UNIVERSITY OF CALIFORNIA
RIVERSIDE

Existence and Structure of P-Area Minimizing Surfaces

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

Alexander Stephen Rowell

December 2022

Dissertation Committee:

Dr. Amir Moradifam, Chairperson
Dr. James Kelliher
Dr. Yat Tin Chow

The Dissertation of Alexander Stephen Rowell is approved:

Committee Chairperson

University of California, Riverside

Acknowledgments

I would like to express my deepest gratitude to my advisor, Dr. Amir Moradifam. He inspired me with his lectures and thoughtful examples as an instructor, then continued to provide patient mentorship in the research process. I would be remiss in not mentioning my spouse, parents, and extended family for helping with child care, meals, and moral support.

The text of this dissertation, in part or in full, is a reprint of the material as it appears in Existence and structure of P-area minimizing surfaces in the Heisenberg group, 2023. The co-author, Amir Moradifam, listed in that publication directed and supervised the research which forms the basis for this dissertation.

To my wife, kids, and parents, who have given me unconditional love and support.

ABSTRACT OF THE DISSERTATION

Existence and Structure of P-Area Minimizing Surfaces

by

Alexander Stephen Rowell

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, December 2022
Dr. Amir Moradifam, Chairperson

This dissertation uses methods from convex analysis and calculus of variations to find solutions to partial differential equations by proving existence of minimizers for the associated energy functionals. In the first problem, we study existence and structure of $P-$area minimizing surfaces in the Heisenberg group under Dirichlet and Neumann boundary conditions. We show that there exists an underlying vector field, N, that characterizes existence and structure of P-area minimizing surfaces. This vector field exists even if there is no P-area minimizing surface satisfying the prescribed boundary conditions. We prove that if $\partial\Omega$ satisfies a so-called Barrier condition, it is sufficient to guarantee existence of such surfaces. Our approach is completely different from previous methods in the literature and makes major progress in understanding existence of P-area minimizing surfaces.

The work on the energy functional associated to the P-mean curvature partial differential equation can be generalized to a class of functionals $I(u) = \int_\Omega (\varphi(x, Du + F) + Hu)\, dx$, where $\varphi(x, \xi)$ is convex, continuous, and homogeneous with respect to the second argument. Using the Rockafellar-Fenchel duality, we prove existence and deduce structure of solutions to the Dirichlet and Neumann boundary problems associated with minimizers of the functionals. The case when φ is not strictly convex is a highly non-trivial problem. We prove the existence of an underlying vector field N, that always exists, and characterizes the structure of minimizers of $I(u)$.

Contents

Chapter 1

Introduction

The motivation of this thesis arose from interest in two areas of conductivity imaging, and generalizing the associated least gradient problem. In particular, Electrical Impedance Tomography (EIT) and its extension to the hybrid method Current Density Impedance Imaging (CDII). In EIT we seek to determine the electrical conductivity at every point interior to a body, given the voltage potential at the boundary, following a corresponding induced current on the boundary. The resulting images of conductivity are very low resolution due to the EIT problem being severely ill-posed. This inspired the introduction of a hybrid method in which EIT is combined with Magnetic Resonance Imaging (MRI) to gather high accuracy data interior to the body of interest. The CDII problem reconstructs the conductivity at every point in the interior, after inducing a current at the boundary and measuring the corresponding magnitude of the current density vector field in the interior.

We are now ready for the statement of the problem. Consider an object represented by an open, bounded, and connected domain $\Omega \subset \mathbb{R}$, with isotropic conductivity σ. For a given voltage f on $\partial\Omega$, J is the current density vector field. The resulting voltage potential v satisfies the equation

$$\nabla \cdot (\sigma \nabla v) = 0, \qquad v|_{\partial\Omega} = f. \tag{1.1}$$

A substitution is applied using Ohm's law $J = -\sigma \nabla v$, yielding equation

$$\nabla \cdot \left(\frac{|J|}{|\nabla v|} \nabla v \right) = 0, \qquad v|_{\partial\Omega} = f. \tag{1.2}$$

Equation (1.2) has the associated energy functional

$$E(v) = \int_\Omega |J||\nabla v|. \tag{1.3}$$

The voltage potential v that minimizes $E(v)$, solves equation (1.2). Then the conductivity σ is uniquely determined by $|J|$ and $v|_{\partial\Omega} = f$.

In research, these problems in EIT and CDII were thoroughly studied in [6, 10, 11, 28, 27], with many interesting results. Furthermore, in [26, 37, 38, 41] the authors worked on finding minimizers for classes of more generalized version of functional (1.3). They are often referred to as the least gradient problem, expressed as

$$\int_\Omega a|Dv| \qquad \text{or} \qquad \int_\Omega \varphi(x, Dv),$$

where $a \in L^\infty(\Omega)$ is a positive function, and $\varphi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a convex function sharing the same properties as norm $|\cdot|$.

In exploring such problems, we discovered geometers were interested in solving the P-mean curvature partial differential equation in the rough form

$$\nabla \cdot \left(a \frac{\nabla u + F}{|\nabla u + F|} \right) = H, \tag{1.4}$$

where $a \in L^\infty(\Omega)$ is a positive function, $F \in (L^2(\Omega))^n$, and $H \in L^2(\Omega)$. Note that equation (1.4) is itself a slight generalization of the actual P-mean curvature equation, explored in greater depth in section 1.1. The associated energy functional to the Euler-Lagrange equation (1.4) is

$$I(u) = \int_\Omega (a|\nabla u + F| + Hu)\ dx. \tag{1.5}$$

One can see that equations (1.2) and (1.3) are special cases of (1.4) and (1.5) respectively, by setting $F \equiv 0, H \equiv 0$, and $a = |J|$. The main substance of this thesis is in proving functional (1.5) has minimizers under prescribed Dirichlet and Neumann boundary conditions. It can be shown that minimizers of $I(u)$ solve the associated partial differential equation by the standard method in Calculus of Variations. For any fixed $v \in C_c^\infty(\Omega)$ we define

$$i(\tau) := I[u + \tau v] \qquad \text{for all } \tau \in \mathbb{R}.$$

Function i has a minimum at zero and $i'(0) = 0$. Then

$$i(\tau) = \int_\Omega a|\nabla u + \tau\nabla v + F| + Hu + \tau Hv,$$

and we compute the component-wise derivative with respect to τ yielding

$$i'(\tau) = \int_\Omega a\frac{(\nabla u + \tau\nabla v + F)}{|\nabla u + \tau\nabla v + F|} \cdot \nabla v + Hv.$$

Consequently, we use integration by parts

$$\begin{aligned} 0 = i'(0) &= \int_\Omega a\frac{(\nabla u + F)}{|\nabla u + F|} \cdot \nabla v + Hv \\ &= \int_\Omega \left[-\nabla \cdot \left(a\frac{\nabla u + F}{|\nabla u + F|}\right) + H\right] v. \end{aligned}$$

Since this holds for all $v \in C_c^\infty(\Omega)$, minimizers of $I(u)$ are solutions to equation (1.4) in Ω.

In a further step of generalization, we endeavor to find existence of minimizers to a class of integral functionals of the form

$$\mathcal{I}(u) = \int_\Omega \varphi(x, Du + F) + Hu, \tag{1.6}$$

where $\varphi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is convex, continuous, and homogeneous function of degree 1 with respect to the second argument. Moreover, Ω is a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary, $F \in (L^2(\Omega))^n$, $H \in L^2(\Omega)$, and φ satisfies conditions (C_1) and (C_2), that provide it with similar properties to the norm $|\cdot|$. This class of functionals is explored further in Section 1.2 and Chapter 3.

1.1 P-area minimizing surfaces in the Heisenberg group

Let's start with some definitions and background of previous research on P-area minimizing surfaces. The $2m+1$-dimensional Heisenberg group is the manifold $\mathbb{H}^m = \mathbb{C}^m \times \mathbb{R}$, endowed with the group product

$$(z,t) \cdot (\zeta, \tau) = \left(z + \zeta, t + \tau + 2\mathrm{Im}\langle z, \overline{\zeta}\rangle\right),$$

where $t, \tau \in \mathbb{R}$, $z, \zeta \in \mathbb{C}^m$ and $\langle z, \overline{\zeta}\rangle = z_1\overline{\zeta}_1 + \cdots + z_m\overline{\zeta}_m$. Suppose that Ω is a bounded region in $\mathbb{R}^{2m}$, and $X = (x_1, x_1', x_2, x_2', \cdots, x_m, x_m') \in \Omega$. Let $u : \mathbb{R}^{2m} \to \mathbb{R}$, and consider the graph $(X, u(X))$ in the Heisenberg group of dimension $2m+1$ with prescribed p-mean curvature $H(X)$. Then u satisfies the equation

$$\nabla \cdot \left(\frac{\nabla u - X^*}{|\nabla u - X^*|} \right) = H, \tag{1.7}$$

where $X^* = (x_1', -x_1, x_2', -x_2, \cdots, x_m', -x_m)$. The p-minimal surfaces are the case when $H \equiv 0$ (also called H-minimal or X-minimal surfaces [20, 21, 48]), and have been studied by many authors. Numerous interesting results have been presented about existence, uniqueness, and regularity of p-minimal surfaces [5, 12, 13, 14, 15, 20, 21, 48]. A summary of the most important results are shown below. Note that we solve this problem in Chapter 2 from the perspective of PDE and Calculus of Variations. In that spirit, notice that equation 1.7 is the Euler-Lagrange equation to the energy functional

$$\mathbb{E}(u) = \int_\Omega (|\nabla u - X^*| + Hu)\, dx_1 \wedge dx_1' \wedge \cdots \wedge dx_n \wedge dx_n'. \tag{1.8}$$

One of the main challenges in studying the equation (1.7) is to deal with the singular set of solutions, i.e.

$$\{X \in \Omega : |\nabla u(X) - X^*| = 0\}.$$

On the other hand, since the energy functional $\mathbb{E}$ is not strictly convex, analysis of existence and uniqueness of minimizers is also a highly non-trivial problem.

In [5] the author studied the size of the singular set of solutions, and showed the existence of solutions with large singular sets. In Theorem A of [15], the authors proved existence of minimizers of (1.8) in the special case $H \equiv 0$, and under the assumption that Ω is a p-convex domain. Consider the condition on $\vec{F}$, for C^1-smooth functions f_K's:

$$\partial_K F_I = \partial_I f_K, \qquad I, K = 1, ..., n \tag{1.9}$$

Theorem 1 ([15]) *Let Ω be a p-convex bounded domain in $\mathbb{R}^n, n \geq 2$, with $\partial\Omega \in C^{2,\alpha}(0 < \alpha < 1)$. Let $\varphi \in C^{2,\alpha}(\overline{\Omega})$. Suppose $\vec{F} \in C^{1,\alpha}(\overline{\Omega})$ satisfies the condition (1.9) for $C^{1,\alpha}$-smooth and bounded f_K's in Ω. Then there exists a Lipschitz continuous minimizer $u \in C^{0,1}(\overline{\Omega})$ for $\mathbb{E}(\cdot)$ with $H = 0$ such that $u = \varphi$ on $\partial\Omega$.*

They also proved interesting uniqueness and comparison results for minimizers of Theorem B and C. Namely,

Theorem 2 ([15]) *Let Ω be a bounded domain in $\mathbb{R}^{2m}$. Let $u, v \in W^{1,2}(\Omega)$ be two minimizers for $\mathbb{E}(\cdot)$ such that $u - v \in W_0^{1,2}(\Omega)$. Suppose $H \in L^\infty(\Omega)$ and $\vec{F} \in W^{1,2}(\Omega)$ satisfying $div(\vec{F}^*) > 0$ (a.e.). Then $u \equiv v$ in Ω (a.e.).*

Theorem 3 ([15]) *Let Ω be a bounded domain in $\mathbb{R}^{2n}$. Let $F \in W^{1,2}(\Omega)$ satisfy $divF^* > 0$ (a.e.). Suppose $u, v \in W1,2(\Omega)$ satisfy the following conditions:*

$$divN(u) \geq divN(v) \ \text{ in } \Omega \text{(in the weak sense)}.$$

$$u \leq v \ \text{ on } \partial\Omega.$$

Then $u \leq v$ in Ω.

In [49], the authors proved existence and uniqueness of minimizers of $\mathbb{E}$ for the case when $H \equiv 0$ in Ω under the so-called bounded slope condition (as defined in [23]).

Definition 4 *We say that the function $U : \partial\Omega \to \mathbb{R}$ satisfies a bounded slope condition with constant $Q > 0$ if for every $x_0 \in \partial\Omega$ there exists two affine functions $w^+ = w^+_{x_0}$ and $w^- = w^-_{x_0}$ such that*

$$w^-(x) \leq U(x) \leq w^+(x) \ \text{ in } \partial\Omega :$$

$$w^-(x_0) = U(x_0) = w^+(x_0);$$

$$Lip(w^-_{x_0}) \leq Q; \quad Lip(w^+_{x_0}) \leq Q,$$

where $Lip(w)$ denotes the Lipschitz constant of w.

Notice that, if φ satisfies the bounded slope condition, then it is Lipschitz continuous on $\partial\Omega$. In this sense, the assumptions on the boundary datum are stronger than those in Theorem 1. As a result, they were able to obtain uniqueness and Lipschitz regularity of the minimizer on (possibly) less regular domains. The results from [49] were extended in [17] to prove existence and uniqueness of minimizers for the more generalized functional, $\mathbb{G}(u) = \int_\Omega g(\nabla u + X^*) d\mathcal{L}^{2n}$, where $g : \mathbb{R}^{2n} \to \mathbb{R}$ is convex but not necessarily strictly convex. In [13], the authors studied uniqueness of minimizers of the functional $\mathbb{E}$. The interesting results regarding uniqueness have been omitted, as the most relevant research is of existence of solutions.

In Chapter 2, we study existence and structure of minimizers of the energy functional $\mathbb{E}$ from a different point of view, using the Rockafellar-Fenchel duality. We prove various existence results that are new, even for the case $a \equiv 1$. Consider the following weighted form of the functional (1.8)

$$\mathcal{F}(u) = \int_\Omega (a|\nabla u - X^*| + Hu)\, dx_1 \wedge dx_1' \wedge \cdots \wedge dx_n \wedge dx_n', \tag{1.10}$$

where $a \in L^\infty(\Omega)$ is a positive function. Minimizers of this functional will satisfy the Euler-Lagrange equation

$$\nabla \cdot \left(a \frac{\nabla u - X^*}{|\nabla u - X^*|} \right) = H, \tag{1.11}$$

which could be viewed as the p-mean curvature of the function $(X, u(X))$, with respect to the metric $g = a^{\frac{2}{n-1}} dx$, which is conformal to the Euclidean metric. Our approach is completely different from the previous ones in the literature and provides major progress in understanding the existence of P-area minimizing surfaces.

1.2 Minimizers for a class of integral functionals

The scope of this section addresses minimizers of a class of functionals broadly explored in Calculus of Variations. In particular, much research has been published on finding existence, uniqueness, regularity, and continuity of minimizers of functionals of the form $\int_\Omega G(Du(x)) + K(x,u)\,dx$, where G is convex and K is locally Lipschitz or identically zero. For background, one should explore the tree of references stemming from [7, 8, 9, 16, 19, 31, 32, 33, 34]. Our motivation for the research in chapter 3 was inspired by the methods used in [43], the paper that makes up the contents of chapter 2. Therein, we proved existence and structure of minimizers of P-area minimizing surfaces in the Heisenberg group. View the references within [15, 43, 49] on literature about P-minimal surfaces in the Heisenberg group. The statement of the problem was stated last section, where the goal was to minimize function $\mathcal{F}(u)$ from (1.10). Under prescribed Dirichlet and Neumann boundary conditions, in Chapter 2 we prove existence and deduce the structure of minimizers of the altered energy functional,

$$\mathbb{I}(u) = \int_\Omega (a|\nabla u + F| + Hu)\ dx, \tag{1.12}$$

where $a \in L^\infty(\Omega)$ is a positive function and $F \in (L^\infty(\Omega))^n$.

The subject of study in Chapter 3 is a class of functionals that generalize (1.12), namely

$$I(u) = \int_\Omega \varphi(x, Du + F) + Hu, \tag{1.13}$$

where $\varphi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is convex, continuous, and homogeneous function of degree 1 with respect the the second argument. Unless otherwise stated, we assume that Ω is a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary, $F \in (L^2(\Omega))^n$, $H \in L^2(\Omega)$, and φ satisfies the following conditions:

(C_1) There exists $\alpha > 0$ such that $0 \leq \varphi(x, \xi) \leq \alpha\, |\xi|$ for all $\xi \in \mathbb{R}^n$.

(C_2) $\xi \mapsto \varphi(x, \xi)$ is a norm for every x.

While not generally required, at times we specify the additional condition on φ:

(C_3) There exists $\beta > 0$ such that $0 \leq \beta\, |\xi| \leq \varphi(x, \xi)$ for all $\xi \in \mathbb{R}^n$.

This problem is of particular interest since the energy functional $I(u)$ is not strictly convex. Analysis of existence and uniqueness of minimizers is a highly non-trivial problem, in which we address using the Rockafellar-Fenchel duality. In doing so, we prove the existence of an underlying vector field N, that always exists, and characterizes the structure of minimizers of (1.13).

It is of interest to investigate [17], where the problem of unique minimizers in $BV(\Omega)$ of (1.13) are found to be Lipschitz continuous. This was for the case where $H \equiv 0$ and with different assumptions on convex function $g : \mathbb{R}^{2n} \to \mathbb{R}$, with solutions u under the bounded slope condition from Definition 4. Note that the authors have two additional conditions on convex function g, which make it not very dissimilar to a strictly convex function (in some sense). In turn, they provide the basis for more regularity of minimizers. Those conditions are

$$g\left(\frac{\xi_1 + \xi_2}{2}\right) = \frac{g(\xi_1) + g(\xi_2)}{2} \implies \xi_1 = \lambda \xi_2, \quad \text{and} \tag{1.14}$$

$$[p \in \partial g(\xi_2) \text{ and } g^{\infty}(\xi_1) = \langle p, \xi_1 \rangle] \implies \xi_1 = \lambda \xi_2. \tag{1.15}$$

Consider

$$G_\Omega := \inf\left\{\liminf_h \int_\Omega g(\nabla u_h + X^*)d\mathcal{L}^{2n} : u_h \in W^{1,1}(\Omega), u_h \to u \text{ in } L^1(\Omega)\right\}.$$

Then their main Theorem (4.4) is as follows.

Theorem 5 ([17]) *Let $\Omega \subset \mathbb{R}^{2n}$ be open, bounded and with Lipschitz regular boundary, let $f : \partial\Omega \to \mathbb{R}$ satisfy the Q-B.S.C. for some $Q > 0$ and let $g : \mathbb{R}^{2n} \to \mathbb{R}$ be a convex function with linear growth satisfying conditions* (1.14) *and* (1.15)*. Then, the minimization problem*

$$\min\{G_\Omega : u \in BV(\Omega), u|_{\partial\Omega} = f\}$$

admits a unique solution $\hat{u}$. Moreover, $\hat{u}$ is Lipschitz continuous and $Lip(\hat{u}) \leq \overline{Q}(Q, \Omega)$.

Another broad area of study is the least gradient problem, a special case of (1.13) in which $F \equiv 0$, $H \equiv 0$, and $\varphi(x, \cdot) = a|\cdot|$, where $a \in L^\infty(\Omega)$ is a positive function. It has applications in conductivity imaging and has been extensively studied by many authors, see [25, 26, 37, 38, 39, 40, 44, 45, 46, 47, 50, 51, 52, 53]. One such example that provides a use case of φ, when $F \equiv 0$ and $H \equiv 0$, is presented in [25]. Considering $J, \xi \in (L^2(\Omega))^n$ and $\sigma_0 \in C^\alpha(\Omega, Mat(n, \mathbb{R}^n))$, the authors defined convex function

$$\varphi(x, \xi) = a(x)\left(\sum_{i,j=1}^n \sigma_0^{ij}(x)\xi_i\xi_j\right)^{1/2},$$

which satisfies conditions (C_1)-(C_3), with $a = \sqrt{\sigma_0^{-1}J \cdot J}$. They seek unique solutions to the least gradient problem

$$\operatorname{argmin}\left\{\int_\Omega \varphi(x, Dv) : v \in BV(\Omega), v|_{\partial\Omega} = f\right\}.$$

The above example displays a non-trivial use case of φ that comes up in conductivity imaging.

Chapter 3 is outlined as follows. It starts with an introduction of preliminary definitions and concepts, with the aim to set up existence proofs for minimizers of functional (1.13). In Section 3.1, we prove existence under the Neumann boundary condition. In Section 3.2, we study existence of minimizers with Dirichlet boundary condition. Finally, in Section 3.3 we provide existence of P-area minimizing surfaces under a Barrier condition on the boundary $\partial\Omega$.

Chapter 2

Existence and structure of P-area minimizing surfaces in the Heisenberg group

This chapter is dedicated to the study of the existence and structure of minimizers of the energy functional

$$I(u) = \int_\Omega \left(a|\nabla u + F| + Hu \right), \tag{2.1}$$

to the Euler-Lagrange equation

$$\nabla \cdot \left(a \frac{\nabla u + F}{|\nabla u + F|} \right) = H, \tag{2.2}$$

where $a \in L^\infty(\Omega)$ is a positive function, $F \in (L^2(\Omega))^n$, and $H \in L^2(\Omega)$. Moreover, we look for solutions u of bounded variation satisfying various Dirichlet and Neumann boundary conditions. Notice that functional (2.1) and equation (2.2) are slight generalizations of the

P-area minimizing problems (1.10) and (1.11) respectively, from Section 1.1. The approach of proving existence of solution is different from others in the previous research, since we use the Rockafellar-Fenchel duality. Also, our last existence result puts a new condition on Ω, the Barrier condition as in Definition (22).

2.1 Existence of P-area minimizing surfaces with Neumann boundary condition

Let Ω be a bounded open region in $\mathbb{R}^n$, $a \in L^\infty(\Omega)$ be a positive function, $F \in (L^2(\Omega))^n$, $H \in L^2(\Omega)$, and consider the minimization problem

$$\inf_{u \in \mathring{BV}(\Omega)} I(u) := \int_\Omega a\,|Du + F| + Hu, \tag{2.3}$$

where

$$\mathring{BV}(\Omega) = \{u \in BV(\Omega) : \int_\Omega u = 0\}.$$

In order to study the minimizers of the least gradient problem (2.3), we first analyze the dual of this problem using Rockafeller-Fenchel duality. Define $E(b) : L^2(\Omega) \to \mathbb{R}$ and $G(u) : \mathring{H}^1(\Omega) \to \mathbb{R}$ as follows

$$E(b) = \int_\Omega a\,|b + F| \quad \text{and} \quad G(u) = \int_\Omega Hu,$$

where $\mathring{H}(\Omega) = \{u \in H^1(\Omega) : \int_\Omega u = 0\}$. Then (2.3) can be rewritten as

$$(P) \qquad \inf_{u \in \mathring{H}^1(\Omega)} \{E(\nabla u) + G(u)\}. \tag{2.4}$$

By Rockafellar-Fenchel duality [18], the dual problem associated to (2.4) is

$$(D) \qquad -\min_{b \in (L^2(\Omega))^n} \{E^*(b) + G^*(-\nabla^* b)\} = \max_{b \in (L^2(\Omega))^n} \{-E^*(b) - G^*(-\nabla^* b)\}, \tag{2.5}$$

where E^* and G^* are the convex conjugates of the convex functions E and G, and ∇^* is the adjoint of the gradient operator $\nabla : \mathring{H}^1(\Omega) \to L^2(\Omega)$. Let us first compute $G^*(-\nabla^* b)$.

$$\begin{aligned}
G^*(-\nabla^* b) &= \sup_{u \in H^1(\Omega)} \left\{ \langle u, -\nabla^* b \rangle_{\mathring{H}^1(\Omega)\times(H^1(\Omega))^*} - G(u) \right\} \\
&= \sup_{u \in \mathring{H}^1(\Omega)} \left\{ \langle u, -\nabla^* b \rangle_{H^1(\Omega)\times(H^1(\Omega))^*} - \int_\Omega Hu \right\} \\
&= \sup_{u \in \mathring{H}^1(\Omega)} \left\{ -\int_\Omega \nabla u \cdot b - \int_\Omega Hu \right\}.
\end{aligned}$$

Since $cu \in \mathring{H}^1$ for any $u \in \mathring{H}^1$ and any $c \in \mathbb{R}$,

$$G^*(-\nabla^* b) = \begin{cases} \infty & \text{if } u \notin \mathcal{D}_0(\Omega) \\ 0 & \text{if } u \in \mathcal{D}_0, \end{cases} \tag{2.6}$$

where

$$\mathcal{D}_0 := \left\{ b \in (L^2(\Omega))^n : \int_\Omega \nabla u \cdot b + Hu = 0, \quad \text{for all} \quad u \in \mathring{H}^1(\Omega) \right\}. \tag{2.7}$$

The computation of $E^*(b)$ is gathered from Lemma 2.1 in [39]. The statement and proof have been included for completeness.

Lemma 6 ([39]) *Let $a \in L^2(\Omega)$ be non-negative and $F \in (L^2(\Omega))^n$. Then*

$$E^*(b) = \begin{cases} -\langle F, b \rangle & \text{if } |b(x)| \leq a(x) \quad a.e. \quad in \quad \Omega \\ \infty & otherwise\ . \end{cases} \tag{2.8}$$

Proof. First, consider the case $|b(x)| > a(x)$, on a set of positive Lebesgue measure $U \subset \Omega$. It follows that

$$
\begin{aligned}
E^*(b) &= \sup_{d \in (L^2(\Omega))^n} \langle d, b\rangle - \int_\Omega a|d+F| \\
&= -\langle b, F\rangle + \sup_{d \in (L^2(\Omega))^n} \left(\langle d, b\rangle - \int_\Omega a|d| dx \right) \\
&\geq -\langle b, F\rangle + \sup_{\lambda \in \mathbb{R}} \lambda \int_U \left(|b|^2 - a(x)|b| \right) dx = \infty,
\end{aligned}
$$

where the second inequality follows from variable substitution. The last inequality is due to the assumption $|b(x)| > a(x)$ and our choice of of d,

$$
d(x) = \begin{cases} \lambda b(x) & \text{for } x \in U \\ 0 & \text{otherwise .} \end{cases}
$$

In the second case, consider $|b(x)| \leq a(x)$, a.e. and we compute

$$
\begin{aligned}
E^*(b) &= -\langle b, F\rangle + \sup_{d \in (L^2(\Omega))^n} \left(\langle d, b\rangle - \int_\Omega a|d| dx \right) \qquad (2.9) \\
&= -\langle b, F\rangle + \sup_{d \in (L^2(\Omega))^n} \int_\Omega \left(b \cdot d - a|d| \right) dx \\
&\leq -\langle b, F\rangle + \sup_{d \in (L^2(\Omega))^n} \int_\Omega |d(x)| \left(|b(x)| - a(x) \right) dx \\
&\leq -\langle b, F\rangle.
\end{aligned}
$$

Choosing $d \equiv 0$ in (2.9) gives

$$
E^*(b) \geq -\langle b, F\rangle.
$$

□

Thus, the dual problem (D) can be written as

$$
(D) \qquad \sup\{\langle F, b\rangle : b \in \mathcal{D}_0 \ \text{ and } \ |b| \leq a \ \text{ a.e. in } \ \Omega\}. \qquad (2.10)
$$

Let ν_Ω denote the outer unit normal vector to $\partial\Omega$. Then for every $b \in (L^\infty(\Omega))^n$ with $\nabla \cdot b \in L^n(\Omega)$ there exists a unique function $[b, \nu_\Omega] \in L^\infty_{\mathcal{H}^{n-1}}(\partial\Omega)$ such that

$$\int_{\partial\Omega} [b, \nu_\Omega] u \, d\mathcal{H}^{n-1} = \int_\Omega u \nabla \cdot b dx + \int_\Omega b \cdot D u dx, \quad u \in C^1(\bar{\Omega}). \tag{2.11}$$

Moreover, for $u \in BV(\Omega)$ and $b \in (L^\infty(\Omega))^n$ with $\nabla \cdot b \in L^n(\Omega)$, the linear functional $u \mapsto (b \cdot Du)$ gives rise to a Radon measure on Ω, and (2.11) is valid for every $u \in BV(\Omega)$ (see [1, 3] for a proof). The following lemma is an immediate consequence of (2.11).

Lemma 7 *Let $b \in (L^\infty(\Omega))^n \cap \mathcal{D}_0$. Then*

$$\nabla \cdot b = H - \int_\Omega H dx \quad a.e. \ in \ \Omega,$$

and

$$[b, \nu_\Omega] = 0 \quad \mathcal{H}^{n-1} - a.e. \quad on \quad \partial\Omega.$$

Indeed, it follows from the above lemma that for any solution N of the dual problem (D), $\nabla \cdot N = H - \int_\Omega H$ a.e. in Ω, and N is orthogonal to the unit normal vector on $\partial\Omega$ in a weak sense. We are now ready to present the main result of this section.

Theorem 8 *Let Ω be a bounded domain in $\mathbb{R}^n$, $F \in (L^2(\Omega))^n$, $H \in L^2(\Omega)$, and $a \in L^2(\Omega)$ be a positive function. Then the duality gap is zero and the dual problem (D) has a solution, i.e. there exists a vector field $N \in \mathcal{D}_0$ with $|N| \le a$, $|Du + F| - a.e.$ in Ω, such that*

$$\inf_{u \in \mathring{H}^1(\Omega)} \int_\Omega (a\,|Du + F| + Hu)\, dx = \langle F, N \rangle \tag{2.12}$$

Moreover

$$a \frac{Du + F}{|Du + F|} = N, \qquad |Du + F| - a.e. \ \ in \ \ \Omega, \tag{2.13}$$

for any minimizer u of (2.4).

Proof. It is easily verified that $I(v) = \int_\Omega a|Dv|$ is convex, and $J : (L^2(\Omega))^n \to \mathbb{R}$ with $J(p) = \int_\Omega a|p|dx$ is continuous at $p = 0$. Hence, it follows from Theorem III.4.1 in [18] that the duality gap is zero and the dual problem (D) has a solution N, and consequently (2.12) holds.

Now let $u \in \mathcal{H}$ be a minimizer of (2.4). Then

$$\begin{aligned}
\langle F, N\rangle &= \int_\Omega a\,|Du + F| + \int_\Omega Hu \\
&\geq \int_\Omega |N||Du + F| + \int_\Omega Hu \\
&\geq \int_\Omega N \cdot (Du + F) + \int_\Omega Hu \\
&= \langle F, N\rangle + \int_\Omega N \cdot Du + Hu \\
&= \langle F, N\rangle,
\end{aligned}$$

since $N \in \mathcal{D}_0$. Therefore, both the inequalities above are equalities, and hence (2.13) holds.

□

Remark 9 *The primal problem (P) may not have a minimizer in $\mathring{H}^1(\Omega)$, but the dual problem (D) always has a solution $N \in (L^2(\Omega))^n$. Note also that the functional $I(u)$ is not strictly convex, and it may have multiple minimizers (see [26]). Theorem 8 asserts that if u_1 and u_2 are both minimizers of (P), then*

$$a\frac{Du_1 + F}{|Du_1 + F|}(x) = a\frac{Du_2 + F}{|Du_2 + F|}(x) = N(x), \tag{2.14}$$

for a.e. point $x \in \Omega$ where $|Du_1 + F|$ and $|Du_2 + F|$ do not vanish.

Next we show that if the primal problem (P) is bounded below, then it has a solution in $BV(\Omega)$. The proof follows from standard facts about BV functions, and we sketch it out for the sake of completeness.

Proposition 10 *There exists a constant C, depending on Ω, such that if*

$$\max_{x\in\overline{\Omega}} |H(x)| < C, \tag{2.15}$$

then the primal problem (P) has a minimizer.

Proof. Let u_n be the minimizing sequence for $I(u)$. Then

$$\int |\nabla u_n| - \int |F| - \int |H||u_n| \leq \int |\nabla u_n| - \int F + \int H u_n \leq \int |\nabla u_n + F| + H u_n < c,$$

for some constant c independent of n. Hence

$$\int |\nabla u_n| \leq C + \int |H||u_n| + \int |F|.$$

It follows from the Poincaré's inequality that there exists a constant C_Ω, independent of n, such that

$$\int |\nabla u_n| \leq C + ||H||_{L^\infty(\Omega)} C_\Omega \int |\nabla u_n| + \int |F|$$
$$\Rightarrow \left(1 - C_\Omega ||H||_{L^\infty(\Omega)}\right) \int |\nabla u_n| \leq C + \int |F|.$$

$$\int |\nabla u_n| \leq C' = \frac{C + \int |F|}{\left(1 - C_\Omega ||H||_{L^\infty(\Omega)}\right)}$$

provided that $1 - C_\Omega ||H||_{L^\infty(\Omega)} > 0$ or equivalently

$$||H||_{L^\infty(\Omega)} \leq C := \frac{1}{C_\Omega}.$$

It follows from standard compactness results for BV functions that u_n has a subsequence, denoted by u_n again, such that u_n converges strongly in L^1 to a function $\hat{u} \in BV$, and Du_n converges to $D\hat{u}$ is the sense of measures. Since the functional $I(u)$ is lower semicontinuous, $\hat{u}$ is a solution of the primal problem (2.3). □

This leads directly to the first existence result. There is a solution to an altered version of PDE (2.2), in the case where $a \equiv 1$.

Corollary 11 *Let Ω be a bounded domain in $\mathbb{R}^n$, $F \in (L^2(\Omega))^n$ and $a \in L^2(\Omega)$ be a positive function. There exists a constant C such that if $||H||_{L^\infty(\Omega)} < C$, then the equation*

$$\nabla \cdot \left(a \frac{Du + F}{|Du + F|} \right) = H - \int_\Omega H$$

has a solution $u \in \overset{\circ}{BV}(\Omega)$, i.e. there exists $N \in \mathcal{D}_0$ such that

$$a \frac{Du + F}{|Du + F|} = N.$$

2.2 Existence of P-area minimizing surfaces with Dirichlet boundary condition

In this section we study existence of p-area minimizing surfaces with a given Dirichlet boundary condition on the boundary $\partial\Omega$. Let Ω be a bounded open region in $\mathbb{R}^n$, $a \in L^\infty(\Omega)$ be a positive function, $f \in L^1(\partial\Omega)$, and consider minimization problem

$$\inf_{u \in BV_f(\Omega)} I(u) := \int_\Omega a\,|Du + F| + Hu, \tag{2.16}$$

where

$$BV_f(\Omega) = \{u \in BV(\Omega) : u|_{\partial\Omega} = f\}.$$

The function $f \in L^1(\partial\Omega)$ can be extended to a function in $W^{1,1}(\Omega)$ (denoted by f again), and the weighted least gradient problem (2.16) can be written as

$$\inf_{u \in BV_0(\Omega)} I(u) := \int_\Omega a \left| Du + \tilde{F} \right| + Hu + \int_\Omega Hf dx,$$

where $\tilde{F} = F + \nabla f$, and $\int_\Omega H f dx$ is a constant. Hence the minimization problem (2.16) is equivalent to the least gradient problem

$$\inf_{u \in BV_0(\Omega)} I(u) := \int_\Omega a\,|Du + F| + Hu. \tag{2.17}$$

It is easy to verify that the minimizers of (2.16) in $BV_0(\Omega)$ satisfy the Euler-Lagrange equation

$$\nabla \cdot \left(a \frac{Du + F}{|Du + F|} \right) = H, \tag{2.18}$$

with $u|_{\partial\Omega} = 0$. However, the minimization problems (2.16) and (2.17) do not necessarily have minimizers even if they are bounded below. This is in contrast with our results in Section 2.1 where boundedness of the functional $I(u)$ in (2.3) from below automatically implies existence of a minimizer. To see this, suppose u_n is a minimizing sequence for (2.17) that converges in $L^1(\Omega)$ to a function $\hat{u} \in BV(\Omega)$. Then it follows from lower semicontinuity of the functional $I(u)$ that

$$I(\hat{u}) \leq \inf_{u \in BV_0(\Omega)} I(u).$$

However, the trace of $\hat{u}$ on $\partial\Omega$ may not necessarily be equal to zero. This is the main reason for nonexistence of minimizers for (2.17). Indeed it is well known that (2.17) may not have a minimizer, and proving existence of minimizers for (2.17) is a challenging problem that we aim to tackle in this section.

Similar to the our approach in Section 2.1, we first analyze the dual of the relaxed minimization problem (2.28) from section 2.2.2, which will be a crucial tool in our analysis.

2.2.1 The dual problem

As in Section 2.1, let $E(b) : (L^2(\Omega))^n \to \mathbb{R}$ and $G(u) : H_0^1(\Omega) \to \mathbb{R}$ as

$$E(b) = \int_\Omega a\,|b+F| \qquad \text{and} \qquad G(u) = \int_\Omega Hu.$$

We can rewrite (2.28) as

$$(P') \qquad \inf_{u\in H_0^1(\Omega)} \{E(\nabla u) + G(u)\}. \tag{2.19}$$

By Rockafellar-Fenchel duality [18], the dual problem associated to (2.19) is

$$(D') \qquad -\min_{b\in(L^2(\Omega))^n} \{E^*(b) + G^*(-\nabla^* b)\} = \sup_{b\in(L^2(\Omega))^n} \{-E^*(b) - G^*(-\nabla^* b)\}, \tag{2.20}$$

where E^* and G^* are the convex conjugates of the convex functions E and G, and ∇^* is the adjoint of the gradient operator $\nabla : H_0^1(\Omega) \to L^2(\Omega)$. Due to the change in our function space, we update the computation of $G^*(-\nabla^* b)$.

$$\begin{aligned}
G^*(-\nabla^* b) &= \sup_{u\in H_0^1(\Omega)} \left\{ \langle u, -\nabla^* b\rangle_{H_0^1(\Omega)\times(H_0^1(\Omega))^*} - G(u) \right\} \\
&= \sup_{u\in H_0^1(\Omega)} \left\{ \langle u, -\nabla^* b\rangle_{H_0^1(\Omega)\times(H_0^1(\Omega))^*} - \int_\Omega Hu \right\} \\
&= \sup_{u\in H_0^1(\Omega)} \left\{ -\int_\Omega \nabla u \cdot b - \int_\Omega Hu \right\}.
\end{aligned}$$

Since $cu \in H_0^1(\Omega)$ for any $u \in H_0^1(\Omega)$ and any $c \in \mathbb{R}$,

$$G^*(-\nabla^* b) = \begin{cases} \infty & \text{if } u \notin \widetilde{\mathcal{D}}_0(\Omega) \\ 0 & \text{if } u \in \widetilde{\mathcal{D}}_0, \end{cases} \tag{2.21}$$

where

$$\widetilde{\mathcal{D}}_0 := \left\{ b \in (L^2(\Omega))^n : \int_\Omega \nabla u \cdot b + Hu = 0, \quad \text{for all} \quad u \in H_0^1(\Omega) \right\} \subseteq \mathcal{D}_0. \tag{2.22}$$

On the other hand, it follows from Lemma 2.1 in [39] (computation shown in Lemma 6) that

$$E^*(b) = \begin{cases} -\langle F, b\rangle & \text{if } |b| \le a \ \text{ a.e. in } \ \Omega \\ \\ \infty & \text{otherwise .} \end{cases} \tag{2.23}$$

Thus the dual problem (D') can be written as

$$(D') \qquad \sup\{\langle F, b\rangle : b \in \widetilde{\mathcal{D}}_0 \ \text{ and } \ |b| \le a \ \text{ a.e. in } \ \Omega\}. \tag{2.24}$$

It follows from the integration by parts formula (2.11) that $b \in (L^\infty(\Omega))^n \cap \widetilde{\mathcal{D}}_0$ if and only if

$$\nabla \cdot b = H \ \text{ a.e. in } \ \Omega.$$

We are now ready to prove the following theorem.

Theorem 12 *Let Ω be a bounded domain in $\mathbb{R}^n$, $F \in (L^2(\Omega))^n$, $H \in L^2(\Omega)$, $a \in L^2(\Omega)$ be a positive function, and assume (P') is bounded below. Then the duality gap is zero and the dual problem (D') has a solution, i.e. there exists a vector field $N \in \widetilde{\mathcal{D}}_0$ with $|N| \le a$, $|Du + F| -$ a.e. in Ω, such that*

$$\inf_{u \in H_0^1(\Omega)} \int_\Omega (a\,|Du + F| + Hu)\, dx = \langle F, N\rangle \tag{2.25}$$

Moreover

$$a\frac{Du + F}{|Du + F|} = N, \qquad |Du + F| - a.e. \ \ in \ \ \Omega, \tag{2.26}$$

for any minimizer u of (2.19).

Proof. It is easily verified that $I(v) = \int_\Omega a|Dv|$ is convex, and $J : (L^2(\Omega))^n \to \mathbb{R}$ with $J(p) = \int_\Omega a|p|dx$ is continuous at $p = 0$. Hence, it follows from Theorem III.4.1 in [18] that the duality gap is zero and the dual problem (D) has a solution N, and consequently (2.25) holds.

Now let $u \in A_0$ be a minimizer of (2.19). Since $N \in \widetilde{\mathcal{D}}_0$, we have

$$\begin{aligned} \langle F, N \rangle &= \int_\Omega a\,|Du + F| + \int_\Omega Hu \\ &\geq \int_\Omega |N||Du + F| + \int_\Omega Hu \\ &\geq \int_\Omega N \cdot (Du + F) + \int_\Omega Hu \\ &= \langle F, N \rangle + \int_\Omega N \cdot Du + Hu \\ &= \langle F, N \rangle. \end{aligned}$$

Therefore, both the inequalities above are equalities, and (2.26) holds. □

Remark 13 *Note that the primal problem (P') may not have a minimizer in $H_0^1(\Omega)$, but the dual problem (D') always has a solution $N \in (L^2(\Omega))^n$. Note also that the functional $I(u)$ is not strictly convex, and it may have multiple minimizers (see [26]). Theorem 12 asserts that if u_1 and u_2 are both minimizers of (P), then*

$$a\frac{Du_1 + F}{|Du_1 + F|}(x) = a\frac{Du_2 + F}{|Du_2 + F|}(x) = N(x), \tag{2.27}$$

for a.e. point $x \in \Omega$ where $|Du_1 + F|$ and $|Du_2 + F|$ do not vanish.

2.2.2 The relaxed problem

Here we study existence of minimizers for the relaxed least gradient problem

$$\inf_{u \in A_0} I(u) = \inf_{u \in A_0} \int_\Omega (a|Du + F| + Hu)dx + \int_{\partial\Omega} a|u|ds, \tag{2.28}$$

where

$$A_0 := \left\{ u \in H^1(\mathbb{R}^n) : u = 0 \text{ in } \Omega^c \right\}.$$

Unlike the problem (2.17), any minimizing sequence for (2.28) converges to a minimizer in A_0. Indeed the following proposition holds.

Proposition 14 *There exists a constant C, depending on Ω, such that if*

$$\max_{x\in\overline{\Omega}} |H(x)| < C, \tag{2.29}$$

then the primal problem (2.17) *has a minimizer in A_0.*

Proof. The proof follows from an argument similar to the one used in the proof of Proposition 10, and the observation that if $u_n \in A_0$ converges to $\hat{u}$ in $L^1(\Omega)$, then $\hat{u} \in A_0$. $\square$

The next theorem characterizes the relationship between these two problems and sheds light on the challenging problem of existence of minimizers for (2.17).

Theorem 15 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, $F \in (L^2(\Omega))^n$, and $H \in L^2(\Omega)$. If the minimization problem* (2.17) *is bounded below, then*

$$\min_{u\in A_0}\left(\int_\Omega (a|Du+F|+Hu)dx + \int_{\partial\Omega} a|u|ds\right) = \inf_{u\in BV_0(\Omega)} \int_\Omega a|Du+F|+Hu \tag{2.30}$$

Moreover, if u is a minimizer of (2.28)*, then*

$$-u[N,\nu_\Omega] = a|u| \quad \mathcal{H}^{n-1} - a.e. \quad on \quad \partial\Omega. \tag{2.31}$$

Proof. Since $BV_0(\Omega)$ can be continuously embedded in A_0, we have

$$\min_{u\in A_0}\left(\int_\Omega (a|\nabla u+F|+Hu)dx + \int_{\partial\Omega} a|u|ds\right) \le \inf_{u\in BV_0(\Omega)} \int_\Omega a|\nabla u+F|+Hu.$$

It follows from Theorem 12 that there exists a vector field N with $|N| \leq a$ a.e. in Ω and

$$N = a\frac{Du + F}{|Du + F|}.$$

Now let u be a minimizer of the relaxed problem with $u|_{\partial\Omega} = g|_{\partial\Omega}$, where $g \in W^{1,1}(\Omega)$. Since $u - g \in \widetilde{\mathcal{D}}_0$, we have

$$\begin{aligned}
\min_{u \in A_0} \left(\int_\Omega (a|Du + F| + Hu)dx + \int_{\partial\Omega} a|u|ds \right) &= \int_\Omega a|\nabla u + F| + Hu + \int_{\partial\Omega} a|u| \\
&\geq \int_\Omega |N||\nabla u + F| + Hu + \int_{\partial\Omega} a|u| \\
&\geq \int_\Omega N(\nabla u + F) + Hu + \int_{\partial\Omega} a|u| \\
&= \int_\Omega N \cdot F + \int_\Omega N \cdot \nabla u + Hu + \int_{\partial\Omega} a|u| \\
&= \langle N, F \rangle + \int_\Omega N \cdot \nabla(u - g) + H(u - g) \\
&+ \int_\Omega N \cdot \nabla g + Hg + \int_{\partial\Omega} a|g| \\
&= \langle N, F \rangle + \int_\Omega N \cdot \nabla g + Hg + \int_{\partial\Omega} a|g| \\
&= \langle N, F \rangle + \int_{\partial\Omega} g[N, \nu_\Omega] + \int_{\partial\Omega} a|g| \\
&\geq \langle N, F \rangle \\
&= \inf_{u \in BV_0(\Omega)} \int_\Omega a|Du + F| + Hu.
\end{aligned}$$

We used integration by parts, and $|N| \leq a$ a.e. in Ω, to obtain the last inequality, and hence (2.30) holds. Moreover, all the inequalities in the above computation are equalities. In particular, (2.31) holds. □

The following theorem is an immediate consequence of both Theorem 12 and Theorem 15.

Theorem 16 *Let Ω be a bounded domain in $\mathbb{R}^n$, $F \in (L^2(\Omega))^n, H \in L^2(\Omega)$, $a \in L^2(\Omega)$ be a positive function, and assume (P') is bounded below. Then there exists a vector field $N \in \widetilde{\mathcal{D}}_0$ with $|N| \leq a$, $|Du + F| -$ a.e. in Ω, such that*

$$a\frac{Du + F}{|Du + F|} = N, \qquad |Du + F| - a.e. \;\; in \;\; \Omega, \tag{2.32}$$

for any minimizer u of (2.17)*. Moreover, every minimizer of* (2.17) *is a minimizer of* (2.28)*, and if u is a minimizer of* (2.28)*, then*

$$sign(-u)[N, \nu_\Omega] = a \;\; \mathcal{H}^{n-1} - a.e. \;\; on \;\; \partial\Omega. \tag{2.33}$$

In particular, $u = 0$ $\mathcal{H}^{n-1}$ a.e. on the set

$$\{x \in \partial\Omega : -|N| < [N, \nu_\Omega] < |N|\}.$$

The next theorem follows immediately from Theorem 16.

Theorem 17 *Let Ω be a bounded domain in $\mathbb{R}^n$, $F \in (L^2(\Omega))^n, H \in L^2(\Omega)$, $a \in L^2(\Omega)$ be a positive function, and assume (P') is bounded below. Let N be the solution of the dual problem guaranteed by Theorem 12 and assume that $-N < [N, \nu_\Omega] < |N|$ almost everywhere on $\partial\Omega$. Then the least gradient problem* (2.17) *has a minimizer in $BV_0(\Omega)$.*

2.3 Existence of minimizers under the Barrier condition

Let $F \in (L^1(\Omega)^n)$ and $a, H \in L^\infty(\Omega)$ with $a > 0$ in Ω, and define $\psi : \mathbb{R}^n \times BV_0(\Omega)$ as follows

$$\psi(x,u) := a(x)|Du + F\chi_{E_u}| + Hu, \tag{2.34}$$

where E_u is the closure of the support of u in Ω.

Define the ψ-perimeter of E in A, as

$$P_\psi(E;A) := \int_A a(x)\,|D\chi_E + F\chi_E| + H\chi_E.$$

Definition 18 *1. A function $u \in BV(\mathbb{R}^n)$ is ψ-total variation minimizing in $\Omega \subset \mathbb{R}^n$ if*

$$\int_\Omega \psi(x,u) \leq \int_\Omega \psi(x,v) \text{ for all } v \in BV(\mathbb{R}^n) \text{ such that } u = v \text{ a.e. in } \Omega^c.$$

2. A set $E \subset \mathbb{R}^n$ of finite perimeter is ψ-area minimizing in Ω if

$$P_\psi(E;\Omega) \leq P_\psi(\tilde{E})$$

for all $\tilde{E} \subset \mathbb{R}^n$ such that $\tilde{E} \cap \Omega^c = E \cap \Omega^c$ a.e..

We will show that the super level sets of ψ-total variation minimizing functions in Ω are ψ-area minimizing in Ω. In order to achieve this, we shall first prove some preliminary lemmas.

Lemma 19 *Let $\chi_{\epsilon,\lambda}$ be defined as in* (2.37)*. Then*

$$P_\psi(E,\Omega) \leq \liminf_{\epsilon\to 0} \int_\Omega a(x)|D\chi_{\epsilon,\lambda} + F\chi_{\epsilon,\lambda}| + H\chi_{\epsilon,\lambda}.$$

Proof. We have

$$\begin{aligned}
& \int_\Omega a(x)|D\chi_{\epsilon,\lambda} + F\chi_{\epsilon,\lambda}| + H\chi_{\epsilon,\lambda} - \int_\Omega a(x)|D\chi_E + F\chi_E| + H\chi_E \\
= & \int_{\Omega\cap\{\lambda-\epsilon<u<\lambda+\epsilon\}} a|D\chi_{\epsilon,\lambda} + F\chi_{\epsilon,\lambda}| + H\chi_{\epsilon,\lambda} - a|D\chi_E + F\chi_E| - H\chi_E \\
\geq & \int_{\Omega\cap\{\lambda-\epsilon<u<\lambda+\epsilon\}} a|D\chi_{\epsilon,\lambda}| - a|F\chi_{\epsilon,\lambda}| + H\chi_{\epsilon,\lambda} - a(x)|D\chi_E| - a|F\chi_E| - H\chi_E \\
= & \int_{\Omega\cap\{\lambda-\epsilon<u<\lambda+\epsilon\}} a|D\chi_{\epsilon,\lambda}| - a(x)|D\chi_E| + H\chi_{\epsilon,\lambda} - H\chi_E - a|F\chi_{\epsilon,\lambda}| - a|F\chi_E| \\
= & \int_\Omega a|D\chi_{\epsilon,\lambda}| - \int_\Omega a(x)|D\chi_E| + \int_\Omega (H\chi_{\epsilon,\lambda} - H\chi_E) \\
& - \int_{\Omega\cap\{\lambda-\epsilon<u<\lambda+\epsilon\}} a|F\chi_{\epsilon,\lambda}| + a|F\chi_E|.
\end{aligned}$$

It is easy to see that the last two integrals converge to zero as $\epsilon \to 0$. Hence

$$\begin{aligned}
& \liminf_{\epsilon\to 0} \int_\Omega a(x)|D\chi_{\epsilon,\lambda} + F\chi_{\epsilon,\lambda}| + H\chi_{\epsilon,\lambda} - P_\psi(E,\Omega) \\
= & \liminf_{\epsilon\to 0} \int_\Omega a(x)|D\chi_{\epsilon,\lambda} + F\chi_{\epsilon,\lambda}| + H\chi_{\epsilon,\lambda} - \int_\Omega a(x)|D\chi_E + F\chi_E| + H\chi_E \\
\geq & \liminf_{\epsilon\to 0} \int_\Omega a|D\chi_{\epsilon,\lambda}| - \int_\Omega a(x)|D\chi_E| \geq 0,
\end{aligned}$$

where we have used the lower semi-continuity of $\int_\Omega a|Dv|$ to obtain the last inequality (see [26]). The proof is complete. □

If $w \in BV(\mathbb{R}^n)$ and Ω is an open set with Lipschitz boundary, we will write w^+ and w^- to denote the outer and inner trace of w on $\partial\Omega$.

Lemma 20 *Let $\Omega \subset \mathbb{R}^n$ be bounded and open, with Lipschitz boundary. Given $g \in L^1(\partial\Omega; \mathcal{H}^{n-1})$, define*

$$I_\psi(v; \Omega, g) := \int_{\partial\Omega} a|g - v^- + F_{\chi_v}| d\mathcal{H}^{n-1} + \int_\Omega \psi(x, Dv).$$

Then $u \in BV(\mathbb{R}^n)$ is ψ-total variation minimizing in Ω if and only if $u|_\Omega$ minimizes $I_\psi(\,\cdot\,; \Omega, g)$ for some g, and moreover $g = u^+$.

Proof: First note that if $v \in BV(\mathbb{R}^n)$ then $v^+, v^- \in L^1(\partial\Omega; \mathcal{H}^{n-1})$, and conversely, for every $g \in L^1(\partial\Omega; \mathcal{H}^{n-1})$ there exists some $v \in BV(\mathbb{R}^n)$ such that $g = v^+$. Also

$$\int_{\partial\Omega} \psi(x, Dv) = \int_{\partial\Omega} a|Dv + F_{\chi_v}| d\mathcal{H}^{n-1} = \int_{\partial\Omega} a|v^+ - v^- + F_{\chi_v}| d\mathcal{H}^{n-1}. \tag{2.35}$$

To see this, note that $|Dv|$ can only concentrate on a set of dimension $n-1$ if that set is a subset of the jump set of v, so (2.35) follows from standard descriptions of the jump part of Dv.

Now if $u, v \in BV(\mathbb{R}^n)$ satisfy $u = v$ a.e. in Ω^c, then $\int_{\bar{\Omega}^c} \varphi(x, Du) = \int_{\bar{\Omega}^c} \varphi(x, Dv)$. In addition, $u^+ = v^+$, so using (2.35) we deduce that

$$\int_{\mathbb{R}^n} \psi(x, Du) - \int_{\mathbb{R}^n} \psi(x, Dv) \;=\; I_\varphi(u; \Omega, u^+) - I_\varphi(v; \Omega, u^+).$$

The lemma easily follows from the above equality. □

Theorem 21 *Consider the bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and a ψ-total variation minimizing function in Ω, $u \in BV(\mathbb{R}^n)$. Let the super level sets of u to be defined as*

$$E_\lambda := \{x \in \mathbb{R}^n : u(x) \geq \lambda\}. \tag{2.36}$$

Then E_λ is ψ-area minimizing in Ω.

Proof. This proof closely mirrors that of Theorem 2.6 in [26]. Consider an arbitrary $\lambda \in \mathbb{R}$, and let $u_1 = \max(u - \lambda, 0), u_2 = u - u_1$. For any $g \in BV(\mathbb{R}^n)$ such that $\text{supp}(g) \subset \overline{\Omega}$, we have

$$
\begin{aligned}
\int_\Omega a\left|Du_1 + F\chi_{\{u\geq\lambda\}}\right| + Hu_1 + \int_\Omega a\left|Du_2 + F\chi_{\{u<\lambda\}}\right| + Hu_2 = \int_\Omega a\left|Du + F\right| + Hu \\
\leq \int_\Omega a\left|D(u+g) + F\right| + H(u+g) \\
= \int_\Omega a\left|Du_1 + D(g\chi_{\{u\geq\lambda\}}) + F\chi_{\{u\geq\lambda\}}\right| + H(u_1+g) \\
+ \int_\Omega a\left|Du_2 + D(g\chi_{\{u<\lambda\}}) + F\chi_{\{u<\lambda\}}\right| + Hu_2 \\
\leq \int_\Omega a\left|Du_1 + D(g\chi_{\{u\geq\lambda\}}) + F\chi_{\{u\geq\lambda\}}\right| + H(u_1+g) \\
+ \int_\Omega a|D(g\chi_{\{u<\lambda\}})| + \int_\Omega a\left|Du_2 + F\chi_{\{u<\lambda\}}\right| + Hu_2 \\
= \int_\Omega a\left|D(u_1+g) + F\chi_{\{u\geq\lambda\}}\right| + H(u_1+g) \\
+ \int_\Omega a\left|Du_2 + F\chi_{\{u<\lambda\}}\right| + Hu_2.
\end{aligned}
$$

Thus

$$\int_\Omega a\left|Du_1 + F\chi_{u_1}\right| + Hu_1 \leq \int_\Omega a\left|D(u_1+g) + F\chi_{u_1}\right| + H(u_1+g),$$

for all $g \in BV(\mathbb{R}^n)$ with $\text{supp}(g) \subset \overline{\Omega}$. Hence u_1 is also ψ-total variation minimizing. By the same process, we can verify that the function defined below is also ψ-total variation minimizing,

$$\chi_{\epsilon,\lambda} := \min\left(1, \frac{1}{\epsilon}u_1\right) = \begin{cases} 0 & \text{if } u \leq \lambda, \\ \frac{1}{\epsilon}(u-\lambda) & \text{if } \lambda < u \leq \lambda+\epsilon, \\ 1 & \text{if } u > \lambda+\epsilon. \end{cases} \tag{2.37}$$

For a.e. $\lambda \in \mathbb{R}$ the boundary of the super level set E_λ is a set of measure zero, that is,

$$\mathcal{L}^n\left(\{x \in \Omega : u(x) = \lambda\}\right) = \mathcal{H}^{n-1}\left(\{x \in \partial\Omega : u^{\pm}(x) = \lambda\}\right) = 0. \tag{2.38}$$

On that account

$$\chi_{\epsilon,\lambda} \to \chi_\lambda := \chi_{E_\lambda} \text{ in } L^1_{\text{loc}}(\mathbb{R}^n), \quad \chi^{\pm}_{\epsilon,\lambda} \to \chi^{\pm}_\lambda \text{ in } L^1(\partial\Omega; \mathcal{H}^{n-1}),$$

as $\epsilon \to 0$.

It follows from Lemma 19 via quite standard arguments that

$$P_\psi(\chi_\lambda, \Omega) \leq \liminf_{\epsilon \to 0} P_\psi(\chi_{\epsilon,\lambda}, \Omega); \tag{2.39}$$

and this, with the L^1 convergence of the traces, implies that

$$I_\varphi(\chi_\lambda; \Omega, \chi^+_\lambda) \leq \liminf_{k \to \infty} I_\varphi(\chi_{\epsilon,\lambda}; \Omega, \chi^+_{\lambda,\epsilon}). \tag{2.40}$$

Now for any $F \subset \mathbb{R}^n$ such that $\chi_\lambda = \chi_F$ a.e. in Ω^c,

$$\begin{aligned} I_\varphi(\chi_{\epsilon,\lambda}; \Omega, \chi^+_{\epsilon,\lambda}) &\leq I_\varphi(\chi_F; \Omega, \chi^+_{\epsilon,\lambda}) \\ &\leq I_\varphi(\chi_F; \Omega, \chi^+_\lambda) + \int_{\partial\Omega} a|\chi^+_\lambda - \chi^+_{\epsilon,\lambda}| \, d\mathcal{H}^{n-1} \\ &\leq I_\varphi(\chi_F; \Omega, {\chi_\lambda}^+) + C \int_{\partial\Omega} |\chi^+_\lambda - \chi^+_{\epsilon,\lambda}| \, d\mathcal{H}^{n-1}. \end{aligned}$$

It follows from this, (2.40), and $\chi^+_{\epsilon,\lambda} \to \chi^+_\lambda$ in $L^1(\partial\Omega; \mathcal{H}^{n-1})$ that

$$I_\varphi(\chi_\lambda; \Omega, \chi^+_\lambda) \leq I_\varphi(\chi_F; \Omega, \chi^+_\lambda),$$

which proves that E_λ is φ-area minimizing in Ω.

In the case where λ does not satisfy (2.38), we can take an increasing sequence $\lambda_k \to \lambda$ as $k \to \infty$, that satisfies (2.38) for each k. This implies that

$$\chi_{\lambda_k} \to \chi_\lambda \text{ in } L^1_{\text{loc}}(\mathbb{R}^n), \quad \chi^{\pm}_{\lambda_k} \to \chi^{\pm}_\lambda \text{ in } L^1(\partial\Omega; \mathcal{H}^{n-1}).$$

This once again leads to the conclusion that E_λ is ψ-area minimizing in Ω in view of Lemma 20. $\square$

Now we are ready to present the main existence results of this section. For any measurable set E define

$$E^{(1)} := \left\{ x \in \mathbb{R}^n : \lim_{r \to 0} \frac{\mathcal{H}^n(B(r,x) \cap E)}{\mathcal{H}^n(B(r))} = 1 \right\}.$$

Definition 22 *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. Then Ω satisfies the barrier condition if for every $x_0 \in \partial\Omega$ and $\epsilon > 0$ sufficiently small, if V minimizes $P_\psi(\cdot;\mathbb{R}^n)$ in*

$$\{W \subset \Omega : W \setminus B(\epsilon, x_0) = \Omega \setminus B(\epsilon, x_0)\}, \tag{2.41}$$

then

$$\partial V^{(1)} \cap \partial\Omega \cap B(\epsilon, x_0) = \emptyset.$$

Lemma 23 *Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ that satisfies the barrier condition from Definition 22, and suppose $E \subset \mathbb{R}^n$ minimizes $P_\psi(\cdot;\Omega)$. Then*

$$\left\{ x \in \partial\Omega \cap \partial E^{(1)} : B(\epsilon, x) \cap \partial E^{(1)} \subset \overline{\Omega} \textit{ for some } \epsilon > 0 \right\} = \emptyset.$$

Proof. Assume there exists $x_0 \in \partial\Omega \cap \partial E^{(1)}$ such that $B(\epsilon, x_0) \cap \partial E^{(1)} \subset \bar{\Omega}$ for some $\epsilon > 0$. Then $\tilde{V} = E \cap \Omega$ is a minimizer of $P_\psi(\,\cdot\,;\mathbb{R}^n)$ in (2.41), and

$$x_0 \in \partial\tilde{V}^{(1)} \cap \partial\Omega \cap B(\epsilon, x_0) \neq \emptyset.$$

This contradicts the barrier condition and finishes the proof. $\square$

In the last theorem we use the definition

$$BV_f(\Omega) := \left\{ u \in BV(\Omega) : \lim_{r \to 0} \operatorname*{ess\,sup}_{y \in \Omega, |x-y|<r} |u(y) - f(y)| = 0 \text{ for } x \in \partial\Omega \right\}.$$

Theorem 24 *Let* $\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *be defined as in* (2.34), *and* $\Omega \subset \mathbb{R}^n$ *be a bounded Lipschitz domain. Suppose* $||H||_{L^\infty(\overline{\Omega})}$ *is small enough such that Proposition 14 holds. If* Ω *satisfies the barrier condition with respect to* ψ, *as given in Definition 22, then for every* $f \in C(\partial\Omega)$ *the minimization problem* (2.16) *has a minimizer in* $BV_f(\Omega)$.

Proof. Since every $\mathcal{H}^{n-1}$ integrable function on Ω is the trace of some (continuous) function in $BV(\Omega^c)$, without loss of generality we may assume that $f \in BV(\mathbb{R}^n)$.

Define

$$\mathcal{A}_f := \{v \in BV(\mathbb{R}^n) : \quad v = f \ \text{ on } \ \Omega^c\},$$

and note that $BV_f(\Omega) \hookrightarrow \mathcal{A}_f$, in the sense that any element v of $BV_f(\Omega)$ is the restriction to Ω of a unique element of $\mathcal{A}_f$. An argument similar to that of Proposition 14 implies that $\int_{\mathbb{R}^n} \psi(x, v)$ has as a minimizer $u \in \mathcal{A}_f$.

We next use the barrier condition to show that $u \in BV_f(\Omega)$. If not, there exists some $x \in \partial\Omega$ and $\delta > 0$ such that

$$\operatorname*{ess\,sup}_{y\in\Omega, |x-y|<r} \big(f(x) - u(y)\big) \geq \delta \qquad \text{or} \qquad \operatorname*{ess\,sup}_{y\in\Omega, |x-y|<r} \big(u(y) - f(x)\big) \geq \delta \tag{2.42}$$

for every $r > 0$. Assume that the latter condition holds. It follows from this and the continuity of f, that $x \in \partial E^{(1)}$ for $E := E_{f(x)+\delta/2}$. By Theorem 21 E is ψ-area minimizing in Ω. However, since f is continuous in Ω^c and $u \in \mathcal{A}_f$, it is clear that $u < f(x) + \delta/2$ in $B(\varepsilon, x) \setminus \Omega$ for all sufficiently small ε. This contradicts Lemma 23. If the first alternative holds in (2.42), then we set $E := \{y \in \mathbb{R}^n : u(y) \leq f(x) - \delta/2\}$ and reach a similar contradiction. Hence $u \in BV_f(\Omega)$, and in view of Theorem 15, it is ψ-total variation minimizing in $BV_f(\Omega)$. $\square$

Chapter 3

Existence and structure of minimizers for a class of integral functionals

A few preliminaries are required to interpret and find minimizers of the class of functionals represented by (1.13). As a brief reminder, the assumptions on these functionals are that $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, $F \in (L^2(\Omega))^n$, $H \in L^2(\Omega)$, and φ satisfies (C_1) and (C_2) where

$$I(u) = \int_\Omega \varphi(x, Du + F) + Hu.$$

Recall that (C_1) creates bounds on φ and (C_2) states that it possesses the properties of a norm.

For an arbitrary $u \in BV_{loc}(\mathbb{R}^n)$, an associated measure $\varphi(x, Du + F)$ is defined by

$$\int_A \varphi(x, Du + F) = \int_A \varphi(x, v^u(x))|Du + F| \quad \text{for each bounded Borel set } A, \tag{3.1}$$

with the vector-valued measure $Du + F$ having a corresponding total variation measure $|Du + F|$, and $v^u(x) = \frac{dDu+F}{d|Du+F|}$ is the Radon-Nikodym derivative. We use standard facts about functions of bounded variation as in [2], and follow the outline in [37]. For any open set U, we write

$$\int_U \varphi(x, Du + F) = \sup \left\{ \int_U (u\nabla \cdot Y - Y \cdot F)dx : Y \in C_c^\infty(U; \mathbb{R}^n), \sup \varphi^0(x, Y(x)) \leq 1 \right\}, \tag{3.2}$$

where $\varphi(x, \cdot)$ has a dual norm on $\mathbb{R}^n$, $\varphi^0(x, \cdot)$, defined by

$$\varphi^0(x, \xi) := \sup \{\xi \cdot p : \varphi(x, p) \leq 1\}.$$

As a consequence of condition (C_1), the dual norm $\varphi^0(x, \cdot)$ has the equivalent definition

$$\varphi^0(x, \xi) = \sup \left\{ \frac{\xi \cdot p}{\varphi(x, p)} : p \in \mathbb{R}^n \right\}. \tag{3.3}$$

Remark 25 *In* (3.2), *we define a notion of integration on* $\varphi(x, Du + F)$ *which applies to functions* u *of bounded variation, which may not even be continuous. Furthermore, this formula still works when the gradient of* u *has been shifted by a vector within convex function* φ, *by utilizing the norm* $\varphi^0(x, \cdot)$, *dual to* $\varphi(x, \cdot)$. *The following computations show* *the motivation for* (3.2)*. Since* $\varphi^0(x,Y) \leq 1$*, we take* $p = \frac{Du+F}{|Du+F|}$ *and* $\xi = -Y$ *in* (3.3)*, which yields*

$$-Y \cdot \frac{Du+F}{|Du+F|} \leq \varphi\left(x, \frac{Du+F}{|Du+F|}\right).$$

This implies

$$\begin{aligned}
\int_\Omega \varphi(x, Du+F) &= \int_\Omega \varphi\left(x, \frac{Du+F}{|Du+F|}\right)|Du+F| \\
&\geq \int_\Omega -Y \cdot \frac{Du+F}{|Du+F|}|Du+F| \\
&= \int_\Omega -Y \cdot Du - Y \cdot F \\
&= \int_\Omega (u\nabla \cdot Y - Y \cdot F).
\end{aligned}$$

3.1 Existence of minimizers with Neumann boundary condition

The purpose of this section is to solve the minimization problem as stated in (3.4), where the solution set is restricted to BV functions whose integral is zero. Let Ω, F, H, and φ be as previously defined, such that φ is not necessarily strictly convex, and consider

$$\inf_{u \in \overset{\circ}{BV}(\Omega)} I(u) := \int_\Omega \varphi(x, Du+F) + Hu, \tag{3.4}$$

where

$$\overset{\circ}{BV}(\Omega) = \left\{u \in BV(\Omega) : \int_\Omega u = 0\right\}.$$

3.1.1 The dual problem

We commence our study of minimizers of (3.4) by applying the Rockefeller-Fenchel duality to the problem. It would benefit the reader to view [43], where one can find calculations that have been omitted. Consider the functions $E : (L^2(\Omega))^n \to \mathbb{R}$ and $G : \mathring{H}^1(\Omega) \to \mathbb{R}$ defined as

$$E(b) = \int_\Omega \varphi\,(x, b + F) \qquad \text{and} \qquad G(u) = \int_\Omega Hu,$$

where $\mathring{H}(\Omega) = \{u \in H^1(\Omega) : \int_\Omega u = 0\}$. Then (3.4) can be equivalently written as

$$(P) \qquad \inf_{u \in \mathring{H}^1(\Omega)} \{E(\nabla u) + G(u)\}. \tag{3.5}$$

The dual problem corresponding to (3.5), as defined by Rockafellar-Fenchel duality [18], is

$$(D) \qquad \max_{b \in (L^2(\Omega))^n} \{-E^*(b) - G^*(-\nabla^* b)\}. \tag{3.6}$$

Note that convex functions E and G have convex conjugates E^* and G^*. Furthermore, gradient operator $\nabla : \mathring{H}^1(\Omega) \to L^2(\Omega)$ has a corresponding adjoint operator ∇^*. As in [43],

$$G^*(-\nabla^* b) = \sup_{u \in \mathring{H}^1(\Omega)} \left\{ -\int_\Omega \nabla u \cdot b - \int_\Omega Hu \right\}.$$

This can be more explicitly calculated by noting that for all real numbers c, $cu \in \mathring{H}^1(\Omega)$ whenever $u \in \mathring{H}^1(\Omega)$. Thus,

$$G^*(-\nabla^* b) = \begin{cases} 0 & \text{if } u \in \mathcal{D}_0, \\ \infty & \text{if } u \notin \mathcal{D}_0 \end{cases} \tag{3.7}$$

where

$$\mathcal{D}_0 := \left\{ b \in (L^2(\Omega))^n : \int_\Omega \nabla u \cdot b + Hu = 0, \quad \text{for all} \ \ u \in \mathring{H}^1(\Omega) \right\}. \tag{3.8}$$

The computations of $E^*(b)$ is shown in Lemma 2.1 of [37], which yields

$$E^*(b) = \begin{cases} -\langle F, b \rangle & \text{if } \varphi^0(x, b(x)) \le 1 \ \text{ in } \ \Omega \\ \infty & \text{otherwise }. \end{cases} \tag{3.9}$$

The dual problem can be rewritten as

$$(D) \qquad \sup\{\langle F, b\rangle : b \in \mathcal{D}_0 \ \text{ and } \ \varphi^0(x, b(x)) \le 1 \ \text{ in } \ \Omega\}. \tag{3.10}$$

The outer unit normal vector to $\partial\Omega$ is denoted by ν_Ω. There is a unique function $[b, \nu_\Omega] \in L^\infty_{\mathcal{H}^{n-1}}(\partial\Omega)$, whenever $\nabla \cdot b \in L^n(\Omega)$ for every $b \in (L^\infty(\Omega))^n$, such that

$$\int_{\partial\Omega} [b, \nu_\Omega] u \, d\mathcal{H}^{n-1} = \int_\Omega u \nabla \cdot b \, dx + \int_\Omega b \cdot Du \, dx, \quad u \in C^1(\bar{\Omega}). \tag{3.11}$$

In [1, 3] it was proved that equation (3.11) holds for every $u \in BV(\Omega)$, since $u \mapsto (b \cdot Du)$ gives rise to a Radon measure on Ω for $u \in BV(\Omega)$, $b \in (L^\infty(\Omega))^n$, and $\nabla \cdot b \in L^n(\Omega)$.

Lemma 26 *Let $b \in (L^\infty(\Omega))^n \cap \mathcal{D}_0$. Then*

$$\nabla \cdot b = H - \int_\Omega H dx \quad \textit{a.e. in} \ \ \Omega,$$

and

$$[b, \nu_\Omega] = 0 \ \ \mathcal{H}^{n-1} - \textit{a.e.} \quad \textit{on} \quad \partial\Omega.$$

The above lemma follows directly from equation (3.11) and the definition of D_0. It also provides the insight that every solution N to the dual problem (D) satisfies equation $\nabla \cdot N = H - \int_\Omega H dx$ a.e. in Ω. Moreover, at every point on $\partial\Omega$, the unit normal vector is orthogonal to N in a weak sense.

Theorem 27 *Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, $F \in (L^2(\Omega))^n$, $H \in L^2(\Omega)$, and $\varphi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a convex function satisfying (C_1) and (C_2). Then the duality gap is zero and the dual problem (D) has a solution, i.e. there exists a vector field $N \in \mathcal{D}_0$ with $\varphi^0(x, N) \leq 1$ such that*

$$\inf_{u \in \mathring{H}^1(\Omega)} \int_\Omega \left(\varphi\left(x, Du + F\right) + Hu\right) dx = \langle F, N \rangle. \tag{3.12}$$

Moreover

$$\varphi\left(x, \frac{Du + F}{|Du + F|}\right) = N \cdot \frac{Du + F}{|Du + F|}, \qquad |Du + F| - a.e. \ \ in \ \ \Omega, \tag{3.13}$$

for any minimizer u of (3.5).

Proof. It is trivial to show $I(v) = \int_\Omega (\varphi(x, Dv + F) + Hv)$ is convex, and $J : (L^2(\Omega))^n \to \mathbb{R}$ with $J(p) = \int_\Omega (\varphi(x, p + F) + Hu_0)dx$ is continuous at $p = 0$, for a fixed u_0, due to (C_2). Thus, the conditions of Theorem III.4.1 in [18] are satisfied. We infer that $(D) = (P)$ and the dual problem is assured a solution N such that (3.12) holds.

Now let $u \in \mathring{H}^1(\Omega)$ be a minimizer of (3.5). Then

$$\begin{aligned}
\langle N, F \rangle &= \int_\Omega \varphi(x, Du + F) + Hu \\
&= \int_\Omega \varphi\left(x, \frac{Du + F}{|Du + F|}\right) |Du + F| + \int_\Omega Hu \\
&\geq \int_\Omega N \cdot \frac{Du + F}{|Du + F|} |Du + F| + \int_\Omega Hu \\
&= \int_\Omega N \cdot (Du + F) + Hu \\
&= \int_\Omega N \cdot F + \int_\Omega N \cdot Du + Hu \\
&= \langle N, F \rangle
\end{aligned}$$

since $N \in \mathcal{D}_0$. Hence, the inequality above becomes an equality and (3.13) holds. $\square$

Remark 28 *The primal problem* (P) *may not have a minimizer in* $u \in \mathring{H}^1(\Omega)$*, but the dual problem* (D) *always has a solution* $N \in (L^2(\Omega))^n$*. Note also that the functional* $I(u)$ *is not strictly convex, and it may have multiple minimizers (see [26]). Furthermore, Theorem 27 asserts that* N *determines* $\frac{Du+F}{|Du+F|}$*,* $|Du+F|-$*a.e. in* Ω*, for all minimizers* u *of* (P)*. It does so since a.e. in* Ω*,*

$$\varphi^0(x,N) \leq 1 \implies \varphi(x,p) \geq N \cdot p$$

for every $p \in S^{n-1}$*. Therefore, the equality in* (3.13) *indicates that*

$$\frac{N \cdot p}{\varphi(x,p)}$$

is maximized by $p = \frac{Du+F}{|Du+F|}$*,* $|Du+F|-$*a.e. In the case that* $F \equiv 0$*,* N *determines the structure of the level sets of minimizers to* (P)*.*

We proceed to show that a solution to primal problem (P) exists in $BV(\Omega)$, provided that it is bounded below. The proof that follow depends on standard facts about BV functions.

Proposition 29 *Let* $\varphi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ *be a convex function satisfying* (C_1)*,* (C_2)*, and* (C_3)*. If there exists a constant* C*, depending on* Ω*, such that*

$$\max_{x \in \overline{\Omega}} |H(x)| < C, \tag{3.14}$$

then the primal problem (P) has a minimizer.

Proof. Consider the minimizing sequence u_n of functional $I(u)$. By condition (C_3) we have

$$\int_\Omega \beta|\nabla u_n + F| + Hu_n \leq \int_\Omega \varphi(x, \nabla u_n + F) + Hu_n < c,$$

for some constant c independent of n. Moreover, the triangle inequality implies

$$\int \beta|\nabla u_n| - \int \beta|F| - \int |H||u_n| \leq \int \beta|\nabla u_n| - \int \beta|F| + \int Hu_n \leq \int \beta|\nabla u_n + F| + Hu_n < c$$

and

$$\int \beta|\nabla u_n| \leq C + \int |H||u_n| + \int \beta|F|.$$

Applying Poincaré's inequality implies that there exists a constant C_Ω, independent of n, where

$$\int \beta|\nabla u_n| \leq C + ||H||_{L^\infty(\Omega)} C_\Omega \int |\nabla u_n| + \int \beta|F|$$

$$\Rightarrow \left(\beta - C_\Omega ||H||_{L^\infty(\Omega)}\right) \int |\nabla u_n| \leq C + \int \beta|F|.$$

Finally,

$$\int |\nabla u_n| \leq C' = \frac{C + \int \beta|F|}{\left(\beta - C_\Omega ||H||_{L^\infty(\Omega)}\right)}$$

provided that $\beta - C_\Omega ||H||_{L^\infty(\Omega)} > 0$ or equivalently

$$||H||_{L^\infty(\Omega)} \leq C := \frac{\beta}{C_\Omega}.$$

It follows from standard compactness results for BV functions that u_n has a subsequence, denoted by u_n again, such that u_n converges strongly in L^1 to a function $\hat{u} \in BV$, and Du_n converges to $D\hat{u}$ is the sense of measures. Since the functional $I(u)$ is lower semicontinuous, $\hat{u}$ is a solution of the primal problem (3.4). □

3.2 Existence of minimizers with Dirichlet boundary condition

Now, let us consider minimizers of the main functional with a given Dirichlet boundary condition on $\partial\Omega$. Let Ω be a bounded region in $\mathbb{R}^n$ with Lipschitz boundary, $F \in (L^2(\Omega))^n$, $H \in L^2(\Omega)$, $f \in L^1(\partial\Omega)$, $\varphi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ a convex function satisfying (C_1) and (C_2), and the minimization problem becomes

$$\inf_{u \in BV_f(\Omega)} I(u) := \int_\Omega \varphi\left(x, Du + F\right) + Hu, \tag{3.15}$$

where

$$BV_f(\Omega) = \{u \in BV(\Omega) : u|_{\partial\Omega} = f\}.$$

We perform the substitution $\tilde{F} = F + \nabla f$ to rewrite (3.15) in terms of BV functions that are zero on $\partial\Omega$. Since there always exists a function $f \in W^{1,1}(\Omega)$ that is an extension of any function in $L^1(\Omega)$, we have.

$$\inf_{u \in BV_0(\Omega)} I(u) := \int_\Omega \varphi\left(x, Du + \tilde{F}\right) + Hu + \int_\Omega Hf dx.$$

Note that $\int_\Omega Hf dx$ is a constant, which implies (3.15) can be represented by the minimization problem

$$\inf_{u \in BV_0(\Omega)} I(u) := \int_\Omega \varphi\left(x, Du + F\right) + Hu. \tag{3.16}$$

In section 3.1 boundedness from below of functional $I(u)$ was sufficient to provide existence of minimizers in $\overset{\circ}{BV}(\Omega)$. This is not the case for (3.15), nor (3.16). The main reason for nonexistence of minimizers is that for a given minimizing sequence such that $u_n \to \hat{u}$ in $L^1(\Omega)$ and $\hat{u} \in BV(\Omega)$, we have

$$I(\hat{u}) \leq \inf_{u \in BV_0(\Omega)} I(u),$$

by the lower semicontinuity of $I(u)$. However, since $\partial\Omega$ is a set of measure zero, the trace of $\hat{u}$ is not guaranteed to be zero. Our aim in this section is to find existence of minimizers for the highly nontrivial problem (3.16), and in turn (3.15).

3.2.1 The dual problem

The setup of the dual problem here is identical to that of Section 2, with the exception of the function space of potential solutions. We plan to analyze solution to (3.16) by first undertaking the relaxed problem (3.25) from Section 3.2.2. With this in mind, let $E : (L^2(\Omega))^n \to \mathbb{R}$ and $G : H_0^1(\Omega) \to \mathbb{R}$ be defined as

$$E(b) = \int_\Omega \varphi\,(x, b + F) \quad \text{and} \quad G(u) = \int_\Omega Hu.$$

Then (3.25) can be equivalently written as

$$(P') \qquad \inf_{u \in H_0^1(\Omega)} \{E(\nabla u) + G(u)\}. \tag{3.17}$$

The dual problem corresponding to (3.17), as defined by Rockafellar-Fenchel duality [18], is

$$(D') \qquad \sup_{b \in (L^2(\Omega))^n} \{-E^*(b) - G^*(-\nabla^* b)\}. \tag{3.18}$$

The updated the computation of $G^*(-\nabla^* b)$ is similarly

$$G^*(-\nabla^* b) = \sup_{u\in H_0^1(\Omega)} \left\{ -\int_\Omega \nabla u \cdot b - \int_\Omega Hu \right\},$$

and more explicitly

$$G^*(-\nabla^* b) = \begin{cases} 0 & \text{if } u \in \widetilde{\mathcal{D}}_0 \\ \\ \infty & \text{if } u \notin \widetilde{\mathcal{D}}_0(\Omega), \end{cases} \tag{3.19}$$

where

$$\widetilde{\mathcal{D}}_0 := \left\{ b \in (L^2(\Omega))^n : \int_\Omega \nabla u \cdot b + Hu = 0, \quad \text{for all} \quad u \in H_0^1(\Omega) \right\} \subseteq \mathcal{D}_0. \tag{3.20}$$

Finally, we use Lemma 2.1 in [37] to get

$$E^*(b) = \begin{cases} -\langle F, b \rangle & \text{if } \varphi^0(x, b(x)) \leq 1 \quad \text{in} \quad \Omega \\ \\ \infty & \text{otherwise .} \end{cases} \tag{3.21}$$

We can therefore rewrite the dual problem as

$$(D') \qquad \sup\{\langle F, b\rangle : b \in \widetilde{\mathcal{D}}_0 \;\; \text{and} \;\; \varphi^0(x, b(x)) \leq 1 \;\; \text{in} \;\; \Omega\}. \tag{3.22}$$

A direct application of the integration by parts formula (3.11) implies that $b \in (L^\infty(\Omega))^n \cap \widetilde{\mathcal{D}}_0$ if and only if

$$\nabla \cdot b = H \quad \text{a.e. in} \quad \Omega.$$

We proceed to prove the analog to Theorem 27.

Theorem 30 *Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, $F \in (L^2(\Omega))^n$, $H \in L^2(\Omega)$, $\varphi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ a convex function satisfying (C_1), (C_2), and assume (P') is bounded below. Then the duality gap is zero and the dual problem (D') has a solution, i.e. there exists a vector field $N \in \widetilde{\mathcal{D}}_0$ with $\varphi^0(x, N) \le 1$ such that*

$$\inf_{u \in H_0^1(\Omega)} \int_\Omega (\varphi(x, Du + F) + Hu)\, dx = \langle F, N \rangle. \tag{3.23}$$

Moreover

$$\varphi\left(x, \frac{Du + F}{|Du + F|}\right) = N \cdot \frac{Du + F}{|Du + F|}, \qquad |Du + F| - a.e. \text{ in } \Omega, \tag{3.24}$$

for any minimizer u of (3.17).

Proof. It is trivial to show $I(v) = \int_\Omega (\varphi(x, Dv + F) + Hv)$ is convex, and $J : (L^2(\Omega))^n \to \mathbb{R}$ with $J(p) = \int_\Omega (\varphi(x, p + F) + Hu_0) dx$ is continuous at $p = 0$, for a fixed u_0, due to (C_2). Thus, the conditions of Theorem III.4.1 in [18] are satisfied. We infer that $(D) = (P)$ and the dual problem is assured a solution N such that (3.23) holds.

Now let $u \in A_0$ be a minimizer of (3.17). Then

$$\begin{aligned}
\langle N, F \rangle &= \int_\Omega \varphi(x, Du + F) + Hu \\
&= \int_\Omega \varphi\left(x, \frac{Du + F}{|Du + F|}\right) |Du + F| + \int_\Omega Hu \\
&\ge \int_\Omega N \cdot \frac{Du + F}{|Du + F|} |Du + F| + \int_\Omega Hu \\
&= \int_\Omega N \cdot (Du + F) + Hu \\
&= \int_\Omega N \cdot F + \int_\Omega N \cdot Du + Hu \\
&= \langle N, F \rangle
\end{aligned}$$

since $N \in \widetilde{\mathcal{D}}_0$. Hence, the inequality becomes an equality and (3.24) holds. □

Remark 31 *The primal problem* (P') *may not have a minimizer in* $H_0^1(\Omega)$*, but the dual problem* (D') *always has a solution* $N \in (L^2(\Omega))^n$*. Note also that the functional* $I(u)$ *is not strictly convex, and it may have multiple minimizers (see [26]). Furthermore, Theorem 30 asserts that* N *determines* $\frac{Du+F}{|Du+F|}$*,* $|Du + F|-$*a.e. in* Ω*, for all minimizers* u *of* (P')*. It does so since a.e. in* Ω*,*

$$\varphi^0(x, N) \leq 1 \implies \varphi(x, p) \geq N \cdot p$$

for every $p \in S^{n-1}$*. Therefore, the equality in* (3.24) *indicates that*

$$\frac{N \cdot p}{\varphi(x, p)}$$

is maximized by $p = \frac{Du+F}{|Du+F|}$*,* $|Du + F|-$*a.e. In the case that* $F \equiv 0$*,* N *determines the structure of the level sets of minimizers to* (P')*.*

3.2.2 The relaxed problem

Now we investigate the existence of minimizer for the relaxed problem associated to (3.16), namely

$$\inf_{u \in A_0} I(u) = \inf_{u \in A_0} \int_\Omega (\varphi(x, Du + F) + Hu)dx + \int_{\partial\Omega} \varphi(x, \nu_\Omega)|u|ds, \tag{3.25}$$

where

$$A_0 := \left\{u \in H^1(\mathbb{R}^n) : u = 0 \text{ in } \Omega^c\right\}.$$

The benefit of the relaxed problem is that any minimizing sequence of (3.25) converges to a minimizer in A_0. This convergence result is not guaranteed for (3.16). It is easily verified that Proposition 29 can be adapted to the relaxed problem, and (3.16) has a solution in A_0 when bounded below.

Proposition 32 *Let $\varphi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a convex function satisfying (C_1), (C_2), and (C_3). If there exists a constant C, depending on Ω, such that*

$$\max_{x \in \overline{\Omega}} |H(x)| < C, \tag{3.26}$$

then the primal problem (3.16) *has a minimizer in A_0.*

Proof. Note that $\hat{u} \in A_0$ whenever $u_n \in A_0$ converges to $\hat{u}$ in $L^1(\Omega)$. Then the proof follows as outlined in Proposition 29. □

The stage is now set for the major result of this section. We are able to show the difficulty of proving existence of minimizers to (3.16), while demonstrating how problems (3.16) and (3.25) are related.

Theorem 33 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, $F \in (L^2(\Omega))^n$, $H \in L^2(\Omega)$, and $\varphi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ a convex function satisfying (C_1), (C_2), and (C_3). If the minimization problem* (3.16) *is bounded below, then*

$$\min_{u \in A_0} \left(\int_\Omega (\varphi(x, Du + F) + Hu)dx + \int_{\partial\Omega} \varphi(x, \nu_\Omega)|u|ds \right) = \inf_{u \in BV_0(\Omega)} \int_\Omega \varphi(x, Du + F) + Hu \tag{3.27}$$

Moreover, if u is a minimizer of (3.25), *then*

$$\varphi(x, \nu_\Omega) = [N, sign(-u)\nu_\Omega] \quad \mathcal{H}^{n-1} - a.e. \quad on \quad \partial\Omega. \tag{3.28}$$

Proof. It can be easily shown that $BV_0(\Omega)$ has a continuous embedding into A_0, which implies

$$\min_{u\in A_0}\left(\int_\Omega (\varphi(x,Du+F)+Hu)dx+\int_{\partial\Omega}\varphi(x,\nu_\Omega)|u|ds\right)\leq \inf_{u\in BV_0(\Omega)}\int_\Omega \varphi(x,Du+F)+Hu.$$

It follows from Theorem 30 that there exists a vector field $N\in\widetilde{\mathcal{D}}_0$ with

$$\varphi\left(x,\frac{Du+F}{|Du+F|}\right)=N\cdot\frac{Du+F}{|Du+F|},\qquad |Du+F|-a.e.\ \text{ in }\ \Omega.$$

Consider minimizer u of the relaxed problem with $u|_{\partial\Omega}=h|_{\partial\Omega}$, where $h\in W^{1,1}(\Omega)$. Since $u-h\in\widetilde{\mathcal{D}}_0$, we have

$$\begin{aligned}
\min_{u\in A_0}(\int_\Omega (\varphi(x,Du+F)+Hu)dx+\int_{\partial\Omega}\varphi(x,\nu_\Omega)|u|ds) &= \int_\Omega \varphi(x,Du+F)+Hu+\int_{\partial\Omega}\varphi(x,\nu_\Omega)|u| \\
&= \int_\Omega \varphi\left(x,\frac{Du+F}{|Du+F|}\right)|Du+F|+\int_\Omega Hu+\int_{\partial\Omega}\varphi(x,\nu_\Omega)|u| \\
&\geq \int_\Omega N\cdot\frac{Du+F}{|Du+F|}|Du+F|+\int_\Omega Hu+\int_{\partial\Omega}\varphi(x,\nu_\Omega)|u| \\
&= \int_\Omega N\cdot(Du+F)+Hu+\int_{\partial\Omega}\varphi(x,\nu_\Omega)|u| \\
&= \int_\Omega N\cdot F+\int_\Omega N\cdot Du+Hu+\int_{\partial\Omega}\varphi(x,\nu_\Omega)|u| \\
&= \langle N,F\rangle+\int_\Omega N\cdot D(u-h)+H(u-h) \\
&+\int_\Omega N\cdot Dh+Hh+\int_{\partial\Omega}\varphi(x,\nu_\Omega)|h| \\
&= \langle N,F\rangle+\int_\Omega N\cdot Dh+Hh+\int_{\partial\Omega}\varphi(x,\nu_\Omega)|h| \\
&= \langle N,F\rangle+\int_{\partial\Omega}[N,\nu_\Omega]h+\int_{\partial\Omega}\varphi(x,\nu_\Omega)|h| \\
&\geq \langle N,F\rangle \\
&= \inf_{u\in BV_0(\Omega)}\int_\Omega\varphi(x,Du+F)+Hu.
\end{aligned}$$

The last inequality was achieved using integration by parts and the fact that $\varphi^0(x,N) \leq 1 \implies [N,\nu_\Omega] \leq \varphi(x,\nu_\Omega)$. Therefore, (3.27) holds and all the inequalities in the above computation are equalities. This provides the relationship $\int_{\partial\Omega}[N,\nu_\Omega]h + \int_{\partial\Omega}\varphi(x,\nu_\Omega)|h| = 0$, which implies that (3.28) holds. □

The last theorem follows directly from Theorem 30 and Theorem 33.

Theorem 34 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, $F \in (L^2(\Omega))^n$, $H \in L^2(\Omega)$, $\varphi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ a convex function satisfying (C_1), (C_2), (C_3), and assume (P') is bounded below. Then there exists a vector field $N \in \widetilde{\mathcal{D}}_0$ with $\varphi^0(x,N) \leq 1$ such that*

$$\varphi\left(x, \frac{Du+F}{|Du+F|}\right) = N \cdot \frac{Du+F}{|Du+F|}, \quad |Du+F| - a.e. \ \ in \ \ \Omega, \tag{3.29}$$

for any minimizer u of (3.16). *Moreover, every minimizer of* (3.16) *is a minimizer of* (3.25)*, and if u is a minimizer of* (3.25)*, then*

$$\varphi(x,\nu_\Omega) = [N, sign(-u)\nu_\Omega] \quad \mathcal{H}^{n-1} - a.e. \ \ on \ \ \partial\Omega. \tag{3.30}$$

Remark 35 *Equation* (3.30) *asserts where a minimizer u along the $\partial\Omega$ will have all its jumps $\mathcal{H}^{n-1} - a.e.$ If $N_{tr} \in (L^\infty(\partial\Omega))^n$ denotes the trace of N, then*

$$\{x \in \partial\Omega : u|_{\partial\Omega} < 0\} \subseteq \{x \in \partial\Omega : \varphi(x,\nu_\Omega(x)) = N_{tr}\cdot\nu_\Omega\} \quad and$$

$$\{x \in \partial\Omega : u|_{\partial\Omega} > 0\} \subseteq \{x \in \partial\Omega : \varphi(x,\nu_\Omega(x)) = -N_{tr}\cdot\nu_\Omega\},$$

for all minimizers of (3.25)*, up to a set of $\mathcal{H}^{n-1}$-measure zero.*

3.3 Existence of minimizers under the Barrier condition

Under the adapted assumptions and definitions that follow, many of the proofs are similar to those in [43], and are included to aide in understanding. Consider $F \in (L^1(\Omega)^n)$, $H \in L^\infty(\Omega)$, and $\psi : \mathbb{R}^n \times BV_0(\Omega)$ given to be

$$\psi(x, u) := \varphi(x, Du + F\chi_{E_u}) + Hu, \tag{3.31}$$

with E_u representing the closure of the support of u in Ω.

The ψ-perimeter of E in A is denoted by

$$P_\psi(E; A) := \int_A \varphi\left(x, D\chi_E + F\chi_E\right) + H\chi_E.$$

Definition 36 *1. A function $u \in BV(\mathbb{R}^n)$ is ψ-total variation minimizing in $\Omega \subset \mathbb{R}^n$ if*

$$\int_\Omega \psi(x, u) \leq \int_\Omega \psi(x, v) \text{ for all } v \in BV(\mathbb{R}^n) \text{ such that } u = v \text{ a.e. in } \Omega^c.$$

2. A set $E \subset \mathbb{R}^n$ of finite perimeter is ψ-area minimizing in Ω if

$$P_\psi(E; \Omega) \leq P_\psi(\tilde{E})$$

for all $\tilde{E} \subset \mathbb{R}^n$ such that $\tilde{E} \cap \Omega^c = E \cap \Omega^c$ a.e..

We set up for the two major results of Section 4, Theorems 39 and 42, by undertaking preliminary lemmas. For a given function $u \in BV(\Omega)$, it is useful to define functions

$$u_1 = \max(u - \lambda, 0) \text{ and } u_2 = u - u_1, \tag{3.32}$$

for an arbitrary $\lambda \in \mathbb{R}$. Moving forward we use the function

$$\chi_{\epsilon,\lambda} := \min\left(1, \frac{1}{\epsilon}u_1\right) = \begin{cases} 0 & \text{if } u \leq \lambda, \\ \frac{1}{\epsilon}(u-\lambda) & \text{if } \lambda < u \leq \lambda+\epsilon, \\ 1 & \text{if } u > \lambda+\epsilon. \end{cases} \tag{3.33}$$

which is shown to be ψ-total variation minimizing in Theorem 39.

Lemma 37 *For $\chi_{\epsilon,\lambda}$ as defined in* (3.33),

$$P_\psi(E,\Omega) \leq \liminf_{\epsilon\to 0} \int_\Omega \varphi(x, D\chi_{\epsilon,\lambda} + F\chi_{\epsilon,\lambda}) + H\chi_{\epsilon,\lambda}.$$

Proof. Due to condition (C_2) we have

$$\begin{aligned} & \int_\Omega \varphi(x, D\chi_{\epsilon,\lambda} + F\chi_{\epsilon,\lambda}) + H\chi_{\epsilon,\lambda} - \int_\Omega \varphi(x, D\chi_E + F\chi_E) + H\chi_E \\ = & \int_{\Omega\cap\{\lambda-\epsilon<u<\lambda+\epsilon\}} \varphi(x, D\chi_{\epsilon,\lambda} + F\chi_{\epsilon,\lambda}) + H\chi_{\epsilon,\lambda} - \varphi(x, D\chi_E + F\chi_E) - H\chi_E \\ \geq & \int_{\Omega\cap\{\lambda-\epsilon<u<\lambda+\epsilon\}} \varphi(x, D\chi_{\epsilon,\lambda}) - \varphi(x, F\chi_{\epsilon,\lambda}) + H\chi_{\epsilon,\lambda} - \varphi(x, D\chi_E) - \varphi(x, F\chi_E) - H\chi_E \\ = & \int_{\Omega\cap\{\lambda-\epsilon<u<\lambda+\epsilon\}} \varphi(x, D\chi_{\epsilon,\lambda}) - \varphi(x, D\chi_E) + H\chi_{\epsilon,\lambda} - H\chi_E - \varphi(x, F\chi_{\epsilon,\lambda}) - \varphi(x, F\chi_E) \\ = & \int_\Omega \varphi(x, D\chi_{\epsilon,\lambda}) - \int_\Omega \varphi(x, D\chi_E) + \int_\Omega (H\chi_{\epsilon,\lambda} - H\chi_E) \\ & - \int_{\Omega\cap\{\lambda-\epsilon<u<\lambda+\epsilon\}} \varphi(x, F\chi_{\epsilon,\lambda}) + \varphi(x, F\chi_E). \end{aligned}$$

Since the last two integrals converge to zero as $\epsilon \to 0$,

$$\begin{aligned} & \liminf_{\epsilon\to 0} \int_\Omega \varphi(x, D\chi_{\epsilon,\lambda} + F\chi_{\epsilon,\lambda}) + H\chi_{\epsilon,\lambda} - P_\psi(E,\Omega) \\ = & \liminf_{\epsilon\to 0} \int_\Omega \varphi(x, D\chi_{\epsilon,\lambda} + F\chi_{\epsilon,\lambda}) + H\chi_{\epsilon,\lambda} - \int_\Omega \varphi(x, D\chi_E + F\chi_E) + H\chi_E \\ \geq & \liminf_{\epsilon\to 0} \int_\Omega \varphi(x, D\chi_{\epsilon,\lambda}) - \int_\Omega \varphi(x, D\chi_E) \geq 0, \end{aligned}$$

where the lower semi-continuity of $\int_\Omega \varphi(x, Dv)$ justifies the last inequality (see [26]). □

The outer and inner trace of w on $\partial\Omega$ are denoted by w^+ and w^- respectively, under the assumptions that Ω is an open set with Lipschitz boundary and $w \in BV(\mathbb{R}^n)$.

Lemma 38 *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded open region with Lipschitz boundary, $g \in L^1(\partial\Omega; \mathcal{H}^{n-1})$, and define*

$$I_\psi(v; \Omega, g) := \int_{\partial\Omega} \varphi(x, g - v^- + F_{\chi_v}) d\mathcal{H}^{n-1} + \int_\Omega \psi(x, Dv).$$

Then $u \in BV(\mathbb{R}^n)$ is ψ-total variation minimizing in Ω if and only if $u|_\Omega$ minimizes $I_\psi(\,\cdot\,; \Omega, g)$ for some g, and moreover $g = u^+$.

Proof: Note that $v^+, v^- \in L^1(\partial\Omega; \mathcal{H}^{n-1})$ whenever $v \in BV(\mathbb{R}^n)$. Conversely, there is a $v \in BV(\mathbb{R}^n)$ with $g = v^+$ for each $g \in L^1(\partial\Omega; \mathcal{H}^{n-1})$. Additionally

$$\int_{\partial\Omega} \psi(x, Dv) = \int_{\partial\Omega} \varphi(x, Dv + F_{\chi_v}) d\mathcal{H}^{n-1} = \int_{\partial\Omega} \varphi(x, v^+ - v^- + F_{\chi_v}) d\mathcal{H}^{n-1}. \qquad (3.34)$$

To see this, note that $|Dv|$ can only concentrate on a set of dimension $n-1$ if that set is a subset of the jump set of v, so (3.34) follows from standard descriptions of the jump part of Dv.

Now if $u, v \in BV(\mathbb{R}^n)$ satisfy $u = v$ a.e. in Ω^c, then $\int_{\bar{\Omega}^c} \varphi(x, Du) = \int_{\bar{\Omega}^c} \varphi(x, Dv)$. In addition, $u^+ = v^+$, so using (3.34) we deduce that

$$\int_{\mathbb{R}^n} \psi(x, Du) - \int_{\mathbb{R}^n} \psi(x, Dv) \;=\; I_\varphi(u; \Omega, u^+) - I_\varphi(v; \Omega, u^+).$$

The lemma easily follows from the above equality. □

The theorem that follows shows super level sets of ψ-total variation minimizing functions in Ω are ψ-area minimizing in Ω.

Theorem 39 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $u \in BV(\mathbb{R}^n)$ a ψ-total variation minimizing function in Ω. The super level sets of u are written as*

$$E_\lambda := \{x \in \mathbb{R}^n : u(x) \geq \lambda\}. \tag{3.35}$$

Then E_λ is ψ-area minimizing in Ω.

Proof. For a fixed $\lambda \in \mathbb{R}$, let u_1 and u_2 be as defined in (3.32). Consider $g \in BV(\mathbb{R}^n)$ with $\mathrm{supp}(g) \subset \overline{\Omega}$. Then

$$\begin{aligned}
\int_\Omega \varphi\left(x, Du_1 + F\chi_{\{u\geq\lambda\}}\right) + Hu_1 + \int_\Omega \varphi\left(x, Du_2 + F\chi_{\{u<\lambda\}}\right) + Hu_2 &= \int_\Omega \varphi\left(x, Du + F\right) + Hu \\
&\leq \int_\Omega \varphi\left(x, D(u+g) + F\right) + H(u+g) \\
&= \int_\Omega \varphi\left(x, Du_1 + D(g\chi_{\{u\geq\lambda\}}) + F\chi_{\{u\geq\lambda\}}\right) + H(u_1+g) \\
&+ \int_\Omega \varphi\left(x, Du_2 + D(g\chi_{\{u<\lambda\}}) + F\chi_{\{u<\lambda\}}\right) + Hu_2 \\
&\leq \int_\Omega \varphi\left(x, Du_1 + D(g\chi_{\{u\geq\lambda\}}) + F\chi_{\{u\geq\lambda\}}\right) + H(u_1+g) \\
&+ \int_\Omega \varphi\left(x, D(g\chi_{\{u<\lambda\}})\right) + \int_\Omega \varphi\left(x, Du_2 + F\chi_{\{u<\lambda\}}\right) + Hu_2 \\
&= \int_\Omega \varphi\left(x, D(u_1+g) + F\chi_{\{u\geq\lambda\}}\right) + H(u_1+g) \\
&+ \int_\Omega \varphi\left(x, Du_2 + F\chi_{\{u<\lambda\}}\right) + Hu_2.
\end{aligned}$$

This implies

$$\int_\Omega \varphi\left(x, Du_1 + F\chi_{u_1}\right) + Hu_1 \leq \int_\Omega \varphi\left(x, D(u_1+g) + F\chi_{u_1}\right) + H(u_1+g),$$

for any $g \in BV(\mathbb{R}^n)$ such that $\mathrm{supp}(g) \subset \overline{\Omega}$. By definition, u_1 is ψ-total variation minimizing. Using the argument outlined above $\chi_{\epsilon,\lambda}$, as defined in (3.33), is also ψ-total variation minimizing.

The boundary of E_λ has measure zero for a.e. $\lambda \in \mathbb{R}$, which is represented by

$$\mathcal{L}^n\left(\{x \in \Omega : u(x) = \lambda\}\right) = \mathcal{H}^{n-1}\left(\{x \in \partial\Omega : u^{\pm}(x) = \lambda\}\right) = 0. \tag{3.36}$$

Thus

$$\chi_{\epsilon,\lambda} \to \chi_\lambda := \chi_{E_\lambda} \text{ in } L^1_{\text{loc}}(\mathbb{R}^n), \quad \chi^{\pm}_{\epsilon,\lambda} \to \chi^{\pm}_\lambda \text{ in } L^1(\partial\Omega; \mathcal{H}^{n-1}),$$

as $\epsilon \to 0$.

We apply Lemma 37 to get

$$P_\psi(\chi_\lambda, \Omega) \leq \liminf_{\epsilon \to 0} P_\psi(\chi_{\epsilon,\lambda}, \Omega). \tag{3.37}$$

It follows from the L^1 convergence of the traces that

$$I_\varphi(\chi_\lambda; \Omega, \chi^+_\lambda) \leq \liminf_{k \to \infty} I_\varphi(\chi_{\epsilon,\lambda}; \Omega, \chi^+_{\lambda,\epsilon}). \tag{3.38}$$

For an arbitrary $F \subset \mathbb{R}^n$ with $\chi_\lambda = \chi_F$ a.e. in Ω^c,

$$\begin{aligned}
I_\varphi(\chi_{\epsilon,\lambda}; \Omega, \chi^+_{\epsilon,\lambda}) &\leq I_\varphi(\chi_F; \Omega, \chi^+_{\epsilon,\lambda}) \\
&\leq I_\varphi(\chi_F; \Omega, \chi^+_\lambda) + \int_{\partial\Omega} \varphi(x, \chi^+_\lambda - \chi^+_{\epsilon,\lambda}) \, d\mathcal{H}^{n-1} \\
&\leq I_\varphi(\chi_F; \Omega, \chi^+_\lambda) + \int_{\partial\Omega} \alpha|\chi^+_\lambda - \chi^+_{\epsilon,\lambda}| \, d\mathcal{H}^{n-1} \\
&\leq I_\varphi(\chi_F; \Omega, {\chi_\lambda}^+) + C \int_{\partial\Omega} |\chi^+_\lambda - \chi^+_{\epsilon,\lambda}| \, d\mathcal{H}^{n-1}.
\end{aligned}$$

The inequality that follows is justified by the above, (3.38), and $\chi^+_{\epsilon,\lambda} \to \chi^+_\lambda$ in $L^1(\partial\Omega; \mathcal{H}^{n-1})$,

$$I_\varphi(\chi_\lambda; \Omega, \chi^+_\lambda) \leq I_\varphi(\chi_F; \Omega, \chi^+_\lambda).$$

This establishes that E_λ is φ-area minimizing in Ω.

If λ does not satisfy (3.36), then there exists an increasing sequence λ_k that converges to λ and satisfies (3.36) for each k. In which case,

$$\chi_{\lambda_k} \to \chi_\lambda \text{ in } L^1_{\text{loc}}(\mathbb{R}^n), \qquad \chi^{\pm}_{\lambda_k} \to \chi^{\pm}_\lambda \text{ in } L^1(\partial\Omega; \mathcal{H}^{n-1}).$$

Thus, by Lemma 38, E_λ is ψ-area minimizing in Ω. □

It remains to lay out a few more definitions, which are key conditions for the last Lemma and Theorem. Let

$$BV_f(\Omega) := \left\{ u \in BV(\Omega) : \lim_{r\to 0} \operatorname*{ess\,sup}_{y\in\Omega, |x-y|<r} |u(y) - f(y)| = 0 \text{ for } x \in \partial\Omega \right\}.$$

For any measurable set E, consider

$$E^{(1)} := \left\{ x \in \mathbb{R}^n : \lim_{r\to 0} \frac{\mathcal{H}^n(B(r,x)\cap E)}{\mathcal{H}^n(B(r))} = 1 \right\}.$$

Definition 40 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. The barrier condition is satisfied for Ω if for every $x_0 \in \partial\Omega$ and $\epsilon > 0$ sufficiently small, V minimizes $P_\psi(\cdot;\mathbb{R}^n)$ in*

$$\{W \subset \Omega : W \setminus B(\epsilon, x_0) = \Omega \setminus B(\epsilon, x_0)\}, \tag{3.39}$$

implies

$$\partial V^{(1)} \cap \partial\Omega \cap B(\epsilon, x_0) = \emptyset.$$

Lemma 41 *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain satisfying the barrier condition, and $E \subset \mathbb{R}^n$ minimizes $P_\psi(\cdot;\Omega)$. Then*

$$\left\{ x \in \partial\Omega \cap \partial E^{(1)} : B(\epsilon, x) \cap \partial E^{(1)} \subset \overline{\Omega} \textit{ for some } \epsilon > 0 \right\} = \emptyset.$$

Proof. We proceed by contradiction. Suppose there exists $x_0 \in \partial\Omega \cap \partial E^{(1)}$ such that $B(\epsilon, x_0) \cap \partial E^{(1)} \subset \bar{\Omega}$ for some $\epsilon > 0$. Then $\tilde{V} = E \cap \Omega$ is a minimizer of $P_\psi(\,\cdot\,; \mathbb{R}^n)$ in (3.39), and

$$x_0 \in \partial \tilde{V}^{(1)} \cap \partial\Omega \cap B(\epsilon, x_0) \neq \emptyset.$$

This is inconsistent with the conclusion of the barrier condition from Definition 40. □

Finally, we are ready to prove the main existence results of the current section.

Theorem 42 *Consider* $\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *as defined in* (3.31) *and bounded Lipschitz domain* $\Omega \subset \mathbb{R}^n$*. Let* $||H||_{L^\infty(\overline{\Omega})}$ *be small enough that Proposition 32 holds. If* Ω *satisfies the barrier condition with respect to* ψ*, as given in Definition 40, then for every* $f \in C(\partial\Omega)$ *the minimization problem* (3.15) *has a minimizer in* $BV_f(\Omega)$*.*

Proof. For a given $f \in C(\partial\Omega)$, it can be extended to $f \in C(\Omega^c)$. Furthermore, we can assume $f \in BV(\mathbb{R}^n)$ since every $\mathcal{H}^{n-1}$ integrable function on Ω is the trace of some (continuous) function in $BV(\Omega^c)$. We then consider a set of such functions

$$\mathcal{A}_f := \{v \in BV(\mathbb{R}^n) : \quad v = f \quad \text{on} \quad \Omega^c\},$$

where any element v of $BV_f(\Omega)$ is the restriction to Ω of a unique element of $\mathcal{A}_f$. Then $\int_{\mathbb{R}^n} \psi(x, v)$ has as a minimizer $u \in \mathcal{A}_f$ due to the assumed condition of Proposition 32 being satisfied. Only slight adjustments are needed to adapt Proposition 32 for $u \in \mathcal{A}_f$.

The argument to follow proves that $u \in BV_f(\Omega)$ with the use of the Barrier condition. Assuming the opposite implies there is an $x \in \partial\Omega$ and $\delta > 0$ such that

$$\operatorname*{ess\,sup}_{y \in \Omega, |x-y|<r} \big(f(x) - u(y)\big) \geq \delta \qquad \text{or} \qquad \operatorname*{ess\,sup}_{y \in \Omega, |x-y|<r} \big(u(y) - f(x)\big) \geq \delta \tag{3.40}$$

for every $r > 0$. First, suppose that the latter condition holds. For $E := E_{f(x)+\delta/2}$ we have that $x \in \partial E^{(1)}$, justified by the second alternative of (3.40) and the continuity of f. Note that Theorem 39 implies E is ψ-area minimizing in Ω. For some small ϵ, $u < f(x) + \delta/2$ in $B(\varepsilon, x) \setminus \Omega$, since $u \in \mathcal{A}_f$ and f is continuous in Ω^c. However, Lemma 41 shows this is impossible. In the case of the first alternative of (3.40), a similar contradiction arises when $E := \{y \in \mathbb{R}^n : u(y) \leq f(x) - \delta/2\}$. We conclude that $u \in BV_f(\Omega)$. Moreover, u is ψ-total variation minimizing in $BV_f(\Omega)$ by Theorem 33. $\square$

Chapter 4

Conclusion

This section will serve as a concluding summary of Chapters 2 and 3, followed by possible future research, which may expand upon the results presented in this Thesis.

4.1 Summary of P-area minimizing surfaces

The problem of finding the existence of P-area minimizing surfaces in the Heisenberg Group is of interest for two reasons. The first, is as a generalization of the well studied least gradient problem, which has many applications in conductivity imaging. The second, is to gain understanding of P-area minimizing surfaces and to apply a method not previously used in the literature. The approach of Rockafeller-Fenchel duality yields insight about the structure of solutions not gathered when approaching the problem from the point of view of differential geometry.

We began the study of P-area minimizing surfaces under the Neumann boundary condition. The goal was to find minimizers of $I(u) = \int_\Omega a|Du+F|+Hu$ in the set $\mathring{BV}(\Omega) =$

$\{u \in BV(\Omega) : \int_\Omega u = 0\}$. In applying the Rockafeller-Fenchel duality to the primal problem, the dual problem was found to always have solution N. However, the primal problem needed to be bounded below to guarantee a solution exists in $\mathring{H}^1(\Omega)$, in which case the dual and primal problems were equal. Moreover, when a solution exists, $N = a\frac{Du+F}{|Du+F|}$ determines the structure of solutions. This is most evident in the case where $F \equiv 0$ and N determines the shape of level sets, as $N = \frac{Du}{|Du|}$ is a unit vector orthogonal to a corresponding level set.

Later, we followed the same outline detailed in the preceding paragraph to find existence of minimizers to $I'(u) = \int_\Omega (a|Du + F| + Hu) + \int_{\partial\Omega} a|u|$ in the set $A_0 = \{u \in H^1(\mathbb{R}^n) : u = 0 \text{ in } \Omega^c\}$. All the same results were achieved, including the guarantee of solutions to the primal problem in A_0 when it is bounded below. On the other hand, no such minimizer could be assured for $I(u)$ in the function space $BV_0(\Omega) = \{u \in BV(\Omega) : u|_{\partial\Omega} = 0\}$, even if bounded below. Indeed, a minimizing sequence of $I(u)$ could converge to a function of bounded variation, whose trace on $\partial\Omega$ is not the zero function. A useful theorem was proved showing the relation of these two problems,

$$\min_{u \in A_0} I'(u) = \inf_{u \in BV_0\Omega} I(u).$$

Furthermore, any minimizer of the relaxed problem (on the left) satisfies condition $u[N, \nu_\Omega] = |u|$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$. This implies that any minimizer of the primal problem (on the right) is also a minimizer of the relaxed problem and $u = 0$ $\mathcal{H}^{n-1}$-a.e. on the set $\{x \in \partial\Omega : [N, \nu_\Omega] < |N|\}$. As before, N always exists and determines the structure of solutions. This time it can be used as a condition to achieve an existence result for $\inf_{u \in BV_0\Omega} I(u)$. That is, a minimizer exists if N is not in the same direction as the normal vector ν_Ω. It should be noted that minimization problem $\inf_{u\in BV_f\Omega} I(u)$ is equivalent to $\inf_{u\in BV_0\Omega} I(u)$. Therefore, we have an existence result to the problem with the Dirichlet boundary condition.

Lastly, we proved existence of minimizers under the barrier condition. We started by generalizing the notion of total variation, perimeter, and area minimizing sets, relative to the integrand $\psi(x,u) = a(x)|Du + F\chi_{E_u}| + Hu$. All the desired properties generalized nicely to work with ψ-total variation, ψ-perimeter, and ψ-area minimizing sets. Then we defined what it means for Ω to satisfy the barrier condition. Intuitively, it is a generalized notion of mean curvature related to ψ, in which the boundary of Ω satisfies a positivity condition. The main existence result is that $\inf_{u\in BV_f(\Omega)} I(u)$ has a minimizer. To prove it, we outlined that $\int_{\mathbb{R}^n} \psi(x,u)$ has a minimizer in $A_f = \{u \in BV(\mathbb{R}^n) : u = f \text{ on } \Omega^c\}$ due to assumed bounds of the functional and H. Finally, it is shown that minimizer $u \in BV_f(\Omega)$, otherwise a construction of a super level set of u would contradict the barrier condition on Ω.

Chapter 3 generally follows the same outline as summarized above. That is, we proved existence of minimizers for $\int_\Omega \varphi(x, Du + F) + Hu$ under prescribed Dirichlet and Neumann boundary conditions. The exception being that as we generalized $a|\cdot|$ to convex function $\varphi(x,\cdot)$, we lost some of the specific concluding Corollaries and Theorems. Namely, Corollary (11) and Theorem (17), which gave us nice information about existence of solutions. The reason for this loss is a more vague solution to the dual problem N. Rather than directly impacting the structure of minimizers, $N = \frac{Du+F}{|Du+F|}$, we got the relation described in Remark (31). It asserts that N determines $\frac{Du+F}{|Du+F|}$ by equality

$$\varphi\left(x, \frac{Du + F}{|Du + F|}\right) = N \cdot \frac{Du + F}{|Du + F|},$$

which indicates $\frac{N \cdot p}{\varphi(x,p)}$ is maximized by $p = \frac{Du+F}{|Du+F|}$, $|Du + F|$−a.e. Furthermore, N has a far more vague relation to u on the boundary of Ω, in the form of $\varphi(x, \nu_\Omega) = [N, sign(-u)\nu_\Omega] \quad \mathcal{H}^{n-1} - a.e.$ on $\partial\Omega$. This still provides us with the jumps of u $\mathcal{H}^{n-1}$-a.e. along the boundary of Ω.

4.1.1 Stability in two and three dimensions

Much work has been done on finding uniqueness of solutions to the P-mean curvature equation. It was proven in [49] under the bounded slope condition. Furthermore, the authors of [13] also achieved uniqueness results. On the other hand, stability has proven quite difficult. Together with the authors of [30], we began the initial steps in adapting their stability results for the CDII problem to the P-mean curvature equation from Chapter 2. See concluding remarks for the difficulty of adapting the problem. Consider the restatement of the equation followed by the desired outcome. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open region with connected boundary, $a \in L^\infty(\Omega)$ is a positive function, $F \in (L^2(\Omega))^n$, and $H \in L^2(\Omega)$. Then u satisfies the equation

$$\nabla \cdot \left(a \frac{\nabla u + F}{|\nabla u + F|} \right) = H, \quad u|_{\partial\Omega} = f, \tag{4.1}$$

with $\sigma = \frac{a}{|\nabla u+F|}$ and $J = -a\frac{\nabla u+F}{|\nabla u+F|}$. Moreover, u minimizes

$$I(w) = \min_{w \in BV_f(\Omega)} \int_\Omega a|\nabla w + F| + Hw\, dx, \tag{4.2}$$

where $a = |J|$, and $BV_f(\Omega) = \{w \in BV(\Omega), w|_{\partial\Omega} = f\}$.

In light of Theorem 12 from Section 2.2, we are guaranteed the solution to the dual problem $N \in \widetilde{\mathcal{D}}_0$, with $|N| \leq a$ such that $a\frac{Du+F}{|Du+F|} = N$, $|Du + F|$-a.e. in Ω, for any minimizer u. For dimensions $n = 2, 3$, suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega} = \tilde{u}|_{\partial\Omega} = f$ and corresponding current density vector fields N and $\tilde{N}$, respectively. We aspire to show

$$\|u - \tilde{u}\|_{L^1(\Omega)} \leq C\||N| - |\tilde{N}|\|^{\frac{1}{2}}_{L^\infty(\Omega)}.$$

With additional assumption on ∇u and the level sets of u, we would also want to show

$$\|\nabla u - \nabla \tilde{u}\|_{L^1(\Omega)} \leq C \parallel |N| - |\tilde{N}| \parallel^{\frac{1}{4}}_{L^\infty(\Omega)} .$$

Under the assumption that $a, \tilde{a} \in C(\Omega)$ with $0 < m \leq a(x), \tilde{a}(x) \leq M, \quad \forall x \in \Omega$, for some constants m, M, we were able to achieve the following three results.

Lemma 43 ([29]) *Let $f \in L^1(\partial\Omega)$, and assume u and $\tilde{u}$ are minimizers of* (4.2) *with the weights a and $\tilde{a}$, respectively. Then*

$$\left| \int_\Omega a|Du + F| + Hu\,dx - \int_\Omega \tilde{a}|D\tilde{u} + F| + H\tilde{u}\,dx \right| \leq C\|a - \tilde{a}\|_{L^\infty(\Omega)}, \tag{4.3}$$

for some constant $C = C(m, M, \Omega, f)$ independent of u and $\tilde{u}$.

Lemma 44 ([29]) *Let $f \in L^1(\partial\Omega)$, and assume u and $\tilde{u}$ are minimizers of* (4.2) *with the weights a and $\tilde{a}$, respectively. Let J and $\tilde{J}$ be the divergence free vector fields guaranteed by Theorem 12. Suppose $0 \leq \sigma(x) = \frac{a(x)}{|Du+F|} \leq \sigma_1 = \frac{\|a\|_{L^\infty(\Omega)}}{\delta}$ in Ω for some constant δ, such that $|Du + F| > \delta > 0$, where σ is the Radon-Nikodym derivative of $|J|\,dx$ with respect to $|Du + F|$. Then*

$$\int_\Omega |J||\tilde{J}| - J \cdot \tilde{J}\,dx \leq C\|a - \tilde{a}\|_{L^\infty(\Omega)}, \tag{4.4}$$

where $C = C(m, M, \sigma_1, \Omega, f, u)$ is a constant independent of $\tilde{a}$.

A way to think about Lemma 44 is that as $a \to \tilde{a}$, $\frac{Du+F}{|Du+F|}(x)$ becomes roughly parallel to $\frac{D\tilde{u}+F}{|D\tilde{u}+F|}(x)$.

Theorem 45 ([29]) *Let $f \in L^1(\partial\Omega)$, and assume u and $\tilde{u}$ are minimizers of* (4.2) *with the weights a and $\tilde{a}$, respectively. Let J and $\tilde{J}$ be the divergence free vector fields guaranteed by Theorem 12. Suppose $0 \le \sigma(x) = \frac{a(x)}{|Du+F|} \le \sigma_1 = \frac{\|a\|_{L^\infty(\Omega)}}{\delta}$ in Ω for some constant δ, such that $|Du+F| > \delta > 0$, where σ is the Radon-Nikodym derivative of $|J|dx$ with respect to $|Du+F|$. Then*

$$\|J - \tilde{J}\|_{L^1(\Omega)} \le C\|a - \tilde{a}\|^{\frac{1}{2}}_{L^\infty(\Omega)}, \tag{4.5}$$

where $C = C(m, M, \sigma_1, \Omega, f, u)$ is a constant independent of $\tilde{a}$.

Note that Theorem 12 tells us $\frac{Du+F}{|Du+F|}$ and $\frac{D\tilde{u}+F}{|D\tilde{u}+F|}$ are parallel to J and $\tilde{J}$, respectively. Thus, if $\tilde{a}$ is close to a, then J is close to $\tilde{J}$, by Theorem 45. Unfortunately, this is where our result differs from those in [30]. We do not share the result that $\tilde{a}$ being close to a implies the structure of level sets of $\tilde{u}$ is close to that of u. This is due to vector field F serving as a sort of perturbation of the normal vector to the level sets of u and $\tilde{u}$. In the case $F \equiv 0$ we maintain the desired structure of level sets. And so, we shift our attention to find stability of minimizers, given perturbations on a and H. Suppose u and $\widetilde{u}$ are minimizers of

$$\inf_w \left(\int_\Omega a|\nabla w| + Hw \right) \qquad \text{and} \qquad \inf_w \left(\int_\Omega \widetilde{a}|\nabla w| + \widetilde{H}w \right)$$

respectively. We endeavor to show

$$\|u - \widetilde{u}\| \le C\left(\|a - \widetilde{a}\| + \|H - \widetilde{H}\| \right),$$

for appropriate norms. This concludes the current state of our work on the problem of stability as applied to the P-mean curvature equation.

4.1.2 Future direction of P-area minimizing surfaces

In consideration of functional (1.13), under the restraints of condition (C_1), it was desirable to allow for convex functions φ of higher growth. To still get an existence result, we could consider the case where φ is strictly convex. It is well established in literature that existence results are easier to come by with such an assumption, and even uniqueness often follows. In fact, the authors of [17] found if convex function g satisfies conditions (1.14) and (1.15), it is not too far from a strictly convex function (in some sense). In the case that $H \equiv 0$, Theorem 5 proves a unique minimizer exists when convex function g grows linearly, as stated in Section 1.2. Furthermore, the authors have developed a similar theorem for the case of superlinear growth of g.

Theorem 46 ([17]) *Let $g : \mathbb{R}^{2n} \to \mathbb{R}$ be a convex function satisfying condition* (1.14) *and let $f : \Omega \to \mathbb{R}$ satisfy the Bounded Slope Condition (Definition 4) of order Q on the boundary of Ω. Assume also that g has superlinear growth, i.e., $g(\xi) \geq \psi(|\xi|)$ for a suitable $\psi : [0, +\infty) \to \mathbb{R}$ such that*

$$\lim_{t \to +\infty} \frac{\psi(t)}{t} = +\infty.$$

Then the functional

$$\mathscr{G}_\Omega(u) = \int_\Omega g(\nabla u + X^*) d\mathcal{L}^{2n}, \qquad u \in f + W_0^{1,1}(\Omega) \tag{4.6}$$

has a unique Lipschitz minimizer, i.e.: there exists $u \in f + W_0^{1,\infty}(\Omega)$ such that $\mathscr{G}_\Omega(u) \leq \mathscr{G}_\Omega(v)$ for every $v \in f + W_0^{1,1}(\Omega)$.

A natural next step would be to develop numerical algorithms to find a minimizer to (4.6), or the more general version (1.13). The approach to finding such a minimizer to the weighted least gradient problem (1.5) was outlined in [39] by my advisor and his collaborators. Although, it was for the case where $F \equiv 0$ and $H \equiv 0$. They proved an alternating split Bregman algorithm will converge to a minimizer with a given non-negative $a \in L^2(\Omega)$ and under the boundary condition $f \in H^{1/2}(\partial\Omega)$. Furthermore, the dual problem provides a way to recover N if first given $|N|$ in the interior and f on the boundary. While this has real applications to conductivity imaging, others in geometry may be interested in similar generalized results for the P-mean curvature equation.

Bibliography

[1] G. Alberti, *A Lusin type theorem for gradients*, J. Funct. Anal., 100 (1991), no.1, 110-118.

[2] M. Amar and G. Bellettini, *A notion of total variation depending on a metric with discontinuous coefficients*, Annales de l'institut Henri Poincaré Analyse non linéaire, 11 (1994), no. 1, 91-133.

[3] G. Anzellotti, *Pairings between measures and bounded functions and compensated compactness*, Ann. Mat. Pura Appl. (4) 135 (1983), 293-318 (1984).

[4] F. Andreu-Vaillo, V. Caselles, and J. M. Mazón, *Parabolic quasilinear equations minimizing linear growth functionals*, Progress in Mathematics, 223. Birkhäuser Verlag, Basel, 2004.

[5] Z.M. Balogh, *Size of characteristic sets and functions with prescribed gradient.* J. Reine Angew. Math. 564 (2003), 63-83.

[6] L. Borcea, *Electrical impedance tomography*, Inverse Problems 18(2002), R99-R136.

[7] P. Bousquet, *Boundary continuity of solutions to a basic problem in the calculus of variations*, Adv. Calc. Var. 3 (2010), 1-27.

[8] P. Bousquet and F. Clarke, *Local Lipschitz continuity of solutions to a problem in the calculus of variations*, J. Differential Equations 243 (2007), 489–503.

[9] A. Cellina, *On the bounded slope condition and the validity of the Euler Lagrange equation*, SIAM J. Control Optim. 40 (2001/02), 1270–1279 (electronic).

[10] M. Cheney and D. Isaacson, *An overview of inversion algorithms for impedance imaging*, Contemp. Math. (1991) 122, 29-39.

[11] M. Cheney, D. Isaacson, and J. C. Newell, *Electrical Impedance Tomography*, SIAM Rev. 41(1999), no.1, 85-101.

[12] J.-H. Cheng and J.-F. Hwang, *Properly embedded and immersed minimal surfaces in the Heisenberg group.* Bull. Aus. Math. Soc., 70 (2004), 507-520.

[13] J.-H. Cheng and J.-F. Hwang, *Uniqueness of generalized P-area minimizers and integrability of a horizontal normal in the Heisenberg group.* Calc. Var. Partial Differential Equations, 50 (2014), no. 3-4, 579-597.

[14] J.-H. Cheng, J.-F. Hwang, A. Malchiodi, and P. Yang, *Minimal surfaces in pseudohermitian geometry.* Annali della Scuola Normale Superiore di Pisa, Classe di Scienze, 4(5) (2005), 129-177.

[15] J.-H. Cheng, J.-F. Hwang, and P. Yang, *Existence and uniqueness for P-area minimizers in the Heisenberg group.* Math. Ann. 337 (2007), no. 2, 253-293.

[16] F. Clarke, *Continuity of solutions to a basic problem in the calculus of variations*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 4 (2005), 511–530.

[17] S. Don, L. Lussardi, A. Pinamonti, and G. Treu, *Lipschitz minimizers for a class of integral functionals under the bounded slope condition.* Nonlinear Analysis, Theory, Methods and Applications, 216 (2022), 112689.

[18] I. Ekeland and R. Témam, *Convex analysis and variational problems*, North-Holland-Elsevier, 1976.

[19] A. Fiaschi and G. Treu, *The bounded slope condition for functionals depending on x, u, and ∇u*, SIAM J. Control Optim., 50 (2012), 991–1011.

[20] B. Franchi, R. Serapioni, and F. Serra Cassano, *Rectifiability and perimeter in the Heisenberg group.* Math. Ann. 321, 479-531 (2001).

[21] N. Garofalo and D.-M Nhie, *Isoperimetric and Sobolev inequalities for Carnot-Caratheodory spaces and the existence of minimal surfaces.* Comm. Pure Appl. Math. 49, (1996), 1081-1144.

[22] E. Giusti, *Minimal Surfaces and Functions of Bounded Variations*, Birkhäuser, Boston, 1984.

[23] E. Giusti, *Direct Methods in the Calculus of Variations*, World Scientific Pub Co Inc, 2003.

[24] W. Górny, *Planar least gradient problem: existence, regularity and anisotropic case*, https://arxiv.org/abs/1608.02617.

[25] N. Hoell, A. Moradifam, and A. Nachman, *Current Density Impedance Imaging with an Anisotropic Conductivity in a Known Conformal Class*, SIAM J. Math. Anal., 46 (2014), 3969-3990.

[26] R.L. Jerrard, A. Moradifam, and A. Nachman, *Existence and uniqueness of minimizers of general least gradient problems*, J. Rein Angew. Math., to appear.

[27] M. L. Joy, G. C. Scott, and M. Henkelman, *In vivo detection of applied electric currents by magnetic resonance imaging*, Magnetic Resonance Imaging, 7 (1989), pp. 89-94.

[28] M. J. Joy, A. I. Nachman, K. F. Hasanov, R. S. Yoon, and A. W. Ma, *A new approach to Current Density Impedance Imaging (CDII)*, Proceedings ISMRM, No. 356, Kyoto, Japan, 2004.

[29] R. Lopez, *Stability of Current Density Impedance Imaging and Uniqueness for the Inverse Sturm-Liouville Problem*, (2021), UC Riverside, PhD dissertation.

[30] R. Lopez and A. Moradifam, *Stability of Current Density Impedance Imaging*, SIAM J. Math. Anal. 52 (2020), no. 5, 4506-4523.

[31] L. Lussardi and E.Mascolo, *A uniqueness result for a class of non strictly convex variational problems*, J. Math. Anal. Appl. 446 (2017), no. 2, 1687–1694.

[32] C. Mariconda and G. Treu, *Existence and Lipschitz regularity for minima*, Proc. Amer. Math. Soc. 130 (2002), 395–404 (electronic).

[33] C. Mariconda and G. Treu, *Lipschitz regularity for minima without strict convexity of the Lagrangian*, J. Differential Equations 243 (2007), 388–413.

[34] C. Mariconda and G. Tre, *Local Lipschitz regularity of minima for a scalar problem of the calculus of variations*, Commun. Contemp. Math. 10 (2008), 1129–1149.

[35] J. M. Mazón, *The Euler–Lagrange equation for the Anisotropic least gradient problem*, Nonlinear Analysis: Real World Applications, 31 (2016) 452-472.

[36] J. M. Mazón, J.D. Rossi, and S.S. De León, *Functions of Least Gradient and 1-Harmonic Functions*, Indiana University Mathematics Journal, 63 (2013) (4): 1067-1084.

[37] A. Moradifam, *Existence and structure of minimizers of least gradient problems*, Indiana University Mathematics Journal 63 (2014), no. 6, 1819-1837.

[38] A. Moradifam, *Least gradient problems with Neumann boundary condition*, J. Differential Equations 263 (2017), no. 11, 7900-7918.

[39] A. Moradifam, A. Nachman, and A. Timonov, *A convergent algorithm for the hybrid problem of reconstructing conductivity from minimal interior data*, Inverse Problems, 28 (2012) 084003.

[40] A. Moradifam, A. Nachman, and A. Tamasan, *Conductivity imaging from one interior measurement in the presence of perfectly conducting and insulating inclusions*, SIAM J. Math. Anal., 44 (2012) (6), 3969-3990.

[41] A. Moradifam, A. Nachman, A. Tamasan, *Uniqueness of minimizers of weighted least gradient problems arising in hybrid inverse problems*, Calc. Var. Partial Differential Equations **57** (2018), no. 1, Art. 6, 14 pp.

[42] A. Moradifam, A. Rowell, *Existence and structure of solutions for general P-area minimizing surfaces*, Calc. Var. Partial Differential Equations, (2022). Submitted.

[43] A. Moradifam, A. Rowell, *Existence and structure of P-area minimizing surfaces in the Heisenberg group*, Journal of Differential Equations, 342 (2023), 325-342.

[44] A. Nachman, A. Tamasan, and A. Timonov, *Conductivity imaging with a single measurement of boundary and interior data*, Inverse Problems, 23 (2007), pp. 2551–2563.

[45] A. Nachman, A. Tamasan, and A. Timonov, *Recovering the conductivity from a single measurement of interior data*, Inverse Problems, 25 (2009) 035014 (16pp).

[46] A. Nachman, A. Tamasan, and A. Timonov, *Reconstruction of Planar Conductivities in Subdomains from Incomplete Data*, SIAM J. Appl. Math., 70(2010), Issue 8, pp. 3342–3362.

[47] A. Nachman, A. Tamasan, and A. Timonov, *Current density impedance imaging*, Tomography and inverse transport theory, 135-149, Contemp. Math. 559, AMS, 2011.

[48] S.D. Pauls, *Minimal surfaces in the Heisenberg group.* Geometric Dedicata, 104 (2004), 201-231.

[49] A. Pinamonti, F. Serra Cassano, G. Treu, and D. Vittone, *BV minimizers of the area functional in the Heisenberg group under the bounded slope condition.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 14 (2015), no. 3, 907-935.

[50] P. Sternberg, G. Williams, and W. P. Ziemer, *Existence, uniqueness and regularity for functions of least gradient*, J. Rein Angew. Math. 430 (1992), 35-60.

[51] P. Sternberg and W. P. Ziemer, *Generalized motion by curvature with a Dirichlet condition*, J. Differ. Eq., 114(1994), pp. 580–600.

[52] P. Sternberg and W. P. Ziemer, *The Dirichlet problem for functions of least gradient. Degenerate diffusions* (Minneapolis, MN, 1991), 197–214, in IMA Vol. Math. Appl., 47, Springer, New York, 1993.

[53] G. S. Spradlin and A. Tamasan, *Not all traces on the circle come from functions of least gradient in the disk*, Indiana University Mathematics Journal 63 (2014), no. 6, 1819-1837.