Eq. (4), we consider also the expected value of the product of random gradients in Eq. (4).

To develop the differential equation governing the perturbations of hydraulic head, h, the mean Eq. (4) is subtracted from Eq. (2), after using Eqs. (1) and (3) into Eq. (2), and simplifying the resulting expression to yield:

$$\frac{\partial^2 h}{\partial x_i \partial x_i} + \frac{\partial T}{\partial x_i}\frac{\partial h}{\partial x_i} + \frac{\partial f}{\partial x_i}\frac{\partial H}{\partial x_i} + \left\{\frac{\partial f}{\partial x_i}\frac{\partial h}{\partial x_i} - E\left(\frac{\partial f}{\partial x_i}\frac{\partial h}{\partial x_i}\right)\right\} = 0 \qquad (5)$$

Equation (5) is the differential equation for the perturbation of hydraulic head, h. Unlike the classical, stationary analysis, the log-hydraulic conductivity trend, T, introduces the second term in the left-hand side of Eq. (5). The term in brackets in Eq. (5), that involves the product of random gradients, has been studied by Loaiciga and Mariño (1990). It has a zero mean and a standard deviation (and, hence, an order of magnitude) that depends on the second derivatives of the covariance and cross-covariance of log-hydraulic conductivity and hydraulic head perturbations (see Eq. (35) of Loaiciga and Mariño, 1990). In solving Eq. (5), the zero-mean term in brackets will be neglected. The conditions under which the order of magnitude of that term becomes negligible will be established below. It is shown below that the conditions for the smallness of the nonlinear perturbation term in (5) are compatible with the stability conditions needed for the solution of (4), thereby justifying our treatment of nonlinear terms.

The approach followed is first to solve the perturbations Eq. (5), and, then, solve the mean equation, to fully characterize the first two moments of hydraulic head. The perturbations Eq. (5) must be solved first in order to develop an expression for the expected value of the product of random gradients in Eq. (4). Strictly, the solutions of the mean and perturbations equations are coupled in the sense that, to solve Eq. (5), one must know the spatial behavior of the mean hydraulic head, and, conversely, to discern the behavior of mean hydraulic head, further knowledge of the product of random gradients is needed. That knowledge, however, is locked-up in the structure of Eq. (5). This circular dependence can be broken if it is assumed that the mean hydraulic head, H, has some special pattern of spatial variability. Upon solution of Eqs. (4) and (5) one can examine how strong or weak this critical assumption is.

SOLUTION OF THE PERTURBATION EQUATION

In attempting to arrive at a solution of Eq. (5), which means deriving the covariance of the hydraulic head perturbations from (5), one must decouple Eq. (5) from the mean Eq. (4). In other words, the mean gradient, $\partial H/\partial x_i$, will be assumed known in solving Eq. (5); henceforth, upon solution of Eq. (5), the mean Eq. (4) can be approached straightforwardly. As will be seen below, the critical conditions needed in the solution of the perturbations Eq. (5) is that the mean hydraulic gradient $\partial H/\partial x_i$, and the gradient of the trend, $\partial T/\partial x_i$, must be approximately constant; otherwise, the spectral representation of the hydraulic head perturbations is not feasible.

In order to develop closed-form solutions to the perturbations Eq. (5) and the mean Eq. (4), we simplify the analysis to the one-dimensional case. Closed-form analytical solutions of both equations do not exist for the general three-dimensional case. The one-dimensional results provide a clear insight of the role of trends in hydraulic conductivity on the behavior of hydraulic head. (Three-dimensional results will be provided also for the variance of hydraulic head and effective hydraulic conductivity, although, unlike the one-dimensional case, closed-form expressions in terms of elementary functions are not generally available in three-dimensions.) Consider the spectral representations for the hydraulic head and log-hydraulic conductivity (Priestley, 1981)

$$h = \int_{-\infty}^{\infty} e^{jkx} dZ_h(k) \tag{6}$$

$$f = \int_{-\infty}^{\infty} e^{jkx} dZ_f(k) \tag{7}$$

in which $E[dZ(k)] = 0$ and $E[dZ(k)\, dZ^*(k)] = \Phi(k)$, for both h and f; the asterisk denotes a complex conjugate; $\Phi(k)$ represents the spectrum and $j^2 = -1$. Equations (6) and (7) are substituted into Eq. (5) to yield

$$dZ_h(k) = J\left[\frac{kj - b}{(b^2 + k^2)}\right] dZ_f(k) \tag{8}$$

in which J is the mean hydraulic gradient,

$$J = \frac{dH}{dx} \tag{9}$$

and b is the trend gradient,

$$b = dT/dx \tag{10}$$

From Eq. (8) and taking the expectation, the spectrum of hydraulic head Φ_{hh} follows at once as a function of the spectrum of log-hydraulic conductivity, Φ_{ff}:

$$\Phi_{hh}(k) = \frac{J^2}{(b^2 + k^2)} \Phi_{ff} \tag{11}$$

Notice that by introducing the effect of a trend in log-hydraulic conductivity

($b = dT/dx$), there are no restrictions on the nature of the spectrum of log-hydraulic conductivity, Φ_{ff}. The standard analysis (see, e.g., Bakr et al., 1978) requires a spectrum Φ_{ff} proportional to k^n, $n > 1$. In this work, we use the popular exponential covariance model of log-hydraulic conductivity, σ_{ff}, to obtain the log-hydraulic conductivity spectrum:

$$\sigma_{ff}(\tau) = \sigma_f^2 \exp(-|\tau|/\lambda) \tag{12}$$

in which σ_f^2 is the variance of log-hydraulic conductivity, τ is the separation distance between any two points, and λ is the correlation scale of log-hydraulic conductivity. The spectrum of log-hydraulic conductivity, Φ_{ff}, is derived by taking the Fourier transform of the covariance σ_{ff} in Eq. (12) to yield,

$$\Phi_{ff}(k) = \frac{\sigma_f^2\lambda}{\pi(1 + k^2\lambda^2)} \tag{13}$$

from which, based on Eq. (11), the spectrum of hydraulic-head perturbations is:

$$\Phi_{hh}(k) = \frac{J^2\sigma_f^2\lambda}{\pi(b^2 + k^2)(1 + k^2\lambda^2)} \tag{14}$$

The covariance of hydraulic head, σ_{hh}, is the Fourier transform of the spectrum of hydraulic head

$$\sigma_{hh}(\tau) = \int_{-\infty}^{\infty} e^{jk\tau}\Phi_{hh}(k)\, dk \tag{15}$$

that, based on Eq. (14), yields (the integration was carried out by the method of residues (Churchill and Ward, 1990))

$$\sigma_{hh}(\tau) = \frac{J^2\sigma_f^2\lambda}{(b^2\lambda^2 - 1)b} [b\lambda \exp(-|\tau|/\lambda) - \exp(-b|\tau|] \tag{16}$$

For $\tau = 0$, Eq. (16) becomes the variance of hydraulic head, σ_h^2,

$$\sigma_h^2 = \frac{J^2\sigma_f^2\lambda}{b(b\lambda + 1)} \tag{17}$$

Equation (16), giving the covariance of hydraulic head, provides the solution to the perturbations Eq. (5), along with the fact that the mean of the head perturbations is zero, $E(h) = 0$. Notice that for the variance in Eq. (17) to be constant, the mean hydraulic gradient, J, and the gradient of the trend, b, must be constant, or approximately so, to validate the spectral representation of hydraulic head perturbations in Eq. (6). The variance of hydraulic head, as seen from Eq. (17), is proportional to the squared mean hydraulic gradient and to the variance of log-hydraulic conductivity, while inversely proportional to the square of the gradient of the log-hydraulic conductivity trend. The role of the correlation scale is somewhat obscured by the gradient of the trend; however, the larger the trend gradient, the less significant the correlation scale is in influencing the magnitude of the variance of the hydraulic head. Equation (17) also shows that the variance of hydraulic head becomes arbitrarily large when there is no trend (i.e., when $b = dT/dx \rightarrow 0$). This is a well-documented anomaly of the one-dimensional spectral analysis of groundwater flow (see, e.g., Bakr et al., 1978), where, in the absence of a trend, the covariance of hydraulic head becomes undefined unless the spectrum of log-conductivity is proportional to k^n, with $n > 1$. It is seen, then, that when a trend exists, such restriction on the admissible spectra of hydraulic head (a function of the log-conductivity spectrum) is no longer binding. Notice also that the variance of hydraulic head becomes undefined when $b\lambda \rightarrow -1$, according to Eq. (17). Since the correlation scale is positive, the condition $b\lambda \rightarrow -1$ could only arise when the trend gradient is negative (i.e., when $b = dT/dx < 0$) and equal to $1/\lambda$ in magnitude. There is a physical analogy in deterministic, heterogeneous, groundwater flow that sheds some light on the instability condition associated with $b\lambda = -1$. Consider the case of confined, steady-state, one-dimensional flow in an aquifer of constant thickness d and variable hydraulic conductivity $K = a + bx$, with $a, x > 0$. For a fixed coordinate location x, the (deterministic) hydraulic head as a function of the slope of hydraulic conductivity is $h(b) = h_0 - (Q'/d \cdot b) \ln [(a + bx)/a]$, where Q' is the groundwater flow in the positive x direction per unit thickness of aquifer and h_0 is the hydraulic head at $x = 0$. When $b \rightarrow 0$ the hydraulic head becomes $h_0 - (Q'x/d \cdot a)$; for $b \rightarrow \infty$, the head tends to h_0; for $b \rightarrow -a/x$, the head becomes undefined. In this latter case the negative slope of hydraulic conductivity, as it approaches its critical value, "chokes off" the flow. The role of correlation scale is played by the coordinate-dependent factor $-a/x$ in the deterministic case.

ERROR ANALYSIS IN THE SOLUTION OF THE PERTURBATION EQUATION

In solving the perturbations Eq. (5), the zero-mean term involving products of random gradients of hydraulic head and log-hydraulic conductivity perturbations was neglected. Loaiciga and Mariño (1990) established that the standard deviation and, hence, the order of magnitude σ_z, of the term in brackets in Eq. (5) is given by:

$$\sigma_z = \{\sigma''_{fh}(0)^2 + \sigma''_{ff}(0)\sigma''_{hh}(0)\}^{1/2} \tag{18}$$

in which $\sigma''_{fh}(0)$ denotes the second derivative of the cross-covariance of the perturbation of log-hydraulic conductivity, f, and hydraulic head, h, evaluated at $\tau = 0$; analogous definitions hold for the second derivatives of the covariances of log-hydraulic conductivity (σ_{ff}, see Eq. 12) and hydraulic head (σ_{hh}, see Eq. 16). The cross-covariance of perturbations, σ_{fh}, is obtained by the Fourier method:

$$\sigma_{fh}(\tau) = \int_{-\infty}^{\infty} \Phi_{fh}(k) e^{jk\tau}\, dk \tag{19}$$

The cross-spectrum Φ_{fh} is defined by

$$\Phi_{fh}(k) = E[dZ_f(k)\, dZ_h^*(k)] \tag{20}$$

that, from Eq. (8), is found to equal

$$\Phi_{fh}(k) = -\frac{J}{(b^2 + k^2)} (b + kj)\Phi_{ff}(k) \tag{21}$$

where the spectrum Φ_{ff} is given by Eq. (13). Substitution of Eq. (21) into Eq. (19), and carrying out the integration, yields the cross-covariance of perturbations:

$$\sigma_{fh} = -\frac{J\sigma_f^2\lambda}{b\lambda + 1} e^{-|\tau|/\lambda}, \qquad \tau \geq 0 \tag{22}$$

Differentiating the covariances of Eqs. (12), (16), and (22) twice with respect to τ, evaluating the second derivatives at $\tau = 0$, and substituting the resulting expressions into Eq. (18) yields the standard deviation of the product of random gradients of perturbations, σ_z (this standard deviation is the order of magnitude of the term in brackets in Eq. 5):

$$\sigma_z = \frac{J\sigma_f^2}{(b\lambda + 1)\lambda} [-b\lambda]^{1/2}, \qquad b < 0 \tag{23}$$

Equation (23) shows that the standard deviation of the term in brackets (involving the product of random gradients) in Eq. (5) vanishes when either $\sigma_f^2 \to 0$, or when $b = 0$. Notice again how the stability condition $b\lambda \neq -1$ holds for the perturbations term also.

MEAN SPECIFIC DISCHARGE AND EQUIVALENT HYDRAULIC CONDUCTIVITY IN THE PRESENCE OF TRENDS

Consider the specific discharge, q_i, as given by Darcy's law:

$$q_i = -K \frac{\partial \phi}{\partial x_i} \tag{24}$$

Substitution of Eqs. (1) and (3) into Eq. (24), and taking the expected value of the resulting expression, leads to

$$E(q_i) = -e^{T+F}\left\{E(e^f)\frac{\partial H}{\partial x_i} + E\left(e^f\frac{\partial h}{\partial x_i}\right)\right\} \tag{25}$$

The function e^f is expanded as a Taylor series in Eq. (25) to yield

$$E(q_i) = -e^{T+F}\left\{E\left[1 + f + \frac{f^2}{2} + \ldots\right]\frac{\partial H}{\partial x_i} + E\left[\left(1 + f + \frac{f^2}{2} + \ldots\right)\frac{\partial h}{\partial x_i}\right]\right\} \tag{26}$$

The Taylor series in Eq. (26) are approximated by keeping those terms of at most order one, i.e., terms with exponents of two or larger are neglected (such as $f^2/2, f^3/6$, etc.). Carrying on the expectation operation in Eq. (26) leads to the following approximate expression for the mean specific discharge:

$$E(q_i) \approx -e^{T+F}\left\{\frac{\partial H}{\partial x_i} + E\left(f\frac{\partial h}{\partial x_i}\right)\right\} \tag{27}$$

where T and F have been previously defined as being the trend and constant level, respectively, of log-hydraulic conductivity (shown in Fig. 1). It is emphasized that Eq. (27) represents a first-order approximation, since the function e^f was truncated after the first-order term, f. Gutjahr et al. (1978) have truncated the Taylor series after the second-order term, $f^2/2$, when multiplying by the gradient, $\partial H/\partial x_i$, in Eq. (26), and after the first-order term, f, when multiplying by the random gradient, $\partial h/\partial x_i$, in that same equation. The product of perturbations in Eq. (27) is evaluated as follows (using now a one-dimensional domain)

$$E\left(f\frac{\partial h}{\partial x}\right) = \int_{-\infty}^{\infty} \Phi_{fh'}(k)\, dk \tag{28}$$

where

$$\Phi_{fh'}(k) = E[dZ_f(k)\, dZ^*_{h'}(k)] \tag{29}$$

in which

$$dZ^*_{h'}(k) = -jk\, dZ^*_h(k) \tag{30}$$

From Eq. (8), the cross-spectrum of Eq. (29) becomes

$$\Phi_{fh'}(k) = \frac{Jjk(b + kj)}{(b^2 + k^2)}\Phi_{ff}(k) \tag{31}$$

in which the spectrum $\Phi_{ff}(k)$ is given by Eq. (13). Substitution of Eq. (31) into Eq. (28) and integrating yields:

$$E\left[f\frac{dh}{dx}\right] = -\frac{J\sigma_f^2}{b\lambda + 1} \tag{32}$$

Based upon the result of Eq. (32), the mean specific discharge in Eq. (27) becomes:

$$E(q_i) \approx -e^{T+F}\left[1 - \frac{\sigma_f^2}{b\lambda + 1}\right]J \tag{33}$$

From Eq. (33), it is seen that the equivalent hydraulic conductivity, K_e, is then

$$K_e = e^{T+F}\left[1 - \frac{\sigma_f^2}{b\lambda + 1}\right] \tag{34}$$

Notice that the effective hydraulic conductivity in Eq. (34) is nonconstant, since the trend, T, by definition depends on the space variable. Notice that to make the equivalent hydraulic conductivity in Eq. (34) physically meaningful (i.e., non-negative), the condition $\sigma_f^2 < b\lambda + 1$ must hold. When Eq. (34) is simplified to the case of no trend ($T = 0$), it becomes

$$K_e = e^F[1 - \sigma_f^2],\ T = 0 \tag{35}$$

in contrast with the result of Gutjahr et al. (1978) (denoting their effective hydraulic conductivity by K'):

$$K' = e^F\left[1 - \frac{\sigma_f^2}{2}\right] \tag{36}$$

The discrepancy between the effective conductivities in Eqs. (35) and (36) arises, as explained before, from the fact K_e in Eq. (35) is based on a first-order approximation of the function e^f, whereas, K' in Eq. (36) is based on first- and second-order approximations to e^f. The effective conductivity is a macroscopic parameter of interest, since it represents the average conductivity under which mean specific discharge takes place. Our results pointing out to the plausibility of a space-dependent effective hydraulic conductivity seem to be novel. They are appealing for the purpose of distributed groundwater model calibration, where average zonal values of hydraulic conductivity are required. Equation (34) shows that the effective hydraulic conductivity is expressed in terms of measurable or estimable macroscopic parameters (b, λ, σ_f^2) and conductivity trend (T, F).

SOLUTION OF THE MEAN EQUATION

The analyses of the previous sections hinge on the nature of the mean hydraulic head, H. It has been already stated that the mean hydraulic gradient, J, may be only weakly space-dependent to validate the spectral representation of hydraulic-head perturbations. In one dimension, the mean hydraulic-head equation becomes

$$\frac{d^2H}{dx^2} + b\frac{dH}{dx} + E\left[\frac{df}{dx}\frac{dh}{dx}\right] = 0 \tag{37}$$

The expected value of the product of random gradients in Eq. (37) can be shown to be given by (Loaiciga and Mariño, 1990):

$$E\left[\frac{df}{dx}\frac{dh}{dx}\right] = -\sigma''_{fh}(0) \tag{38}$$

where the second derivative of the cross-covariance σ_{fh} is readily obtained by differentiating twice in Eq. (22) and evaluating the resulting expression at $\tau = 0$; therefore,

$$E\left[\frac{df}{dx}\frac{dh}{dx}\right] = \frac{dH}{dx}\frac{\sigma_f^2}{\lambda(b\lambda + 1)} \tag{39}$$

In view of Eq. (39), Eq. (37) can be rewritten as

$$\frac{d^2H}{dx^2} + \left[b + \frac{\sigma_f^2}{\lambda(b\lambda + 1)}\right]\frac{dH}{dx} = 0 \tag{40}$$

It is straightforward to verify that a solution to Eq. (40) is

$$H(x) = C_1 + C_2 \int \exp\left[-\int p(x)\, dx\right] dx \tag{41}$$

in which C_1 and C_2 are constants of integration (that depend on the geometry of the flow domain and boundary conditions, that, theoretically, must be defined in an infinite domain), and

$$p(x) = b + \frac{\sigma_f^2}{\lambda(b\lambda + 1)} \tag{42}$$

If the trend gradient, b, is constant, as when the trend is linear (i.e., $T = a + bx$), the mean hydraulic head in Eq. (41) becomes

$$H(x) = C_1 - \frac{C_2}{p} e^{-px} \tag{43}$$

This solution can be approximated around any point, x_0, in the flow domain by a Taylor series:

$$H(x) = C_1 - C_2 \frac{e^{-px_0}}{\rho} \sum_{r=0}^{\infty} \frac{(-p)^r (x - x_0)^r}{r!} \tag{44}$$

If $\sigma_f^2 << b\lambda + 1$ and the slope of the trend is also small (these are conditions approximating those imposed in classical spectral analysis of groundwater flow), then the mean hydraulic head can be approximated by a first-order expansion in Eq. (44):

$$H(x) \approx C_1' + C_2' x \tag{45}$$

where

$$C_1' = C_1 - C_2 \left(\frac{1}{p} + x_0\right) e^{-px_0} \tag{46}$$

and

$$C_2' = C_2 e^{-px_0} \tag{47}$$

From Eq. (45), the mean hydraulic gradient, J, is constant (and equal to C_2'); the trend gradient, b, is assumed constant and mild (i.e., the slope of the trend is small). Under these conditions our analysis of stochastic groundwater flow with trends in hydraulic conductivity is fully justified, and it provides an accurate insight of the role of spatial conductivity trends on hydraulic head, already exposed in previous sections. Figure 3 shows a qualitative description of the

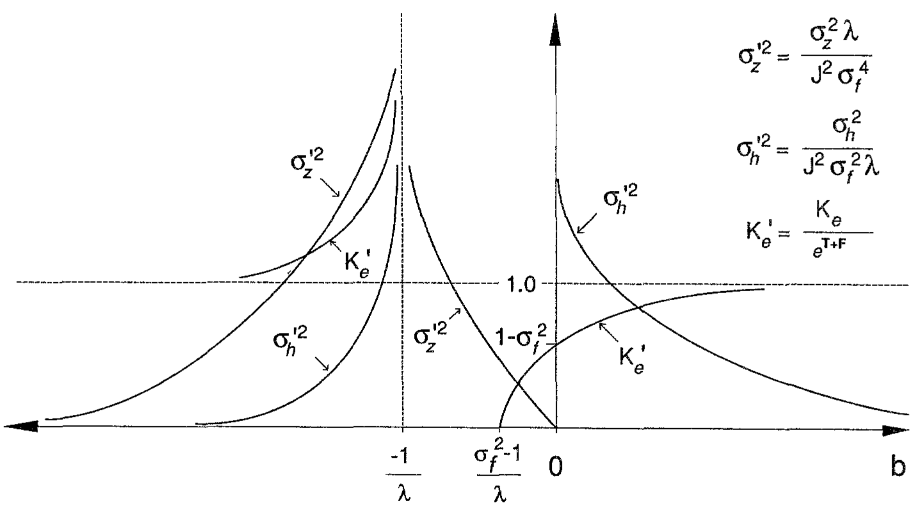

Fig. 3. Qualitative behavior of (scaled) variance of hydraulic head, variance of the product of perturbation gradients, and effective hydraulic conductivity.

dependence of (1) the head variance, (2) effective hydraulic conductivity, and (3) order of magnitude of the product of perturbation gradients on the trend slope b.

THREE-DIMENSIONAL RESULTS

The methods of the previous sections can be extended to two and three dimensions straightforwardly by using multidimensional spectral representation of conductivity and head perturbations (Priestley, 1981). For example, the variance of hydraulic head becomes in three dimensions (where R represents the complete three-dimensional space, and tensorial index notation is used again):

$$\sigma_h^2 = \frac{\sigma_f^2 \lambda^3}{\pi^2} \int_R \frac{(k_i J_i)^2}{[(k_i k_i)^2 + (k_i a_i)^2][1 + (k_i k_i)\lambda^2]^2} \, dk \tag{48}$$

in which J_i is the mean hydraulic gradient in the ith-direction, a_i is the ith component of the trend gradient ($a_i = \partial T/\partial x_i$), and k_i is the ith component of the wave number vector k. The above expression cannot be integrated in terms of elementary functions as in the one-dimensional case. It is possible to show that the right-hand side of Eq. (48) is integrable for all a_i. The stability conditions encountered in the one-dimensional case are removed in three dimensions. (The only possibility to quantify the integral in (48) is numerically or, with significantly more effort, in terms of a class of infinite series called polylogarithms (Lewin, 1981)).

The three-dimensional effective hydraulic conductivity can be shown to be:

$$K_e = -e^{T+F}\left\{ J_i + \frac{\sigma_f^2 \lambda^3}{\pi^2} \int_R j k_i \frac{[(k_i a_i) + j\,(k_i k_i)]\,[k_i J_i]}{[(k_i k_i)^2 + (k_i a_i)^2][1 + (k_i k_i)\,\lambda^2]^2} \, dk \right\} \tag{49}$$

for $i = 1, 2, 3$. One immediate conclusion to be drawn from Eq. (49) is that the effective hydraulic conductivity in three dimensions is isotropic only if the mean hydraulic gradient is the same in all directions. (Some authors consider only one non-zero gradient direction, that of mean specific discharge.) Otherwise, the conductivity is anisotropic. The right-hand side of (49) is integrable for all a_i. No attempt has been made in this paper to integrate the expressions in Eqs. (48)–(49), although this is feasible for specified values of the intervening parameters.

SUMMARY AND CONCLUSIONS

The analysis of steady-state stochastic groundwater flow in the presence of trends in log-hydraulic conductivity has resulted in the following findings:

(1) The perturbations and mean equations of hydraulic head are coupled, and, in general, their separate solution can be derived only if the mean hydraulic gradient is weakly space dependent.

(2) By introducing trends in log-hydraulic conductivity, there are no restrictions on the structure of the log-hydraulic conductivity covariance function.

(3) The stability of the variance of hydraulic head depends on the magnitude of the gradient of the trend and on the product of the trend gradient times the correlation scale of log-conductivity ($b\lambda$); poles exist at $b = 0$ and $b\lambda = -1$ in the one-dimensional case.

(4) The error analysis of the perturbations equation established the order of magnitude of the nonlinear term involving products of random gradients: that analysis showed that such nonlinear term is negligible for small values of the variance of log-conductivity.

(5) The equivalent hydraulic conductivity is nonconstant, and it depends on the trend, trend slope, constant level of the trend, and the variance and correlation scale of log-hydraulic conductivity; in the one-dimensional case a pole exists at $b\lambda = -1$.

(6) The solution of the one-dimensional equation for mean hydraulic head in the presence of log-conductivity trends results in an exponential function, that can be approximated by a linear function when the trend gradient and the variance of log-hydraulic conductivity are small.

(7) In the one-dimensional case, the necessary and sufficient conditions for the validity of the analysis of hydraulic head in the presence of trends in log-conductivity are that $\sigma_f^2 < b\lambda + 1$, b be small, and $b\lambda \neq -1$.

(8) Three-dimensional expressions for the variance of hydraulic head and for effective hydraulic conductivity were derived. It was shown that in three dimensions the effective hydraulic conductivity is anisotropic except when the mean hydraulic gradient is the same in all directions.

(9) The condition $b\lambda \neq -1$ in the one-dimensional flow problem can be related to the deterministic case where flow is choked off when the slope of hydraulic conductivity reaches a critical negative value.

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REFERENCES

Bakr, A. A., Gelhar, L. W., Gutjahr, A. L., and MacMillan, J. R., 1978, Stochastic Analysis of Spatial Variability in Subsurface Flows, I, Comparison of One- and Three-Dimensional Flows: Water Res. Res., v. 14, p. 263–271.

Churchill, R. V., and Ward, J., 1990, Complex Variables and Applications, 5th ed.: McGraw-Hill, New York.

Dagan, G., 1985, A Note on the Higher-Order Corrections of the Head Covariances in Steady Aquifer Flow: Water Res. Res., v. 21, p. 573–578.

Dagan, G., 1989, Flow and Transport in Porous Formations: Springer-Verlag, Berlin.

Farvolden, R. N., Pfannkuch, O., Pearson, R., and Fritz, P., 1988, Region 12, Precambrian Shield, *in* W. Back, J. S. Rosenshein, and P. R. Seaber (Eds.), The Geology in North America, Vol. O-2, Hydrogeology: The Geological Society of America, Boulder, Colorado.

Gelhar, L. W., 1986, Stochastic Subsurface Hydrology: From Theory to Applications: Water Res. Res, v. 22, p. 135S–145S.

Gelhar, L. W., and Axness, C. L., 1983, Three-Dimensional Stochastic Analysis of Macrodispersion in Aquifers: Water Res. Res., v. 19, p. 161–180.

Gutjahr, A. L., Gelhar, L. W., Bakr, A. A., and MacMillan, J. R., 1978, Stochastic Analysis of Spatial Variability in Subsurface Flows, 2. Evaluation and Application: Water Res. Res., v. 14, p. 953–959.

Lewin, L., 1981, Polylogarithms and Associated Functions: North Holland, New York.

Loaiciga, H. A., and Mariño, M. A., 1990, Error Analysis and Stochastic Differentiability in Subsurface Flow Modeling: Water Res. Res., v. 26, p. 2897–2902.

Priestley, M. B., 1981, Spectral Analysis of Time Series, Vol. 1: Academic Press, London.

Rajaram, H., and MacLaughlin, D., 1990, Identification of Large-Scale Trends in Hydrologic Data: Water Res. Res., v. 26, p. 2411–2423.

Smith, L., and Freeze, R. A., 1979, Stochastic Analysis of Steady-State Groundwater Flow in Bounded Domain: Two-Dimensional Simulations: Water Res. Res., v. 18, p. 1543–1559.